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Global Supply Sets in Graphs

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Global Supply Sets in Graphs

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ABSTRACT

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by

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For a graph $G = (V, E)$, a set $S \subseteq V$ is a *global supply set* if every vertex $v \in V \backslash S$ has at least one neighbor, say u, in S such that u has at least as many neighbors in S as v has in $V \backslash S$. The *global supply number* is the minimum cardinality of a global supply set, denoted $\gamma_{gs}(G)$. We introduce *global supply sets* and determine the *global supply number* for selected families of graphs. Also, we give bounds on the *global supply number* for general graphs, trees, and grid graphs.

DEDICATION

This thesis is dedicated to my parents, Deana and Jason Moore. Without their sacrifices, love, and support this would not have been possible.

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I would like to thank Dr. Stephen Hedetniemi for proposing such an interesting problem which became the topic of this thesis.

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1 INTRODUCTION

1.1 Introduction to Graph Theory

A *graph* $G = (V, E)$ consists of a finite vertex set, $V(G)$, and a finite edge set, $E(G)$. If G is clear from the context, then we generally use V and E. The *order* of a graph G, denoted n, is the number of vertices in G, that is, $n = |V|$. The *size* of a graph G, denoted m, is the number of edges in G, that is, $m = |E|$. Two vertices u and v are *adjacent* if there is an edge in E, typically denoted as $uv \in E$, connecting the two. We say that the vertices $u, v \in V$ are *incident* with edge $uv \in E$. We only consider graphs that do not have directions on the edges and do not have multiple edges connecting the same two vertices.

A vertex and an edge are said to *cover each other* in G if they are incident in G. A *vertex cover* in G is a set of vertices that covers all the edges of G. The minimum cardinality of a vertex cover in G is called the *vertex covering number* $\beta(G)$. A set $S \subset V$ is said to be *independent* if for all $u, v \in S$, $uv \notin E$. The *independence number* $\alpha(G)$ of a graph G is the maximum cardinality of an independent set in G.

The *complement* $\overline{G}$ of G is a graph with vertex set V and where two vertices are adjacent if and only if they are not adjacent in G. So, $E\left(\overline{G}\right) = \overline{E\left(G\right)}$.

For a vertex $v \in V$, the set $N\left(v\right) = \{u \in V \mid uv \in E\}$ is called the *open neighborhood* of v. That is, $N\left(v\right)$ is the set of all vertices that v is adjacent to in G. Each vertex $u \in N\left(v\right)$ is called a *neighbor* of v. The *closed neighborhood* of a vertex v is $N[v] = N\left(v\right) \cup \{v\}$. The *open neighborhood of a set* $S \subseteq V$ is $N(S) = \bigcup_{v \in S} N(v)$, and the *closed neighborhood of a set* $S \subseteq V$ is $N[S] = \bigcup_{v \in S} N[v]$. The *degree* in G of a vertex v is $deg_G(v) = |N(v)|$. If G is clear from context, then we use $deg(v)$. A vertex v with $deg(v) = 1$ is called a *leaf*. The neighbor of a leaf is called a *support vertex*. A support vertex with more than one leaf neighbor is called a *strong support vertex*. If $S \subseteq V$, then the *boundary* of S is $\partial(S) = N(S) \cap (V \backslash S)$. In other words, the boundary of S is the set of vertices in $V \backslash S$ that are adjacent to at least one vertex in S.

For two vertices $u, v \in V$, a $u - v$ *walk* W is a sequence of vertices in G, beginning with u and ending with v such that the consecutive vertices in W are adjacent in G. A *path* is a walk in which no vertex is repeated. The *distance d(u,v)* between two vertices $u, v \in V$ is the minimum length of any $u - v$ path in G. The maximum distance from v to all other vertices is called the *eccentricity* of v. The *diameter* of G, *diam(G)*, is defined as the greatest eccentricity among all the vertices of G. A vertex that has eccentricity equal to *diam(G)* is called a *peripheral vertex*. A graph that has a $u - v$ path for all $u, v \in V$ is a *connected graph*.

A *path* P_n is a graph with $V = \{v_1, v_2, ..., v_n\}$ and $E = \{v_i v_{i+1} \mid i = 1, 2, ..., n-1\}$. A *cycle* C_n of order $n \geq 3$ is a graph with $V = \{v_1, v_2, ..., v_n\}$ and $E = \{v_i v_{i+1} \mid i = 1, 2, ..., n\}$. A connected graph that contains no cycles is a *tree* T. We *root* a tree at vertex $r \in V$, such that for all $v \in V$ where $v \neq r$, the *parent* of v is the neighbor of v on the unique $r - v$ path, and a *child* of v is any other neighbor of v. A *star* $S_{1,n-1}$ is a tree with exactly one support vertex and $n - 1$ leaves, that is, a star $S_{1,n-1}$ is a tree with diameter 2. A *double star* $S_{r,s}$ is a tree with diameter 3, that is, $S_{r,s}$ has two support vertices $u, v \in V$ such that $uv \in E$ and u has r leaf neighbors while v has s leaf neighbors.

