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Lisa M. Dollar
Southern Illinois University Edwardsville, lisamariedollar@gmail.com

G. Stacey Staples
Southern Illinois University Edwardsville, sstaple@siue.edu

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Lisa M. Dollar, G. Stacey Staples*

Abstract

Zeon algebras can be thought of as commutative analogues of fermion algebras, and they can be constructed as subalgebras within Clifford algebras of appropriate signature. Their inherent combinatorial properties make them useful for applications in graph enumeration problems and evaluating functions defined on partitions. In this paper, kth roots of invertible zeon elements are considered. More specifically, conditions for existence of roots are established, numbers of existing roots are determined, and computational methods for constructing roots are developed. Recursive and closed formulas are presented, and specific low-dimensional examples are computed with *Mathematica*. Interestingly, Stirling numbers of the first kind appear among coefficients in the closed formulas.
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1 Introduction

Zeon algebras were first defined and applied to graph theory in [13], although the name "zeon algebra" was first used by Feinsilver [2, 3]. They arise as commutative subalgebras of fermions (generated by disjoint pairs of fermions), and can be constructed as subalgebras of Clifford algebras.

Combinatorial properties of zeons have been shown to generate Stirling numbers of the second kind, Bell numbers, Catalan numbers, and Bessel numbers [8]. Further, they have been useful in defining partition-dependent

*Department of Mathematics and Statistics, Southern Illinois University Edwardsville, Edwardsville, IL 62026-1653,USA, email: sstaple@siue.edu

stochastic integrals. In fact, expanding powers of zeon elements is equivalent to summing over partitions [9].

Weighting the vertices of a graph with zeon generators allows one to construct a *nilpotent adjacency matrix*, $\mathfrak{A}$, whose entries are generators of the algebra. The matrix is very convenient for performing symbolic computations and allows enumeration of cycles by considering traces of matrix powers. This idea has led to a number of applications to graph enumeration problems and even routing problems in communication networks [10].

Recently, combinatorial identities involving zeons have been studied in a number of works by A.F. Neto [4, 5, 6, 7]. In these works, Bernoulli numbers, m-Stirling numbers of the second kind, higher order derivatives of trigonometric functions, and representations of Bernoulli polynomials are presented in the context of zeon algebras.

The current paper is an extension of work begun in Dollar's master's thesis [1]. It is the first work exploring the basic algebraic properties of the abelian multiplicative group formed by the algebra's non-nilpotent elements. Necessary and sufficient conditions are established for the existence of kth roots, recursive and closed formulas are given for their construction, and a number of examples are provided using *Mathematica*.

The remainder of the paper is laid out thusly. Terminology and notational conventions are established in subsection 1.1. Group-theoretic properties of invertible zeons, including conditions for invertibility and formulas for computing inverses, are established in Section 2. Existence of kth roots and recursive formulations of those roots are established in Section 3 before explicit closed formulas for kth roots are established in Section 4. The paper closes with concluding remarks in Section 5.

Examples appearing throughout the paper were computed using *Mathematica* with the **CliffMath** [11] and **CliffSymNil** [12] packages. These packages, along with more examples, can be found through the *Research* link at `http://www.siue.edu/~sstaple`.

1.1 Preliminaries

Let ${\mathcal{C}\ell_n}^{\mathrm{nil}}$ denote the real abelian algebra generated by the collection $\{\zeta_i\}$ $(1 \leq i \leq n)$ along with the scalar $1 = \zeta_0$ subject to the following multiplication rules:

$$\zeta_i\,\zeta_j = \zeta_j\,\zeta_i \;\text{ for } i \neq j, \text{ and}$$
$${\zeta_i}^2 = 0 \;\text{ for } 1 \leq i \leq n.$$

It is evident that a general element $\alpha \in {\mathcal{C}\ell_n}^{\rm nil}$ can be expanded as

$$\alpha = \sum_{I \in 2^{[n]}} \alpha_I\, \zeta_I\,,$$

where $I \in 2^{[n]}$ is a subset of the *n-set*, $[n] := \{1, 2, \ldots, n\}$, used as a multi-index, $\alpha_I \in \mathbb{R}$, and $\zeta_I = \prod_{\iota \in I} \zeta_\iota$. The algebra ${\mathcal{C}\ell_n}^{\rm nil}$ is called the (n-particle) *zeon algebra*.

As a vector space, this 2^n-dimensional algebra has a canonical basis of *basis blades* of the form $\{\zeta_I : I \subseteq [n]\}$. The null-square property of the generators $\{\zeta_i : 1 \leq i \leq n\}$ guarantees that the product of two basis blades satisfies the following:

$$\zeta_I \zeta_J = \begin{cases} \zeta_{I \cup J} & I \cap J = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

An inner product is defined on ${\mathcal{C}\ell_n}^{\rm nil}$ by linear extension of

$$\left\langle \sum_{I \in 2^{[n]}} u_I \zeta_I, \zeta_J \right\rangle = u_J.$$

Hence, any elements $u \in {\mathcal{C}\ell_n}^{\rm nil}$ can be expanded as $u = \sum_{I \in 2^{[n]}} \langle u, \zeta_I \rangle\, \zeta_I$.

For convenience, arbitrary elements of ${\mathcal{C}\ell_n}^{\rm nil}$ will be referred to simply as "zeons." Using the basic notions above, some computational tools can be developed and some properties can be established.

