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Infinite-Dimensional Measure Spaces And Frame Analysis.

Palle E. T. Jorgensen · Myung-Sin Song

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Abstract We study certain infinite-dimensional probability measures in connection with frame analysis. Earlier work on frame-measures has so far focused on the case of finite-dimensional frames. We point out that there are good reasons for a sharp distinction between stochastic analysis involving frames in finite vs infinite dimensions. For the case of infinite-dimensional Hilbert space \mathcal{H} , we study three cases of measures. We first show that, for \mathcal{H} infinite dimensional, one must resort to infinite dimensional measure spaces which properly contain \mathcal{H} . The three cases we consider are: (i) Gaussian frame measures, (ii) Markov path-space measures, and (iii) determinantal measures.

Keywords Hilbert space, frames, reproducing kernel, Karhunen-Loève

1 Introduction and Setting.

Over the past two decades, frames have proved to be powerful tools in signal processing for a number of reasons, especially on account of their resilience to additive noise, to quantization; and because of their numerical stability in their use in the reconstruction step, they have improved our ability to capture significant signal characteristics. Frame theory is now a dynamic subject with applications that include variety of areas in both mathematics and engineering: operator theory, harmonic analysis, wavelet theory, sampling theory, nonlinear sparse approximation, wireless communication, data transmission with erasures, filter banks, signal

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processing, image processing, geophysics, quantum computing, sensor networks, and more.

In a host of applications, starting with signal and image processing, one makes use of inner-product spaces (in our case, choices of suitable Hilbert spaces) in achieving efficient signal-representations. In general, orthogonal expansions are not available. Nonetheless, in many signal processing problems, it is still possible to find overcomplete basis expansions, called frame expansions, see the references cited below. For example, in analysis/synthesis problems, when we sample an analog signal above the Nyquist rate, the amplitudes will be coefficients in a suitable frame expansion. Such decompositions have received extensive attention in the literature when the decomposition parameter is assumed discrete; but, nonetheless, the frame-decompositions take a probabilistic form which we shall emphasize below. More generally, staying with the overcomplete framework, we present here instead a continuous, and a more versatile, probability space approach to these expansion problems. Our applications include determinantal measures, Gaussian frame measures; and, more generally, to the setting of Markov path-space measures.

We study frames in Hilbert space \mathcal{H} , i.e., systems of vectors in \mathcal{H} which allow computable representations of arbitrary vectors in \mathcal{H} , (especially the case when \mathcal{H} is infinite-dimensional) in a way that is analogous to more familiar basis expansions. Frames are also called “over complete” systems, and they generalize the better known orthonormal bases (O.N.B.s). Their applications include signal processing and wavelet theory, to name only a few.

Frames (Definition 2) are systems of vectors in Hilbert space \mathcal{H} which allow for “effective” analysis and reconstruction for vectors in \mathcal{H} ; details below. As is known from the literature on frames, both pure and applied (see e.g., [Pe15], [FJMP15], [AYB15], [HoLa15], [PeHaMo15], [QuHiSh15]), there is a scale of “basis-like” properties that frames may have: in one end of the spectrum, there are the orthonormal bases (O.N.B.s), then Parseval frames, and in the other end of the spectrum, there are systems of vectors with frame-like properties, but where we may lack one of the two bounds, lower or upper, but nonetheless, for some other reason, we may still get analysis and reconstruction formulas.

Since the early days of Hilbert space axioms and quantum theory and potential theory, probability has always played an important role, for example such tools as balayage; but it is not until relatively recently that the role of probability has been studied systematically in connection with frame analysis. In our present approach, we have been especially inspired by the important paper by Ehler and Okoudjou [Eh12], [EhOk12], [EhOk13]; but the work in [Eh12], [EhOk12], [EhOk13]; and in related papers, has so far focused on the case of finite-dimensional frames. As we point out below, there are good reasons for a sharp distinction between stochastic analysis involving frames in finite vs infinite dimensions. The three cases, of measures in infinite dimensions we shall consider are the Gaussian measures, Markov path-space measures, and determinantal measures; but our present emphasis will be on certain Gaussian families (section 4).

In our paper, we shall adopt a general notion of probabilistic frames, referring simply to methods in frame theory involving probability and stochastic analysis. By “frame analysis” we shall refer to a setup where it is possible to construct the dual pair of operators, an *analysis operator*, and an associated *synthesis operator*. For technical details, see the next section.

Frames let us formulate a harmonic analysis of practical problems, but, so far, there are only few harmonic analysis tools available for the analysis of the frames themselves. Nonetheless, there are beginnings to a theory of “frame measures,” but so far only covering the case when \mathcal{H} is *finite-dimensional*. Our first result (section 3) shows that, unless one passes to a larger measure space, the notions of frame measures in finite dimension simply do not go over to infinite-dimensional \mathcal{H} . On the other hand, we show (section 4) that there is a way to build ambient measure spaces in such a way that we arrive at a rich family of *Gaussian wavelet measures*, covering the case when \mathcal{H} is infinite-dimensional.

We also study other families of measures associated with frames in infinite dimensions, e.g. Markov measures, and determinantal measures which seem promising. This endeavor takes advantage of the probabilistic features already inherent in the axioms of Hilbert space as they were developed in the foundations of quantum theory; i.e., the study of transition probability, referring to transition between states, for example states of different energy levels in atomic models.

The applied mathematicians who use frames have, so far, only developed very few quantitative gauges which will tell us how “different” two given frames might be; or will allow us to make precise “how much” better one frame is as compared to anyone in a set of alternatives.

1.1 Frame Measures

To help readers appreciate some key features regarding frame measures in the finite dimensional case, and their applications, we review some highpoints from [EhOk13].

