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Locating and Total Dominating Sets in Trees.

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Locating and Total Dominating Sets in Trees

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by

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ABSTRACT

Locating and Total Dominating Sets in Trees

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A set S of vertices in a graph $G = (V, E)$ is a total dominating set of G if every vertex of V is adjacent to some vertex in S. In this thesis, we consider total dominating sets of minimum cardinality which have the additional property that distinct vertices of V are totally dominated by distinct subsets of the total dominating set.

DEDICATION

I dedicate this thesis to all of the many people who have supported me throughout my educational journey. I first thank my family for always encouraging me to do my very best. I also thank all of my friends I have met along the way. Each has contributed a special part to my journey and without all of the fun times and late nights we've shared, my experience would not be the same. A big thank you goes to my boyfriend, David Atkins, for his support, understanding, and love over the past four years. He brings so much joy to my life and has made me laugh even during the times when I felt like crying. I love you very much! Finally, I would like to thank God. He is the source of all wisdom and has given me strength to make it through each day of my life.

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1 Introduction

1.1 Domination and Total Domination

The concept of domination in graphs began in the 1850s with the game of chess. The goal of the problem was to use certain chess pieces to dominate the squares of a chessboard. Knowing that a queen can move horizontally, vertically, or diagonally, de Jaenish, in 1862, considered the problem of finding the minimum number of queens that can be placed on a chessboard such that every square is either occupied by a queen or can be occupied by a queen in a single move. It turns out that the minimum number of queens needed is five, and this became known as the Five Queens Problem [4, 7].

The connection between the Five Queens problem and domination can be seen if we let each vertex of a graph represent a square of the 64 squares of a chessboard. Then, two vertices are adjacent in G if each corresponding square can be reached by a queen on the other square in a single move. This graph is referred to as the Queen's graph. Hence, the minimum number of queens that can dominate the entire chessboard forms a dominating set in G [7]. We now consider the formal definitions and concepts of domination.

A set S of vertices of a graph $G = (V, E)$ is a *dominating set* of G if every vertex in $V - S$ is adjacent to some vertex of S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. Consider the example of a P_6 shown in Figure 1.

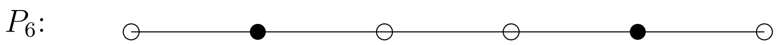


Figure 1: Domination of a Path

Notice that if we let our set S be the two darkened vertices, then each of the remaining vertices is adjacent to a vertex in S. Hence $\gamma(P_6) \leq 2$. Since no single vertex can dominate all of the remaining vertices, we have $\gamma(P_6) \geq 2$. Therefore, it follows that $\gamma(P_6) = 2$.

Considering a different example, suppose we have the following graph shown in Figure 2.

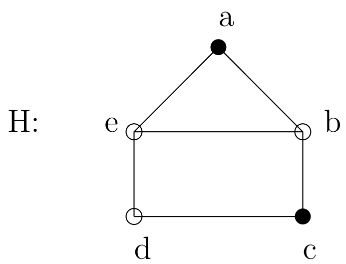


Figure 2: Domination Example

Then the set $S = \{a, c\}$ forms a dominating set of H, and since this set is of minimum cardinality, we have $\gamma(H) = 2$.

A vertex u is said to be *connected* to a vertex v in a graph G if there exists a $u - v$ path in G. A graph G is connected if every pair of its vertices is connected. A *tree* T is a connected graph with no cycles. In a tree, a vertex of degree one is referred to as a *leaf* and a vertex which is adjacent to a leaf is a *support vertex*. If a support vertex is adjacent to two or more leaves, it is called a *strong support vertex.* In many cases, we look at instances of domination dealing with trees. For example, consider the tree shown in Figure 3.

T :

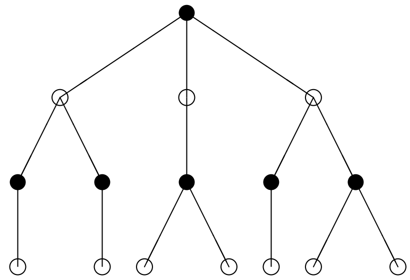


Figure 3: Domination of a Tree

Notice that, in this case, $\gamma(T) = 6$ since we must include at least the number of support vertices to dominate the leaves as well as an additional vertex to dominate the root of the tree.

In order for a set of vertices, S to be dominating, every vertex not in the set must be adjacent to at least one vertex in the set. If we tighten the condition and require every vertex of a graph G to be adjacent to some vertex in S, then we have a total dominating set of G. Formally, a set S of vertices of a graph $G = (V, E)$ is a *total dominating set* of G if every vertex in V is adjacent to some vertex in S. The minimum cardinality of a total dominating set of G is the *total domination number* $\gamma_t(G)$. Note that $\gamma_t(G)$ is defined only for graphs with no isolated vertices. Since every total dominating set is a dominating set, we have $\gamma(G) \leq \gamma_t(G)$ for all graphs G with no isolated vertices.

Consider again the path P_6 shown in Figure 1. The darkened vertices form a dominating set S of P_6, but, these two vertices are not adjacent to a vertex in S. A total dominating set S' of P_6 is illustrated in Figure 4.

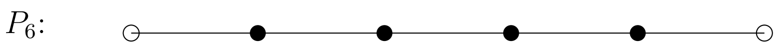


Figure 4: Total Domination of a Path

Letting S' represent the darkened vertices, notice that every vertex of P_6 is now adjacent to a vertex of S'. Hence, $\gamma_t(P_6) \leq 4$. Since no three vertices forming a P_3 dominates P_6, $\gamma_t(P_6) \geq 4$. Therefore, we conclude that $\gamma_t(G) = 4$. In general, the total domination number for paths is $\gamma_t(P_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$.

Consider again the graph shown in Figure 2. The set $S = \{a, c\}$ forms a dominating set of H and we have $\gamma(H) = 2$. Since these two vertices are not also adjacent to each other, S is not a total dominating set of H. However, the set $S' = \{b, c\}$ forms a total dominating set of H shown in Figure 5 below.

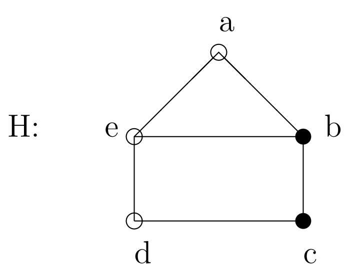


Figure 5: Total Domination Example

Therefore, $\gamma_t(H) \leq 2$. Since $2 = \gamma(H) \leq \gamma_t(H)$, we have $\gamma_t(H) = 2$. Notice this is an example of a graph for which the domination and total domination numbers are equal.

Finally, consider again the tree shown in Figure 3. The support vertices as well as the root of the tree are a dominating set of T. However, since the root and the support vertices are not adjacent, this cannot also be a total dominating set of T. A total dominating set of T is shown below by the set of darkened vertices and it can be shown that $\gamma_t(T) = 8$.

