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Radical *p*-chains in $L_3(2)$.

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RADICAL p-CHAINS IN $L_3(2)$

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ABSTRACT

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The McKay-Alperin-Dade Conjecture, which has not been finally verified, predicts the number of complex irreducible characters in various p-blocks of a finite group G as an alternating sum of the numbers of characters in related p-blocks of certain subgroups of G. The subgroups involved are the normalizers of representatives of conjugacy classes of radical p-chains of G. For this reason, it is of interest to study radical p-chains. In this thesis, we examine the group $L_3(2)$ and determine representatives of the conjugacy classes of radical p-subgroups and radical p-chains for the primes $p = 2, 3$, and 7. We then determine the structure of the normalizers of these subgroups and chains.

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CHAPTER 1

INTRODUCTION

The McKay-Alperin-Dade Conjecture, which has not been finally verified, predicts the number of complex irreducible characters in various p-blocks of a finite group G as an alternating sum of the numbers of characters in related p-blocks of certain subgroups of G. The subgroups involved are the normalizers of representatives of conjugacy classes of radical p-chains of G. For this reason, it is of interest to study radical p-chains.

We will begin by defining some terms which will be referred to throughout the thesis, with the main definitions being that of a radical p-subgroup and radical p-chain. This will lead to some minor results concerning radical p-subgroups. Then we will look at an example group and its radical p-subgroups and radical p-chains. Next, we examine the group $L_3(2)$ and determine representatives of the conjugacy classes of radical p-subgroups and radical p-chains for the primes $p = 2, 3$, and 7. In addition, we will determine the structure of the normalizers of these subgroups and chains. Finally, we will summarize the results.

1.1 Definitions and Minor Results

We begin with some definitions. Let G be any group and p be any prime. Let $|G|$ be the order of G. We define $H \leq G$ as "H is a subgroup of G". We call H a **normal** subgroup of G if $g^{-1}hg \in H$ for all $h \in H$ and $g \in G$. From this we can say that G is a normal subgroup of itself, since $g_1^{-1}g_2g_1 \in G$ for $g_1, g_2 \in G$. We will call the product $g_1^{-1}g_2g_1$ the **conjugate** of g_2 by g_1. If $H, K \leq G$, then we define the **normalizer** of H in K as $N_K(H) = \{k \in K | k^{-1}hk \in H, \forall h \in H\}$. We will call G the **semi-direct product** of H and K, denoted $H \propto K$, if K is a normal subgroup of G, $G = HK = \{hk | h \in H, k \in K\}$, where multiplication is defined by $(h_1k_1)(h_2k_2) = h_1h_2 \underbrace{h_2^{-1}k_1h_2}_{\in K} k_2 = h_3k_3 \in HK$, and $H \cap K = \mathbf{1}$, the trivial subgroup. A **p-subgroup** of **G** is a subgroup of **G** with order $p^n, n = 0, 1, 2, \ldots$, such that p^n divides $|G|$. A **Sylow-p subgroup** of G is a p-subgroup of order p^m, where m is the largest exponent such that p^m divides $|G|$. We will use the notation $O_p(G)$ to denote the largest **normal** p-subgroup of G. We may now define a **radical** p-**subgroup** of G to be a subgroup $P \leq G$ such that $P = O_p(N_G(P))$. That is, P is the largest normal p-subgroup of its normalizer in G. For this thesis, we will use the notation $P_{p,n}$ to denote a radical p-subgroup of order p^n, with an interesting exception we will see later. A **p-chain** C of G is any non-empty, strictly increasing chain $C : P_0 < P_1 < P_2 < \cdots < P_n$ of p-subgroups P_i of G. The **stabilizer** of C in any $K \leq G$ is the "normalizer" $N_K(C) = N_K(P_0) \cap N_K(P_1) \cap \cdots \cap N_K(P_n)$. A **radical p-chain** of G is a p-chain $C : P_0 < P_1 < \cdots < P_n$ of G satisfying $P_0 = O_p(G)$ and $P_i = O_p(N_G(C_i))$ for $i = 1, \ldots, n$, where $C_i : P_0 < \cdots < P_i$.

