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Abstract. This paper investigates the class of k-universal finite graphs, a local analog of the class of universal graphs, which arises naturally in the study of finite variable logics. The main results of the paper, which are due to Shelah, establish that the class of k-universal graphs is not definable by an infinite disjunction of first-order existential sentences with a finite number of variables and that there exist k-universal graphs with no k-extendible induced subgraphs.

1. Introduction

This paper continues the investigation of the existential fragment of $L^{\omega}_{\infty\omega}$ from the point of view of finite model theory initiated in [**RW95**] and [**Ros95**]. In particular, we further study an analog of universal structures, namely, k-universal structures, which arise naturally in the context of finite variable logics. The main results of this paper, Theorems 2.1 and 2.4, which are due to Shelah, apply techniques from the theory of sparse random graphs as developed in [**SS88**] and [**BS95**] to answer some questions about k-universal structures left open in these earlier works. In order to make the current paper more or less self-contained, we recall some notions and notations from the papers cited above, which may be consulted for further background and references.

We restrict our attention to languages which contain only relation symbols. We let L^k denote the fragment of first-order logic consisting of those formulas all of whose variables both free and bound are among $x_1, \ldots, x_k$, and similarly, $L^k_{\infty\omega}$ is the k-variable fragment of the infinitary language $L_{\infty\omega}$. We let $L^k(\exists)$ denote the collection of existential formulas of L^k, that is, those formulas obtained by closing the set of atomic formulas and negated atomic formulas of L^k under the operations of conjunction, disjunction, and existential quantification, and we let $L^k_{\infty\omega}(\exists)$ be the existential fragment of $L^k_{\infty\omega}$. The fragments $\bigwedge L^k(\exists)$ and $\bigvee L^k(\exists)$ of $L^k_{\infty\omega}(\exists)$ consist of the countable conjunctions and the countable disjunctions of formulas of $L^k(\exists)$ respectively. We write $\mathrm{qr}(\theta)$ for the quantifier rank of the formula θ, which is defined as usual.

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DEFINITION 1.1. Let A and B be structures of the same relational signature. $A \preceq^k B$ ($A \preceq^{k,n} B$) ($A \preceq^k_{\infty\omega} B$), if and only if, for all $\theta \in L^k(\exists)$ (with $\mathrm{qr}(\theta) \leq n$) (for all $\theta \in L^k_{\infty\omega}(\exists)$), if $A \models \theta$, then $B \models \theta$.

These relations may be usefully characterized in terms of the following non-alternating, local variants of the Ehrenfeucht-Fraisse game. The *n-round*, $\exists^k$-game from A to B is played between two players, Spoiler and Duplicator, with k pairs of pebbles, $(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)$. The Spoiler begins each round by choosing a pebble α_i that may or may not be in play and placing it on an element of A. The Duplicator then plays β_i onto an element of B. The Spoiler wins the game if after any round $m \leq n$ the function f from A to B, which sends the element pebbled by α_i to the element pebbled by β_i is not a partial isomorphism; otherwise, the Duplicator wins the game. The *eternal* $\exists^k$-game is an infinite version of the n-round game in which the play continues through a sequence of rounds of order type ω. The Spoiler wins the game, if and only if, he wins at the n^{th}-round for some $n \in \omega$ as above; otherwise, the Duplicator wins. The following proposition provides the link between the $\exists^k$-game and logical definability.

PROPOSITION 1.2 ([**KV90**]).
1. *For all structures A and B, the following conditions are equivalent.*
   (a) $A \preceq^{k,n} B$.
   (b) *The Duplicator has a winning strategy for the n-round $\exists^k$-game from A to B.*
2. *For all structures A and B, the following conditions are equivalent.*
   (a) $A \preceq^k_{\infty\omega} B$.
   (b) *The Duplicator has a winning strategy for the eternal $\exists^k$-game from A to B.*
3. *For all structures A and finite structures B, the following conditions are equivalent.*
   (a) $A \preceq^k_{\infty\omega} B$.
   (b) $A \preceq^k B$.

In this paper, we will focus our attention on the class of finite simple graphs, that is, finite structures with one binary relation which is irreflexive and symmetric. We will use the term graph to refer to such structures. In general, we let $A, B, \ldots$ refer both to graphs and to their underlying vertex sets and we let $|A|$ denote the cardinality of A. We use E for the edge relation of a graph. $\mathrm{Edges}(A)$ is the edge set of the graph A, that is, $\mathrm{Edges}(A) = \{\{a, b\} \subseteq A : E(a, b)\}$.

2. k-Universal Graphs: Definability and Structure

We say that a graph G is *k-universal*, if and only if, for all graphs H, $H \preceq^k G$. By Proposition 1.2, this is equivalent to G satisfying every sentence of $L^k_{\infty\omega}(\exists)$ which is satisfied by some (possibly infinite) graph. We say that a graph G is *k-extendible*, if and only if, $k \leq |G|$ and for each $1 \leq l \leq k$

$$G \models \forall x_1 \ldots \forall x_{k-1} \exists x_k \Big(\bigwedge_{1 \leq i < j \leq k-1} x_i \neq x_j \rightarrow \Big(\bigwedge_{1 \leq i \leq k-1} x_i \neq x_k \wedge \bigwedge_{1 \leq i < l} E(x_i, x_k) \wedge \bigwedge_{l \leq i < k} \neg E(x_i, x_k) \Big) \Big).$$

It is easy to verify, by applying Proposition 1.2, that every k-extendible graph is k-universal. The class of k-extendible graphs plays an important role in the study of $0-1$ laws for certain infinitary logics and logics with fixed point operators (see [**KV92**]). Indeed, the existence of k-universal finite graphs follows immediately from the fact that for every k, the random graph $G = G(n,p)$ with constant edge probability $0 < p < 1$ is almost surely k-extendible (see, for example, [**Bol79**]).