The *cartesian product* $H = G_1 \Box G_2$ has a vertex set $V(H) = V(G_1) \times V(G_2)$. If $u, x \in V(G_1)$ and $v, y \in V(G_2)$ then the two distinct vertices $(u, v), (x, y) \in H$ are adjacent if either $u = x$ and $vy \in E(G_2)$ or $v = y$ and $ux \in E(G_1)$. A *grid graph* $G_{r,c} = P_r \Box P_c$ where $r, c \geq 2$. The *join* of two graphs $H = G_1 \vee G_2$ has a vertex set $V(H) = V(G_1) \cup V(G_2)$ and edge set $E(H) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. A *wheel graph* with order n is the join of a single vertex and a cycle of order $n-1$, that is, $W_{1,n-1} = P_1 \vee C_{n-1}$.

A graph in which every two distinct vertices are adjacent is called a *complete graph* K_n. A graph is a *complete bipartite graph* $K_{r,s}$ if the vertex set can be partitioned into two disjoint independent sets, say R and S, where for all $r \in R$ and $s \in S$ we have that $rs \in E$.

A set $S \subseteq V$ is a *dominating set* of G if every vertex $v \in V$ is either in S or $\partial(S)$, that is, $\partial(S) = V \backslash S$. The minimum cardinality of any dominating set of G is called the *domination number* $\gamma(G)$ of G. A set $S \subseteq V$ is a *total dominating set* of G if S is a dominating set and every $v \in S$ has a neighbor in S. The minimum cardinality of a total dominating set is called the *total domination number* $\gamma_t(G)$ of G.

For terminology not defined here, refer to *Graphs and Digraphs* by Chartrand, Lesniak, and Zhang [2].

1.2 Problem Statement

Consider the following new definition, which was suggested by Professor S. T. Hedetniemi [7]. A set $S \subseteq V$ is a *supply set* if every vertex $v \in \partial(S)$ is adjacent to a vertex $u \in S$ with $deg_S(u) = |N(u) \cap S| \geq deg_{V \backslash S}(v) = |N(v) \cap (V \backslash S)|$.

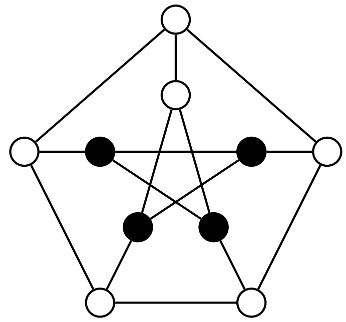

Figure 1: Petersen Graph

One way to look at supply sets is to think of a vertex $v \in \partial(S)$ and its neighbors in $V \backslash S$ as needing some number of units of some resource, one unit per vertex. Vertex v can ask a vertex $u \in (S \cap N(v))$ to deliver $|N[v] \cap (V \backslash S)|$ units. Vertex u can provide this amount only if vertex u can receive from itself and its neighbors in S at least this number, that is, $|N[u] \cap S| \geq |N[v] \cap (V \backslash S)|$. We can think of a supply set as a set that is capable of providing *two-day delivery* to any vertex in $\partial(S)$. On day-one each neighbor of u in S ships a unit of resource to u. Then on day-two, vertex u ships all of the resources to vertex $v \in \partial(S)$ over the edge uv. Figure 1 demonstrates a supply set for the well-known Petersen Graph.

If a supply set S is also a dominating set, then S is called a *global supply set.* We denote a global supply set as a *gs-set.* The *global supply number* $\gamma_{gs}(G)$ is the minimum cardinality of a global supply set of G. We let $\gamma_{gs}(G)$*-set* denote a gs-set S of G where $|S| = \gamma_{gs}(G)$. Figure 2 illustrates a global supply set for the Petersen Graph.

In this thesis, we are introducing the study of supply sets, focusing on global supply sets. Thus, our main goal is to determine the global supply number for families of

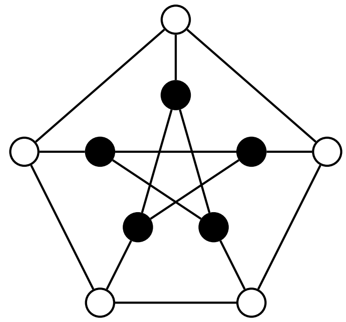

Figure 2: Petersen Graph, $\gamma_{gs}(G) = 5$

graphs and to establish bounds on $\gamma_{gs}(G)$.

To begin, we look at a related subject, global alliances in graphs. This was first introduced by Hedetniemi, Hedetniemi, and Kristiansen in [3]. In Section 2, we discuss two types of global alliances that are closely related to global supply sets.

Preliminary results for k-day supply sets will be given in Section 3. In Section 4, we will determine the global supply number for select families of graphs.

We found it was difficult to establish bounds on the global supply number for some select families of graphs and for graphs in general. In Section 5, we provide a few bounds on the global supply number and discuss the comparability of global supply sets with similar parameters. In particular, we discuss bounds in terms of the order, n, of a graph. We finish with a discussion of a few conjectures and open problems for the global supply number in Section 6.

2 ALLIANCES IN GRAPHS

An alliance is a pact, coalition, or friendship between two or more parties, made in order to advance common goals and to secure common interests. In this section we consider two different types of graph models for alliances, that is, sets of entities uniting for a common cause. The applications of these sets are widespread from social and business associations to national defense coalitions. We focus primarily on global alliances.