We may now discuss some minor results. First, $N_K(H) \leq G$. If $K = G$, then $H \leq N_G(H)$, since $H \leq G$ and $h_1^{-1}h_2h_1 \in H$ for any $h_1, h_2 \in H$. In fact, H is a normal subgroup of $N_G(H)$ by the definition of a normalizer. With this information we can conclude that H is a normal subgroup of G if and only if $N_G(H) = G$. Now suppose H is a Sylow-p subgroup of G. Then H is a normal subgroup of $N_G(H)$ and, since there can be no p-subgroup larger than H, $H = O_p(N_G(H))$. Therefore, if H is a Sylow-p subgroup of G, then H is a radical p-subgroup of G. Finally, we note that the trivial subgroup of G, denoted by $\mathbf{1}$, is a p-subgroup for any prime p which divides $|G|$, since $|\mathbf{1}| = 1 = p^0$, while $N_G(\mathbf{1}) = G$.

1.2 Examples

As an example, let us examine A_5, the set of all even permutations of five elements. The order of this group is $|A_5| = 60 = 2^2 \cdot 3 \cdot 5$. Note that this group is simple. That is, the only normal subgroups are A_5 and $\mathbf{1}$. For this reason, we may conclude that $\mathbf{1}$ is a radical p-subgroup for $p = 2, 3$, and 5. It must also be noted that in each case we only need to find one representative radical p-subgroup of each conjugacy class, for the others can then be found by conjugation. That is, radical p-subgroups of the same order are in the same conjugacy class for each p, unless otherwise noted. This is true of their normalizers as well.

For $p = 2$, the 2-subgroups of A_5 have orders $2^2 = 4, 2^1 = 2$, and $2^0 = 1$. The subgroups of order 4 are the Sylow-2 subgroups. These are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. An example of such a group is $\{1, (12)(45), (14)(25), (15)(24)\}$, and we will denote these groups as $P_{2,2}$. The normalizers of these subgroups in A_5 are subgroups of A_5 isomorphic to A_4. For an example of this subgroup, consider all the even permutations of the set $\{1, 2, 4, 5\}$. That is, take all the even permutations of any four of the five elements of $\{1, 2, 3, 4, 5\}$, and you will have a subgroup of A_5 isomorphic to A_4. No subgroup of A_5 with order 4 can be isomorphic to $\mathbb{Z}_4$ because there is no element of order 4 in A_5. That is, no element of A_5 can generate a subgroup of order 4. The next subgroups we will look at have order 2, and are isomorphic to $\mathbb{Z}_2$, an example of which is $\{1, (12)(34)\}$. The normalizers of these subgroups in A_5 are the $P_{2,2}$ subgroups, so no subgroup of order 2 can be a radical 2-subgroup. The last subgroup we consider is the trivial subgroup $\mathbf{1}$, whose normalizer is $N_{A_5}(\mathbf{1}) = A_5$. No other 2-subgroups of A_5 are normal subgroups of A_5, so $\mathbf{1}$ is a radical 2-subgroup, denoted $P_{2,0}$. The radical 2-chains are then $C_{21} : P_{2,0}$, and $C_{22} : P_{2,0} < P_{2,2}$. The stabilizers of these chains are $N_{A_5}(C_{21}) = A_5$ and $N_{A_5}(C_{22}) \cong A_4$.

The p-subgroups of A_5 for $p = 3$ have orders $3^1 = 3$ and $3^0 = 1$. The subgroups of order 3 are the Sylow-3 subgroups, which we will denote $P_{3,1}$, and are isomorphic to $\mathbb{Z}_3$. An example of such a subgroup is $< (124) >= \{1, (124), (142)\}$. The normalizers in A_5 of these subgroups are isomorphic to $\mathbb{Z}_2 \propto \mathbb{Z}_3$. As an example of this subgroup, let $\mathbb{Z}_2 = \{1, (24)(35)\}$ and let $\mathbb{Z}_3$ be as above. Then $\mathbb{Z}_2 \propto \mathbb{Z}_3$ would consist of the products of the elements of $\mathbb{Z}_2$ with the elements of $\mathbb{Z}_3$. As a side note, this subgroup is also isomorphic to S_3, the set of all permutations of three elements. Now we consider the only other 3-group, the trivial subgroup $\mathbf{1}$, whose normalizer in A_5 is A_5. No other 3-subgroup is normal in A_5, so $\mathbf{1}$ is a radical 3-subgroup, denoted $P_{3,0}$. The radical 3-chains are $C_{31} : P_{3,0}$ and $C_{32} : P_{3,0} < P_{3,1}$. The stabilizers of these chains are $N_{A_5}(C_{31}) = A_5$ and $N_{A_5}(C_{32}) \cong \mathbb{Z}_2 \propto \mathbb{Z}_3$.

