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Discrete Morse Functions, Vector Fields, and Homological Sequences on Trees

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Abstract

The goal of this project is to construct a discrete Morse function which induces both a specified gradient vector field and homological sequence on a given tree. After reviewing the basics of discrete Morse theory, we will show that the two standard notions of equivalence of discrete Morse functions, Forman and homological equivalence, are independent of one another. We then show through a constructive algorithm the existence of a discrete Morse function on a tree inducing a desired gradient vector field and homological sequence. After proving that our algorithm is correct, we give an example to illustrate its use.

1 Introduction

Discrete Morse theory was invented by Robin Forman [8] as an analogue of "smooth" Morse theory popularized by Milnor [11]. Many classical results in Morse theory, such as the Morse inequalities, carry over into the discrete setting [10]. Applications of discrete Morse theory are vast, ranging from applications in configuration spaces [12] to computer science search problems [9].

Let f, g be two discrete Morse functions defined on a 1-dimensional simplicial complex, i.e., a graph. Inspired by Nicolaescu [1], R. Ayala et al. [2] studied the homological sequence of a discrete Morse function by introducing the notion of f and g being homologically equivalent, and they counted the number of excellent discrete Morse functions on all graphs [6]. In addition, the same authors counted the number of discrete Morse functions on a graph up to Forman equivalence in [4]. In this paper, we desire to combine these two questions by investigating the combination of the two notions of equivalence. To this end, let $\mathcal{V}$ be a fixed gradient vector field on a graph with $m > 1$ critical values. How many homological sequences can have $\mathcal{V}$ as their gradient vector field? Conversely, fix a homological sequence B with $m > 1$ critical values. How many discrete vector fields can have B as homological sequence? In the case that G is a tree, we show in Proposition 3.0.10 that any two gradient vector fields and homological sequences with the same number of critical values can be realized by a discrete Morse function. This is done in algorithmic form, the details of which are given in Algorithm 3. The layout of the paper is as follows. Section 2 is devoted to introducing the notation and terminology that will be used, as well as background results. Section 3 is the main section where we give an algorithm that yields a desired homological sequence and gradient vector field on a tree. Finally, in section 4, we will give an example illustrating the utility of our algorithm.

2 Background

In this section, we establish notation and terminology that will be used in the body of this paper. Our main reference for graph theory is [7] while we refer the reader to [10] for the basics of discrete Morse theory. We begin with graphs.

2.1 Graphs

Definition 2.1.1. Let $V \neq \emptyset$ be a set called the **vertex set**. A **graph** $G = (V, E)$ is a collection of distinct subsets of V of size 2, denoted E, called the **edge set**. Elements of V are **vertices** while elements of E are called **edges**. If $e = \{a, b\}$ is any edge $\in E$, a and b are **endpoints** of the edge. We could also say that a and b are **incident** with the edge. Usually we write $ab = \{a, b\}$ for an edge when there is no possibility of confusion.

Definition 2.1.2. Let u, v be two distinct vertices of G. A **(u, v)-path** is a sequence

$$u = u_0, e_0, u_1, e_1, ..., e_n, u_n + 1 = v$$

of distinct vertices and edges such that u_i, u_{i+1} are the endpoints of e_i. Suppose G is a graph such that for every pair of distinct vertices u, v there exists a unique (u, v)-path. Then G is a **tree**. A disconnected graph in which each component is a tree is called a **forest**.

2.2 Discrete Morse theory

We now define the basics of discrete Morse theory, which was originally introduced by Robin Forman [8].

Definition 2.2.1. Let G be a graph. A **discrete Morse function** G is a function $f\colon G \to \mathbb{R}$ such that for every $v \in G$, we have

$$|\{e \in E : f(e) \geq f(v), v \text{ an endpoint of } e\}| \leq 1$$

and for every $e \in G$

$$|\{v \in V : f(v) \geq f(e), v \text{ an endpoint of } e\}| \leq 1.$$

Of particular interest are those vertices and edges which admit of no exception, the so-called critical values.

Definition 2.2.2. If vertex v satisfies,

$$|\{e \in E : f(e) \geq f(v), v \text{ an endpoint of } e\}| = 0$$

then v is a **critical vertex** and the value $f(v)$ is a **critical value**. If an edge e satisfies

$$|\{v \in V : f(v) \geq f(e), v \text{ an endpoint of } e\}| = 0$$

then e is a **critical edge** and $f(e)$ is a **critical value**.

The critical values of a discrete Morse function tell us how to "build" the graph G in stages. This is formally accomplished through the level subcomplexes.

Definition 2.2.3. For any $c \in \mathbb{R}$, the **level subcomplex** $G(c)$ is defined as the smallest graph satisfying $\{e, v \in G : f(e), f(v) \leq c\}$.

