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From Sets to Metric Spaces to Topological Spaces

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1 Introduction

Felix Hausdorff (1868–1942) is known as one of the founders of modern topology, due in part to his systematic treatment of topological spaces in the influential textbook *Grundzüge der Mengenlehre* (*Fundamentals of Set Theory*) [Hausdorff, 1914]. This text, one of the first book-length treatments of set theory, was instrumental in establishing set theory as a foundation for the study of other mathematical fields—a viewpoint that is very familiar to us today, but was still gaining ground in the early 20th century.[1] After devoting the first seven chapters to general set theory, Hausdorff devoted the remainder of the text to a treatment of "point set theory," as the study of topology was called at that time. One of the significant contributions he made in this treatment was to clearly lay out for the reader the differences and similarities between sets, metric spaces, and topological spaces. It is easy to see how metric and topological spaces are built upon sets as a foundation, while also clearly seeing what is "added" to sets in order to obtain metric and topological spaces. We will follow Hausdorff as he built topology "from the ground up" with sets as his starting point.

2 Set theory

Part of what makes Hausdorff's work so interesting is that he did an excellent job of explaining how sets act as a foundation for other mathematical systems. He began by discussing how set theory is the basis for varied and diverse branches of mathematics[2].

> Set theory has celebrated its loveliest triumphs in the application to point sets in space, and in the clarification and sharpening of foundational geometric concepts, which will be conceded even by those who have a skeptical attitude towards abstract set theory.
>
> First we will be clear about the position of point set theory in the system of general set theory. One can treat a set purely as a system of its elements, without considering relationships between these elements. [Hausdorff, 1914, p. 209]

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[1]Hausdorff fittingly dedicated his 1914 text to Georg Cantor (1845–1918), whom he described as "the creator of the theory of aggregates (sets)." Cantor's earliest work in set theory appeared in a series of papers published between 1874 and 1884, in which several early topological ideas also appeared in connection with his work in real analysis. The author has written several projects in topology based on Cantor's work; the interested reader can find these additional projects at https://digitalcommons.ursinus.edu/triumphs_topology/.

[2]The English translation of this and all other excerpts in this project are due to David Pengelley.

Task 1 What did Hausdorff mean when he described treating a set as "a system of its elements without considering relationships between these elements?" What are some purely set-theoretic notions that he might have had in mind?

Hausdorff was interested in starting with the basics of set theory, with which we are now all familiar, and adding additional structure. He gave the example below.

Secondly however one can consider relations between the elements. . . This concerns itself, for any two elements, with one of the three relations $a \lesseqgtr b$, and we could interpret that as a function $f(a, b)$ of (ordered) pairs of the set being given, which however, can only take on three values (only two when restricted to pairs of different elements). For partially ordered sets a fourth relation or a fourth possible function value was also added. [Hausdorff, 1914, p. 209]

Let's investigate this further. Today we would add the adjective "totally" in front of Hausdorff's phrase "ordered set," so when Hausdorff writes "ordered set," we understand this as "totally ordered set." We will contrast this with a partially ordered set. A set X is partially ordered if it has a reflexive, antisymmetric, transitive relation $\leq$. The set is ordered (or totally ordered) if it has a partial order $\leq$ such that for every $a, b \in X$, either $a \leq b$ or $b \leq a$.

Task 2
(a) Explain an ordered set is in your own words.
(b) Give an example of an ordered set.
(c) What are these "three values" that Hausdorff mentioned above? In the case of a partially ordered set, what is the "fourth value"?

Next Hausdorff began to generalize the particular phenomena that we saw above into a more general framework. This is a very common thing for mathematicians to do; that is, once a concrete, particular notion proves useful, to generalize it or abstract away the particulars in order to develop a more general, comprehensive framework.

Now there is nothing to prevent a generalization of this idea, and we can imagine that an arbitrary function of pairs of a set is defined; that is, to each pair (a, b) of elements of a set M, a specific element $n = f(a, b)$ of a second set N is assigned. [Hausdorff, 1914, p. 210]

For simplicity, let's call Hausdorff's above definition a **2-association on the set** M.

Task 3 Using modern notation, write down a definition of what it means to have a 2-association on the set M. Explain how this is a generalization of a totally ordered set. Equivalently, explain how a totally ordered set is just a special case of the above.

In yet further generalizations we can take into consideration a function of a triple of elements, a sequence of elements, complexes of elements, subsets and the like of M. [Hausdorff, 1914, p. 210]

Task 4 Define using modern notation what it would mean to have an n-association on the set M.

The definition you gave in Task 4 generalizes Hausdorff's "triple of elements" idea. His next three suggestions concern functions on a set with a certain structure or that satisfy a certain property. Let us make the following definition.

Let M be a set and P some property, M_P the set of all collections of elements in M that satisfy property P. We call any function $f\colon M_P \to N$ a **property P function**.

