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Persistence Equivalence of Discrete Morse Functions on Trees

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Abstract

We introduce a new notion of equivalence of discrete Morse functions on graphs called persistence equivalence. Two functions are considered persistence equivalent if and only if they induce the same persistence diagram. We compare this notion of equivalence to other notions of equivalent discrete Morse functions. We then compute an upper bound for the number of persistence equivalent discrete Morse functions on a fixed graph and show that this upper bound is sharp in the case where our graph is a tree. We conclude with an example illustrating our construction.

1 Introduction

Since its inception in the early 2000s, persistent homology has almost single handedly brought topology to the forefront of mathematics. A panacea of sorts, persistence has been used to study statistical mechanics [13], hypothesis testing [6], image analysis [7], complex networks [12], and many other phenomena. Part of the utility of persistent homology is that it can be used to recreate or estimate some unknown. For example, there have been several recent results allowing one to reconstruct a simplicial complex from certain collections of persistence diagrams [14, 5].

A close cousin of persistent homology [4], discrete Morse theory is a topological tool due to Robin Forman [10, 11] that can be used to simplify a simplicial complex. Among other things, a discrete Morse function on a simplicial complex naturally gives rise to a filtration, and a filtration gives rise to a persistence diagram. The purpose, then, of this paper is to introduce and study a new notion of equivalence of discrete Morse functions on graphs (although this may be defined on any simplicial complex) called **persistence equivalence**. Two discrete Morse functions on a graph G are called persistence equivalent if $D_f = D_g$ where D_f is the persistence diagram induced by f. We then count and construct all discrete Morse functions up to persistence equivalence on a fixed tree. This is a version of the realization problem, recently studied in the smooth case by Curry [9] where he investigated a notion that he called graph-equivalence (not

to be confused with the notion of graph equivalence in Section 3.3).

The idea is that it is easy to compute the persistent homology of a discrete Morse function on a tree without having to resort to a matrix. This is done in Lemma 4.2.2. The critical vertices birth new components while a critical edge connecting two trees kills a bar corresponding to the tree whose minimum value is greater than the minimum value of the other tree. We give the background in discrete Morse theory and persistence homology and introduce persistence equivalence in Section 2. In Section 3, we compare our new notion of equivalence with ones currently found in the literature. Using a slight variation of the standard definition of a discrete Morse functions in order to ensure finiteness, we will count the total number of discrete Morse functions with a fixed number of critical values up to persistence equivalence on a tree. This is accomplished by providing a combinatorial upper bound in Corollary 4.1.3. Theorem 4.2.5 then provides a method to construct any such discrete Morse function with a desired barcode. We end in Section 4.3 with an example illustrating our construction.

2 Graphs, discrete Morse theory, and persistence

In this section, we introduce the background and notation that is needed throughout the body of this paper. We begin by reviewing the basics of graph theory.

2.1 Graphs

A **graph** $G = (V, E)$ is a non-empty finite set V along with a symmetric, irreflexive relation E on V. The set $V = V(G)$ is called the **vertex set** of G while $E = E(G)$ is called the **edge set**. If $(u, v) \in E$, we write $e = uv$ or $u, v < e$ to denote the edge e with **endpoints** u and v. In this case, we say that u and v are **adjacent** while e and u are **incident**. We will use **simplex** (plural: **simplices**) to refer to a vertex or an edge of G, and use a Greek letter such as σ to denote either a vertex or an edge.

A **path** in G is a list $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$ of vertices and edges such that the edge e_i has endpoints v_i and v_{i+1} for $1 \le i \le k$. We further require that no edge is repeated. If there is a path between any two vertices of G, we define G to be **connected**.

A **cycle** in a graph is a path with at least three edges that begins and ends at the same vertex and never repeats a vertex (other than the starting and ending vertex). In other words, a **cycle** is a path $v_1, e_1, v_2, \ldots, e_k, v_{k+1}$ such that no vertex is repeated other than $v_1 = v_{k+1}$.

Because our main construction in Theorem 4.2.5 is on a special kind of graph called a tree, we recall several important characterizations of trees. They will

be utilized without further reference.

Theorem 2.1.1. *(Characterization of trees) Let G be a connected graph with v vertices and e edges. The following are equivalent:*

  a) Every two vertices of G are connected by a unique path.

  b) $v = e + 1$

  c) G contains no cycles

  d) $b_1(G) = 0$

  e) The removal of any edge from G results in a disconnected graph.

A connected graph that satisfies any of the above characterizations is called a **tree**. Proofs of the equivalence of the statements may be found in any graph theory textbook(e.g. [8, Chapter 2.2]).

2.2 Discrete Morse theory on graphs

We use a slightly more restrictive definition of a discrete Morse function than is normally given in the literature. For a discussion on the reason for the choices made in the definition, see Remark 2.3.2.