Definition 2.2.4. Let G be a forest. The number of components of a graph is denoted $b_0(G)$ and is called the **Betti number** of G. Now let $f : G \to \mathbb{R}$ be a discrete Morse function with critical values: $c_0, c_1, c_2, ..., c_m$. The **homological sequence** of f, denoted B_f, is given by $b_0(G(c_0)), b_0(G(c_1)), ..., b_0(G(c_m))$. To discrete Morse functions $f, g \colon G \to \mathbb{R}$ are said to be **homologically equivalent** if both f and g induce the same homological sequence.

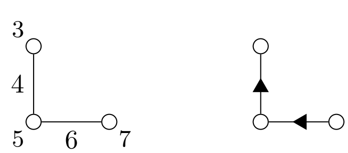
There is an notion of equivalence of discrete Morse functions due to Robin Forman.

Definition 2.2.5. Let $f, g \colon G \to \mathbb{R}$ be discrete Morse functions. Then f and g are **Forman equivalent** if for every pair (v, e) consisting of a vertex v and an incident edge e, we have $f(v) < f(e)$ if and only if $g(v) < g(e)$.

It turns out that this definition has a nice geometric characterization defined in terms of a gradient vector field.

Definition 2.2.6. Let $f : G \to \mathbb{R}$ be a discrete Morse function. The **induced gradient vector field** is defined by $\mathcal{V}_f := \{(v, e) : f(v) \geq f(e), v \text{incident with } e\}$.

Example 2.2.7. Let f be the discrete Morse function defined on the graph to the left. The induced gradient vector field is shown on the right.



The following Theorem, due to Ayala et al., shows that Forman equivalence can be characterized in terms of the gradient vector field.

Theorem 2.2.8. [5, Thm. 3.1] *Two discrete Morse functions f and g defined on G are equivalent if and only if $V_f = V_g$.*

An immediate corollary of this result is that a gradient vector field is completely determined by the critical simplices. We will use this fact throughout the rest of the paper without further comment.

Corollary 2.2.9. *f and g are equivalent if and only if they have the same critical simplices.*

3 Algorithm

Ayala et al. counted the number of discrete Morse functions on a graph up to Forman equivalence in [4]. In addition, the same authors began a count counting the number of discrete Morse functions on graphs up to homological equivalence in [3] and completed their work in [6]. As mentioned above, we combine these two questions in this paper by investigating the combination of the two notions of equivalence. That is, suppose we fix a gradient vector field on a graph. How many homological sequences can have the given gradient vector field? Conversely, suppose we fix a homological sequence. How many discrete vector fields can have the given homological sequence? Interestingly, the result in the case of trees turns out to be all of them in both cases! That is, we will prove below in our algorithm that given a tree, a fixed gradient vector field, and a homological sequence with as many critical values as the gradient vector field, there exists a discrete Morse function on the tree inducing both the desired gradient vector field and homological sequence. This result is now stated using an algorithm.

Let G be a given tree, $\mathcal{V}$ a gradient vector field on $\mathcal{V}$, and B a homological sequence with the same number of critical values that $\mathcal{V}$ induces. We fix the following notation for Algorithm 3. Let v denote a vertex, cv a critical vertex, and ncv a non-critical. In the algorithm we will label vertices and edges, so in order to keep track of this, let V denote the set of all labeled vertices, and V_0 set of all unlabeled vertices. Similarly, e denotes an edge, ce critical edge, and nce a non-critical edge. The set E is the set of all labeled edges, and E_0 set of all unlabeled edges.

Algorithm 1 Forman Numbering Algorithm

`Input:` A tree G with a given gradient vector field $\mathcal{V}$, and homological sequence B

`Output:` A discrete Morse function $f\colon G \to \mathbb{R}$ such that $\mathcal{V}_f = \mathcal{V}$ and $B_f = B$.

(a) Initiate $n = 1$.

(b) Choose any two vertices $cv_t, cv_u \in V_0$ such that there exists a unique critical edge ce in the (cv_t, cv_u)-path. Label $f(cv_t) = n$ and $f(cv_u) = n + 1$. For any $v \in V$ incident with any $nce \in E_0$ label nce and its other endpoint $n + 1$. Continue until there is no $v \in V$ such that v is incident with any $nce \in E_0$.

(c) If you wish to increase the value of b_n by 1, proceed to step d. If you wish to decrease b_n by 1, proceed to step e. If $V_0, E_0 = \emptyset$, exit the algorithm.

(d) Let v be any $v \in V$. Select one $cv \in V_0$ such that there exists exactly one ce the (v, cv)-path. Label $f(cv) = n + 1$, then label any $nce \in E_0$ that is incident with any $v \in V$ and its opposite endpoint as $n + 1$. Continue until there does not exist any $v \in V$ with v incident to any $nce \in E_0$. Repeat step d until you no longer wish to increase b_0. Then, return to step c.

