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Explorations in the Classification of Vertices as Good or Bad.

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Explorations in the Classification of
Vertices as Good or Bad

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ABSTRACT

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For a graph G, a set S is a dominating set if every vertex in $V - S$ has a neighbor in S. A vertex contained in some minimum dominating set is called good; otherwise it is bad. A graph G has $g(G)$ good vertices and $b(G)$ bad vertices. The relationship between the order of G and $g(G)$ assigns the graph to one of four classes.

Our results include a method of classifying caterpillars. Further, we develop realizability conditions for a graph G given a triple of nonnegative integers representing $\gamma(G)$, $g(G)$, and $b(G)$, respectively, and provide constructions of graphs meeting those conditions. We define the goodness index of a vertex v in a graph G as the ratio of distinct $\gamma(G)$-sets containing v to the total number of $\gamma(G)$-sets, and provide formulas that yield the goodness index of any vertex in a given path.

DEDICATION

To my husband Rhydon.

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I would like to take this space to thank several people who have been most helpful and encouraging throughout the preparation of this thesis. Naturally, Teresa Haynes tops the list. Two years ago I could not have imagined that such a professor existed. Her support and enthusiasm (not to mention brain power) must be experienced to be believed. Pete Slater also receives my heartfelt thanks. Through conversations with this esteemed mathematician and through his writing, I have been the beneficiary of much inspiration and encouragement. I would also like to express my appreciation for my family, both the Staskos and the Jacksons, whose support has made this process far easier for me.

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CHAPTER 1

INTRODUCTION

Many situations worthy of study may be modeled quite effectively by graphs. Applications abound in areas such as management, the social sciences, and DNA research, to name a few. In this thesis, we are inspired by questions raised by such applications. We begin by presenting an elementary overview of graph theory with some specialized definitions the reader will find useful.

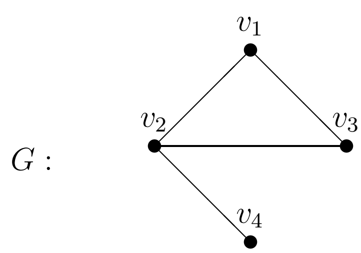

Figure 1: An example of a graph G.

A *graph* G is a set of objects, or *vertices* $V(G)$, together with a subset of $(V \times V)$, or *edges* $E(G)$, each element of which represents a specified relation between two vertices. In this thesis we assume that $E(G)$ is symmetric but not reflexive. Usually the vertices of a graph are illustrated as points and the edges as appropriately placed line segments. Figure 1 shows a graph G with $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{v_1v_2, v_1v_3, v_2v_3, v_2v_4\}$.

The number of vertices of a graph is its *order*, denoted n, and the number of edges its *size*, denoted m. If a graph G has $n = 1$, then $m = 0$ and G is called the *trivial graph*. For the most part we concern ourselves with nontrivial graphs.

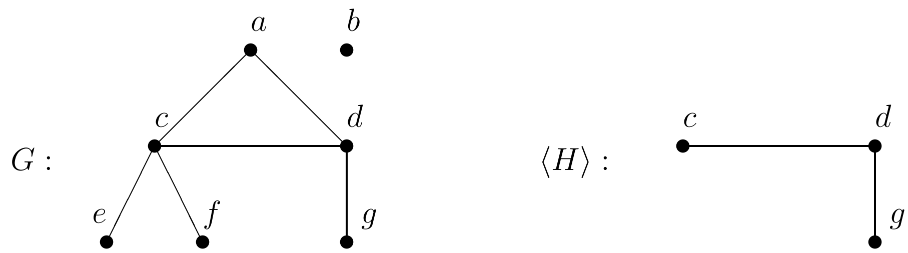

Figure 2: A graph G and an induced subgraph $\langle H \rangle$ where $H = \{c, d, g\}$.

If $uv \in E(G)$ for some graph G, then we say u is *adjacent* to v, and vice versa. The number of distinct vertices to which a vertex v is adjacent is its *degree*, denoted $deg(v)$. If $deg(v) = 0$, then v is called an *isolate*. If $deg(v) = 1$, then v is called an *endvertex*. A vertex that is adjacent to at least one endvertex is a *support vertex*. In particular we say that if a vertex is adjacent to exactly one endvertex, then it is a *weak support vertex*; but if a vertex is adjacent to more than one endvertex, it is a *strong support vertex*. In Figure 2 for the graph G, the vertex b is an isolate since $deg(b) = 0$ and g is an endvertex since $deg(g) = 1$. Also in G, c is a strong support vertex and d is a weak support vertex.

The set of all vertices to which a vertex v is adjacent is called the *open neighborhood of* v, denoted $N(v)$. The *closed neighborhood of* v is $N(v) \cup \{v\} = N[v]$. Let $S \subseteq V(G)$ and $v \in S$. The *private neighborhood of* v *with respect to* S is the set of vertices in the closed neighborhood of v that have no other neighbors in S. This set is denoted $pn[v, S]$ and is given by $pn[v, S] = \{u : N[u] \cap S = \{v\}\}$. In Figure 2 for the graph G, $N(a) = \{c, d\}$ and $N[a] = \{a, c, d\}$. If $S = \{c, d, b\}$, then $pn[c, S] = \{e, f\}$.

Frequently it is necessary to examine an *induced subgraph* $\langle H \rangle$ of a graph G. This is defined as a set of vertices $H \subseteq V(G)$ with $u, v \in H$ adjacent if and only if u is adjacent to v in G. In Figure 2, if $H = \{c, d, g\}$, then the graph $\langle H \rangle$ is as shown.

A type of graph in which we have a special interest here are paths. A *path* P_n has $V(P_n) = \{v_1, \ldots, v_n\}$ where $e \in E(P_n)$ if and only if $e = v_i v_{i+1}$, for $1 \leq i \leq n-1$. Note that $m = n-1$. A path $(u = v_1, \ldots, v_n = v)$ is called a *u-v path*. We say that a graph G is *connected* if for every pair of vertices $u, v \in V(G)$ there exists a *u-v path*. Note that in Figure 2, we have $\langle H \rangle = P_3$. Furthermore, G is not connected since, for example, there is no b-a path in G.

Figure 3: A tree T and the complete graph on five vertices K_5.

Other types of graphs we must mention are trees and complete graphs. *Trees* are connected graphs of order n and size $m = n - 1$. Paths are examples of trees. An endvertex of a tree may be called a *leaf*. A *complete graph on n vertices*, or K_n, is a graph in which all possible edges are present, that is, a graph G where $v_i v_j \in E(G)$ for all $v_i, v_j \in V(G)$ for $i \neq j$. See Figure 3 for examples of these graphs. A *clique* is a maximal complete subgraph. In Figure 2 for the graph G, the subgraph $\langle \{a, c, d\} \rangle$ is a clique, as is the subgraph $\langle \{c, e\} \rangle$.

Now let us consider an application. Suppose the upperclass students at a college are represented as vertices of a graph. For any two students having the same major, their corresponding vertices are adjacent. Assuming the unlikely case that there are no double majors among the students, the resulting graph will be a union of disjoint complete graphs, where each complete graph represents students in the same program. But if we have a double major, then there is an edge between a vertex of a clique, where the vertex represents the double major, and every vertex of another clique. Obviously, there are quite a few possibilities regarding what this graph may look like.

Some questions this graph may help us answer are:

- If only people with the same major communicate, is it possible for a piece of news given to one person to reach all students in the class? (Is the graph connected?)

- If the graph is connected and we associate the same amount of time to each edge, how quickly can we expect the news to spread? (What is the longest of all shortest u-v paths?)

- Suppose a committee were to be formed requiring a representative from each major. Are there reasons to choose one student over another inherent in the graph?

Before we restate the last question in terms of our graph we must define a few more concepts which are central to this thesis. A *dominating set* $D \subseteq V(G)$ is a set of vertices such that every vertex in $V - D$ has a neighbor in D. We say that D *dominates* G. The smallest such set in a graph G is called a $\gamma(G)$*-set*, or simply a γ*-set* when G is clear from the context. The cardinality of a smallest dominating set of G is the *domination number* of the graph G, denoted $\gamma(G)$. We note that a graph

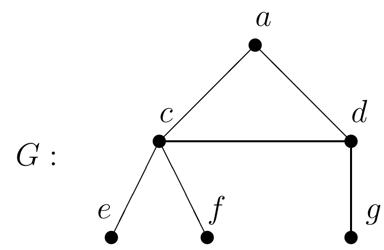

Figure 4: $D = \{e, f, d\}$ is a dominating set of G but is not a $\gamma(G)$-set.

may have many γ-sets but only one domination number. For example, in Figure 4, $\gamma(G) = 2$ where $S_1 = \{c, d\}$ and $S_2 = \{c, g\}$ are both $\gamma(G)$-sets. Also note that $\{c, a\}$ is not a $\gamma(G)$-set as $N[g] \cap \{c, a\} = \emptyset$. We note that a dominating set is minimal if and only if $pn[v, S] \neq \emptyset$ for each $v \in S$.

A vertex that is contained in some $\gamma(G)$-set is called a *good* vertex, otherwise it is *bad*. We let $g(G)$ denote the number of good vertices in a graph G. Similarly, the number of bad vertices is denoted $b(G)$. In Figure 4, $g(G) = 3$ and $b(G) = 3$, where c, d, and g are good vertices and a, e, and f are bad vertices.

With these terms defined, we return to our last question and ask: Are there committees that are preferable to others? If it were necessary to minimize the size of the committee, we would be looking for a γ-set. How many γ-sets are there from which to choose? What are other considerations that must be made when choosing such a set? For example, do we also need to choose alternates? If there is to be compensation for serving on the committee, should everyone receive the same pay? Or perhaps those whose corresponding vertices are contained in only one γ-set should be paid less than one whose corresponding vertex is contained in all γ-sets. If so, how much less?