Definition 2.2.1. Let G be a graph with n vertices and edges, $f \colon G \to [0, n]$ a function. Then f is **monotone** if whenever $v < e$, then $f(v) \leq f(e)$. We say f is a **discrete Morse function** if f is a monotone function with $\min(f) = 0$ which is at most $2 - 1$ where if $f(v) = f(e)$, then $v < e$. Furthermore, we require that if f is 1-1 on $f(\sigma)$, then $f(\sigma) \in \mathbb{N}$. Such a value is called a **critical value** and σ is a **critical vertex** or **critical edge.** If σ is not critical, σ is called **regular**. If a vertex v is incident with exactly one edge e, the pair $\{v, e\}$ is called a **free pair**.

Under this definition, it is easy to see that $\min\{f\} = 0$ will always be a critical value. It is also not difficult to show [10, Lemma 2.5] that regular simplices come in pairs. Embedded in Definition 2.2.1 is the fact that the vertex/edge regular pair is given the same value under the discrete Morse function. This condition is called **flat**. For those concerned that our definition is too restrictive, Uli Bauer has shown that every discrete Morse function is homologically equivalent to one which is flat [4, Proposition 2.19].

One of the fundamental results in discrete Morse theory is the (weak) discrete Morse inequalities, relating the number of critical simplices of a discrete Morse function to the Betti numbers. We will utilize this theorem in Lemma 4.1.1.

Theorem 2.2.2. *[10, Cor 3.7](Weak Morse inequalities for graphs) Let G be a graph and f a discrete Morse function of G with the number of critical i-simplices of f denoted by m_i, $i = 0, 1$. Then*

(i) $m_0 \geq b_0$ and $m_1 \geq b_1$ where b_i denotes the ith Betti number of G.
(ii) $b_0 - b_1 = m_0 - m_1$.

Let $f\colon G \to [0, n]$ be a discrete Morse function. For any $a \in \mathbb{R}$, we define **level subcomplex** a by $G_a := \{\sigma \in G\colon f(\sigma) \leq a\}$. Note that by the fact that f is flat, G_a is always a subgraph. If $0 = c_0 < c_1 < \ldots < c_{m-1}$ are the critical values of f, we consider the sequence of subcomplexes

$$\{v\} = G_{c_0} \subseteq G_{c_1} \subseteq \ldots \subseteq G_{c_{m-1}}$$

called the **filtration of G induced by the discrete Morse function f**. This induced filtration will be used in the next section.

2.3   Persistent homology

Let G be a graph. Suppose we have a filtration

$$G_0 \subseteq \ldots \subseteq G_{m-1}.$$

For $i \leq j$, there is an inclusion function $f^{i,j}\colon G_i \to G_j$. Passing to homology, we obtain a linear transformation $f_p^{i,j}\colon H_p(G_i) \to H_p(G_j)$. The pth **persistent homology groups**, denoted $H_p^{i,j}$, is defined by $H_p^{i,j} := \mathrm{im}(f_p^{i,j})$. The pth-**persistent Betti numbers** are the corresponding Betti numbers, $\beta_p^{i,j} := \mathrm{rank} H_p^{i,j}$.

A class $[\alpha] \in H_p(G_i)$ is said to be **born at G_i or at time i** if $[\alpha]$ is not in the image of $f_p^{i-1,i}$. A class $[\alpha] \in H_p(G_i)$ is said to **die at G_{i+1} or at time $i+1$** if $f_p^{i,j}([\alpha])$ is not in the image of $f_p^{i-1,j}$ but $f_p^{i,j+1}([\alpha])$ is in the image of $f_p^{i-1,j+1}$. If α is born at i and dies at j, we call (i, j) a **persistence pair**. If σ is born and never dies, then σ is called a **point at infinity**. Plotting all persistence pairs in the Euclidean plane along with all points at infinity (represented by a y-value greater than the maximum death time) yields the **persistence diagram**, denoted D.

The persistence diagram thus records the lifetime of a topological feature via the plotting of a single point. The value on the x-axis represents the time the topological feature is born; the value on the y-axis represents its time of death. A point at height $y = \infty$ means that the topological feature was born and never died. Note that it is not possible to have any points beneath the diagonal, since we can't have a point born later than it died. Hence it is customary to include the diagonal $x = y$.

As noted above, discrete Morse function with critical values $c_0 < c_1 < \ldots c_{m-1}$ induces a filtration

$$G_{c_0} \subseteq G_{c_1} \subseteq \ldots G_{c_{m-1}}$$

where G_{c_i} is the level subcomplex of G at level c_i. Hence, a discrete Morse function f induces a persistence diagram D_f.

We are now ready to give the main object of study in this paper.

Definition 2.3.1. Two discrete Morse functions $f, g \colon G \to [0, n]$ are **persistence equivalent** if $D_f = D_g$.

Another viewpoint that we will adopt in this paper is to consider two function persistence equivalent if their corresponding barcodes are equal, where equality is given up to permutation of the vertical stacking of the bars.

