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Abstract

The distributions $P(X)$ of singular thermodynamic quantities, on an ensemble of d-dimensional quenched random samples of linear size L near a critical point, are analyzed using the renormalization group. For L much larger than the correlation length ξ, we recover strong self-averaging (SA): $P(X)$ approaches a Gaussian with relative squared width $R_X \sim (L/\xi)^{-d}$. For $L \ll \xi$ we show weak SA (R_X decays with a small power of L) or *no* SA [$P(X)$ approaches a non-Gaussian, with universal L-independent relative cumulants], when the randomness is irrelevant or relevant, respectively.

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Absence of Self-Averaging and Universal Fluctuations in Random Systems near Critical Points

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(Received 1 August 1996)

The distributions $P(X)$ of singular thermodynamic quantities, on an ensemble of d-dimensional quenched random samples of linear size L near a critical point, are analyzed using the renormalization group. For L much larger than the correlation length ξ, we recover strong self-averaging (SA): $P(X)$ approaches a Gaussian with relative squared width $R_X \sim (L/\xi)^{-d}$. For $L \ll \xi$ we show weak SA (R_X decays with a small power of L) or *no* SA [$P(X)$ approaches a non-Gaussian, with universal L-independent relative cumulants], when the randomness is irrelevant or relevant, respectively. [S0031-9007(96)01491-3]

Systems with frozen randomness have been studied for several decades [1]. A useful early result [2], now the so-called Harris criterion, applies to randomness in the local transition temperature T_c (or in the local "exchange" coupling constants). It shows that weak randomness does not change the critical behavior near a second order phase transition in d dimensions if the specific heat exponent α_{pure} of the nonrandom ("pure") system is negative, or equivalently if the correlation length exponent ν_{pure} obeys $\nu > 2/d$. We denote this case by "P"; it applies, e.g., to the random XY or Heisenberg spin models at $d = 3$. This criterion has been supported by subsequent renormalization group (RG) analyses based on the $\epsilon = 4 - d$ expansion [3–5] and by a scaling analysis of a perturbation expansion in the randomness [6]. It has also been proven [7] to hold in the presence of strong randomness subject to certain mild assumptions. When $\alpha_{\text{pure}} > 0$, the RG calculations [3–6] exhibit a flow from the pure fixed point (which is then unstable) towards a new, stable, "random" fixed point. Although the stability exponent of the latter, ϕ_{random}, is not equal to the value of α_{random} associated with it [8], the proof of Ref. [7] seems to indicate that $\alpha_{\text{random}} < 0$, namely, $\nu_{\text{random}} > 2/d$, as found by the RG [3,5]. The random behavior (denoted below by "R") is known to describe, e.g., the random Ising model at $d = 3$, as confirmed experimentally [9].

It is usually not easy to test these ideas in a completely convincing way. Experimental and numerical investigations are usually hampered by difficulties, having to do with fluctuations in and extrapolation of data from finite samples. In a random hypercubic sample of linear dimension L and volume $N = L^d$, any observable singular property X has different values for different realizations of the randomness and is thus a stochastic variable described by a probability distribution function $P(X, L)$. We may characterize P by the average $[X]$ and variance $(\Delta X)^2 \equiv [X^2] - [X]^2$, where $[\ ]$ indicates an average over all realizations of the randomness. If X corresponds to a macroscopic variable (e.g., the susceptibility χ, the mag-

