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On t-Restricted Optimal Rubbling of Graphs

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On t-Restricted Optimal Rubbling of Graphs

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by

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ABSTRACT

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For a graph $G = (V, E)$, a pebble distribution f is defined as $f : V \to \mathbb{N} \cup \{0\}$, where each vertex $v \in V$ begins with $f(v)$ pebbles. A pebbling move takes two pebbles from some vertex adjacent to v and places one pebble on v. A rubbling move takes one pebble from each of two vertices that are adjacent to v and places one pebble on v. A vertex x is reachable under a pebbling distribution f if there exists some sequence of rubbling and pebbling moves that places a pebble on x. A pebbling distribution where every vertex is reachable is called a rubbling configuration. The t-restricted optimal rubbling number of G is the minimum number of pebbles required for a rubbling configuration where no vertex is initially assigned more than t pebbles. Here we present results on the 1-restricted optimal rubbling number and the 2-restricted optimal rubbling number.

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1 INTRODUCTION

We begin with some basic definitions in Section 1.1 and give definitions specific to our problem in Section 1.2. Our problem statement is also given in Section 1.2.

1.1 General Graph Theory Definitions

Our definitions are consistent with [3]. A *graph* $G = (V, E)$ is a set of *vertices* $V(G)$ and a set of *edges* $E(G)$ drawn between distinct pairs of vertices. For the purposes of this paper, we will assume that each pair of vertices is connected by at most one edge. Figure 1 gives an example of a graph with $u, v \in V(G)$ and $uv \in E(G)$. For two sets A and B, the *set difference* $A - B$ is defined as $A - B = \{x : x \in A \text{ and } x \notin B\}$. Two vertices $u, v \in V(G)$ that are connected by an edge $uv \in E(G)$ are said to be *adjacent*. We also refer to u and v as *neighbors*. The set of all neighbors of some vertex v is called the *open neighborhood* of v, and is denoted $N(v)$. The *closed neighborhood* of v, denoted $N[v]$, is the set $N(v) \cup \{v\}$. For some set $S \subset V(G)$, we define $N(S)$ as the union of the open neighborhoods of the vertices of S. We analogously define $N[S]$ as the union of the closed neighborhoods of the vertices of S. Let $S \subseteq V$ and $v \in S$. Vertex u is called a *private neighbor*, denoted *S-pn*, of v with respect to S if $u \in N[v]-N[S-\{v\}]$. An *S-pn* of v is *external* if it is a vertex in $V - S$. The *external private* *neighborhood* of v, denoted $epn(v, S)$, is the set of all of the external private neighbors of v. In Figure 1, vertex u is a private neighbor of v with respect to $V(G)$.

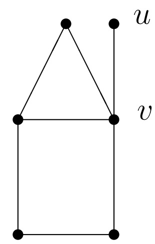

Figure 1: House graph with a chimney

The *order* n of a graph G is the cardinality of the vertex set $V(G)$. If $|V(G)| = 1$, then we say that G is the *trivial graph.* The *size* m of a graph is the cardinality of the edge set $E(G)$. The *degree* of a vertex v, denoted $deg(v)$, is the cardinality of $N(v)$. If $deg(v) = 1$, then we say that v is a *leaf.* If $deg(v) \geq 2$, then v is an *internal vertex.* A vertex that is adjacent to a leaf is called a *support vertex.* A vertex adjacent to two or more leaves is called a *strong support vertex.* The largest degree among the vertices of G, denoted $\Delta(G)$, is called the *maximum degree* of G. The *minimum* degree of G, denoted $\delta(G)$, is the smallest degree among the vertices of G. If $E(G) = \emptyset$, then we say that G is an *empty graph.* An *r-regular* graph is a graph whose vertices all have degree equal to r. For two graphs H and G, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then we say that H is a *subgraph* of G. If H is a subgraph of G such that $V(G) = V(H)$, then H is a *spanning subgraph* of G. For a nonempty subset S of $V(G)$, the subgraph $G[S]$ of G, *induced* by S has S as its vertex set and two vertices u and v are adjacent in $G[S]$ if and only if they are adjacent in G. A subgraph H of a graph G is called an *induced subgraph* if there is a nonempty subset S of $V(G)$ such that $H = G[S]$.

Now we will define specific classes of graphs that are of interest. A graph G on n vertices is a *complete graph* K_n if every pair of vertices of G are adjacent. In other words, all possible edges are present in G. For an integer $n \geq 1$, a *path* P_n is a graph of order n and size $n-1$ whose vertices can be labeled by $p_1, p_2, ..., p_n$ and whose edges are $p_i p_{i+1}$ for $i = 1, 2, ..., n-1$. Vertices p_1 and p_n are called the *terminal vertices* of P_n. Figure 2 is the path P_3. For an integer $n \geq 3$, a *cycle* C_n is a graph of order n and size n whose vertices can be labeled by $v_1, v_2, ..., v_n$ and whose edges are $v_1 v_n$ and $v_i v_{i+1}$ for $i = 1, 2, ..., n-1$. Figure 3 is the cycle C_4. A *wheel* W_n is a cycle on n vertices, with an extra vertex adjacent to each vertex on the cycle.

A graph G is *bipartite* if $V(G)$ can be partitioned into two sets U and W, such that every edge of $E(G)$ joins a vertex of U and a vertex of W. We call U and W *partite sets*. A graph G is a *complete bipartite graph* if $V(G)$ can be partitioned in to two sets U and W so that $uw \in E(G)$ if and only if $u \in U$ and $w \in W$. A complete bipartite graph with $|U| = r$ and $|W| = s$ is denoted $K_{r,s}$, where $1 \leq r \leq s$. In particular, if $r = 1$ in a complete bipartite graph, the graph is called a *star*. For an example, see Figure 4. The vertex of maximum degree in a star is called the *center*.

Figure 2: The path P_3

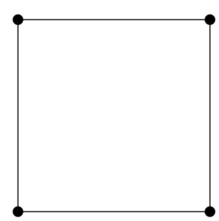

Figure 3: The cycle C_4

For two vertices u and v in a graph G, we say that u and v are *connected* if there exists a path subgraph of G where u and v are terminal vertices. We refer to the path between u and v as a *u-v path*. If every pair of vertices in $V(G)$ are connected, then G is a *connected graph*. The *distance* $d(u, v)$ from a vertex u to a vertex v is the minimum length of all the paths between u and v. For a vertex v in a connected graph G, the *eccentricity* of v is the largest value of $d(v, u)$ for all $u \in V(G)$. The *diameter* of a graph G, denoted $diam(G)$, is the maximum eccentricity of all the vertices in $V(G)$. A vertex

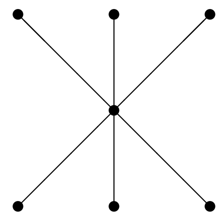

Figure 4: The star $K_{1,6}$

whose eccentricity equals the diameter of G is called a *peripheral vertex* of G. The subgraph induced by the peripheral vertices of G is called the *periphery* of G.

A *tree* is a connected graph with no cycle subgraphs. A *double star* $S_{r,s}$ is a tree with exactly two adjacent nonleaf vertices, one of which is adjacent to r leaves and the other adjacent to s leaves. A tree is said to be *rooted* if a single vertex r is singled out as the *root*, and the other vertices are classified in terms of their distance from r. A *parent* of a vertex v is a vertex who lies on the unique path from v to r. If v lies on the unique path from some vertex u to the root r, then u is said to be a *child* of v. A *branch* is defined as a unique path from the root to some vertex of T.

The *complement* of a graph G is defined as the graph $\overline{G}$, such that $V(\overline{G}) = V(G)$, and $uv \in E(G)$ if and only if $uv \notin E(\overline{G})$. The *Cartesian product* of two graphs G_1 and G_2 is denoted by $G_1 \Box G_2$ and defined as follows: the graph $G_1 \Box G_2$ has vertex set $V(G_1 \Box G_2) = V(G_1) \times V(G_2)$, and two distinct vertices (u, v) and (x, y) are adjacent if either: (1) $u = x$ and $vy \in E(G_2)$ or (2) $v = y$ and $ux \in E(G_1)$. A *prism* is the Cartesian product of a P_2 and a C_n. The *n-cube* Q_n, is defined as K_2 if $n = 1$. If $n \geq 2$, then Q_n is defined recursively as the Cartesian product $Q_{n-1} \square K_2$.

A set $S \subseteq V(G)$ is a dominating set of G if each vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality among all dominating sets of G is called the *domination number* of G, and is denoted $\gamma(G)$. The *k-domination number* of a graph G is the smallest cardinality among all subsets S of $V(G)$ such that each vertex in $V(G) - S$ is adjacent to at least k vertices in S, and is denoted $\gamma_k(G)$.

1.2 Pebbling and Rubbling Definitions

We will now define terms specifically associated with graph rubbling. Let G be a graph. A *pebble distribution* on G is defined as a whole number of pebbles placed on the vertices of G (see [1]). Figure 5 gives an example of a pebble distribution where two pebbles are placed on u, one pebble is placed on x, and zero pebbles are placed on each other vertex in $V(G)$.

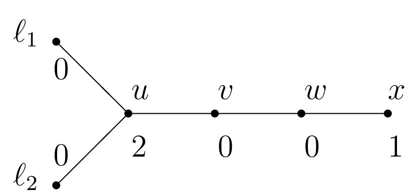

Figure 5: A pebble distribution

Using the pebbles placed on a graph by a pebble distribution, we define a *pebbling move* and a *rubbling move.*

Definition 1.1 *Let $G = (V, E)$ be a graph with adjacent vertices u and v in $V(G)$. Let f be a pebble distribution such that $f(u) \geq 2$. Then a pebbling move, denoted $p(u \to v)$, removes two pebbles from u and places one on v. This defines a new pebble distribution, f' such that: $f'(u) = f(u) - 2$, $f'(v) = f(v) + 1$, and $f'(z) = f(z)$ for all other $z \in V(G)$.*

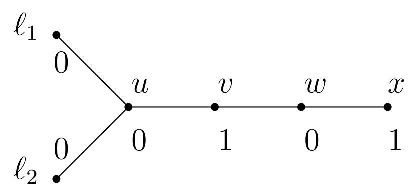

Figure 6: A pebbling move

Observe that Figure 6 depicts the graph from Figure 5 after the pebbling move $p(u \to v)$.

Definition 1.2 *Let $G = (V, E)$ be a graph with some vertex $w \in V(G)$ adjacent to distinct vertices $v \in V(G)$ and $x \in V(G)$. Let f be a pebble distribution such that $f(v) \geq 1$ and $f(x) \geq 1$. Then a rubbling move, denoted $r(v, x \to w)$, removes one pebble each from v and x and places one new pebble on w. In the new pebble distribution f' we have: $f'(v) = f(v) - 1$, $f'(x) = f(x) - 1$, $f'(w) = f(w) + 1$, and $f'(z) = f(z)$ for all other $z \in V(G)$.*

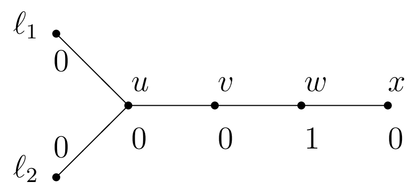

Figure 7: A rubbling move

Figure 7 depicts the graph in Figure 6 after the rubbling move $r(v, x \to w)$. A vertex v is said to be *reachable* if there exists some sequence of rubbling and pebbling moves that can place a pebble on v. A vertex that begins with zero pebbles in an initial pebble distribution is said to be *open*.

Graph pebbling, which preceded graph rubbling, only allows the pebbling move. Hence, for the following definitions concerning pebbling, it is assumed that only the pebbling move is allowed. As such, we will say that a vertex v is *reachable by pebbling* if there is a way to place a pebble on v using only pebbling moves. The pebbling number $\pi(G)$ of a graph G is defined as the smallest k such that for every pebble distribution of k pebbles, every vertex in $V(G)$ is reachable by pebbling.

The *optimal pebbling number* π_{opt} of a graph G is the least k such that there exists some distribution of k pebbles where every vertex in $V(G)$ is reachable by pebbling. The *optimal-t pebbling number* is defined as the minimum number of pebbles needed to place at least t pebbles on each vertex using only pebbling moves. Finally, the *t-restricted optimal pebbling number* is defined as the least k such that there exists some pebble distribution f on k pebbles where for each $v \in V(G)$, $f(v) = \{0, 1, ..., t-1, t\}$ and every vertex is reachable by pebbling.

In graph rubbling both the pebbling and rubbling moves are allowed. Hence, this is assumed in the following definitions concerning rubbling. If every vertex of a graph G is reachable under some pebble distribution f, then we say that f is a *rubbling configuration*. The *rubbling number* of a graph G, denoted $\rho(G)$, is the smallest k such that every pebble distribution of k pebbles results in a rubbling configuration (see [1]). The *optimal rubbling number* $\rho_{opt}(G)$ is the smallest number of pebbles required for some rubbling configuration on a graph G. Finally, the *optimal-t rubbling number* is defined as the minimum number of pebbles needed to place at least t pebbles on each vertex.

We now define the *t-restricted optimal rubbling number*, which is the focus of this paper.

Definition 1.3 *Let $G = (V, E)$ be a graph. Then the t-restricted optimal rubbling number $\rho_t^*(G)$ of G is the least k such that there exists some rubbling configuration f on k pebbles where for each $v \in V(G)$, $f(v) = \{0, 1, ..., t-1, t\}$.*

For convenience, we refer to an optimal rubbling configuration in which each vertex begins with less than t pebbles as a ρ_t^*-configuration. It is important to note that the restriction to t pebbles only applies to the initial pebble distribution. In many cases, vertices are required to obtain more than t pebbles after some sequence of moves. This is allowed, as long as no vertex begins with t pebbles.

The t-restricted optimal pebbling number has been studied, but the t-restricted optimal rubbling number has not. The goal of this thesis is to introduce and investigate t-restricted optimal rubbling. Chapter 2 will give a brief history of pebbling and rubbling. Our results are given in Chapters 3, 4, and 5. In particular, Chapter 3 gives a detailed problem statement. In Chapter 4 we will present results on the 1-restricted optimal rubbling number. This will include characterizations for specific families of graphs, as well as upper and lower bounds. Chapter 5 contains results on the 2-restricted optimal rubbling number, comparing some of these results to those obtained in Chapter 4. Finally, we conclude with open problems.

2 LITERATURE REVIEW

2.1 Graph Pebbling

As graph rubbling originates from graph pebbling, we will first discuss some notable results concerning graph pebbling. According to Hurlbert [16], Lagarias and Saks first introduced pebbling as an approach to a theorem in number theory. In 1989, Chung's results on pebbling the n-cube were used to give a proof to the theorem originally proposed by Lagarias and Saks [5]. Numerous results on graph pebbling followed this initial discovery. Some of these results are found in [5, 15, 21, 24].