Let $\mathcal{U}^k$ be the class of k-universal graphs and let

$$\Xi^k = \{\theta \in L^k(\exists) : \exists G(G \text{ is a graph and } G \models \theta)\}.$$

Note that for all graphs $G, G \in \mathcal{U}^k$, if and only if, $G \models \bigwedge \Xi^k$. Thus, $\mathcal{U}^k$ is definable in $\bigwedge L^k(\exists)$ over the class of graphs. In [**RW95**], we established via an explicit construction that for all $2 \leq k$, $\mathcal{U}^k$ is not definable in $\bigvee L^k(\exists)$. The following theorem significantly strengthens this result for large enough k; its proof involves a probabilistic construction employing techniques from the theory of sparse random graphs.

THEOREM 2.1. *For all $k \geq 7$ and $k' \in \omega$, $\mathcal{U}^k$ is not definable in $\bigvee L^{k'}(\exists)$ over the class of graphs.*

We call a class of structures $\mathcal{C}$ finitely based, if and only if, there is a finite set of structures $\{A_1, \ldots, A_n\} \subseteq \mathcal{C}$ such that for every structure $B \in \mathcal{C}$, $A_i \subseteq B$ for some $1 \leq i \leq n$. We obtain the following result as a corollary to the proof of Theorem 2.1.

COROLLARY 2.2. *For all $k \geq 7$,*

1. *$\mathcal{U}^k$ is not finitely based, and*
2. *the class of k-extendible graphs is not finitely based.*

In [**RW95**], we observed that for all k, $\mathcal{U}^k$ is decidable in deterministic polynomial time. The following theorem gives a stronger "descriptive complexity" result.

THEOREM 2.3. *For all k, $\mathcal{U}^k$ is definable in least fixed point logic.*

It is clear that if G is k-extendible and $G \subseteq H$, then H is k-universal. The question naturally arises whether there are k-universal graphs which contain no k-extendible subgraph. The following theorem answers this question affirmatively.

THEOREM 2.4. *For each $k \geq 4$, there is a graph G such that*

1. *G is k-universal, and*
2. *$\forall H \subseteq G, H$ is not k-extendible.*

The next theorem is a strengthening of the first part of Corollary 2.2. The proof of this theorem expands on the construction developed to prove Theorem 2.4. We say a graph G is a *minimal* k-universal graph just in case G is k-universal and contains no proper induced subgraph which is k-universal.

THEOREM 2.5. *For all $k \geq 6$, there is an infinite set of pairwise L^k-inequivalent minimal k-universal graphs.*

We proceed to prove the above results. Theorem 2.1 is an immediate corollary of the following lemma which is due to Shelah.

LEMMA 2.6. *For all $k \geq 7$ and $k' \in \omega$, there is a graph N such that*

1. *N is k-extendible and*

2. *for every* $\theta \in L^{k'}(\exists)$, *if* $N \models \theta$, *then there is a structure* M *such that* $M \models \theta$ *and* M *is not* k*-universal.*

We approach the proof of Lemma 2.6 through a sequence of sublemmas. We first introduce some graph-theoretic concepts which play a central role in the argument.

DEFINITION 2.7. Let A be a finite graph.

1. We say $\overline{a} = \langle a_1, \ldots, a_n \rangle$ is a *t-witness* for A, if and only if, $\overline{a}$ is an injective enumeration of A and for each $i \leq n, |\{j < i : E(a_j, a_i)\}| \leq t$.
2. $\chi^*(A) =$ the least t such that there is a t-witness for A. ($\chi^*(A)$ is the *coloring number* of A.)
3. $K_t^\infty = \{A : \chi^*(A) \leq t\}$.
4. $A \leq_t^\otimes B$, if and only if, $A \subseteq B, B \in K_t^\infty$ and every t-witness for A can be extended to a t-witness for B, that is, if $\overline{a}$ is a t-witness for A, then there is a $\overline{b}$ such that $\overline{a}\overline{b}$ is a t-witness for B.

The coloring number was introduced and extensively studied in [**EH66**]. The following sublemma states a free amalgamation property of $\leq_t^\otimes$.

DEFINITION 2.8. Let A and B be finite graphs.

1. A is *compatible* with B, if and only if, the subgraph of A induced by $A \cap B$ is identical to the subgraph of B induced by $A \cap B$.
2. Suppose A is compatible with B and let C be the subgraph of A induced by $A \cap B$. The *free join* of A and B over C, denoted by $A \otimes_C B$, is the graph whose vertex set is $A \cup B$ and whose edge set is $\text{Edges}(A) \cup \text{Edges}(B)$.

SUBLEMMA 2.9. *Suppose* $A, B \in K_t^\infty$, A *is compatible with* B, C *is the subgraph of* A *induced by* $A \cap B$, $C \leq_t^\otimes A$, *and* $C \leq_t^\otimes B$. *Then,* $A \otimes_C B \in K_t^\infty$, $A \leq_t^\otimes A \otimes_C B$, *and* $B \leq_t^\otimes A \otimes_C B$.

PROOF. The sublemma follows immediately from the definitions. ∎

The next sublemma establishes a lower bound on $\chi^*(G)$ when G is k-universal. For the proof of the sublemma we extend the definition of k-universality to apply also to tuples. We also introduce a refinement of the concept that will be used in the proof of Theorem 2. An m-tuple $\overline{a} = (a_1, \ldots, a_m)$ is *proper* iff for all $i < j \leq m, a_i \neq a_j$. For all models A and B, and j-tuples $\overline{a} \subseteq A, \overline{b} \subseteq B$, we write $(A, \overline{a}) \preceq^k (B, \overline{b})$ ($(A, \overline{a}) \preceq^{k,n} (B, \overline{b})$) iff for all formulas $\theta(\overline{x}) \in L^k(\exists)$ (with $qr(\theta) \leq n$), with j free variables, if $A \models \theta[\overline{a}]$, then $B \models \theta[\overline{b}]$.

