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Paired and Total Domination on the Queen's Graph.

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Paired and Total Domination on the Queen's graph

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ABSTRACT

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The Queen's domination problem has a long and rich history. The problem can be simply stated as: What is the minimum number of queens that can be placed on a chessboard so that all squares are attacked or occupied by a queen? The problem has been expanded to include not only the standard 8x8 board, but any rectangular $m \times n$ sized board. In this thesis, we consider both paired and total domination versions of this renowned problem.

DEDICATION

This thesis is dedicated to Mr. Pete Shaw and many others too numerous to mention. Without their time spent over the many years this thesis wouldn't have been possible.

ACKNOWLEDGEMENTS

It is a pleasure to thank the many people that made this thesis possible. I would first like to thank my advisors. Without their immeasurable patience and kind words this project wouldn't have been possible. Much thanks also goes to Travis Coake for help on the software. Thanks also to Steve Lane for verifying many of the constructions. I would also like to thank my family for their emotional support. They also have exercised immeasurable patience with me through my college endeavors.

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1 INTRODUCTION

1.1 Queen's Domination

The Five Queen's Problem can be simply stated as the following: What is the minimum number of queens that can be placed on a chessboard so that every square is attacked or occupied? The problem has been generalized to include not only the standard 8×8 board, but also any square, $n \times n$ sized board. This more general problem is known as the Queen's domination problem. The Queen's domination problem has been generalized even further to include rectangular, $m \times n$ sized boards. Much work has been done on rectangular boards for this problem, however, in this thesis we will only consider square boards.

It is often helpful in studying this problem to conceptualize the Queen's domination problem in terms of graph theory. The board itself can be represented as a set of vertices (or squares). Edges are added between any two vertices if it is possible to move from one of the corresponding squares to the other by a single move of the queen. Recall a queen can move any distance vertically, horizontally, or diagonally. Hence a pair of vertices have an edge between them if their corresponding squares share a common row, column, or diagonal. An $n \times n$ board can be represented by a graph having exactly n^2 vertices, with edges added using the above rule. This corresponding graph is called the *Queen's graph*, and is denoted Q_n.

On any graph, two vertices are said to be *adjacent* if they are joined by an edge. By definition, a given vertex is said to *dominate* itself and any adjacent vertices. A graph G is said to be *dominated* by a subset of vertices, say D, if any vertex in G is dominated by a vertex in D.

Applying the above to the Queen's graph, a board is dominated by a set of queens if every square on the board is either occupied or attacked by a queen. The minimum number of queens needed to dominate a given $n \times n$ board, denoted $\gamma(Q_n)$, is known as the *domination number* of the Queen's graph. For the standard, 8×8 chessboard, it has been proven that $\gamma(Q_8) = 5$. In 1964, Yaglom and Yaglom [25] showed that there are exactly 4860 unique placements of five dominating queens on the standard 8×8 chessboard. One of these solutions is given below.

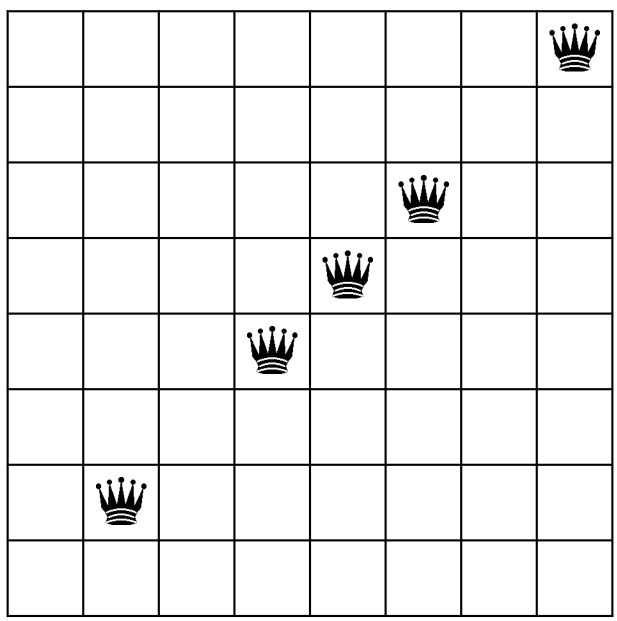

Figure 1: A Dominating Set for $n = 8$

The Queen's domination problem was formally proposed by de Jaenisch in 1862 [17]. The problem's significance lies partly in the fact that it was the first known problem which considered domination. When the mathematical concept of domination was formalized in 1958 with Berg and Ore [15], the problem itself was already 95 years old. With its rich history many have turned their attention to the problem and, as mentioned by Cockayne [8], it helped facilitate a "revival" in the study of domination type problems in the 1970's.

Since the inception of the Queen's domination problem, much progress has been made. In 1892, Rouse Ball [3] provided minimum dominating sets of Q_n for $n \leq 8$. Ahrens [1] followed this in 1910 by providing minimum dominating sets of Q_n for $9 \leq n \leq 13$ and $n = 17$. Many of the proofs that these were actually minimum dominating sets came more recently when work began on lower bounds. Beginning with Spencer [8] in 1990, work on lower bounds followed from Burger, Mynhardt, Cockayne, Weakley, Gibbons, Webb, and Kearse [3, 4, 5, 6, 11, 18, 24]. Spencer's lower bound is especially important to the contents of this thesis and will be considered further.

The necessity of lower bounds for the Queen's domination problem should be noted. In 1964, Yaglom and Yaglom [25], as mentioned above, showed there are exactly 4860 placements of five queens on the standard 8×8 chessboard that dominate the board. Their method was exhaustive and is simply not plausible for large values of n. With the Queen's domination problem classified as NP-complete, even computer searches are limited for large board sizes. Thus, lower bounds for $\gamma(Q_n)$ are necessary for large values of n to show a given dominating set is minimum.