An interesting area of study in pebbling is Graham's conjecture, which deals with the Cartesian product of graphs. If A and B are graphs with pebbling numbers $\pi(A)$ and $\pi(B)$, then Graham's conjecture states that $\pi(A \square B) \leq \pi(A) \times \pi(B)$. While this bound has yet to be proved in general, it has been shown to hold for several classes of graphs. Some of the current results on Graham's conjecture are found in [6, 7, 13, 27].

A likely application of graph pebbling is observed in modeling the transportation of materials. If a truck must transport fuel along a road, then some of that fuel must be used in order for the truck to reach its destination [1]. This loss is reflected by the pebbling move. In order to transport two pebbles to a new vertex, one pebble must be sacrificed along the way.

Optimal pebbling was first introduced by Pachter, Snevily and Voxman [24]. Since then, there has been extensive work done on optimal pebbling [2, 8, 10, 24, 26]. Similar to the general pebbling number, the optimal pebbling number for the hypercube has been studied [9, 22]. Milans and Clark studied the computational complexity of pebbling and optimal pebbling [20]. Optimal pebbling has also been studied in [14, 23].

The optimal t-pebbling number has also been widely studied [11, 12, 19]. Bounds for paths and cycles have been found on the optimal-t pebbling number [25, 28]. The t-restricted optimal pebbling number was introduced by Chellali, Haynes, Hedetniemi, and Lewis [4]. Their work included characterizations of those graphs with small t-restricted optimal pebbling numbers, as well as common graphs like cycles and paths. In addition, they found upper bounds related to trees, some of which serve as inspiration for our results on trees.

2.2 Graph Rubbling

The rubbling move was defined by Belford and Sieben [1]. In addition to the rubbling move, they also defined the rubbling number and the optimal rubbling number of a graph. Since our results focus mainly on t-restricted optimal rubbling, we present some of their notable results concerning the optimal rubbling number ρ_{opt}.

Theorem 2.1 (Theorem 3.3 in [1]) *Let K_n be the complete graph on n vertices with $n \geq 2$. Then $\rho_{opt}(K_n) = 2$.*

Theorem 2.2 (Theorem 3.4 in [1]) *Let W_n be the wheel graph on n vertices, with $n \geq 5$. Then $\rho_{opt}(W_n) = 2$.*

Theorem 2.3 (Theorem 3.5 in [1]) *Let $K_{m_1,m_2,\ldots,m_\ell}$ be the complete ℓ partite graph on $m_1+m_2+\ldots+m_\ell$ vertices. If there exists $j \in \{1, 2, \ldots\ell\}$ such that $m_j = 1$ or $m_j = 2$, then $\rho_{opt}(K_{m_1,m_2,\ldots,m_\ell}) = 2$. Otherwise, $\rho_{opt}(K_{m_1,m_2,\ldots,m_\ell}) = 3$.*

Theorem 2.4 (Theorem 3.13 in [1]) *Let P_n be the path on n vertices. Then $\rho_{opt}(P_n) = \lfloor \frac{n}{2} + 1 \rfloor$.*

Theorem 2.5 (Theorem 3.14 in [1]) *Let C_n be the cycle on n vertices. Then $\rho_{opt}(C_n) = \lfloor \frac{n+1}{2} \rfloor$.*

Following these initial results, Sieben and Katona discovered several upper and lower bounds on the optimal rubbling number [18]. This includes the following general upper bound and lower bound for connected graphs.

Theorem 2.6 (Corollary 6.2 and Proposition 6.3 in [18]) *Let G be a connected graph with diameter d and order n. Then $\lceil \frac{d+2}{2} \rceil \leq \rho_{opt}(G) \leq \lceil \frac{n+1}{2} \rceil$.*

More bounds on the optimal rubbling number were studied by Katona and Papp in [17]. We present one of the main results from their work:

Theorem 2.7 (Theorem 5.5 in [17]) *The optimal rubbling numbers for the n-prisms are:*

$$\rho_{opt}(C_{3k-1}\square P_2) = 2k,$$

$$\rho_{opt}(C_{3k}\square P_2) = 2k,$$

$$\rho_{opt}(C_{3k+1}\square P_2) = 2k+1,$$

$$\textit{except } \rho_{opt}(C_3\square P_2) = 3.$$

Recall the application mentioned for pebbling. Allowing the rubbling move expands the model in order to consider the situation in which two trucks may be traveling to the same location where they will combine their remaining fuel upon intersection [1].

3 DETAILED PROBLEM STATEMENT

While the t-restricted optimal pebbling number has been studied, the effects of a similar restriction applied to the optimal rubbling number have yet to be investigated. The following lemma will allow many results previously shown for $\rho_{opt}(G)$ to be extended to $\rho_t^*(G)$.

Lemma 3.1 *For any graph G, $\rho_{opt}(G) \leq \rho_t^*(G)$.*

Proof. Since any 1-restricted optimal rubbling configuration is also an optimal rubbling configuration, $\rho_{opt}(G) \leq \rho_t^*(G)$. ■

For our research we will focus on the specific cases where $t = 1$ and $t = 2$. By studying t-restricted optimal rubbling, we can provide insight to the most fuel efficient configurations when trucks have a strict limit to how much fuel they can hold, while still allowing each desired location to be reached.

4 1-RESTRICTED OPTIMAL RUBBLING

Our first result gives a lower bound on $\rho_1^*(G)$ in terms of the diameter of G. While the reasoning behind the following theorem will become more apparent as we explore various types of graphs, our previous results on ρ_{opt} leave us well-equipped to prove the result.

Theorem 4.1 *If T is a tree of diameter d, then $\rho_1^*(T) \geq \lceil \frac{d+2}{2} \rceil$.*

Proof. This follows from Theorem 2.6 and Lemma 3.1. ■

Here we see the first of many results that rely on Lemma 3.1. The next result provides another lower bound that deals with the effects of deleting a vertex from a graph.

Lemma 4.2 *For any graph G and any vertex $v \in V(G)$, $\rho_1^*(G) - 1 \leq \rho_1^*(G - v)$.*

Proof. Let f_v be a ρ_1^*-configuration of $G - v$ for some $v \in V(G)$. Then f_v can be extended to a 1-rubbling configuration of G by placing a pebble on v. Hence, $\rho_1^*(G) \leq \rho_1^*(G - v) + 1$, and the result follows. ■

It should be noted that Lemma 4.2 only provides a lower bound for $G - v$. It is possible that the removal of a vertex could increase the 1-restricted optimal rubbling number of a graph, especially if that vertex has a high degree, or the removal results in a disconnected graph. Our next theorem will make use of the previous lemma, but first we need to make a quick observation.

Observation 4.3 *For any graph G, $\rho_1^*(G) = 1$ if and only if G is the trivial graph.*

This idea is very straightforward, as an isolated pebble cannot move from its current vertex. Thus, any graph with more than one vertex will require more than one pebble. We are now ready to present our next theorem, dealing with the 1-restricted optimal rubbling number of induced subgraphs.

Theorem 4.4 *Let G be a connected graph with $\rho_1^*(G) = p$. Then for each value of $q = 1, 2, ...p$, there exists an induced subgraph H of G, such that $\rho_1^*(H) = q$.*

Proof. Let $G = G_n, G_{n-1}, ..., G_1 = K_1$ be any sequence of induced subgraphs created by removing one vertex at a time from some graph G. Note, the order of $G_k = k$ for all $1 \leq k \leq n$. By assumption, $\rho_1^*(G_n) = p$. By Observation 4.3, $\rho_1^*(G_1) = 1$. Lemma 4.2 guarantees that ρ_1^* decreases by at most one each time a vertex is removed. Thus, for each integer $1 < q < p$, there must exist a graph in the sequence whose 1-restricted optimal rubbling number equals q. Otherwise, the removal of some vertex would cause ρ_1^* to decrease by more than one, contradicting Lemma 4.2. ■

Note that Theorem 4.4 requires that we allow for the possibility of disconnected graphs. By allowing this however, Lemma 4.2 guarantees that the order in which we remove each vertex does not matter.

4.1 Small Values

We have already seen that K_1 is the only graph whose 1-restricted optimal rubbling number equals one. In this subsection we proceed to determine which non-trivial graphs have small values of ρ_1^*. It is important to note that in any initial 1-restricted optimal rubbling configuration, a rubbling move must be the first move made in any sequence. Furthermore, each time a move is made, a pebble is lost. Keeping these ideas in mind, we now characterize those graphs whose 1-restricted optimal rubbling number equals two.

Theorem 4.5 *For any graph G, $\rho_1^*(G) = 2$ if and only if there exist two vertices, $u, v \in V(G)$ such that $N(u) \cap N(v) = V(G) - \{u, v\}$.*

Proof. If $\rho_1^*(G) = 2$, then there is at most one possible rubbling move, and no possible pebbling moves. It follows that every vertex must be reached with the single rubbling move. Thus, there must exist two vertices $u, v \in V(G)$ such that $N(u) \cap N(v) = V(G) - \{u, v\}$. Conversely, if there are two vertices $u, v \in V(G)$ such that $N(u) \cap N(v) = V(G) - \{u, v\}$, then place one pebble on each of u and v. Every other vertex in G is adjacent to both u and v, so a rubbling move can place a pebble on any vertex. Since G is not trivial, $\rho_1^*(G) = 2$. ■

Given the results of the previous theorem, we now proceed to characterize the graphs that have a 1-restricted optimal rubbling number equal to three.

Theorem 4.6 *Let G be a graph. Then $\rho_1^*(G) = 3$ if and only if the following two conditions hold*

(i) $\rho_1^*(G) > 2$*;*

(ii) There exists a set $S \in V(G)$ with $|S| = 3$, such that for every vertex $v \in V(G) - S$ either v has at least two neighbors in S, or v has exactly one neighbor in S and a neighbor that is adjacent to every vertex of $S - N(v)$.

Proof. If condition (ii) is satisfied by a set $S = \{x, y, z\}$, then place a pebble on each vertex in S. Let $v \in V(G) - S$. Clearly, if v has at least two neighbors in S, then a rubbling move will place a pebble on v. Suppose that $N(v) \cap S = \{x\}$. Let u be a neighbor of v that is adjacent to both y and z. Note that u could be x. If $u = x$, then the rubbling move $r(y, z \to x)$ results in two pebbles on x. The pebbling move $p(x \to v)$ places a pebble on v. If $u \neq x$, then $r(y, z \to u)$ places a pebble on u, and $r(u, x \to v)$ places a pebble on v. In any case, v is pebbled, so $\rho_1^*(G) \leq 3$. Condition (i) guarantees that $\rho_1^*(G) \geq 3$, and so, $\rho_1^*(G) = 3$.

Now assume that $\rho_1^*(G) = 3$. It follows that condition (i) must be true. Let $S = \{x, y, z\}$ be a set such that placing one pebble on each of x, y, and z results in a rubbling configuration of G. Consider vertex $v \in V - S$. Since $\rho_1^*(G) = 3$, and a pebble is lost with each move, at most two moves are allowed as the result of any ρ_1^*-configuration. Furthermore, immediately prior to v receiving a pebble, there must be at least two pebbles in $N(v)$. It follows that there are only three ways to place a pebble on v: a rubbling move, two rubbling moves, or a rubbling move followed by a pebbling move. If a single rubbling move will reach v, then clearly v must have two neighbors in S. If v has no neighbors in S, then it would take at least two moves to place two pebbles in $N(v)$. Since only two moves are allowed, v cannot be reached. Hence, we may assume, without loss of generality, that v is adjacent to x and some vertex $w \in N(v)$ is adjacent to y and z. This implies that (ii) must be true. ∎

4.2 Specific Families of Graphs

We will now turn our focus to common families of graphs, including cycles, paths, stars, and bipartite graphs. Many results in this section will make use of Lemma 3.1. Our first result characterizes the 1-restricted optimal rubbling number for complete graphs. While our result is the same as that of Belford and Sieben [1], we provide a proof in order to account for the added initial restriction.

Theorem 4.7 *Let K_n be the complete graph of order $n \geq 2$. Then $\rho_1^*(K_n) = 2$.*

Proof. By Observation 4.3, we have that $\rho_1^*(K_n) \geq 2$. From the definition of K_n, every pair of vertices must satisfy the conditions of Theorem 4.5. Hence, $\rho_1^*(K_n) = 2$. ■

The following three theorems will deal with different families of bipartite graphs. We begin by determining ρ_1^* for the stars $K_{1,s}$ when $s \geq 3$. Stars are of particular interest because they will reappear frequently throughout this paper.

Theorem 4.8 *Let $K_{1,s}$ be a star with $s \geq 3$. It follows that $\rho_1^*(K_{1,s}) = 3$.*

Proof. From Observation 4.3 and Theorem 4.5, we see that $\rho_1^*(K_{1,s}) \geq 3$. Using three pebbles, place one on the center c, and one on each of two leaves ℓ_1 and ℓ_2. A rubbling move $r(\ell_1, \ell_2 \to c)$ places two pebbles on c. A subsequent pebbling move will reach any vertex in $V(K_{1,s}) - \{\ell_1, \ell_2, c\}$. Thus, $\rho_1^*(K_{1,s}) \leq 3$, and it follows that $\rho_1^*(K_{1,s}) = 3$. ■

Theorem 4.9 *Let $G = S_{r,s}$ be a double star with $1 \leq r \leq s$. Then $\rho_1^*(G) = 3$ if $r = 1$ and $s \leq 2$, and $\rho_1^*(G) = 4$ otherwise.*

Proof. Let c_1 and c_2 be the two nonleaf vertices of G. If $r = 1$ and $s = 1$, then place one pebble on each of the two leaves and one pebble on any internal vertex to form a rubbling configuration. If $r = 1$ and $s = 2$, then place one pebble on each of leaves of G to form a rubbling configuration. Both of these configurations are optimal from Theorem 4.5. Thus, we may assume that $2 \leq r \leq s$. From Observation 4.3 and Theorem 4.5, we see that $\rho_1^*(G) \geq 3$. Since $r \geq 2$, any set of three vertices that is also a dominating set must contain c_1, c_2, and some leaf ℓ. Without loss of generality, consider a dominating set S where ℓ is adjacent to c_1. Then each vertex in $N(c_2)$ has only one neighbor in S, and c_2 is not adjacent to $V(G) - S = \{c_1, \ell\}$. Thus from Theorem 4.6, we see that $\rho_1^*(G) \geq 4$. Now place a pebble on c_1, c_2, and any two leaves ℓ_1 and ℓ_2 adjacent to c_1. A rubbling move $r(\ell_1, \ell_2 \to c_1)$ places two pebbles on c_1, allowing each of its leaves to be reached. Finally, a pebbling move $p(c_1 \to c_2)$ places two pebbles on c_2, allowing each of its leaves to be reached. Thus, $\rho_1^*(G) \leq 4$. It follows that $\rho_1^*(G) = 4$. ■

Our next result generalizes Theorem 4.8. Note that the ρ_1^*-configurations for these graphs are occasionally different, but use the same number of pebbles as those results in the proof of Theorem 2.3 (Theorem 3.5 in [1]). This is due to our restriction of one pebble placed on each vertex.