2 Group Properties

Since ${\mathcal{C}\ell_n}^{\rm nil}$ is an algebra, its elements form a commutative multiplicative semigroup. It is not difficult to establish convenient formulas for expanding products of zeons.

Lemma 2.1. *Let* $\alpha, \beta \in \mathcal{C}\ell_n{}^{\text{nil}}$ *and write* $\alpha = \sum_{I \in 2^{[n]}} \alpha_I \zeta_I$ *and* $\beta = \sum_{I \in 2^{[n]}} \beta_I \zeta_I$*. Let the product* $\gamma = \alpha\beta$ *be written* $\gamma = \sum_{I \in 2^{[n]}} \gamma_I \zeta_I$*. For fixed multi-index* I*, the corresponding coefficient of* ζ_I *in* γ *is given by*

$$\gamma_I = \sum_{K \subseteq I} \alpha_K \beta_{I \setminus K}.$$

Proof. The result follows immediately from $\gamma = \left(\sum_K \alpha_K \zeta_K\right)\left(\sum_J \beta_J \zeta_J\right)$ in light of (1.1). □

For convenience, a collection of pairwise-disjoint subsets of $2^{[n]}$ is denoted by $\{I_1|\ldots|I_k\}$. Such a collection is said to be *independent* if its elements are pairwise disjoint.

Lemma 2.2. *Let* $\alpha \in \mathcal{C}\ell_n{}^{\text{nil}}$ *and write* $\alpha = \sum_{I \in 2^{[n]}} \alpha_I \zeta_I$*. For positive integer* k*, let* $\gamma = \alpha^k$ *be written* $\gamma = \sum_{I \in 2^{[n]}} \gamma_I \zeta_I$*. For fixed multi-index* I*, the corresponding coefficient of* ζ_I *in* γ *is given by* [1]

$$\gamma_I = \sum_{j=0}^{k} \frac{k!}{j!} {\alpha_\emptyset}^j \sum_{\substack{\pi \in \mathcal{P}(I) \\ |\pi| = k-j}} a_\pi.$$

Proof. Applying the multinomial theorem, the null-square property of zeon

[1] By convention, define ${\alpha_\emptyset}^0 = 1$ when $\alpha_\emptyset = 0$.

generators yields

$$\begin{aligned}
\left(\sum_{I\in 2^{[n]}} \alpha_I \zeta_I\right)^k &= \sum_{\substack{0\le \ell_\emptyset,\ldots,\ell_{[n]}\\ \ell_0+\cdots+\ell_{2^{[n]}}=k}} \binom{k}{\ell_\emptyset,\ldots,\ell_{[n]}} \prod_{I\in 2^{[n]}} {\alpha_I}^{\ell_I} {\zeta_I}^{\ell_I}\\
&= \sum_{j=0}^{k} \binom{k}{j} {\alpha_\emptyset}^j \sum_{\substack{\{I_1|\cdots|I_{k-j}\}\\ \text{independent}}} (k-j)!\zeta_{I_1\cup\cdots\cup I_{k-j}} \prod_{\ell=1}^{k-j} a_{I_\ell}\\
&= \sum_{j=0}^{k} \frac{k!}{j!} {\alpha_\emptyset}^j \sum_{\substack{\{I_1|\cdots|I_{k-j}\}\\ \text{independent}}} \zeta_{I_1\cup\cdots\cup I_{k-j}} \prod_{\ell=1}^{k-j} a_{I_\ell}.
\end{aligned}$$

Evaluating the coefficient of a particular basis blade ζ_J is thereby accomplished by considering a sum over partitions of the multi-index J. More specifically, letting $\mathcal{P}(J)$ denote the collection of partitions of J,

$$\begin{aligned}
\left\langle \left(\sum_{I\in 2^{[n]}} \alpha_I \zeta_I\right)^k, \zeta_J \right\rangle &= \sum_{j=0}^{k} \frac{k!}{j!} {\alpha_\emptyset}^j \sum_{\substack{\pi\in\mathcal{P}(J)\\ |\pi|=k-j}} \prod_{\ell=1}^{k-j} a_{\pi_\ell}\\
&= \sum_{j=0}^{k} \frac{k!}{j!} {\alpha_\emptyset}^j \sum_{\substack{\pi\in\mathcal{P}(J)\\ |\pi|=k-j}} a_\pi.
\end{aligned}$$

□

Proposition 2.3. *Let* $\alpha = \sum_{I\in 2^{[n]}} \alpha_I\zeta_I \in {\mathcal{C}\ell_n}^{\text{nil}}$. *Write* $\alpha = \alpha_\emptyset + \beta$, *where* $\beta = \sum_{\emptyset\neq I\in 2^{[n]}} \alpha_I\zeta_I = \alpha - \alpha_\emptyset$ *is nilpotent of index* k. *It follows that* α *is invertible if and only if* $\alpha_\emptyset \neq 0$, *and setting*

$$\alpha' = \sum_{j=1}^{k} (-1)^{j-1} {\alpha_\emptyset}^{-j} \beta^{j-1}$$

one sees that $\alpha\alpha' = \alpha'\alpha = 1$.