2.1 Global Offensive Alliances

As defined in [3], a non-empty set S is an *offensive alliance* if for every vertex $v \in \partial(S)$, $|N(v) \cap S| \geq |N[v] \cap (V \backslash S)|$. If set S is also a dominating set, then S is a *global offensive alliance*. The *global offensive alliance number* $\gamma_{oa}(G)$ is the minimum cardinality of a global offensive alliance of G. Many applications of alliances, including the coalition of nations for defense purposes, were also stated in [3]. Considering this application for an offensive alliance S, it is reasonable to imagine each vertex in S is in alliance with its neighbors in S (assuming strength in numbers) against its neighbors in $\partial(S)$. For the set S as a whole, since an attack by a offensive alliance S on the vertices of $\partial(S)$ can result in no worse than a "tie," the vertices in S can "successfully" attack $\partial(S)$. Two examples of global offensive alliances are given in Figure 3 and Figure 4. For graphs G and H, we have $\gamma_{oa}(G) = 4 = \gamma_{oa}(H)$.

Global offensive alliances have been studied since 2004. In one particular research paper [5], the authors explored both upper and lower bounds on the global supply number of graphs. Their results include the following two theorems.

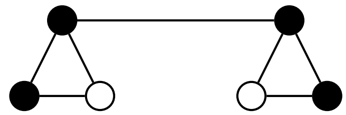

Figure 3: $\gamma_{oa}(G) = 4$

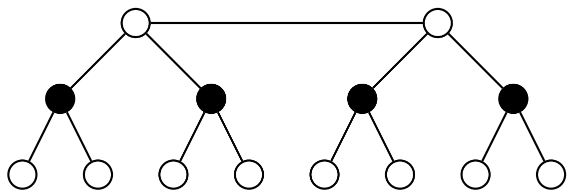

Figure 4: $\gamma_{oa}(H) = 4$

Theorem 2.1 [5] *For all connected graphs G of order $n \geq 2$,*

- $\gamma_{oa}(G) \leq min\{n - \alpha(G), \lfloor \frac{n+\alpha(G)}{2} \rfloor\}$;
- $\gamma_{oa}(G) \leq \lfloor \frac{2n}{3} \rfloor$;
- $\gamma_{oa}(G) \leq \lfloor \frac{\gamma(G)+n}{2} \rfloor$.

Theorem 2.2 [5] *For all graphs G of order n and size m,*

$$\gamma_{oa}(G) \geq \left\lfloor \frac{3n - \sqrt{9n^2 - 8n - 16m}}{4} \right\rfloor$$

2.2 Global Defensive Alliances

A non-empty set of vertices $S \subseteq V$ is called a *defensive alliance* if for every $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V \backslash S)|$. We say that S is a *global defensive alliance* if it is also a dominating set. The *global defensive alliance number*, denoted $\gamma_a(G)$, is the minimum cardinality of a global defensive alliance of G. Similar to an offensive alliance, it is reasonable to imagine each vertex in S is in alliance with its neighbors in S (assuming strength in numbers) against its neighbors in $\partial(S)$. For the set S as a whole, since an attack by the vertices of $\partial(S)$ on a defensive alliance S can result in no worse than a "tie," the vertices in S can "successfully" defend against $\partial(S)$. Defensive alliances were also first introduced in [3].

Two examples of a global defensive alliance are given in Figure 5 and Figure 6.

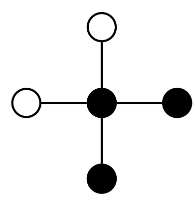

Figure 5: $\gamma_a(S_{1,4}) = 3$

In 2003, Haynes, Hedetniemi, and Henning determined the global defensive alliance number for several families of graphs in [4]. The following is a selection of their results.

Proposition 2.3 [4] *For the complete graph K_n, $\gamma_a(K_n) = \lfloor \frac{n+1}{2} \rfloor$.*

Proposition 2.4 [4] *For the complete bipartite graph $K_{r,s}$,*

1. $\gamma_a(K_{1,s}) = \lfloor \frac{s}{2} \rfloor + 1$.

2. $\gamma_a(K_{r,s}) = \lfloor \frac{r}{2} \rfloor + \lfloor \frac{s}{2} \rfloor$ if $r, s \geq 2$.

Proposition 2.5 [4] *For cycles C_n, $n \geq 3$, $\gamma_a(C_n) = \gamma_t(C_n)$.*

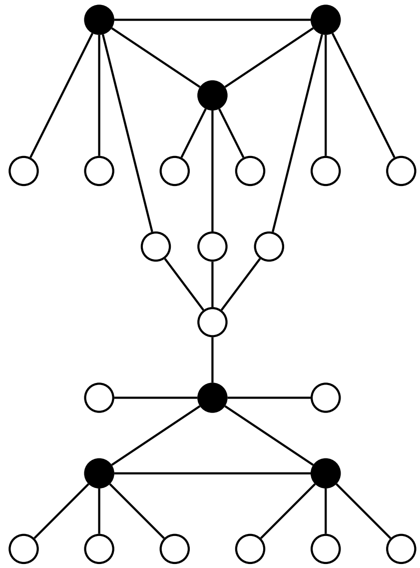

Figure 6: $\gamma_a(G) = 6$

Proposition 2.6 [4] *For $n \geq 2$, $\gamma_a(P_n) = \gamma_t(P_n)$ unless $n \equiv 2$ (mod 4), in which $\gamma_a(P_n) = \gamma_t(P_n) - 1$.*