Theorem 4.10 *The complete bipartite graph $K_{r,s}$, where $2 \leq r \leq s$, has $\rho_1^*(K_{r,s}) = 2$ if $r = 2$ and $\rho_1^*(K_{r,s}) = 3$ otherwise.*

Proof. Let R and S be the partite sets of $K_{r,s}$ such that $|S| = s$ and $|R| = r$. If $r > 2$, then place one pebble on each of two vertices $s_1, s_2 \in S$ and one pebble on some vertex $r_1 \in R$. A rubbling move $r(s_1, s_2 \to r_i)$ places a pebble on any $r_i \in R$. Another rubbling move $r(r_i, r_1 \to s_j)$ places a pebble on any $s_j \in S$. Since every vertex is reachable, this is a rubbling configuration, and $\rho_1^*(K_{r,s}) \leq 3$. It follows from Theorem 4.5 that $\rho_1^*(G) \geq 3$. Hence, $\rho_1^*(K_{r,s}) = 3$. Now assume that $r = 2$. Place a pebble on each vertex in R. A rubbling move will place a pebble on any vertex in S, so $\rho_1^*(G) \leq 2$. Clearly, $K_{r,s}$ is not trivial, so $\rho_1^*(K_{r,s}) = 2$. ■

We will now proceed to paths and cycles. They provide two interesting classes of graphs whose 1-restricted optimal rubbling number and optimal rubbling number are identical.

Theorem 4.11 *For any path P_n on n vertices, $\rho_1^*(P_n) = \left\lfloor \frac{n}{2} + 1 \right\rfloor$.*

Proof. Using the configuration described by Belford and Sieben in their proof of Theorem 2.4 (Theorem 3.13 in [1]), place one pebble on each p_i, where i is odd. If n is odd, then we are done. If n is even, then place one more pebble on p_n to complete the rubbling configuration using $\left\lfloor \frac{n}{2} + 1 \right\rfloor$ pebbles. By Lemma 3.1 and Theorem 2.4, this is optimal. ■

We will see later on that the upper bound on P_n from the proof is, in fact, a general upper bound for all connected graphs.

Theorem 4.12 *Let C_n be a cycle on n vertices. Then $\rho_1^*(C_n) = \left\lfloor \frac{n+1}{2} \right\rfloor$.*

Proof. Using the configuration described by Belford and Sieben in their proof of Theorem 2.5 (Theorem 3.14 in [1]), place one pebble on each v_i where i is odd. Each vertex is either adjacent to two pebbles, or begins with a pebble. Thus, this is a rubbling configuration using $\left\lfloor \frac{n+1}{2} \right\rfloor$ pebbles. From Lemma 3.1 and Theorem 2.5, this must be optimal. ■

The following theorem gives the 1-restricted optimal rubbling number of prisms, $C_n \square P_2$. It should be noted that the ρ_1^*-configurations we give will be identical or very similar to those given in the proof of Theorem 2.7 (Theorem 5.5 in [17]) by Katona and Papp. In fact, we will again make use of Lemma 3.1 in order to show that our configurations are optimal.

Theorem 4.13 *The* 1*-restricted optimal rubbling number of prism is as follows:*

(i) $\rho_1^*(C_3 \square P_2) = 3$.

(ii) $\rho_1^*(C_4 \square P_2) = 4$.

(iii) $\rho_1^*(C_n \square P_2) = \left\lceil \frac{2n}{3} \right\rceil$.

Proof. *(i)* From Theorem 4.5, observe that $\rho_1^*(C_3 \square P_2) \geq 3$. A rubbling configuration is obtained from placing one pebble on each of two vertices in one copy of C_3 and one pebble on any vertex in the other copy. Thus, $\rho_1^*(C_3 \square P_2) \leq 3$, and it follows that $\rho_1^*(C_3 \square P_2) = 3$.

(ii) Now, if $n = 4$, then Theorem 4.6 proves that $\rho_1^*(C_4 \square P_2) \geq 4$. A rubbling configuration is obtained by placing pebbles on each copy of C_4 in the method of Theorem 4.12. Thus, $\rho_1^*(C_4 \square P_2) \leq 4$, and it follows that $\rho_1^*(C_4 \square P_2) = 4$.

(iii) Assume that $n \geq 5$. Let $v_0, v_1, ..., v_{n-1}$ be the vertices of one copy of C_n and let $u_0, u_1, ...,\ u_{n-1}$ be the vertices of the other copy of C_n such that $u_i v_i \in E(G)$ for $1 \leq i \leq n-1$. Partition the vertices into three sets X_j for $0 \leq j \leq 2$ where $u_i, v_i \in X_j$ when $i \equiv j \pmod 3$. For convenience, we compute the indices using mod three arithmetic. Place pebbles on each $v_i \in X_0$ and each $u_i \in X_1$. Consider a vertex u_i that has no pebble. We see that $u_i \in X_0 \cup X_2$. If $u_i \in X_0$, then a rubbling move $r(u_{i+1}, v_i \to u_i)$ will reach u_i. It should be noted that in some cases u_{i+1} will not begin with a pebble in the initial pebbling distribution. This can be remedied, however, with a rubbling move $r(v_{i+1}, u_{i+2} \to u_{i+1})$. Since this does not use the pebble on v_i, the result above still holds. If $u_i \in X_2$, then the rubbling move $r(u_{i+2}, v_{i+1} \to u_{i+1})$ places a pebble on u_{i+1}. A subsequent rubbling move $r(u_{i-1}, u_{i+1} \to u_i)$ will reach u_i. Again, in some cases, u_{i+2} will not begin with a pebble. A rubbling move $r(v_{i+2}, u_{i+3} \to u_{i+2})$ will allow u_{i+2} to receive a pebble without disturbing any other vertex used in the sequence above.

Next consider a vertex v_i that has no pebble. Then $v_i \in X_1 \cup X_2$.

If $v_i \in X_1$, then the rubbling move $r(v_{i-1}, u_i \to v_i)$ places a pebble on v_i. If $v_i \in X_2$, then the rubbling move $r(v_{i-2}, u_{i-1} \to v_{i-1})$, followed by the rubbling move $r(v_{i-1}, v_{i+1} \to v_i)$ reaches v_i.

Since at most $\lceil \frac{n}{3} \rceil$ pebbles were used on each copy of C_n, it follows that $\rho_1^*(C_n \square P_2) \leq \lceil \frac{2n}{3} \rceil$. Now, for some $k \geq 2$, if $n = 3k - 1$, then $\lceil \frac{2n}{3} \rceil = \left\lceil \frac{2(3k-1)}{3} \right\rceil = \lceil 2k - \frac{1}{3} \rceil = 2k$. If $n = 3k$, then $\lceil \frac{2n}{3} \rceil = \left\lceil \frac{2(3k)}{3} \right\rceil = \lceil 2k \rceil = 2k$. Finally, if $n = 3k + 1$, then $\lceil \frac{2n}{3} \rceil = \left\lceil \frac{2(3k+1)}{3} \right\rceil = \lceil 2k + \frac{1}{3} \rceil = 2k + 1$. Hence, from Lemma 3.1 and Theorem 2.7, we have that $\rho_1^*(C_n \square P_2) \geq \lceil \frac{2n}{3} \rceil$, and it follows that $\rho_1^*(C_n \square P_2) = \lceil \frac{2n}{3} \rceil$. ■

It is quite fascinating that along with paths and cycles, $\rho_{opt}(C_n \square P_2) = \rho_1^*(C_n \square P_2)$. Our next two results will make use of the following lemma concerning vertices of degree two.

Lemma 4.14 *For any graph G, no vertex of degree two will be the first vertex to receive two pebbles as the result of any 1-restricted optimal rubbling configuration.*

Proof. Let $v \in V(G)$ such that $deg(v) = 2$. Assume to the contrary that v is the first vertex to receive two pebbles under a ρ_1^*-configuration f. Since $f(v) \leq 1$, at least one pebble must be moved to v through a rubbling move $r(u, w \to v)$, where u and w are the neighbors of v. From our initial assumptions, no pebbles can be placed on v through a pebbling move prior to v receiving two pebbles. Hence, immediately prior to v receiving a second pebble, there must be one pebble on each of v, u, and w. Since this must be the case regardless of the configuration, we can assume, without loss of generality, that $f(v) = f(u) = f(w) = 1$. After the rubbling move $r(u, w \to v)$, in the resulting pebbling distribution f' we have $f'(v) = 2$, $f'(u) = 0$, $f'(w) = 0$, and $f'(x) = f(x)$ for all other $x \in V(G)$. Under f' there are only two possible pebbling moves from v, namely, $p(v \to u)$ and $p(v \to w)$. But each of these moves just returns one pebble to either u or v. Vertex v is still reachable if $f(v) = 0$ and $f(u) = f(w) = 1$. Thus, the same three vertices can be reached using one less pebble, and it follows v will not be the first vertex to receive two pebbles under any ρ_1^*-configuration of G. ■

We now define two more graphs closely related to stars that are of interest. Namely, we define brooms and dumbbells. The *broom*, denoted $B(n, m)$, can be obtained from the path on n vertices by appending m leaves to one endpoint of the path. We will label the vertices of the path as p_1,...,p_n and the leaves as ℓ_1,...,ℓ_m. Without loss of generality, we will assume that the leaves are adjacent to p_n and that $m \geq 2$. The *dumbbell* (a.k.a., the *double broom*), denoted $D(n, r, q)$, is obtained from the path on the vertices p_1,...,p_n by appending r leaves to p_1 and q leaves to p_n. The leaves adjacent to p_1 will be denoted ℓ_1,...,ℓ_r. The leaves adjacent to p_n will be denoted x_1,...,x_q. Without loss of generality, we will assume that $r \geq q \geq 2$ and that $n \geq 3$.

Theorem 4.15 *Let k be a positive integer. The 1-rubbling number for brooms* *is as follows:*

(i) $\rho_1^*(B(2k+1,2)) = k+2$*.*

(ii) $\rho_1^*(B(2k+1,m)) = k+3$*, for* $m \geq 3$*.*

(iii) $\rho_1^*(B(2k,m)) = k+2$*.*

Proof. *(i)* In order to construct a rubbling configuration of $B(2k+1,2)$, place one pebble on each of ℓ_1 and ℓ_2, and p_{2i+1}, where $i = 1,...,k$. A rubbling move $r(\ell_1, \ell_2 \to p_{2k+1})$ places a pebble on p_{2k+1} and every other vertex is reachable by the proof of Theorem 4.11. Hence, $\rho_1^*(B(2k+1,2)) \leq k+2$. Observe that $diam(B(2k+1,2)) = 2k+1$. Thus, by Theorem 4.1, $\rho_1^*(B(2k+1,2)) \geq \lceil \frac{2k+3}{2} \rceil = k+2$. It follows that $\rho_1^*(B(2k+1,2)) = k+2$.

(ii) Consider $B(2k+1,m)$, where $m \geq 3$. Place one pebble on each of ℓ_1, ℓ_2, p_1, p_{2k+1}, and p_{2i+2}, where $i = 1,...,k-1$. From Theorem 4.11 and Theorem 4.8 this is a rubbling configuration. Thus, $\rho_1^*(B(2k+1,m)) \leq k+3$.

To see that at least $k+3$ pebbles are necessary, first consider $N[p_{2k+1}]$. By Theorem 4.8, the most efficient way to reach the leaves $\ell_1, \ell_2, ..., \ell_m$ is to place two pebbles on p_{2k+1}. However, by Lemma 4.14, p_{2k+1} must be the first vertex on the path $p_1, p_2, ..., p_{2k+1}$ to receive two pebbles. Thus, three pebbles must be placed in $N[p_{2k+1}]$ in the form of Theorem 4.8. Now assume to the contrary that the vertices $p_1, p_2, ..., p_{2k-2}$ are reached in such a way that uses less than k pebbles. Then at least two open vertices, say p_i and

p_{i-1} for $1 \leq i \leq 2k-2$, are each adjacent to at most one vertex with a pebble. Clearly p_{i-1} cannot receive a pebble before p_i. Thus, the only way left to reach p_i is through a pebbling move $p(p_{i+1} \to p_i)$. But this requires p_{i+1} to receive two pebbles. By placing one pebble on each of p_{i-1} and p_{i+1}, we have necessarily used less pebbles and reach both p_{i-1} and p_i. Since p_i and p_{i-1} were arbitrary vertices, we have placed pebbles in the exact form of the proof of Theorem 4.11. This contradicts our initial assumption that less than k pebbles were used for the p_1-p_{2k-2} subpath. We now have that at least $k+3$ pebbles are required, and so, $\rho_1^*(B(2k+1, m)) = k+3$.

(iii) Consider $B(2k, m)$. Place one pebble on each of ℓ_1, ℓ_2, p_{2k}, and p_{2i-1}, where $i = 1, ..., k-1$. Then this is a rubbling configuration by Theorem 4.8 and Theorem 4.11.

To see that this configuration is optimal, first consider the vertices p_1, $..., p_{2k-3}$. Using an argument analogous to that of part (ii) of this proof, we observe that at least $\left\lfloor \frac{2k-3}{2} \right\rfloor + 1 = k-1$ pebbles are required to reach these vertices. Thus, we wish to show that at least three more pebbles are necessary. If at most two pebbles are used for the rest of the graph, then we claim that they must be placed on the leaves $\ell_1, ..., \ell_m$. If we are only allowed two pebbles, then there is no way for v_{2k} to receive two pebbles. This is true since Lemma 4.14 guarantees v_{2k} must be the first vertex to receive two pebbles, requiring at least three pebbles in $N[v_{2k}]$. If $m \geq 3$, then the claim holds. If $m = 2$, then v_{2k-1} is still unreachable, and so the claim holds. Thus, at least three more pebbles are required, and it follows that $k + 2$ pebbles is optimal. ■

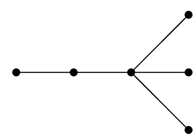

Figure 8: The broom, $B(3, 3)$

Theorem 4.16 *Let k be a positive integer. The 1-rubbling number for dumbbells is as follows:*

(i) $\rho_1^*(D(2k + 1, 2, 2)) = k + 3$.

(ii) $\rho_1^*(D(2k + 1, p, q)) = k + 4$*, for* $p \geq 3$.

(iii) $\rho_1^*(D(2k, p, 2)) = k + 3$.

(iv) $\rho_1^*(D(2k, p, q)) = k + 4$*, for* $q \geq 3$.

Proof. *(i)* Consider $D(2k+1, 2, 2)$. To create a rubbling configuration, place one pebble on each of ℓ_1, ℓ_2, x_1, x_2, and p_{2i+1} for $i = 1, ..., k - 1$. Note that this uses $k + 3$ pebbles, and so, $\rho_1^*(D(2k + 1, 2, 2)) \leq k + 3$.