Remark 2.3.2. *When computing persistence, the definition of a discrete Morse function (Definition 2.2.1) ensures that all births and deaths (corresponding to critical values) occur only at integer values and that furthermore, the first birth occurs at time 0 and that the barcode is completed by time n, where n is the number of simplices of G. In addition, there can be at most one event (either a birth or a death) at any time. So for example, the following could be a barcode induced by a discrete Morse function on a graph with 11 simplices*
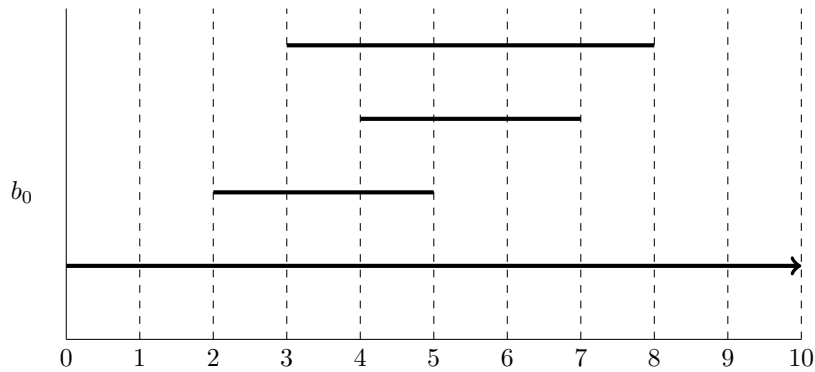
but this one could not

5
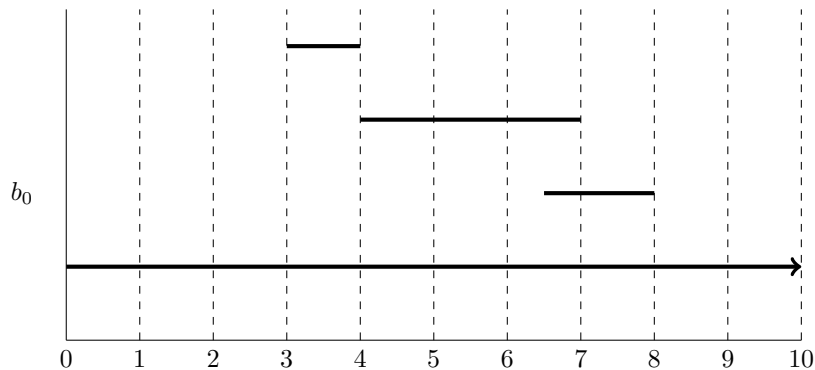
since there is a birth and a death at $t = 4$ and furthermore, there is a birth at $t = 6.5 \notin \mathbb{Z}$.

Hence, given these conditions, the total number barcodes that one can obtain from a discrete Morse function on a fixed graph is finite. In Section 4, we will give an upper bound for the number of persistence equivalent discrete Morse functions on a graph, and show that this estimate is sharp in the special case where $G = T$ is a tree. Bur first, we compare persistence equivalence with other notions of equivalence.

# 3   Relation with other notions of equivalence

3.1   Forman equivalence

Recall that two discrete Morse functions $f, g \colon G \to [0, n]$ defined on a graph are **Forman equivalent** if and only if $V_f = V_g$, where V_f is the induced gradient vector field of f [10]. It is easy to see that neither persistence nor Forman equivalence imply each other.

Example 3.1.1. *The following two discrete Morse functions are persistence equivalent but not Forman equivalent.*
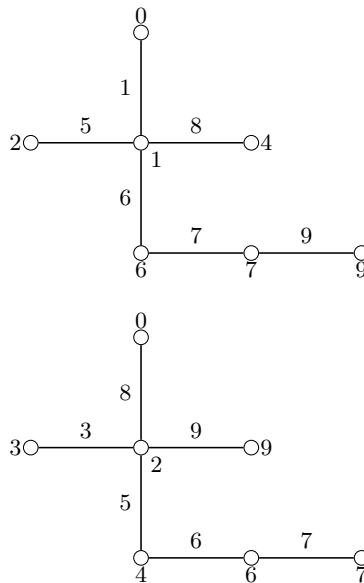
Using the exact same graphs, the example below also shows that Forman equiv-

alence does not imply persistence equivalence.
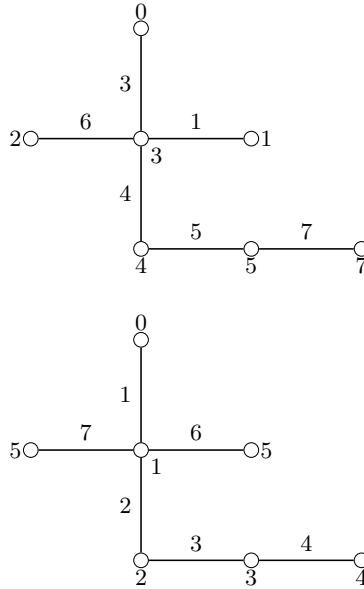
3.2 Homological equivalence

In [2], Ayala et al. introduced the notion of horological equivalence and counted
the number of discrete Morse functions up to homological equivalence in [3] on
all graphs.

Definition 3.2.1. Two discrete Morse functions f and g defined on a graph G
with critical values $a_0 < a_1 < \ldots < a_{m-1}$ and $c_0 < c_1 < \ldots < c_{m-1}$ respectively
are **homologically equivalent** if $b_0(a_i) = b_0(c_i)$ and $b_1(a_i) = b_1(c_i)$ for all
$0 \leq i \leq m-1$.