Proof. Note that writing $\alpha = \alpha_\emptyset + \beta$, where $\beta = \sum_{\emptyset \neq I \in 2^{[n]}} \alpha_I \zeta_I$, one sees immediately that β is nilpotent; i.e., $\beta^k = 0$ for some positive integer $k \leq n+1$. Consequently, α is not invertible if $\alpha_\emptyset = 0$.
Claim: For positive integer k,

$$\alpha \left(\sum_{j=1}^{k} (-1)^{j-1} {\alpha_\emptyset}^{-j} \beta^{j-1} \right) = 1 + (-1)^{k-1} {\alpha_\emptyset}^{-k} \beta^k.$$

Proof of the claim proceeds by induction on k. When $k = 1$, one finds

$$\alpha {\alpha_\emptyset}^{-1} = (\alpha_\emptyset + \beta) {\alpha_\emptyset}^{-1} = 1 - {\alpha_\emptyset}^{-1} \beta.$$

Assuming the result holds for some $k \geq 1$, one finds

$$\begin{aligned} &\alpha \left(\sum_{j=1}^{k+1} (-1)^{j-1} {\alpha_\emptyset}^{-j} \beta^{j-1} \right) \\ &\quad = \alpha \left(\sum_{j=1}^{k} (-1)^{j-1} {\alpha_\emptyset}^{-j} \beta^{j-1} \right) + (-1)^k \alpha \left({\alpha_\emptyset}^{-(k+1)} \beta^k \right) \\ &\quad = 1 + (-1)^{k-1} {\alpha_\emptyset}^{-k} \beta^k + (-1)^k (\alpha^{-k} \beta^k + \alpha^{-(k+1)} \beta^{k+1}) \\ &\quad = 1 + (-1)^k {\alpha_\emptyset}^{-(k+1)} \beta^{k+1}. \end{aligned}$$

This establishes the claim. It follows immediately that when β is nilpotent of index k,

$$\begin{aligned} \alpha \left(\sum_{j=1}^{k} (-1)^{j-1} {\alpha_\emptyset}^{-j} \beta^{j-1} \right) &= 1 + (-1)^{k-1} {\alpha_\emptyset}^{-k} \beta^k \\ &= 1. \end{aligned}$$

□

Lemma 2.4. *If $\alpha \in {\mathcal{C}\ell_n}^{\mathrm{nil}}$ is invertible, then the inverse is unique.*

Proof. Suppose $\alpha\alpha' = \alpha\gamma = 1$. Then, $\alpha' = (\alpha\gamma)\alpha' = (\gamma\alpha)\alpha' = \gamma(\alpha\alpha') = \gamma$. □

Lemmas 2.1 and 2.4 imply that for any positive integer n, the invertible elements of ${\mathcal{C}\ell_n}^{\text{nil}}$ form an abelian multiplicative group. For convenience, this group is denoted by ${\mathcal{C}\ell_n}^{\text{nil}\star}$. In particular,

$$ {\mathcal{C}\ell_n}^{\text{nil}\star} := \left\{ \alpha \in {\mathcal{C}\ell_n}^{\text{nil}} : \alpha_\emptyset \neq 0 \right\}. $$

Lemma 2.5. *Every element of ${\mathcal{C}\ell_n}^{\text{nil}}$ is either nilpotent or invertible.*

Proof. Let $\alpha \in {\mathcal{C}\ell_n}^{\text{nil}}$. If $\alpha_\emptyset \neq 0$, then α^{-1} exists. If $\alpha_\emptyset = 0$, then a simple application of the multinomial theorem shows that $\alpha^{n+1} = 0$. □

Lemma 2.6. *The nilpotent elements of ${\mathcal{C}\ell_n}^{\text{nil}}$ form a maximal ideal, henceforth denoted ${\mathcal{C}\ell_n}^{\text{nil}_0}$.*

Proof. Closure under addition and multiplication is obvious. To see that the nilpotent elements form an ideal, let $\alpha = \alpha_\emptyset + \beta$, where $\beta = \sum_{\emptyset \neq I \in 2^{[n]}} \alpha_I \zeta_I$ and $\alpha_\emptyset \neq 0$. Let $\gamma \in {\mathcal{C}\ell_n}^{\text{nil}}$ be nilpotent, i.e., assume $\gamma_\emptyset = 0$. Then,

$$ \langle \alpha\gamma \rangle_0 = \alpha_\emptyset 0 = 0. $$

Denoting this ideal by ${\mathcal{C}\ell_n}^{\text{nil}_0}$, maximality is established by noting that the quotient ${\mathcal{C}\ell_n}^{\text{nil}}/{\mathcal{C}\ell_n}^{\text{nil}_0}$ is isomorphic to the field of real numbers. □

Definition 2.7. For $n \in \mathbb{N}$, ${\mathcal{C}\ell_n}^{\text{nil}\star}$ is defined to be the collection of invertible elements in ${\mathcal{C}\ell_n}^{\text{nil}}$. That is,

$$ {\mathcal{C}\ell_n}^{\text{nil}\star} = \{ u \in {\mathcal{C}\ell_n}^{\text{nil}} : u_\emptyset \neq 0 \}. $$

Clearly, ${\mathcal{C}\ell_n}^{\text{nil}\star}$ is closed under (commutative) zeon multiplication, has multiplicative identity 1, and every element is invertible by Proposition 2.3. Hence, the following lemma is established.

Lemma 2.8. *The invertible zeons ${\mathcal{C}\ell_n}^{\text{nil}\star}$ form an abelian group under multiplication.*

3 Existence and Recursive Formulations of Zeon Roots

As will be shown, invertible zeons have roots of all odd orders and roots of all even orders when their scalar parts are positive. A recursive algorithm establishes their existence and provides a convenient method for their computation.