Task 5 Let $M = N = \mathbb{N}^{\geq 2}$ and let P be the property "is a prime." Observe that M_P is the set of all subsets of prime numbers. Find a property P function (there are many, many options).

3 Metric Spaces

After having attempted to generalize to the n^{th} degree in the definition of property P, Hausdorff reeled us back in, reminding us that sometimes we can be so general that very little can be said. Instead, Hausdorff suggested an example that serves as a "golden mean" between too general and too specific.

A quite generally worded theory of this nature would of course cause considerable complications, and deliver few positive results. But among the special examples that occupy a heightened interest belongs, apart from the theory of ordered sets, especially the theory of point sets in space, in fact here the foundational relationship is again a function of pairs of elements, namely the distance between two points: a function which however now is capable of infinitely many values. [Hausdorff, 1914, p. 210]

Task 6 Given a set M, give the 2-association on M which associates a distance to pairs of elements in M.

As you have probably guessed, a distance isn't simply any function that associates a real number to a pair of points. For example, it seems like we should not allow negative distances. It is also probably the case that we would want the distance from point A to point B to be the same distance as from point B to point A. Now we consider the conditions for axioms that Hausdorff required of a distance.

By a metric space we understand a set E, in which to each two elements (points) x, y a non-negative real number is assigned, their distance $\overline{xy} \geq 0$; in fact we demand moreover the validity of the following

Distance axioms:

(α) **(Symmetry axiom).** Always $\overline{yx} = \overline{xy}$.

(β) **(Coincidence axiom).** $\overline{xy} = 0$ if and only if $x = y$.

(γ) **(Triangle axiom).** Always $\overline{xy} + \overline{yz} \geq \overline{xz}$.

We denote specially as the Euclidean n-dimensional number space E_n the set of complexes of real numbers

$$x = (x_1, x_2, \ldots, x_n),$$

in which the distance is defined by

$$\overline{xy} = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} \geq 0,$$

and as Euclidean n-dimensional space a metric space... [Hausdorff, 1914, p. 211]

Task 7 Prove that in Euclidean space, (α) and (β) are always satisfied.

Task 8 Let $X = \{a, b, c, d\}$. Define a 2-association on X which is a metric structure. That is, find a function on X which associates to each pair of point in X some real number that satisfies Hausdorff's distance axioms. Since X is a finite set, we call X along with its associated metric a **finite metric space**.

In a metric space E we understand by a neighborhood U_x of the point x the set of points y, whose distance from x is less then a specific positive number ρ ($\overline{xy} < \rho$). Such a neighborhood depends on the center point x and on the radius ρ; a point x has, when one varies the radius, infinitely many neighborhoods, which however as sets need not in general all be different. [Hausdorff, 1914, p. 212]

Task 9 Use your finite metric space in Task 8 to compute all the neighborhoods of the point a. Even though there are infinitely many radii you may pick, only finitely many neighborhoods should differ and hence, you have provided an example of neighborhoods which are not all different.

4 Topological Spaces

Now that Hausdorff had a definition for a metric space (i.e. a set together with a 2-association satisfying some properties), he took away the 2-association itself and instead focused on the properties of "neighborhoods" to arrive at a precise definition of the structure of a general topological space. As you read that definition below, be aware that Hausdorff was searching for axioms, trying to find the best way to, as we have made note of before, abstract away the particulars and identify the essential elements of what makes a topological space a topological space.

> One can make this system of neighborhoods be the foundation of the whole theory, with elimination of the concept of distance... Thereby we will change our standpoint, as announced earlier, that we will abstain from distances, with whose help we defined neighborhoods, and accordingly place the mentioned properties as axioms in the lead.
>
> By a topological space we understand a set E, in which to the elements (points) x certain subsets U_x are assigned, which we call neighborhoods of x, in fact subject to the following
>
> **Neighborhood axioms:**
>
> **(A) To each point x there corresponds at least one neighborhood U_x; each neighborhood U_x contains the point x.**
>
> **(B) If U_x, V_x are two neighborhoods of the same point x, then there is a neighborhood W_x, which is a subset of both ($W_x \subseteqq \mathfrak{D}(U_x, V_x)$).**[3]
>
> **(C) If the point y lies in U_x, then there is a neighborhood U_y, which is a subset of U_x ($U_y \subseteqq U_x$).** [Hausdorff, 1914, p. 2132–14]

This definition of a topological space may look quite different than the one to which you have been exposed. Nevertheless, we will work through a proof to show that the neighborhood axioms are equivalent with the open set axioms that have become standard in today's definition of a topological space. We first recall that definition:

> By a topological space we understand a set E along with a collection of subsets of E, called open sets, that satisfy the following conditions:
>
> **Open set axioms**
>
> 1. The space E and the null set $\emptyset$ are open.
> 2. The intersection of two open sets is open.
> 3. The union of any number of open sets is open.