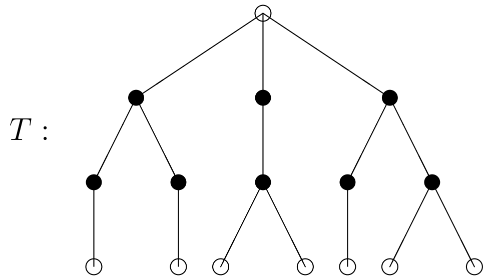


Figure 6: Total Domination of a Tree

In general, we follow the terminology of [4]. A more extensive study of domination in graphs can be found in [4, 5].

1.2 Locating-Dominating and Differentiating Dominating Sets

Consider the floor plan of a building as modelled by a graph where a vertex represents a room in the building and two vertices are adjacent if the corresponding rooms are adjacent. Suppose we wish to install expensive sensors in the building which will transmit a signal at the detection of an intruder (fire, burglar, etc.). Since the sensors are expensive we wish to optimize their usage. This safeguards facility analysis of the corresponding graph motivated the concept of locating sets and further the idea of locating-dominating sets, which in turn gave rise to the problem considered in this thesis.

Before presenting the work of this thesis, we first discuss these motivating concepts. Let $S = \{v_1, v_2, ..., v_k\}$ be a set of vertices in a connected graph $G = (V, E)$ and let $v \in V$. The k-vector (ordered k-tuple) $c_s(v)$, of v with respect to S is defined by

$$c_s(v) = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$$

where $d(v, v_i)$ is the distance between v and v_i $(1 \leq i \leq k)$. The set S is called a *locating set* if the k-vectors $c_s(v)$, for all vertices $v \in V$, are distinct. This concept is studied in [9, 10].

For example, suppose we have the graph H given in Figure 7.

In order for the set $S = \{b, e\}$ to be a locating set of H, the 2-vectors $c_s(v)$ must be distinct for all $v \in V$. Notice

$$c_s(a) = \{1, 1\}$$

$$c_s(b) = \{0, 1\}$$

$$c_s(c) = \{1, 2\}$$

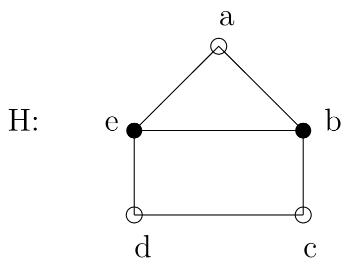


Figure 7: Locating Set Example

$$c_s(d) = \{2, 1\}$$

$$c_s(e) = \{1, 0\}$$

Since all of the 2-vectors are distinct, we conclude that S is a locating set of H.

Slater [10, 11] defined a *locating-dominating set* in a connected graph G to be a dominating set S of G such that for every two vertices u and v in $V(G) - S$, $N(u) \cap S \neq N(v) \cap S$. The minimum cardinality of a locating-dominating set of G is the *location-domination number* $\gamma^L(G)$. Notice that the location-domination number is defined for every connected graph G since V is such a set. This concept is studied in [1, 2, 8, 10, 11, 12] and elsewhere. To illustrate, suppose we have $G = K_4$, a complete graph on four vertices, as shown in Figure 8 below.

Notice that to dominate this graph, we need only one vertex, say vertex a, since all edges are present between any pair of vertices. Therefore $\gamma(K_4) = 1$. However, the dominating set $S = \{a\}$ cannot be a locating-dominating set since the remaining vertices are all adjacent to a, and hence $N(b) \cap S = \{a\}, N(c) \cap S = \{a\}, N(d) \cap S = \{a\}$. In fact, any locating-dominating set of a complete graph must include all the vertices, except one. For instance $S' = \{a, b, c\}$ is a locating-dominating set of K_4 as illustrated in Figure 9.

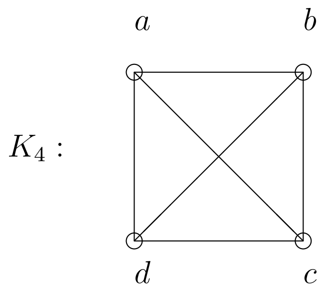


Figure 8: Complete Graph, K_4

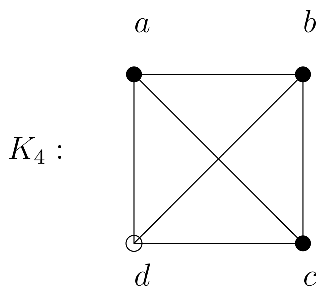


Figure 9: Locating-Dominating Set

Hence $\gamma^L(K_4) \leq 3$. Notice that since K_4 is a complete graph, any locating-dominating set must include all the vertices except one, and so $\gamma^L(G) \geq 3$. Therefore, $\gamma^L(G) = 3$. In general, for all complete graphs K_n, $\gamma^L(K_n) = n - 1$.

In order for a set S to be a locating set, every vertex in $V(G) - S$ must be distinguished in terms of its open neighborhood intersecting S. If we require all of the vertices of G to be distinguished, then we have a differentiating set of G. Gimbel et al. [3] defined a set S to be a *differentiating dominating set* if S is a dominating set and for every pair of vertices u and v in V, $N[u] \cap S \neq N[v] \cap S$. The *differentiating* *domination number* $\gamma_D(G)$, is the minimum cardinality of a differentiating dominating set of G. Since every differentiating dominating set is a dominating set, we have $\gamma(G) \leq \gamma_D(G)$. Consider the path P_8 shown in Figure 10

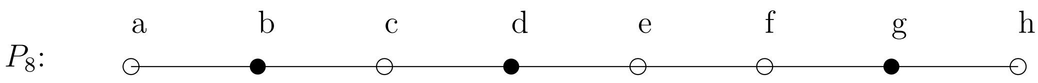
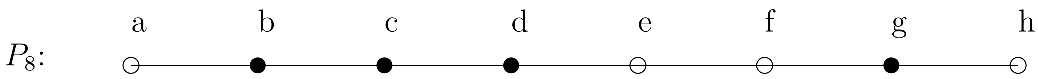
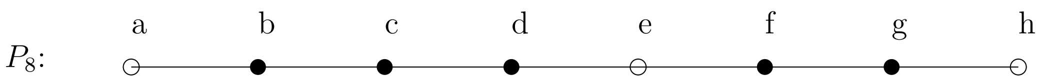


Figure 10: Path on Eight Vertices

Notice that the set $S = \{b, d, g\}$ forms a dominating set of P_8, and it can be shown that $\gamma(P_8) = 3$. However, notice that $N[a] \cap S = \{b\}$ and $N[b] \cap S = \{b\}$, and so S is not differentiating. Suppose we include the vertex c in our set S. That is, $S = \{b, c, d, g\}$. So, we have



Now, the set of vertices $\{a, b, c, d, e\}$ are all differentiated in terms of their closed neighborhoods intersecting S. However, $N[f] \cap S = \{g\}, N[g] \cap S = \{g\}$, and $N[h] \cap S = \{g\}$. Therefore S is not a differentiating dominating set of P_8. Suppose we add the vertex f to our set S, that is, $S = \{b, c, d, f, g\}$.