In [Eh12], [EhOk12], [EhOk13]; finite frames in \mathbb{R}^N are considered, where frame vectors are viewed as discrete mass distributions on \mathbb{R}^N , the frame concepts are extended to probability measures, and the properties of probabilistic frames are summarized. Let $\mathcal{P} := \mathcal{P}(\mathcal{B}, \mathbb{R}^N)$ denote the collection of *probability measures* on \mathbb{R}^N with respect to the Borel σ -algebra \mathcal{B} . The support of $\mu \in \mathcal{P}$, denoted by $\text{supp}(\mu)$, is the set of all $x \in \mathbb{R}^N$ such that for all open neighborhoods $U_x \subset \mathbb{R}^N$ of x , we have $\mu(U_x) > 0$. Set $\mathcal{P}(K) := \mathcal{P}(\mathcal{B}, K)$ for those probability measures in \mathcal{P} whose support is contained in $K \subset \mathbb{R}^N$. The linear span of $\text{supp}(\mu)$ in \mathbb{R}^N is denoted by E_μ .

Definition 1 [EhOk13] A Borel probability measure $\mu \in \mathcal{P}$ is a *probabilistic frame* if there exists $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \int_{\mathbb{R}^N} |\langle x, y \rangle|^2 d\mu(y) \leq B\|x\|^2, \quad \text{for all } x \in \mathbb{R}^N. \quad (1)$$

The constants A and B are called lower and upper probabilistic frame bounds, respectively. When $A = B$, μ is called a tight probabilistic frame.

Let

$$\mathcal{P}_2 := \mathcal{P}_2(\mathbb{R}^N) = \{\mu \in \mathcal{P} : M_2^2(\mu) := \int_{\mathbb{R}^N} \|x\|^2 d\mu(x) < \infty\} \quad (2)$$

be the (convex) set of all probability measures with finite second moments. Frame measures in \mathbb{R}^N are in \mathcal{P}_2 , and they satisfy $E_\mu = \mathbb{R}^N$. There exists a natural metric

on \mathcal{P}_2 called the 2-*Wasserstein metric*, which is given by

$$W_2^2(\mu, \nu) := \min\left\{\int_{\mathbb{R}^N \times \mathbb{R}^N} \|x - y\|^2 d\gamma(x, y), \gamma \in \Gamma(\mu, \nu)\right\}, \quad (3)$$

where $\Gamma(\mu, \nu)$ is the set of all Borel probability measures γ on $\mathbb{R}^N \times \mathbb{R}^N$ whose marginals are μ and ν , respectively, i.e., $\gamma(A \times \mathbb{R}^N) = \mu(A)$ and $\gamma(\mathbb{R}^N \times B) = \nu(B)$ for all Borel subsets A, B in \mathbb{R}^N . [EhOk13]

Let $\mu \in \mathcal{P}$ be a probabilistic frame. The probabilistic *analysis operator* is given by

$$T_\mu : \mathbb{R}^N \rightarrow L^2(\mathbb{R}^N, \mu), \quad x \mapsto \langle x, \cdot \rangle. \quad (4)$$

Its adjoint operator is defined by

$$T_\mu^* : L^2(\mathbb{R}^N, \mu) \rightarrow \mathbb{R}^N, \quad f \mapsto \int_{\mathbb{R}^N} f(x)x d\mu(x) \quad (5)$$

and is called the probabilistic *synthesis operator*. The probabilistic Gramian operator of μ is $G_\mu = T_\mu T_\mu^*$. [EhOk13] The probabilistic frame operator of μ is $S_\mu = T_\mu^* T_\mu$,

$$S_\mu : \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad S_\mu(x) = \int_{\mathbb{R}^N} \langle x, y \rangle y d\mu(y).$$

The Gramian of μ , G_μ is the integral operator defined on $L^2(\mathbb{R}^N, \mu)$ by

$$G_\mu f(x) = T_\mu T_\mu^* f(x) = \int_{\mathbb{R}^N} \langle x, y \rangle f(y) d\mu(y). \quad (6)$$

Note that $\mathcal{H} = \mathbb{R}^N$ and $N < \infty$ in [EhOk13]. In the next section we extend these tools to infinite dimensions, pointing out a number of subtleties, and differences between the two cases, finite vs infinite.

1.2 Infinite Dimensions

If \mathcal{H} is an *infinite dimensional* Hilbert space, we shall show that the formulas (2), (3) and (5) carry over from \mathbb{R}^N , $N < \infty$, to $\dim \mathcal{H} = \aleph_0$; but it will be necessary to create an ambient measure space (Ω, \mathcal{F}) where Ω is a certain vector space containing \mathcal{H} . We will show that in this case, the four formulas carry over with the following modifications: In (4), we show that $y \mapsto \langle x, y \rangle$ extends from \mathcal{H} to Ω ; and the integral in (5) will then be

$$x = \int_{\Omega} \langle x, \tilde{\omega} \rangle \tilde{\omega} d\mu(\omega) \quad \text{in } \mathcal{H} \quad (7)$$

where $\langle \cdot, \tilde{\cdot} \rangle$ refers to this extension. But appropriate generalizations of (6) are much more subtle.

The extension of the results in (4) and (5) will involve this $\mathcal{H} \rightarrow \Omega$ extension \sim : In (4), we will consider an analysis operator $T_\mu : \mathcal{H} \rightarrow L^2(\Omega, \mu)$,

$$\mathcal{H} \ni x \rightarrow \langle x, \tilde{\cdot} \rangle \quad (\text{on } \Omega);$$

and then (5) will read as follows:

$$L^2(\Omega, \mu) \ni f \xrightarrow{T_\mu^*} \int_{\Omega} f(\omega) \omega d\mu(\omega) \in \mathcal{H}. \quad (8)$$

Note that, since $\mathcal{H} \subsetneq \Omega$, it is a non-trivial assertion that the RHS in (8) is a vector in \mathcal{H} . We now turn to the technical details.