From the definitions, the following is immediate.

Proposition 3.2.2. *If f and g are persistence equivalent, then f and g are
homologically equivalent.*

Of course, the converse is clearly false, as the following simple example illus-
trates.

Example 3.2.1.
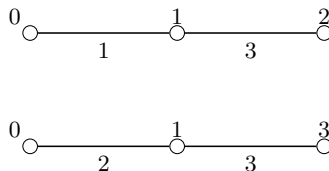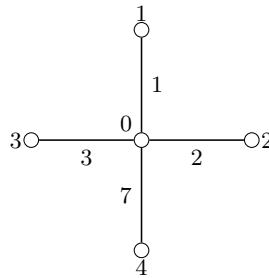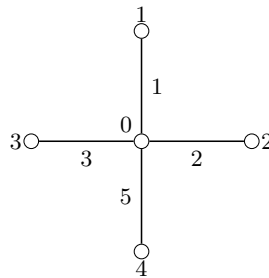
3.3 Graph equivalence

The second author introduced the following notion of equivalence of discrete Morse functions on graphs in [1].

Definition 3.3.1. Let $f, g\colon G \to [0, n]$ be two discrete Morse functions on a graph G with critical values $a_0, a_1, ..., a_{m-1}$ and $c_0, c_1, ..., c_{m-1}$ respectively. The functions f and g are said to be **graph equivalent** if $G(a_i) \cong G(c_i)$ for every $0 \leq i \leq m-1$; that is, each level subcomplex is isomorphic as graphs.

As noted in the introduction, this is not to be confused with the notion of graph equivalence recently introduced by Curry [9].
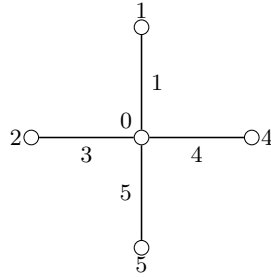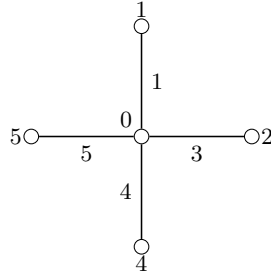
Although graph equivalence is quite stringent, two discrete Morse functions which are graph equivalent are not necessarily persistence equivalent.

Example 3.3.1.

Conversely, persistence equivalence does not imply graph equivalence.

Example 3.3.2.

4 Counting persistence equivalence classes

4.1 An upper bound

We first prove an upper bound for the number of persistence equivalence classes for any connected graph. In general, this upper bound is not sharp, as we illustrate in Example 4.1.1. However, we will see in Theorem 4.2.5 that this upper bound is sharp on a certain class of graphs, namely, trees. First, a lemma.

Lemma 4.1.1. *Let $f\colon G \to [0, n]$ be a discrete Morse function with m critical values on a connected graph G. Then $m = 1 + b_1(G) + 2k$ for some $k \in \mathbb{Z}$.*

Proof. Let $m = m_0 + m_1$, and suppose that $m = 2j + 1$, as the case when m is even is similar. By the Theorem 2.2.2 (i), $m_0 \geq b_0$ and $m_1 \geq b_1$ so that $m = b_0 + b_1 + h$. By part (ii) of that same theorem, $b_0 - b_1 = m_0 - m_1$. Adding this to $2j + 1 = m_0 + m_1$, we obtain $2j + 2 - b_1 = 2m_0$ so that $b_1 = 2\ell$ is even. But if G is connected, $b_0 = 1$ and $2j + 1 = m = 1 + 2\ell + h$. Hence h is even. $\square$

Proposition 4.1.2. *Let G be a connected graph on n vertices, and let $m := 1 + b_1 + 2k$ where $m \leq n$. Then there are at most*

$$\frac{\binom{n-1}{b_1}\binom{n-1-b_1}{2}\binom{n-1-b_1-2}{2}\binom{n-1-b_1-4}{2}\cdots\binom{n-1-b_1-2k+2}{2}}{k!}$$

persistence equivalence classes of discrete Morse functions with m critical values on G.

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Proof. We compute the upper bound by counting all possible barcodes that could be obtained. The minimum value (in this case 0) of every discrete Morse function corresponds to a critical vertex, which in turn induces a birth at $t = 0$, leaving $n - 1$ other times for births and deaths. Since each independent cycle is born and never dies, we have $\binom{n-1}{b_1}$ choices of times to birth cycles. The other $2k$ critical values correspond to persistence pairs. For the first persistence pair, we choose a birth and death time from the remaining $n - 1 - b_1$ options for $\binom{n-1-b_1}{2}$ options. There are then $\binom{n-1-b_1-2}{2}$ options for the next persistence pair. Continuing in this manner, we obtain

$$\binom{n-1}{b_1}\binom{n-1-b_1}{2}\binom{n-1-b_1-2}{2}\binom{n-1-b_1-4}{2}\cdots\binom{n-1-b_1-2k+2}{2}.$$

However, the order in which we choose birth death pairs does not matter, so we must divide by the number of permutations on the number of persistence pairs chosen i.e., divide by $k!$. Thus the result. $\square$

Unfortunately this result is not sharp for all graphs.