DEFINITION 2.10. For $j \leq k$, a proper j-tuple $\overline{a} \subseteq A$ is *k-universal in A* (*k,n-universal in A*) iff for all B, and proper j-tuples $\overline{b} \subseteq B$ such that the partial function $f(x)$ from A to B that maps a_i to b_i is a partial isomorphism, $(B, \overline{b}) \preceq^k (A, \overline{a})$ ($(B, \overline{b}) \preceq^{k,n} (A, \overline{a})$). The *rank* of $\overline{a} \subseteq A$ is ω if it is k-universal, and the greatest n such that it is k, n-universal, otherwise.

SUBLEMMA 2.11. *If* $\chi^*(G) < 2^{k-2}$, *then* G *is not* k*-universal.*

PROOF. Suppose $\chi^*(G) < 2^{k-2}$, and, for *reductio*, that G is k-universal. Suppose $G = \{a_i : i < n\}$, and let

$$I = \{\langle i_1, \ldots, i_k \rangle : i_1 < i_2 < \ldots < i_k < n \text{ and } \langle a_{i_1}, \ldots, a_{i_k} \rangle \text{ is } k-\text{universal in } G\}.$$

Since G is k-universal, it follows that $I \neq \emptyset$. Let $\langle i_1, \ldots, i_k \rangle \in I$ with i_k maximal. Let $w = \{j < i_k : E(a_j, a_{i_k})\}$, and for each $j \in w$, let $u_j = \{l : l \in$

$\{1,\ldots,k-1\}$ and $E(a_j,a_{i_l})\}$. Choose $l^*\in\{1,\ldots,k-1\}$. As $|w|<2^{k-2}$, there is $u\subseteq\{1,\ldots,k-1\}-\{l^*\}$ such that for every $j\in w, u\neq u_j-\{l^*\}$.

Now, let H be a k-extendible graph with edge relation E'. Since $\langle a_{i_1},\ldots,a_{i_k}\rangle$ is k-universal in G, we may choose $b_1,\ldots,b_k\in H$ such that the Duplicator has a winning strategy for the $\exists^k$-game played from H to G with the j^{th} pair of pebbles placed on b_j and a_{i_j}. We show that, in fact, the Spoiler can force a win from this position, which yields the desired contradiction. The Spoiler picks up the pebble resting on b_{l^*} and places it on a point $b\in H-\{b_1,\ldots,b_k\}$ such that $E'(b,b_k)$ and $E'(b,b_l)$ for each $l\in u$ while $\neg E'(b,b_l)$ for each $l\in\{1,\ldots,k-1\}-(u\cup\{l^*\})$. In order to successfully answer the Spoiler's move, the Duplicator must move the pebble now resting on $a_{i_{l^*}}$ and place it on a point $a_m\in G$ such that $E(a_m,a_{i_k})$ and $a_m\neq a_{i_k}$. In order to achieve this, she must choose a_m so that either $i_k<m$ or $m\in w$. But in the first case we would have that the position $\langle\ldots,\langle b_j,a_{i_j}\rangle,\ldots,\langle b_k,a_{i_k}\rangle,\langle b,a_m\rangle : j\neq l^*\rangle$ is a winning position for the Duplicator in the $\exists^k$-game from H to G. This implies that $\langle\ldots,a_{i_j},\ldots,a_{i_k},a_m : j\neq l^*\rangle$ is k-universal in A. But then, since $i_k<m$, we have $\langle\ldots,i_j,\ldots,i_k,m : j\neq l^*\rangle\in I$. But, this contradicts the choice of i_k to be maximal with this property. Therefore, it suffices to show that $m\notin w$. But this follows immediately from the fact that $m<i_k$ and the construction of u. ■

The next sublemmas deal with the theory of the random graph $G=G(n,n^{-\alpha})$, α an irrational between 0 and 1, as developed in [**SS88**] (see also [**BS95**] for connections with model theory). We say a property holds almost surely (abbreviated a.s.) in $G(n,n^{-\alpha})$, if and only if, its probability approaches 1 as n increases. Shelah and Spencer showed (see [**SS88**]) that for any first-order property θ and any irrational α between 0 and 1, either θ holds a.s. in $G(n,n^{-\alpha})$ or $\neg\theta$ holds a.s. in $G(n,n^{-\alpha})$. For each such α, we let $T^\alpha=\{\theta : \theta \text{ holds a.s. in } G(n,n^{-\alpha})\}$ and we let K^α_∞ be the set of finite graphs each of which is embeddable in every model of T^α. We will suppress the superscripts on these notations, when no confusion is likely to result; in general, we will use notations which leave reference to a particular α implicit, as in the following definition.

Definition 2.12 ([**SS88**]). Let G and H be graphs with $G\subseteq H$, and let α be a fixed irrational between 0 and 1.

1. (G,H) is *sparse*, if and only if, $|\text{Edges}(H)-\text{Edges}(G)|/|H-G|<1/\alpha$.
2. (G,H) is *dense*, if and only if, $|\text{Edges}(H)-\text{Edges}(G)|/|H-G|>1/\alpha$.
3. $G\leq_s H$, if and only if, for every I, if $G\subset I\subseteq H$, then (G,I) is sparse.
4. $G\leq_i H$, if and only if, for every I, if $G\subseteq I\subset H$, then (I,H) is dense.

We say G is *sparse* (*dense*), if and only if, $(\emptyset,G)$ is sparse (dense).

Note that since α is irrational every (G,H) as above is either sparse or dense.