Theorem 3.1. *Let $w \in \mathcal{C}\ell_n{}^{\text{nil}\star}$, and let $k \in \mathbb{N}$. Then, $\exists u \in \mathcal{C}\ell_n{}^{\text{nil}\star}$ such that $u^k = w$, provided $w_\emptyset > 0$ when k is even. Further, writing $w = \varphi + \zeta_{\{n\}}\psi$, where $\varphi, \psi \in \mathcal{C}\ell_{n-1}{}^{\text{nil}}$, u is computed recursively by*

$$u = w^{1/k} = \varphi^{1/k} + \zeta_{\{n\}}\frac{1}{k}\varphi^{-(k-1)/k}\psi.$$

Proof. Assuming $w \in \mathcal{C}\ell_n{}^{\text{nil}\star}$ guarantees $w_\emptyset \neq 0$, so the scalar part of w has odd roots of all orders. Even-order roots ${w_\emptyset}^{1/k}$ exist for positive values of $w_\emptyset$.

Proof is by induction on n. When $n = 1$, let $w = a + b\zeta_{\{1\}}$, where $w_\emptyset = a$. Applying the binomial theorem and null-square properties of zeon generators, one finds

$$\left(a^{1/k} + \frac{b}{ka^{(k-1)/k}}\zeta_{\{1\}}\right)^k = a + ka^{(k-1)/k}\frac{b}{ka^{(k-1)/k}}\zeta_{\{1\}} = a + b\zeta_{\{1\}}.$$

Next, suppose the result holds for some $n - 1 \geq 1$ and let $w \in \mathcal{C}\ell_n{}^{\text{nil}}$ be written $w = \varphi + \zeta_{\{n\}}\psi$, where $\varphi, \psi \in \mathcal{C}\ell_{n-1}{}^{\text{nil}}$. In particular, this implies $\varphi \in \mathcal{C}\ell_n{}^{\text{nil}\star}$. Let $\alpha = \varphi^{1/k}$, and let $u = \alpha + \frac{1}{k}\zeta_{\{n\}}\alpha^{-(k-1)}\psi$. Then

$$\begin{aligned} u^k = \left(\alpha + \zeta_{\{n\}}\frac{1}{k}\alpha^{-(k-1)}\psi\right)^k &= \varphi + k\alpha^{(k-1)}\frac{1}{k}\zeta_{\{n\}}\alpha^{-(k-1)}\psi \\ &= \varphi + \zeta_{\{n\}}\psi \\ &= w. \end{aligned}$$

□

3.1 Counting Zeon Roots

Whenever an element $u \in \mathcal{C}\ell_n{}^{\text{nil}}$ has a kth root, a natural question that arises is "How many kth roots does u have?" Perhaps not surprisingly, the answer depends on whether u is nilpotent or invertible.

For example, an element of the form $u = a\zeta_I \in \mathcal{C}\ell_n{}^{\text{nil}_0}$, where $a \neq 0$ and $|I| \geq 2$, has infinitely many square roots of the form $x = b\zeta_J + \frac{a}{b}\zeta_{I\setminus J}$, where $\emptyset \neq J \subsetneq I$ and $b \neq 0$, since

$$x^2 = \left(b\zeta_J + \frac{a}{2b}\zeta_{I\setminus J}\right)^2 = 2b\frac{a}{2b}\zeta_{J\cup(I\setminus J)} = a\zeta_I = u.$$

This result generalizes to *kth* roots of zeon monomials as follows.

Lemma 3.2 (Roots of nilpotent monomials)**.** *Let $u = a\zeta_I \in \mathcal{C}\ell_n{}^{\text{nil}_0}$, where $a \neq 0$ and $I \neq \emptyset$. For positive integer k ($2 \leq k \leq |I|$), the element u has infinitely many kth roots of the form*

$$x = \sum_{\ell=1}^{k} b_\ell \zeta_{J_\ell},$$

where $I = J_1 \sqcup \cdots \sqcup J_k$ is any k-block partition of I and $\prod_{\ell=1}^{k} b_\ell = \frac{a}{k!}$. Moreover, no kth roots exist when $k > |I|$.

Proof. Applying Lemma 2.2,

$$x^k = \left(\sum_{\ell=1}^{k} b_\ell \zeta_{J_\ell} \right)^k = k! \prod_{\ell=1}^{k} b_\ell \zeta_{J_\ell} = k! \frac{a}{k!} \zeta_{J_1 \cup \cdots \cup J_\ell} = a\zeta_I = u.$$

It is not difficult to see that the monomial $a\zeta_I$ has no kth roots when $k > |I|$ because the necessary partitioning of I is impossible. $\square$

Constructing roots of more general nilpotent elements lies beyond the scope of the current work. Of particular interest here is determining numbers of kth roots of invertible elements of $\mathcal{C}\ell_n{}^{\text{nil}}$.