Work on upper bounds has also seen recent progress. In 1990, L. Welsh [22] provided a formation of queens that showed for n divisible by 3, $\gamma(Q_n) \geq \dfrac{2n}{3}$. Welsh's construction is also of significant importance here and will be considered in detail.

The necessity of upper bounds should, likewise, be discussed. Finding minimum dominating sets, even for relatively small board sizes, can be quite difficult. With an exhaustive method not feasible for larger values of n, constructions are given for specified board sizes. In this way, these constructions are done "in bulk", yielding upper bounds on $\gamma(Q_n)$. An example of this is Welch's construction. The specified board size is for $n \equiv 0 \pmod 3$. Constructions have followed for board sizes of $n \equiv 3 \pmod{24}$ and $n \equiv 26 \pmod{46}$ [10, 12]. It should be noted that more recent upper bounds have been given by considering specific types of coverings, the Parallelogram Law, and an algorithm developed by Knuth as cited in [20]. Though similar types of work may prove to be fruitful for both paired and total domination on the Queen's graph, for now they are left for future work.

The dominating set illustrated in Figure 1 has two interesting characteristics. First, it is a minimum dominating set of queens. Second, the queens have all been placed along one of the main diagonals of the board. This leads to an obvious question: Can one always find a minimum dominating set of queens that are all placed along one of the main diagonals of the board? Clearly one can dominate the $n \times n$ board by placing queens in every square along the main diagonal. However, limiting the placement of queens to the main diagonal may not allow for a *minimum* dominating set of queens. It should be noted that although not possible in general, it is possible for many small values of n to find a minimum dominating set using a placement of queens along the main diagonal. To study precisely when a minimum dominating set can be constructed by placing queens along the main diagonal of the board, the diagonal number has been introduced.

The *diagonal number* is defined as the minimum number of queens placed along the main diagonal of the board so that the board is dominated. For a given $n \times n$ board, this number is denoted as $\gamma_{diag}(Q_n)$. For any $n \times n$ board, with $n \geq 3$, a diagonally dominating set may be constructed by $n-2$ queens. To see this, simply form a 3×3 subboard in one of the corners of the board. Place queens in all squares on the main diagonal not on this 3×3 subboard. A queen is then placed in the center square of the 3×3 subboard. These $n-2$ queens form a diagonally dominating set as can be seen in Figure 2. It follows that $\gamma_{diag}(Q_n) \leq n-2$ for any $n \geq 3$.

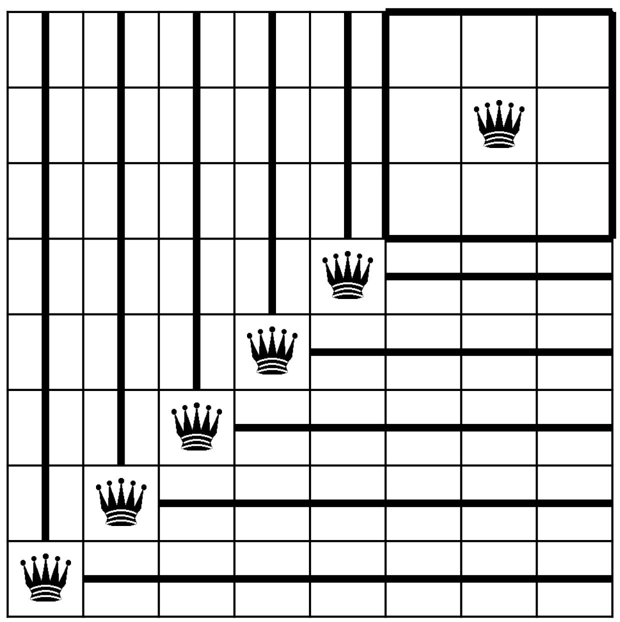

Figure 2: Constructing a Diagonally Dominating Set with $n-2$ Queens for $n \geq 3$

The diagonal number has been reduced by Cockayne and Hedetniemi [9] to a well studied, number-theoretic function. Also important for both paired and total domination on the Queen's graph, the diagonal number will be explored more in the next section.

1.2 Paired and Total Domination

Since work began on combinatorial chessboard problems, interest in many different domination parameters has been expressed. In 1910, Ahrens [1] posed two different questions in addition to the standard queen's domination problem. These two problems can be stated as:

1. What is the minimum number of queens that can be placed on a board so that every square is attacked or occupied and no two queens attack one another?

2. What is the minimum number of queens that can be placed on a board so that every square is attacked and not simply occupied?

The first question has been studied alongside the standard Queen's domination problem and much progress has been made on it. It deals with the domination parameter known as *independent domination.* A set of vertices is defined as *independent* if no two vertices in the set are adjacent. A set D of vertices is said to *independently dominate* a graph G if D dominates G and D is an independent set. The minimum cardinality among all independent dominating sets for a graph G is known as the *independent domination number* of G. On the Queen's graph this number is denoted $i(Q_n)$. Because any independent dominating set must also be a dominating set, it follows that $\gamma(Q_n) \leq i(Q_n)$.

Further relating this parameter to the standard domination parameter on the Queen's graph, upper bounds for $\gamma(Q_n)$ have been improved, in part, by reducing the size (number of edges) of the subgraph induced by the dominating set. Since the size of the subgraph induced by any independently dominating set is zero, it would seem $i(Q_n)$ would provide a very good upper bound for $\gamma(Q_n)$. In fact, it has been recently shown that $\lim_{n\to\infty} \dfrac{\gamma(Q_n) - i(Q_n)}{n} < 0.031$ [20].