Example 4.1.1. *Let C_6 be a cycle of length 6, and consider the barcode*
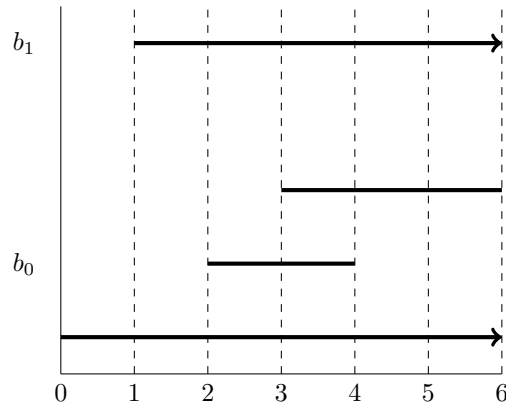
This barcode is certainly counted as a possibility in Proposition 4.1.2. However, it cannot be obtained on C_6 since in order to have the cycle born at time 1, the entire graph must be built and hence, it is impossible to have any more births and deaths after the cycle is born. Thus, there is only one barcode with the cycle born at $t = 1$ on a cycle of any length.

In the special case of trees, we obtain

Corollary 4.1.3. *Let T be a tree on n simplices, $m = 2k + 1$ an integer $1 \leq m \leq n$. Then there are at most*

$$\frac{(n-1)(n-2)(n-3)\ldots(n-2k)}{2^k k!}$$

persistence equivalence classes of discrete Morse functions on T with m critical values.

Proof. By Theorem 2.1.1, $b_1(T) = 0$ so that Proposition 4.1.2 becomes

$$\frac{\binom{n-1}{2}\binom{n-3}{2}\dots\binom{n-2k+1}{2}}{k!}.$$

Observe that $\binom{n-\ell}{2} = \frac{(n-\ell)(n-\ell-1)}{2}$. Hence, replace each factor in the product and simplify to obtain

$$\frac{(n-1)(n-2)(n-3)\dots(n-m+2)(n-2k)}{2^k k!}.$$

$\square$

As we will see in Theorem 4.2.5, this upper bound is attained. The next section is devoted to constructing such discrete Morse functions.

4.2 Counting on trees

We begin by fixing some notation. Let T be a tree on n simplices. We say that a persistence diagram $D = \{(c_i, d_i)\}$ is **consistent with** T if $(0, \infty) \in D$ and for all c_i, d_j appearing in ordered pairs in $D - \{(0, \infty)\}$, we have

1. $c_i, d_j \in \mathbb{Z}$

2. $1 \le c_i, d_j \le n$

3. $c_i < d_i$, $c_i < c_j$, and $d_i < d_j$ for all $i < j$

4. All c_i, d_j are distinct.

Given a tree T and a persistence diagram D consistent with T, we will show that there exists a discrete Morse function on T such that $D_f = D$.

Definition 4.2.1. Let $f \colon T \to [0, n]$ be a discrete Morse function. For a fixed level subcomplex T_{c_j}, let $S[v]$ denote the tree of T_{c_j} whose minimum critical vertex is v.

The following Lemma allows us to compute the persistence diagram using the critical values of the discrete Morse function.

Lemma 4.2.2. *Let $f \colon T \to [0, n]$ be a discrete Morse function on a tree. Then v is born at c_i if and only if $f(v) = c_i$ is critical.*

Furthermore, v dies at c_j if and only if there exists a critical edge e with $f(e) = c_j$ where e joins trees $S[v], S[u]$ in T_{c_j} with $f(u) < f(v)$.

In other words, when two trees are joined by a critical edge, the vertex that dies is the one that belongs to the tree with larger value.

Proof. Suppose vertex v is born at T_{c_i}. Then $[v] \notin \operatorname{im}(f_0^{c_{i-1},c_i})$. We claim that v is critical. If not, then v is regular and hence part of a free pair, $v < e$. Write $e = uv$, and suppose that $u \in T_{c_{i-1}}$. Then $f_0^{c_{i-1},c_i}([u]) = [v]$, a contradiction. If $u \notin T_{c_{i-1}}$, then u is part of a free pair, and the result follows inductively. For the converse, we clearly have that $[v] \notin \operatorname{im}(f_0^{c_{i-1},c_i})$ since $v \notin T_{c_{i-1}}$ and there does not exist a path from v to any other vertex in T_{c_i}.

Suppose that v dies at T_{c_j} and write $f(e) = c_j$. For the same reason as above, e must be critical. Now e connects two trees $S(v)$ and $S(u)$ which were disconnected in $T_{c_{j-1}}$. By definition of v dying at c_j, $f_0^{c_i,c_j}(v)$ is the class $[u]$ that existed at $T_{c_{i-1}}$. Since $f(\sigma) \leq c_{i-1}$ for every $\sigma \in T_{c_{i-1}}$, $f(u) \leq c_{i-1} < f(v)$. $\qquad\square$

Our construction will break a tree down into a forest and build a discrete Morse function with a single critical value on each tree of the forest. The following lemma allows us to accomplish this. For the purposes of this lemma, we allow a discrete Morse function to take on any non-negative value for its minimum.