Definition 2 Let \mathcal{H} be a Hilbert space (over \mathbb{R} , but \mathbb{C} will work also with small modifications). Let $\alpha, \beta \in \mathbb{R}_+$, $0 < \alpha \leq \beta < \infty$.

Set

$$F(\alpha, \beta) := \{ \{ \varphi_n \}_{n \in \mathbb{N}}; \alpha \|x\|^2 \leq \sum_n |\langle x, \varphi_n \rangle|^2 \leq \beta \|x\|^2, \forall \varphi_n \in \mathcal{H} \}. \quad (9)$$

Let (Ω, \mathcal{F}) be a measure space, Ω a set, \mathcal{F} a σ -algebra.

We assume further that Ω is a vector space equipped with a weak*-topology such that the dual Ω' satisfies

$$\Omega' \subset \mathcal{H} \subset \Omega, \quad \text{and} \quad (10)$$

the inclusion mappings in (10) are assumed continuous with respect to the respective topologies; and Ω' is dense in \mathcal{H} . Equation (10) is an example of a *Gelfand triple*. Hence, for all $x \in \mathcal{H}$, $\langle x, \cdot \rangle$ on \mathcal{H} , extends uniquely to a measurable function $\langle x, \tilde{\cdot} \rangle$ on Ω .

Set

$$FM_{\Omega}(\alpha, \beta) := \{ \text{finite positive measures } \mu \text{ on } (\Omega, \mathcal{F}) ; \quad (11) \\ \alpha \|x\|^2 \leq \int_{\Omega} |\langle x, \omega \rangle|^2 d\mu(\omega) \leq \beta \|x\|^2, \forall x \in \mathcal{H} \}.$$

2 Measures Constructed Directly from Frames.

2.1 Markov measures from frames.

The purpose of the below is to make the connection between frames with discrete index on the one hand, and Markov chains on the other. This in turn allows us to take advantage of tools from Markov chains, and to make the connection to continuous Markov processes (see section 4 below.)

Proposition 1 Let \mathcal{H} be a Hilbert space, and let $\{ \varphi_n \}_{n \in \mathbb{N}}$ be a frame in \mathcal{H} with frame bounds α, β , $0 < \alpha \leq \beta < \infty$, i.e.,

$$\alpha \|x\|^2 \leq \sum_{n \in \mathbb{N}} |\langle x, \varphi_n \rangle|^2 \leq \beta \|x\|^2 \quad (12)$$

holds for all $x \in \mathcal{H}$.

We then get a system of transition probabilities

$$p_{x,y} = \frac{|\langle x, y \rangle|^2}{c(x)} \quad (13)$$

with

$$c(x) := \sum_{n \in \mathbb{N}} |\langle x, \varphi_n \rangle|^2, \quad (14)$$

having the following properties: For $x, y \in \mathcal{H} \setminus \{0\}$, we have:

(i) *Reversible:*

$$c(x)p_{x,y} = c(y)p_{y,x}$$

(ii) *Markov-rules:* $p_{x,\varphi_n} \leq 1$ for $n \in \mathbb{N}$,

$$\sum_{n \in \mathbb{N}} p_{x,\varphi_n} = 1.$$

(iii) *Normalization:*

$$p_{x,y} \leq \|y\|^2/\alpha$$

where α is the lower frame bound from (12).

Proof Rule (i) is immediate from the definition in (13). It is also clear from (14) that $p_{x,\varphi_n} \leq 1$, for all $n \in \mathbb{N}$. As for (ii), we have:

$$\sum_n p_{x,\varphi_n} = \sum_n \frac{|\langle x, \varphi_n \rangle|^2}{c(x)} = \sum_n \frac{|\langle x, \varphi_n \rangle|^2}{\sum_k |\langle x, \varphi_k \rangle|^2} = 1,$$

which is the desired property.

The last property (iii) follows from Schwarz and the lower frame bound as follows:

$$p_{x,y} = \frac{|\langle x, y \rangle|^2}{c(x)} \leq \frac{\|x\|^2 \|y\|^2}{\alpha \|x\|^2} = \|y\|^2/\alpha.$$

We now give the path space measures \mathbb{P}_x :

Corollary 1 *Every frame $\{\varphi_n\}_{n \in \mathbb{N}}$ defines a Markov process $\{X_k\}_{k \in \mathbb{N}_0}$ as follows:*

$$\mathbb{P}_x(\{\omega : X_1(\omega) = n_1, \dots, X_k(\omega) = n_k\}) = p_{x,\varphi_{n_1}} p_{\varphi_{n_1}, \varphi_{n_2}} \cdots p_{\varphi_{n_{k-1}}, \varphi_{n_k}}. \quad (15)$$

From the proposition, it follows that (14) defines a consistent system of Markov transitions. Existence of the corresponding Markov process rule follows from Kolmogorov's theorem.

2.2 Determinantal measures from frames.

Starting with a frame \mathcal{F} , we arrive at an associated Grammian. Hence for each finite subset of \mathcal{F} , we get a finite Grammian, and its determinant is non-negative, and it induces an n associated determinantal measure. The relevance of these measures is discussed below, as well as the continuous-index analogues.