Sublemma 2.13. *If $G\in K_\infty$, then $\emptyset\leq_s G$.*

Proof. The reader may find a proof of this sublemma in [**Spe90**]. ■

Sublemma 2.14. *If α is irrational and $1/(k+1)<\alpha<1$, then*

1. *$K_\infty\subseteq K^\infty_{(2k+1)}$ and*
2. *if $A\leq_s B$, then $A\leq^\otimes_{2k+1} B$.*

Proof. 1. By Sublemma 2.13, it suffices to show that if $\emptyset\leq_s G$, then $G\in K^\infty_{2k+1}$. So suppose $\emptyset\leq_s G$. We inductively define a $2k+1$-witness for G proceeding

from the top down. Since G is sparse, $|\text{Edges}(G)|/|G| < k+1$, from which it follows immediately that there is a point $a \in G$ whose degree is $< 2k+2$. We let $a = a_{|G|}$ be the last element of our $2k+1$-witness for G. Now, since $\emptyset \leq_s G$, $G' = G - \{a\}$ is sparse, so we may find an $a' \in G'$ whose degree (in G') is $< 2k+2$ as before. We let $a' = a_{|G|-1}$ be the next to last element of our $2k+1$-witness for G. Proceeding in this way, we may complete the construction of a $2k+1$-witness for G.

2. Suppose $A \leq_s B$ and suppose $\overline{a}$ is a $2k+1$-witness for A. Just as above we may inductively construct an enumeration $\overline{b}$ of $B - A$ so that $\overline{a}\overline{b}$ is a $2k+1$-witness for B. ∎

The following closure operator plays an important role in the proof of Lemma 2.6.

DEFINITION 2.15. We define for graphs G, H with $G \subseteq H$ and natural numbers l, a closure operator $\text{cl}^{l,m}(G,H)$ by recursion on m.

1. $\text{cl}^{l,0}(G,H) = G$;
2. $\text{cl}^{l,m+1}(G,H) = \bigcup\{B : B \subseteq H \text{ and } |B| \leq l \text{ and } B \cap \text{cl}^{l,m}(G,H) \leq_i B\}$.

We let $\text{cl}^{l,\infty}(G,H) = \bigcup_{m\in\omega} \text{cl}^{l,m}(G,H)$. We say that H is *l-small*, if and only if, there is a $G \subseteq H$ such that $|G| \leq l$ and $\text{cl}^{l,\infty}(G,H) = H$.

The following lemma gives the crucial property of closures we will exploit – for a fixed l there is almost surely in $G(n, n^{-\alpha})$ a uniform bound on the cardinality of the closure of a set of size at most l.

SUBLEMMA 2.16. *For every l there is an l^* such that* a.s. *for every $A \subseteq G(= G(n,n^{-\alpha}))$, if $|A| \leq l$, then $|\text{cl}^{l,\infty}(A,G)| \leq l^*$.*

PROOF. Note that if $B \leq_i B'$ and $B \subseteq C \subseteq B'$, then $C \leq_i B'$. It follows that we may represent $\text{cl}^{l,\infty}(A,G)$ as $A \cup \bigcup_{i<i^*} B_i$ where $|B_i| \leq l$ and $(A \cup \bigcup_{j<i} B_j) \cap B_i \leq_i B_i$. Moreover, we may suppose, without loss of generality, that this last extension is strict, for otherwise B_i could be omitted from the representation. Next we argue that there is an m (depending on l) which a.s. uniformly bounds i^*, that is, there is an m such that

> (†) a.s. in $G = G(n,n^{-\alpha})$ for all $A \subseteq G, |A| \leq l$, there is an $i^* \leq m$ such that $\text{cl}^{l,\infty}(A,G)$ may be represented as $A \cup \bigcup_{i<i^*} B_i$ where $|B_i| \leq l$ and $(A \cup \bigcup_{j<i} B_j) \cap B_i \leq_i B_i$.

The sublemma follows immediately from this, for then $l^* = m \cdot l$ is an a.s. uniform bound on $|\text{cl}^{l,\infty}(A,G)|$.

Let

$$\varepsilon = \text{Min}(\{(\alpha \cdot |\text{Edges}(B) - \text{Edges}(C)|) - (|B - C|) : B \subseteq G, |B| \leq l, A \cap B \leq_i B, A \cap B \subseteq C \subset B\}).$$

It follows from the definition of $\leq_i$ that $\varepsilon > 0$. Let $m = 1 + l/\varepsilon$. We claim that m satisfies condition (†). Let

$$w_i = |A \cup \bigcup_{j<i} B_j| - \alpha \cdot |\text{Edges}(A \cup \textstyle\bigcup_{j<i} B_j)|.$$

Then, by hypothesis, $w_0 \leq |A| \leq l$. Moreover, $w_{i+1} \leq (w_i - \varepsilon)$. To see this, let $C = B_i \cap (A \cup \bigcup_{j<i} B_j))$. Then, $A \cap B_i \subseteq C \subset B_i$. Hence, $w_{i+1} = |(A \cup \bigcup_{j<i} B_j) \cup B_i| - \alpha \cdot |\text{Edges}((A \cup \bigcup_{j<i} B_j) \cup B_i)| \leq (|A \cup \bigcup_{j<i} B_j| + |B_i - C|) -$

$\alpha \cdot (|\text{Edges}(A \cup \bigcup_{j<i} B_j)| + (|\text{Edges}(B_i)| - |\text{Edges}(C)|)) \leq (w_i - \varepsilon)$. It follows, by induction, that $w_i \leq l - i \cdot \varepsilon$. Therefore, if $i > l/\varepsilon$, then $w_i < 0$. So, by Sublemma 2.13, if $i^* \geq m$, then $\text{cl}^{l,\infty}(A, G) = A \cup \bigcup_{i<i^*} B_i \notin K_\infty$. Therefore, a.s. $i^* < m$. ■

For the purposes of the next sublemma and beyond, we introduce the following notational convention: we write $A \leq^{\otimes} B$ for $A \leq^{\otimes}_t B$, when $t = 2^{k-2} - 1$.