Lemma 4.2.3. *Given a tree T and a vertex $v \in T$, a single value $f \colon \{v\} \to [f(v), n]$, and some number n_0 satisfying $n \geq n_0 > f(v) \geq 0$, we can extend $f \colon T \to [f(v), n]$ such that f is a discrete Morse function with unique critical simplex v and $\max(f) < n_0$.*

Proof. If T is a tree with a single vertex, it follows that $f(v) < n_0$. Otherwise, let T be a tree with ℓ vertices and let n_0 be as above. Define $\alpha_0 := f(v) < \alpha_1 < \ldots < \alpha_\ell$ recursively by $\alpha_i := \frac{n_0 + \alpha_{i-1}}{2}$ for $1 \leq i \leq \ell$. We show that $f(v) < \alpha_i < n_0$. We prove this by induction on i. When $i = 1$, $\alpha_1 = \frac{f(v)+n}{2}$. So it is obvious that $f(v) < \alpha_1 < n_0$. When $i > 1$, assume that our claim holds for $i = k$, we show it holds for $i = k + 1$. Now we have $f(v) < \alpha_k < n_0$, and $\alpha_{k+1} = \frac{n_0 + \alpha_k}{2}$, so $\alpha_k < \alpha_{k+1} < n$, and hence $f(v) < \alpha_{k+1} < n_0$. Therefore by induction, we have $f(v) < \alpha_i < n_0$.

Now we wish to extend f to all of T. Define f by the following:

    i. for any vertex $u \neq v$, label $f(u) := \alpha_{d(v,u)} + f(v)$

    ii. for any edge $e = uw$, label $f(e) := \max\{f(u), f(w)\}$.

We still need to verify that f is a discrete Morse function with unique critical simplex v and $\max(f) < n_0$.

Clearly, v is critical since for any edge $e = uv$, we have $f(e) = \max\{f(u), f(v)\} = \alpha_{(v,u)} + f(v) > f(v)$.

Next, we show that any vertex $u \neq v$ is regular. Let $v_1, \ldots, v_j$ be the neighbors of u. We claim that $f(u) < f(v_i)$ for all i other than exactly one value. Since T is a tree, there is a unique path from v to u. Moreover, this path must pass through exactly one of the v_i. Now the unique path from v to any other

neighbor v_k of u, must go through u. Otherwise, we obtain a cycle. That indicates $d(v, v_i) < d(v, u) < d(v, v_k)$ for some i with $1 \leq i \leq j$ and for all k with $1 \leq k \leq j$, $k \neq i$. It follows that $f(v_i) < f(u) < f(v_k)$. Hence $f(u) < f(v_i)$ for all but exactly one value and we know that all vertices $u \neq v$ are regular.

Finally, for any edge $e = uw$ in T, $f(e) = \max\{f(u), f(w)\}$. Hence we need to show that $f(u) \neq f(w)$. By contradiction, assume that $f(u) = f(w)$. Then $d(u, v) = d(u, w)$, which implies that the path from v to u and the path from v to w along with edge e, is a cycle, a contradiction. We conclude that f is a discrete Morse function with the desired properties. $\qquad\square$

The following lemma is clear.

Lemma 4.2.4. *Let T be a tree, $E := \{e_1, e_2, \ldots, e_k\}$ a set of edges of T, and $F := T - E$ the resulting forest. Let $\widetilde{T}$ be any tree in F. Then there exists an edge $e \in E$ with one endpoint in $\widetilde{T}$, and the other endpoint in a different tree of the forest.*

We now come to our main result.

Theorem 4.2.5. *Let T be a tree, $D := \{(c_i, d_i)\}$ a persistence diagram consistent with T. Then there exists a discrete Morse function $f : T \to [0, n]$ such that $D_f = D$.*

Proof. Let T be a tree and $D := \{(c_i, d_i)\}$ a persistence diagram consistent with T. Order and label the values in each persistence pair of the persistence diagram as $0 = a_0 < a_1 < \ldots < a_{m-1} < a_m = \infty$. Remove any $\frac{m-1}{2}$ edges E from T and write $F := T - E$. We will label T with a discrete Morse function by inducing on c_i, the birth times, and continually extending the function f on subgraphs of T until it is defined on all of T.

For $c_0 = 0$, pick any tree T_0 in F and any vertex v_0 of T_0. Define $f(v_0) := 0$. Applying Lemma 4.2.3 on T_0 with $n_0 = a_1$, we obtain a discrete Morse function on T_0 with v_0 the unique critical vertex. In the case where $a_1 = \infty$, pick any finite $n > 0$ to obtain a labeling of the entire tree T.