Notice that the addition of vertex f to our set S still does not make S differentiating since $N[f] \cap S = \{f, g\}$ and $N[g] \cap S = \{f, g\}$. Therefore, we must include vertex h to our set S. That is, $S = \{b, c, d, f, g, h\}$, and we have

P_8:

Now any pair of distinct vertices u and v in V can be differentiated and so $\gamma_D(P_8) \leq 6$. Since no five vertices forming a P_5 is a differentiating dominating set for P_8, $\gamma_D(P_8) \geq 6$. Therefore, $\gamma_D(P_8) = 6$.

1.3 Locating-Total Dominating and Differentiating Total Dominating Sets

The location of monitoring devices, such as surveillance cameras or fire alarms, to safeguard a system also serves as a motivation for my thesis work. Specifically, I am focusing on the problem of placing monitoring devices in a system in such a way that every site in the system (including the monitors) is adjacent to a monitor site. This problem can be modelled using total domination in graphs. As before, we let the floor plan of a building be modelled by a graph $G = (V, E)$ where a vertex represents a room in the building and two vertices are adjacent if the corresponding rooms are adjacent. If we find a total dominating set S for the graph, then we can place monitoring devices in each of the rooms corresponding to the vertices of S. Since S is a total dominating set, if a monitor goes down, the adjacent monitor can still protect the room. For example, suppose we have the following floor plan of a building as modelled by the graph shown in Figure 11, where L, P, W, MO, VP, M represent the Lounge, President's office, Women's restroom, Main Office, Vice President's office, and Men's restroom, respectively.

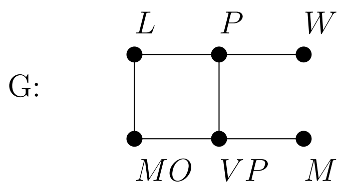
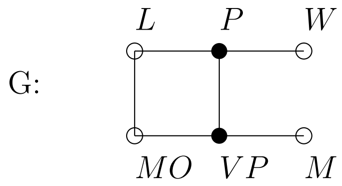


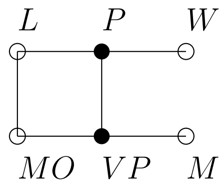
Figure 11: Floor Plan Model

Notice that if we place monitoring devices in the President's office and the Vice President's office, then all of the rooms are adjacent to a monitor site, including the President's and Vice President's office, and the building is guarded, that is, these two vertices form a total dominating set for the graph.



Suppose there is a "problem" at one of the facilities in the building such as a fire, burglar, or any other intruder. Then we want to be able to uniquely identify the location of the "problem" by a set of monitors. This concept serves as further motivation for my thesis work, as it combines the two concepts, total domination and location in graphs. In my thesis, I merge the concepts of a locating set and a total dominating set by defining two new sets, namely a locating-total dominating set and a differentiating total dominating set. In addition, I establish bounds on these parameters in a tree and investigate the ratio of these parameters in trees.

A *locating-total dominating set* (LTDS) S of a graph G is a total dominating set S of G such that for every two vertices u and v in $V - S$, $N(u) \cap S \neq N(v) \cap S$. The *locating-total domination number* $\gamma_t^L(G)$ is the minimum cardinality of a LTDS of G. A LTDS of cardinality $\gamma_t^L(G)$ we call a $\gamma_t^L(G)$-set. Note that the location total-domination number is defined for every graph G with no isolated vertex, since V is such a set. Consider again the graph of the floor plan of the building in Figure 11.

G: 

The set $S = \{P, VP\}$ is a total dominating set of G. To determine if this set is a LTDS of G, we must consider the vertices in $V - S$ or $\{L, MO, W, M\}$. Notice that $N(W) \cap S = \{P\}$ and $N(L) \cap S = \{P\}$. Therefore, if a "problem" exists in either of these two rooms, then only the monitor in the President's office will sound, and we will not know exactly where the problem is located. Suppose we place an additional monitor in the main office, and our set becomes $S = \{P, VP, MO\}$ as shown in Figure 12.

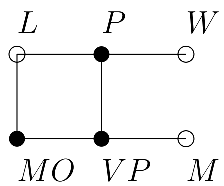
G: 

Figure 12: Locating-Total Dominating Set

For the remaining vertices in $V - S$ we have

$$N(L) \cap S = \{P, MO\}$$

$$N(W) \cap S = \{P\}$$

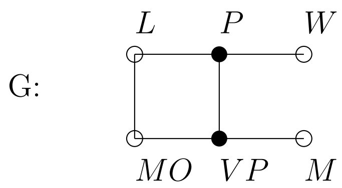
$$N(M) \cap S = \{VP\}$$

Since all of these sets are different, we conclude that S is a LTDS of G and $\gamma_t^L(G) \leq 3$. Since the vertices W and M have degree 1, we must include the vertices

P and VP in our LTDS. As shown above, these vertices alone do not form a LTDS of G and so we have $\gamma_t^L(G) \geq 3$. Hence, it follows that $\gamma_t^L(G) = 3$ and $S = \{P, VP, MO\}$ is a $\gamma_t^L(G)$-set.

A set S of vertices of a graph G is called a *differentiating total dominating set* (DTDS) of G if S is a total dominating set and for every pair of distinct vertices u and v in V, $N[u] \cap S \neq N[v] \cap S$. The *differentiating total domination number* $\gamma_t^D(G)$ is the minimum cardinality of a DTDS of G. A DTDS of minimum cardinality $\gamma_t^D(G)$ is called a $\gamma_t^D(G)$-set. Notice that every DTDS is a LTDS and so it follows that $\gamma_t^L(G) \leq \gamma_t^D(G)$ for every graph G without isolated vertices.

Let's revisit the floor plan of the building one last time.



If our set $S = \{P, VP\}$, then $N[P] \cap S = \{P, VP\}$ and $N[VP] \cap S = \{P, VP\}$ and so a "problem" in either of these two rooms would not be uniquely located. However, including the Main office in S as before yields the graph of Figure 13.

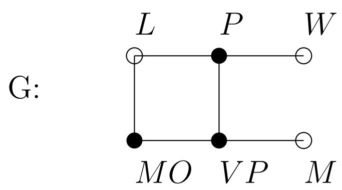


Figure 13: Differentiating Total Dominating Set

From this graph, we have

$$N[L] \cap S = \{P, MO\}$$

$$N[MO] \cap S = \{MO, VP\}$$

$$N[P] \cap S = \{P, VP\}$$

$$N[VP] \cap S = \{MO, VP, P\}$$

$$N[M] \cap S = \{VP\}$$

$$N[W] \cap S = \{P\}$$

Since all of these sets are different, we conclude that the set $S = \{MO, P, VP\}$ is a DTDS of G and so $\gamma_t^D(G) \leq 3$. Since $\gamma_t^L(G) \leq \gamma_t^D(G)$, we have $\gamma_t^D(G) \geq 3$. Hence, it follows that $\gamma_t^D(G) = 3$ and $S = \{P, VP, MO\}$ is a $\gamma_t^D(G)$-set.