To see that this configuration is optimal, first consider the leaves ℓ_1, ℓ_2, x_1, and x_2. In any configuration, they must begin with a pebble, or be reached through a rubbling move from p_1 or p_{2k+1}. Either way this requires at least two pebbles in $N[p_1]$ and $N[p_{2k+1}]$. Now consider the vertices $p_3, ..., p_{2k-1}$. If less than $k-1$ pebbles are used to reach these vertices, then some open pair of vertices, say p_i and p_{i+1} for $3 \leq i \leq 2k-2$, must be each adjacent to at most one vertex with a pebble. It follows that at least one of these vertices, say p_i, must receive a pebble through the pebbling move $p(p_{i-1} \to p_i)$. But this requires at least two pebbles on p_{i-1}. Observe that both p_i and p_{i+1} are reachable simply by placing one pebble on each of p_{i-1} and p_{i+1}, and this necessarily uses less pebbles. But this just places pebbles in the form of Theorem 4.11, which contradicts our assumption that less than $k-1$ pebbles were used. Hence, at least $k+3$ pebbles are required, and it follows that $k+3$ pebbles is optimal.

(ii) Consider $D(3, r, q)$, where $r \geq 3$. Place one pebble on each of ℓ_1, ℓ_2, p_1, p_3, and x_1 to form a rubbling configuration. To see that this is optimal, first consider the two sets of leaves. Either $\ell_1, \ell_2, ..., \ell_r$ must each begin with a pebble, or p_1 must receive two pebbles at some point. Similarly, either $x_1, ..., x_q$ must each begin with a pebble, or p_3 must receive two pebbles at some point. Since there are at least five total leaves, we must consider a scenario where either p_1 or p_3 receives two pebbles. Note that it takes three pebbles placed initially in $N[p_1]$ to move two pebbles to p_1, and since $r \geq 3$, this is optimal. Knowing this, we cannot place two pebbles on p_3 using only

one more pebble. Thus, at least two more are required. It follows that five pebbles is optimal.

Consider $D(2k+1, r, q)$, where $k \geq 2$ and $r \geq 3$. Place one pebble on each of ℓ_1, ℓ_2, x_1, x_2, p_1, p_{2k+1} and p_{2i} for $i = 2, ..., k-1$ to form a rubbling configuration using $k+4$ pebbles. Thus, $\rho_1^*(D(2k+1, r, q)) \leq k+4$.

To see that this is optimal, first consider $N[p_1]$. By Theorem 4.8 and Lemma 4.14 we see that at least three pebbles are required in $N[p_1]$. Next, consider the vertices $p_4, ..., p_{2k-2}$. Using an argument completely analogous to part (i) of this proof, we observe that at least $k-2$ pebbles are required for these vertices. Now if $q \geq 3$, then it follows from Theorem 4.8 and Lemma 4.14 that at least three pebbles are required for the remaining vertices. Hence, assume $q = 2$. If only two pebbles are used, then they must be placed on x_1 and x_2, leaving p_{2k-1} unreachable. Thus, three pebbles are still required. It follows that at least $k+4$ pebbles are needed, and so, $\rho_1^*(D(2k+1, r, q)) = k+4$.

(iii) Consider $D(2k, r, 2)$. Place one pebble on each of ℓ_1, ℓ_2, x_1, x_2, p_1, and p_{2i+2} for $i = 1, ..., k-2$ to obtain a rubbling configuration using $k+3$ pebbles. Thus, $\rho_1^*(D(2k, r, 2)) \leq k+3$.

To see that this is optimal, we first note that at least three pebbles are needed for $N[p_1]$ by Theorem 4.8 and Lemma 4.14. Using an argument analogous to parts (i) and (ii) of this proof, we observe that at least $k-2$

pebbles are required for the vertices $p_4, ..., p_{2k-2}$. At this point we have used $k+1$ pebbles. Since neither x_1, nor x_2 have been reached, it follows that we need at least two more pebbles. Thus, at least $k+3$ pebbles are required, and so, $\rho_1^*(D(2k, r, 2)) = k+3$.

(iv) Consider $D(2k, r, q)$, where $q \geq 3$. Place one pebble on each of ℓ_1, ℓ_2, x_1, x_2, p_1, and p_{2i+2} for $i = 1, ..., k-2$ to create a rubbling configuration using $k+4$ pebbles. Thus, $\rho_1^*(D(2k, r, q)) \leq k+4$.

To see this is optimal, note that Theorem 4.8 and Lemma 4.14 guarantee that we must place three pebbles in each of $N[p_1]$ and $N[p_{2k}]$. Now consider the vertices $p_4, ..., p_{2k-2}$. Using an argument entirely analogous with the previous parts of this proof, we observe that at least $k-2$ pebbles are required for these vertices. Hence, $\rho_1^*(D(2k, r, q)) \geq k+4$. It follows that $\rho_1^*(D(2k, r, q)) = k+4$. ■

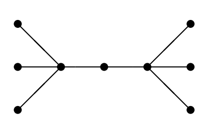

Figure 9: The dumbbell, $D(3, 3, 3)$

Recall that the diameter of a graph is the length of the longest subpath path in $V(G)$. From our previous results on paths, brooms, and dumbbells, the 1-restricted optimal rubbling number was found for different families of trees of varying diameter. However, these families only encompass a small portion of all trees. Our next result will determine ρ_1^* for all trees of diameter four.

Any tree of diameter four can be obtained by appending leaves to the existing vertices of $K_{1,s}$, where $s \geq 2$. Label the center of the star as x and the vertices adjacent to x as $y_1, ..., y_s$. Suppose that we append c leaves to x, namely $x_1, ..., x_c$ and a_i leaves to y_i, where $a_i \geq 1$. These leaves are denoted $y_{i,1}, ..., y_{i,a_i}$ for $i = 1, ..., s$. Note that for $i \neq j$ and for any ℓ and m, the vertices $y_{i,\ell}$, y_i, x, y_j, and $y_{j,m}$ induce a path of length four. Thus, this construction gives all trees of diameter four. The resulting graph will be denoted $K_{1,s}(c; a_1, ..., a_s)$. Without loss of generality, we will assume that $a_1 \geq \cdots \geq a_s \geq 1$.

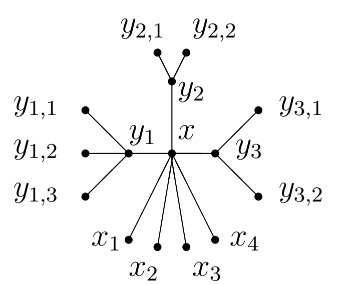

Figure 10: The graph $K_{1,3}(4; 3, 2, 2)$

Theorem 4.17 *For $s \geq 2$, let $K_{1,s}(c; a_1, ..., a_n)$ be a tree of diameter four. The 1-rubbling number on trees of diameter of four is as follows:*

(i) $\rho_1^*(K_{1,s}(c; a_1, ..., a_s)) = 3$ if and only if $s = 2$, $c = 0$, and $a_1 = a_2 = 1$.

(ii) $\rho_1^*(K_{1,s}(c;a_1,...,a_s)) = 4$ *if and only if it is one of the following:* $K_{1,2}(c;a_1,a_2)$ *(where* $a_1 \geq a_2 \geq 2$ *and* $c \geq 1$*),* $K_{1,2}(0;a_1,1)$ *(where* $a_1 \geq 2$*),* $K_{1,2}(0;2,2)$*, or* $K_{1,3}(c;a_1,a_2,a_3)$.

(iii) $\rho_1^*(K_{1,s}(c;a_1,...,a_s)) = 5$ *if and only if* $s = 4, 5$ *or* $s = 2$*,* $c = 0$*,* $a_1 \geq 3$*, and* $a_2 \geq 2$.

(iv) $\rho_1^*(K_{1,s}(c;a_1,...,a_s)) = 6$ *if and only if* $s = 6$.

(v) $\rho_1^*(K_{1,s}(c;a_1,...,a_s)) = 7$ *if and only if* $s \geq 7$.

Proof.

(i) Note that $K_{1,2}(0;1,1) \cong P_5$. Hence $\rho_1^*(P_5) = 3$ by Theorem 4.11.

(ii) Consider $K_{1,2}(c;a_1,a_2)$, where $a_1 \geq a_2 \geq 2$ and $c \geq 1$. A rubbling configuration is obtained by placing one pebble on each of y_1, y_2, x, and x_1. This is minimum as we require at least two pebbles to reach $y_{1,1}$,...,y_{1,a_1} and at least two pebbles to reach $y_{2,1}$,...,y_{2,a_2}.

Consider $K_{1,2}(0;a_1,1)$, where $a_1 \geq 2$. A rubbling configuration is obtained by placing one pebble on each of $y_{1,1}$, $y_{1,2}$, y_1, and $y_{2,1}$. Note that we require at least two pebbles to reach $y_{1,1}$,...,y_{1,a_1}. We require at least two additional pebbles to reach x and $y_{2,1}$.

Consider $K_{1,2}(0;2,2)$. A rubbling configuration is obtained by placing one pebble on each of $y_{1,1}$, $y_{1,2}$, $y_{2,1}$, and $y_{2,2}$. This is minimum as we require at least two pebbles to reach $y_{1,1}$ and $y_{1,2}$ and at least two pebbles to reach $y_{2,1}$ and $y_{2,2}$.

Let $s = 3$. A rubbling configuration is obtained by placing one pebble on each of y_1, y_2, y_3, and x. Note that it takes at least one pebble to reach each of $y_{1,1}$, $y_{2,1}$, and $y_{3,1}$. We require an additional pebble to reach either $y_{1,2}$ or x.

(iii) Let $s = 2$, $c = 0$, $a_1 \geq 3$, and $a_2 \geq 2$. A rubbling configuration is obtained by placing one pebble on each of $y_{1,1}$, $y_{1,2}$, y_1, y_2, and x. Note that we require at least three pebbles to reach $y_{1,1}$,...,y_{1,a_1} and at least two pebbles to reach $y_{2,1}$,...,y_{2,a_2}. Hence this is minimum.

Let $s = 4$. A rubbling configuration is obtained by placing one pebble on each of y_1,...,y_4, and x. To see that this is minimum, to have a pebble on $y_{i,1}$, we may either have a pebble there in the initial configuration (this requires at least four pebbles), we may have four pebbles on x (this requires at least six pebbles), or we may have two pebbles on y_i. In this last case, we may have two pebbles on y_i initially (which would require eight pebbles) or move two to y_i. To move two on to y_i, we need to have one on y_i and two on x. To place two on x, we may either have two there initially, or create two by having one on each of x, y_j, and y_k, where $i \neq j$, $i \neq k$, and $j \neq k$ (recall that this requires one pebble to remain on y_i). Again, this requires at least four pebbles. However, we need at least one additional pebble for the final branch of the tree.

Let $s = 5$. A rubbling configuration is obtained by placing one pebble on each of y_1,...,y_5. The rest of the argument is analogous to the case where $s = 4$.

(iv) Let $s = 6$. A rubbling configuration is obtained by placing one pebble on each of y_1,...,y_6. The rest of the argument is analogous to the case where $s = 4$.

(v) Let $s \geq 7$. A rubbling configuration is obtained by placing one pebble on each of y_1,...,y_6 and x. To see that this is minimum, note that we require at least seven pebbles to reach $y_{1,1}$,...,$y_{7,1}$. Thus, the minimum is achieved by moving four pebbles to x and then moving to the leaves using pebbling moves. This requires one pebble on x and at least six pebbles on its neighbors. ■

Theorem 4.17 has some interesting implications. Note that $K_{1,2}(0; a_1, a_2)$ is a proper subgraph of $K_{1,2}(c; a_1, a_2)$, where $c \geq 1$. However, if $a_1 \geq 3$ and $a_2 \geq 2$, then $\rho_1^*(K_{1,2}(0; a_1, a_2)) = 5 > 4 = \rho_1^*(K_{1,2}(c; a_1, a_2))$.

4.3 Upper and Lower Bounds

Our final section on ρ_1^* will consist of upper and lower bounds. A bound is considered *sharp* if there exists a graph for which equality holds. Thus, the bound may not be constrained further without excluding the aforementioned graph. First we will compare the 1-restricted optimal rubbling number to the 2-domination number.

Theorem 4.18 *For every graph G, $\rho_1^*(G) \leq \gamma_2(G)$.*

Proof. Let S be a γ_2-set of G and place a pebble on each vertex in S. Then each vertex in $V(G) - S$ must be adjacent to two pebbles, resulting in a rubbling configuration. ■

Like many bounds we will encounter, the bound in Theorem 4.18 can be sharp. The 2-domination number of a P_3 must be two since P_3 contains two leaves. By Theorem 4.11, $\rho_1^*(P_3) = 2$ as well. We will now observe that the difference between $\rho_1^*(G)$ and $\gamma_2(G)$ can also be made arbitrarily large.

Observation 4.19 *The difference $\rho_1^*(G) - \gamma_2(G)$ can be made arbitrarily large.*

Proof. Let G be star. Note that each leaf must be in the γ_2-set, so $\gamma_2(G) = n - 1$. By Theorem 4.8, $\rho_1^*(G) = 3$, regardless of the order of G. So the difference $\rho_1^*(G) - \gamma_2(G)$ can be made arbitrarily large simply by appending more leaves to the center. ■

While γ_2 provides us with a very straightforward bound, the relationship between the domination number γ and ρ_1^* is not as trivial. The following two results will deal with the domination number.

Theorem 4.20 *For every graph G, $\rho_1^*(G) \leq 3\gamma(G)$.*

Proof. Let S be a γ-set of G. Place a pebble on each vertex in S. Let $v \in S$. If $|epn(v, S)| = 1$, then place a pebble on the external private neighbor of v. If $|epn(v, S)| \geq 2$, then choose exactly two external private neighbors of v and place a pebble on each. Clearly, this has at most three pebbles for each $v \in S$, implying that at most $3\gamma(G)$ pebbles are used. Furthermore this is a rubbling configuration. To see this, let $x \in V(G)$ be an open vertex. If x has at least two neighbors in S, then a rubbling move from these two neighbors places a pebble on x. Thus, we can assume $x \in epn(v, S)$ for some $v \in S$. By the way our configuration is built, $|epn(v, S)| \geq 3$ and two vertices, say w and y in $epn(v, S) - \{x\}$ have one pebble each. Then the rubbling move $r(w, y \to v)$ results in two pebbles on v, and a pebbling move $p(v \to x)$ places a pebble on x, and so, $\rho_1^*(G) \leq 3\gamma(G)$. ■

Interestingly, while stars provide a case where $\rho_1^*(G)$ is smaller than $\gamma_2(G)$, $\rho_1^*(G) = 3\gamma(G)$ for stars. While we do not provide a characterization of all graphs for which the bound in Theorem 4.20 is sharp, the following theorem presents a necessary condition for equality.