Proposition 2.7 [4] *For $r, s \geq 1$, $\gamma_a(S_{r,s}) = \lfloor \frac{r-1}{2} \rfloor + \lfloor \frac{s-1}{2} \rfloor + 2$.*

The authors also introduce a family of trees $\mathcal{T}_1$ as follows: Let $T = P_5$ or $T = K_{1,4}$ or let T be the tree obtained from $tK_{1,4}$ (the disjoint union of t copies of $K_{1,4}$) by adding $t-1$ edges between leaves of these copies of $K_{1,4}$ in such a way that the center of each $K_{1,4}$ is adjacent to exactly three leaves in T. An example of such trees can be seen in Figure 7. Let $\mathcal{T}_1$ be the family of all such trees T. They then go on to give the following theorem.

Theorem 2.8 [4] *If T is a tree of order $n \geq 4$, then*

$$\gamma_a(T) \leq \frac{3n}{5}$$

with equality if and only if $T \in \mathcal{T}_1$.

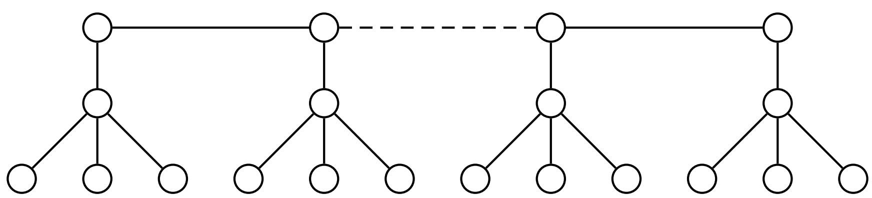

Figure 7: An example of a tree in $\mathcal{T}_1$

3 GLOBAL SUPPLY NUMBER

We now turn our attention to global supply sets and give the main results of this thesis. In this section we provide the global supply supply number for several families of graphs. Since a global supply set is a dominating set and the vertex set of any graph is a global supply set, we make the following observation.

Observation 3.1. *For any graph G of order n, $1 \leq \gamma(G) \leq \gamma_{gs}(G) \leq n$.*

Now we show that the only graph G having $\gamma_{gs}(G) = 1$ is the star, $S_{1,n-1}$.

Theorem 3.2. *A graph G of order n has $\gamma_{gs}(G) = 1$ if and only if $G = S_{1,n-1}$.*

Proof. $\Rightarrow$ Let S be a $\gamma_{gs}(G)$-set with $|S| = 1$. Let $u \in S$. Therefore, any $v \in V \backslash \{u\}$ must have $deg(v) = 1$ and $uv \in E$. Thus, $G = S_{1,n-1}$.

$\Leftarrow$ Let $G = S_{1,n-1}$. Clearly, $\gamma_{gs}(S_{1,n-1}) = 1$. ■

Next, we show that the only graph G having $\gamma_{gs}(G) = n$ is the complexment of the complete graph.

Theorem 3.3. *A graph G of order n has $\gamma_{gs}(G) = n$ if and only if $G = \overline{K_n}$.*

Proof. Clearly, $\gamma_{gs}\left(\overline{K_n}\right) = n$.

Therefore, suppose $G \neq \overline{K_n}$. Then G has at least one edge, say uv. Thus, $V \backslash \{u\}$ is a gs-set of G. Hence, $\gamma_{gs}(G) \leq |V \backslash \{u\}| = n - 1$. ■

Theorem 3.4. *For a non-trivial path P_n of order n, $\gamma_{gs}(P_n) = \lfloor \frac{n}{2} \rfloor$.*

Proof. For $n = 2$ or $n = 3$, $\gamma_{gs}(P_n) = 1 = \lfloor \frac{n}{2} \rfloor$. So, let $n \geq 4$. Label the vertices of P_n as $v_1, v_2, ..., v_n$. Let S be a $\gamma_{gs}(P_n)$-set. Since S is a dominating set, no three consecutive vertices are in $V \backslash S$. Also, since S is a $\gamma_{gs}(P_n)$-set, for every four consecutive vertices of P_n at most two can be in $V \backslash S$. Therefore at least two out of any four consecutive vertices are in S. Thus, $|S| \geq \lfloor \frac{n}{2} \rfloor$.

To achieve equality, let $S = \{v_i | \ i \text{ is even}\}$. Clearly, S is a gs-set of P_n and $|S| = \lfloor \frac{n}{2} \rfloor$. Thus, we conclude that $\gamma_{gs}(P_n) = \lfloor \frac{n}{2} \rfloor$. ∎

Theorem 3.5. *For a cycle C_n of order n, $\gamma_{gs}(C_n) = \lceil \frac{n}{2} \rceil$.*

Proof. For $n = 3$, $\gamma_{gs}(C_n) = 2 = \lceil \frac{n}{2} \rceil$. So, let $n \geq 4$. Label the vertices of C_n as $v_1, v_2, ..., v_n$. Let S be a $\gamma_{gs}(C_n)$-set. Since S is a dominating set, no three consecutive vertices are in $V \backslash S$. Also, since S is a $\gamma_{gs}(C_n)$-set, for every four consecutive vertices of C_n at most two can be in $V \backslash S$. Therefore at least two out of any four consecutive vertices are in S. Thus, $|S| \geq \lceil \frac{n}{2} \rceil$.