netization $\mathbf{M}$, the singular part of the energy E, or that of the specific heat C), then the system is said to exhibit self-averaging (SA) if $R_X(L) \equiv (\Delta X)^2/[X]^2 \to 0$ as $L \to \infty$. Off criticality, where L is much larger than the correlation length ξ, the central limit theorem indicates "strong" SA, i.e., $R_X \sim L^{-d}$. SA is less obvious near the critical point, where $L \ll \xi$. We say that there is no SA when R_X has an L-independent finite value. This is known to happen, with universal finite cumulants, at $L \ll \xi$ with *strong* disorder, e.g., at the percolation threshold, for the mass of the spanning cluster [10] and the conductance of classical dilute resistor networks [11], and at the mobility edge of quantum mesoscopic systems, for the conductance [12]. Recently Wiseman and Domany (WD) [13] investigated SA near criticality for weak randomness. Generalizing the Harris criterion, they presented a heuristic scaling theory which predicted that at criticality, $R_X \sim C \sim L^{\alpha/\nu}$, for *both* P and R behavior. Since α/ν is small and nonpositive for both cases at the stable fixed point [7], this implies "weak" SA. WD then tested these predictions numerically on the bond-diluted Ashkin-Teller models in $d = 2$, which have a variable (positive and negative) α_{pure} and a R logarithmic specific heat, and found strong SA for $L \gg \xi$, and consistency with their scaling predictions for $L \ll \xi$ at P. When $\alpha_{\text{pure}} > 0$, both their heuristic theory and numerics showed an increase in R_X for small L, and a very slow (possibly logarithmic) variation of R_X for larger L, which could be consistent with either *no* SA or weak SA. As WD emphasize, when the system does *not* exhibit SA, numerical studies of the asymptotic random critical behavior will be quite difficult: Even if one fixes the temperature to be the correct transition temperature for $L \to \infty$, $[T_c(\infty)]$, *letting L become larger does not lead to improved statistics.* However, the theoretical basis for these results remained unclear. Particularly, the status of $P(X)$ for R, and its implications concerning the issue of SA, has not been conclusively studied, in spite of the fact that it has a crucial bearing on experimental and numerical studies.

In the present Letter we use RG techniques to study these questions for several types of quenched local randomness. We first show that $P(X)$ is invariant under the RG flow, which means that it can be calculated at any stage of the flow. We then evaluate the cumulants of $P(X)$ perturbatively after eliminating the critical fluctuations and use information on the RG flow to prove strong SA for $L \gg \xi$ for both P and R, and weak SA for $L \ll \xi$ in P. For $L \ll \xi$ in R, we show (in contrast to WD) that $P(X)$ flows towards a universal scale-independent non-Gaussian distribution, characterized by the random fixed point.

Consider first a random ferromagnetic spin model: $\mathcal{H} = -2\sum_{\langle ij \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j$, where $\mathbf{S}_i$ is an m-component unit vector, $\langle ij \rangle$ indicates summation over pairs of nearest neighbors on a d-dimensional hypercubic lattice, and each J_{ij} is an independent random variable, with average $J = [J_{ij}]$ and variance $(\Delta J)^2 = [(J - J_{ij})^2]$. Introducing Fourier transforms via $\mathbf{S}_i = N^{-1}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}_i}\boldsymbol{\sigma}(\mathbf{q})$, the RG is conveniently done on a Ginzburg-Landau-Wilson version of the free energy [3–6],

$$-F = (2N)^{-1}\sum_{\mathbf{q}} (r + q^2)\boldsymbol{\sigma}(\mathbf{q}) \cdot \boldsymbol{\sigma}(-\mathbf{q})$$
$$+ uN^{-3}\sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3} \boldsymbol{\sigma}(\mathbf{q}_1) \cdot \boldsymbol{\sigma}(\mathbf{q}_2)\boldsymbol{\sigma}(\mathbf{q}_3) \cdot \boldsymbol{\sigma}(-\textstyle\sum_{i=1}^{3}\mathbf{q}_i)$$
$$+ (2N^2)^{-1}\sum_{\mathbf{q}_1,\mathbf{q}_2} \delta\hat{r}(\mathbf{q}_1)\boldsymbol{\sigma}(\mathbf{q}_2) \cdot \boldsymbol{\sigma}(-\mathbf{q}_1 - \mathbf{q}_2), \quad (1)$$

where $r \propto T - [T_c]$ (the proportionality coefficient involves trivial scale factors [6]) while $\delta r_i = N^{-1}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}_i}\delta\hat{r}(\mathbf{q})$ represents the local fluctuation in T_c, with variance $w \equiv [(\delta r)^2] \propto (\Delta J)^2 \propto (\Delta T_c)^2$. The RG iterations now involve integration over large $\mathbf{q}$, rescaling lengths by factors e^{-l} and spins by factors $\zeta = \exp[(d + 2 - \eta)l/2]$ [14], so that the renormalized F maintains its form as above [6,15].