Given a Hilbert space \mathcal{H} , $\dim \mathcal{H} = \aleph_0$. Let $\{\varphi_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ be a system of vectors in \mathcal{H} such that the following *a priori* estimate holds:

$$\exists \beta < \infty \quad \text{such that} \quad \sum_n \sum_m c_n c_m \langle \varphi_n, \varphi_m \rangle \leq \beta \sum_n |c_n|^2. \quad (16)$$

Remark 1 The estimate (16) is known to be implied by the upper bound estimate (1) in Definition 1. When (16) holds, we talk about a Riesz basis sequence. Also, see [Ly03].

We shall make use of the following ideas from the setting of determinantal measures, see e.g., [Buf16]. More generally, a determinantal point process is a stochastic point process, where the local the probability distributions may be represented by determinants of suitable kernel functions. In our case, we shall consider the case where the local determinants are computed from Grammians computed from overcomplete frame systems (details below). Determinantal processes arise as important tools in random matrix theory, in combinatorics, and in physics, see e.g., [GoOl15], [OlGr11].

Proof Suppose the upper bound in (1) holds for some $\beta < \infty$, and some system $\{\varphi_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$. Then set $T : \mathcal{H} \rightarrow l^2$,

$$Tx = (\langle x, \varphi_n \rangle_{n \in \mathbb{N}}), \quad x \in \mathcal{H}. \quad (17)$$

The upper bound in (1) is then equivalent to the following estimate in the ordering of Hermitian operators

$$T^*T \leq \beta I_{\mathcal{H}}; \quad (18)$$

and so $\|T^*T\| \leq \beta$. It follows that

$$\|T\|^2 = \|T^*\|^2 = \|T^*T\| \leq \beta,$$

and therefore $\|T^*\| \leq \sqrt{\beta}$.

But, $c = (c_n) \in l^2(\mathbb{N})$, then $T^*c = \sum_n c_n \varphi_n$, and

$$\|T^*c\|_{\mathcal{H}}^2 = \sum_n \sum_m c_n c_m \langle \varphi_n, \varphi_m \rangle_{\mathcal{H}} \leq \beta \|c\|_{l^2}^2$$

which is the desired estimate (16).

Then note that

$$\det(\langle \varphi_j, \varphi_k \rangle_{j,k=1}^n) \geq 0 \quad \text{for all } n.$$

Then there is a measure $\mu = \mu^{(\varphi)}$ defined on a point configurations in \mathbb{N} as follows: Let Φ be a random point configuration in \mathbb{N} , then $\mu = \mu^{(\varphi)}$ is determined to be

$$\mu(\Phi \supset \{1, 2, \dots, n\}) = \det(\langle \varphi_j, \varphi_k \rangle_{j,k=1}^n).$$

This measure is called the associated *determinantal measure*.

3 A Negative Result.

Theorem 1 *Let $\dim \mathcal{H} = \aleph_0$, and given $\alpha > 0$, $\beta < \infty$, then there is no Borel measure μ on \mathcal{H} satisfying*

$$\alpha \|x\|^2 \leq \int_{\mathcal{H}} |\langle x, y \rangle|^2 d\mu(y) \leq \beta \|x\|^2, \quad (19)$$

in other words, $FM_{\mathcal{H}}(\alpha, \beta) = \emptyset$. Also, see [GiSk74].

Proof Indirect. Suppose some finite positive Borel measure μ exists and satisfies the condition (19) for α, β fixed. Pick an O.N.B. (orthonormal bases) b_1, b_2, \dots in \mathcal{H} , then

$$\sum_n |\langle x, b_n \rangle|^2 = \|x\|^2 \quad \text{by Parseval,}$$

so $\lim_{n \rightarrow \infty} \langle x, b_n \rangle = 0$, pointwise for all $x \in \mathcal{H}$. Consequently,

$$\begin{aligned} \langle b_n, x \rangle^2 &\rightarrow 0 \quad n \rightarrow \infty \\ \alpha &\leq \int_{\mathcal{H}} |\langle b_n, y \rangle|^2 d\mu(y) \leq \beta. \\ \int \langle b_n, \cdot \rangle^2 &\rightarrow 0 \quad \text{and domination holds.} \end{aligned}$$

In summary, the sequence of functions on \mathcal{H} , $\langle \cdot, b_n \rangle \rightarrow 0$ as $n \rightarrow \infty$ pointwise convergence, and we have the domination, since

$$\int |\langle \cdot, b_n \rangle|^2 d\mu(\cdot) \leq \beta \quad \forall n.$$

So by the Lebesgue dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_{\mathcal{H}} |\langle \cdot, b_n \rangle|^2 d\mu(\cdot) = 0$$

contradicting the lower bound $0 < \alpha \leq \int_{\mathcal{H}} |\langle \cdot, b_n \rangle|^2 d\mu(\cdot)$ in (19). Since $0 < \alpha \leq 0$, we get a contradiction. We have proved that $FM_{\mathcal{H}}(\alpha, \beta) = \emptyset$, whenever $0 < \alpha \leq \beta < \infty$.

In [EhOk13], suppose $N < \infty$, $\mathcal{H} = \mathbb{R}^N$, the authors study $\mu \in FM(\alpha, \beta)$

$$\alpha \|x\|^2 \leq \int_{\mathcal{H}} |\langle x, y \rangle|^2 d\mu(y) \leq \beta \|x\|^2, \quad \alpha > 0, \quad \beta < \infty.$$

The theorem shows that new techniques are needed when $\dim \mathcal{H} = \infty$.

4 Gaussian Frame Measures.

Starting with a separable Hilbert space \mathcal{H} , we shall need an associated framework from the construction of Gaussian probability measures. We shall then discuss how from this we get associated families of probability-frames. Starting with \mathcal{H} , we first show in Lemma 1, that there is a triple of containments (see Definition 3), with \mathcal{H} contained in S' and a Gaussian probability space where the events is the sigma-algebra of subsets of S' generated by the cylinder sets. This in turn is based on an application of Minlos' theorem, see also section 5 below, and [HiSi08], [LuZu12], [OkBe08], [TITa15].