Theorem 4.21 *Let G be a graph and S a γ-set of G. A necessary condition for $\rho_1^*(G) = 3\gamma(G)$ is that for all $x \in S$, $deg(x) \geq 3$ and for all $u, v \in S$, $N[u] \cap N[v] = \emptyset$.*

Proof. Assume that $\rho_1^*(G) = 3\gamma(G)$. The proof of Theorem 4.20 implies that $|epn(u, S)| \geq 2$ for all $u \in S$, and so, $deg(u) \geq 3$ for all $u \in S$. Suppose, to the contrary, that S is a γ-set of G and that there exist $u, v \in S$ such that $N[u] \cap N[v] \neq \emptyset$. For equality to hold in Theorem 4.20, we need to have three distinct pebbles in $N[u] - N[v]$ and $N[v] - N[u]$.

First suppose that $\{u, v\} \subseteq N[u] \cap N[v]$. In other words, $uv \in E(G)$. To see that fewer than $3\gamma(G)$ pebbles are needed in a rubbling configuration of G, we form a ρ_1^*-configuration as follows. Place a pebble on each vertex in S (note that this places one pebble on each of u and v), and following the method used in the proof of Theorem 4.20, place pebbles on the neighbors of the vertices in $S - \{u, v\}$. To complete the configuration, place a pebble on each of two neighbors of u, say u_1 and u_2. Clearly, each vertex in $V(G) - N(u)$ is reachable. Now, the rubbling move $r(u_1, u_2 \to u)$ places a second pebble on u, and a subsequent pebbling move $p(u \to v)$ places a second pebble in v. It follows that each vertex in $N(v)$ is reachable and that the initial configuration must be a ρ_1^*-configuration. Since a total of at most $3(|S| - 2) + 4 = 3\gamma(G) - 2$ pebbles were used, it follows that $\rho_1^*(G) < 3\gamma(G)$.

Next suppose that $w \in N[u] \cap N[v]$, that is, $uw, vw \in E(G)$. If we place a pebble on w, then this pebble is adjacent to both u and v. Thus, we would need at most two additional pebbles, one on each of u and v, and one on each of $N(v) - \{w\}$ and $N(u) - \{w\}$ to reach the neighbors of u and v. Place pebbles on the rest of the graph in the configuration described in Theorem 4.20 to obtain a rubbling configuration using at most $3\gamma(G) - 1$

pebbles. Hence, there is a contradiction. ■

Note that the condition given in Theorem 4.21 is not sufficient. As an example, consider a tree T with center c. The tree T has n "branches" off of the center c, where $n \geq 30$. The ith branch is a path on the vertices c, x_i, y_i, z_i, with three leaves $z_{i,1}$, $z_{i,2}$, and $z_{i,3}$ appended to z_i (see Figure 11). The γ-set of T is $S = \{c, z_i : i = 1, ..., n\}$. Note that for all $u, v \in S$, $deg(u) \geq 3$ and $N[u] \cap N[v] = \emptyset$ and that $\gamma(G) = n + 1 \geq 31$. Consider the following set $C = \{c, x_i : i = 1, ..., 30\}$. We claim that C is a 1-rubbling configuration of T. To see this, use rubbling moves to move all pebbles from the x_i onto c. This results in sixteen pebbles on c. Suppose that our destination is $z_{i,j}$. Use pebbling moves to move all pebbles to x_i. This results in eight pebbles on x_i. Now, use pebbling moves to move all pebbles to y_i. This results in four pebbles on y_i. Again, we move all pebbles to z_i. This results in two pebbles on z_i. Finally, use a pebbling move to move a pebble to $z_{i,j}$. Hence, $\rho_1^*(T) \leq 31 \leq \gamma(T)$.

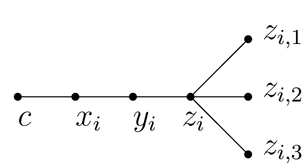

Figure 11: The i^{th} branch of the tree T

The following theorem relates the maximum degree of G and $\rho_1^*(G)$.

Theorem 4.22 *Let G be any connected graph of order $n \geq 3$. If $\Delta(G) = n - k$, where k is an integer $1 \leq k \leq n-2$, then $\rho_1^*(G) \leq k + 2$.*

Proof. Let $v \in V(G)$ such that $deg(v) = n-k$. Then $|V(G) - N[v]| = k-1$. First place a pebble on each vertex in $V(G) - N[v]$. Now place one pebble on v and one pebble on any two vertices in $N(v)$. Then a rubbling move will place two pebbles on v allowing a pebbling move to place a pebble on any vertex in $N(v)$. Since we have used $k+2$ pebbles, $\rho_1^*(G) \leq k+2$. ■

Similar to Theorem 4.20, we see that sharpness in Theorem 4.22 can be found in stars.

We have stated that Graham's pebbling conjecture has not yet been proved in general. Interestingly, the same bound does hold in general for ρ_1^*, as is shown in the following theorem.

Recall that we denote the Cartesian product of two graphs G and H as $G \square H$. Let (g, h) be the vertex of $G \square H$ corresponding to $g \in V(G)$ and $h \in V(H)$. Let G_h denote the copy of G in $G \square H$ induced by the set of vertices $\{(g, h) : g \in V(G)\}$. We denote the graph of G induced by some specific vertex $h_1 \in H$ as G_{h_1} Analogously, H_g denotes the copy of H in $G \square H$ induced by the set of vertices $\{(g, h) : h \in V(H)\}$. Additionally, we denote the graph of H induced by some specific vertex $g_1 \in G$ as H_{g_1}.

Theorem 4.23 *For graphs G and H, $\rho_1^*(G \square H) \leq \rho_1^*(G)\rho_1^*(H)$.*

Proof. Let $\rho_1^*(G) = k$ and $\rho_1^*(H) = m$. Suppose that placing one pebble on each vertex in the subset $A = \{g_1, ..., g_k\}$ of $V(G)$ results in a 1-restricted optimal rubbling configuration of G. Suppose that placing one pebble on each vertex in the subset $B = \{h_1, ..., h_m\}$ of $V(H)$ is a 1-restricted optimal rubbling configuration of H.

Consider the pebble distribution on $G\Box H$ obtained by placing one pebble on each vertex in the set,

$$\{(g_i, h_j) : i = 1, ..., k, j = 1, ..., m\}.$$

Clearly, G_{h_j} is reached for all j as $\{(g_i, h_j) : i = 1, ..., k\}$ is a rubbling configuration on G_{h_j}. Similarly, H_{g_i} is reached for all i. Suppose that $(g, h) \in V(G\Box H)$ such that $g \notin \{g_1, ..., g_k\}$ and $h \notin \{h_1, ..., h_m\}$. Since $B = \{h_1, ..., h_m\}$ is a rubbling configuration on H, we can perform rubbling moves on H_{g_i} to place a pebble on (g_i, h) for each $i = 1, ..., k$. Since $A = \{g_1, ..., g_k\}$ is a rubbling configuration on G, we can perform rubbling moves on G_h to place a pebble on (g, h).

Hence, $\{(g_i, h_j) : i = 1, ..., k, j = 1, ..., m\}$ is a rubbling configuration. This requires $km = \rho_1^*(G)\rho_1^*(H)$ pebbles, and so it follows that $\rho_1^*(G\Box H) \leq \rho_1^*(G)\rho_1^*(H)$. ■

From Observation 4.13 this bound is observed to be sharp in the graph

$C_4 \square P_2$. It is also possible to make the difference, $\rho_1^*(G)\rho_1^*(H) - \rho_1^*(G \square H)$ arbitrarily large. Recall that in Theorem 4.13 we found that if $n \geq 5$, then $\rho_1^*(C_n \square P_2) = \lceil \frac{2n}{3} \rceil$. For large values of n, the difference $2(\frac{n}{2}) - \frac{2n}{3}$ gets arbitrarily large. Our next result deals with a more specific case of Cartesian products.

Theorem 4.24 *Let G be a graph with 1-restricted optimal rubbling number $\rho_1^*(G)$ and domination number $\gamma(G)$. Then $\rho_1^*(G \square P_2) \leq \gamma(G) + \rho_1^*(G)$.*

Proof. Label the two copies of G created by $G \square K_2$ as G_1 and G_2. Place pebbles on G_1 in the ρ_1^*-configuration of G. Then each vertex in G_1 is reachable. By definition, each vertex in G_2 is adjacent to exactly one vertex in G_1. Hence, using only moves on the vertices on G_1, we can place one pebble in the neighborhood of any vertex in G_2. Now place one pebble on each vertex in some minimal dominating set of G_2. It follows that it is possible to place two pebbles in the neighborhood of any open vertex in G_2. This gives us a rubbling configuration using $\gamma(G) + \rho_1^*(G)$ pebbles, and so, $\rho_1^*(G \square P_2) \leq \gamma(G) + \rho_1^*(G)$. ■

The remaining theorems in this section will deal with upper bounds on trees. The method used in this next proof bears resemblance to a concept Belford and Sieben define as rolling [1]. Although we will present the formal definition (as we do not use it in its exact form), rolling involves moving a pebble along a path of vertices each with at least one pebble.

Theorem 4.25 *If T is a tree with order n and $\ell \geq 3$ leaves, then $\rho_1^*(T) \leq n - \ell + 2$. Further, this bound is sharp.*

Proof. Let T be a tree of order n with $\ell \geq 3$ leaves. Begin by placing a pebble on each internal vertex. If T is a star, then

$$\rho_1^*(T) = 3 = n - (n-1) + 2 = n - \ell + 2.$$

Thus, this bound is sharp. We may assume that T is not a star, that is, T has at least two internal vertices. We consider two cases:

Case 1. T has a support vertex v adjacent to two internal vertices.

Let x and y be two internal vertex neighbors of v, and let w be a support vertex of T different from v. Complete the initial configuration by placing a pebble on a leaf neighbor, say u, of v. Since there is a unique path via internal vertices from v to any other support vertex, there exists a v-w path that does not include at least one of x and y, say x. Hence, a rubbling move $r(u, x \to v)$ places two pebbles on v. Then a sequence of pebbling moves on the v-w path will result in two pebbles on w. Hence, the leaves adjacent to w are reachable. Since w is an arbitrary support vertex, it follows that this is a 1-rubbling configuration of T. Hence, $\rho_1^*(T) \leq n - \ell + 1 < n - \ell + 2$.

Case 2. Every support vertex is adjacent to exactly one internal vertex. First assume that T has a strong support vertex. In this case, complete the initial configuration by placing one pebble on each of two leaf neighbors of v. Then a rubbling move between these two leaves results in two pebbles on v. Repeating the argument in case one shows that this is a 1-rubbling configuration. Hence, $\rho_1^*(T) \leq n - \ell + 2$.

Henceforth, we may assume that every support vertex of T is adjacent to exactly one leaf and exactly one internal vertex, that is, every support vertex of T has degree two. If T is the path P_n, then by Theorem 4.11, $\rho_1^*(T) \leq n = n - 2 + 2$, and the bound holds. Thus, we may assume that at least one internal vertex, say x, has degree three or more. Note that x is not a support vertex by assumption. Hence, every neighbor of x is an internal vertex of T. Recall that there is a unique path from x to any support vertex, say v. Since x has degree three or more, at least two neighbors of x, say y and z, are not on the x-v path. Moreover, y and z are internal vertices, so they each have one pebble. A rubbling move $r(y, z \rightarrow x)$ places two pebbles on x, and a sequence of pebbling moves results in two pebbles on v. Hence, the leaf neighbors of v can be pebbled. Since v is an arbitrary support vertex, it follows that this is a 1-rubbling configuration of T. Hence, $\rho_1^*(T) \leq n - \ell < n - \ell + 2$. ■

The following two results directly follow Theorem 4.25.

Corollary 4.26 *Let T be a tree with order $n \geq 4$ and $\ell \geq 3$ leaves. If T has at least one non-support vertex of degree three, then $\rho_1^*(T) \leq n - \ell$.*

Proof. Place one pebble on each internal vertex of T, and let v be some non-support vertex of degree three or more. Then a rubbling move will place two pebbles on v. Since each leaf must be on a unique path from v containing only one of the neighbors of v, this move can be done separate of that particular neighbor. Thus, each vertex on the graph can be reached in a manner identical to the one described in Theorem 4.25. So, $\rho_1^*(T) \leq n - \ell$.

■

Observation 4.27 *The bound from Theorem 4.25 is only sharp in stars, double stars, in the broom $B(3, m)$, in the dumbbell $D(3, p, q)$ where $p \geq 3$ and $q \geq 2$, and in the dumbbell $D(4, p, q)$ where $p \geq q \geq 3$.*

Proof. As a result of Corollary 4.26, the bound presented in Theorem 4.25 can only be sharp in paths, stars, brooms and dumbbells. From Theorem 4.11, this bound is not sharp for paths. From Theorem 4.8, the bound is sharp on stars. From Theorem 4.9, the bound is sharp for double stars. Furthermore, from Theorems 4.15 and 4.16, observe that the bound is sharp for the broom $B(3, m)$, the dumbbell $D(3, p, q)$ where $p \geq 3$ and $q \geq 2$, and the dumbbell $D(4, p, q)$ where $p \geq q \geq 3$. ■

We have already seen that the diameter is closely related to the 1-restricted optimal rubbling number. The following two theorems will establish an upper bound for trees based on the diameter. We consider even and odd diameter as seperate cases.

Theorem 4.28 *Suppose that T is a tree of diameter $2k$ whose center has degree at least $2^{k+1} - 5$. It follows that $\rho_1^*(T) \leq 2^{k+1} - 1$.*

Proof. Let T be a tree with diameter $2k$. Then the center of T is a unique vertex u. Let $deg(u) = d$ and the neighbors of u be $v_1,...,v_d$.

Suppose that $2^{k+1} - 5 \leq d \leq 2^{k+1} - 2$. For our initial configuration, place one pebble on each of u, $v_1,...,v_d$. Without loss of generality, assume that our destination is a leaf w on the periphery such that the shortest path from u to w passes through v_d. We use rubbling moves to move pebbles from $v_1,...,v_{d-1}$ onto u. Vertex u now has $\frac{d+1}{2}$ pebbles. Using pebbling moves, move all of these pebbles to v_d. We now have $\frac{d+1}{4} + 1$ pebbles on v_d. Since $d \geq 2^{k+1} - 5$, it follows that $\frac{d+1}{4} + 1 \geq 2^{k-1}$. Hence, we can reach any leaf on the branch rooted at v_d. Note that our initial configuration used $1 + d \leq 1 + 2^{k+1} - 2 = 2^{k+1} - 1$ pebbles. Therefore, $\rho_1^*(T) \leq 2^{k+1} - 1$.