It is these types of questions that fuel our interest in domination theory. Broadly speaking, our problem considers the set of good vertices in a graph and attempts to determine other characteristics not previously considered that may help us with applications like the one described above. Throughout this thesis we emphasize the quality of the vertex itself in relation to the graph as a whole. We hope to learn something of the nature of domination on a local scale and its effect on the graph globally.

We begin with a literature survey of some work that has either been done in this area or has served as an catalyst for new ideas in this area. In Chapter 3 we determine $g(G)$ and $b(G)$ for a family of trees known as caterpillars. In Chapter 4 we answer the question of whether, given $\gamma(G)$, there exists such a graph G with specified $g(G)$ and $b(G)$. In Chapter 5 we define a number called the goodness index for each vertex of a path, we look closely at the individual vertices of paths P_n, and we determine the number of $\gamma(P_n)$-sets that contain each vertex. We close the thesis with a short chapter listing some open questions that have arisen during the course of this study.

CHAPTER 2

LITERATURE SURVEY

In this thesis, our concern is with vertices that are contained in $\gamma(G)$-sets for a graph G. Here we present a brief survey of literature that has led to this interest.

Gunther, Hartnell, Marcus, and Rall [4] study graphs with unique minimum dominating sets. This is of special interest to us as such graphs have $\gamma(G) = g(G)$. They present the following result for trees.

Theorem 2.1 [4] *Let T be a tree of order at least three. Then the following conditions are equivalent:*

1. *T has a unique γ-set D.*

2. *T has a γ-set D for which every vertex $x \in D$ has at least two private neighbors other than itself.*

3. *T has a γ-set D for which every vertex $x \in D$ has the property that $\gamma(T - x) > \gamma(T)$.*

In addition to this result, the authors present operations involving addition of edges and/or vertices which may be performed on two connected graphs, each of which has a unique minimum dominating set, to create a new connected graph with a minimum dominating set.

Mynhardt [8] furthers their results by considering the class of all trees. Here she identifies the vertices of a tree that are contained in every minimum dominating set as well as those vertices that are contained in no minimum dominating set.

An *independent set* of vertices is one in which no two vertices are adjacent. If a dominating set of a graph G is independent it is called an $i(G)$-set, or just an i-set, if G is clear from the context. The *independent domination number* of a graph, denoted $i(G)$, is the smallest cardinality of any maximal independent set of G. Since any maximal independent set is a dominating set we can restste the definition of the independent domination number as the smallest cardinality of any independent dominating set.

Cockayne, Favaron, Mynhardt, and Puech, [2], characterize trees for which $\gamma(T) = i(T)$ in terms of the set of vertices of the tree which are contained in all its minimum dominating and minimum independent dominating sets.

Exactly how many minimum dominating sets a graph has is a challenging question. In [9], Slater provides some tools for counting dominating sets for graphs, paths in particular. Although he does not formally ask how many $\gamma(P_n)$-sets exist for a path P_n, the question seems to be implied. In Chapter 5 of this thesis, that question is answered.

Finally, we come to the work that has been the biggest impetus for this thesis. Fricke, Haynes, Hedetniemi, Hedetniemi, and Laskar [3] define a *good* vertex $v \in V(G)$ to be one which is contained in some $\gamma(G)$-set. Otherwise, the vertex is said to be *bad*. They let $g(G)$ denote the number of good vertices and $b(G)$ denote the number of bad vertices in G, and give the following classification scheme for graphs of order n:

1. G is γ-*excellent* if $g(G) = n$ and $b(G) = 0$.

2. G is γ-*commendable* if $\frac{n}{2} < g(G) < n$ and $0 < b(G) < \frac{n}{2}$.

3. G is γ-*fair* if $g(G) = \frac{n}{2}$ and $b(G) = \frac{n}{2}$.

4. G is γ-*poor* if $g(G) < \frac{n}{2}$ and $b(G) > \frac{n}{2}$.

They expand on the work of Mynhardt by concentrating on trees and consider those trees for which every vertex v is contained in some γ-set; that is, trees which are γ-excellent.

Two observations made by the authors which we will use throughout the thesis are the following.

Observation 2.2 [3] *For any connected graph $G \neq K_2$, every support vertex is γ-good and there exists a $\gamma(G)$-set containing all the support vertices of G.*

Observation 2.3 [3] *For any γ-excellent graph G, every endvertex is in some $\gamma(G)$-set and no endvertex is in every $\gamma(G)$-set of G.*

Note also that if a support vertex u is adjacent to two or more endvertices of G, then u is in every $\gamma(G)$-set, and hence the endvertices in $N(u)$ are not in any $\gamma(G)$-set.

Observation 2.4 [3] *For any γ-excellent graph G, any support vertex is adjacent to exactly one endvertex.*

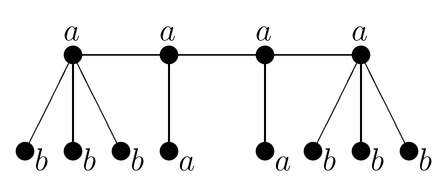

Figure 5: A γ-fair graph. The good vertices are labeled a and the bad vertices b.

Another result from the authors which is elementary to our arguments is the following.

Proposition 2.5 [3] *The path P_n is γ-excellent if and only if $P_n = P_2$ or $n \equiv 1 \pmod 3$.*

A graph G that is *i-excellent* is one in which every vertex $v \in V(G)$ is contained in some i-set. Fricke, et al. [3] relate γ-excellent and i-excellent trees as follows.

Theorem 2.6 [3] *If T is a γ-excellent tree, then $\gamma(T) = i(T)$ and T is an i-excellent tree.*

Haynes and Henning [7] give a constructive characterization for i-excellent trees. Even though all γ-excellent trees are i-excellent, note that not all γ-sets of a γ-excellent tree are independent as we see in the following lemma.

Lemma 2.7 [3] *If T is a γ-excellent tree of order $n \geq 4$, then there exists a $\gamma(T)$-set S such that S is not independent.*

Using Lemma 2.7 Fricke, et al. [3] give a construction for γ-excellent trees.

Construction $\mathcal{A}$ [3] To construct a γ-excellent tree T

(1) Let T_1 and T_2 be γ-excellent trees (each of order at least 4). By Lemma 2.7, we can assume that S_1 and S_2 are γ-sets, but not i-sets of T_1 and T_2, respectively. Further let $u \in S_1$ and $v \in S_2$ where u (respectively, v) is not an isolate in $\langle S_1 \rangle$ (respectively, $\langle S_2 \rangle$).

(2) Let $T = T_1 \cup T_2 + uv$.

These results and the questions they raise lead us to our problems. First, we want to classify all trees as γ-excellent, γ-commendable, γ-fair, or γ-poor. We examine a subclass of trees known as caterpillars in Chapter 3 and establish a method for their classification.

Second, we ask if given any triple of nonnegative integers (x, y, z) with $1 \leq x \leq y$ corresponding to $\gamma(G)$, $g(G)$, and $b(G)$, respectively, does such a graph G exist? The somewhat surprising answer is no. Conditions for realizability are given in Chapter 4, as well as constructions that validate the realizability.

Finally, we answer the question of how many distinct $\gamma(P_n)$-sets there are for a given path on n vertices, and we answer the question of how many of these distinct $\gamma(P_n)$-sets contain a particular vertex. These results are found in Chapter 5.

CHAPTER 3

CATERPILLARS

The problem of partitioning the set of all graphs into the classes γ-excellent, γ-commendable, γ-fair, and γ-poor seems to be a very difficult one. In this chapter, we will concentrate on a very simple class of trees known as caterpillars. A *caterpillar* T_c is a nontrivial tree of order $n > 2$ for which the removal of all endvertices yields a path $P_k = \{v_1, v_2, \ldots, v_k\}$, which is called the *spine* of the caterpillar. A *spine vertex* is simply a vertex on the spine of a caterpillar. The *code* of a caterpillar T_c is $c(T_c) = (c_1, c_2, \ldots, c_k)$ where c_i is the number of endvertices adjacent to the vertex v_i. Note that $c_1 \neq 0$ and $c_k \neq 0$. Also, by convention, $c_1 \geq c_k$. For an example, see Figure 6.

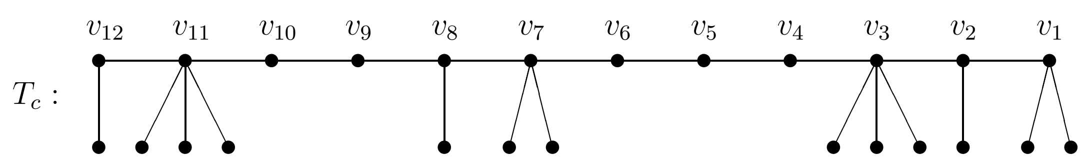

Figure 6: $c(T_c) = (2, 1, 3, 0, 0, 0, 2, 1, 0, 0, 3, 1)$

Much of the work in this thesis relies on the following.