Remark 2 Let \mathcal{H} be a Hilbert space, and assume $\dim \mathcal{H} = \aleph_0$. There exist S, S' where S is a Fréchet space, S' is the dual space of S such that $S \subset \mathcal{H} \subset S'$, continuous inclusions, and a Gaussian measure μ on S' such that $\mu \in FM_{S'}(1, 1)$. We can take μ to be Gaussian.

s is the space of sequences $\mathbf{x} = (x_n)$ which fall off at infinity faster than any polynomial in n . A sequence $\mathbf{y} = (y_n)$ is in s' if and only if there is a positive M so that (y_n) grows at most like $\mathcal{O}(n^M)$. We identify a system of seminorms on s which turns it into a Fréchet space. The space of continuous linear functionals on s will then coincide with s' .

Definition 3 The spaces s and s' . Both s and its dual s' are sequence spaces $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$, $\mathbf{y} = (y_n)_{n \in \mathbb{N}}$; indexed by \mathbb{N} or by \mathbb{Z} , and we have

$$\mathbf{x} \in s \stackrel{Def}{\iff} \forall k \in \mathbb{N}, \exists C_k, \text{ such that } |n|^k |x_n| \leq C_k, \quad \forall n \in \mathbb{N}.$$

$$\mathbf{y} \in s' \stackrel{Def}{\iff} \exists M \in \mathbb{N}, \exists C < \infty \text{ such that } |y_n| \leq C(1 + |n|^M), \quad \forall n \in \mathbb{N}. \quad (20)$$

With the seminorms

$$|\mathbf{x}|_k = \sup_n |n|^k |x_n|,$$

we note that s becomes a Fréchet space, and its dual is s' . We have

$$s \subset l^2(\mathbb{N}) \subset s'. \quad (21)$$

Lemma 1 If \mathcal{H} is a fixed Hilbert space, we pick an O.N.B. $\{b_n\}_{n \in \mathbb{N}}$ and set $x = \sum_n x_n b_n$, $x_n = \langle x, b_n \rangle_{\mathcal{H}}$, and via the isomorphism $\mathcal{H} \longleftrightarrow l^2(\mathbb{N})$, we get

$$s \subset \mathcal{H} \subset s'$$

where

$$\mathbf{x} = (x_n) \quad x_n = \langle x, b_n \rangle \quad \mathbf{x} \in s, \quad \mathbf{y} \in s', \quad \langle \mathbf{x}, \mathbf{y} \rangle = \sum_n x_n y_n.$$

Proof Note that if $\mathbf{y} \in S'$ satisfies (20) for some M , then

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq Const |\mathbf{x}|_{M+2} \quad \text{for all } \mathbf{x} \in s.$$

Since s is dense in $l^2(\mathbb{N})$, the “inclusion” $l^2(\mathbb{N}) \hookrightarrow s'$ is indeed $1 - 1$.

The topology, and the σ -algebra, on s' is generated by the following subsets of s' , the cylinder-sets. They are indexed by $k \in \mathbb{N}$, open subsets $\mathcal{O} \subset \mathbb{R}^k$, and subsets $\{x_i\}_{i=1}^k \subset S$ with

$$Cyl(\{x_i\}, \mathcal{O}) = \{\omega \in s'; (\langle x_i, \omega \rangle)_1^k \in \mathcal{O}\}. \quad (22)$$

The cylinder-sets form a basis for both a topology on s' (making it the dual of S), and of a σ -algebra. We shall use both.

We now verify why we need the space S' with μ a positive measure defined on the cylinder Borel σ -algebra of subsets of S' .

Let $\dim \mathcal{H} = \infty \rightarrow \aleph_0$ and (b_n) O.N.B. We have $s \subset \mathcal{H} \subset s'$. We use the σ -algebra subsets of s' generated by the cylinder sets.

We shall adopt the following standard terminology from probability theory: By a probability space we mean a triple $(\Omega, \mathcal{F}, \mu)$, i.e., sample space, sigma-algebra, and probability measure. The \mathcal{F} -measurable functions f on Ω are the random variables, The integral of f with respect with μ is called the expectation, and it is denoted $\mathbb{E}(f)$.

Now, for the Gaussian measures: There exists a measure μ , Gaussian on S' with $\Omega = S'$, we have

$$\mathbb{E}(f) := \mathbb{E}(\Omega, \mathcal{F}, \mu, f) = \int_{\Omega} f d\mu, \quad \int_{\mathbf{y} \in S'} |\langle \mathbf{x}, \mathbf{y} \rangle|^2 d\mu(\mathbf{y}) = \|\mathbf{x}\|^2 \quad (23)$$

or $\mathbb{E}(|\langle x, \cdot \rangle|^2) = \|x\|^2$. See also Lemma 2. More generally, we have the following theorem:

Given \mathcal{H} Hilbert space under inner product $\langle \cdot, \cdot \rangle$. How to extend for $\mathcal{H} \times \mathcal{H}$ to $\mathcal{H} \times S'$:

Step 1. Pick a Gelfand triple

$$S \subset \mathcal{H} \subset S'. \quad (24)$$

Step 2. By Minlos, we get a probability space $(S', \mathcal{C}$ Cylinder σ -algebra, μ) where μ is a probability measure, on \mathcal{C} . Here \mathcal{C} is the cylinder σ - algebra of subsets in S' .