We now use the susceptibility χ to demonstrate our results. For each random realization, in the disordered phase, $\chi = \langle\sum_{i,j} S_i^\mu S_j^\mu\rangle/N = \langle\sigma^\mu(\mathbf{q} = 0)\sigma^\mu(\mathbf{q} = 0)\rangle/N$, where $\langle\rangle$ denotes a thermal average with the random F, and S^μ is any component of $\mathbf{S}$. After l iterations, the above rescaling of $\sigma^\mu(0)$ and of $N(l) = Ne^{-dl}$ yield $\chi = \zeta^2 e^{-dl}\chi(l) = e^{(2-\eta)l}\chi(l)$, where $\chi(l)$ is to be calculated with the random renormalized $F(l)$. From this we conclude that both $[\chi^2]$ and $[\chi]^2$ are multiplied under renormalization by the same prefactor $e^{2(2-\eta)l}$, and therefore these prefactors drop out from ratios like R_χ. Hence our *first main result*,

$$R_\chi(L, \xi, w, \dots) \equiv R_\chi\big(Le^{-l}, \xi e^{-l}, w(l), \dots\big) \equiv R_\chi(l), \quad (2)$$

without any rescale prefactors. The dots represent all the other parameters, including, e.g., $u(l)$. In fact, every macroscopic singular physical quantity X (like $\mathbf{M}$ or the singular part in E) can also be expressed as some operator in Fourier space, with $\mathbf{q} = 0$, which has a rescale factor like ζ. For example, $M^\mu = \langle\sum_i S_i^\mu\rangle/N = \langle\sigma^\mu(0)\rangle/N$, with the rescale factor ζe^{-dl}. These factors will appear in the same way in $(\Delta X)^2$ and in $[X]^2$, and will drop out in the ratio R_X. Thus we expect Eq. (2) to hold for all the singular quantities. The same arguments also apply to higher relative cumulants of the form $R_{p,X} \equiv [X^p]_c/[X]^p$; these also involve the same prefactors in the numerator and denominator, and are therefore invariant under the RG, i.e., they can be calculated at any stage l of the RG flow.

We now continue the RG iterations until either $\xi(l^*) = 1$ (when $L \gg \xi$), or $L(l^*) = Le^{-l^*} = 1$ (when $L \ll \xi$). In the former case, $R_\chi(l^*) = R_\chi(L/\xi, 1, w(l^*), \dots)$. Since $\xi(l^*) = 1$, there are no critical fluctuations left, and therefore we can use the central limit theorem and find that $R_\chi(l^*) = R_\chi^{(1)}/N(l^*) = R_\chi^{(1)}(\xi/L)^d$, where $R_\chi^{(1)}$ denotes the variance of χ for a single renormalized spin [of size $\xi(l^*) = 1$]. At this point, $[\chi(l^*)] = 1/r(l^*) = 1$ [r appears in Eq. (1)], and $[\Delta\chi(l^*)]^2 = w(l^*)$. Thus $R_\chi^{(1)} = w(l^*)$. In the latter case, $R_\chi(l^*)$ measures the relative variance of χ for a single renormalized spin of size $L(l^*) = 1$, and we have already identified this as $R_\chi^{(1)} = w(l^*)$ [16]. Both cases can be combined into our *second main result*,