For $p = 5$, the 5-subgroups of A_5 have orders $5^1 = 5$ and $5^0 = 1$. The subgroups of order 5 are the Sylow-5 subgroups, denoted $P_{5,1}$, and are isomorphic to $\mathbb{Z}_5$. An example of this subgroup is

$$< (12345) >= \{1, (12345), (13524), (14253), (15432)\}.$$

The normalizer of this group in A_5 is isomorphic to $\mathbb{Z}_2 \propto \mathbb{Z}_5$. Using the above example as $\mathbb{Z}_5$, we let $\mathbb{Z}_2$ be the group $\{1, (12)(35)\}$. Then $\mathbb{Z}_2 \propto \mathbb{Z}_5$ will consist of the product of the elements of $\mathbb{Z}_2$ with the elements of $\mathbb{Z}_5$. The only other subgroup to consider is $\mathbf{1}$, with $N_{A_5}(\mathbf{1}) = A_5$. No other 5-subgroup is normal in A_5, so $\mathbf{1}$ is a radical 5-subgroup, denoted $P_{5,0}$. The radical 5-chains are then $C_{51} : P_{5,0}$ and $C_{52} : P_{5,0} < P_{5,1}$. The stabilizers of these chains are $N_{A_5}(C_{51}) = A_5$ and $N_{A_5}(C_{52}) \cong \mathbb{Z}_2 \propto \mathbb{Z}_5$.

Table 1: Radical p-chain Summary for A_5

p	Radical p-subgroups	Normalizers	Radical p-chains	Stabilizers
2	$P_{2,2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$N_{A_5}(P_{2,2}) \cong A_4$	$C_{21} : P_{2,0}$	$N_{A_5}(C_{21}) = A_5$
	$P_{2,0} = 1$	$N_{A_5}(P_{2,0}) = A_5$	$C_{22} : P_{2,0} < P_{2,2}$	$N_{A_5}(C_{22}) \cong A_4$
3	$P_{3,1} \cong \mathbb{Z}_3$	$N_{A_5}(P_{3,1}) \cong \mathbb{Z}_2 \propto \mathbb{Z}_3$	$C_{31} : P_{3,0}$	$N_{A_5}(C_{31}) = A_5$
	$P_{3,0} = 1$	$N_{A_5}(P_{3,0}) = A_5$	$C_{32} : P_{3,0} < P_{3,1}$	$N_{A_5}(C_{32}) \cong \mathbb{Z}_2 \propto \mathbb{Z}_3$
5	$P_{5,1} \cong \mathbb{Z}_5$	$N_{A_5}(P_{5,1}) \cong \mathbb{Z}_2 \propto \mathbb{Z}_5$	$C_{51} : P_{5,0}$	$N_{A_5}(C_{51}) = A_5$
	$P_{5,0} = 1$	$N_{A_5}(P_{5,0}) = A_5$	$C_{52} : P_{5,0} < P_{5,1}$	$N_{A_5}(C_{52}) \cong \mathbb{Z}_2 \propto \mathbb{Z}_5$

CHAPTER 2

THE GROUP $L_3(2)$

We will begin our examination of $L_3(2)$ by discussing some properties of this group. Most of this information is provided by the Atlas of Finite Groups [1]. First, $L_3(2)$ is the group of invertible three by three matrices whose entries come from a field of order two. We will represent an element of this group by a matrix whose entries are either 1 or 0. For example, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in L_3(2)$. The order of this group is $|L_3(2)| = 168 = 2^3 \cdot 3 \cdot 7$.

$L_3(2)$ is a simple group, just as A_5 is. This means, of course, that the trivial subgroup $\mathsf{1}$ is a radical p-subgroup for $p = 2, 3$, and 7, with normalizer $N_{\mathsf{L_3(2)}}(\mathsf{1}) = L_3(2)$.

There are three types of maximal subgroups in $L_3(2)$, that is, there are no subgroups of $L_3(2)$ which contain them as subgroups. One such subgroup is isomorphic to $\mathbb{Z}_3 \propto \mathbb{Z}_7$. The other two are isomorphic to S_4. However, these two subgroups are not conjugate. This means that all the elements of one of these S_4 subgroups cannot be found by conjugating the elements from the other S_4 subgroup by the same element.