Proposition 3.8 *For any path P_n, $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$, and if $n \equiv 0(mod\ 3)$, then P_n has a unique γ-set consisting of vertices v_i where $i \equiv 2(mod\ 3)$.*

Observation 3.9 *Every strong support vertex is in every $\gamma(G)$-set. Moreover, there exists a $\gamma(G)$-set which includes every support vertex of G. We note that every vertex v_i with $c_i \geq 2$ is in every $\gamma(T_c)$-set, and there exists a $\gamma(T_c)$-set including every vertex v_i such that $c_i \geq 1$.*

To aid in our discussion, we call a maximal sequence of zero entries in the code $c(T_c)$ a *zero string*, denoted $\mathcal{Z}_p$ where p indicates the pth such string in a code. For example, the caterpillar in Figure 6 has two zero strings, namely $\mathcal{Z}_1 = (c_4, c_5, c_6)$ and $\mathcal{Z}_2 = (c_9, c_{10})$. We let z_p denote the number of zeroes in the zero string. In the above example, $|\mathcal{Z}_1| = z_1 = 3$. Finally, for a zero string $\mathcal{Z}_p$, let $g(\mathcal{Z}_p)$ denote the number of good vertices v_i such that $c_i \in \mathcal{Z}_p$.

Determining the class of a given caterpillar relies in part on knowing the length of each zero string. We first determine the number of good vertices corresponding to codes in each zero string.

Lemma 3.10 *For any zero string $\mathcal{Z}_p$,*

$$g(\mathcal{Z}_p) = \begin{cases} \left\lfloor \frac{z_p}{3} \right\rfloor & \text{for } z_p \equiv 2 (mod\ 3) \\ z_p & \text{for } z_p \equiv 0 (mod\ 3) \\ 2\left\lfloor \frac{z_p}{3} \right\rfloor & \text{for } z_p \equiv 1 (mod\ 3). \end{cases}$$

Proof: Let T_c be a caterpillar with a zero string $\mathcal{Z}_p = (c_i, c_{i+1}, \ldots, c_{j-1}, c_j)$. Let S be a $\gamma(T_c)$-set such that S includes all vertices v_i with $c_i \geq 1$. The existence of such a set S is guaranteed by Observation 3.9. Let $S_1 \subseteq S$ be the subset of vertices with $c_i \geq 1$.

We consider the cases where $z_p = 1$ and $z_p = 2$ separately.

Let $1 \leq z_p \leq 2$. Then, $c_{i-1} \geq 1$ and $c_{j+1} \geq 1$, and Observation 3.9 implies that v_{i-1} and v_{j+1} are in S, and hence v_i and v_j (possibly $v_i = v_j$) are dominated by S_1. By the minimality of a $\gamma(T_c)$-set, v_i and v_j will never be in any $\gamma(T_c)$-set, so v_i and v_j (possibly $v_i = v_j$) are bad. Hence, $g(\mathcal{Z}_p) = 0 = 2\left\lfloor \frac{1}{3} \right\rfloor$ for $z_p = 1$ and $g(\mathcal{Z}_p) = 0 = \left\lfloor \frac{2}{3} \right\rfloor$ for $z_p = 2$.

Thus we assume $z_p \geq 3$, and let $l = z_p - 2 \geq 1$. Since $c_{i-1} > 0$ and $c_{j+1} > 0$, vertices v_i and v_j are dominated by S_1, leaving the path $P_l = \langle v_{i+1} \dots v_{j-1} \rangle$ to be dominated by vertices corresponding to codes on $\mathcal{Z}_p$.

If $z_p \equiv 2 (\text{mod } 3)$, then $l \equiv 0 (\text{mod } 3)$ and by Proposition 3.8, the path P_l has a unique γ-set with $\gamma(P_l) = \frac{l}{3}$. Note that neither v_{i+1} nor v_{j-1} is in the unique $\gamma(P_l)$-set. If $v_i \in S$, $pn[v_i, S] = \emptyset$, contradicting the minimality of S. Therefore $v_i \notin S$, and by symmetry, $v_j \notin S$. Since S is an arbitrary $\gamma(T_c)$-set, none of v_i, v_{i+1}, v_{j-1}, and v_j are in any $\gamma(T_c)$-set. Hence if $z_p \equiv 2 (\text{mod } 3)$, then $g(\mathcal{Z}_p) = \frac{l}{3} = \frac{z_p - 2}{3} = \left\lfloor \frac{z_p}{3} \right\rfloor$.

If $z_p \equiv 0 (\text{mod } 3)$, then $l \equiv 1 (\text{mod } 3)$ and by Proposition 2.5 every vertex in P_l is good with $\gamma(P_l) = \left\lceil \frac{l}{3} \right\rceil$. Moreover, if vertex $v_i \in S$, then vertices v_i, v_{i+1}, and v_j are dominated by $S_1 \cup v_i$, leaving a path P_{l-1} to be dominated. This can be accomplished by using exactly $\frac{l-1}{3}$ vertices. Note that $\left\lceil \frac{l}{3} \right\rceil = 1 + \frac{l-1}{3}$ when $l \equiv 1 (\text{mod } 3)$. Hence, v_i can be in S, that is, v_i is good. Symmetry establishes the goodness of v_j as well. Hence if $z_p \equiv 0 (\text{mod } 3)$, then $g(\mathcal{Z}_p) = z_p$.

Finally, if $z_p \equiv 1 (\text{mod } 3)$, then $l \equiv 2 (\text{mod } 3)$ and $g(P_l) = 2\left\lceil \frac{l}{3} \right\rceil$ by Corollary

5.27 with $\gamma(P_l) = \left\lceil \frac{l}{3} \right\rceil$. Note that the endvertices of P_l are good. Now, if vertex $v_i \in S$, then vertices v_i, v_{i+1}, and v_j are dominated by $S_1 \cup \{v_i\}$, leaving a path P_{l-1} to be dominated. This implies that at least $\left\lceil \frac{l-1}{3} \right\rceil$ additional vertices are in S to dominate the vertices of Z_p. But notice that $1 + \left\lceil \frac{l-1}{3} \right\rceil = 1 + \frac{l+1}{3} > \left\lceil \frac{l}{3} \right\rceil$ when $l \equiv 2(\text{mod } 3)$. Thus, $v_i \notin S$ and by symmetry, $v_j \notin S$. Hence, if $z_p \equiv 1(\text{mod } 3)$, then $g(\mathcal{Z}_p) = 2\left\lceil \frac{l}{3} \right\rceil = 2\left\lceil \frac{z_p-2}{3} \right\rceil = 2\left\lfloor \frac{z_p}{3} \right\rfloor$.

$\square$

Another consideration in determining the classification of a caterpillar is the relationship between the weak support vertices and zero strings $\mathcal{Z}_p$ where $z_p \equiv 2(\text{mod } 3)$. We will find the following lemma useful:

Lemma 3.11 *Let v_i be a spine vertex of a caterpillar T_c with $c_i = 1$. Then v_i is in every $\gamma(T_c)$-set if and only if c_{i-1} or c_{i+1} is part of a zero string $\mathcal{Z}_p$ with $z_p \equiv 2(\text{mod } 3)$.*

Proof: Let v_i be a spine vertex of a caterpillar T_c where $c_i = 1$ and u_i is the endvertex adjacent to v_i.

Suppose v_i is in every $\gamma(T_c)$-set. Thus no $\gamma(T_c)$-set includes u_i, implying that at least one of v_{i-1} and v_{i+1} is in $pn[v_i, S]$, for any $\gamma(T_c)$-set S. Suppose that there exist $\gamma(T_c)$-sets S' and S'' such that $v_{i+1} \in pn[v_i, S']$ and $v_{i-1} \notin pn[v_i, S']$ and $v_{i+1} \notin pn[v_i, S'']$ and $v_{i-1} \in pn[v_i, S'']$. Then there exists a $\gamma(T_c)$-set S such that $S = (S' \cap \{v_1, \ldots, v_{i-1}\}) \cup \{u_i\} \cup (S'' \cap \{v_{i+1}, \ldots, v_k\})$, contradicting that v_i is in every

$\gamma(T_c)$-set. Hence, either $v_{i+1} \in pn[v_i, S]$ for all $\gamma(T_c)$-sets S, or $v_{i-1} \in pn[v_i, S]$ for all $\gamma(T_c)$-sets S.

Without loss of generality, we assume that $v_{i+1} \in pn[v_i, S]$, for all $\gamma(T_c)$-sets S. Then v_{i+1} and v_{i+2} are bad, and c_{i+1} and c_{i+2} are part of a zero string. From the proof of Lemma 3.10, the only zero strings having the property that the vertices corresponding to the first two elements in the string are not in any $\gamma(T_c)$-set are those $\mathcal{Z}_p$ with $z_p \equiv 2(\text{mod } 3)$.

Conversely, we may assume without loss of generality that c_{i+1} is part of a zero string $\mathcal{Z}_p$ with $z_p \equiv 2(\text{mod } 3)$. Let S be a $\gamma(T_c)$-set such that $v_i \notin S$. Then $u_i \in S$ and either $v_{i+1} \in S$ or $v_{i+2} \in S$ to dominate v_{i+1}. But by the proof of Lemma 3.10, $v_{i+1} \notin S$ and $v_{i+2} \notin S$. Hence, $v_i \in S$ and v_i is in every $\gamma(T_c)$-set.

□

Corollary 3.12 *In a caterpillar T_c, an endvertex adjacent to a vertex v_i with $c_i = 1$ is bad if and only if c_{i-1} or c_{i+1} is part of a zero string $\mathcal{Z}_p$ with $z_p \equiv 2(mod\ 3)$.*

The above results allow us to establish formulas for finding $g(T_c)$ and $b(T_c)$ given $c(T_c) = (c_1, c_2, \ldots, c_k)$ for $k \geq 3$. Let us first define the following.