Step 3. We get a pairing

$$\langle s, \omega \rangle, \quad \omega \in S', \quad s \in S \text{ such that} \quad (25)$$

$$\mathbb{E}_{\mu}(e^{i\langle s, \cdot \rangle}) = e^{-\frac{1}{2}\|s\|_{\mathcal{H}}^2} \quad (26)$$

See (29).

Step 4. Let

$$\langle s, \cdot \rangle : S \rightarrow L^2(S', \mu) \quad (27)$$

be an isometry, and let S be dense in \mathcal{H} . Then (27) extends by limit to an isometry

$$\langle x, \cdot \rangle : \mathcal{H} \rightarrow L^2(S', \mu). \quad (28)$$

By abuse of a notation, we write $\langle \cdot, \cdot \rangle$ also for the $\mathcal{H} \times S'$ pairing, but $\langle x, \cdot \rangle$ is only defined almost everywhere on S' with respect to μ .

Theorem 2 [TlTa15], [OkBe08], [HiSi08] (Minlos' theorem) *There exists a unique Gaussian measure μ on S' such that*

$$\mathbb{E}(e^{i\langle \mathbf{x}, \cdot \rangle}) = e^{-\frac{1}{2}\|\mathbf{x}\|^2} \quad \text{holds for all } \mathbf{x} \in \mathcal{H}. \quad (29)$$

Where (23) is applied to $\omega \rightarrow e^{i\langle x, \omega \rangle}$ on the LHS in (29). The RHS is called a Gaussian covariance function.

Consider $S \subset \mathcal{H} \subset S'$, S is a Fréchet space with a nuclear embedding, and a Gelfand triple. (See [Jo14], [LuZu12], [JoPe11].)

$$\mathbb{E}(\langle \mathbf{x}, \cdot \rangle^{2k+1}) = \int \langle \mathbf{x}, \cdot \rangle^{2k+1} d\mu = 0,$$

$$\mathbb{E}(\langle \mathbf{x}, \cdot \rangle^{2k}) = \int \langle \mathbf{x}, \cdot \rangle^{2k} d\mu = (2k-1)!! \|\mathbf{x}\|^{2k},$$

where $(2k-1)!! = \frac{(2k)!}{2^k \cdot k!} = 1 \cdot 3 \cdot 5 \cdots (2k-1)$, starting with

$$\int \langle x, \cdot \rangle d\mu = 0.$$

Note that since μ is Gaussian, it is determined by its first two moments.

We now turn to the Gaussian process associated with a fixed frame:

Corollary 2 *Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a fixed frame in \mathcal{H} , $\dim \mathcal{H} = \aleph_0$; see Definition 2. Let μ denote the Gaussian measure in Theorem 2; and let $T_\mu : \mathcal{H} \rightarrow L^2(S', \mu)$ be the canonical isometry $T_\mu x = \langle x, \cdot \rangle$, $x \in \mathcal{H}$. Then the Gaussian covariance matrix for the Gaussian process $\{T_\mu \varphi_k\}_{k \in \mathbb{N}}$ is $(\langle \varphi_k, \varphi_n \rangle_{\mathcal{H}})$, i.e., the Gramian of the frame.*

Proof Given a fixed frame $\{\varphi_n\}_{n \in \mathbb{N}}$, with frame constants $\alpha, \beta \in \mathbb{R}_+$, $0 < \alpha \leq \beta < \infty$. Set $G = (\langle \varphi_j, \varphi_k \rangle)_{j,k \in \mathbb{N}}$, and $G_n = (\langle \varphi_j, \varphi_k \rangle)_{j,k=1}^n$, $n \times n$ matrix. Let μ be the Gaussian measure from Theorem 2. Then, for every $n \in \mathbb{N}$, the joint distribution of the system $\{T_\mu \varphi_k\}_{k=1}^n$ of Gaussian random variables is

$$(\det G_n)^{-\frac{1}{2}} e^{-\frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n x_j x_k (G_n^{-1})_{j,k}} \underset{n\text{-dimensional Lebesgue measure}}{dx_1 \cdots dx_n}. \quad (30)$$

Indeed, a direct computation, using (30) shows that, when $j, k \in \{1, 2, \dots, n\}$, we get

$$\begin{aligned} (\det G_n)^{-\frac{1}{2}} \int_{\mathbb{R}^n} x_j x_k e^{-\frac{1}{2} \langle x, G_n^{-1} x \rangle} dx_1 \cdots dx_n &= \langle T_\mu \varphi_j, T_\mu \varphi_k \rangle_{L^2(\mu)} \\ &= \langle \varphi_j, \varphi_k \rangle = G_{j,k} \end{aligned}$$

which is the desired conclusion.

5 Analysis and Synthesis From Gaussian Frame Measures.

We recall Minlos' theorem. Construct $S \subset \mathcal{H} \subset S'$. On the cylinder σ -algebra of subsets of S' with μ a probability measure.

A good reference for the present discussion regarding infinite-dimensional Gaussian distributions is [Bo88]. Nonetheless we have included below enough details in order to make our paper readable for a general audience. This reference addresses in detail such subtleties as measurability; and the fact that in the infinite-dimensional case, the measure of \mathcal{H} is zero.

Lemma 2 *For all $x \in \mathcal{H}$, $\langle x, \cdot \rangle$ on \mathcal{H} has an extension to S' (denote also by $\langle x, \omega \rangle$, $\omega \in S'$ such that*

$$\int_{S'} e^{i \langle x, \omega \rangle} d\mu(\omega) = \mathbb{E}(e^{iT_\mu x}) = e^{-\frac{1}{2} \|x\|^2}. \quad (31)$$

Also, see [Bo88].