In the case where $c_1 = a_1 \neq \infty$, let $(c_1, d_1) \in D$. Apply Lemma 4.2.4 on T and E with $\widetilde{T} = T_0$ to obtain an edge e_1 joining T_0 and an unlabeled tree T_1. Pick any $v_1 \in T_1$ and again applying Lemma 4.2.3 on T_1 with $f(v_1) := c_1$ and $n_0 = a_2$. Then we extend f to obtain a discrete Morse function on $T_0 \cup T_1$ with v_1 also critical. Furthermore, since $\max f < a_2$ and $a_2 \leq d_1$, if we label $f(e_1) := d_1$, it follows that e_1 is a critical edge.

In general, let $(c_i, d_i) \in D$ with $i > 1$ and suppose that $c_i = a_j$. We again apply Lemma 4.2.4 to T on $\widetilde{E}$, the subset of E consisting of currently unlabeled edges, and let $\widetilde{F} := T - \widetilde{E}$ be the resulting forest. Then there is a unique tree $\widetilde{T}$ of $\widetilde{F}$ which is labeled by f. The Lemma then guarantees that there is an edge e_i with

one endpoint in $\widetilde{T}$ and the the other in a different (unlabeled) tree T_i. Choosing a vertex $v_i \in T_i$, label $f(v_i) := c_i$, and apply Lemma 4.2.3 with $n := a_{j+1}$ to obtain a discrete Morse function on $\widetilde{T} \cup T_i$. Furthermore, since $\max\{f\} < a_j$ and $a_j \leq d_i$, if we label $f(e_i) := d_i$, it follows that e_i is a critical edge. In this way, we obtain a discrete Morse function on all of T. Using the language of Definition 4.2.1, note that $T_i = S[v_i]$ by the above construction.

Finally, we need to show that $D_f = D$. For this, we apply Lemma 4.2.2. The critical vertices are exactly those labeled c_i so that the birth times are correct. Consider level subcomplex T_{d_i}. By construction, $d_i = f(e_i)$ with edge e_i joining trees $S[v_i]$ and $S[v_j]$ and $f(v_j) < f(v_i)$. Hence v_i dies at d_i i.e. (c_i, d_i) is a persistence pair of D_f. $\qquad\square$

4.3   An example

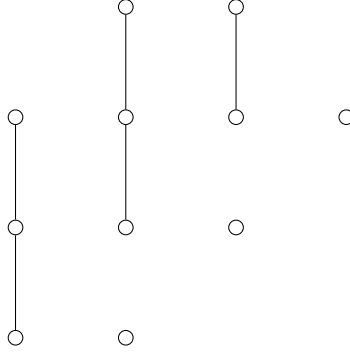We illustrate the construction given in Theorem 4.2.5. Let T be the tree
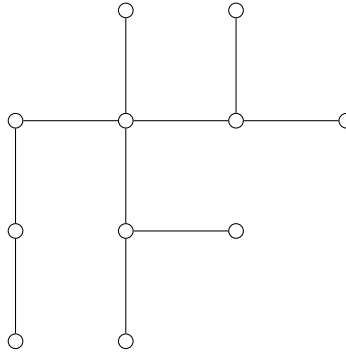

and barcode given by
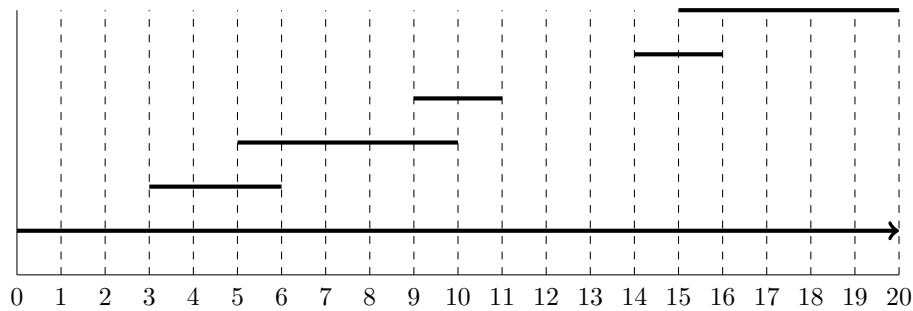
We will use the method of Theorem 4.2.5 to construct a discrete Morse function f of T that induces the above barcode. Following the proof, we first order the critical values from the barcode

$$0 < 3 < 5 < 6 < 9 < 10 < 11 < 14 < 15 < 16 < 20 < \infty.$$

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Next we remove $\frac{m-1}{2} = \frac{11-1}{2} = 5$ edges from T to obtain the forest F below.