A summation which counts the good vertices on the p zero strings of T_c:

$$g_z = \sum_{i=1}^{p} g(\mathcal{Z}_i)$$

A function which assigns a one to each code element corresponding to a strong support vertex and a zero to all other code elements:

$$f_1(c_i) = \begin{cases} 1 & \text{for } c_i \geq 2 \\ 0 & \text{for } c_i \leq 1 \end{cases}$$

A summation that gives the total number of strong support vertices, all of which are good:

$$g_s = \sum_{i=1}^{k} f_1(c_i)$$

A set which contains the vertices of every zero string $\mathcal{Z}_i$, where $z_i \equiv 2(\text{mod } 3)$:

$$Z_2 = \cup\{\mathcal{Z}_i | z_i \equiv 2(\text{mod } 3)\}$$

A set function that returns the code for the entire set of spine vertices adjacent to a given spine vertex v_i, particularly v_1 and v_k:

$$R_i = \begin{cases} \{c_{i+1}\} & \text{for } i = 1 \\ \{c_{i-1}, c_{i+1}\} & \text{for } 1 < i < k \\ \{c_{i-1}\} & \text{for } i = k \end{cases}$$

A function that assigns to each code element c_i corresponding to a weak support vertex a one or a two, depending on whether the endvertex adjacent to v_i is included in any $\gamma(T_c)$-set:

$$f_2(c_i) = \begin{cases} 0 & \text{for } c_i \neq 1 \\ 1 & \text{for } c_i = 1 \text{ and } R_i \cap Z_2 \neq \emptyset \\ 2 & \text{for } c_i = 1 \text{ and } R_i \cap Z_2 = \emptyset \end{cases}$$

A summation that gives the total number of weak support vertices, each of which is good, along with the total number of good endvertices adjacent to these:

$$g_w = \sum_{i=1}^{k} f_2(c_i)$$

A summation that gives the total number of vertices in the caterpillar:

$$n = k + \sum_{i=1}^{k} c_i$$

With the above defined for caterpillars with at least two spine vertices, the formulas are given as follows:

$$g(T_c) = g_z + g_s + g_w$$

$$b(T_c) = n - g(T_c)$$

Now we can easily classify the caterpillars T_c in the following manner.

T_c is γ-excellent if $g(T_c) = g_z + g_s + g_w = n$ and $b(T_c) = 0$.

T_c is γ-commendable if $\frac{n}{2} < g(T_c) = g_z + g_s + g_w < n$ and $0 < b(T_c) < \frac{n}{2}$.

T_c is γ-fair if $g(T_c) = g_z + g_s + g_w = \frac{n}{2}$ and $b(T_c) = \frac{n}{2}$.

T_c is γ-poor if $g(T_c) = g_z + g_s + g_w < \frac{n}{2}$ and $b(T_c) > \frac{n}{2}$.

As an example, we can return to the caterpillar shown in Figure 6. We calculate:

$$g_z = \sum_{i=1}^{p} g(\mathcal{Z}_i) = 3$$

$$g_s = \sum_{i=1}^{k} f_1(c_i) = 4$$

$$g_w = \sum_{i=1}^{k} f_2(c_i) = 5$$

$$n = k + \sum_{i=1}^{k} c_i = 12 + 13 = 25$$

$$g(T_c) = g_z + g_s + g_w = 3 + 4 + 5 = 12$$

$$b(T_c) = n - g(T_c) = 25 - 12 = 13$$

Hence, our caterpillar is γ-poor.

CHAPTER 4

REALIZABILITY

Our aim in this chapter is to determine for which triples (x, y, z) there exists a graph G such that $\gamma(G) = x$, $g(G) = y$, and $b(G) = z$. We observe that no graph G has $\gamma(G) = g(G) = b(G)$.

Observation 4.13 *No graph G has $\gamma(G) = g(G) = b(G)$.*

Proof. Suppose there exists a graph G of order n with $\gamma(G) = g(G) = b(G) = \frac{n}{2}$. However, it is established [3] that the only graphs G with $\gamma(G) = \frac{n}{2}$ are also γ-excellent graphs. Thus, $\gamma(G) = \frac{n}{2}$, $b(G) = 0$, and $g(G) = n$, contradicting our assumption that such a graph exists.

□

Note that $g(G) \geq \gamma(G)$ for all graphs G. Our next theorem establishes a bound on $\gamma(G)$ for a given $g(G)$ and $b(G)$.

Theorem 4.14 *If G is a connected graph with $g(G) = \gamma(G)+k$ and $b(G) = \gamma(G)+j$, where j and k are nonnegative integers such that $j + k \geq 1$, then $\gamma(G) \leq j + 2k$.*

Proof. Let G be a connected graph with $g(G) = \gamma(G) + k$ and $b(G) = \gamma(G) + j$, where $j + k \geq 1$. Let S be a $\gamma(G)$-set.

To prove the theorem, we first show that every vertex of S, except possibly k of them, has at least two private neighbors in $V - S$. Obviously, the $\gamma(G)$ vertices of S

are good vertices. Let A_k denote the set of k good vertices in $V - S$, and let $A \subseteq S$ such that for each vertex $u \in A$, u has at most one private neighbor in $V - S$. If $A = \emptyset$, then we are finished, so assume that $A \neq \emptyset$.

If $u \in A$ has no private neighbors in $V - S$, then u is an isolate in $\langle S \rangle$ by the minimality of S. Since G is connected, u must have a neighbor x in $V - S$. Furthermore, since $S - \{u\} \cup \{x\}$ is a $\gamma(G)$-set, it follows that x is a good vertex for all $x \in N(u)$. Thus, $N(u) \subseteq A_k$. Moreover, suppose $v \neq u \in A$ has no private neighbor in $V - S$. If u and v have a common neighbor, say x, in A_k, then $S - \{u, v\} \cup \{x\}$ is a dominating set of G with cardinality less than $\gamma(G)$, a contradiction. Hence, for each pair of vertices $u, v \in A$ with no private neighbors in $V - S$, $N(u) \cap N(v) = \emptyset$.

If, on the other hand, $u \in A$ has exactly one private neighbor x in $V - S$, then $S - \{u\} \cup \{x\}$ is a $\gamma(G)$-set, implying that $x \in A_k$.

We have just shown that for each vertex in A we can associate a unique vertex in A_k. Thus, it follows that $|A| \leq |A_k| = k$. (Note that these associated neighbors cannot be private neighbors of the vertices of $S - A$.)

Let $|A| = t$. Since every vertex in $S - A$ has at least two private neighbors in $V - S$, it follows that

$$\begin{aligned} 2(\gamma(G) - t) &\leq |V - S| - t \\ &= b(G) + g(G) - \gamma(G) - t \end{aligned}$$

$$= \gamma(G) + j + k - t.$$

Thus, $\gamma(G) \leq j + k + t$ and since $t \leq k$, we have

$$\gamma(G) \leq j + 2k.$$

□

Corollary 4.15 *If G is a connected graph and $g(G) = \gamma(G) + k$, then $b(G) \geq 2(\gamma(G) - k)$.*

Proof. Let G be a connected graph with $g(G) = \gamma(G) + k$. Let S be a $\gamma(G)$-set where t vertices of S have at most one private neighbor in $V - S$. Then from the proof of Theorem 4.14, we have

$$\begin{aligned} 2(\gamma(G) - t) &\leq |V - S| - t \\ &= b(G) + k - t \end{aligned}$$

implying $b(G) \geq 2\gamma(G) - t - k$.

Since $t \leq k$, we have

$$b(G) \geq 2(\gamma(G) - k).$$

□

Note that the bound in Corollary 4.15 is only meaningful for $k < \gamma(G)$ since $b(G) \geq 0$ for any graph G. We also observe that if G is a nontrivial graph and $k = 0$, then $b(G) \geq 2\gamma(G)$.

4.1 Constructions

We have established that for a nontrivial connected graph G, the only possibilities are

Property 1 $g(G) = \gamma(G) + k$, for $0 \le k < \gamma(G)$ and $b(G) \ge 2(\gamma(G) - k)$, and

Property 2 $g(G) = \gamma(G) + k$, for $k \ge \gamma(G)$ and $b(G) \ge 0$.

We recall the following result.

Observation 4.16 *If u is an endvertex of G, then either u or its support vertex is in any $\gamma(G)$-set. A strong support vertex is in every $\gamma(G)$-set. Moreover, there exists a $\gamma(G)$-set that contains all the support vertices of G.*

Our first theorem shows that graphs with Property 1 are realizable.

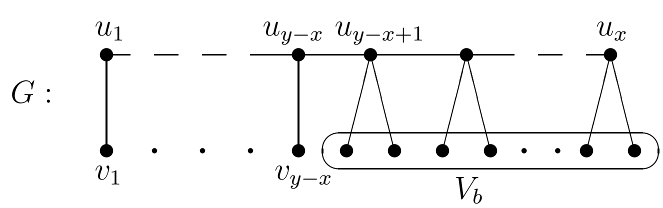

Figure 7: Construction of G as described in the proof of Theorem 4.17.

Theorem 4.17 *For nonnegative integers (x, y, z) with $x \le y < 2x$ and $z \ge 2(2x-y)$, there exists a connected graph G such that $\gamma(G) = x$, $g(G) = y$ and $b(G) = z$.*

Proof: Let (x, y, z) be nonnegative integers with $x \leq y < 2x$ and $z \geq 2(2x - y)$. We construct a graph G as follows:

Begin with a path $P_x = u_1, u_2, \ldots, u_x$. If $y > x$, then add a new vertex v_i and edge $u_i v_i$ for $1 \leq i \leq y - x$. For each u_i, $y - x < i \leq x$, we add at least two new vertices adjacent to u_i while requiring that the total number of those new vertices is z. Note that this construction is possible since $x \leq y < 2x$ and $z \geq 2(2x - y)$. In Figure 7, this set of vertices is denoted V_b.