Proof By comparing the two power series, we then get

$$\int_{S'} |\langle x, \omega \rangle|^2 d\mu(\omega) = \|x\|^2 \quad (32)$$

where (32) or equivalently T_μ is called an Ito-isometry. See [All06], [Kam96].

Corollary 3 *The inner product $\langle x, \cdot \rangle$ on \mathcal{H} extends to S' as follows:*

$$T_\mu x = \langle x, \cdot \rangle \quad \text{on } S' \quad (33)$$

such that T_μ is an isometry, see (32), from \mathcal{H} into $L^2(S', \mu)$. Also, see [Bo88].

The Gaussian property μ is part of Minlos' theorem.
From the corollary we get the adjoint operator,

$$T_\mu^* : L^2(S', \mu) \longrightarrow \mathcal{H}$$

as a co-isometry.

An important technical point is that the operator mapping x to the measurable extension of the specified linear functional. We leave to the reader checking that indeed we get agreement between the following two: (i) the infinite-dimensional integration with the measurable linear functionals; and (ii) the finite dimensional inner product from Lemma 3. This consistency issue is important for Corollaries 4 and 5.

Lemma 3 For T_μ^* we have

$$(T_\mu^* f) = \int_{S'} f(\omega) \omega d\mu(\omega), \quad \forall f \in L^2(S', \mu), \quad \text{and} \quad (34)$$

RHS in (34) is in \mathcal{H} .

Suppose $x \in \mathcal{H}$, then $T_\mu x \in L^2(S', \mu)$; and if $f \in L^2(\mu)$, then $T_\mu^* f \in \mathcal{H}$. In summary we have a dual pair of operators:

$$\mathcal{H} \xrightleftharpoons[T_\mu^*]{T_\mu} L^2(S', \mu) \quad (35)$$

Proof To show that $\int_{S'} f(\omega) \omega d\mu(\omega) \in \mathcal{H}$, we can use Riesz, and instead prove the following a priori estimate:

$$\exists \text{Const} < \infty, \quad \text{such that} \quad \left| \int_{S'} f(\omega) \langle x, \omega \rangle d\mu(\omega) \right|^2 \leq \text{Const} \|x\|^2 \quad \forall x \in \mathcal{H}. \quad (36)$$

Then we conclude that $\int_{S'} f(\omega) \omega d\mu(\omega) \in \mathcal{H}$.

Details:

$$\begin{aligned} LHS(36) &\leq \int_{S'} |f|^2 d\mu \int_{S'} |\langle x, \omega \rangle|^2 d\mu(\omega) \\ &= \|f\|_{L^2(S', \mu)}^2 \| \langle x, \cdot \rangle \|_{L^2(S', \mu)}^2 \\ &\stackrel{\text{by (32)}}{=} \|f\|_{L^2(S', \mu)}^2 \|x\|_{\mathcal{H}}^2, \quad \forall x \in \mathcal{H}, \quad C = \|f\|_{L^2(S', \mu)}^2 \end{aligned}$$

T_μ is called the Ito-isometry. See [All06], [Kam96].

Apply Riesz to the Hilbert space \mathcal{H} , and we conclude that the integral below is a vector in \mathcal{H} , i.e., that

$$T_\mu^* f = \int_{S'} f(\omega) \omega d\mu(\omega) \in \mathcal{H}.$$

Corollary 4 For every $x \in \mathcal{H}$, we have the following frame decomposition

$$x = \int_{S'} \langle x, \omega \rangle \omega d\mu(\omega). \quad (37)$$

Proof We showed that T_μ is isometry (see (36)), and that T_μ^* is given by (34). Hence

$$T_\mu^* T_\mu = I_{\mathcal{H}}, \quad \text{so} \quad (38)$$

$$x = T_\mu^* T_\mu x. \quad (39)$$

We write out the RHS in (39) as $\int_{S'} \langle x, \omega \rangle \omega d\mu(\omega) = x$, since $\int_{S'} f(\omega) \omega d\mu(\omega) \in \mathcal{H}$ if $f \in L^2(S', \mu)$.

Corollary 5 *Let \mathcal{H} , S' and μ be as above, and let T_μ and T_μ^* be the corresponding operators in Lemma 2; then $T_\mu T_\mu^*$ is the projection onto the range of T_μ ; i.e.,*

$$Q_\mu = T_\mu T_\mu^* = \text{proj}\{T_\mu x; x \in \mathcal{H}\}.$$

Proof By (38) above, we have

$$\begin{aligned} Q_\mu^2 &= (T_\mu T_\mu^*)(T_\mu T_\mu^*) \\ &= T_\mu (T_\mu^* T_\mu) T_\mu^* \\ &=_{\text{by (38)}} T_\mu T_\mu^* = Q_\mu. \end{aligned}$$

Definition 4 Let $x \in \mathcal{H}$, and let μ and μ^x be Gaussian measures, then

$$\begin{aligned} \int_{S'} \varphi d\mu^x &= \int \varphi(\cdot + x) d\mu(\cdot), \\ \mu^x(E) &= \mu(E - x) \quad \text{where } E \subset S'. \end{aligned}$$

There are several candidates for frame measures in the case of infinite-dimensional separable Hilbert space \mathcal{H} , i.e., $\mathcal{H} \simeq l^2(\mathbb{N})$, one is the case of

1. Gaussian measures μ supported in a measure space S' which is derived from a Gelfand triple $S \subset \mathcal{H} \subset S'$ where S is a Fréchet space, $S \hookrightarrow \mathcal{H}$ is continuous on S' . Here S' is the dual of S . If μ is determined from

$$\int_{S'} e^{i\langle x, \cdot \rangle} d\mu(\cdot) = e^{-\frac{1}{2}\|x\|^2}$$

then

$$\int_{S'} |\langle x, y \rangle|^2 d\mu(y) = \|x\|^2$$

holds for all $x \in \mathcal{H}$. See [Jo14], [LuZu12], [JoPe11].