Suppose that $d \geq 2^{k+1} - 1$. For our initial configuration, place a pebble on each of u, $v_1,...,v_{2^{k+1}-2}$. We use rubbling moves to move all pebbles to u. Now u has 2^k pebbles. Hence, it is possible to reach any vertex in the graph using a series of pebbling moves. Again, we used $2^{k+1} - 2$ pebbles in our initial configuration, so the bound still holds. ■

Theorem 4.29 *Suppose that T is a tree of diameter $2k + 1$ with center vertices u and v such that $deg(u) \geq 2^{k+1} - 5$ and $deg(v) \geq 2^k - 2$. It follows that $\rho_1^*(T) \leq 2^{k+1} + 2^k - 4$.*

Proof. Suppose that the non-center neighbors of u are u_1,...,u_n. Suppose that the non-center neighbors of v are v_1,...,v_m. If $2^{k+1} - 6 \leq n \leq 2^{k+1} - 2$ and $m \geq 2^k - 2$, then begin by placing a pebble on each of u, v, u_1,...,u_n, v_1,...,v_m. If our destination is a leaf on the periphery rooted at u_n, then move all pebbles from u_1,...,u_{n-1}, and v to u using rubbling moves. We now have $\frac{n+2}{2}$ pebbles on u. Move all pebbles from u to u_n using pebbling moves. There are now $\frac{n+6}{4}$ pebbles on u_n. Since $n \geq 2^{k+1} - 6$, there are at least 2^{k-1} pebbles on u_n. Hence, we can reach any leaf rooted at u_d using pebbling moves. Similarly, if our destination is a leaf on the periphery rooted at v_m, we move all pebbles from u_1,...,u_n to u using rubbling moves. We then move all pebbles from u to v using pebbling moves. We also move all pebbles from v_1,...,v_{m-1} to v using rubbling moves. There are now $\frac{n+2m+4}{4}$ pebbles on v. Move all pebbles from v to v_m using pebbling moves. We now have $\frac{n+2m+12}{8}$ pebbles on v_m. Since $n \geq 2^{k+1} - 6$ and $m \geq 2^{k-1} - 3$, there are at least 2^{k-1} pebbles on v_m. Hence, we can reach any leaf rooted at v_m. Note that we used $2 + n + m$ pebbles in our initial configuration. Since $n \leq 2^{k+1} - 2$ and $m \leq 2^k - 2$, it follows that $\rho_1^*(T) \leq 2^{k+1} + 2^k - 4$.

For the remaining cases, use a similar argument to the above argument as well as the proof of Theorem 4.28. ■

In their work on optimal rubbling, Katona and Sieben proved that for any graph G, $\rho_{opt}(G) \leq \left\lceil \frac{n+1}{2} \right\rceil$ (see [18]). We will do the same for the 1-

restricted optimal rubbling number. Our proof will rely on induction in a very similar manner to the proof of Katona and Sieben, however we must account for the fact that each vertex can begin with at most one pebble.

Theorem 4.30 *For any tree T of order n, $\rho_1^*(T) \leq \lceil \frac{n+1}{2} \rceil$.*

Proof. Let T be a tree of order n. We proceed by induction on n. By Theorem 4.11, the bound holds when $n = 2$ or $n = 3$. Furthermore, by Theorem 4.8 and Theorem 4.9, the result holds for all for stars and double stars. Hence, we may assume that $diam(T) \geq 4$ and $n \geq 5$.

Assume for any tree T' with order $n' < n$, that $\rho_1^*(T') \leq \lceil \frac{n'+1}{2} \rceil$. Root T at some leaf r, and let leaf w be of maximum distance from r. Let v be the unique parent of w, and let u be the unique parent of v. We consider three cases.

Case 1. Vertex v has exactly one child. Let $T' = T - \{v, w\}$. Then $n' = n-2$. From our inductive hypothesis, there is some rubbling configuration, $f(T')$, such that $|f(T')| \leq \lceil \frac{(n-2)+1}{2} \rceil$. In the rubbling configuration of T', u can obtain a pebble. Hence, we construct a rubbling configuration of T using $f(T')$ and placing a pebble on w. A rubbling move $r(u, w \to v)$ places a pebble on v. Then this is a rubbling configuration of T, and $\rho_1^*(T) \leq \left\lceil \frac{(n-2)+1}{2} \right\rceil + 1 = \lceil \frac{n+1}{2} \rceil$.

Case 2. Vertex v has exactly two children. Let x be the other child of v. Then x is a leaf. Define $T' = T - \{w, x\}$. Then from our inductive hypothesis, there exists a rubbling configuration $f(T')$ such that $f(T') \leq \left\lceil \frac{(n-2)+1}{2} \right\rceil$. We are now presented with two subcases.

(a) In $f(T')$, v begins with a pebble. Place pebbles on T in the form of $f(T')$. Remove the pebble from v and place one pebble on each of w and x. A rubbling move $r(w, x \to v)$ places a pebble on v, giving the previous configuration $|f(T')|$. Recall that $T' = T - \{w, x\}$ and each of w and x began with a pebble. Hence, this is a rubbling configuration, and $\rho_1^*(T) \leq \left\lceil \frac{(n-2)+1}{2} + 1 \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil$.

(b) In $f(T')$, v does not begin with a pebble. Thus, there must exist some sequence of moves using only the vertices of T' that places a pebble on v. Place pebbles on T in the form of $f(T')$ and place one additional pebble on v. Then v can receive a second pebble, and a pebbling from v move reaches w and x. Clearly, this is a rubbling configuration of T, and so, $\rho_1^*(T) \leq \left\lceil \frac{(n-2)+1}{2} + 1 \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil$.

Case 3: Vertex v has three or more children. Let $\ell_1, \ell_2, ..., \ell_k$ be the leaves adjacent to v, where $k \geq 3$. Define $T' = T - \{v, \ell_1, \ell_2, ..., \ell_k\}$. By our inductive hypothesis, there exists some rubbling configuration $f(T')$, such that $|f(T')| \leq \left\lceil \frac{(n-k-1)+1}{2} \right\rceil$. We consider two subcases

(a) Vertex u begins with a pebble in $f(T')$. Note that $n' \leq n-4$. Place pebbles on T in the form of $f(T')$ and remove the pebble from u. Place one pebble on each of v, ℓ_1, and ℓ_2. Then a rubbling move $r(\ell_1, \ell_2 \to v)$

places a second pebble on v. It follows that every other leaf neighbor of v is reachable. Furthermore, the pebbling move $p(v \to u)$ places one pebble on u, leaving us with $f(T')$. Then we have a rubbling configuration, and $\rho_1^*(T) \leq \left\lceil \frac{(n-k)+1}{2} + 2 \right\rceil \leq \lceil \frac{n+1}{2} \rceil$.

(b) Vertex u does not begin with a pebble in $f(T')$. Place pebbles on T in the form of $f(T')$ and place one pebble on each of u and v. In a manner identical to subcase 2(b), u can now obtain two pebbles. Then the pebbling move $p(u \to v)$ places a second pebble on v. Finally, a pebbling move from v will reach each of the leaves of v. Once again, $\rho_1^*(T) \leq \left\lceil \frac{(n-\ell)+1}{2} + 2 \right\rceil \leq \left\lceil \frac{n+1}{2} \right\rceil$. We have now shown inductively that $\rho_1^*(T) \leq \left\lceil \frac{n+1}{2} \right\rceil$ for every tree T. ■

Since each connected graph must contain a spanning tree, the bound of Theorem 4.30 can be generalized to all graphs. In fact, the bound can be sharp for non-trees. Define graph G as the graph in Figure 12. Then by Theorem 4.5, $\rho_1^*(G) < 2$. But $\lceil \frac{4+1}{2} \rceil = 3$, and so, the bound is sharp. The following corollary uses the result of Theorem 4.30. Here we will consider disconnected graphs.

Corollary 4.31 *Let G be a graph of order n with complement $\overline{G}$. Then $\rho_1^*(G) + \rho_1^*(\overline{G}) \leq n + 2$. Further, this bound is sharp.*

Proof. If both G and $\overline{G}$ are connected, then from Theorem 4.30,

$$2\left\{\left\lceil \frac{n+1}{2} \right\rceil\right\} \leq n + 2.$$

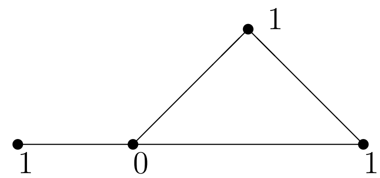

Figure 12: Sharpness in Theorem 4.30

Assume, without loss of generality, that $\overline{G}$ is not connected. This presents two cases.

Case 1: Each component of $\overline{G}$ has at most two vertices. In this case, $\rho_1^*(\overline{G}) = n$. If each component of $\overline{G}$ is a single vertex, then G is a complete graph and $\rho_1^*(G) = 2$. If there exists at least one component in $\overline{G}$ that consists of two adjacent vertices, say a and b, then in G, $N(a) \cap N(b)$ contains every vertex in $G - \{a, b\}$. Hence, by Theorem 4.5, $\rho_1^*(G) \leq 2$.

Case 2: Some component of $\overline{G}$ has at least three vertices. We will begin by constructing a rubbling configuration on G. Let x and y be vertices of some component $X \in \overline{G}$ such that X contains at least three vertices. Let z be a vertex not in X. Then in G, x and y are adjacent to every vertex not in X, and z is adjacent to each of x and y. Place a pebble on each of x, y, and z. Then in G, the rubbling move $r(x, y \to v)$ will reach each $v \notin X$. Thus, a rubbling move $r(x, y \to z)$ will place two pebbles on z. A subsequent pebbling move from z will reach each vertex in X. Hence, each vertex in G is reachable, and so $\rho_1^*(G) \leq 3$. Now, by Theorem 4.30, $\rho_1^*(\overline{G}) \leq \left\lceil \frac{n(X)+1}{2} \right\rceil + n - n(X)$. Thus, $\rho_1^*(\overline{G}) + \rho_1^*(G) \leq 3 + \left\lceil \frac{n(X)+1}{2} \right\rceil + n - n(X) \leq n + 2$. ■

We see from Theorem 4.7 that this bound can be sharp. If $G = K_n$, then $\rho_1^*(\overline{G}) = n$ since $\overline{G}$ must be an empty graph. Our final result will show that for each integer value $2 \leq k \leq \lceil \frac{n+1}{2} \rceil$, there exists a graph of order n whose 1-restricted optimal rubbling number equals k. To do so, we define the *double pencil* $P_{(k,x)}$ on n vertices as a path on $k = n - 2$ vertices labelled $p_1, p_2, ..., p_{n-2}$, and two more vertices u and v each adjacent to $p_1, p_2, ..., p_x$ for $1 \leq x \leq k$. We will also say that $uv \notin E(G)$, so that $deg(u) = deg(v) = x$.

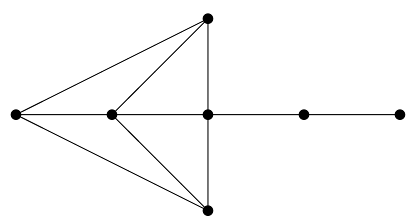

Figure 13: The graph $P_{5,3}$

Theorem 4.32 *An ordered pair (n, b) is realizable as the order and the 1-restricted optimal rubbling number of a connected graph G if and only if $n \geq 3$ and $2 \leq b \leq \lceil \frac{n+1}{2} \rceil$*

Proof. Consider the double pencil $P_{k,x}$ as defined above. If $n = 3$, then $\lceil \frac{n+1}{2} \rceil = 2$. By Theorem 4.5, every connected graph on three vertices has a

1-restricted optimal rubbling number equal to two. Since the result holds if $n = 3$, we may assume that $n \geq 4$.

For each value of x, we can create a rubbling configuration by placing one pebble on each of u and v and placing pebbles on the vertices $p_{x+2}, ..., p_k$ in the form of Theorem 4.11. A rubbling move will reach each vertex in $N(u) \cap N(v)$, and every vertex on the $p_{x+2} - p_k$ path is reachable by Theorem 4.11. Finally, p_{x+1} is reachable by the rubbling move $r(p_x, p_{x+2} \to p_{x+1})$. Thus,

$$\rho_1^*(P_{k,x}) \leq 2 + \left\lfloor \frac{k-(x+1)}{2} + 1 \right\rfloor.$$

To see that this configuration is optimal, first note that $diam(P_{k,x}) = k - x + 2$. Thus, by Theorem 4.1 we have that $\left\lceil \frac{(k-x+2)+2}{2} \right\rceil \leq \rho_1^*(P_{k,x})$. Observe that $\left\lceil \frac{(k-x+2)+2}{2} \right\rceil = \left\lceil \frac{k-x}{2} \right\rceil + 2 = \left\lfloor \frac{k-x+1}{2} \right\rfloor + 2 = \left\lfloor \frac{k-x-1}{2} + 1 \right\rfloor + 2$. Hence, $\left\lfloor \frac{k-x-1}{2} + 1 \right\rfloor + 2 \leq \rho_1^*(P_{k,x})$. It follows that $\left\lfloor \frac{k-x-1}{2} + 1 \right\rfloor + 2 = \rho_1^*(P_{k,x})$. The value of $\left\lfloor \frac{k-x-1}{2} + 1 \right\rfloor + 2$ can be made to equal any integer value on the interval $[2, \left\lceil \frac{n+1}{2} \right\rceil]$ simply by increasing or decreasing the value of x. Thus, the result holds. ■

Our focus will now shift to the 2-restricted optimal rubbling number and its relationship with the 1-restricted optimal rubbling number.

5 2-RESTRICTED OPTIMAL RUBBLING

Our remaining results study the effects of loosening the initial restriction from one to two pebbles. Since every 2-restricted optimal rubbling configuration must be a 1-restricted optimal rubbling configuration by definition, we observe the somewhat trivial, yet useful upper bound.

Observation 5.1 *Let G be a graph. Then $\rho_2^*(G) \leq \rho_1^*(G)$.*

While it is expected that this bound can be strict, it is less obvious that it can be sharp. Recall that Lemma 3.1 guarantees that $\rho_{opt}(G) \leq \rho_2^*(G)$ for all graphs G. Using this fact, and Observation 5.1, we now list results for which $\rho_1^*(G) = \rho_2^*(G)$.

Theorem 5.2 *For any path P_n on $n \geq 2$ vertices, $\rho_2^*(P_n) = \left\lfloor \frac{n}{2} + 1 \right\rfloor$.*

Proof. From Lemma 3.1, $\rho_2^*(P_n) \geq \left\lfloor \frac{n}{2} + 1 \right\rfloor$. We see from Observation 5.1 and Theorem 4.11 that $\rho_2^*(P_n) \leq \left\lfloor \frac{n}{2} + 1 \right\rfloor$. Thus, $\rho_2^*(P_n) = \left\lfloor \frac{n}{2} + 1 \right\rfloor$. ■

Theorem 5.3 *For any cycle C_n of order $n \geq 3$, $\rho_2^*(C_n) = \lceil \frac{n}{2} \rceil$.*

Proof. This result similarly follows from Lemma 3.1, Observation 5.1 and Theorem 4.12. ■

Theorem 5.4 *Let $n \in \mathbb{Z}$ such that $n \geq 5$, and let $G = C_n \square K_2$. Then $\rho_2^*(G) = \left\lceil \frac{2n}{3} \right\rceil$.*

Proof. Again, this follows from Lemma 3.1, Observation 5.1, and Theorem 4.13. ■

Surely these three results do not encompass every graph for which $\rho_1^*(G) = \rho_2^*(G)$. A common trait among these graphs is that no vertex has a degree larger than three. Perhaps this does have some effect, as many graphs we will see for which $\rho_2^*(G) < \rho_1^*(G)$ have some vertex of a high degree.