In many cases, the locating-total domination number is smaller than the differentiating total domination number since every vertex in the graph must be differentiated, but only the vertices in $V - S$ must be located. Consider the graph H shown below in Figure 14 examined earlier.

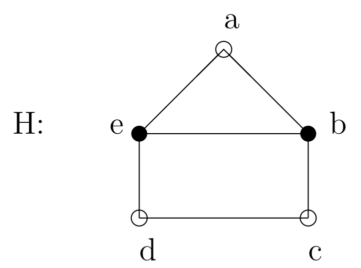


Figure 14: Locating-Total Domination Example

Notice that the set $S = \{b, e\}$ forms a locating-total dominating set of H, and it follows that $\gamma_t^L(H) = 2$. However, S is not a differentiating total dominating set of H

since vertices b and e cannot be differentiated in terms of their closed neighborhoods intersected with S. Furthermore, no other pair of vertices form a differentiating total dominating set. Hence, $\gamma_t^D(H) \geq 3$.

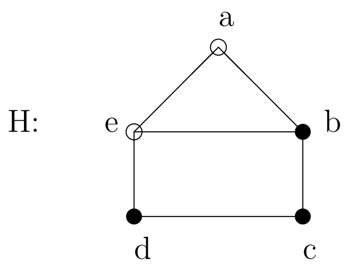


Figure 15: Locating-Total Domination Example

Notice that $S = \{b, c, d\}$ is a differentiating total dominating set of H since all of the vertices of H have different closed neighborhoods intersected with the set S. Hence, it follows that $\gamma_t^D(H) = 3$. Therefore, for this example, we have $\gamma_t^L(H) < \gamma_t^D(H)$. In general, no two adjacent vertices are differentiating. Thus, $\gamma_t^D(G) \geq 3$ for all graphs G.

We note that the differentiating total domination number is not defined for every graph. Consider the complete graph K_5 in Figure 16.

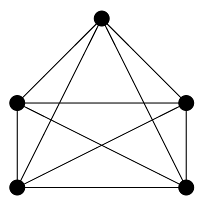


Figure 16: Complete graph, K_5

Since all edges between any two vertices u and v exist in this graph, $N[u] \cap S = N[v] \cap S$ for all $u, v \in V$. Therefore, no DTDS exists. In general, for any complete graph K_n, there does not exist a DTDS.

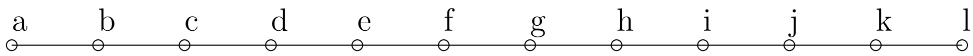
2 New Results-Locating-Total Domination

In this chapter, I will present the new results we have obtained for locating-total domination for trees. We know that every locating-total dominating set of a graph is also a total-dominating set of the graph, and so it follows that $\gamma_t^L(G) \geq \gamma_t(G)$ for every graph G.

Note that a special kind of tree is a path. In the case when G is a path, every total-dominating set of G is also a locating-total dominating set of G. Thus, the locating-total domination number of a path is precisely its total domination number and we have the following theorem.

Theorem 1 *For $n \geq 2$, $\gamma_t^L(P_n) = \gamma_t(P_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$.*

To illustrate, consider the following path P_{12} shown below.



From every four vertices in the path, if we put the middle two vertices in the set, we will form a locating-total dominating set of P_{12} as shown in Figure 17.

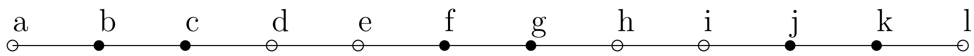
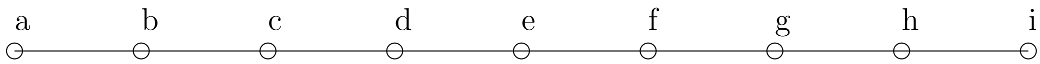


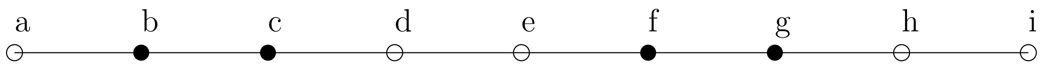
Figure 17: Locating-total domination number of a path

Hence, we have $\gamma_t^L(P_{12}) = \lfloor \frac{12}{2} \rfloor + \lceil \frac{12}{4} \rceil - \lfloor \frac{12}{4} \rfloor = 6 + 3 - 3 = 6$ and the theorem holds.

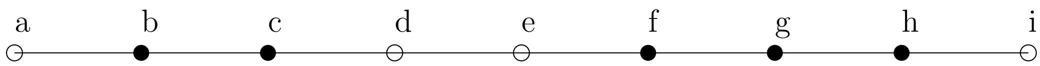
Next consider the path P_9 shown below.



Following the pattern, our locating-total dominating set would include the darkened vertices below.



However, notice that vertex i is not dominated, and so we must include vertex h in the set in order to totally dominate vertex i. Therefore, a locating-total dominating set of P_9 is shown below.



Hence, we have $\gamma_t^L(P_9) = \lfloor \frac{9}{2} \rfloor + \lceil \frac{9}{4} \rceil - \lfloor \frac{9}{4} \rfloor = 4 + 3 - 2 = 5$ and the theorem holds.

The next result presents a lower bound on the locating-total domination number of a tree in terms of its order and characterizes those trees T for which equality is achieved. Let $\mathcal{T}_1$ be the family of trees that can be obtained from k disjoint copies of P_4 by first adding $k-1$ edges in such a manner that they are incident only with support vertices and the resulting graph is connected. Then subdivide each new edge exactly once.

Theorem 2 *If T is a tree of order $n \geq 2$, then*

$$\gamma_t^L(T) \geq \frac{2}{5}(n+1),$$

with equality if and only if $T \in \mathcal{T}_1$.

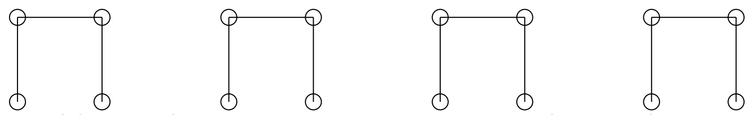
Proof. Let T be a tree of order n, and let S be a $\gamma_t^L(T)$-set. Let $T_1, T_2, ..., T_k$ be the components of $T[S]$. Notice that every component of $T[S]$ contains at least two vertices in S. Thus, we have $|S| \geq 2k$, and so $k \leq |S|/2$.

Let P be the set of all external private neighbors of vertices in S. Thus, if $v \in P$, then $|N(v) \cap S| = 1$. Since each vertex of S has at most one external private neighbor, $|P| \leq |S|$. Let $R = V - S - P$ and let $|R| = r$. Note that each vertex in R is adjacent to at least two vertices in S.