To achieve equality, we must examine two cases.

Case 1. n is even.

Let $S = \{v_i | \ i \text{ is even}\}$. Clearly, S is a gs-set of C_n and $|S| = \frac{n}{2}$.

Case 2. n is odd.

Let $S = \{v_i | \ i \text{ is even}\} \cup \{v_n\}$. Clearly, S is a gs-set of C_n and $|S| = \frac{n+1}{2}$.

Thus, we can conclude that $\gamma_{gs}(C_n) = \lceil \frac{n}{2} \rceil$. ∎

Notice that $\gamma_t(C_n) = \lceil \frac{n}{2} \rceil$. Thus, the previous result along with Proposition 2.5 gives the following corollary.

Corollary 3.6. *For a cycle C_n of order $n \not\equiv 2 \pmod 4$, $\gamma_{gs}(C_n) = \gamma_t(C_n) = \gamma_a(C_n)$.*

Theorem 3.7. *For the complete graph K_n of order n, $\gamma_{gs}(K_n) = \lceil \frac{n}{2} \rceil$.*

Proof. Let S be a $\gamma_{gs}(K_n)$-set. By definition of K_n and $\gamma_{gs}(K_n)$-set, we have that $|S| \geq |V \backslash S|$. If n is even, then $|S| = \frac{n}{2} = |V \backslash S|$. If n is odd, then $|S| = \frac{n+1}{2} > |V \backslash S| = \frac{n-1}{2}$. Thus, $|S| = \lceil \frac{n}{2} \rceil$ and $\gamma_{gs}(K_n) = \lceil \frac{n}{2} \rceil$. ∎

Theorem 3.8. *For the complete bipartite graph $K_{r,s}$ where $1 \leq r \leq s$, $\gamma_{gs}(K_{r,s}) = r$.*

Proof. Let R and S be the partite sets with $|R| = r$ and $|S| = s$. If $r = 1$, then $\gamma_{gs}(K_{r,s}) = 1$. Thus, assume that $r \geq 2$. Notice that R is a gs-set. Hence, $\gamma_{gs}(K_{r,s}) \leq |R| = r$.

To see that $\gamma_{gs}(K_{r,s}) \geq r$, suppose to the contrary that D is a gs-set such that $|D| = \gamma_{gs}(K_{r,s}) < r$. Thus there exists a vertex $v \in R$ such that $v \notin D$. Hence, to dominate v, there must be a vertex $u \in S$ such that $u \in D$. That is, $D \cap S \neq \emptyset$. Further, since $\gamma_{gs}(K_{r,s}) < r \leq s$, then there exists a vertex $w \in S$ such that $w \notin D$. Moreover, w has a neighbor in $R \cap D$. So, we know that $D \cap S \neq \emptyset$, $D \cap R \neq \emptyset$, $S \backslash D \neq \emptyset$, and $R \backslash D \neq \emptyset$. Let $R_1 = R \backslash D$, $R_2 = D \cap R$, $|R_1| = r_1$, and $|R_2| = r_2$. We define S_1, S_2, s_1, and s_2 in an analogous way. Note that $r_1, r_2, s_1, s_2 \geq 1$. Now, every vertex in R_1 has a demand of $s_1 + 1$ and a supply of $r_2 + 1$. Similarly, a vertex of S_1 has a demand of $r_1 + 1$ and a supply of $s_2 + 1$. Thus, we can deduce that $r_2 + 1 \geq s_1 + 1$ and $s_2 + 1 \geq r_1 + 1$. That is, $r_2 \geq s_1$ and $s_2 \geq r_1$. Recall that $|D| = \gamma_{gs}(K_{r,s}) = r_2 + s_2 \geq r_2 + r_1 = r$. Thus, we have a contradiction.

Hence, we can conclude that $\gamma_{gs}(K_{r,s}) = r$. ∎

Theorem 3.9. *For the wheel graph $W_{1,n-1}$ where $n \geq 5$, $\gamma_{gs}(W_{1,n-1}) = 3$.*

Proof. Note that $W_{1,n-1} = K_1 \vee C_{n-1}$. Label the vertices of $W_{1,n-1}$ as v and u_i where $i = 1, 2, ..., n-1$ such that $v \in V(K_1)$ and $u_i \in V(C_{n-1})$.

Let $S = \{v, u_i, u_j \mid i \neq j\}$. Clearly, S is a dominating set. Notice that for $u_k \in V(W_{1,n-1}) \setminus S$ such that $k \neq i, j$, we have $|N(u_k) \cap (V \setminus S)| \leq 2 = |N(v) \cap S|$. Since $v \in N(u_k)$, S is a gs-set with $|S| = 3$. Thus, $\gamma_{gs}(W_{1,n-1}) \leq 3$.