2.1 Radical 7-chains

We will now determine the radical 7-chains of $L_3(2)$, as well as their stabilizers, by proving the following theorem.

Theorem 2.1 The radical 7-subgroups of $L_3(2)$ are $P_{7,0} = \mathsf{1}$ and $P_{7,1} \cong \mathbb{Z}_7$. The radical 7-chains of $L_3(2)$ are $C_{71} : P_{7,0}$, and $C_{72} : P_{7,0} < P_{7,1}$. The stabilizers are $N_{\mathsf{L_3(2)}}(C_{71}) = L_3(2)$ and $N_{\mathsf{L_3(2)}}(C_{72}) \cong \mathbb{Z}_3 \propto \mathbb{Z}_7$.

Proof: The only possible radical 7-subgroups must have order seven or order one. The subgroups of order seven are the Sylow-7 subgroups, which we have already determined to be radical 7-subgroups. The only subgroup of order one is the trivial subgroup $\mathbf{1}$. We have already concluded that $\mathbf{1}$ is a radical 7-subgroup with $N_{L_3(2)}(\mathbf{1}) = L_3(2)$. Thus it is clear that $C_{71} : P_{7,0} = \mathbf{1}$ is a radical 7-chain. The stabilizer of this chain is $N_{L_3(2)}(C_{71}) = N_{L_3(2)}(\mathbf{1}) = L_3(2)$.

It is only left to determine the structure of the Sylow-7 subgroups and their normalizers. Since the order of the Sylow-7 subgroups is seven, a prime, they must all be isomorphic to $\mathbb{Z}_7$. To demonstrate this we need only to find one example of such a group, and the others may be found by conjugation. Consider the matrix $A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \in L_3(2)$. The group it generates is

$$< A_1 >= \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$

which has order seven, and must be isomorphic to $\mathbb{Z}_7$. Thus $P_{7,1} \cong \mathbb{Z}_7$.

We will now use $< A_1 >$ to determine $N_{L_3(2)}(P_{7,1})$. Consider $A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in L_3(2)$. This element has order three, that is, this element generates a subgroup of order three. It is simple to check that $A_2, A_2^{-1} \in N_{L_3(2)}(< A_1 >)$. In particular,

$$A_2 : A_1 \to (A_1)^4 \to (A_1)^2 \to A_1,$$

$$A_2 : (A_1)^3 \to (A_1)^5 \to (A_1)^6 \to (A_1)^3,$$

and sends the identity element to itself under conjugation. By this we can tell that a subgroup isomorphic to $\mathbb{Z}_3 \propto \mathbb{Z}_7$ is contained in $N_{L_3(2)}(< A_1 >)$. Since $\mathbb{Z}_3 \propto \mathbb{Z}_7$ is a maximal subgroup structure in $L_3(2)$, either $N_{L_3(2)}(< A_1 >) \cong \mathbb{Z}_3 \propto \mathbb{Z}_7$, or $N_{L_3(2)}(< A_1 >) = L_3(2)$. However, $L_3(2)$ is a simple group, which means $< A_1 >$ is not a normal subgroup of $L_3(2)$. Thus $N_{L_3(2)}(< A_1 >) \neq L_3(2)$, and $N_{L_3(2)}(< A_1 >) \cong \mathbb{Z}_3 \propto \mathbb{Z}_7$. Since $< A_1 >$ is a representative of the conjugacy class of radical all 7-subgroups of $L_3(2)$, we can say $N_{L_3(2)}(P_{7,1}) \cong \mathbb{Z}_3 \propto \mathbb{Z}_7$, for any radical 7-subgroup $P_{7,1}$ of $L_3(2)$.

This gives us the radical 7-chain $C_{72} : P_{7,0} < P_{7,1}$, with stabilizer $N_{L_3(2)}(C_{72}) = N_{L_3(2)}(\mathbf{1}) \cap N_{L_3(2)}(P_{7,1}) = N_{L_3(2)}(P_{7,1}) \cong \mathbb{Z}_3 \propto \mathbb{Z}_7$. 2

2.2 Radical 3-chains

We will prove the following theorem for the radical 3-chains of $L_3(2)$:

Theorem 2.2 The radical 3-subgroups of $L_3(2)$ are $P_{3,0} = \mathbf{1}$ and $P_{3,1} \cong \mathbb{Z}_3$. The radical 3-chains of $L_3(2)$ are $C_{31} : P_{3,0}$ and $C_{32} : P_{3,0} < P_{3,1}$. The stabilizers are $N_{L_3(2)}(C_{31}) = L_3(2)$ and $N_{L_3(2)}(C_{32}) \cong S_3$.