Let each component $T_1, T_2, ..., T_k$ of $T[S]$ be represented by a single vertex $u_1, u_2, ..., u_k$, and let $K = \{u_1, u_2, ..., u_k\}$. Let F be a forest of order $k + r$ with $V(F) = K \cup R$. Then, a vertex $u \in K$ is adjacent to a vertex $v \in R$ in F if and only if the vertex v is adjacent in T to a vertex in the component of $T[S]$ corresponding to the vertex u. Then, $|E(F)| \geq 2|R| = 2r$, and so, $k + r = |V(F)| \geq |E(F)| + 1 \geq 2r + 1$. Thus, $r \leq k - 1$. Hence, $n - |S| = |V - S| = |P| + |R| \leq |S| + (k - 1) \leq 3|S|/2 - 1$, and so $n \leq 5|S|/2 - 1$. Consequently, $\gamma_t^D(T) = |S| \geq 2(n + 1)/5$.

This bound is sharp if and only if equality is achieved in each of the above inequalities. In particular, $k = |S|/2$ implying that each component of $T[S]$ is a K_2. Also, $V - S - P = R$ and $r = k - 1$. It follows that $T[R \cup S]$ is a tree in which each vertex in R has degree two. Moreover, $|P| = |S|$, and so, since $T[R \cup S]$ is a tree, $T[P \cup S]$ is the union of k disjoint paths P_4 where each vertex of P is a leaf of T. Hence, $T \in \mathcal{T}_1$. $\square$

To illustrate the sharpness of this bound, we look at a tree $T \in \mathcal{T}_1$. Consider the case where $k = 4$, that is, the tree T consists of 4 disjoint copies of a P_4 as shown below.



We will now add $r = k - 1 = 3$ new vertices and join them with the support vertices of each P_4 to make the resulting graph connected as shown in Figure 18. We note that there are other ways to connect the P_4's and hence T is not unique.

T:

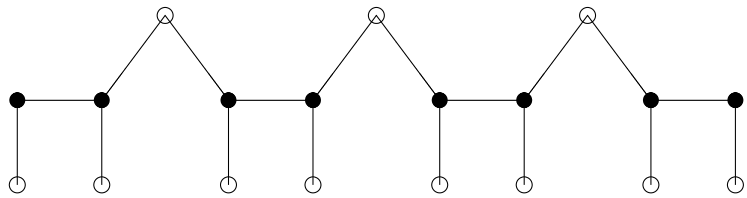


Figure 18: Locating-total dominating set for $T \in \mathcal{T}_1$

Notice that the darkened vertices form a locating-total dominating set for T and so, we have $\gamma_t^L(T) = \frac{2}{5}(19+1) = 8$ and the theorem holds.

The final result establishes a lower bound on the locating-total domination number of a tree in terms of its order and its number of leaves and support vertices. Let $\mathcal{T}_2$ be the family of trees T that can be obtained from any tree T' by attaching at least two leaves to each vertex of T' and, if T' is nontrivial, subdividing each edge of T' exactly once.

Theorem 3 *If T is a tree of order $n \geq 3$ with l leaves and s support vertices, then*

$$\gamma_t^L(T) \geq \frac{n + 2(l-s) + 1}{3},$$

with equality if and only if $T \in \mathcal{T}_2$.

Proof. Let T be a tree of order n. If $n = 3$, then $\gamma_t^L(T) = 2 = (n+2(l-s)+1)/3$ and $T \in \mathcal{T}_2$. If $n = 4$, then either $T = K_{1,3}$, in which case $\gamma_t^L(T) = 3 = (n+2(l-s)+1)/3$ and $T \in \mathcal{T}_2$, or $T = P_4$, in which case $\gamma_t^L(T) = 2 > (n+2(l-s)+1)/3$. Suppose then that $n \geq 5$.

Let S be a $\gamma_t^L(T)$-set that contains a minimum number of leaves. At most one leaf neighbor of every support vertex is not in S. Assume that for some support vertex v, every leaf neighbor of v is in S. If v has a non-leaf neighbor x such that $N[x] \cap S = \{v\}$, then adding x to the set S and removing a leaf neighbor of v from S produces a new $\gamma_t^L(T)$-set containing fewer leaves that does S, a contradiction. Hence, every neighbor of v in $V - S$ has another neighbor in S. If v has two or more neighbors in S, then removing a leaf neighbor of v from S produces a locating-total dominating set with cardinality less than $\gamma_t^L(T)$, a contradiction. Hence, v has exactly one leaf neighbor u and $N[v] \cap S = \{u, v\} \subset S$. Then $(S - \{u\}) \cup \{x\}$, where x is a non-leaf neighbor of v, is a new $\gamma_t^L(T)$-set containing fewer leaves than does S, a contradiction. Hence, for every support vertex v, exactly one leaf neighbor of v is not in S.

Let $T_1, T_2, ..., T_k$ be the components of $T[S]$. Notice that any support vertex and its leaves that are in S are in the same component of $T[S]$. Hence the number of components of $T[S]$ is bounded above by the number of vertices in S that are not leaves of T. Thus our choice of S implies that $k \leq |S| - l + s$.

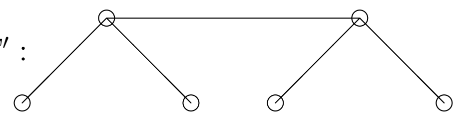
Let P be the set of all external private neighbors of vertices in S. Thus, if $w \in P$, then $|N(w) \cap S| = 1$. Since no leaf of T in the set S has any external private neighbors, and since each vertex of S has at most one external private neighbor, $|P| \leq |S| - l + s$.

Let $R = V - S - P$ and let $|R| = r$. Note that each vertex in R is adjacent to at least two vertices in S. Let K be a set of k vertices corresponding to the k components of $T[S]$. Let F be the forest of order $k + r$ as defined in the proof of Theorem 2. Then, as before, $|E(F)| \geq 2|R| = 2r$ and $r \leq k - 1 \leq |S| - l + s - 1$. Hence, $n - |S| = |V - S| = |P| + |R| \leq (|S| - l + s) + (|S| - l + s - 1)$, and so $n \leq 3|S| - 2(l - s) - 1$. Consequently, $\gamma_t^L(T) = |S| \geq (n + 2(l - s) + 1)/3$.