Then G is a caterpillar, and each u_i, $1 \leq i \leq x$, is a support vertex. Observation 4.16 implies that there is a $\gamma(G)$-set containing all the vertices of P_x, so $\gamma(G) \geq x$. And since $\{u_i \mid 1 \leq i \leq x\}$ dominates G, we have $\gamma(G) \leq x$. Thus, $\gamma(G) = x$ as desired. Also, it follows that the vertices of P_x are good. From Observation 4.16, we know that no leaf adjacent to a strong support vertex is in any $\gamma(G)$-set, implying that the z endvertices adjacent to the strong support vertices u_i, for $y - x < i \leq x$, are bad. Moreover, for $1 \leq i \leq y - x$, each u_i is adjacent to exactly one endvertex v_i, so $P_x - \{u_i\} \cup \{v_i\}$ is a $\gamma(G)$-set. Therefore, $\{v_i \mid 1 \leq i \leq y - x\}$ is a set of good vertices. Hence, $g(G) = x + y - x = y$, and $b(G) = z$ as desired.

□

Before showing that graphs with property 2 are realizable, we establish a lemma and define two additional terms.

Lemma 4.18 *If G is a connected graph and $b(G) = 1$, then $\gamma(G) \geq 2$.*

Proof: Let G be a connected graph of order n with $b(G) = 1$. Then $g(G) = n - 1$. Let v denote the bad vertex of G, and let u_i for $1 \leq i \leq n-1$, denote the good vertices of G. Suppose to the contrary that $\gamma(G) = 1$. Thus each u_i is a $\gamma(G)$-set because each u_i is good. Therefore G is complete. But then v dominates G, so v is good, a contradiction. Hence, $\gamma(G) \geq 2$.

□

The *join* operation on two graphs G and H, denoted $G + H$, creates a new graph with $E(G+H) = \{e_G \,|\, e_G \in E(G)\} \cup \{e_H \,|\, e_H \in E(H)\} \cup \{uv \,|\, u \in V(G) \text{ and } v \in V(H)\}$. The *subdivision* operation on an edge uv of a graph G introduces a new vertex w such that $E(G \cup \{w\}) = E(G) - \{uv\} \cup \{uw, wv\}$.

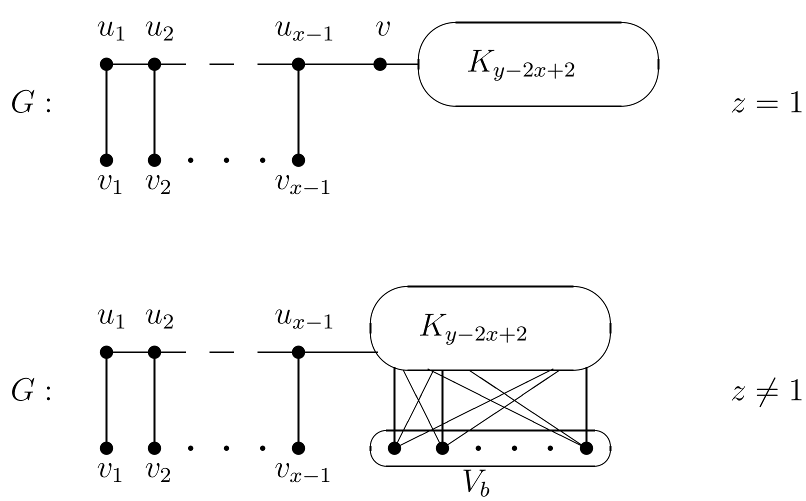

Figure 8: Construction of G as described in the proof of Theorem 4.19.

Theorem 4.19 *For the triple (x, y, z) of nonnegative integers, where $2 \leq 2x \leq y$ and*

$z = 1$ *only if* $x \geq 2$, *there exists a connected graph* G *such that* $\gamma(G) = x$, $g(G) = y$ *and* $b(G) = z$.

Proof: Let (x, y, z) be a triple of nonnegative integers where $2 \leq 2x \leq y$ and $z = 1$ only if $x \geq 2$. We construct a graph G as follows:

Begin with a complete graph K_{y-2x+2}. If $z \neq 1$, let V_b be a set of isolates where $|V_b| = z$ and join V_b to K_{y-2x+2}. (Note that if $z = 0$, we have $V_b = \emptyset$.) Now, $x \geq 1$ since $z \neq 1$, by Lemma 4.18. If $x = 1$, then we may stop our construction here and note that any one of the vertices in K_{y-2x+2} dominates $G = K_{y-2x+2} + V_b$. Thus, $\gamma(G) = 1 = x$ as desired. Further, no vertex in V_b dominates G, so $g(G) = y-2x+2 = y-2(1)+2 = y$ and $b(G) = z$.

Next, let $x \geq 2$ with $z \neq 1$. Introduce a path $P_{x-1} = u_1, u_2, \ldots, u_{x-1}$ into the graph constructed above with edge $u_{x-1}w$, for some $w \in V(K_{y-2x+2})$. For each vertex u_i, $1 \leq i \leq x-1$, add a new vertex v_i and edge u_iv_i. Then each u_i is a support vertex. Observation 4.16 implies that there exists a $\gamma(G)$-set containing all the support vertices, so $\gamma(G) \geq x-1$. Now, $\{u_i \mid 1 \leq i \leq x-1\}$ does not dominate G, so $\gamma(G) \geq x$. And since $\{u_i \mid 1 \leq i \leq x-1\} \cup \{w\}$ dominates G, for any vertex $w \in K_{y-2x+2}$, we have $\gamma(G) \leq x$. Thus, $\gamma(G) = x$ as desired. Also, it follows that the vertices of P_{x-1} and K_{y-2x+2} are good. Since no vertex in V_b dominates K_{y-2x+2} but every vertex of K_{y-2x+2} dominates $K_{y-2x+2} + V_b$, we have no good vertices in V_b. Moreover, each $u_i \in P_{x-1}$ is adjacent to exactly one endvertex v_i. Letting S denote the $\gamma(G)$-set that contains all the support vertices u_i, we also have $S - \{u_i\} \cup \{v_i\}$ a $\gamma(G)$-set. Hence, $\{v_i \,|\, 1 \leq i \leq x-1\}$ is a set of good vertices. Therefore, $g(G) = 2(x-1) + y - 2x + 2 = y$ and $b(G) = z$ as desired.

Finally, we consider the case where $z = 1$. Then $x \geq 2$ by Lemma 4.18. We begin as before with the complete graph K_{y-2x+2} and introduce a path $P_{x-1} = u_1, u_2, \ldots, u_{x-1}$ with edge $u_{x-1}w$, where $w \in V(K_{y-2x+2})$. Again, for each vertex u_i, add a new vertex v_i and edge u_iv_i. Now subdivide the edge $u_{x-1}w$, labeling the new vertex v. We show that v is the only bad vertex in G.

From Observation 4.16, we know that there exists a $\gamma(G)$-set containing all the support vertices u_i. Thus, $\gamma(G) \geq x - 1$. But $\{u_i \,|\, 1 \leq i \leq x-1\}$ does not dominate G, so $\gamma(G) \geq x$. Let S be a $\gamma(G)$-set that contains all the support vertices u_i. Then v is dominated by S. Further, any vertex $w \in K_{y-2x+2}$ will dominate the clique. Henc,e $\gamma(G) = x$ as desired. It follows then that all vertices $u_i \in P_{x-1}$ and all vertices $w \in K_{y-2x+2}$ are good.

Now, let $S' = \{u_i \,|\, 1 \leq i \leq x-1\} \cup \{w\}$. Then v is dominated by w, and therefore $S' - \{u_i\} \cup \{v_i\}$ is also a $\gamma(G)$-set. Thus all vertices v_i, $1 \leq i \leq x-1$, are good.

To show v is bad, suppose there exists S'', a $\gamma(G)$-set, such that $v \in S''$. Then S'' dominates u_{x-1} and w_1. But to dominate $P_{x-2} \cup \{v_i \mid 1 \leq i \leq x-1\}$ requires $x-1$ more vertices and to dominate $K_{y-2x+2} - \{w_1\}$ requires one more vertex. Hence, $|S''| = 1 + (x-1) + 1 = x + 1 > \gamma(G)$. Therefore, v is bad, and we have $g(G) = 2(x-1) + y - 2x + 1 = y$ and $b(G) = 1 = z$ as desired.

□

CHAPTER 5

THE GOODNESS INDEX

Every graph G has a finite number of $\gamma(G)$-sets. Slater [9] denotes the number of distinct $\gamma(G)$-sets as $\#\gamma(G)$. We define the *goodness index* of a vertex v, denoted $g(v)$, to be the ratio of the number of distinct $\gamma(G)$-sets that contain v, denoted $a(v)$, to the total number of distinct $\gamma(G)$-sets, that is,

$$g(v) = \frac{a(v)}{\#\gamma(G)}.$$

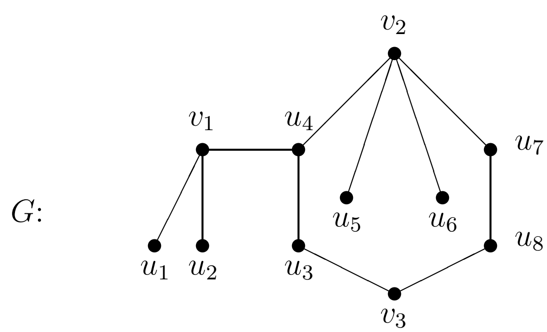

Figure 9: The graph G has a unique $\gamma(G)$-set $= \{v_1, v_2, v_3\}$. Then $g(v_i) = 1$, for $1 \leq i \leq 3$, and $g(u_i) = 0$, for $1 \leq i \leq 8$.