To see that $\gamma_{gs}(W_{1,n-1}) \geq 3$, assume to the contrary that $\gamma_{gs}(W_{1,n-1}) < 3$. Let S be a $\gamma_{gs}(W_{1,n-1})$-set. Then for $x \in S$, $|N(x) \cap S| \leq 1$. If $v \notin S$, then $|N(v) \cap (V \setminus S)| \geq 2 > 1 \geq |N(x) \cap S|$, a contradiction. Hence, $v \in S$. If $|S| = 1$, then $|N(v) \cap S| = 0$. Notice that v has a neighbor $u_i \in \partial(S)$ such that $|N(u_i) \cap (V \setminus S)| = 2 > 0 = |N(v) \cap S|$, a contradiction. Thus, $S = \{v, u_i\}$, $|S| = 2$ and $|N(v) \cap S| = 1$. Again, v has a neighbor $u_j \in \partial(S)$ $(i \neq j)$ such that $|N(u_j) \cap (V \setminus S)| = 2 > 1 = |N(v) \cap S|$, a contradiction.

Hence, $S = \{v, u_i, u_j\}$ and $|S| \geq 3$. Thus, $\gamma_{gs}(W_{1,n-1}) = 3$. ■

4 BOUNDS ON THE GLOBAL SUPPLY NUMBER

Here, we discuss bounds on the global supply number. We also make observations about the comparability of global supply sets with global alliances and other parameters. Our first bound is for grid graphs $G_{r,c} = P_r \square P_c$.

Theorem 4.1. *Let the grid graph $G_{r,c}$ be such that $2 \leq r \leq c$. Then*

1. *If r is even and c is odd, then $\gamma_{gs}(G_{r,c}) \leq \lfloor \frac{c}{2} \rfloor \cdot r$*

2. *$\gamma_{gs}(G_{r,c}) \leq \lfloor \frac{r}{2} \rfloor \cdot c$, otherwise.*

Proof. Label the vertices of $G_{r,c}$ as (x_i, y_j) where $i = 1, 2, ..., r$ and $j = 1, 2, ..., c$. Then, two vertices (x_i, y_j) and (x_k, y_l) are adjacent if $i = k$ and $j = l \pm 1$ or if $i = k \pm 1$ and $j = l$. If r is even and c is odd, let $S = \{(x_i, y_j) \mid i = 1, 2, ..., r$ and j even$\}$. Clearly, S is a gs-set and $|S| = \lfloor \frac{c}{2} \rfloor \cdot r$.

If r is even and c is even, let $S = \{(x_i, y_j) \mid i$ even and $j = 1, 2, ..., c\}$. Clearly, S is a gs-set and $|S| = \frac{r \cdot c}{2}$. If r is odd, let $S = \{(x_i, y_j) \mid i$ even and $j = 1, 2, ..., c\}$. Clearly, S is a gs-set and $|S| = \lfloor \frac{r}{2} \rfloor \cdot c$. ■

Illustrations for this proof can be found in Figures 8 and 9. We next give an upper bound on the global supply number in terms of the vertex cover number.

Theorem 4.2. *A graph G with no isolates and order n with vertex cover number $\beta(G)$ has $\gamma_{gs}(G) \leq \beta(G)$.*

Proof. Let S be a $\alpha(G)$-set. Then $V \backslash S$ is a vertex cover with $|V \backslash S| = \beta(G)$ and so every edge of G has at least one end vertex in $V \backslash S$. Moreover, since G has no

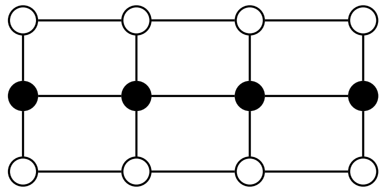

Figure 8: $\gamma_{gs}(G_{3,4}) = 4$.

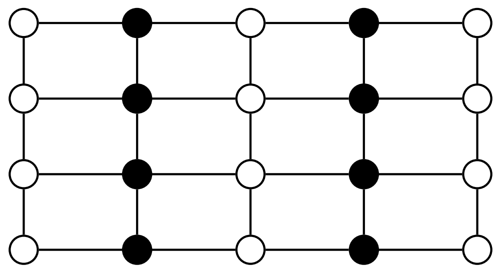

Figure 9: $\gamma_{gs}(G_{4,5}) = 8$.

isolates, every vertex in S has a neighbor in $V \backslash S$. Thus, $V \backslash S$ is a dominating set. Since S is an indepedent set, $V \backslash S$ is a gs-set and so $\gamma_{gs}(G) \leq |V \backslash S| = \beta(G)$. ■

Next we give an upper bound on the global supply number of trees.

Theorem 4.3. *For any tree T of order n, $\gamma_{gs}(T) \leq \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. Let T be a tree of order n. We proceed by induction on n. Note that if T is the star $K_{1,n-1}$, then $\gamma(K_{1,n-1}) = 1 \leq \left\lfloor \frac{n}{2} \right\rfloor$. If T is the double star $S_{r,s}$ (where $1 \leq r \leq s$), and $\gamma_{gs}(S_{r,s}) = 2 \leq \left\lfloor \frac{n}{2} \right\rfloor$. Hence, we may assume that the the diameter of T is at least 4, and so, $n \geq 5$.