Theorem 3.3. *Let $\alpha \in \mathcal{C}\ell_n{}^{\text{nil}\star}$, and let $k \in \mathbb{N}$. Then, assuming $\alpha_\emptyset > 0$ when k is even,*

$$\sharp\{u : u^k = \alpha\} = \begin{cases} 1 & \text{when } k \equiv 1 \pmod 2, \\ 2 & \text{when } k \equiv 0 \pmod 2. \end{cases}$$

Proof. Note that the choice of scalar term, $u_\emptyset$, is unique when k is odd. Now suppose $u^k = \alpha = v^k$, and observe that $u - v$ is nilpotent. Writing $u = a + \beta$ for some nilpotent β, it follows that $v = a + \gamma$ for nilpotent γ. Observe that the product $\alpha\delta$ of an invertible element α and a nilpotent δ, is zero if and only if $\delta = 0$, since $0 = \alpha^{-1}0 = \delta$. Hence, assuming $u^k = v^k$, one finds

$$\begin{aligned} u^k - v^k &= (u - v)(u^{k-1} + u^{k-2}v + \cdots + v^{k-1}) \\ &= (u - v)\left[(a^{k-1} + \delta_1) + (a^{k-1} + \delta_2) + \cdots + (a^{k-1} + \delta_k)\right] \\ &= (u - v)\left[ka^{k-1} + \delta\right], \end{aligned}$$

where $\delta = \delta_1 + \cdots + \delta_k$ is nilpotent by the ideal property of ${\mathcal{C}\ell_n}^{\rm nil_0}$ established in Lemma 2.6. It is clear that $ka^{k-1}+\delta$ is invertible, so $(u-v)(ka^{k-1}+\delta) = 0$ implies $(u-v) = 0$.

In the case of even k, there are two possible choices for the scalar term, $\pm u_\emptyset$. In one of these cases, $u^k = v^k$ implies $u - v$ is nilpotent and the proof proceeds as above. In the other case, $u^k = v^k$ implies $u + v$ is nilpotent. Considering this case in detail, one writes $u = a + \beta$ and $v = -a + \gamma$ for nilpotent elements β and γ. For even values of k, a little algebra thereby yields

$$\begin{aligned} u^k - v^k &= (u+v)\left(u^{k-1} - u^{k-2}v + \cdots + (-1)^{k-1}v^{k-1}\right) \\ &= (u+v)\left[(a^{k-1}+\delta_1) - (-a^{k-1}+\delta_2) + \cdots - (-a^{k-1}+\delta_k)\right] \\ &= (u+v)\left[ka^{k-1} + (\delta_1 - \delta_2 + \cdots - \delta_k)\right]. \end{aligned}$$

Letting $\delta = \delta_1 - \delta_2 + \cdots - \delta_k$, one sees that $(u+v)(ka^{k-1}+\delta) = 0$ implies $(u+v) = 0$ as before. Hence, $u = -v$. □

Given an invertible zeon u and even positive integer k, it now makes sense to define the *principal kth root* of u as the zeon w satisfying $w_\emptyset > 0$ and $w^k = u$. All roots of odd order can be considered principal.

Example 3.4. Consider the following zeon element of ${\mathcal{C}\ell_5}^{\rm nil}$:

$$\begin{aligned} 396\zeta_{\{1,2\}} - 108\zeta_{\{1,3\}} - 108\zeta_{\{1,4\}} - 396\zeta_{\{1,5\}} + 324\zeta_{\{2,3\}} + 324\zeta_{\{2,4\}} \\ -1332\zeta_{\{2,5\}} - 432\zeta_{\{3,4\}} - 324\zeta_{\{3,5\}} - 324\zeta_{\{4,5\}} + 72\zeta_{\{1,2,3\}} - 36\zeta_{\{1,2,4\}} \\ +1014\zeta_{\{1,2,5\}} + 144\zeta_{\{1,3,4\}} - 72\zeta_{\{1,3,5\}} - 72\zeta_{\{1,4,5\}} + 720\zeta_{\{2,3,4\}} \\ +1080\zeta_{\{2,3,5\}} + 1620\zeta_{\{2,4,5\}} - 720\zeta_{\{3,4,5\}} - 732\zeta_{\{1,2,3,4\}} - 318\zeta_{\{1,2,3,5\}} \\ -624\zeta_{\{1,2,4,5\}} + 624\zeta_{\{1,3,4,5\}} + 204\zeta_{\{2,3,4,5\}} - 864\zeta_{\{1,2,3,4,5\}} - 108\zeta_{\{1\}} \\ -540\zeta_{\{2\}} + 540\zeta_{\{5\}} + 216. \end{aligned}$$

Applying the result of Theorem 3.1, the principal eighth root of u is determined to be