Consider a graph G with a unique $\gamma(G)$-set S, in other words, $\#\gamma(G) = 1$. Since any vertex $v \in S$ is in exactly one $\gamma(G)$-set, we have $g(v) = 1$ for all $v \in S$. Alternatively, if $u \in V - S$, then u is contained in no $\gamma(G)$-sets and $g(u) = 0$. In such cases, we say v is 1-good and u is 0-good. For an example of a graph G with a unique $\gamma(G)$-set, see Figure 9.

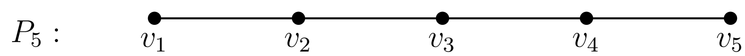

Figure 10: The path P_5 has $\gamma(P_5) = 2$ with three distinct γ-sets.

An example of a graph that has more than one γ-set is P_5, the path on five vertices, illustrated in Figure 10. Note that $\gamma(P_5) = 2$ and $\#\gamma(P_5) = 3$. Specifically, $S_1 = \{v_1, v_4\}$, $S_2 = \{v_2, v_4\}$, and $S_3 = \{v_2, v_5\}$ are $\gamma(P_5)$-sets. Thus, $g(v_1) = g(v_5) = \frac{1}{3}$, $g(v_2) = g(v_4) = \frac{2}{3}$, and $g(v_3) = 0$. We say that v_1 is $\frac{1}{3}$-good, v_2 is $\frac{2}{3}$-good, and so on.

Finally, let us emphasize the distinction between $g(G)$, the number of good vertices in a graph G, and $g(v)$, the goodness index of a vertex v. Context will make the distinction clear as well as the fact that $g(G)$ is always an integer greater than or equal to one, while $g(v)$ is always a rational number between zero and one, inclusive.

Observation 5.20 *For all $v_i \in V(G)$,*

$$\sum_{i=1}^{n} g(v_i) = \gamma(G) \textit{ for all } v_i \in V(G).$$

5.1 Paths

Our goal in this section is to establish the goodness index for every vertex of a path P_n. We introduce additional notation specific to our discussion and make some observations, all of which will be useful to our arguments.

We label the vertices of a path P_n as $v_1, v_2, \ldots, v_n$ where v_i and v_{i+1} are adjacent for all $i \in \{1, 2, \ldots, n-1\}$.

The symmetry of a path allows us to make the following observation.

Observation 5.21 *For a path P_n, $a(v_i) = a(v_{n-i+1})$.*

Since a leaf or its adjacent support vertex must be in every $\gamma(G)$-set, we observe the following.

Observation 5.22 *Each $\gamma(P_n)$-set contains exactly one of v_1 and v_2, and exactly one of v_{n-1} and v_n.*

Our next observation follows directly from Proposition 3.8.

Proposition 5.23 *For a path P_n where $n \equiv 0(mod\ 3)$,*

$$g(v_i) = \begin{cases} 1 & \text{for } i \equiv 2(mod\ 3) \\ 0 & \text{otherwise.} \end{cases}$$

5.1.1 Paths P_n, $n \equiv 2(\textbf{mod } 3)$

In this section we determine $g(v_i)$ for each $v_i \in V(P_n)$ where $n \equiv 2(\text{mod } 3)$.

Table 1: Number of distinct $\gamma(P_n)$-sets containing v_i, where $n \equiv 2(\text{mod } 3)$ and $2 \leq n \leq 17$.

	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
n																		
2		1	1															
5		1	2	0	2	1												
8		1	3	0	2	2	0	3	1									
11		1	4	0	2	3	0	3	2	0	4	1						
14		1	5	0	2	4	0	3	3	0	4	2	0	5	1			
17		1	6	0	2	5	0	3	4	0	4	3	0	5	2	0	6	1

Simple observation reveals the data shown in Table 1. We will now find the goodness index of each vertex v_i for a path P_n, $n \equiv 2(\text{mod } 3)$.

Lemma 5.24 *For the path P_n where $n \equiv 2(mod\ 3)$, a vertex v_i where $i \equiv 0(mod\ 3)$ is not in any $\gamma(P_n)$-set.*

Proof: Let $i \equiv 0(\text{mod } 3)$, $n \equiv 2(\text{mod } 3)$, and assume that $v_i \in S$, some $\gamma(P_n)$-set S. Then v_i dominates v_{i-1} and v_{i+1}, leaving subpaths $P_{i-2} = \langle v_i, \ldots, v_{i-2} \rangle$ and $P_{n-(i+1)} = \langle v_{i+2}, \ldots, v_n \rangle$ to be dominated be $S - \{v_i\}$. Since $i \equiv 0(\text{mod } 3)$, it follows that $i-2 \equiv 1(\text{mod } 3)$ and $n-(i+1) \equiv 1(\text{mod } 3)$. Thus S contains $\left\lceil \frac{i-2}{3} \right\rceil + \left\lceil \frac{n-(i+1)}{3} \right\rceil = \frac{i}{3} + \frac{n-i+1}{3} = \frac{n+1}{3}$ vertices in addition to v_i. But $1 + \frac{n+1}{3} = \frac{n+4}{3} > \left\lceil \frac{n}{3} \right\rceil = \gamma(P_n)$, a contradiction. Hence, $v_i \notin S$ for any $\gamma(P_n)$-set S. □

Lemma 5.25 *For the path P_n where $n \equiv 2(mod\ 3)$, the number of $\gamma(P_n)$-sets is $\#\gamma(P_n) = \frac{n+4}{3}$.*

Proof: We begin by noting that $\#\gamma(P_2) = 2$. Thus, let $n \geq 5$. Assume $v_1 \in S$, for a $\gamma(P_n)$-set. Then, by Observation 5.22, $v_2 \notin S$. Thus S dominates $\{v_3, \ldots v_n\}$ with exactly $\frac{n-2}{3}$ vertices since $|\{v_3, \ldots v_n\}| \equiv 0 (\text{mod } 3)$. Further, since $\{v_3, \ldots v_n\}$ induces a path with a unique γ-set, it follows that v_1 is contained in exactly one $\gamma(P_n)$-set.

On the other hand, suppose $v_2 \in S$. Then v_2 dominates v_1 and v_3, leaving $\{v_4, \ldots v_n\}$ to be dominated by $S - \{v_2\}$. These vertices induce a path of order $n \equiv 2 (\text{mod } 3)$, so v_2 is contained in at least $\#\gamma(P_{n-3})$ $\gamma(P_n)$-sets. But by Lemma 5.24, $v_3 \notin S$, for all $\gamma(P_n)$-sets S. Hence, v_2 occurs in exactly $\#\gamma(P_{n-3})$ $\gamma(P_n)$-sets. Thus, the following recurrence relation occurs.

$$
\begin{aligned}
\#\gamma(P_n) &= 1 + \#\gamma(P_{n-3}) \\
&= 1 + 1 + \#\gamma(P_{n-6}) \\
&= 1 + 1 + 1 + \#\gamma(P_{n-9}) \\
&\quad \vdots \\
&= \frac{n-2}{3} + \#\gamma(P_{n-(n-2)}) \\
&= \frac{n-2}{3} + \#\gamma(P_2) \\
&= \frac{n-2}{3} + 2 \\
&= \frac{n+4}{3}.
\end{aligned}
$$

□

Theorem 5.26 *For a path P_n with $n \equiv 2(mod\ 3)$ and vertex v_i,*

$$g(v_i) = \begin{cases} 0 & \text{for } i \equiv 0(mod\ 3) \\ \frac{i+2}{n+4} & \text{for } i \equiv 1(mod\ 3) \\ \frac{n+3-i}{n+4} & \text{for } i \equiv 2(mod\ 3). \end{cases}$$

Proof: Let S be a $\gamma(P_n)$-set. From the proof of Lemma 5.25, we have that $a(v_1) = 1$ and so by Observation 5.21, $a(v_n) = 1$. This result along with Observation 5.22 implies that $a(v_{n-1}) = \#\gamma(P_n) - 1$.

Now suppose S is a $\gamma(P_n)$-set with $v_{n-1} \in S$. Then a subpath P_{n-3} remains to be dominated. By the argument in the preceding paragraph and since v_{n-4} and v_{n-3} are not both in a $\gamma(P_n)$-set S, it follows that $a(v_{n-4}) = \#\gamma(P_{n-3}) - 1$. In general, if $i = n - j$, where $j \equiv 1(\text{mod } 3)$, we have

$$\begin{aligned} a(v_i) &= a(v_{n-j}) \\ &= \#\gamma(P_{n-j+1}) - 1 \\ &= \frac{(n-j+1)+4}{3} - 1 \\ &= \frac{n-j+2}{3} \\ &= \frac{i+2}{3} \end{aligned}$$

and $g(v_i) = \frac{i+2}{3} \div \frac{n+4}{3} = \frac{i+2}{n+4}$, where $i \equiv 1(\text{mod } 3)$.

Now, let $i \equiv 2(\text{mod } 3)$. By Observation 5.21, $g(v_i) = g(v_{n-i+1})$, and note that $n - i + 1 \equiv 1(\text{mod } 3)$, since $n \equiv 2(\text{mod } 3)$. Thus, $g(v_i) = \frac{(n-i+1)+2}{n+4} = \frac{n+3-i}{n+4}$, where $i \equiv 2(\text{mod } 3)$.

Finally, Lemma 5.24 established that $g(v_i) = 0$ for $i \equiv 0(\text{mod } 3)$. $\square$

Note that the numbers given in Table 1 divided by $\#\gamma(P_n)$ are confirmed in the resulting equations.