Proof: The only possible radical 3-subgroups must have order three or order one. The subgroups of order three are the Sylow-3 subgroups, which we have already determined to be radical 3-subgroups. The only subgroup of order one is the trivial subgroup $\mathbf{1}$. We have already concluded that $\mathbf{1}$ is a radical 3-subgroup with $N_{L_3(2)}(\mathbf{1}) = L_3(2)$. Thus it is clear that $C_{31} : P_{3,0} = \mathbf{1}$ is a radical 3-chain. The stabilizer of this chain is $N_{L_3(2)}(C_{31}) = N_{L_3(2)}(\mathbf{1}) = L_3(2)$.

Of course, it only remains to determine the structure of the Sylow-3 subgroups and their normalizers. Since the order of the Sylow-3 subgroups is three, a prime, the subgroups must be isomorphic to $\mathbb{Z}_3$. As proof, consider the element $A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. The subgroup it generates is

$$< A_3 >= \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$

which has order three and is isomorphic to $\mathbb{Z}_3$. Thus $P_{3,1} \cong \mathbb{Z}_3$.

To determine the structure of $N_{\mathsf{L}_3(2)}(P_{3,1})$, we will use $< A_3 >$. Consider the element $A_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. This element has order two, and $A_4 \in N_{\mathsf{L}_3(2)}(< A_3 >)$ since $A_4 : A_3 \rightarrow (A_3)^2 \rightarrow A_3$ by conjugation. This gives us a subgroup isomorphic to $\mathbb{Z}_2 \propto \mathbb{Z}_3 \cong S_3$. We will show that this is, in fact, equal to $N_{\mathsf{L}_3(2)}(< A_3 >)$.

To show this, we must look at the **centralizer** of A_3, which is the subgroup $C(A_3) = \{A \in L_3(2) | AA_3 = A_3A\}$. According to [1], the order of its centralizer is $|C(A_3)| = 3$. This means $C(A_3) =< A_3 >$.

Consider an element of order seven. In order for it to be in $N_{\mathsf{L}_3(2)}(< A_3 >)$, it must fix each element of $< A_3 >$ by conjugation. This is because of its odd order and the fact that there are only two non-trivial elements in $< A_3 >$. In other words, this element must be in the centralizer of A_3. Since this is not the case, no element of order seven can be in $N_{\mathsf{L}_3(2)}(< A_3 >)$. The same argument can be made for elements of order three which are not in $< A_3 >$.

Now consider an element of order four. In order for it to be in $N_{L_3(2)}(< A_3 >)$, its square must fix each element of $< A_3 >$ by conjugation. The square of an order four element is an element of order two. Thus, for an element of order four to be in $N_{L_3(2)}(< A_3 >)$, its square must be in $C(A_3)$. Again, this is not the case, so no element of order four can be in $N_{L_3(2)}(< A_3 >)$. This case rules out the possibility of $N_{L_3(2)}(< A_3 >) \cong S_4$.

Finally, we note that S_3 is a maximal subgroup structure of S_4. Since we have ruled out S_4 and all elements of order seven, we can conclude that $N_{L_3(2)}(< A_3 >) \cong S_3$. Since $< A_3 >$ is a representative of the conjugacy class of all radical 3-subgroups of $L_3(2)$, we can say $N_{L_3(2)}(P_{3,1}) \cong S_3$.

Hence we have the radical 3-chain $C_{32} : P_{3,0} < P_{3,1}$, which has stabilizer $N_{L_3(2)}(C_{32}) = N_{L_3(2)}(1) \cap N_{L_3(2)}(P_{3,1}) = N_{L_3(2)}(P_{3,1}) \cong S_3$. 2

2.3 Radical 2-chains

We will now determine the radical 2-chains for $L_3(2)$. In the following theorem, note that there are two conjugacy classes for the radical 2-subgroups of order 4.