$$
\begin{aligned}
6^{3/8} - \frac{3^{3/8}\zeta_{\{1\}}}{8\ 2^{5/8}} + \frac{5\ 3^{3/8}\zeta_{\{5\}}}{8\ 2^{5/8}} + \frac{71\zeta_{\{1,2\}}}{128\ 6^{5/8}} + \frac{35\ 3^{3/8}\zeta_{\{1,5\}}}{128\ 2^{5/8}} + \frac{3\ 3^{3/8}\zeta_{\{2,3\}}}{8\ 2^{5/8}} \\
+\frac{3\ 3^{3/8}\zeta_{\{2,4\}}}{8\ 2^{5/8}} + \frac{175\ 3^{3/8}\zeta_{\{2,5\}}}{128\ 2^{5/8}} + \frac{7\ 3^{3/8}\zeta_{\{1,3,5\}}}{64\ 2^{5/8}} + \frac{7\ 3^{3/8}\zeta_{\{1,4,5\}}}{64\ 2^{5/8}} \\
+\frac{15\ 3^{3/8}\zeta_{\{2,4,5\}}}{64\ 2^{5/8}} + \frac{35\ 3^{3/8}\zeta_{\{3,4,5\}}}{32\ 2^{5/8}} + \frac{2009\ 3^{3/8}\zeta_{\{1,2,3,5\}}}{2048\ 2^{5/8}} + \frac{2569\ 3^{3/8}\zeta_{\{1,2,4,5\}}}{2048\ 2^{5/8}} \\
+\frac{997\zeta_{\{1,3,4,5\}}}{1536\ 6^{5/8}} + \frac{63\ 3^{3/8}\zeta_{\{2,3,4,5\}}}{64\ 2^{5/8}} + \frac{13\ 3^{3/8}\zeta_{\{1,2,3,4,5\}}}{512\ 2^{5/8}} - \frac{3^{3/8}\zeta_{\{3,4\}}}{2\ 2^{5/8}} \\
-\frac{5\ 3^{3/8}\zeta_{\{2\}}}{8\ 2^{5/8}} - \frac{3^{3/8}\zeta_{\{1,3\}}}{8\ 2^{5/8}} - \frac{3^{3/8}\zeta_{\{1,4\}}}{8\ 2^{5/8}} - \frac{3\ 3^{3/8}\zeta_{\{3,5\}}}{8\ 2^{5/8}} - \frac{3\ 3^{3/8}\zeta_{\{4,5\}}}{8\ 2^{5/8}} \\
-\frac{25\ 3^{3/8}\zeta_{\{2,3,5\}}}{64\ 2^{5/8}} - \frac{21\ 3^{3/8}\zeta_{\{1,3,4,5\}}}{64\ 2^{5/8}} - \frac{5\zeta_{\{3,4,5\}}}{2\ 6^{5/8}} - \frac{\zeta_{\{1,3,5\}}}{4\ 6^{5/8}} - \frac{\zeta_{\{1,4,5\}}}{4\ 6^{5/8}} \\
-\frac{11\zeta_{\{1,5\}}}{8\ 6^{5/8}} - \frac{37\zeta_{\{2,5\}}}{8\ 6^{5/8}} - \frac{5\zeta_{\{1,3,4\}}}{32\ 6^{5/8}} - \frac{25\zeta_{\{2,3,4\}}}{32\ 6^{5/8}} - \frac{5\zeta_{\{1,2,3\}}}{64\ 6^{5/8}} \\
-\frac{29\zeta_{\{1,2,4\}}}{64\ 6^{5/8}} - \frac{61\zeta_{\{1,2,3,4\}}}{1536\ 6^{5/8}} - \frac{4519\zeta_{\{2,3,4,5\}}}{1536\ 6^{5/8}} - \frac{8243\zeta_{\{1,2,3,5\}}}{3072\ 6^{5/8}} \\
-\frac{8987\zeta_{\{1,2,4,5\}}}{3072\ 6^{5/8}} - \frac{4135\zeta_{\{1,2,5\}}}{6144\ 6^{5/8}} - \frac{13321\zeta_{\{1,2,3,4,5\}}}{24576\ 6^{5/8}}.
\end{aligned}
$$

4 Explicit kth Root Formulas

While recursive constructions are convenient for proving existence of roots, they do not give a clear picture of the algebraic structure. The goal now is to give a more explicit formulation of zeon roots.

Given a polynomial function $f(x)$ and $a \in \mathbb{R}$, recall the *Taylor series expansion of f about a*:

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!}(x-a)^j.$$

Writing arbitrary $\alpha \in \mathcal{C}\ell_n{}^{\rm nil}$ in the form $\alpha = a_\emptyset + \beta$, where $\beta \in \mathcal{C}\ell_n{}^{\rm nil_0}$, the formal Taylor series of $f(\alpha)$ about $a_\emptyset$ is defined by

$$\begin{aligned} f(\alpha) &= \sum_{j=0}^{\infty} \frac{f^{(j)}(a_\emptyset)}{j!}(\alpha - a_\emptyset)^j \\ &= \sum_{j=0}^{\infty} \frac{f^{(j)}(a_\emptyset)}{j!}\beta^j. \end{aligned}$$

With the formal series in hand, an explicit formula for the principal kth root is within reach.

Theorem 4.1. *Let $\alpha \in \mathcal{C}\ell_n{}^{\text{nil}_\star}$, where $n \geq 1$, and let $k \geq 2$ be a positive integer. The principal kth root of α is given by*

$$u^{1/k} = a_\emptyset{}^{1/k} + \sum_{I \neq \emptyset} \left(\sum_{j=1}^{|I|} a_\emptyset{}^{-j+\frac{1}{k}} \sum_{\ell=0}^{j} \frac{S_1(j,\ell)}{k^\ell} \sum_{\substack{\pi \in \mathcal{P}(I) \\ |\pi| = j}} a_\pi \right) \zeta_I.$$

Proof. Let $u \in \mathcal{C}\ell_n{}^{\text{nil}}$ be written in the form $a_\emptyset + \beta$, where $a_\emptyset \in \mathbb{R}$ is nonzero and β is nilpotent of index $m+1$. For fixed $k \in \mathbb{N}$, let $f(x) = x^{1/k}$ and consider the Taylor series expansion of $f(u)$ expanded about $a_\emptyset$. In particular,

$$\begin{aligned} u^{1/k} &= \sum_{j=0}^{\infty} \frac{f^{(j)}(a_\emptyset)}{j!}\beta^j \\ &= \sum_{j=0}^{m} \frac{f^{(j)}(a_\emptyset)}{j!}\beta^j. \end{aligned}$$