[3] The script D here stands for the German word "durchschnitt," meaning average or intersection.

We also need to define what we mean by “open” in terms of neighborhoods. Call a subset U of E **open in** E if U can be written as the union of neighborhoods.

Task 10 Prove if a collection of sets of E satisfies the neighborhood axioms, then the induced open sets (defined above) satisfy the open set axioms.

Task 11 Now assume that E has a collection of open sets. Show that if we define a neighborhood of a point to be any open set containing that point, then this collection of neighborhoods satisfies the neighborhood axioms.

It is interesting to note that Hausdorff also had a fourth neighborhood axiom:

(D) For two different points x, y, there are two neighborhoods U_x, U_y without any points in common ($\mathfrak{D}(U_x, U_y) = 0$). [Hausdorff, 1914, p. 214]

Today this property is referred to as, appropriately enough, the Hausdorff property.

Task 12 Prove that every metric space satisfies the Hausdorff property.

Task 13 Give an example of a topological space which does not satisfy the Hausdorff property. (Hint: Try a finite topological space which is not a metric space.)

References

F. Hausdorff. *Grundzüge der Mengenlehre (Fundamentals of Set Theory)*. Leipzig, Von Veit, 1914.

Notes to Instructors

Primary Source Project Content: Topics and Goals

This project is intended for students in a topology course after they have had some familiarity and experience with a topological space. In particular, this is essential to understand the thrust of Task 10 and more generally, all of section 4, as the student is asked in this section to prove that Hausdorff's neighborhood axioms are equivalent to the axioms that the student is familiar with. Students will start with sets and begin to add structure to these sets with a (hopefully) familiar example of an ordered set in Task 2. From there the student will be led through an abstraction of an ordered set to what we are calling, for lack of a better term, a 2-association in the remaining tasks in this section. In particular, expect a mess from your students in Task 3. This of course is by design, as coming up with "the right" definition or the right way to think about something is very challenging. After students have come up with answers to this question, discussing their answers, along with "the right" answer, can be a good opportunity for an in-class discussion. In section 3, a metric space is then seen as simply a special case of the more general abstraction that was developed in section 2. It is then Hausdorff's observation about neighborhoods being defined in a metric space (quote starting "In a metric space E, we understand ...") as the key transition from the metric space to the topological space. Hausdorff accomplished this by taking the neighborhoods, defined from a metric space, and simply decreeing that these "neighborhoods" satisfy certain properties. The culmination of the project is then in Tasks 10 and 11, where the student works through a proof that these new axioms of Hausdorff are equivalent to the ones that they are familiar with. As an interesting historical side note, the project concludes by noting that Hausdorff included a fourth axiom in his neighborhood axioms, one which has been dropped as an axiom from the modern point of view. This is the property of being Hausdorff.

One or days class periods devoted to this project should suffice. The first half of the project, especially section 2, is more exploratory and hence, students may benefit from working in groups and discussing these questions among the group and then as a class. Another option is to have students read the first two sections for homework and come to class prepared to discuss the project and ask questions. The material quickly becomes abstract and challenging, which is by design. This is to encourage students to ask questions and bounce ideas off of each other. The material and tasks in section 4 is much more "standard" in terms of presentation and what the student is expected to do, but also more challenging. In all, Tasks 10 and 11 have 6 different claims to prove. Hence, some can be assigned as homework, some in groups, and some presented in class by the instructor. It will be a rare student who is able to successfully work through both these tasks completely on his own without either seeing at least 1 done by the professor or seeing how to start such tasks. Part of the difficulty for students is knowing what you get to assume and what you need to show. So it is worth reminding the student that in Task 10, they get to assume that, for example, given any point x, there exists a neighborhood of x (not to mention axioms (B) and (C)). But after working through these tasks, students should have a deeper appreciation for axiomatic systems and the position of topology in relation to other kinds of systems.

Connections to other Primary Source Projects

There are several other projects in topology written by the author. Project titles along with links are given below. The last two of these are full-lengths PSPs; all others are mini-PSPs that are intended to be completed in 1–2 class days.

- *Topology from Analysis*
  https://digitalcommons.ursinus.edu/triumphs_topology/1/
- *The Cantor Set before Cantor*
  https://digitalcommons.ursinus.edu/triumphs_topology/2/
- *Connecting Connectedness*
  https://digitalcommons.ursinus.edu/triumphs_topology/3/
- *The Closure Operation as the Foundation of Topology*
  https://digitalcommons.ursinus.edu/triumphs_topology/4/
- *A Compact Introduction to a Generalized Extreme Value Theorem*
  https://digitalcommons.ursinus.edu/triumphs_topology/5/
- *Nearness Without Distance*
  https://digitalcommons.ursinus.edu/triumphs_topology/7/
- *Connectedness: Its Evolution and Applications*
  https://digitalcommons.ursinus.edu/triumphs_topology/8/

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