The second question deals with the domination parameter known as *total domination*. A set D of vertices is said to *totally dominate* G if D dominates G and every vertex in D is adjacent to another vertex in D. The minimum cardinality among all total dominating sets for a graph G is known as the *total domination number* of G, denoted as $\gamma_t(G)$. For the Queen's graph this is denoted as $\gamma_t(Q_n)$. Note that $\gamma_t(G)$ exists only for graphs without isolated vertices. On the Queen's graph, a value for $\gamma_t(Q_1)$ doesn't exist since the graph for Q_1 is one vertex. Results for $\gamma_t(Q_n)$ have not been produced since 1910 when Ahrens [1] provided $\gamma_t(Q_n)$ values for $n \leq 9$.

Similar to the way in which $\gamma(Q_n)$ and $i(Q_n)$ are studied side by side, we introduce the study of paired domination on the Queen's graph alongside of total domination. For any graph G, the set of vertices D is defined as a *paired dominating set* if D is a dominating set and the subgraph induced by D has a *perfect matching*. The minimum cardinality among all paired dominating sets, for a graph G, is known as the *paired domination number* of G. For the Queen's graph, we say there exists a *perfect matching* among a set of queens if they can be placed on the board, two at a time, in attacking pairs. The paired domination number for a $n\times n$ board is denoted $\gamma_{pr}(Q_n)$. The existence of a perfect matching implies $\gamma_{pr}(G)$ must be even for any graph G. It should be noted that, like the total domination parameter, $\gamma_{pr}(G)$ exists only for graphs without isolated vertices. Hence a value for $\gamma_{pr}(Q_1)$ does not exist.

Paired domination was introduced in 1998 by Haynes and Slater [13]. Work has followed on paired domination, including, a close look at the relationship between total domination and paired domination parameters [13, 14, 21]. Note that any paired dominating set of a graph G is also a total dominating set. Thus $\gamma(G) \leq \gamma_t(G) \leq \gamma_{pr}(G)$ for any graph G without isolates. It also follows that since no vertex can be adjacent to itself, any total dominating set must have at least two vertices. Thus $2 \leq \gamma_t(G) \leq \gamma_{pr}(G)$.

There is also a relationship between paired and total domination that might prove to be of particular interest on the Queen's graph. When $\gamma_t(G)$ is even, the subgraph induced by the total dominating set has a minimum size of $\gamma_t(G)/2$. Similarly, the subgraph induced by any paired dominating set has a minimum size of $\gamma_{pr}(G)/2$. As noted previously, upper bounds for $\gamma(Q_n)$ have been improved, in part, by reducing the size of the subgraph induced by the dominating set. Similar to the way in which $i(Q_n)$ has provided a good upper bound for $\gamma(Q_n)$, $\gamma_{pr}(Q_n)$ may provide a good upper bound for $\gamma_t(Q_n)$.

As mentioned previously, there are relationships that exist between both paired and total domination numbers with the diagonal number. Recall the diagonal number is defined as the minimum number of queens placed along the main diagonal of the board so that the board is dominated. Note if there is more than one queen placed along the main diagonal, then all queens along the main diagonal are attacked. Thus any diagonally dominating set of at least two queens is also a total dominating set of queens. Hence if $\gamma_{diag}(Q_n) > 1$, then $\gamma_t(Q_n) \leq \gamma_{diag}(Q_n)$.

Similarly, consider a placement of an even number of queens along the main diagonal. A perfect matching among these squares occupied by queens can be defined in obvious fashion. It follows that a diagonally dominating set of even cardinality is a paired dominating set. Thus if $\gamma_{diag}(Q_n)$ is even, then $\gamma_{pr}(Q_n) \leq \gamma_{diag}(Q_n)$.

Now consider a placement of an odd number of queens along the main diagonal. Note first that for $n = 1$, $\gamma_{pr}(Q_n)$ doesn't exist. Note also that for $n \geq 2$, $\gamma_{diag}(Q_n) \leq n-1$. It follows that, for $n \geq 2$, there is at least one empty square on the main diagonal. Adding another queen to the main diagonal would provide a set of diagonally dominate queens whose corresponding squares could be perfectly matched. Hence if $\gamma_{diag}(Q_n)$ is odd and $n \neq 1$, then $\gamma_{pr}(Q_n) \leq \gamma_{diag}(Q_n) + 1$.

2 UPPER BOUNDS

Much of the recent work on the Queen's domination problem has focused on improving existing upper bounds. This has been done, in part, by finding particular formations of queens that dominate various board sizes. One such formation in particular has implications for both paired and total domination. In 1990 Welsch [22] provided a formation of queens that produced the theorem below.

Theorem 1 *Welsch [22]: Let $n = 3q + r$ where $0 \leq r < 3$. Then $\gamma(Q_n) \leq 2q + r$.*

To see the general idea behind the proof, suppose $n \equiv 0 \pmod 3$. Begin by splitting the board into 9 regions of equal size. Label the bottom regions of the board I-III from left to right, the middle regions IV-VI, and the top regions of the board VII-IX. Queens are then placed in the bottom-left corner of region I, along the diagonal to the immediate right of the main diagonal in region I, and along the main diagonal of region IX. In this formation, it can be seen there is exactly one queen in each column and row of regions I and IX. It follows there are exactly $\frac{2}{3}n$ queens in this placement. Figure 3 illustrates Welsch's formation for a 12×12 chessboard.