Our first step in labeling is to pick a tree in F and label a vertex 0. We then apply Lemma 4.2.3 with $n_0 = a_1 = 3$ to obtain
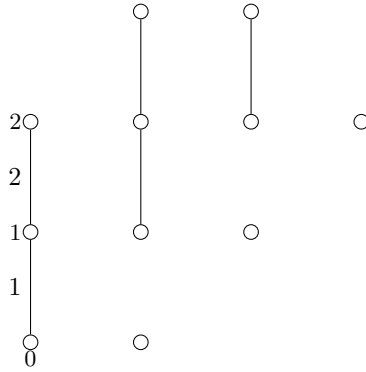

The first persistence pair with both values finite is $(3, 6)$. By Lemma 4.2.4, there is a removed edge connected the labeled tree with an unlabeled tree (in this case, there is only one such edge). We pick a vertex in the unlabeled tree and label it 3. Applying the same Lemma as above with $n_0 = a_2 = 5$ and furthermore labeling the connecting edge by 6, we have
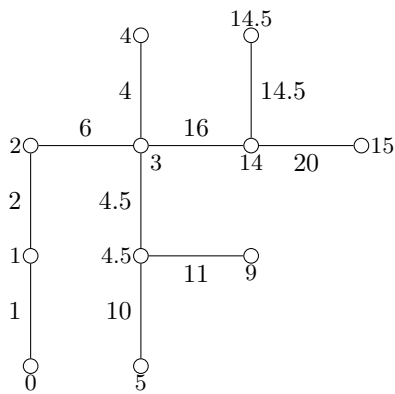
4○

4

6
2○———○
3

2   4.5

1○   4.5○

1

○
0

○

○

○

○
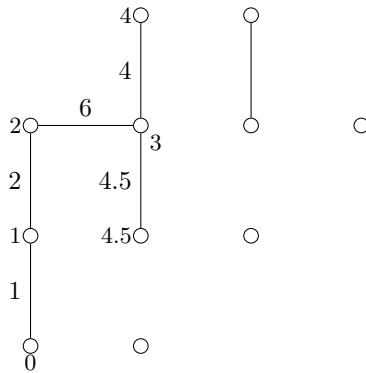
We continue in this manner. The next persistence pair is $(5, 10)$. Choosing the lowest tree, we label the sole vertex 5 and the edge 10.

4○   ○

4

6
2○———○
3

2   4.5

1○   4.5○   ○

1   10

○   ○
0   5
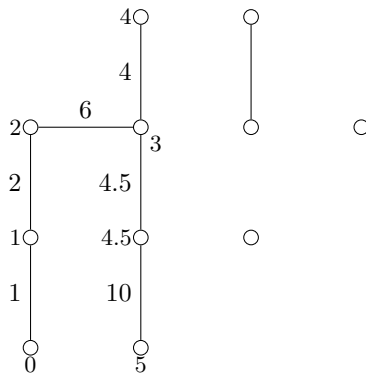
Three more iterations of this step yields the following discrete Morse function.

14.5
○

4○   14.5

4   16
6
2○———○———○———○15
3   14   20

2   4.5

1○   4.5○———○
11   9

1   10

○   ○
0   5

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While Theorem 4.2.5 gurantees this discrete Morse function induces the desired barcode, it can alos be checked by hand.

References

[1] S. E. Aaronson, M. E. Meyer, N. A. Scoville, M. T. Smith, and L. M. Stibich, *Graph isomorphisms in discrete Morse theory*, AKCE Int. J. Graphs Comb. **11** (2014), no. 2, 163–176. MR 3243115

[2] R. Ayala, L. M. Fernández, D. Fernández-Ternero, and J. A. Vilches, *Discrete Morse theory on graphs*, Topology Appl. **156** (2009), no. 18, 3091–3100.

[3] R. Ayala, D. Fernández-Ternero, and J. A. Vilches, *The number of excellent discrete Morse functions on graphs*, Discrete Appl. Math. **159** (2011), no. 16, 1676–1688.

[4] U. Bauer, *Persistence in discrete morse theory*, Ph.D. thesis, Göttigen, 2011.

[5] R. L. Belton, B. T. Fasy, R. Mertz, S. Micka, D. L. Millman, D. Salinas, A. Schenfisch, J. Schupbach, and L. Williams, *Learning Simplicial Complexes from Persistence Diagrams*, ArXiv e-prints (2018).

[6] A. J. Blumberg, I. Gal, M. A. Mandell, and M. Pancia, *Robust statistics, hypothesis testing, and confidence intervals for persistent homology on metric measure spaces*, Found. Comput. Math. **14** (2014), no. 4, 745–789.

[7] G. Carlsson, *Topology and data*, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 2, 255–308.

[8] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & digraphs*, sixth ed., Textbooks in Mathematics, CRC Press, Boca Raton, FL, 2016.

[9] J. Curry, *The Fiber of the Persistence Map*, ArXiv e-prints (2017).

[10] R. Forman, *Morse theory for cell complexes*, Adv. Math. **134** (1998), no. 1, 90–145.

[11] _____, *A user's guide to discrete Morse theory*, Sém. Lothar. Combin. **48** (2002), Art. B48c, 35.

[12] N. A. Scoville and K. Yegnesh, *A Persistent Homological Analysis of Network Data Flow Malfunctions*, Journal of Complex Networks **6** (December 2017), no. 1, 884–892.

[13] G. Spreemann, B. Dunn, M. B. Botnan, and N. A. Baas, *Using persistent homology to reveal hidden covariates in systems governed by the kinetic Ising model*, Phys. Rev. E **97** (2018), no. 3, 032313, 14.

[14] K. Turner, S. Mukherjee, and D. M. Boyer, *Persistent homology transform for modeling shapes and surfaces*, Inf. Inference **3** (2014), no. 4, 310–344.