Assume that any tree T' with order $n' < n$ has $\gamma_{gs}(T') \leq \left\lfloor \frac{n'}{2} \right\rfloor$. Among all leaves of T, choose r and v to be two leaves such that the distance between r and v is the diameter of T. We root the tree T at vertex r. Let w be the unique neighbor of v. Note that by our choice of v, every child of w is a leaf of T. Let $T' = T - T_w$. Then T' is a non-trivial tree of order $n' \leq n-2$. Now any $\gamma_{gs}(T')$-set S' can be extended to a gs-set of T by adding w. Hence, $\gamma_{gs}(T) \leq |S'|+1 = \gamma_{gs}(T')+1$. Now, applying our inductive hypothesis, we have $\gamma_{gs}(T) \leq \gamma_{gs}(T')+1 \leq \left\lfloor \frac{n'}{2} \right\rfloor + 1 \leq \left\lfloor \frac{n-2}{2} \right\rfloor + 1 = \left\lfloor \frac{n}{2} \right\rfloor$. ■

Theorem 4.3 is not true for graphs in general. Consider the graph in Figure 10 where a gs-set is illustrated. Thus, $\gamma_{gs}(G) \leq 8$.

To see that $\gamma_{gs}(G) \geq 8$, let S be a $\gamma_{gs}(G)$-set. First suppose that one of the C_5 subgraphs has at most two vertices in S. In this case, it follows that the vertex of degree 3, say v, in the C_5 is not dominated by vertices in the C_5. Thus, the neighbor of v not on the C_5, say u, must be in S. Since $|N(v) \cap (V \backslash S)| = 2$, we have that the two neighbors of u on the C_4 subgraph are in S. This argument holds for both C_5 subgraphs, so in this case $\gamma_{gs}(G) \geq 8$.

Therefore, assume that each of the C_5 subgraphs have three vertices in S. The best case leaves two degree two vertices in the C_4 subgraph that are not dominated. Further, at least two more vertices from the C_4 are needed for any gs-set. Again, $\gamma_{gs}(G) \geq 8$.

Observation 4.4. *The values $\gamma_t(G)$ and $\gamma_{gs}(G)$ are incomparable.*

We have already seen an example of $\gamma_{gs}(G) = \gamma_t(G)$ in Corollary 3.6. Now, notice for P_n where $n \not\equiv 0 \pmod 4$ we have $\gamma_{gs}(P_n) < \gamma_t(P_n)$. Next, consider the wheel graph where $n \geq 5$, then $\gamma_{gs}(W_{1,n-1}) = 3 > \gamma_t(W_{1,n-1}) = 2$.

Observation 4.5. *The values $\gamma_{oa}(G)$ and $\gamma_{gs}(G)$ are incomparable.*

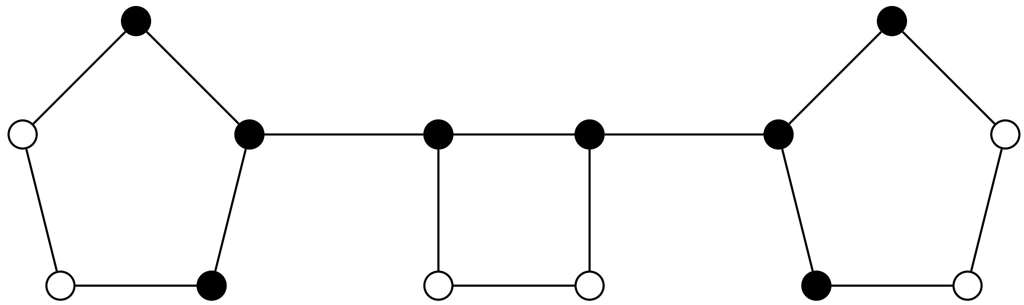

Figure 10: $\gamma_{gs}(G) > \lceil \frac{n}{2} \rceil$.

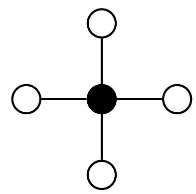

Figure 11: $\gamma_{gs}(S_{1,4}) = 1$

Consider for example, the cycle C_n, where $\gamma_{oa}(C_n) = \gamma_{gs}(C_n) = \lfloor \frac{n}{2} \rfloor$. For the graph G in Figure 3, we have $\gamma_{oa}(G) = 4$ and $\gamma_{gs}(G) = 2$; while for the graph H in Figure 4, we have $\gamma_{oa}(H) = 4$ and $\gamma_{gs}(H) = 5$.

Observation 4.6. *The values $\gamma_a(G)$ and $\gamma_{gs}(G)$ are incomparable.*

Again, we have already seen an example of $\gamma_{gs}(G) = \gamma_a(G)$ in Corollary 3.6. To see $\gamma_{gs}(G) < \gamma_a(G)$, notice that Figure 11 shows that $\gamma_{gs}(S_{1,4}) = 1$ and Figure 5 shows that $\gamma_a(S_{1,4}) = 3$. Figure 6 and Figure 12 can be used to see an example of a case where $\gamma_{gs}(G) > \gamma_a(G)$.

This is also true for trees. Figures 13 and 14 illustrate an example of the case where $\gamma_{gs}(T) > \gamma_a(T)$. If we take i copies of T and make adjacent v_i and v_{i-1} where v_i is a leaf in the i^{th} copy of T (similarly for v_{i-1}), then we get $\gamma_{gs}(T)$ is arbitrarily larger than $\gamma_a(T)$.