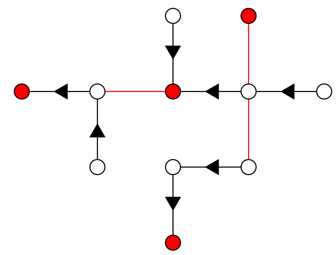
(e) Label any $ce \in E_0$ incident with any two $v \in V$ as $n + 1$. Repeat until you no longer wish to decrease b_0, then return to step c.

Proposition 3.0.10. *Algorithm 2 is correct.*

Proof. Suppose G is a tree, $\mathcal{V}$ a gradient vector field, and B a homological sequence with the same number of critical values induced by $\mathcal{V}$ (Recall by Corollary 2.2.9 that a gradient vector field is completely determined by its critical simplices). Algorithm 3 yields a labeling f of G. We need to show that f is a discrete Morse function, that $\mathcal{V}_f = \mathcal{V}$, and that $B_f = B$. To see that the algorithm yields a discrete Morse function, let v be a vertex of G. If v is critical, then v is labeled less than all of its incident edges by step 4. If v is non-critical, then v is part of a pair (v, e) in the gradient vector field $\mathcal{V}$ and $f(v) = f(e)$, while any incident edge with e is labeled greater than $f(v)$. A similar argument shows that e is critical in $\mathcal{V}$ if and only if e is critical under f. This shows that f is a discrete Morse function with precisely the desired critical values. By Corollary 2.2.9, $\mathcal{V}_f = \mathcal{V}$. That $B_f = B$ is obvious given the nature of the algorithm. □

4 Example

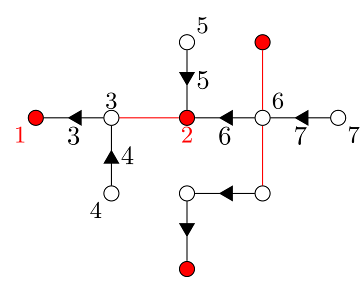
We illustrate the algorithm with an example. Let G be the following graph along with the following gradient vector field (note that the critical values are red).



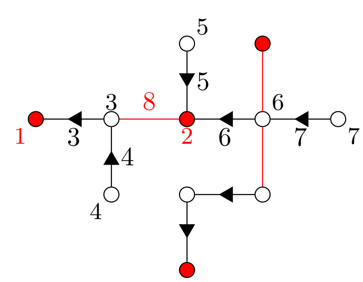
There are seven critical values in this discrete Morse function. Consider the following homological sequence.

	c_0	c_1	c_2	c_3	c_4	c_5	c_6
B_0 :	1	2	1	2	3	2	1

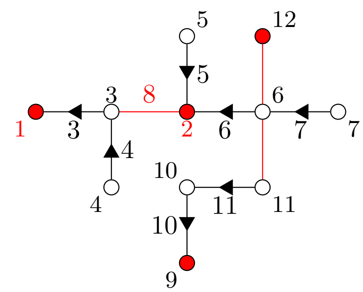
We wish to construct $f: G \to \mathbb{R}$ with critical values $c_0, \ldots, c_6$ that induce the above gradient vector field as well as the homological sequence. Begin with steps a and b.



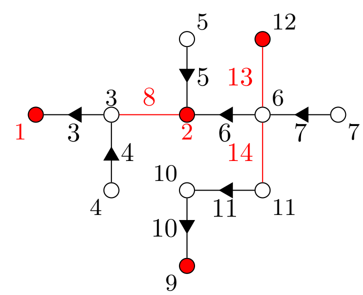
Now, seeing that we wish to decrease the Betti number by one, proceed with step e exactly once.



Next, we wish to increase the Betti number two times consecutively, so utilize step d twice.



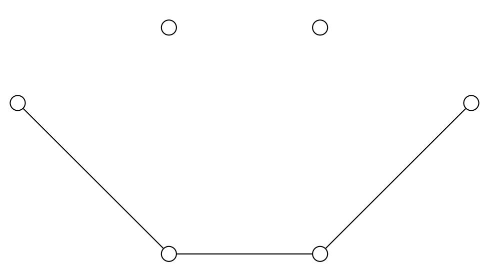
Lastly, we wish to decrease b_0 from 3 to 1, so repeat step e twice.



Now that the values of $V_0, E_0 = \emptyset$, the algorithm is complete.

5 Conclusion

This algorithm can match any gradient vector field of ≥ 1 critical value. Any possible homological sequence can be obtained. The process doesn't break the discrete Morse function, and it always preserves the initial Gradient Vector Field. The algorithm is suited for practical use.



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