SUBLEMMA 2.17. *If α is irrational, $1/(k+1) < \alpha < 1, k \geq 7$ and $k+1 < k'$ then the following condition holds* a.s. *in $G = G(n, n^{-\alpha})$. For all $a_1, \ldots, a_{k'} \in G$, if $A = \text{cl}^{k',\infty}(\{a_1, \ldots, a_{k'-1}\}, G)$ and $B = \text{cl}^{k',\infty}(\{a_1, \ldots, a_{k'}\}, G)$, then*

1. *$B \in K_\infty$ and*
2. *$A \leq^{\otimes} B$.*

PROOF. 1. This is an immediate consequence of the preceding Sublemma. By the first-order 0-1 law for $G(n, n^{-\alpha})$, given any fixed bound l^*, a.s. for all $A \subseteq G$, if $|A| \leq l^*$, then $A \in K_\infty$.
2. First observe that our closure operator is monotone in $\subseteq$, hence $A \subseteq B$ and also, by the definition of the closure operator, that for no $C \subseteq B, C \not\subseteq A, |C| \leq k'$ do we have $A \cap C \leq_i C$. We argue that $A \leq^{\otimes} B$ as follows. Suppose $\overline{a} = \langle a_1, \ldots, a_{|A|} \rangle$ is a $2^{k-2} - 1$-witness for A, and let $\overline{b} = \langle b_1, \ldots, b_{|B|} \rangle$ be a $2k+1$-witness for B. The latter exists by Sublemma 2.14 since $B \in K_\infty$. Now, for every $b \in B - A, |\{a \in A : E(a, b)\}| \leq k$, for otherwise we could find a set $C \subseteq B, C \not\subseteq A, |C| = k+2$, such that $A \cap C \leq_i C$. Let $w = \{i : 1 \leq i \leq |B| \text{ and } b_i \notin A\}$, and let $\overline{b'} = \langle b_i : i \in w \rangle$ be the restriction of $\overline{b}$ to an enumeration of $B - A$. By hypothesis, $k \geq 7$, so $(2k+1) + k \leq 2^{k-2} - 1$; hence, we may conclude that $\overline{a}\overline{b'}$ is a $2^{k-2} - 1$-witness for B. ■

SUBLEMMA 2.18. *If $0 < \alpha < 1/k$, then $G(n, n^{-\alpha})$ is* a.s. *k-extendible.*

PROOF. The reader may find a proof of this sublemma in [**McA95**]. ■

We are now in a position to proceed to the proof of Lemma 2.6.

PROOF OF LEMMA 2.6. Let $k \geq 7$ and, without loss of generality, let $k' > k+1$. Fix α to be an irrational number between $1/(k+1)$ and $1/k$. It then follows from Sublemmas 2.17 and 2.18 that there is a finite graph N such that

(N1) N is k-extendible;
(N2) for all $a_1, \ldots, a_{k'} \in N$, if $A = \text{cl}^{k',\infty}(\{a_1, \ldots, a_{k'-1}\}, N)$ and $B = \text{cl}^{k',\infty}(\{a_1, \ldots, a_{k'}\}, N)$, then $B \in K_\infty$ and $A \leq^{\otimes} B$.

To complete the proof we must construct for each $\theta \in L^{k'}(\exists)$, a graph M such that M is not k-universal and if $N \models \theta$, then $M \models \theta$. By Sublemma 2.11 and Proposition 1.2, it suffices to construct for each $d \in \omega$ a graph M such that

(M1) $\chi^*(M) < 2^{k-2}$, and
(M2) the Duplicator has a winning strategy for the d-move $\exists^{k'}$-game from N to M.

We proceed to construct a structure M that satisfies conditions (M1) and (M2). We first define chains of structures $\langle M_i : i \leq d+1 \rangle$ and $\langle M_{i,j} : i \leq d, j \leq j_i \rangle$, satisfying the following conditions.

1. If $A \subseteq M_i, A \leq^{\otimes} B, B \in K_\infty$, and B is k'-small, then for some $j < j_i, A = A_{i,j}$ and B and $B_{i,j}$ are isomorphic over A.

2. $M_0 = \emptyset$.
3. For all $i \leq d+1, \chi^*(M_i) < 2^{k-2}$.
4. For each $i \leq d, M_{i,0} = M_i$ and $M_{i,j_i} = M_{i+1}$.
5. For each $j < j_i$, there are $A_{i,j}, B_{i,j}$ with
   (a) $B_{i,j}$ is k'-small;
   (b) $B_{i,j} \in K_\infty$;
   (c) $A_{i,j} \subseteq M_i$;
   (d) $A_{i,j} \leq^{\otimes} B_{i,j}$;
   (e) $B_{i,j}$ is compatible with $M_{i,j}$ and $A_{i,j}$ is the subgraph of $M_{i,j}$ induced by $B_{i,j} \cap M_{i,j}$;
   (f) $M_{i,j+1} = M_{i,j} \otimes_{A_{i,j}} B_{i,j}$;

By Sublemma 2.16, there are only finitely many k'-small $B \in K_\infty$. The existence of chains satisfying the above conditions then follows immediately from the free amalgamation property for $\leq^{\otimes}$ stated in Sublemma 2.9.

We now let $M = M_{d+1}$. It follows immediately from the construction that M satisfies condition (M1) above. Thus, it only remains to show that M satisfies condition (M2). In order to do so, it suffices to verify the following claim which supplies a winning strategy for the Duplicator in the d-move $\exists^{k'}$-game from N to M.

> *Claim*: Suppose $A = \{a_1, \ldots, a_{k'}\} \subseteq N, A' = \mathrm{cl}^{k',\infty}(A, N)$ and f is an embedding of A' (the subgraph of N induced by A') into $M_{(d+1)-i}$. Then the pebble position with α_r on a_r and β_r on $f(a_r)$, for $1 \leq r \leq k'$ is a winning position for the Duplicator in the i-move $\exists^{k'}$-game from N to M.