Theorem 2.3 The radical 2-subgroups of $L_3(2)$ are $P_{2,0} = L_3(2)$, $P_{2,2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2(\mathrm{I})$, $P'_{2,2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2(\mathrm{II})$, and $P_{2,3} \cong \mathbb{Z}_2 \propto \mathbb{Z}_4 \cong D_4$. The radical 2-chains are $C_{21} : P_{2,0}$, $C_{22} : P_{2,0} < P_{2,2}$, $C_{23} : P_{2,0} < P'_{2,2}$, $C_{24} : P_{2,0} < P_{2,2} < P_{2,3}$, $C_{25} : P_{2,0} < P'_{2,2} < P_{2,3}$, and $C_{26} : P_{2,0} < P_{2,3}$. The stabilizers are $N_{L_3(2)}(C_{21}) = L_3(2)$, $N_{L_3(2)}(C_{22}) \cong S_4(\mathrm{I})$, $N_{L_3(2)}(C_{23}) \cong S_4(\mathrm{II})$, $N_{L_3(2)}(C_{24}) \cong D_4$, $N_{L_3(2)}(C_{25}) \cong D_4$, and $N_{L_3(2)}(C_{26}) \cong D_4$. We use (I) and (II) to denote the two non-conjugate elementary abelian 2-subgroups of order 4 and their respective normalizers.

Proof: The only possible radical 2-subgroups of $L_3(2)$ have order $2^3 = 8$, $2^2 = 4$, $2^1 = 2$, and $2^0 = 1$. The only subgroup of order one is the trivial subgroup $\mathbf{1}$, which is a radical 2-subgroup. This, of course, gives us the radical 2-chain $C_{21} : P_{2,0}$ with stabilizer $N_{L_3(2)}(C_{21}) = N_{L_3(2)}(P_{2,0}) = L_3(2)$.

The radical 2-subgroups of order eight are the Sylow-2 subgroups. They are isomorphic to $\mathbb{Z}_2 \propto \mathbb{Z}_4 \cong D_4$, a dihedral subgroup. To demonstrate such a group let us consider the elements $A_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, which has order 2, and $A_6 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, which has order 4. Then $< A_5 > \cong \mathbb{Z}_2$ and $< A_6 > \cong \mathbb{Z}_4$. We get $\mathbb{Z}_2 \propto \mathbb{Z}_4$ by taking products of elements from these two groups. In this case, the group is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

and is isomorphic to $\mathbb{Z}_2 \propto \mathbb{Z}_4 \cong D_4$. Thus $P_{2,3} \cong D_4$.

Before we can determine the normalizer of this group, we must first determine possible radical 2-subgroups of order 4. Using our D_4 subgroup as a guide, we can find two possible structures. One is isomorphic to $\mathbb{Z}_4$, an example of which is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

This group is $< A_6 >$, and is a subgroup of our D_4 subgroup.

Any other possible subgroup is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, two examples of which are

$$H_1 = \left\{ \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : b, c \in \mathsf{F}_2 \right\}$$

and

$$H_2 = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : a, b \in \mathsf{F}_2 \right\}$$

where F_2 denotes a field of order two. Both are subgroups of our D_4 subgroup, and with tedious calculations it can be shown that they are not conjugate.

As an example of such a calculation, consider $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in H_2$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \in H_1$.

For a matrix $\beta = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ to send A to D under conjugation, it must be true that

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

or

$$\begin{bmatrix} a & a+b & c \\ d & d+e & f \\ g & g+h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d+g & e+h & f+i \\ g & h & i \end{bmatrix}.$$

Under this condition and the condition that $\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = 1$, we get $a = g = i = 0$, $c = d = h = 1$, and our matrix becomes $\beta = \begin{bmatrix} 0 & b & 1 \\ 1 & e & f \\ 0 & 1 & 0 \end{bmatrix}$.

Now consider $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in H_2$, and $E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in H_1$. In order to send B to E under conjugation by β, it must be true that $\beta B = E\beta$, or

$$\begin{bmatrix} 0 & b & 1 \\ 0 & e & 1+f \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1+b & 1 \\ 0 & 1+e & f \\ 0 & 1 & 0 \end{bmatrix}.$$

This is impossible. In order to send B to F under conjugation by β, it must be true that $\beta B = F\beta$, or

$$\begin{bmatrix} 0 & b & 1 \\ 1 & e & 1+f \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1+b & 1 \\ 0 & e & f \\ 0 & 1 & 0 \end{bmatrix}.$$

This is impossible as well. In this way, we determine that it is not possible to conjugate the elements of H_2 by β and get elements of H_1.