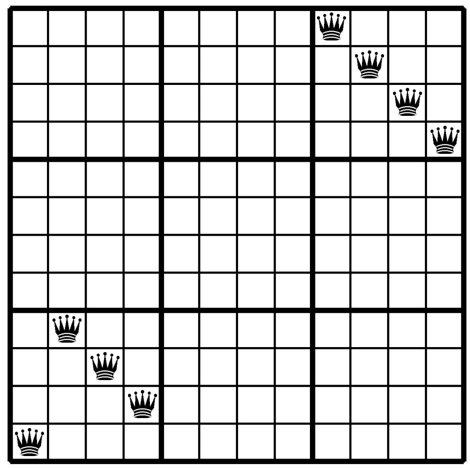

Figure 3: Welsch's Formation for $n = 12$

This set of queens has been shown to dominate the board for any n, where $n \equiv 0 \pmod 3$. To see this, one can simply note that the squares in region I-III and regions VII-IX are all dominated row-wise by the queens in regions I and IX respectively. Regions IV and VI are dominated column-wise by the queens in regions I and IX respectively. This leaves region V which is diagonally dominated by the queens in regions I and IX. A slight modification of this formation will yield a dominating set for other values of n. In these cases, use Welsch's formation to dominate a $m \times m$ subboard, where m is the largest value for which $m \equiv 0 \pmod 3$ and $m \leq n$. Depending upon whether $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$, there are either one or two rows and columns not entirely dominated. Queens are then added to the board at the intersection of these remaining rows and columns, as illustrated for $n = 13$ and $n = 14$ in Figures 4 and 5 respectively.

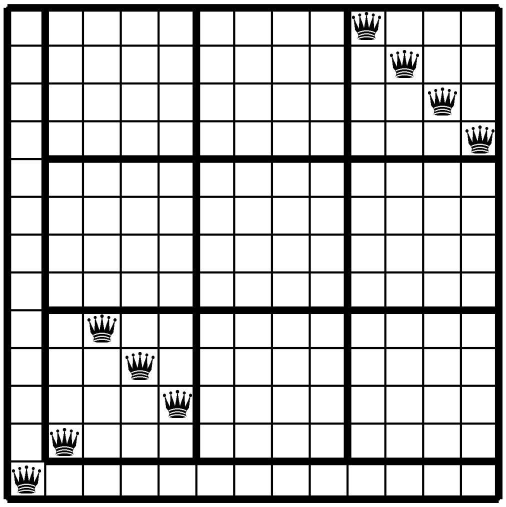

Figure 4: Welsch's Formation for $n = 13$

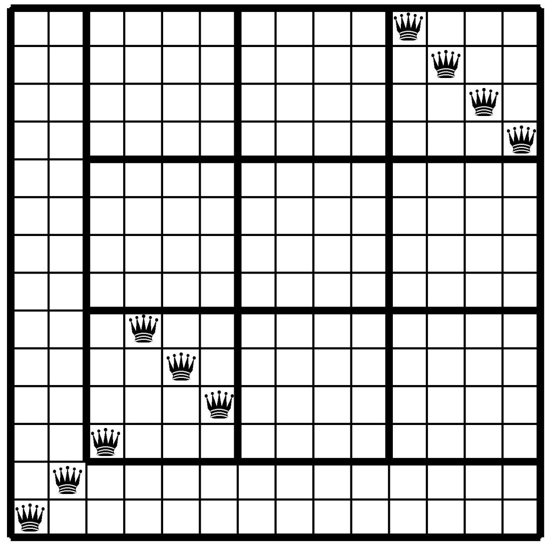

Figure 5: Welsch's Formation for $n = 14$

We are now ready to state our first result.

Theorem 2 *Let n=3q + r where* $0 \le r < 3$ *and* $q \ge 1$. *Then* $\gamma_t(Q_n) \le 2q + r$.

Proof: To show this, we use the same formation as in Welsch's. Recall that for a set of queens to be a total dominating set, the squares occupied by queens must also be attacked. Since Welsh's formation is a dominating set, the only squares to consider are those that are occupied by queens in this formation.

First, suppose $n \equiv 0 \pmod 3$.

Define A as a set consisting of the square occupied by the queen in the lower-left hand corner of the board. Define B as the set of squares to the immediate right of the main diagonal in Region I. Note if $n = 3$, set B is empty. Define C as the set of squares along the main diagonal of region IX.

The constructions for $n = 3$ and $n = 6$ are provided for these two trivial cases in figure 6. It is straightforward to see from these constructions that the sets of queens are total dominating sets.

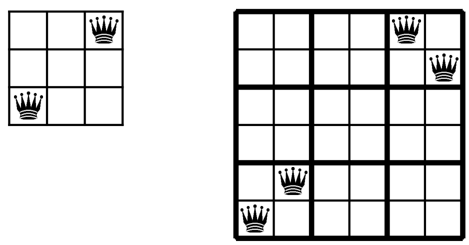

Figure 6: Welsch's Formation for $n = 3$ and $n = 6$

Suppose $n \geq 9$. It follows that there are at least two queens placed on squares in each of the sets B and C. Since the squares in B and C lie along two diagonals, then any squares occupied by these queens are attacked. For this case, we are only left to consider the square in set A.

Suppose now n is odd. Set up an x-y coordinate system with the origin placed at the center of the middle square. As is standard, define the coordinates of a given square as the coordinates at the center of that square. A given square with coordinates (x,y) is defined as having a positive diagonal value of $y - x$. This value corresponds to the y-intercept of a line with slope 1 passing through (x,y). Similarly, define the negative diagonal value of a square (x,y) as the sum $x + y$. Likewise, this corresponds to the y-intercept of a line with slope -1 passing through (x,y). It can be easily seen that any two squares with the same diagonal number, whether a positive or negative diagonal number, lie on a common diagonal.