We proceed to establish the claim by induction. Given $1 \leq i \leq d$, suppose that A, A', f, and the pebble position are as described. It suffices to show that given any move by the Spoiler, the Duplicator can respond with a move into $M_{(d+1)-(i-1)}$ which will allow the conditions of the claim to be preserved. Suppose, without loss of generality, that the Spoiler moves $\alpha_{k'}$ onto a vertex $a \in N$. Let $A'' = \mathrm{cl}^{k',\infty}(\{a_1, \ldots, a_{k'-1}\}, N)$ and let $A''' = \mathrm{cl}^{k',\infty}(\{a_1, \ldots, a_{k'-1}, a\}, N)$. Then, by condition (N2), $A''' \in K_\infty$ and $A'' \leq^{\otimes} A'''$. Then, by condition 5 on the construction of our chains defining M, there is a $B \subseteq M_{(d+1)-(i-1)}$ and an isomorphism f' from A''' onto B with f' and f having identical restrictions to A''. Therefore, the conditions of the claim will be preserved, if the Duplicator plays pebble $\beta_{k'}$ onto $f'(a)$. ∎

Proof of Corollary 2.2. Let $k \geq 7$. 1. Suppose, for *reductio*, that $\mathcal{U}^k$ is finitely based with "basis" $\{A_1, \ldots, A_n\}$. Let k' be the maximum of the cardinalities of the A_i. Then, there is a sentence of $L^{k'}(\exists)$ which defines $\mathcal{U}^k$, contradicting Theorem 2.1.

2. Suppose for *reductio* that the class of k-extendible structures is finitely based and choose k' as above with respect to a "basis" for this class. As in the proof of Lemma 2.6, there is a k-extendible graph N such that each $L^{k'}(\exists)$ sentence true in N has a model which is not k-universal and hence not k-extendible. This implies that every submodel of N of size at most k' is not k-extendible, which yields the desired contradiction. ∎

Proof of Theorem 2.3. We show that the complement of $\mathcal{U}^k$ is definable in least fixed point logic, which is sufficient since the language is closed under negation.

In fact, it is defined by a purely universal sentence. The main idea is to show that for all A, $A \notin \mathcal{U}^k$ iff either $card(A) < k-1$ or for all proper $k-1$-tuples $\overline{a} \subseteq A$, $\overline{a}$ is not k,m-universal for some $m \in \omega$. Equivalently, every proper $k-1$-tuple has finite rank. This follows easily from the following sequence of observations.

1. For all A, A is k-universal iff there is a proper $k-1$-tuple $\overline{a} \subseteq A$ such that $\overline{a}$ is k-universal in A.
2. For all A, and every proper $k-1$-tuple $\overline{a} \subseteq A$, $\overline{a}$ is k-universal in A iff $\overline{a}$ is k,m-universal in A, for all $m \in \omega$.
3. For every A and proper $k-1$-tuple $\overline{a}$, if $\overline{a}$ has rank $m+1$ in A, then there is some set $S \subseteq \{1,\ldots,k-1\}$ and formula $\varphi(x_1,\ldots,x_k) = \bigwedge_{i<k} x_i \neq x_k \wedge \bigwedge_{i\in S} E(x_i,x_k) \wedge \bigwedge_{i\notin S} \neg E(x_i,x_k)$, such that for all $a' \in A$, if $A \models \varphi(\overline{a}a')$, then $\overline{a}a'$ has rank $\leq m$.

Observations 1 and 2 essentially follow immediately from the definitions. Observation 3 may be verified by considering the k-extendible models.

The above conditions yield an easy inductive definition of all the proper $k-1$-tuples that are not k-universal. Call a formula of the form of φ above a *k-extension formula*. Let $\varphi_1,\ldots,\varphi_t$ be the set of k-extension formulas. By observation 3, a proper $k-1$-tuple $\overline{a}$ has rank 0 iff there is some k-extension formula φ such that there is no a' such that $A \models \varphi(\overline{a}a')$; and $\overline{a}$ has rank $\leq m+1$ iff there is some k-extension formula φ such that for all a', if $A \models \varphi(\overline{a}a')$, then $\overline{a}a'$ has rank $\leq m$.

We now show how to express this definition by a least fixed point formula. Let $\theta(x_1,\ldots,x_{k-1})$ be the following formula:

$$\bigvee_{i<j\leq k-1} x_i = x_j \vee \bigvee_{s\leq t} \forall x_k(\neg\varphi_s(\overline{x}x_k) \vee \bigvee_{j\leq k} R(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_k)).$$

R appears positively in the formula, so that θ defines an inductive operator on each graph $G, \Theta_G(X)$, that maps $k-1$-ary relations P to $k-1$-ary relations $\Theta_G(P)$. Let $\Theta_G^0 = \Theta_G(\emptyset)$, and let $\Theta_G^{n+1} = \Theta_G(\Theta_G^n)$. If $\Theta_G^{n+1} = \Theta_G^n$, then Θ_G^n is a fixed point of the operator. In fact, it is the least fixed point, which we denote Θ_G^∞. Observe that for all proper $k-1$-tuples $\overline{a},\overline{a} \in \Theta_G^{n+1} - \Theta_G^n$ iff the rank of $\overline{a}$ is n. By the above observation, G is k-universal iff $\Theta_G^\infty = A^{k-1}$. Therefore, the following formula defines the class of graphs that are not in $\mathcal{U}^k$.

$$\forall x_1 \ldots x_{k-1} \bigvee_{i<j\leq k-1} x_i = x_j \vee \forall x_1 \ldots x_{k-1} \Theta_G^\infty(x_1,\ldots,x_{k-1})$$

This completes the proof. ∎

Proof of Theorem 2.4. Let $k \geq 4$. We construct G as follows. Let V be the set of binary sequences of length k, that is, V is the set of $0,1$-valued functions with domain $\{1,\ldots,k\}$. For each $1 \leq i \leq k$, let $V_i = V \times \{i\}$ and let $U = \bigcup_{1\leq i\leq k} V_i$. U is the set of vertices of the graph G. The edge relation E of G is defined as follows:

$$E((f,i),(g,j)) \longleftrightarrow (i \neq j \wedge f(j) = g(i)).$$

We proceed to verify that G satisfies the conditions of the theorem.