We now refer to Proposition 1.48(iv), page 40-41, in The Classification of Finite Simple Groups [2], which states, "If X is a group with dihedral Sylow 2-subgroup S, then we have ... According as $|S| = 4$ or $|S| > 4$, X has one or two conjugacy classes of four-subgroups." Since the Sylow-2 subgroups are dihedral with order greater than four, there are two conjugacy classes of 4-subgroups. This confirms our calculations. We can conclude that these two conjugacy classes have subgroups isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, since both H_1 and H_2 are both isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, yet are not conjugate. We will denote the conjugacy class which contains H_1 as $\mathbb{Z}_2 \times \mathbb{Z}_2$(I) and the class which contains H_2 as $\mathbb{Z}_2 \times \mathbb{Z}_2$(II).

We will now look at the normalizers of these groups, beginning with H_1. It can be verified that A_6, the element of order four in D_4, is in $N_{\mathsf{L}_3(2)}(H_1)$, so we can conclude that $D_4 \leq N_{\mathsf{L}_3(2)}(H_1)$. Since D_4 is a maximal subgroup structure of S_4 and S_4 is maximal in $L_3(2)$, either $N_{\mathsf{L}_3(2)}(H_1) \cong D_4$ or $N_{\mathsf{L}_3(2)}(H_1) \cong S_4$. However, the element $A_7 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \notin D_4$, of order three, is an element of $N_{\mathsf{L}_3(2)}(H_1)$. Thus $N_{\mathsf{L}_3(2)}(H_1) \cong S_4$. The same holds for $N_{\mathsf{L}_3(2)}(H_2)$, using the element $A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, also of order three. Since there are two conjugacy classes of S_4 and two conjugacy classes of $\mathbb{Z}_2 \times \mathbb{Z}_2$, we can conclude $N_{\mathsf{L}_3(2)}(H_1) \cong S_4(I)$ and $N_{\mathsf{L}_3(2)}(H_2) \cong S_4(II)$.

With this information, we can now determine $N_{\mathsf{L}_3(2)}(P_{2,3})$. We know $A_7 \in N_{\mathsf{L}_3(2)}(H_1)$, but $A_7 \notin N_{\mathsf{L}_3(2)}(D_4)$, and also $A_2 \in N_{\mathsf{L}_3(2)}(H_2)$, but $A_2 \notin N_{\mathsf{L}_3(2)}(D_4)$. Hence $N_{\mathsf{L}_3(2)}(D_4) \ncong S_4$. We must conclude that $N_{\mathsf{L}_3(2)}(D_4) \cong D_4$. Thus $N_{\mathsf{L}_3(2)}(P_{2,3}) = P_{2,3}$.

Now we will determine $N_{\mathsf{L}_3(2)}(\mathbb{Z}_4)$ by finding $N_{\mathsf{L}_3(2)}(< A_6 >)$, where again A_6 is our element of order four in D_4. It can be verified that $A_5 \in N_{\mathsf{L}_3(2)}(< A_6 >)$, where A_5 has order two and is in D_4, so $D_4 \leq N_{\mathsf{L}_3(2)}(< A_6 >)$. However, no element of order three can be an element of $N_{\mathsf{L}_3(2)}(< A_6 >)$, since sending $A_6 \to (A_6)^2 \to (A_6)^3$ is impossible because $(A_6)^2$ has order two. The order of $C(A_6)$ is 4, however, which means $C(A_6) =< A_6 >$, and does not have any elements of order three. Thus $N_{\mathsf{L}_3(2)}(< A_6 >) \ncong S_4$, and we can conclude $N_{\mathsf{L}_3(2)}(< A_6 >) \cong D_4$. Hence $N_{\mathsf{L}_3(2)}(\mathbb{Z}_4) = P_{2,3}$.

The last possible radical 2-subgroups have order two, and are isomorphic to $\mathbb{Z}_2$. As an example, consider the element $A_8 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The subgroup it generates is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

which is isomorphic to $\mathbb{Z}_2$. For any element A to be in $N_{\mathsf{L}_3(2)}(< A_8 >)$, $A^{-1}A_8A$ must equal A_8, since the conjugate of the identity by any other element is itself. Thus $N_{\mathsf{L}_3(2)}(< A_8 >)=$ $C(A_8)$. Well, $|C(A_8)| = 8$ ([1]), so $N_{\mathsf{L}_3(2)}(< A_8 >) \cong D_4$. Hence $N_{\mathsf{L}_3(2)}(\mathbb{Z}_2) \cong D_4$.