The coordinates of the squares in set C can be defined as the set of coordinates $\{(\frac{n-1}{2}-i, \frac{n+3}{6}+i) \mid i \in \mathbb{Z} \text{ and } 0 \leq i \leq \frac{n-3}{3}\}$. Note that if n is odd and $n \equiv 0 \pmod 3$, then $\frac{n-3}{6}$ is an integer. Also for $n \geq 0$, $\frac{n-3}{6} \leq \frac{n-3}{3}$. Thus, taking $i = \frac{n-3}{6}$, we can see that $(\frac{n}{3},\frac{n}{3})$ is in the above set. Moreover, the square in set A has coordinates $(\frac{1-n}{2}, \frac{1-n}{2})$. It can be seen that both these coordinates lie on the positive diagonal with value zero. Thus, the square in set A is attacked by the indicated queen in set C. For an illustrated example see figure 7.

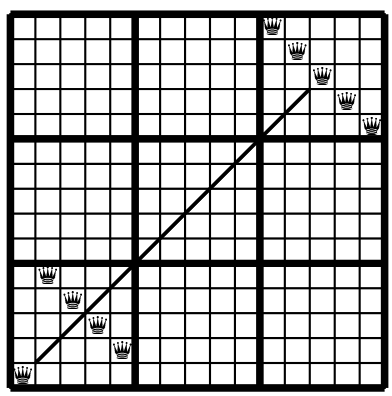

Figure 7: Welsch's Formation for $n = 15$

Suppose n is even. Again, using a coordinate system, let the origin be placed in the middle of the square in set A. The coordinates of the squares in set B can be defined as the set of coordinates $\{(\frac{n}{3} - 1 - i, 1 + i) \mid i \in \mathbb{Z} \text{ and } 0 \leq i \leq \frac{n-6}{3}\}$. It follows that if n is even and $n \equiv 0 \pmod 3$, then $\frac{n-6}{6}$ is an integer. Also for $n \geq 0$,

$\frac{n-6}{6} \leq \frac{n-6}{3}$. Thus, taking $i = \frac{n-6}{6}$, we can see that the square with coordinates $(\frac{n}{6}, \frac{n}{6})$ is in set B. Note that the square in set A has coordinates (0,0). It can be seen that both these coordinates lie on the positive diagonal with value zero. Thus, the square in set A is attacked by the indicated queen in set B. An illustrated example can be see in figure 8.

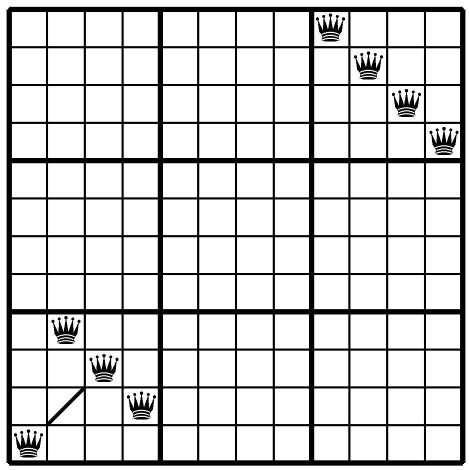

Figure 8: Welsch's Formation for $n = 12$

Next, consider the cases for $n \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$. Use the same placement of queens for these values as in Welsch's formation. Since these formations are dominating sets, all that is left to consider are the squares that have queens placed on them. In a similar fashion, consider an $m \times m$ subboard, where m is the largest value for which $m \equiv 0 \pmod 3$ and $m \leq n$. The above proof for the case of $n \equiv 0 \pmod 3$ also shows that all squares on the $m \times m$ subboard are totally dominated. For the case of $n \equiv 1 \pmod 3$, it is easy to see the added queen is attacked by the queen occupying the square in set A. For the case of $n \equiv 2 \pmod 3$, it is easy to see the additional queen is attacked by the queen added for the case of $n \equiv 1 \pmod 3$.

For illustrations see figures 9 and 10.