First we show that G is k–universal. Let H be an arbitrary graph. We describe a winning strategy for the Duplicator in the $\exists^k$-game from H to G. At each round the Duplicator plays so as to pebble at most one element of each V_i. We may suppose without loss of generality that all k pebbles are on the board at round s, that the Duplicator has played β_i on an element of V_i, and that the map from the elements

pebbled in H to the corresponding elements pebbled in G is a partial isomorphism. Suppose the Spoiler plays α_j onto an element $b \in H$ at round $s+1$ and let X be the set of i such that there is an edge between b and the vertex of H pebbled by α_i. Let (f_i, i) be the vertex of G pebbled by β_i at round s. We must show that the Duplicator may play β_j at round $s+1$ onto a vertex $(g, j) \in V_j$ such that for all $1 \leq i \leq k$,

$$E((g,j),(f_i,i)) \longleftrightarrow i \in X.$$

It is clear that (g, j) satisfies this condition when g is defined as follows: $g(i) = f_i(j)$, if $i \in X$; $g(i) = 1 - f_i(j)$, if $i \notin X$. This completes the proof that G is k-universal.

Let $H \subseteq G$, and suppose, for *reductio*, that H is k-extendible. It is easy to verify that any graph H is k-extendible iff for all j-tuples $\overline{a}$ in H, $j \leq k$, $\overline{a}$ is k-universal in H. To establish the contradiction, we show that there are $a_1, a_2 \in H$ such that (a_1, a_2) is not k-universal in G, which immediately implies that (a_1, a_2) is not k-universal in H either.

The cardinality of any k-extendible graph is $\geq k+1$, so there is an $l \leq k$ such that H contains two vertices, $(f_1, l), (f_2, l)$, in V_l. Let $w' = \{j \mid j \neq l \text{ and } f_1(j) \neq f_2(j)\}$ and let $w'' = \{j \mid j \neq l \text{ and } f_1(j) = f_2(j)\}$. Let $w = w'$, if $|w'| \leq |w''|$, and let $w = w''$, otherwise. Observe that $|w| \leq (k-1)/2$, which is $< k-2$ for all $k \geq 4$. We now show that $(f_1, l), (f_2, l)$ is not $k, |w|+1$-universal in G. Suppose that $w = w'$. Let $\theta(x_1, \ldots, x_{|w|+3}) =$

$$\bigwedge_{1 \leq i < j \leq |w|+3} x_i \neq x_j \wedge \bigwedge_{3 \leq i \leq |w|+3} (E(x_1, x_i) \wedge \neg E(x_2, x_i)) \wedge \bigwedge_{3 \leq i < j \leq |w|+3} E(x_i, x_j).$$

(Note that $|w|+3 \leq k$, since $k \geq 4$.) Observe that for any $|w|+3$-tuple $\overline{a} = (a_1, \ldots, a_{|w|+3})$ such that $a_1 = (f_1, l)$ and $a_2 = (f_2, l)$, $G \not\models \theta(\overline{a})$. If we let

$$\varphi(x_1, x_2) = \exists x_3 \ldots x_{|w|+3} \theta(x_1, \ldots, x_{|w|+3}),$$

then it follows that $G \not\models \varphi((f_1, l), (f_2, l))$. Therefore $((f_1, l), (f_2, l))$ is not $k, |w|+1$-universal in G. The argument for $w = w''$ is similar. ∎

The above construction may be extended to arbitrary finite relational signatures.

Proof of Theorem 2.5. Let $k \geq 6$. For all $n \geq 4k$, we construct graphs G_n such that:

1. G_n is k-universal.
2. For all $H \subseteq G_n$, if H is k-universal, then the diameter of H is $\geq \lfloor (n-1)/2 \rfloor / (k-1)$.

(Recall that the diameter of a graph is the maximum distance between any two vertices if it is connected, and ω otherwise. It is an easy exercise to show that for $k \geq 3$, every minimal k-universal graph is connected.) This immediately yields the fact that there are minimal k-universal models of arbitarily large finite diameter. It is easy to check that the property of having finite diameter $= d$ is expressible in L^3, which implies that any two graphs with different diameters are L^k-inequivalent.

The graphs G_n are based on a modification of the construction from the proof of Theorem 3. Let V be the set of functions from the interval $\{-(k-2), \ldots, k-2\}$ into $\{0, 1\}$. For each $m, 0 \leq m \leq n-1$, let $V_m = \{0,1\} \times V \times \{m\}$. The set of vertices of G_n is $\bigcup_m V_m$. The edge relation on G_n is defined as follows. For all m, m', $a \in V_m, a' \in V_{m'}$, if $m = m'$ or $k \leq m - m' \leq n - k (mod\ n)$, then $\neg E(a, a')$.

If $0 < m - m' < k - 1 (mod\ n)$, and $a = (\delta, f, m), a' = (\delta', f', m')$, with $\delta, \delta' \in \{0, 1\}$ and $f, f' \in V$, then $E(a, a')$ iff $f'(m - m') = f(m' - m)$. (Here, subtraction is *modulo* n.) Finally, if $m - m' = k - 1 (mod\ n)$, then $E(a, a')$ iff $\delta = 1$. In this case, each $a \in V_m$ is either adjacent to every vertex in $V_{m'}$ or to none of them. If $m' = m + \lfloor (n-1)/2 \rfloor$, then the distance $d(a, a') \geq \lfloor (n-1)/2 \rfloor / (k-1)$. Observe also that for all $l \leq n - 1$, there is an automorphism of G_n taking each V_m to V_{m+l}. (All indices are *modulo* n.)