This bound is sharp if and only if equality is achieved in each of the above inequalities. In particular, $k = |S| - l + s$ implying that each component of $T[S]$ is a star of order at least 2. Also, $V - S - P = R$ and $r = k - 1$. It follows that every vertex in R has degree exactly 2. Hence $T \in \mathcal{T}_2$. $\square$

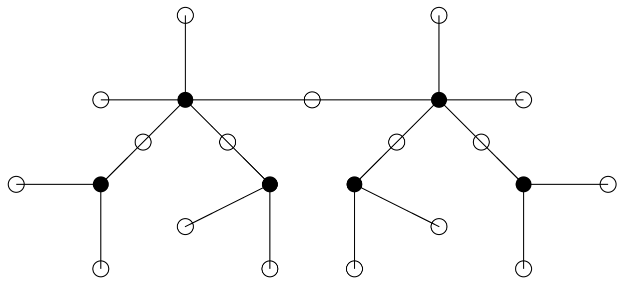
To illustrate the sharpness of this bound, we look at a tree $T \in \mathcal{T}_2$. We start with any tree T' such as the one shown below.

T' :



To each vertex of T', we attach at least two leaves and subdivide each edge of T' once to produce the tree T shown below.

T :



To form a locating-total dominating set of T, we must include all of the vertices of the original tree T' as well as all but one leaf attached to each vertex of T'. The darkened vertices in Figure 19 below represent the locating-total dominating set S of T.

T :

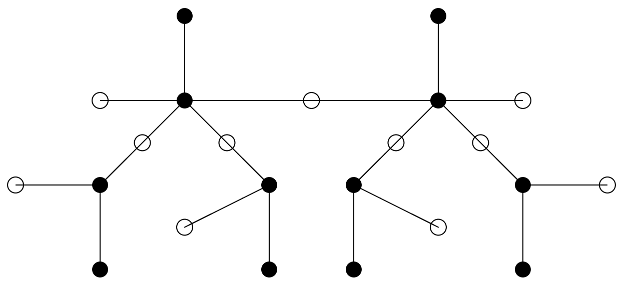


Figure 19: Locating-total dominating set for $T \in \mathcal{T}_2$

Notice that each component of $T[S]$ is a star of order 2 and the number of components is $k = |S| - l + s = 12 - 12 + 6 = 6$. Furthermore, $R = V - S - P = 23 - 6 - 12 = 5$ implying $r = k - 1 = 6 - 1 = 5$. Therefore, every vertex of R has degree 2, and we have $\gamma_t^L(T) = \frac{n+2(l-s)+1}{3} = \frac{23+2(12-6)+1}{3} = \frac{36}{3} = 12$, and so the theorem holds.

3 New Results-Differentiating Total Domination in Trees

Recall that a *differentiating total dominating set* is a set S of minimum cardinality satisfying two properties

(1) S is a total dominating set.

(2) For all distinct vertices u and v, $S \cap N[u] \neq S \cap N[v]$.

Since every differentiating total dominating set is a locating total dominating set, we have $\gamma_t^L(G) \leq \gamma_t^D(G)$.

Gimbel et. al. [3] defined two vertices u and v to be *redundant* if $N[u] = N[v]$. Furthermore, they define a graph to be *distinguishable* if the differentiating number is defined. They showed that a graph with no isolated vertex is distinguishable if and only if it contains no pair of redundant vertices. In addition, they showed that almost every graph is distinguishable and all trees of order at least three are distinguishable. For our research on differentiating total domination, we investigated the differentiating total domination number in trees. We begin with the simplest of trees, namely paths.

Theorem 4 *For $n \geq 3$,*

$$\gamma_t^d(P_n) = \begin{cases} \lceil \frac{3n}{5} \rceil & \text{if } n \not\equiv 3 (\text{mod } 5) \\ \lceil \frac{3n}{5} \rceil + 1 & \text{if } n \equiv 3 (\text{mod } 5) \end{cases}$$

Proof. We proceed by induction on n. Clearly, the result can be verified for small values of n, $3 \leq n \leq 7$. Let $n \geq 8$ and suppose the result holds for all paths of order n' where $3 \leq n' < n$. Let $T : v_1, v_2, ..., v_n$ be a path of order n. Let S be a $\gamma_t^D(T)$-set.

Notice we can choose S so that $v_1 \notin S$. For if $v_1 \in S$, let v_j be the vertex of smallest subscript that is not in S, and replace v_1 in S by v_j to get a new $\gamma_t^D(T)$-set. Since $v_1 \notin S$, it follows that $\{v_2, v_3, v_4\} \subset S$. Similarly, we can choose $v_5 \notin S$. For if $v_5 \in S$, we can replace v_5 with v_k in S, where v_k is the vertex of smallest subscript such that $k > 5$ and $v_k \notin S$, to form a new $\gamma_t^D(T)$-set. Let $T' = T - \{v_1, v_2, v_3, v_4, v_5\}$. Then T' is a path of order $n' = n - 5 \geq 3$ and $S - \{v_2, v_3, v_4\}$ is a differentiating total dominating set of T'. Thus, $\gamma_t^D(T') \leq |S| - 3 = \gamma_t^D(T) - 3$, or, equivalently, $\gamma_t^D(T) \geq \gamma_t^D(T') + 3$. Let

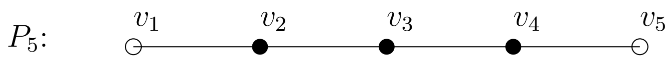
$$D = \bigcup_{i=0}^{\lfloor \frac{n-4}{5} \rfloor} \{v_{5i+2}, v_{5i+3}, v_{5i+4}\}.$$

We consider two cases.

Case 1. $n \equiv 3 (\text{mod } 5)$. Then $n' \equiv 3 (\text{mod } 5)$. Applying the inductive hypothesis to $T', \gamma_t^D(T') = \lceil \frac{3n'}{5} \rceil + 1 = \lceil \frac{3n}{5} \rceil$ - 2. Hence, $\gamma_t^D(T) \geq \gamma_t^D(T') + 3 = \lceil \frac{3n}{5} \rceil + 1$. On the other hand the set $S = D \cup \{v_{n-2}, v_{n-1}, v_n\}$ is a differentiating total dominating set of T, and so $\gamma_t^D(T) \leq |S| = \lceil \frac{3n}{5} \rceil + 1$. Consequently, $\gamma_t^D(T) = \lceil \frac{3n}{5} \rceil + 1$.

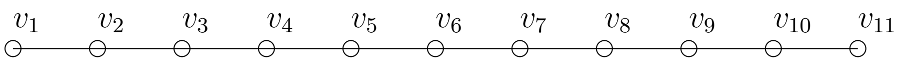
Case 2. $n \not\equiv 3 (\text{mod } 5)$. Then $n' \not\equiv 3 (\text{mod } 5)$. Applying the inductive hypothesis to $T', \gamma_t^D(T') = \lceil \frac{3n'}{5} \rceil = \lceil \frac{3n}{5} \rceil$ - 3. Hence, $\gamma_t^D(T) \geq \gamma_t^D(T') + 3 = \lceil \frac{3n}{5} \rceil$. If $n \equiv 0$ or $4 (\text{mod } 5)$, set $S = D$. If $n \equiv 1 (\text{mod } 5)$, let $S = D \cup \{v_{n-1}\}$. If $n \equiv 2 (\text{mod } 5)$, let $S = D \cup \{v_{n-2}, v_{n-1}\}$. Then, S is a differentiating total dominating set of T, and so $\gamma_t^D(T) \leq |S| = \lceil \frac{3n}{5} \rceil$. Consequently, $\gamma_t^D(T) = \lceil \frac{3n}{5} \rceil$. $\square$

To illustrate, consider the path P_5 shown below.