QED

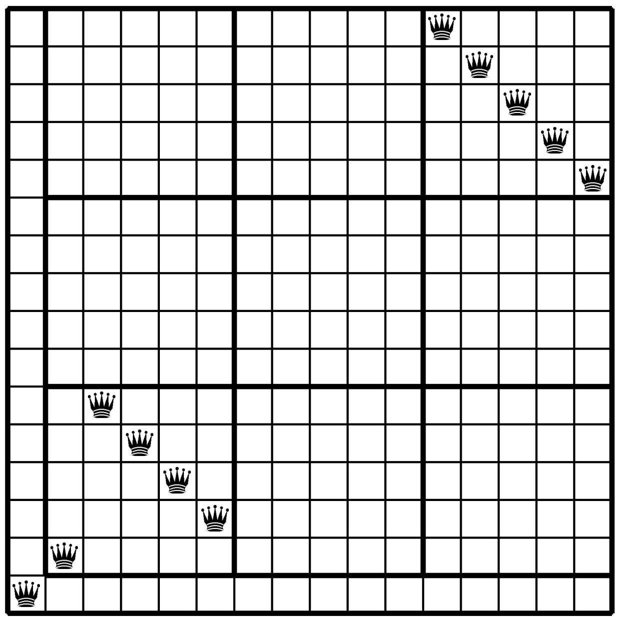

Figure 9: Welsch's Formation for $n = 16$

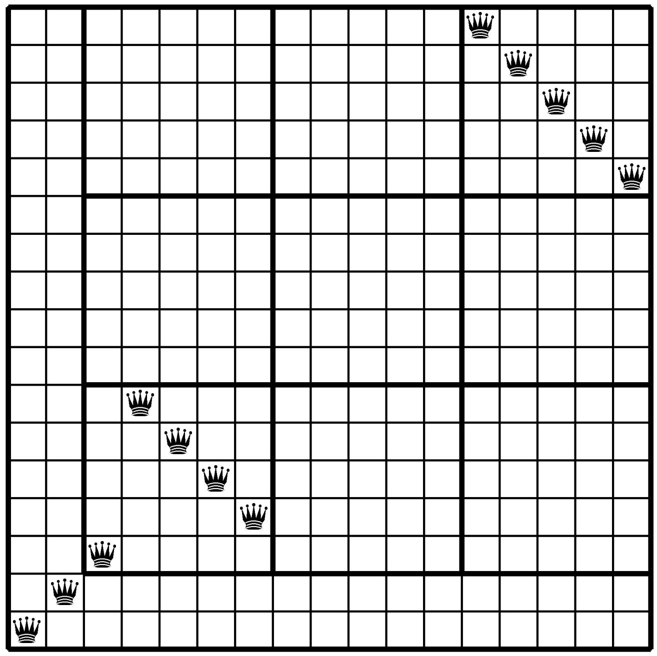

Figure 10: Welsch's Formation for $n = 17$

Corollary 3 *For the Queen's graph, Q_n,*

$$\lim_{n\to\infty} \frac{\gamma_t(Q_n)}{n} \leq \frac{2}{3}$$

.

Theorem 4 *Let $n = 3q + r$ where $0 \leq r < 3$ and $q \geq 1$. If $r = 0$ or $r = 2$, then $\gamma_{pr}(Q_n) \leq 2q + r$. If $r = 1$, then $\gamma_{pr}(Q_n) \leq 2q + 2$.*

Proof: Because Welsch's formation is a dominating set, then all that needs to be shown is the existence of a perfect matching. To show this, we use the same formation as in Welsch's, except when $n \equiv 1 \pmod 3$. For this case, a queen is added to the formation to form a perfect matching.

Assume first $n \equiv 0 \pmod 3$.

Suppose n is even. Since n is even, the cardinality of set C is even. A perfect matching among these squares easily can be seen. Since n is even, the cardinality of set B is odd. Note, however, the queen with coordinates $(\frac{n}{6}, \frac{n}{6})$ is in set B. As shown previously, this square is attacked by the queen on the square in set A. Hence, this square can be paired with the square in set A. This leaves an even number of squares remaining in set B. Since the squares of B are on a common diagonal, the remaining squares in set B can be matched.

Suppose n is odd. This case is similar to the above, except for the fact that set B is of even cardinality and set C is of odd cardinality. However, for this case the square in set A is adjacent to a square in set C, as previously shown. Hence, we can use the same argument as the case where n is even.

Next, we must consider the cases for which $n \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$. For the case where $n \equiv 2 \pmod 3$, Welsch's formation has, using the previous argument for $n \equiv 0 \pmod 3$, a perfect matching defined on a $m \times m$ subboard (where $m = n - 2$). The two remaining queens are on a common diagonal. Hence, their squares can be paired. Since all squares can be paired using the above matching, then a perfect matching has been defined for $n \equiv 2 \pmod 3$.

The case for $n \equiv 1 \pmod 3$ is similar to the above case. On the $m \times m$ subboard (where $m = n - 1$) part of a perfect matching has been defined. There is one remaining square in the dominating set not part of the perfect matching. This square is occupied by the queen not on the $m \times m$ subboard. For this case, place a queen adjacent to the occupied square not on the subboard. This would form a set of occupied squares on which a perfect matching could be defined. An example is illustrated in Figure 11.

QED

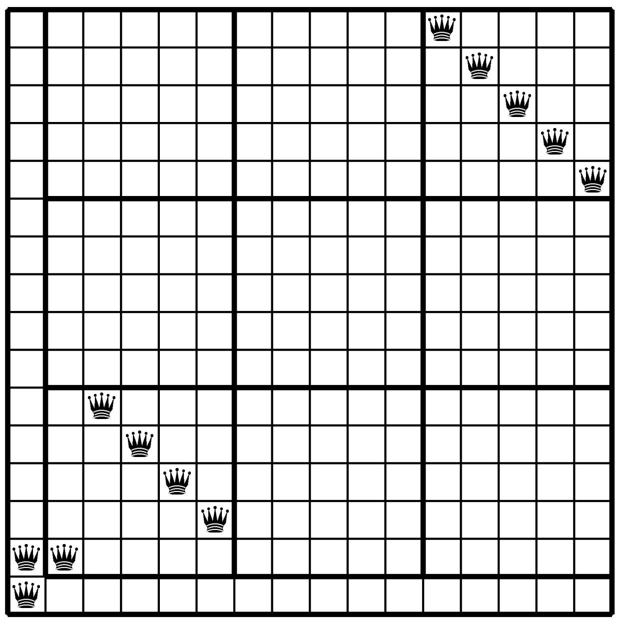

Figure 11: Welsch's Formation for $n = 17$, Modified for Paired Domination

Corollary 5 *For the Queen's graph, Q_n,*

$$\lim_{n \to \infty} \frac{\gamma_{pr}(Q_n)}{n} \leq \frac{2}{3}$$

Recently many of the values for the standard domination problem were established, in part, by new lower bounds. The first of these lower bounds was given by Spencer in 1990 [22]. This lower bound is as follows:

Theorem 6 *For the Queen's graph, Q_n,*

$$\gamma(Q_n) \geq \frac{(n-1)}{2}.$$

We give lower bounds for both paired and total domination on the Queen's graph.