$$R_\chi = w(l^*)/N(l^*) = w(l^* = \ln x)(x/L)^d, \quad (3)$$

with $x = e^{l^*} = \min\{L, \xi\}$. All that now remains is to use explicit expressions for the dependence of w on l. These have been known for more than 20 years [3–5]: In case P, w scales as [6] $w(l) = w\exp[l(\alpha/\nu)_{\text{pure}}]$, hence $w(\ln x) = wx^{(\alpha/\nu)_{\text{pure}}}$. This implies weak SA for $L \ll \xi$, $R_\chi = wL^{(\alpha/\nu)_{\text{pure}}}$, and strong SA, $R_\chi = w\xi^{2/\nu_{\text{pure}}}/L^d \sim (T - [T_c])^{-2}/L^d$ for $L \gg \xi$. In case R, $w(l)$ first increases and then approaches the random fixed point, with a finite fixed point value w^*_{random} and with $w(l) - w^*_{\text{random}} \propto \exp[l(\phi/\nu)_{\text{random}}]$, where $\phi_{\text{random}} < 0$. In $d = 4 - \epsilon$ dimensions, $w^*_{\text{random}} = O(\epsilon)$ for $m > 1$, or $O(\epsilon^{1/2})$ for the Ising case $m = 1$ [4,17,18]. For R, this maintains strong SA when $L \gg \xi$, now with $R_\chi = w^*_{\text{random}}(\xi/L)^d$. However, when $L \ll \xi$ this yields a *complete asymptotic absence of SA*, with $R_\chi = w^*_{\text{random}} + O(L^{(\phi/\nu)_{\text{random}}})$, contradicting WD's conjecture that $R_X \sim L^{(\alpha/\nu)_{\text{random}}}$. Note that Eq. (3) also applies when $w(l) \sim l^{-\tau}$, in which case $R_\chi \sim (\ln x)^{-\tau}$. For example, the R behavior for the random Ising model in $d = 4$ and for the random dipolar Ising model in $d = 3$ has $w(\ln x) \sim (\ln x)^{-1/2}$ [18]. Such a decay probably also yields logarithmic behavior in $d = 2$ [13,19].

The heuristic argument which led to Eq. (3) can be replaced by an explicit perturbative expansion of $R_\chi(l^*)$, which should be analytic and converge since at l^* all the critical fluctuations are removed. Since $R_\chi = 0$ when $w = 0$, the leading term in this expansion is linear in $w(l^*)$. Keeping track of factors of N in the various Fourier transforms, a simple calculation yields $\delta\chi = \delta\hat{r}(0)(NG)^2/N^3 + [\delta\hat{r}(0)]^2(NG)^3/N^5 + O(\delta\hat{r}(0)^3)$, with

$G^{-1} = r + q^2$. Using also $[(\delta\hat{r}(0))^2] = Nw$ and $[\chi(l^*)] = G(l^*) = 1$ [16], we recover Eq. (3) to leading order in w. Exactly the same results, as well as a more direct proof of Eq. (2), are found from the replica version of this calculation [5], where $[\chi^2]_c$ is replaced by an annealed replica average of the form $\langle\chi(\alpha)\chi(\beta)\rangle$ with different replicas, i.e., $\alpha \neq \beta$.

It is straightforward to extend these results to higher cumulants. Ignoring higher cumulants of the initial distribution of δr [20], $R_{p,\chi}$ is found diagrammatically from the connected part $[(\delta\chi)^p]_c$. For example, using the above expansion of $\delta\chi$ to second order in $\delta\hat{r}$ yields (to leading order) $[\chi^3]_c = 6N^{-11}[NG]^7[Nw]^2 = 6N^{-2}G^7w^2$, which reduces at l^* to $6w(l^*)^2/N(l^*)^2$. The result for general $p > 1$ may be written as

$$R_{p,\chi} = p!\,2^{p-3}[w(l^*)/N(l^*)]^{p-1} = p!\,2^{p-3}R_\chi^{p-1}. \quad (4)$$

This leads to our *third main conclusion:* For strong or weak SA, R_χ vanishes as a power of L, and the distribution of $\chi/[\chi]$ goes to a Gaussian with a width $\delta x \sim R_\chi^{1/2}$; for $L \gg 1$, the pth cumulant becomes much smaller than $(\delta x)^p$. However, in the R critical regime ($L \ll \xi$) we have $R_{p,\chi} = O((w^*_{\text{random}})^{p-1})$ [20], and these are all finite. Thus in this case the limiting distribution function for χ is L independent and *not* Gaussian. It is interesting to note in this connection that the distributions measured by WD do *not* look Gaussian. Thus $P(\chi)$ remains fixed under rescaling and does not become sharper by going to larger L. This is obviously important for simulations or measurements on critical random systems.