First we show that G_n is k-universal. Let G' be an arbitary graph. It suffices to prove that the D wins the $\exists^k$-game from G' to G_n. By an argument similar to the one given in the proof of Theorem 3, it is easy to see that the D can play so that in each round $i \leq k$, she plays a pebble on a vertex in V_i. We now argue by induction that in each subsequent round $j > k$, she can maintain the following condition: there is some $l \leq n$ such that there is exactly one pebble on each V_m, for m such that $0 \leq m - l \leq k - 1 (mod\ n)$. The basis step is already taken care of. Suppose that in round j, the D has a single pebble in each vertex set $V_l, \ldots, V_{l+(k-1)}$. We consider two cases. One, the S replays the pebble α_i whose pair β_i in G_n is on an element of V_l. It is easy to verify that the D can respond by playing β_i on a vertex in V_{l+k}. Observe that the D's pebbles are now on $V_{l+1}, \ldots, V_{l+k}$, as desired. Two, the S replays any other pebble $\alpha_{i'}$, whose pair $\beta_{i'}$ is on some element of $V_{l'}, l \neq l'$. The D can respond by replaying the pebble on some other element of $V_{l'}$. Again, that this is possible essentially follows from the proof of Theorem 3.

Next we argue that any k-universal $H \subseteq G_n$ has diameter $\geq \lfloor (n-1)/2 \rfloor / (k-1)$. In particular, it is sufficient to prove H must contain a vertex from each $V_m, m \leq n - 1$. Let A be any k-extendible graph. The argument proceeds by establishing that, in the $\exists^k$-game from A to H, the S can eventually force the D to play a pebble on a vertex in each $V_m \cap H$. If $V_m \cap H = \emptyset$, for some m, then the D loses.

In rounds 1 through k, the S plays on a k-clique in A. For every k-clique in G_n, and hence also in H, there is an $m \leq n - 1$ such that each $V_{m'}, 0 \leq m' - m \leq k - 1 (mod\ n)$, contains exactly one element from the clique. Therefore, after k rounds, the D must have a single pebble on each of $V_m, \ldots, V_{m+(k-1)}$, for some m. It suffices to show that the S can force the D to play so that exactly one pebble occupies a vertex in each set $V_{m+1}, \ldots, V_{m+k}$, since by iterating this strategy, he can force the D to play onto each V_l.

To simplify the notation, we assume $m = 0$ and that each pebble $\beta_i, 0 \leq i \leq k - 1$, is on a vertex in V_i. Let $b_i = (\delta_i, f_i, i), \delta_i \in \{0, 1\}, f_i \in V$, be the element pebbled by β_i. In round $k + 1$, the S replays pebble α_0 and places it on an element $a \in A$ such that $E(a, \alpha_1)$ and for $i \in \{2, \ldots, k - 1\}$, $E(a, \alpha_i)$ iff $\delta_i = 0$. (Here we abuse notation and use α_j to refer also to the element on which the pebble is located.) Since α_0 and α_1 are now adjacent in A, the D has to play β_0 on some element in a set V_l, for $-(k-2) \leq l \leq k (mod\ n)$, so that it is adjacent to β_1.

By the condition that for $i \in \{2, \ldots, k - 1\}$, $E(a, \alpha_i)$ iff $\delta_i = 0$, the D cannot play in V_l, for $-(k-3) \leq l \leq 0 (mod\ n)$. If the D plays the pebble in V_k, then the S has succeeded. Suppose that the D plays β_0 on an element of $V_{-(k-2)}$. We now claim that there is no 3-clique in $G_n\ [H]$ each of whose elements is adjacent to both β_{k-1} and β_0. This is because (i) the only elements of G_n that are adjacent to vertices in both $V_{-(k-2)}$ and V_{k-1} are members of either V_0 or V_1, and (ii) there is no 3-clique in $V_0 \cup V_1$. Thus the S can force a win in 3 moves by replaying pebbles $\alpha_1, \alpha_2, \alpha_3$ so that they occupy a 3-clique each of whose elements are adjacent to α_0 and α_{k-1}.

The remaining case occurs when the D plays β_0 on a vertex in V_j, for $1 \leq j \leq k-1$. Without loss of generality, let $j = k-2$, and let b' be the vertex now occupied by β_0. Let $w' = \{i \mid 1 \leq i \leq 3 \text{ and } E(b_{k-2}, b_i) \text{ iff } E(b', b_i)\}$ and $w'' = \{i \mid 1 \leq i \leq 3 \text{ and } E(b_{k-2}, b_i) \text{ iff } \neg E(b', b_i)\}$. Again without loss of generality, suppose that $|w'| \geq 2$ and $w' = \{1, 2\}$. By exploiting the fact that β_0 and β_{k-2} both occupy vertices in V_{k-2}, the S can now force the D to play β_2 onto V_k.

The S first places α_2 on a vertex such that for all $j, 1 \leq j \leq k-1, j \neq 2$, $E(\alpha_2, \alpha_j)$, and $\neg E(\alpha_2, \alpha_0)$. It is easy to see that the D must put β_2 on either V_0 or V_k. In the first case, the S responds by playing α_1 so that for all $j, 2 \leq j \leq k-1$, $E(\alpha_1, \alpha_j)$ and $\neg E(\alpha_1, \alpha_0)$. The D now loses immediately. The only vertices adjacent to each $\beta_j, 2 \leq j \leq k-1$, are elements of V_1 or V_2, but for each $b \in V_1$ or V_2, $E(b, \beta_{k-2})$ iff $E(b, \beta_0)$. In the second case, the S then plays α_0 onto a vertex such that for all $j, 1 \leq j \leq k-1$, $E(\alpha_0, \alpha_j)$. This compels the D to play β_0 in V_2, so that there is a now a single pebble in each $V_1, \ldots, V_k$, as desired. ■

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