Theorem 7 *For the Queen's graph, Q_n,*

$$\gamma_t(Q_n) \geq \frac{4(n-1)}{7}.$$

Proof: The trivial cases for $n = 2$ and $n = 3$ are straightforward because $2 \leq \gamma_t(Q_n) \leq \gamma_{pr}(Q_n)$ for all n.

Let $n \geq 4$, and S be a $\gamma_t(Q_n)$-set. We construct a graph G having vertex set S and edges as follows. Two vertices are adjacent if and only if the queens on these squares can attack one another by moving only on vacant squares (squares unoccupied by queens) of the $n \times n$ board. Note that G is not necessarily the same as the subgraph induced by S in Q_n. For example, if there are three queens in a single column, the topmost queen cannot attack the bottommost queen via unoccupied squares. Hence their corresponding vertices would not be adjacent in G. On the other hand, both these vertices are adjacent to the vertex representing the queen in the middle. Note that a subset of vertices that are on the same column (or, respectively, row or diagonal) induces a path in G, whereas the same subset of vertices induces a complete subgraph in Q_n.

If two vertices are adjacent in G because they can attack along unoccupied squares of a column, we say they are column adjacent. Row and diagonal adjacent are defined as expected. To aid in our proof , we count the edges of G. Let c, r, and d, represent the number of edges among the vertices that are column, row, and diagonal adjacent, respectively. Then, $|\mathrm{E}(G)| = c + r + d$. Note that since S is a total dominating set of Q_n, G has no isolated vertices. Thus, $c + r + d \geq |S|/2 = \gamma_t(Q_n)/2$.

We say a column (or, respectively, row or diagonal) is unoccupied if there is no queen in it. Let a_1 denote the leftmost unoccupied column, a_2 the rightmost unoccupied column, b_1 the bottommost unoccupied row, and b_2 the top-most unoccupied row. These rows and columns exist for $n \geq 4$, since $2 \leq \gamma_t(Q_n) \leq \gamma_{diag}(Q_n) \leq n - 2$. Hence for any $\gamma_t(Q_n)$-set with $n \geq 4$, there are at least two unoccupied rows and two unoccupied columns.

In a_1 and a_2, there are $2(n - \gamma_t(Q_n) + r)$ squares that do not share a common row or column with a queen in S. Likewise, in b_1 and b_2 there are $2(n - \gamma_t(Q_n) + c)$ squares that do not share a common row or column with a queen in S. Note there are four squares which are counted more than once. The four corners where the outtermost, unoccupied rows meet the outtermost, unoccupied columns overlap. Hence, these squares are included in exactly two of the above counts. Thus, the total number of squares that do not share a common row or column with a queen in S can be expressed as:

$$2(n - \gamma_t(Q_n) + r) + 2(n - \gamma_t(Q_n) + c) - 4.$$

Note also any one diagonal, whether a positive or negative diagonal, dominates at most two of the squares in all of a_1, a_2, b_1, and b_2. Also the total number of diagonals occupied by queens is $2\gamma_t(Q_n)$ - d. Because any of the squares in this "outer rim" of squares must be diagonally dominated, it follows that:

$2(n - \gamma_t(Q_n) + r) + 2(n - \gamma_t(Q_n) + c) - 4 \leq 2(2\gamma_t(Q_n) - d)$

or $4n - 4 + 2(c + r + d) \leq 8\gamma_t(Q_n)$.

But since $c + r + d \geq \frac{\gamma_t(Q_n)}{2}$, we have

$4(n-1) + 2(\gamma_t(Q_n)/2) \leq 8\gamma_t(Q_n)$ or

$4(n-1)/7 \leq \gamma_t(Q_n)$.

QED

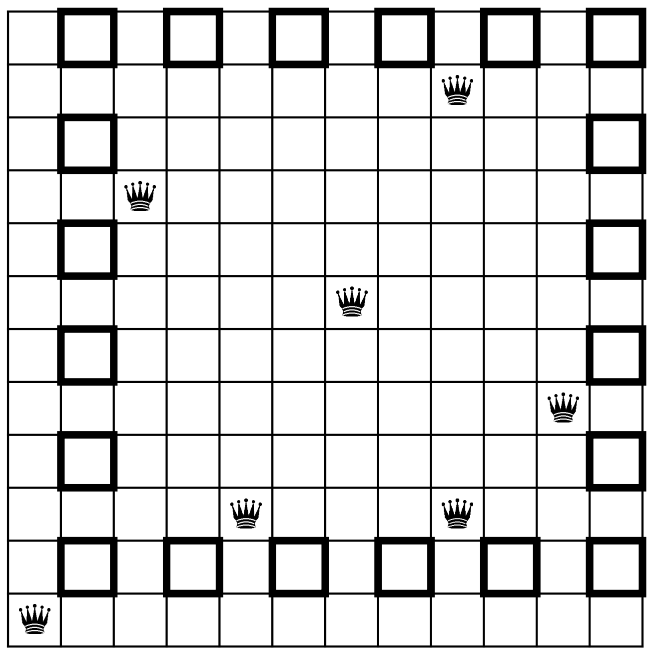

Figure 12: A Minimum Total Dominating Set for $n = 12$

Figure 12 illustrates a minimum total dominating set for Q_{12} of 7 queens. Note here $c = 1$, $r = 1$, and $d = 3$. In this case, the subgraph induced by S is isomorphic to G because there are no more than two queens in any single row, column, or diagonal.

Corollary 8 *For the Queen's graph, Q_n,*

$\frac{4(n-1)}{7} \leq \gamma_{pr}(Q_n)$.