5.1 Small Values

We will mirror those results on $\rho_1^*(G)$ by characterizing the graphs for which $\rho_2^*(G) = 2$ and $\rho_2^*(G) = 3$. We will continue to allow ourselves to exclude the trivial graph, as that is the only graph for which $\rho_2^*(G) = 1$.

Theorem 5.5 *For any non-trivial graph G, $\rho_2^*(G) = 2$ if and only if $\rho_1^*(G) = 2$ or $\gamma(G) = 1$.*

Proof. We consider the case where $\rho_1^*(G) \neq 2$ since the result follows trivially if $\rho_1^*(G) = 2$. Thus, let $\gamma(G) = 1$. Place two pebbles on a single vertex that dominates G. Then a pebbling move reaches every vertex, and it follows that $\rho_2^*(G)$.

Now, if $\rho_2^*(G) = 2$ and $\rho_1^*(G) \neq 2$, then from Theorem 4.5 there cannot exist two vertices $u, v \in V(G)$ such that $N(u) \cap N(v) = G - \{u, v\}$. Thus, no rubbling configuration of two pebbles can use a rubbling move. If a pebbling move must be used, then both pebbles must be placed on a single vertex

u. Since this is a rubbling configuration, u must be adjacent to every other vertex in $V(G)$. This is a equivalent to saying $\gamma(G) = 1$. ■

Regardless of our choice for t, Theorem 4.5 and Theorem 5.5 characterize all graphs for which $\rho_t^*(G) = 2$.

In Theorem 4.6, we characterized all graphs with an 1-restricted optimal rubbling number equal to three. For all of these graphs, if $\gamma \neq 1$, then $\rho_2^*(G) = 3$ from Observation 5.1 and Theorem 5.5. The rubbling configuration for all these graphs is necessarily achieved by placing one pebble on each of three different vertices. However, allowing an extra pebble in the initial configuration gives a second possible rubbling configuration using three pebbles: placing two on a single vertex, and a third on another vertex. For convenience, we refer to this configuration as F. The following theorem characterizes those graphs for which $\rho_2^*(G) = 3$ in the form of F. Some of these graphs will also have a rubbling configuration in the form of Theorem 4.6, however, some will not.

Theorem 5.6 *Let G be a graph. Then $\rho_2^*(G) = 3$ in the form of F if and only if $3 \leq diam(G) \leq 4$ and there exists a minimal dominating set $S = \{u, v\}$, such that*

(i) $uv \in E(G)$ or

(ii) v is a terminal vertex in every maximal subpath.

Proof. Let G be a graph with a set S such that $3 \leq diam(G) \leq 4$. By Lemma 3.1 and Theorem 2.6, $\rho_2^*(G) \geq 3$. Place two pebbles on u and one pebble on v. If (i) is true, then a pebbling move $p(u \to x)$ will reach each vertex $x \in N(u)$. Note that this includes v. Following the pebbling move $p(u \to v)$, a second pebbling move from v will reach each vertex in $N(v)$. Since $\{u, v\}$ dominates G, it follows that this is a rubbling configuration. Hence, assume that $uv \notin E(G)$ and that v is a terminal vertex in every maximal subpath. It follows that $epn(v, V(G)) = 0$. Furthermore, since $\{u, v\}$ is a dominating set, each vertex in $N(v)$ must be adjacent to some vertex in $N[u]$. Thus, a pebbling move from u will place a second pebble in the neighborhood of any vertex in $N(v)$. It follows that each vertex is reachable, and that $\rho_2^*(G) \leq 3$. Hence, $\rho_2^*(G) = 3$.

Now let $\rho_2^*(G) = 3$. By Theorem 5.5, $\gamma(G) \geq 2$. If $\gamma(G) > 2$, then in any initial rubbling configuration, at least one vertex, say x, is not adjacent to any pebbles. It will take at least two moves to place two pebbles in $N(x)$. But we only began with three pebbles. Thus, after two moves there can only be one pebble remaining, which cannot reach x. It follows $\gamma(G) \leq 2$, and so, $\gamma(G) = 2$. Clearly, $diam(G) \geq 2$. Assume to the contrary that $diam(G) = 2$. If $uv \notin E(G)$, then for each minimal dominating set $S = \{u, v\}$, we have that $epn(u, v) = epn(v, u) = 0$. But then placing one pebble on each of u and v results in a rubbling configuration, which is a contradiction. If $uv \in E(G)$ then only one of u or v, say u, has any private neighbors. But then u dominates G, and placing two pebbles on u results in a rubbling configuration, which is a contradiction. Thus, $diam(G) \geq 3$. By Theorem 2.6, $diam(G) \leq 4$, and we observe that $3 \leq diam(G) \leq 4$.

Finally, assume to the contrary that (i) and (ii) are both false. Let F be a rubbling configuration where $F(u) = 2$, $F(v) = 1$, and $F(x) = 0$ for each other $x \in V(G)$. Since $uv \notin E(G)$, each vertex in $N(v)$ and not in $N(u)$ must be reached by a rubbling move. Then each vertex in $N(v)$ must be adjacent to some vertex in $N[u]$ and $epn(v, V(G)) = 0$, contradicting the assumption that claim (ii) is false. Thus, by contradiction, either (i) or (ii) must be true. ■

While we have not considered the case where three pebbles are placed on a single vertex, we can be sure that Theorems 4.6 and 5.6 encompass all such graphs for which $\rho_t^*(G) = 3$. Consider placing three pebbles on a single vertex u. The first move is necessarily a pebbling move to some vertex v, and the resulting configuration results in one pebble on each of u and v. The only remaining move is a rubbling move from u and v to some third vertex in $N(u) \cap N(v)$. But then this vertex was reachable in the initial configuration, and placing two pebbles on u would yield a rubbling configuration.

5.2 Specific Families of Graphs

We now present the 2-restricted optimal rubbling numbers for specific types of graphs. For many of these graphs we will observe that $\rho_2^*(G) < \rho_1^*(G)$.

Theorem 5.7 *Let $K_{1,s}$ be a star with $s \geq 3$. It follows that $\rho_2^*(K_{1,s}) = 2$.*

Proof. Note that the unique center of any star is a dominating set. Hence, from Theorem 5.5, $\rho_2^*(K_{1,s}) = 2$. ■

A similar result on double stars follows.

Theorem 5.8 *Let $G = S_{r,s}$ be a double star with $2 \leq r \leq s$. It follows that $\rho_2^*(G) = 3$.*

Proof. Note that the set $\{r, s\}$ is a dominating set where $rs \in E(G)$ and $diam(G) = 3$. Thus, the conditions from Theorem 5.6 are satisfied and $\rho_2^*(G) = 3$. ■

We now return to the trees of diameter four. Similar to our previous results, we will characterize the 2-restricted optimal rubbling number for these trees.

Theorem 5.9 *For $s \geq 2$, let $K_{1,s}(c; a_1, ..., a_s)$ be a tree of diameter four. The 2-restricted optimal rubbling number for $K_{1,s}(c; a_1, ..., a_s)$ is as follows:*

(i) $\rho_2^*(K_{1,s}(c; a_1, ..., a_s)) = 3$ if and only if $s = 2$, $c = 0$, and $a_2 = 1$.

(ii) $\rho_2^*(K_{1,s}(c;a_1,...,a_s)) = 4$ *if and only if it is either* $K_{1,2}(c;a_1,a_2)$ *(where* $a_2 \geq 2$*) or* $K_{1,3}(c;a_1,a_2,a_3)$*.*

(iii) $\rho_2^*(K_{1,s}(c;a_1,...,a_s)) = 5$ *if and only if* $s = 4$ *or* $s = 5$*.*

(iv) $\rho_2^*(K_{1,s}(c;a_1,...,a_s)) = 6$ *if and only if* $s \geq 6$

Proof. *(i)* We construct a rubbling configuration by placing two pebbles on y_1, one pebble on $y_{2,1}$, and no pebbles on the remaining vertices. To see that this is minimum, first note that this class of graph includes $P_5 \cong K_{1,2}(0;1,1)$ and $\rho_2^*(P_5) = 3$ by Theorem 5.2. Further, for $a_1 \geq 2$, at least two pebbles are needed to reach the vertices $y_{1,1}$,...,y_{1,a_1}. Since none of these pebbles can reach $y_{2,1}$, at least one additional pebble is needed for this vertex.

(ii) Let $s = 2$. We construct a rubbling configuration by placing two pebbles on each of y_1 and y_2 and no pebbles on the remaining vertices. To show that this is minimum, note that two pebbles are needed for the vertices $y_{1,1}$,...,y_{1,a_1}. If $a_2 \geq 2$, then two additional vertices are needed for $y_{2,1}$,...,y_{2,a_2}. Likewise, if $c \geq 1$ and $a_2 = 1$, then two pebbles are needed to reach $y_{1,1}$,...,y_{1,a_1}. However, these pebbles cannot reach $y_{2,1}$, nor can they reach x_1,...,x_c. Thus, we need an additional pebble on x. However, this will still not let us reach $y_{2,1}$. Thus, we will need a total of four pebbles to reach the entire graph.

Let $s = 3$. We construct a rubbling configuration by placing one pebble on each of y_1, y_2, y_3, and x. To see that this is minimum, note that at least one pebble is needed for each of $y_{1,1}$, $y_{2,1}$, and $y_{3,1}$. If $a_1 \geq 2$, then an additional pebble is needed. Likewise, if $c \geq 1$, then an additional pebble is needed to reach x_1,...,x_c.

(iii) Let $s = 4$. We construct a rubbling configuration by placing one pebble on each of y_1,...,y_4, and x. To see that this is minimum, to place a pebble on $y_{i,1}$, we may either have a pebble there in the initial configuration (this requires at least four pebbles), we may have four pebbles on x (this requires at least six pebbles), or we may have two pebbles on y_i. In this last case, we may have two pebbles on y_i initially (which would require eight pebbles) or move two pebbles to y_i. To place two pebbles on y_i, we need to have one pebble on y_i and two pebbles on x. To place two pebbles on x, we may either place two there initially, or move them there by having one pebble on each of x, y_j, and y_k, where $i \neq j$, $i \neq k$, and $j \neq k$ (recall that this requires one pebble to remain on y_i). Again, this requires at least four pebbles. However, we need at least one additional pebble for the final branch of the tree.

Let $s = 5$. We construct a rubbling configuration by placing one pebble on each of y_1,...,y_5. The rest of the argument is analogous to the case where $s = 4$.

(iv) Let $s \geq 6$. We construct a rubbling configuration by placing two

pebbles on each of y_1, y_2, and x. To see that this is minimum, note that we require at least six pebbles to reach $y_{1,1}$,...,$y_{6,1}$. Thus, the minimum is achieved by placing four pebbles on x and moving them to the leaves using pebbling moves. This requires two pebbles on x and at least four pebbles on its neighbors. ■

5.3 Relationships

Our first few results in this section will focus on the relationship between ρ_1^* and ρ_2^*. Using the results from Theorem 5.9 and Theorem 4.17, we will construct a tree for which $\rho_1^*(G) - \rho_2^*(G) = k$ for any $k \geq 1$.

Theorem 5.10 *The difference $\rho_1^*(G) - \rho_2^*(G)$ can be made arbitrarily large.*

Proof. Consider the star $K_{1,s}$ for $s \geq 3$. From Theorem 4.8, $\rho_1^*(K_{1,s}) = 3$ and from Theorem 5.7, $\rho_2^*(K_{1,s}) = 2$. We will now construct a graph G for which $\rho_1^*(G) - \rho_2^*(G) = k$. Begin with the path P_{4k-3} with vertex set $p_1, p_2, ..., p_{4k-3}$. Associate the center of the star $K_{1,s}$ with the vertex p_{4i+1} on the path for $i = 0, 1, 2, ..., k-1$. Using the configurations of Theorem 4.8 and Theorem 5.7, it is possible to reach the vertices of each star with two or three pebbles, depending on the initial restriction. Furthermore, since the distance between center vertices equals four, the remaining vertices of the path are reachable. Hence, $\rho_1^*(G) \leq 3k$ and $\rho_2^*(G) \leq 2k$.

Since each star has at least three leaves, it is not optimal to place one pebble on each leaf of any one star. Thus, if some star K_{1,s_i} does not begin with either two or three pebbles, then it follows from Theorem 4.8 and Theorem 5.7 that vertex p_{4i+1} must receive two pebbles from some other vertex on the graph. Due to the construction of the graph, this must be through pebbling moves from p_{4i} or p_{4i+2}. Without loss of generality, assume this will be from p_{4i}. From Lemma 4.14, p_{4i} (and p_{4i+1}) can only receive two pebbles if each vertex on the p_{i-3}-p_{i-1} path receives two pebbles first. Clearly it is most efficient to begin by placing one pebble on each of $p_{4i-2}, ..., p_{4i+1}$. But this already requires at least four pebbles, which is not optimal. Hence, $\rho_1^*(G) \geq 3k$ and $\rho_2^*(G) \geq 2k$, and so, $\rho_1^*(G) = 3k$ and $\rho_2^*(G) = 2k$. Thus, we have constructed a graph for which $\rho_1^*(G) - \rho_2^*(G) = k$. ■

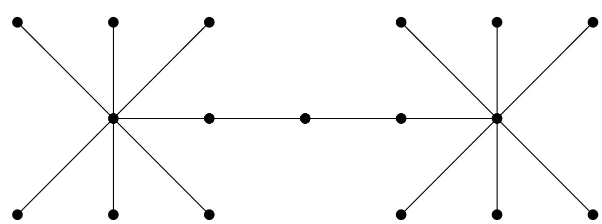

Figure 14: A graph where $\rho_1^* - \rho_2^* = 2$

We will now establish a general bound between ρ_1^* and ρ_2^* for specific graphs. Our proof will construct a 1-rubbling configuration from a 2-rubbling configuration while counting the number of added pebbles. To do so we will use the following definition, which partitions the vertex set of a graph G in to subsets based on the number of pebbles they are assigned by some ρ_t^*-configuration f. Define subsets $V_k \subset V(G)$ for $1 \leq k \leq t$, such that a vertex v is a member of V_k if v is assigned k pebbles by f. So if $f(v) = 1$, then $v \in V_1$.