Based on this information, $\mathbb{Z}_4$ and $\mathbb{Z}_2$ cannot be radical 2-subgroups. Therefore, the radical 2-subgroups are $P_{2,3} \cong D_4$, $P_{2,2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2(\mathrm{I})$, $P'_{2,2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2(\mathrm{II})$, and $P_{2,0} = L_3(2)$.

The radical 2-chains are then $C_{21}1 : P_{2,0}$, $C_{22} : P_{2,0} < P_{2,2}$, $C_{23} : P_{2,0} < P'_{2,2}$, $C_{24} : P_{2,0} < P_{2,2} < P_{2,3}$, $C_{25} : P_{2,0} < P'_{2,2} < P_{2,3}$, and $C_{26} : P_{2,0} < P_{2,3}$, with stablizers $N_{L_3(2)}(C_{21}) = L_3(2)$, $N_{L_3(2)}(C_{22}) = N_{L_3(2)}(P_{2,2}) \cong S_4(\mathrm{I})$, $N_{L_3(2)}(C_{23}) = N_{L_3(2)}(P'_{2,2}) \cong S_4(\mathrm{II})$, $N_{L_3(2)}(C_{24}) = N_{L_3(2)}(P_{2,3}) \cong D_4$, $N_{L_3(2)}(C_{25}) = N_{L_3(2)}(P_{2,3}) \cong D_4$, and $N_{L_3(2)}(C_{26}) \cong D_4$. 2

CHAPTER 3

CONCLUSIONS

The table on the next page summarizes our results.

As you can see, finding radical p-chains and their stabilizers can be an interesting undertaking. As future research, the information gathered here can be applied to the McKay-Alperin-Dade Conjecture to verify the claim for $L_3(2)$, or it could prove to be a counterexample. Only time will tell.

Table 2: Radical p-chain Summary for $L_3(2)$

p	Radical p-subgroups	Normalizers	Radical p-chains	Stabilizers
2	$P_{2,3} \cong \mathbb{Z}_2 \propto \mathbb{Z}_4 \cong D_4$	$N_{L_3(2)}(P_{2,3}) \cong D_4$	$C_{21} : P_{2,0}$	$N_{L_3(2)}(C_{21}) = L_3(2)$
	$P_{2,2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2(I)$	$N_{L_3(2)}(P_{2,2}) \cong S_4(I)$	$C_{22} : P_{2,0} < P_{2,2}$	$N_{L_3(2)}(C_{22}) \cong S_4(I)$
	$P'_{2,2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2(II)$	$N_{L_3(2)}(P'_{2,2}) \cong S_4(II)$	$C_{23} : P_{2,0} < P'_{2,2}$	$N_{L_3(2)}(C_{23}) \cong S_4(II)$
	$P_{2,0} = 1$	$N_{L_3(2)}(P_{2,0}) = L_3(2)$	$C_{24} : P_{2,0} < P_{2,2} < P_{2,3}$	$N_{L_3(2)}(C_{24}) \cong D_4$
			$C_{25} : P_{2,0} < P'_{2,2} < P_{2,3}$	$N_{L_3(2)}(C_{25}) \cong D_4$
			$C_{26} : P_{2,0} < P_{2,3}$	$N_{L_3(2)}(C_{26}) \cong D_4$
3	$P_{3,1} \cong \mathbb{Z}_3$	$N_{L_3(2)}(P_{3,1}) \cong S_3$	$C_{31} : P_{3,0}$	$N_{L_3(2)}(C_{31}) = L_3(2)$
	$P_{3,0} = 1$	$N_{L_3(2)}(P_{3,0}) = L_3(2)$	$C_{32} : P_{3,0} < P_{3,1}$	$N_{L_3(2)}(C_{32}) \cong S_3$
7	$P_{7,1} \cong \mathbb{Z}_7$	$N_{L_3(2)}(P_{7,1}) \cong \mathbb{Z}_3 \propto \mathbb{Z}_7$	$C_{71} : P_{7,0}$	$N_{L_3(2)}(C_{71}) = L_3(2)$
	$P_{7,0} = 1$	$N_{L_3(2)}(P_{7,0}) = L_3(2)$	$C_{72} : P_{7,0} < P_{7,1}$	$N_{L_3(2)}(C_{72}) \cong \mathbb{Z}_3 \propto \mathbb{Z}_7$

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