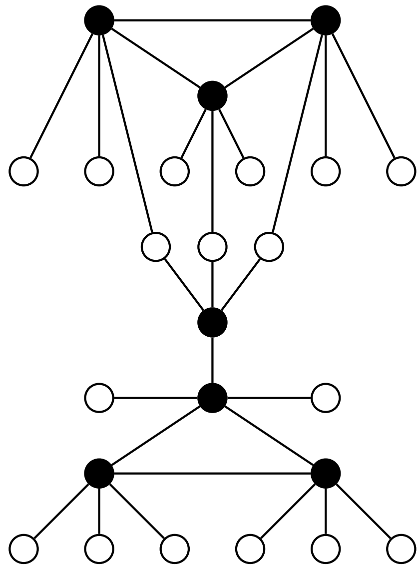

Figure 12: $\gamma_{gs}(G) = 7$.

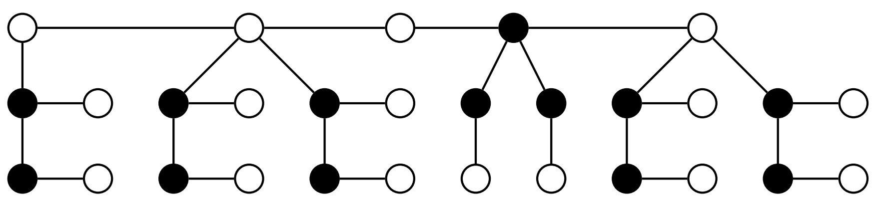

Figure 13: $\gamma_a(T) = 13$.

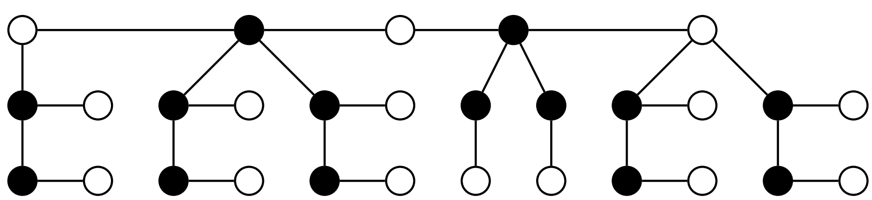

Figure 14: $\gamma_{gs}(T) = 14$.

5 CONCLUDING REMARKS

In this thesis we have introduced and studied global supply sets. Provided herein is the global supply number for several common families of graphs and upper bounds for graphs in general and two other families. Also, we demonstrated the incomparability of $\gamma_{gs}(G)$ with $\gamma_t(G)$, $\gamma_{oa}(G)$, and $\gamma_a(G)$. We conclude with a conjecture and ideas for future study.

The conjecture naturally arises from Theorem 4.1 in which we proved an upper bound on the global supply number for grid graphs $G_{r,c}$.

Conjecture 5.1. *Let the grid graph $G_{r,c}$ be such that $r \leq c$, then*

1. *If r is even and c is odd, then $\gamma_{gs}(G_{r,c}) = \lfloor \frac{c}{2} \rfloor \cdot r$*

2. *$\gamma_{gs}(G_{r,c}) = \lfloor \frac{r}{2} \rfloor \cdot c$, in all other cases.*

For graphs G in general, we often seek bounds on parameters in terms of the order n of a graph G. We saw that we did not have an upper bound on $\gamma_{gs}(G)$ as $\lceil \frac{n}{2} \rceil$. A problem of interest would be to establish a bound on the global supply number in terms of n. Similarly, it would be interesting to study bounds on the global supply number for all graphs G in terms of the minimum and maximum degree of G, $\delta(G)$ and $\Delta(G)$, respectively.

Another particular area of interest would be in classifying graphs for which $\gamma_{gs}(G) = \lceil \frac{n}{2} \rceil$. If this proves to be difficult, relaxing it to trees would be equally as intriguing. Similarly, another open problem is to classify graphs (or relaxing to trees) for which $\gamma_{gs}(G) = \gamma(G)$, $\gamma_{gs}(G) = \gamma_a(G)$, or $\gamma_{gs}(G) = \gamma_{oa}(G)$. This problem would be of great importance since bounds on $\gamma(G)$, $\gamma_a(G)$, and $\gamma_{oa}(G)$ have already been determined in previous work.

BIBLIOGRAPHY

[1] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York (1998).

[2] G. Chartrand, L. Lesniak, P. Zhang, *Graphs and Digraphs, Fifth Edition,* CRC Press, Boca Raton, FL (2011).

[3] S. M. Hedetniemi, S. T. Hedetniemi, and P. Kristiansen, Alliances in graphs. *J. Combin. Math. Combin. Comput.* 48 (2004) 157–177.

[4] T. W. Haynes, S. T. Hedetniemi, and M. A. Henning, Global defensive alliances in graphs. *The Electronic J. Combin.* 10 (2003) R47.

[5] J. A. Rodríguez-Velazquez, J. M. Sigarreta, Global offensive alliances in graphs. *Electronic Notes in Discrete Math.* 25 (2006) 157-164.

[6] M. Chellali and T. W. Haynes, Global alliances and independence in trees. *Discuss. Math. Graph Theory* 27 (2007) 19-27.

[7] S. T. Hedetniemi, Supply sets. Personal Communication (2015).

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