Given a vector x , then the Radon-Nikodym derivative

$\frac{d\mu^x}{d\mu} = e^{(T_\mu x)(\omega) - \frac{1}{2}\|x\|^2}$, will represent a multiplier for an associated Ito-integral; see also (40) below.

$\mu(S') = 1$, $S \subset \mathcal{H} \subset S'$, $y \rightarrow \langle x, y \rangle$,

$$\begin{aligned} \int_{S'} \tilde{x} d\mu &= 0 \\ \int_{S'} |\tilde{x}|^2 d\mu &= \|x\|^2, \quad x \in \mathcal{H}. \end{aligned}$$

The key point is that in infinite dimension, the Gaussian measures cannot be invariant under translation, only quasi-invariant. Hence, for each x , we get a Radon-Nikodym derivative. And the family of these Radon-Nikodym derivatives form a cocycle. Readers will be able to find detailed discussion of this in for example, [Bo88], [GiSk74], [HiSi08], [Jo14], [OkBe08].

6 Translation.

Now, let \mathcal{H} be such that $\dim\mathcal{H} = \aleph_0$ and $S \subset \mathcal{H} \subset S'$. Let μ be Gaussian probability measure in S' , and

$$T_\mu : \mathcal{H} \xrightarrow{\text{isometry}} L^2(S', \mu),$$

$$T_\mu x = \langle x, \cdot \rangle \quad x \in \mathcal{H} \quad \text{extension from } \mathcal{H} \text{ to } S'.$$

Applications of

$$T_\mu : \mathcal{H} \rightarrow L^2(S', \mu).$$

See above for a discussion of the cocycle of Radon-Nikodym derivatives.

Theorem 3 *We can define μ^x*

$$\mu^x(E) := \mu(E - x), \quad x \in \mathcal{H}, \quad E \subset S' \quad \text{Borel}$$

and the Radon-Nikodym derivative is

$$\frac{d\mu^x}{d\mu} \in L^1_+(S', \mu)$$

$$\frac{d\mu^x}{d\mu}(\omega) = e^{(T_\mu x)(\omega) - \frac{1}{2}\|x\|^2}, \quad \omega \in S'. \quad (40)$$

See, [Bo88].

Theorem 4 *Let \mathcal{H} and μ be as above, $\dim\mathcal{H} = \aleph_0$; and let $x, y \in \mathcal{H}$. Set*

$$\mathcal{E}_\mu(x)(\cdot) = e^{(T_\mu x)(\cdot)} e^{-\frac{1}{2}\|x\|^2} \quad \text{on } S', \quad (41)$$

see (40). Then

$$\int_{S'} \mathcal{E}_\mu(x)(\omega) \langle y, \omega \rangle^2 d\mu(\omega) = \langle x, y \rangle^2 + \|y\|^2; \quad (42)$$

and the following co-cycle property holds:

$$\mathcal{E}_\mu(x_1)(\omega) \mathcal{E}_\mu(x_2)(\omega) = e^{-\langle x_1, x_2 \rangle_{\mathcal{H}}} \mathcal{E}_\mu(x_1 + x_2)(\omega), \quad \text{for all } x_1, x_2 \in \mathcal{H}, \text{ and } \omega \in S'. \quad (43)$$

Proof

$$\begin{aligned} \int_{S'} \mathcal{E}_\mu(x)(\omega) \langle y, \omega \rangle^2 d\mu(\omega) &= \int_{S'} \langle y, x + \omega \rangle^2 d\mu(\omega) \\ &= \int_{S'} (\langle y, x \rangle^2 + \langle y, \omega \rangle^2 + 2\langle y, x \rangle \langle y, \omega \rangle) d\mu(\omega) \\ &\stackrel{\text{by Theorem 2}}{=} \langle y, x \rangle^2 + \int_{S'} \langle y, \omega \rangle^2 d\mu(\omega) \\ &= \langle y, x \rangle^2 + \|y\|^2 \end{aligned}$$

which is the desired conclusion. The co-cycle property (43) is immediate from (41).

Corollary 6 For each Parseval frames (φ_n) in \mathcal{H} , there exists an associated i.i.d. $N(0, 1)$, system (Z_n) on $L^2(S', \mu)$ such that

$$(Tx)(\omega) = \sum_n \langle x, \varphi_n \rangle Z_n(\omega), \quad \omega \in S'.$$

Proof There exists an isometry $V : \mathcal{H} \rightarrow l^2(\mathbb{N})$. If $\epsilon_k(n) = \delta_{k,n}$ in $l^2(\mathbb{N})$ then $V^* \epsilon_k = \varphi_k$, for all $k \in \mathbb{N}$. See [Jor08], [JoTi15].

Gaussian Karhunen-Loève Expansion. Suppose $\{\varphi_n\}$ is a Parseval frame. We then have

$$x = \sum_n \langle x, \varphi_n \rangle \varphi_n$$

$$Tx = \sum_n \langle x, \varphi_n \rangle T\varphi_n = \sum_n \langle x, \varphi_n \rangle Z_n(\cdot).$$

By Karhunen-Loève expansion, there exists an i.i.d. Z_n , $N(0, 1)$ system such that $T\varphi_n = TV^* \epsilon_n = Z_n(\cdot)$. Hence

$$\|Tx\|_{L^2(\mu)}^2 = \sum_n |\langle x, \varphi_n \rangle|^2 = \|x\|^2$$

because (φ_n) is a Parseval frame.

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