Corollary 5.27 *For a path P_n with $n \equiv 2(mod\ 3)$, $g(P_n) = 2\left\lceil \frac{n}{3} \right\rceil$.*

Proof: Let P_n be a path with $n \equiv 2(\text{mod } 3)$. Then P_n contains $\left\lceil \frac{n}{3} \right\rceil$ vertices v_i with $i \equiv 1(\text{mod } 3)$ and $\left\lceil \frac{n}{3} \right\rceil$ vertices v_i with $i \equiv 2(\text{mod } 3)$. Since $g(v_i) \neq 0$ for $i \equiv 1(\text{mod } 3)$ and $i \equiv 2(\text{mod } 3)$, and $g(v_i) = 0$ for $i \equiv 0(\text{mod } 3)$, we have $g(P_n) = 2\left\lceil \frac{n}{3} \right\rceil$ where $n \equiv 2(\text{mod } 3)$. $\square$

5.1.2 Paths P_n, $n \equiv 1(\text{mod } 3)$

In the final section of this chapter we determine $g(v_i)$ for each $v_i \in V(P_n)$ where $n \equiv 1(\text{mod } 3)$.

As in Table 1, Table 2 displays data that was gathered from studying the paths. To aid in finding the goodness index of each vertex v_i for a path P_n, $n \equiv 1(\text{mod } 3)$, we will use the following lemma.

Lemma 5.28 *A nontrivial path P_n where $n \equiv 1(mod\ 3)$ has a γ-set including adjacent vertices v_i and v_{i+1} only if $i \equiv 2(mod\ 3)$. A $\gamma(P_n)$-set may have no more than one such pair of adjacent vertices and for every such pair of adjacent vertices in P_n there exists a $\gamma(P_n)$-set that contains them.*

Table 2: Number of distinct $\gamma(P_n)$-sets containing v_i, where $n \equiv 1 (\text{mod } 3)$ and $1 \leq n \leq 19$.

	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
n																				
1		1																		
4		2	2	2	2															
7		3	5	2	4	2	5	3												
10		4	9	2	6	5	5	6	2	9	4									
13		5	14	2	8	9	5	9	5	9	8	2	14	5						
16		6	20	2	10	14	5	12	9	9	12	5	14	10	2	20	6			
19		7	27	2	12	20	5	15	14	9	16	9	14	15	5	20	12	2	27	7

Proof: Suppose that S is a $\gamma(P_n)$-set such that vertices v_j and v_{j+1} are in S, where $j \equiv 1(\text{mod } 3)$. Then $\{v_j, v_{j+1}\}$ dominates $\{v_{j-1}, v_j, v_{j+1}, v_{j+2}\}$, assuming these vertices exist, leaving paths $P_{j-2} = \langle v_1, \ldots, v_{j-2} \rangle$ and $P_{n-(j+2)} = \langle v_{j+3}, \ldots, v_n \rangle$ to be dominated by $S - \{v_j, v_{j+1}\}$. Note that $j - 2 \equiv 2(\text{mod } 3)$ and $n - (j+2) \equiv 1(\text{mod } 3)$. Hence we require at least $\left\lceil \frac{j-2}{3} \right\rceil + \left\lceil \frac{n-(j+2)}{3} \right\rceil$ to dominate $P_{j-2} \cup P_{n-(j+2)}$. But $2 + \left\lceil \frac{j-2}{3} \right\rceil + \left\lceil \frac{n-(j+2)}{3} \right\rceil = 2 + \frac{j-1}{3} + \frac{n-j}{3} = \frac{n+5}{3} > \left\lceil \frac{n}{3} \right\rceil = \frac{n+2}{3} = \gamma(P_n)$, a contradiction. Note that if $j = 1$, then we have only to dominate $P_{n-(j+2)} = P_{n-3}$ requiring at least $\left\lceil \frac{n-3}{3} \right\rceil$ vertices. But $2 + \left\lceil \frac{n-3}{3} \right\rceil = 2 + \frac{n-1}{3}$ which, as we have just seen, leads to a contradiction. Hence, no $\gamma(P_n)$-set contains vertices v_j and v_{j+1} where $j \equiv 1(\text{mod } 3)$.

Similarly, suppose S is a $\gamma(P_n)$-set such that vertices v_j and v_{j+1} are in S, where $j \equiv 0(\text{mod } 3)$. Then again $\{v_j, v_{j+1}\}$ dominates $\{v_{j-1}, v_j, v_{j+1}, v_{j+2}\}$, assuming these vertices exist, leaving paths $P_{j-2} = \langle v_1, \ldots, v_{j-2} \rangle$ and $P_{n-(j+2)} = \langle v_{j+3}, \ldots, v_n \rangle$ to be dominated by $S-\{v_j, v_{j+1}\}$. Note that $j-2 \equiv 1(\text{mod } 3)$ and $n-(j+2) \equiv 2(\text{mod } 3)$. Therefore we require at least $\left\lceil \frac{j-2}{3} \right\rceil + \left\lceil \frac{n-(j+2)}{3} \right\rceil$ to dominate $P_{j-2} \cup P_{n-(j+2)}$. But $2 + \left\lceil \frac{j-2}{3} \right\rceil + \left\lceil \frac{n-(j+2)}{3} \right\rceil = 2 + \frac{j}{3} + \frac{n-j-1}{3} = \frac{n+5}{3} > \left\lceil \frac{n}{3} \right\rceil = \frac{n+2}{3} = \gamma(P_n)$, a contradiction. Note that if $j = n-1$ then we need only concern ourselves with dominating $P_{j-2} = P_{n-3}$. Thus, to dominate the graph would require $2 + \frac{n-1}{3} = \frac{n+5}{3}$ vertices, again contradicting the minimality of S. Hence, no $\gamma(P_n)$-set contains vertices v_j and v_{j+1} where $j \equiv 0(\text{mod } 3)$.

Now we must show that such a $\gamma(P_n)$-set exists for adjacent vertices v_i and v_{i+1} where $i \equiv 2(\text{mod } 3)$. Let S be a $\gamma(P_n)$-set such that vertices v_i and v_{i+1} are in S, where $i \equiv 2(\text{mod } 3)$. Then $\{v_i, v_{i+1}\}$ dominates $\{v_{i-1}, v_i, v_{i+1}, v_{i+2}\}$, leaving paths $P_{i-2} = \langle v_1, \ldots, v_{i-2} \rangle$ and $P_{n-(i+2)} = \langle v_{i+3}, \ldots, v_n \rangle$, $i-2 \geq 1$ and/or $n-(i+2) \geq 1$. Note that $i-2 \equiv 0(\text{mod } 3)$ and $n-(i+2) \equiv 0(\text{mod } 3)$. Therefore we require at least $\left\lceil \frac{i-2}{3} \right\rceil + \left\lceil \frac{n-(i+2)}{3} \right\rceil$ to dominate $P_{i-2} \cup P_{n-(i+2)}$. Note that $2 + \left\lceil \frac{i-2}{3} \right\rceil + \left\lceil \frac{n-(i+2)}{3} \right\rceil = 2 + \frac{i-2}{3} + \frac{n-(i+2)}{3} = \frac{n+2}{3} = \left\lceil \frac{n}{3} \right\rceil = \gamma(P_n)$. Also note that if $i = n-2$ or if $i = 2$ then we have only to dominate $P_{n-(j+2)} = P_{n-4}$, which requires at least $\left\lceil \frac{n-4}{3} \right\rceil$ vertices. $2 + \left\lceil \frac{n-4}{3} \right\rceil = 2 + \frac{n-4}{3} = \frac{n+2}{3}$ as desired. Hence, a path P_n has a γ-set including adjacent vertices v_i and v_{i+1} only if $i \equiv 2(\text{mod } 3)$.

Finally, note that neither γ-set of the induced subpaths P_{i-2} and $P_{n-(i+2)}$ contains adjacent vertices as their orders are both congruent to $0(\text{mod } 3)$. Hence the only adjacent pair of vertices in S is v_i and v_{i+1}. $\square$

Lemma 5.29 *For a path P_n where $n \equiv 1(mod\ 3)$, $a(v_n) = \left\lceil \frac{n}{3} \right\rceil$.*

Proof: Let S be a $\gamma(P_n)$-set such that $v_n \in S$. Then, $v_{n-1} \notin S$ by Observation 5.22 and a subpath P_{n-2} is left to be dominated. Since $n-2 \equiv 2(\text{mod } 3)$, we have $\#\gamma(P_{n-2}) = \frac{(n-2)+4}{3} = \frac{n+2}{3} = \left\lceil \frac{n}{3} \right\rceil$, by Lemma 5.25. Thus, $a = \left\lceil \frac{n}{3} \right\rceil$.

□

Lemma 5.30 *For a path P_n where $n \equiv 1(mod\ 3)$, $a(v_{n-1}) = \#\gamma(P_{n-3}) + 1$.*

Proof: Let S be a $\gamma(P_n)$-set such that $v_{n-1} \in S$ and $v_n, v_{n-2} \notin S$. Then we have a path P_{n-3} to dominate. Therefore, v_{n-1} is in $\#\gamma(P_{n-3})$ $\gamma(P_n)$-sets which contain neither v_n nor v_{n-2}. By Observation 5.22 we have that no $\gamma(P_n)$-set contains both v_n and v_{n-1}, and by Lemma 5.28 exactly one $\gamma(P_n)$-set contains both v_{n-1} and v_{n-2}, since $n-2 \equiv 2(\text{mod } 3)$. Hence, $a(v_{n-1}) = \#\gamma(P_{n-3}) + 1$.