Let $S_1(j,\ell)$ denote the Stirling number of the first kind defined by the following property: $(-1)^{(j-\ell)}S_1(j,\ell)$ is the number of permutations of j elements that contain exactly ℓ cycles. It is well known that Stirling numbers of the first kind are generated by the falling factorial $(x)_n$. In light of this result, it is not difficult to show that

$$\frac{d^j}{dx^j}(x^{1/k}) = \sum_{\ell=0}^{j} \frac{S_1(j,\ell)}{k^\ell x^{j-1/k}}.$$

Hence,

$$\frac{f^{(j)}(a_\emptyset)}{j!} = \frac{1}{j!} \sum_{\ell=0}^{j} \frac{S_1(j,\ell)}{k^\ell a_\emptyset{}^{j-1/k}} = \frac{a_\emptyset{}^{-j+\frac{1}{k}}}{j!} \sum_{\ell=0}^{j} \frac{S_1(j,\ell)}{k^\ell}.$$

Writing $\beta = \sum_{I \neq \emptyset} a_I \zeta_I$, Lemma 2.2 gives $\beta^j = j! \sum_{|I| \geq j} \zeta_I \sum_{\substack{\pi \in \mathcal{P}(I) \\ |\pi| = j}} a_\pi$, so that

$$\begin{aligned}
u^{1/k} &= \sum_{j=0}^{m} \frac{f^{(j)}(a_\emptyset)}{j!} \beta^j \\
&= {a_\emptyset}^{1/k} + \sum_{j=1}^{m} \frac{{a_\emptyset}^{-j+\frac{1}{k}}}{j!} \sum_{\ell=0}^{j} \frac{S_1(j,\ell)}{k^\ell} \beta^j \\
&= {a_\emptyset}^{1/k} + \sum_{j=1}^{m} \frac{{a_\emptyset}^{-j+\frac{1}{k}}}{j!} \sum_{\ell=0}^{j} \frac{S_1(j,\ell)}{k^\ell} j! \sum_{|I| \geq j} \sum_{\substack{\pi \in \mathcal{P}(I) \\ |\pi| = j}} a_\pi \zeta_I \\
&= {a_\emptyset}^{1/k} + \sum_{j=1}^{m} {a_\emptyset}^{-j+\frac{1}{k}} \sum_{\ell=0}^{j} \frac{S_1(j,\ell)}{k^\ell} \sum_{|I| \geq j} \sum_{\substack{\pi \in \mathcal{P}(I) \\ |\pi| = j}} a_\pi \zeta_I.
\end{aligned}$$

For fixed nontrivial multiindex I, the coefficient of ζ_I in $u^{1/k}$ is now given by rearranging the summation: namely,

$$\langle u^{1/k}, \zeta_I \rangle = \sum_{j=1}^{|I|} {a_\emptyset}^{-j+\frac{1}{k}} \sum_{\ell=0}^{j} \frac{S_1(j,\ell)}{k^\ell} \sum_{\substack{\pi \in \mathcal{P}(I) \\ |\pi| = j}} a_\pi.$$

Expanding $u^{1/k}$ in terms of basis blades then reveals the desired result:

$$u^{1/k} = {a_\emptyset}^{1/k} + \sum_{I \neq \emptyset} \left(\sum_{j=1}^{|I|} {a_\emptyset}^{-j+\frac{1}{k}} \sum_{\ell=0}^{j} \frac{S_1(j,\ell)}{k^\ell} \sum_{\substack{\pi \in \mathcal{P}(I) \\ |\pi| = j}} a_\pi \right) \zeta_I.$$

□

Example 4.2. A closed formula for the principal 4th root of $\sum_{I \in 2^{[2]}} a_I \zeta_I \in {\mathcal{C}\ell_2}^{\mathrm{nil}}$, as computed by *Mathematica*, is

$$\frac{5a_\emptyset \left(a_{\{1,2\}}\zeta_{\{1,2\}} + a_{\{1\}}\zeta_{\{1\}} + a_{\{2\}}\zeta_{\{2\}}\right) - 4a_{\{1\}}a_{\{2\}}\zeta_{\{1,2\}} + 25{a_\emptyset}^2}{25{a_\emptyset}^{9/5}}.$$

Collecting terms by basis blade, this is seen to be

$$\sqrt[4]{a_\emptyset} + \frac{a_{\{1\}}\zeta_{\{1\}}}{4{a_\emptyset}^{3/4}} + \frac{a_{\{2\}}\zeta_{\{2\}}}{4{a_\emptyset}^{3/4}} + \frac{\left(4a_\emptyset a_{\{1,2\}} - 3a_{\{1\}}a_{\{2\}}\right)\zeta_{\{1,2\}}}{16{a_\emptyset}^{7/4}}.$$

4.1 Closed formulas for $n = 1, 2, 3$.

This approach leads to a number of dimension-dependent special formulas. For example, in ${\mathcal{C}\ell_1}^{\text{nil}_\star}$, the principal kth root of $\alpha = a_\emptyset + a_{\{1\}}\zeta_{\{1\}}$ is

$$\alpha^{1/k} = {a_\emptyset}^{1/k} + \frac{a_\emptyset^{\frac{1}{k}-1} a_{\{1\}}\zeta_{\{1\}}}{k}.$$