Notice that the darkened vertices form a differentiating total dominating set for the P_5. The proof follows that we can take a path of any length, divide it into smaller paths as shown above, and choose the middle three vertices to be in our differentiating total dominating set, S. If five does not evenly divide the length of the entire path, then we may have to add additional vertices to our set S. This fact is taken care of in the two cases within the proof.

For example, consider the path P_{11} shown below.



Notice that $11 \not\equiv 3 (\text{mod } 5)$. According to the proof, a differentiating total dominating set will consist of the vertices v_2, v_3, v_4 and v_7, v_8, v_9. By Case 2, since $n \equiv 1 (\text{mod } 5)$, we must also include vertex v_{11-1}, that is, v_{10}. This differentiating total dominating set is shown by the darkened vertices in Figure 20.

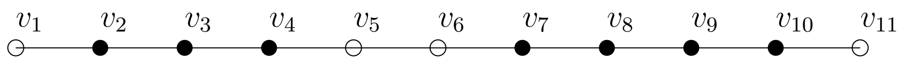


Figure 20: A differentiating total dominating set for P_{11}.

Hence, $\gamma_t^D(P_n) = \lceil \frac{3(11)}{5} \rceil = 7$ and the theorem holds.

For a set $S \subset V$, the set S is a *packing* if the vertices in S are pairwise at distance at least 3 apart in G.

Consider the tree shown below.

T :

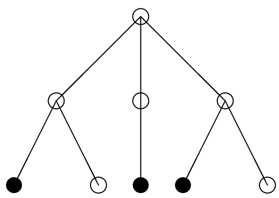


Figure 21: A packing in a tree T

Notice that the darkened vertices form a packing of T, since any two of the darkened vertices are at distance 4 apart.

Using the idea of a packing as well as the differentiating total domination number of a path from the previous theorem, we have the following result which provides an upper bound on the differentiating total domination number of a tree in terms of its order and number of support vertices.

Theorem 5 *If $T \neq P_4$ is a tree of order $n \geq 4$ with s support vertices, then*

$$\gamma_t^D(T) \leq n - s.$$

Proof. Let S be a packing in T consisting of precisely s leaves. Since $T \neq P_4$ and $n \geq 4$, $T[V - S]$ is a tree of order at least 3. It follows that $V - S$ is a differentiating total dominating set of T, and so $\gamma_t^D(T) \leq |V - S| = n - s$. □

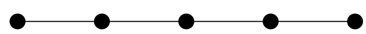
To illustrate, consider the tree T shown in Figure 21. Notice that there are 3 support vertices, and so, $s = 3$. Therefore, we will let S be the set of darkened vertices, which form a packing in T. Now, we consider $T[V - S]$, illustrated by the white vertices. Notice that these vertices form a differentiating total dominating set of T, and so it follows that $\gamma_t^D(T) \leq |V - S| = n - s = 9 - 3 = 6$, and the theorem holds.

As an immediate consequence of Theorem 7, we have the following result.

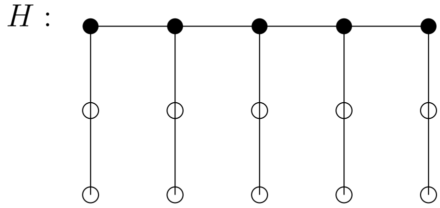
Corollary 6 *If T is a tree of order $n \geq 4$, then $\gamma_t^D(T) \leq n - 1$ with equality if and only if $T = P_4$, or T is a star.*

For $k \geq 1$, the *k-corona* of a graph H is the graph of order $(k+1)|V(H)|$ obtained from H by attaching a path of length k to each vertex of H so that the resulting paths are vertex disjoint. In particular, the 1-corona of H, also called the *corona* of H and denoted by $H \circ K_1$, is obtained from H by adding a pendant edge to each vertex of H. For example, let $H = P_5$ as shown below.

H :



To form the 2-corona of H, denoted $H \circ K_2$, we will attach a path of length 2 to each vertex of H.



Next, we present a lower bound on the differentiating total domination number of a tree in terms of its order and characterize those trees T for which equality is achieved. Let $\mathcal{T}_3$ be the family of trees which can be obtained from k disjoint copies of a $P_3 \circ K_1$ by first adding $k-1$ edges in such a manner that they are incident only with support vertices and the resulting graph is connected, and then subdividing each new edge with a single vertex. Since $\gamma_t^D(G) \geq \gamma_D(G)$ for all distinguishable graphs with no isolated vertex, our next result is an immediate consequence of a theorem in Gimbel et al. [3] However, we include a proof here for completeness.

Theorem 7 *If T is a tree of order $n \geq 3$, then*

$$\gamma_t^D(T) \geq \frac{3}{7}(n+1),$$

with equality if and only if $T \in \mathcal{T}_3$.

Proof. Let T be a tree of order n, and let S be a $\gamma_t^D(T)$-set. Let $T[S]$ be the subgraph of T induced by S, and let $T_1, T_2, ..., T_k$ be the components of $T[S]$. Then $|S| \geq 3k$, and so we have, $k \leq |S|/3$.

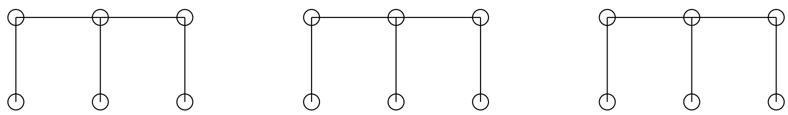
Let P be the set of all external private neighbors of vertices in S. Thus, if $v \in P$, then $|N(v) \cap S| = 1$. Since each vertex of S has at most one external private neighbor,

$|P| \leq |S|$. Let $R = V - S - P$, and $|R| = r$. Notice, each vertex in R is adjacent to at least two vertices in S.

Let each component $T_1, T_2, ..., T_k$ of $T[S]$ be represented by a single vertex $u_1, u_2, ...u_k$ and let $K = \{u_1, u_2, ..., u_k\}$. Let F be a forest of order $k + r$ with $V(F) = \{K \cup R\}$. Then, a vertex $u \in K$ is adjacent to a vertex $v \in R$ in F if and only if the vertex v is adjacent in T to a vertex in the component of $T[S]$ corresponding to the vertex u. Then, $|E(F)| \geq 2|R| = 2r$, and so, $k + r = |V(F)| \geq |E(F)| + 1 \geq 2r + 1$. Thus, $r \leq k - 1$. Hence, $n - |S| = |V - S| = |P| + |R| \leq |S| + (k - 1) \leq 4|S|/3 - 1$, and so, $n \leq 7|S|/3 - 1$. Consequently, $\gamma_t^D(T) = |S| \geq 3(n + 1)/7$.