Theorem 5.11 *If G has a ρ_2^*-configuration $f = (V_0, V_1, V_2)$ such that every vertex in V_2 has at least two private neighbors in V_0, then $\rho_1^*(G) \leq \frac{3}{2}\rho_2^*(G)$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a ρ_2^*-configuration of a graph G such that every vertex in V_2 has at least two private neighbors in V_0.

Now for each $v \in V_2$, select two of those private neighbors, say $u, w \in V_0$. In order to construct a 1-rubbling configuration of G, begin with the configuration f. Place one pebble on each of u and w and remove one pebble from v. Notice that each of v, u, and w each begin with one pebble. Furthermore, a rubbling move $r(u, w \to v)$ will place a second pebble on v. Since v is an arbitrary vertex in V_2, and u and w are private neighbors of v, this can be done for each vertex in V_2. This gives us a rubbling configuration of G, as each vertex begins with at most one pebble, and after a rubbling move with each vertex in V_2 we return to configuration f. We have added at most $|V_2|$ pebbles, so

$$\rho_1^*(G) \leq \rho_2^*(G) - |V_2| + 2|V_2|$$

$$= |V_1| + 3|V_2|$$

$$= \rho_2^*(G) + |V_2|.$$

We use the most pebbles if $V_1 = \emptyset$ and so we have $\rho_1^*(G) \leq 3|V_2| = \frac{3}{2}\rho_2^*(G)$.

■

It is currently unknown if a similar type of bound exists in general for all graphs. This bound could fail if some vertex in V_2 did not have at least two external private neighbors. We now proceed with two theorems relating the diameter of a tree T, and $\rho_2^*(G)$. Similar to before, we will consider even and odd diameters separately.

Theorem 5.12 *Suppose that T is a tree of diameter $2k$ whose center has degree at least $2^k - 5$. It follows that $\rho_2^*(T) \leq 2^{k+1} - 2$.*

Proof. Let T be a tree with diameter $2k$. Then the center of T is a unique vertex u. Let $deg(u) = c$ and the neighbors of u be v_1,...,v_c.

Suppose that $2^k - 5 \leq c \leq 2^k - 2$. For our initial configuration, place two pebbles on each of u, v_1,...,v_c. Without loss of generality, assume that our destination is a leaf w on the periphery such that the shortest path from u to w passes through v_c. Use pebbling moves to move pebbles from v_1,...,v_{c-1} onto u. Vertex u now has $c + 1$ pebbles. Using pebbling moves, move all of these pebbles to v_c. We now have $\frac{c+1}{2} + 2$ pebbles on v_c. Since $c \geq 2^k - 5$, it follows that $\frac{c+1}{2} + 2 \geq 2^{k-1}$. Hence we can reach any leaf on the branch rooted at v_c. Note that our initial configuration used $2+2c \leq 2+2(2^k-2) = 2^{k+1}-2$ pebbles. Therefore, $\rho_2^*(T) \leq 2^{k+1}-2$.

Suppose that $c \geq 2^k-1$. For our initial configuration, place two pebbles on each of u, v_1,...,v_{2^k-2}. Use pebbling moves to move all pebbles to u. Now, u has 2^k pebbles. Hence, it is possible to reach any vertex in the graph using a series of pebbling moves. Again, we used $2^{k+1}-2$ pebbles in our initial configuration, so the bound still holds. ■

Theorem 5.13 *For $k \geq 2$, suppose that T is a tree of diameter $2k+1$ with center vertices u and v such that deg$(u) \geq 2^k-5$ and deg$(v) \geq 2^{k-1}-2$. It follows that $\rho_2^*(T) \leq 2^{k+1}+2^k-4$.*

Proof. Suppose that the non-center neighbors of u are u_1,...,u_n. Suppose that the non-center neighbors of v are v_1,...,v_m. If $2^k-6 \leq n \leq 2^k-2$ and $m = 2^{k-1}-3$, then begin by placing two pebbles on each of u, v, u_1,...,u_n, v_1,...,v_m. If our destination is a leaf on the periphery rooted at u_n, then move all pebbles from u_1,...,u_{n-1}, and v to u using pebbling moves. We now have $n+2$ pebbles on u. Move all pebbles from u to u_n using pebbling moves. There are now $\frac{n+6}{2}$ pebbles on u_n. Since $n \geq 2^k-6$, there are at least 2^{k-1} pebbles on u_n. Hence, we can reach any leaf rooted at u_d using pebbling moves. Similarly, if our destination is a leaf on the periphery rooted at v_m, we move all pebbles from u_1,...,u_n to u using pebbling moves. We then move all pebbles from u, v_1,...,v_{m-1} to v using pebbling moves. There are now

$\frac{n+2}{2}+m+1$ pebbles on v. Move all pebbles from v to v_m using pebbling moves. We now have $\frac{n+2m+12}{4}$ pebbles on v_m. Since $n \geq 2^k - 6$ and $m \geq 2^{k-1} - 3$, there are at least 2^{k-1} pebbles on v_m. Hence, we can reach any leaf rooted at v_m. Note that we use $4 + 2n + 2m$ pebbles in our initial configuration. Since $n \leq 2^k - 2$ and $m \leq 2^{k-1} - 2$, it follows that $\rho_2^*(T) \leq 2^{k+1} + 2^k - 4$.

For the remaining cases, use a similar argument to the above argument as well as the proof of Theorem 5.12. ■

Note that the bound given in Theorem 5.13 does not hold for trees of diameter three (i.e., the double stars) as $\rho_2^*(T) = 3$, but $2^2 + 2 - 4 = 2$. Further, note that we can likely "improve" the bound on the degrees of our centers by moving all of the pebbles (except those on our chosen branch) to the centers. This allows the centers to "share" pebbles more effectively. In this case, the above result holds if $2n+m \geq 2^{k+1}-12$ and $n+2m \geq 2^{k+1}-12$.

Theorems 5.12 and 5.13 depend on specific values for the degree of the center vertex or center vertices. While we will see later that these bounds do not necessarily hold for all trees, in the next theorem we will show that the bound in Theorem 5.12 holds for all trees of diameter eight.

Theorem 5.14 *Let T be a tree of diameter eight. Then $\rho_2^*(T) \leq 30$.*

Proof. First we will define a few terms:

- u: The unique center of T.

- X_1: The set of vertices in $N(u)$.
- X_2: The set of all vertices distance two from u.
- X_2^*: The set of all vertices distance two from u that begin with 2 pebbles. Note that $X_2^* \subset X_2$.

If $|X_1| \geq 11$, then $\rho_2^*(T) \leq 30$ by Theorem 5.12. Hence, assume that $3 \leq |X_1| \leq 10$. First place two pebbles on u and two pebbles on each vertex in X_1. Place the remaining pebbles on vertices in X_2. Those vertices in X_2 that receive pebbles will comprise the set X_2^*. Note that $|X_1 + X_2^*| \leq 14$. We will consider two cases.

Case 1: $|X_1| + |X_2| \leq 14$. Observe that $X_2 = X_2^*$, since every vertex in X_2^* is in X_2. Furthermore, every open vertex in T is at most distance two from some vertex in X_2. Thus, if four pebbles can be placed on every vertex $y \in X_2$ adjacent to some $x_1 \in X_1$, it follows that every vertex in $V(T)$ is reachable. Since y begins with two pebbles in the initial configuration, only two more pebbles must be moved to y. Consider vertices $x_2, x_3 \in V_1$ not adjacent to y. The pebbling moves $p(x_2 \to u)$ and $p(x_3 \to u)$ will place four pebbles on u. Executing the pebbling move $p(u \to x_1)$ twice will place four pebbles on x_1. Finally, executing the pebbling move $p(x_1 \to y)$ twice will place four pebbles on y. Since y was an arbitrary vertex in X_2, it follows that every vertex in X_2 can be pebbled in this way, giving a rubbling configuration of T. We used less than $2 + 28 = 30$ pebbles, and so $\rho_2^*(T) \leq 30$.

Case 2: $|X_1| + |X_2| > 14$. In this case, there must be least one vertex in X_2 that is not in X_2^*. This implies that $|X_1| + |X_2^*| = 14$. Hence, we wish to show that regardless of the number of vertices in X_1, eight pebbles can be placed on any single vertex in X_1. This presents three subcases dependent on the value of $|X_1|$.

Subcase (a): $|X_1| \geq 8$. In this case, there is at least one vertex in X_1, say x_1, such that $|N(x_1) \cap X_2| \geq 1$ and $|N(x_1) \cap X_2^*| = 0$. Note that for every 4 vertices in X_2^* we can place one pebble on x_1. Furthermore, for every 2 vertices in X_1, we can place one pebble on x_1. In our initial configuration, x_1 begins with two pebbles, and u can place one pebble on x_1 through a pebbling move. Thus, if

$$\frac{|X_1| - 1}{2} + \frac{|X_2^*|}{4} + 3 \geq 8$$

we have a rubbling configuration. We can simplify this inequality:

$$\frac{|X_1| - 1}{2} + \frac{|X_2^*|}{4} + 3 \geq 8$$

$$2|X_1| - 2 + |X_2| + 12 \geq 32$$

$$2|X_1| + |X_2^*| \geq 22.$$

Which, when given our initial values of $|X_1|$, and the fact that $|X_1| + |X_2^*| = 14$, it is straightforward to check that the inequalty holds for each $|X_1| \geq 8$.

Subcase (b): $5 \leq |X_1| \leq 7$. Given the number of vertices in X_1, it follows that each vertex in X_1 with a neighbor in X_2 must have at least one neighbor in X_2^*. Since there still must be some vertices in X_2 not in X_2^*, we wish to show that eight pebbles can be placed on any vertex in X_1. Similar to above, x_1 begins with two pebbles, and u contributes one pebble. This time, the vertex in X_2^* adjacent to x_1 will also contribute a pebble. Hence, consider the following equation:

$$\frac{|X_1| - 1}{2} + \frac{|X_2^*| - 1}{4} + 4 \geq 8.$$

If this inequality holds, then we have a rubbling configuration. It simplifies:

$$\frac{|X_1| - 1}{2} + \frac{|X_2^*| - 1}{4} + 4 \geq 8,$$

$$2|X_1| - 2 + |X_2| - 1 + 16 \geq 32,$$

$$2|X_1| + |X_2^*| \geq 19.$$

Which again, one can check that it holds for all values of $|X_1|$.

Subcase (c): $3 \leq |X_1| \leq 4$. In this last case, each vertex in X_1 with more than 2 neighbors in X_2 must have at least 2 neighbors in X_2^*. Using pebbling and rubbling moves similar to before, we can set up the inequality

$$\frac{|X_1| - 1}{2} + \frac{|X_2^*| - 2}{4} + 5 \geq 8.$$

If this inequality holds, then we will prove that the bound holds for this final subcase. It simplifies:

$$\frac{|X_1| - 1}{2} + \frac{|X_2^*| - 2}{4} + 5 \geq 8,$$

$$2|X_1| - 2 + |X_2| - 2 + 20 \geq 32,$$

$$2|X_1| + |X_2^*| \geq 16.$$

Which holds when $|V_1|$ is equal to three or four. Finally, for some vertex in $x_1 \in X_1$ with a single neighbor in $y \in X_2$, observe that y must begin with two pebbles. Hence, we wish to show that y can receive four pebbles. Vertex u can receive two more pebbles independent of x_1. Thus, using pebbling moves, two more pebbles can be moved to y. Either way, we have a rubbling configuration. Thus, we have shown that for every degree of u, the bound holds. ■

Observation 5.15 *The bound in Theorem 5.12 does not hold for trees of diameter ten.*

Proof. Let T be a tree of diameter ten and center c such that $deg(c) =$

23. Now let every other non-leaf vertex have degree equal to 64. For $k = 1, 2, ..., 5$, let X_k be subsets of $V(T)$ such that $v \in X_k$ if $d(v, c) = k$. Then each $v \in X_k$ has exactly one neighbor in X_{k-1} and 63 neighbors in X_{k+1}. As a result, in every possible configuration of 62 pebbles, each $x \in X_1$ must be adjacent to at least one $y \in X_2$ such that y has no pebbles. Furthermore, for at least one of these vertices, each vertex on every $y - \ell$ subpath for $\ell \in X_5$ of length three does not begin with a pebble. It follows that if $\rho_2^*(G) = 62$, then each vertex in X_1 must be able to receive sixteen pebbles in order to reach each vertex in X_5. Furthermore, in any pebbling distribution, at least one vertex in X_1 will be adjacent to at most two extra pebbles, which must be placed on c. Consider some vertex $v_1 \in X_1$ for which this is true. Each rubbling or pebbling move requires the loss of one pebble, so it follows that if the configuration that minimizes the sum of the distance of the remaining pebbles to v_1 will not place sixteen pebbles on v_1, then no configuration will do this. So, with the 62 pebbles, place two on v_1, place two on c, 44 on the remaining vertices in X_1, and 14 on arbitrary vertices in X_2 that are not adjacent to vertex v_1. Pebbling moves will allow us to use the pebbles in X_1 to place 22 pebbles on c. However, the 14 pebbles in X_2 will result in at most one extra pebble on c. This leaves 25 pebbles on c, only allowing for 14 pebbles on v_1. Thus, the tree T must have an 2-restricted optimal rubbling number greater than 62. ■

6 CONCLUSIONS AND OPEN PROBLEMS

In our work we introduced the t-restricted optimal rubbling number as an added restriction to the optimal rubbling number of a graph. While we explored both 1-restricted and 2-restricted optimal rubbling, our main focus was on results in 1-restricted optimal rubbling. Many of the results specifically shown for 1-rubbling could be extended to, or improved upon in 2-rubbling.

In their work on t-restricted optimal pebbling, Chellali, Haynes, Hedenimi, and Lewis studied its relationship to the Roman domination number [4]. A *Roman dominating function* is a mapping of the vertex set $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) = 2$. On the surface, it appears that the 2-restricted optimal rubbling number could have a very close relationship with the Roman domination number. While we did not investigate any potential relationship, this could be an avenue for further study.

Other results associated with pebbling and rubbling have yet to be researched for t-restricted optimal rubbling. While the computational complexity of pebbling was determined in [20], there is no related work for rubbling, or t-restricted optimal rubbling. Pebbling as a subject of study finds its roots in studying the hypergraph. While we have shown some results on the Cartesian product of graphs, there has yet to be any work done on the t-restricted optimal rubbling number of a hypergraph. Additional problems for future study include:

1. Is it possbile to characterize those graphs for which $\rho_1^*(G) = \gamma(G)$?
2. How does the value of t affect the difference of $\rho_t^*(G)$ and $\rho_{opt}(G)$?
3. Characterize those graphs for which $\rho_1^*(G) = \rho_2^*(G)$. Is it possible to characterize the graphs for which ρ_t^* is the same regardless of our choice for t?
4. For what graphs are the t-restricted optimal pebbling and t-restricted optimal rubbling numbers the same? Does our choice for t affect this?
5. Characterize those graphs for which $\rho_1^*(G \square H) = \rho_1^*(G)\rho_1^*(H)$.

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