We can now extend these results to other critical quantities such as the magnetization M. The heuristic estimate of $R_M^{(1)}$ then follows from the fact that at l^* one has roughly the mean-field relation $M(l^*)^2 = -r(l^*)/u(l^*)$ [15] so that $\delta M/M = \delta r/2r$, hence $R_M^{(1)} = w(l^*)/4$, i.e., $R_M = R_\chi/4$. A diagrammatic calculation of $R_M(l^*)$ requires a shift of $\mathbf{S}$ by $\mathbf{M}$ [21,22], and yields the same result. In fact, $(\Delta M)^2$ is equal to the quenched random part of the structure factor $C^{(s)}(q) = [m(q)m(-q)] - M^2\delta(q)$ [21] in the limit $q \to 0$. Using explicit expressions for $C^{(s)}$ from Refs. [21] reproduces our result for R_M. These references also contain information yielding $R_M \sim L^{-d}$ in the ordered phase. The results for M can easily be generalized: At l^*, all singular quantities are of the form $X \sim |r(l^*)|^{\omega_{\text{MF}}}$, where ω_{MF} is the mean-field exponent for X. Hence our *fourth conclusion: All the relative variances of all the measurable quantities are simply related to each other,* via $R_X/R_\chi = \omega_{\text{MF}}^2$. Specifically this yields $R_E = R_\chi$. Note that since the singular part in E does not diverge, using the full energy in the denominator of R_E may lead to deviations. Indeed WD only looked at $(\Delta E)^2$.

The *fifth step in this discussion* involves *universality.* As stated, in case R, $P(X)$ for $L \ll \xi$ is completely determined by the values of the parameters at the random fixed point. Since this point is stable, it will "attract" many random systems with various initial values of the randomness, as well as various short range details (like the lattice structure) [23]. Thus in this case our theory predicts *universal* asymptotic values of all the $R_{p,X}$'s, and thus a universal L-independent distribution of $x = X/[X]$, for $1 \ll L \ll \xi$.

It is interesting to relate these results to the Harris criterion. WD presented a heuristic argument in which they related the dependence of $[(\Delta T_c)^2]$ on L to that of R_X. Define an L-dependent transition temperature $T_c(L)$, which fluctuates among samples. Assuming that there exists a sharp phase transition, one would write $\chi \sim (T - T_c)^{-\gamma}$, and expand about the average $[\chi]$ in the form $\chi(T,L) \approx [\chi(T,L)] + \chi'(T_c,L) \times \{T_c - [T_c(L)]\}$, where $\chi' = \partial[\chi]/\partial T_c$ at $T_c = [T_c(L)]$. For $L \gg \xi$ we have $\chi'/[\chi] = 1/(T - T_c) \sim \xi^{1/\nu}$, and thus $R_\chi \sim (\Delta T_c)^2\xi^{2/\nu}$. Comparing with Eq. (3) then yields that $(\Delta T_c)^2 \sim w(\ln\xi)\xi^{d-2/\nu}/L^d$, and therefore $(\Delta T_c)^2 \sim w/L^d$ for P and $(\Delta T_c)^2 \sim w^*_{\text{random}}\xi^{-(\alpha/\nu)_{\text{random}}}/L^d$ for R. Although the former result agrees with the basic assumption of the Harris criterion and with WD, the latter result is different: For $L \sim \xi$ (and also for all $L < \xi$), the latter result implies the modified behavior $(\Delta T_c)^2 \sim L^{-2/\nu_{\text{random}}}$. This implies that in case R the shift $([T_c(L)] - [T_c(\infty)])$, the typical deviation $\delta T_c = \{T_c - [T_c(L)]\}$, and $(T - T_c) \sim \xi^{-1/\nu}$ all scale in the same way, as $e^{l/\nu_{\text{random}}}$. Heuristically one might say that the strong randomness mixes all these temperature scales together, in contrast to the P case for which the shift is larger than δT_c. It would be interesting to test this prediction numerically.