In ${\mathcal{C}\ell_2}^{\text{nil}_\star}$, the principal kth root of $\alpha = a_\emptyset + a_{\{1\}}\zeta_{\{1\}} + a_{\{2\}}\zeta_{\{2\}} + a_{\{1,2\}}\zeta_{\{1,2\}}$ is

$$\begin{aligned}\alpha^{1/k} &= {a_\emptyset}^{1/k} + \frac{{a_\emptyset}^{\frac{1}{k}-1} a_{\{1\}}\zeta_{\{1\}}}{k} + \frac{{a_\emptyset}^{\frac{1}{k}-1} a_{\{2\}}\zeta_{\{2\}}}{k} \\ &\quad + \left[\frac{{a_\emptyset}^{\frac{1}{k}-1} a_{\{1,2\}}}{k} + \frac{{a_\emptyset}^{\frac{1}{k}-2} a_{\{1\}}a_{\{2\}}}{k^2} - \frac{{a_\emptyset}^{\frac{1}{k}-2} a_{\{1\}}a_{\{2\}}}{k}\right]\zeta_{\{1,2\}}.\end{aligned}$$

In ${\mathcal{C}\ell_3}^{\text{nil}_\star}$, the principal kth root of $\alpha = \sum_{I\in 2^{[3]}} a_I\zeta_I$ is given by

$$\begin{aligned}\alpha^{1/k} = {a_\emptyset}^{1/k} &+ \frac{{a_\emptyset}^{\frac{1}{k}-1} a_{\{1\}}}{k}\zeta_{\{1\}} + \frac{{a_\emptyset}^{\frac{1}{k}-1} a_{\{2\}}}{k}\zeta_{\{2\}} + \frac{{a_\emptyset}^{\frac{1}{k}-1} a_{\{3\}}}{k}\zeta_{\{3\}} \\ &+ \left[\frac{{a_\emptyset}^{\frac{1}{k}-1} a_{\{1,2\}}}{k} + \frac{{a_\emptyset}^{\frac{1}{k}-2} a_{\{1\}}a_{\{2\}}}{k^2} - \frac{{a_\emptyset}^{\frac{1}{k}-2} a_{\{1\}}a_{\{2\}}}{k}\right]\zeta_{\{1,2\}} \\ &+ \left[\frac{{a_\emptyset}^{\frac{1}{k}-1} a_{\{1,3\}}}{k} + \frac{{a_\emptyset}^{\frac{1}{k}-2} a_{\{1\}}a_{\{3\}}}{k^2} - \frac{{a_\emptyset}^{\frac{1}{k}-2} a_{\{1\}}a_{\{3\}}}{k}\right]\zeta_{\{1,3\}} \\ &+ \left[\frac{{a_\emptyset}^{\frac{1}{k}-1} a_{\{2,3\}}}{k} + \frac{{a_\emptyset}^{\frac{1}{k}-2} a_{\{2\}}a_{\{3\}}}{k^2} - \frac{{a_\emptyset}^{\frac{1}{k}-2} a_{\{2\}}a_{\{3\}}}{k}\right]\zeta_{\{2,3\}} \\ + \left[\frac{{a_\emptyset}^{\frac{1}{k}-3} a_{\{1\}}a_{\{2\}}a_{\{3\}}}{k^3} - \frac{3{a_\emptyset}^{\frac{1}{k}-3} a_{\{1\}}a_{\{2\}}a_{\{3\}}}{k^2} + \frac{2{a_\emptyset}^{\frac{1}{k}-3} a_{\{1\}}a_{\{2\}}a_{\{3\}}}{k}\right]\zeta_{\{1,2,3\}} \\ &+ \left[\frac{{a_\emptyset}^{\frac{1}{k}-2} a_{\{3\}}a_{\{1,2\}}}{k^2} + \frac{{a_\emptyset}^{\frac{1}{k}-2} a_{\{2\}}a_{\{1,3\}}}{k^2} - \frac{{a_\emptyset}^{\frac{1}{k}-2} a_{\{3\}}a_{\{1,2\}}}{k}\right]\zeta_{\{1,2,3\}} \\ &+ \left[\frac{{a_\emptyset}^{\frac{1}{k}-2} a_{\{1\}}a_{\{2,3\}}}{k^2} - \frac{{a_\emptyset}^{\frac{1}{k}-2} a_{\{2\}}a_{\{1,3\}}}{k} - \frac{{a_\emptyset}^{\frac{1}{k}-2} a_{\{1\}}a_{\{2,3\}}}{k}\right]\zeta_{\{1,2,3\}} \\ &+ \frac{{a_\emptyset}^{\frac{1}{k}-1} a_{\{1,2,3\}}}{k}\zeta_{\{1,2,3\}}.\end{aligned}$$

5 Concluding Remarks

Combinatorial applications of zeons have been studied in several papers in recent years, primarily in the context of products and integral powers of nilpotent elements. The current paper represents a logical step forward by considering rational powers and group-theoretic properties of invertible zeons. While their structure is easy to define, they offer some interesting revelations (e.g. the discovery of Stirling numbers of the first kind in the closed formula for kth roots) when examined in depth.

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