This bound is sharp if and only if equality is achieved in each of the above inequalities. In particular, $k = |S|/3$ implying that each component $T[S]$ is a P_3. Also, $V - S - P = R$ and $r = k - 1$. It follows that $T[R \cup S]$ is a tree in which each vertex in R has degree 2. Moreover, $|P| = |S|$, and so, since $T[R \cup S]$ is a tree, $T[P \cup S]$ is the union of k disjoint copies of $P_3 \circ K_1$, where each vertex of P is a leaf of T. Hence, $T \in \mathcal{T}_3$. $\square$

To illustrate the sharpness of this bound, we look at a tree $T \in \mathcal{T}_3$. Consider the case where $k = 3$, that is, the tree T consists of 3 disjoint copies of a $P_3 \circ K_1$ as shown below.



We will now add $r = k - 1 = 2$ new vertices and join them with the support vertices of each $P_3 \circ K_1$ to make the resulting graph connected as shown in Figure 22.

$T:$

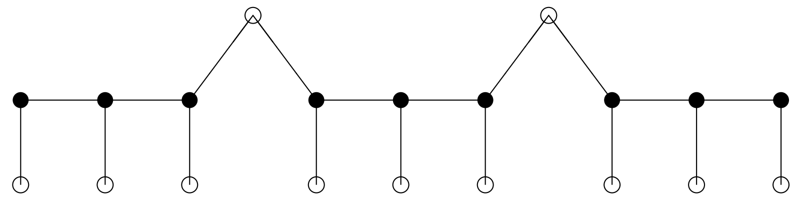


Figure 22: Differentiating-total dominating set for $T \in \mathcal{T}_3$

Notice that the darkened vertices form a differentiating-total dominating set for T and so, we have $\gamma_t^D(T) = \frac{3}{7}(20+1) = 9$ and the theorem holds.

Our final result shows that the ratio $\gamma_t^D(T)/\gamma_t^L(T)$ is bounded below by 1 and above by 3/2 when T is a tree.

Theorem 8 *For any tree* T*,*

$$\gamma_t^L(T) \leq \gamma_t^D(T) \leq \frac{3}{2}\gamma_t^L(T)$$

and these bounds are sharp.

Proof. Since every differentiating total dominating set is a locating-total dominating set, we have $\gamma_t^L(T) \leq \gamma_t^D(T)$ for all trees T. This establishes the lower bound. To establish the upper bound on $\gamma_t^D(T)$, let S be a $\gamma_t^L(T)$-set. Suppose $T[S]$ has k components. Since every component of $T[S]$ has at least two vertices, $|S| \geq 2k$, and so $k \leq |S|/2$. For each 2-component of $T[S]$, add to the set S a vertex in $V - S$ that is adjacent to a vertex in that component. Then the resulting set S' is a differentiating total dominating set of T, and so, we have $\gamma_t^D(T) \leq |S'| \leq |S| + k \leq 3|S|/2 = 3\gamma_t^L(T)/2$.

Equality is achieved in the lower bound by taking, for example, T to be the corona of a tree of order at least 3, while equality is achieved in the upper bound by taking, for example, T to be the 3-corona of any tree. □

To illustrate the sharpness of the lower bound, consider the tree T shown below.

T :

We form the corona of T by attaching a path of length 1 to every vertex in T.

T :

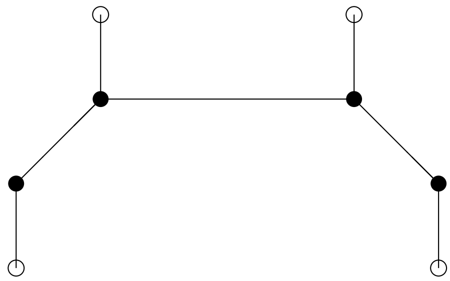
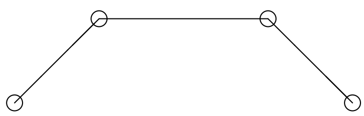


Figure 23: A tree T for which $\gamma_t^L(T) = \gamma_t^D(T)$

Notice that the darkened vertices form a locating-total dominating set as well as a differentiating total dominating set of T. Hence, $\gamma_t^L(T) = \gamma_t^D(T) = 4$.

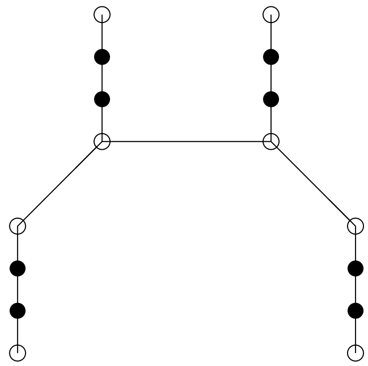
To illustrate the sharpness in the upper bound, consider the tree T as above.

T :



We now form the 3-corona of T by attaching a path of length 3 to all of the vertices in T.

T :



Notice that the darkened vertices form a locating-total dominating set of T. However, they do not form a differentiating total dominating set of T because the darkened vertices cannot be distinguished. In fact, in order to form a differentiating total dominating set of T, we must include at least 3 vertices from each "leg" as shown in Figure 24.

T :

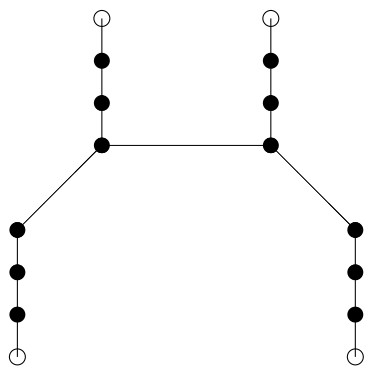


Figure 24: A tree T for which $\gamma_t^D(T) = \frac{3}{2}\gamma_t^L(T)$

Therefore, we have, $\gamma_t^L(T) = 8$ and $\gamma_t^D(T) = 12$, and so it follows that $\gamma_t^D(T) = \frac{3}{2}\gamma_t^L(T)$.

It would be very interesting in the future to try to characterize those trees for which $\gamma_t^D(T) = \frac{3}{2}\gamma_t^L(T)$. For now, we conclude only that the bound is sharp.

There is still much more research to be done on this topic of locating-total dominating sets and differentiating total dominating sets. For this thesis, we only considered the parameters as applied to trees. Further research will consider the parameters applied to other families of graphs. In [6], the concepts of a locating set and a dominating set are merged by defining the *metric-locating-dominating set* in a connected graph G to be a set of vertices of G that is both a dominating set and a locating set in G. This topic could be extended to similarly define a *metric-locating-total dominating set* in a graph G and this new parameter could also be explored. In relation to the work in this thesis, additional research may include finding trees for which there is a unique locating-total dominating set and a unique differentiating total dominating set. Hopefully, we will investigate these and other topics related to locating-total dominating and differentiating total dominating sets in the future.

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