Similar analysis applies for other types of randomness, like random anisotropies [22,24], $D\Sigma_i(\hat{x}_i \cdot \mathbf{S}_i)^2$ ($\hat{x}_i$ is a random unit vector), and random fields [22,25], $\Sigma_i\mathbf{h}_i \cdot \mathbf{S}_i$, with variances $y \sim D^2$, $\lambda \sim [\mathbf{h}_i^2]$. Specifically Eq. (2) still holds, with the additional dependence on $y(l)$ and $\lambda(l)$. We now need to follow the RG flow, and choose l^*. For $m > 1$, both of these perturbations destroy long range order for $d < 4$, and the samples break into Imry-Ma domains of size ξ_R [22,25]. ξ_R becomes of order 1 when either $y(l^*)$, or $\lambda(l^*)$ becomes equal to 1. Thus we stop iterations when the largest of $L(l)^{-1}$, $\xi(l)^{-1}$, $y(l)$, or $\lambda(l)$ reaches 1. At that point perturbation expansions converge, and we can calculate $R_X^{(1)}$. For example, consider R_M. Since $(N\Delta M)^2 = \Sigma_{ij}[\langle S_i^\mu\rangle\langle S_j^\mu\rangle] \equiv \langle\sigma^\mu(0)\sigma^\mu(0)\rangle$, we can calculate this perturbatively to find $(NG)^2\lambda/N$ or $(NG)^2yM^2/N$ [22]. Using also $M(l)^2 \sim -r(l)/u(l)$, we end up with $R_M \sim y(l^*)/N(l^*)$ and $R_M \sim \lambda(l^*)u(l^*)/N(l^*)$. These expressions generalize Eq. (3) for these cases. When the initial values of y and λ are very small, we have $L(l), \xi(l) \ll \xi_R(l)$, and we stay in the vicinity of the previously discussed pure or random fixed points. Both perturbations are strongly relevant near both of these fixed points, and y and λ increase as $\exp(l\phi/\nu)$, with $\phi_y \approx 0.37$ [24] and $\phi_\lambda \approx \gamma$ near both fixed points

[25,26]. Thus $R_M \sim L^{\phi/\nu}$ strongly increases with L when $L \ll \xi, \xi_R$, apparently breaking SA. (This is similar to the initial increase in R_X near the pure fixed point when $\alpha_{\text{pure}} > 0$, but much stronger.) This increase stops, of course, when L or ξ become comparable to ξ_R, when $R_M \sim 1$, and long range order is completely lost. For $m = 1$, the random field system actually has a phase transition at $d = 3$. In $d = 6 - \epsilon$, this transition is described by a fixed point where $\lambda(l^*)u(l^*) = O(\epsilon)$ [25]. Thus even in that case one expects R_M to approach a universal constant with no SA. In fact in this case the divergence of $(N\Delta M)^2/N$ is the same as that of the disconnected susceptibility, i.e., $(T - T_c)^{-\bar{\gamma}}$. Using $\bar{\gamma} = 2\gamma$ and $2\beta + \gamma = (d - 2 + \eta)\nu$ [27], we recover the universal constant value of R_X even in three dimensions. It is interesting to note that these systems are known to equilibrate very slowly, due to metastable states. It is tempting to speculate that these difficulties may be related to the fast increase in the random fluctuations for small initial values of y or λ.

In summary: We analyzed SA near critical points in random systems, and we confirmed strong SA for $L \gg \xi$. For $L \ll \xi$ there are two cases: One has weak SA when randomness is irrelevant, but *no* SA when randomness is relevant. In the latter case the asymptotic distribution function is universal and generically non-Gaussian. Distributions of different measurable critical quantities are simply related to each other through their mean-field exponents. These results, which seem to apply for all the known cases of strong randomness, are important for both experimental and numerical investigations of the critical properties of finite random systems. It would be particularly interesting to test these for the random Ising model in $d = 3$ (e.g., along the critical line of the dilute case) and to understand the connections between the lack of SA in many strongly random critical systems.

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