□

Lemma 5.31 *For a path P_n where $n \equiv 1(mod\ 3)$,*

$$\#\gamma(P_n) = \frac{n^2 + 13n + 4}{18}.$$

Proof: Note that if $n = 1$ the lemma holds. Hence we assume $n = 3k + 1 \geq 4$. Since either (but not both) of v_n and v_{n-1} are in any $\gamma(P_n)$-set, (Observation 5.22),

$\#\gamma(P_n) = \#\gamma(P_{n-3}) + 1 + \left\lceil \frac{n}{3} \right\rceil$, by Lemmas 5.29 and 5.30. Then

$$\begin{aligned}
\#\gamma(P_n) &= \#\gamma(P_{n-3}) + 1 + \left\lceil \frac{n}{3} \right\rceil \\
&= \left(\#\gamma(P_{n-6}) + 1 + \left\lceil \frac{n-3}{3} \right\rceil\right) + 1 + \left\lceil \frac{n}{3} \right\rceil \\
&= \left[\left(\#\gamma(P_{n-9}) + 1 + \left\lceil \frac{n-6}{3} \right\rceil\right) + 1 + \left\lceil \frac{n-3}{3} \right\rceil\right] + 1 + \left\lceil \frac{n}{3} \right\rceil \\
&\vdots \\
&= \#\gamma(P_{n-3k}) + 1 + \left\lceil \frac{n-(3k-3)}{3} \right\rceil + 1 + \cdots + 1 + \left\lceil \frac{n}{3} \right\rceil \\
&= \#\gamma(P_1) + k + \sum_{i=1}^{k}(i+1) \qquad \left(\text{since } \left\lceil \frac{n}{3} \right\rceil = \frac{3k+3}{3} = k+1\right) \\
&= 1 + k + k + \frac{k(k+1)}{2} \\
&= \frac{k^2+5k+2}{2} \\
&= \frac{n^2+13n+4}{18}.
\end{aligned}$$

□

Lemma 5.32 *For a path P_n where $n \equiv 1(mod\ 3)$, $a(v_{n-2}) = 2$.*

Proof: Let S be a $\gamma(P_n)$-set with $v_{n-2} \in S$. Then, since either v_{n-1} or v_n is in S, we have $\{v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$ dominated by $S \cap \{v_{n-2}, v_{n-1}, v_n\}$. This leaves a subpath P_{n-4} to be dominated. Since $n-4 \equiv 0(\text{mod } 3)$, there is a unique γ-set that dominates P_{n-4}. Hence, $a(v_{n-2}) = 2$. □

Lemma 5.33 *For a path P_n where $n \equiv 1(mod\ 3)$, $a(v_{n-1}) = \frac{n^2+7n-8}{18}$.*

Proof: Recall that $\#\gamma(P_n) = \frac{n^2+13n+4}{18}$, by Lemma 5.31. Also, from Lemma 5.30, we have $a(v_{n-1}) = \#\gamma(P_{n-3}) + 1$. Thus,

$$\begin{aligned} a(v_{n-1}) &= \#\gamma(P_{n-3}) + 1 \\ &= \frac{(n-3)^2 + 13(n-3) + 4}{18} + 1 \\ &= \frac{n^2 + 7n - 8}{18}. \end{aligned}$$

□

We are finally ready to prove the main results for this section.

Theorem 5.34 *For a path P_n where $n \equiv 1(mod\ 3)$, if $i \equiv 0(mod\ 3)$, then $g(v_i) = \frac{i^2-9i}{n^2+13n+4}$.*

Proof: Let P_n be a path on n vertices where $n \equiv 1(\text{mod } 3)$ and choose a vertex v_i where $i \equiv 0(\text{mod } 3)$. Let $c = n-i-1$. Note that $c \equiv 0(\text{mod } 3)$ and $i = n-c-1$. Let S be a $\gamma(P_n)$-set. If $v_i \in S$, then v_{n-c} is dominated by v_i, leaving $P_c = \langle v_{n-c+1}, \ldots, v_n \rangle$ to be dominated. Since $c \equiv 0(\text{mod } 3)$, the vertices of the unique $\gamma(P_c)$-set are a subset of S.

Thus, the number of $\gamma(P_n)$-sets containing v_i is independent of how large c is. Hence $a(v_i) = a(v_{n-c-1}) = \frac{(n-c)^2+7(n-c)-8}{18}$, by Lemma 5.33. Then

$$a(v_i) = \frac{(i+1)^2 + 7(i+1) - 8}{18}$$

$$= \frac{i^2 + 9i}{18}$$

and

$$\begin{aligned} g(v_i) &= \frac{i^2 + 9i}{18} \div \frac{n^2 + 13n + 4}{18} \\ &= \frac{i^2 + 9i}{n^2 + 13n + 4}. \end{aligned}$$

□

Corollary 5.35 *For a path P_n where $n \equiv 1(mod\ 3)$, if $i \equiv 2(mod\ 3)$, then $g(v_i) = \frac{(n-i+1)^2+9(n-i+1)}{n^2+13n+4}$.*

Proof: Let P_n be a path on n vertices where $n \equiv 1(\text{mod } 3)$ and choose a vertex v_i where $i \equiv 2(\text{mod } 3)$. Note that $a(v_i) = a(v_{n-i+1})$ and $n - i + 1 \equiv 0(\text{mod } 3)$. Then by Theorem 5.34, $a(v_i) = a(v_{n-i+1}) = \frac{(n-i+1)^2+9(n-i+1)}{18}$. Hence

$$\begin{aligned} g(v_i) &= \frac{(n-i+1)^2 + 9(n-i+1)}{18} \div \frac{n^2 + 13n + 4}{18} \\ &= \frac{(n-i+1)^2 + 9(n-i+1)}{n^2 + 13n + 4}. \end{aligned}$$

□

Theorem 5.36 *For a path P_n where $n \equiv 1(mod\ 3)$, if $i \equiv 1(mod\ 3)$, then $g(v_i) = \frac{2(i+2)(n-i+3)}{n^2+13n+4}$.*

Proof: Let P_n be a path on n vertices where $n \equiv 1(\text{mod } 3)$ and choose a vertex v_i where $i \equiv 1(\text{mod } 3)$. Define $P_i = \langle v_1, \ldots, v_i \rangle$ and $P_{n-i-1} = \langle v_{i+2}, \ldots, v_n \rangle$. Let S be a

$\gamma(P_n)$-set. If $v_i \in S$ then v_{i+1} is dominated by v_i, leaving P_{n-i-1} to be dominated by $S - \{v_i\}$. Now, $n-i-1 \equiv 2 (\text{mod } 3)$, so by Lemma 5.25 $\#\gamma(P_{n-i-1}) = \frac{n-i-1+4}{3} = \frac{n-i+3}{3}$. Lemma 5.29 implies that v_i is in $\left\lceil \frac{i}{3} \right\rceil$ $\gamma(P_i)$-sets, and Lemma 5.28 implies that v_i and v_{i+1} are not in the same $\gamma(P_n)$-set. Hence,

$$\begin{aligned} a_n(v_i) &= \left\lceil \frac{i}{3} \right\rceil \frac{(n-i+3)}{3} \\ &= \frac{(i+2)}{3} \frac{(n-i+3)}{3} \\ &= \frac{(i+2)(n-i+3)}{9} \end{aligned}$$

and

$$\begin{aligned} g(v_i) &= \frac{(i+2)(n-i+3)}{9} \div \frac{n^2+13n+4}{18} \\ &= \frac{2(i+2)(n-i+3)}{n^2+13n+4}. \end{aligned}$$

□

We conclude this section of Chapter 5 with a summary of the above results.

For a path P_n where $n \equiv 1 (\text{mod } 3)$ we have

$$g(v_i) = \begin{cases} \frac{i^2+9i}{n^2+13n+4} & \text{for } i \equiv 0 (\text{mod } 3) \\ \frac{2(i+2)(n+3-i)}{n^2+13n+4} & \text{for } i \equiv 1 (\text{mod } 3) \\ \frac{(n-i+1)^2+9(n-i+1)}{n^2+13n+4} & \text{for } i \equiv 2 (\text{mod } 3). \end{cases}$$

Note that the numbers given in Table 2 divided by $\#\gamma(P_n)$ are confirmed in the resulting equations.

CHAPTER 6

OPEN PROBLEMS

One of the most enjoyable aspects of working on this thesis was the abundance of new questions that seemed to arise as some of the old questions were answered. We list a few.

1. How many nonisomorphic graphs have the property $g(G) = b(G) = \gamma(G) + 1$ and what are they? This is work in progress and it is hoped that we may publish the result soon as part of an upcoming paper.

2. The above question is easy to answer if the answer to the following is affirmative: If $e \in E(\overline{G})$ and $g(G \cup \{e\}) > g(G)$, then is e a domination critical edge? That is, if we introduce a new edge to an existing graph and the number of good vertices decreases, does the domination number also decrease? Again, this is work in progress, as it is intimately related to problem 1.

3. If a set of vertices of a graph is randomly chosen, what is the probability that it is a dominating set? A γ-set? Does knowledge of the goodness index reduce the randomness of the choice? Can this knowledge improve our chances? If so, when and by how much?

4. Let S be a set of vertices of a graph G such that each vertex $v \in S$ has goodness index $g(v) \geq k$, for a fixed k. If a subset of these vertices is randomly chosen,

what is the probability that it is a dominating set? A γ-set?

5. Given the set of all dominating sets of a graph G, can G be reconstructed? What are our limitations?

6. If all dominating sets of a graph can be randomly generated, does the ratio of the number of times a particular vertex appears to the number of sets generated converge to anything? If so, to what? The goodness index?

7. How small can $g(v)$ be if v is a weak support vertex?

8. Given a sequence of rational numbers $\langle x_i \rangle_{i=1}^{n}$, $0 \leq x_i \leq 1$, does a graph G of order n, $v_i \in V(G)$ with $g(v_i) = x_i$ exist?

9. Once more is known about the goodness index, what are some applications of this concept?

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2001 Phi Kappa Phi Honor Society

2000 President's Pride Service Organization
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