Corollary 9 *For the Queen's graph, Q_n,*

$$\frac{4}{7} \leq \lim_{n\to\infty} \frac{\gamma_t(Q_n)}{n} \leq \lim_{n\to\infty} \frac{\gamma_{pr}(Q_n)}{n} \leq \frac{2}{3}$$

4 $\gamma_t(Q_n)$ AND $\gamma_{pr}(Q_n)$ VALUES

Before giving some $\gamma_t(Q_n)$ and γ_{pr} values, a summary of all that is known about bounds on $\gamma_t(Q_n)$ and $\gamma_{pr}(Q_n)$ will be given.

The total domination number has the following lower bounds:

$\gamma_t(Q_n) \geq 2$

$\gamma_t(Q_n) \geq \gamma(Q_n)$

$\gamma_t(Q_n) \geq \frac{4(n-1)}{7}$ as indicated in column labeled L.B. for γ_t. This number has been rounded up.

The total domination number has the following upper bounds:

$\gamma_t(Q_n) \leq \gamma_{pr}(Q_n)$

If $\gamma_{diag} \geq 2$, then $\gamma_t(Q_n) \leq \gamma_{diag}(Q_n)$

Let $n = 3q + r$ and $q \geq 0$. Then $\gamma_t(Q_n) \leq 2q + r$. This is indicated in the column labeled U.B. for γ_t.

The paired domination number has the following lower bounds:

$\gamma_{pr}(Q_n) \geq 2$

$\gamma_{pr}(Q_n) \geq \gamma(Q_n)$

$\gamma_{pr}(Q_n) \geq \frac{4(n-1)}{7}$ as indicated in column labeled L.B. for γ_{pr}. This value has been rounded up to the closest even integer.

$\gamma_{pr}(Q_n) \geq \gamma_t(Q_n)$

The paired domination number has the following upper bounds:

Let $n = 3q + r$ and $q \geq 0$. If $r = 0$ or $r = 2$, then $\gamma_{pr}(Q_n) \leq 2q + r$. If $r = 1$ then $\gamma_{pr}(Q_n) \leq 2q + 2$. This upper bound is indicated in the column U.B. for $\gamma_{pr}(Q_n)$.

If $\gamma_{diag}(Q_n)$ is even, then $\gamma_{pr}(Q_n) \leq \gamma_{diag}(Q_n)$.

If $\gamma_{diag}(Q_n)$ is odd and $n \neq 1$, then $\gamma_{pr}(Q_n) \leq \gamma_{diag}(Q_n) + 1$

Note also that $\gamma_{pr}(Q_n)$ must be an even integer.

The following chart has been compiled with the above bounds and the constructions that follow. Some of the identified $\gamma_{pr}(Q_n)$ and $\gamma_t(Q_n)$ values have letters superscripted. These refer to the constructions that follow. The values for $\gamma(Q_n)$ that were used are found in [20]. The diagonal numbers were verified via computer search by Steve Lane, an ETSU graduate student in mathematics.

Table 1

Some Values for $\gamma_t(Q_n)$ and $\gamma_{pr}(Q_n)$

N	$\gamma(Q_n)$	LB γ_t	$\gamma_t(Q_n)$	UB γ_t	LB γ_{pr}	$\gamma_{pr}(Q_n)$	UB γ_{pr}	$\gamma_{diag}(Q_n)$
2	1	1	2	-	2	2	-	1
3	1	2	2	2	2	2	2	1
4	2	2	2	3	2	2	4	2
5	3	3	3	4	4	4	4	3
6	3	3	$4^{see\ a)}$	4	4	4	4	4
7	4	4	4	5	4	4	6	4
8	5	4	5	6	4	6	6	5
9	5	5	5	6	6	6	6	5
10	5	6	6	7	6	6	8	6
11	5	6	6v7	8	6	6v8	8	7
12	6	7	$7^{see\ b)}$	8	8	8	8	8
13	7	7	7v8	9	8	$8^{see\ c)}$	10	9
14	8	8	8v$9^{see\ d)}$	10	8	8v10	10	10
15	9	8	9v10	10	8	10	10	11
16	9	9	9v10	11	10	$10^{see\ e)}$	12	12
17	9	10	10v$11^{see\ f)}$	12	10	10v12	12	12
18	9	10	10-12	12	10	10v12	12	13
19	10	11	11v12	13	12	$12^{see\ g)}$	14	14
20	10v11	11	11-$13^{see\ h)}$	14	12	12v14	14	
21	11	12	12v$13^{see\ i)}$	14	12	12v14	14	
22	11v12	12	12-$14^{see\ j)}$	15	12	12v14v16	16	
23	12	13	13-$15^{see\ k)}$	16	14	14v16	16	
24	12v13	14	14-16	16	14	14v16	16	
25	13	14	14-17	17	14	14v16v18	18	

a) Verified by Steve Lane via computer search. Also provided by Ahrens in [1].

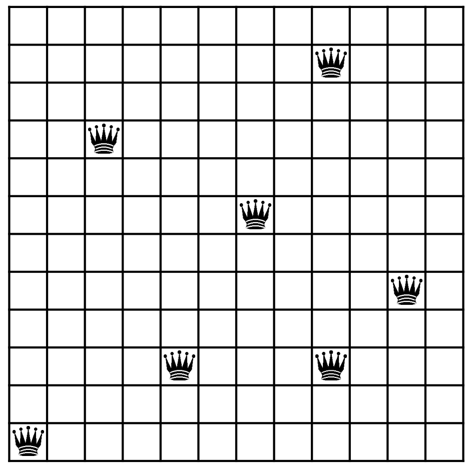

Figure 13: b) A Total Dominating Set for $n = 12$ of 7 Queens

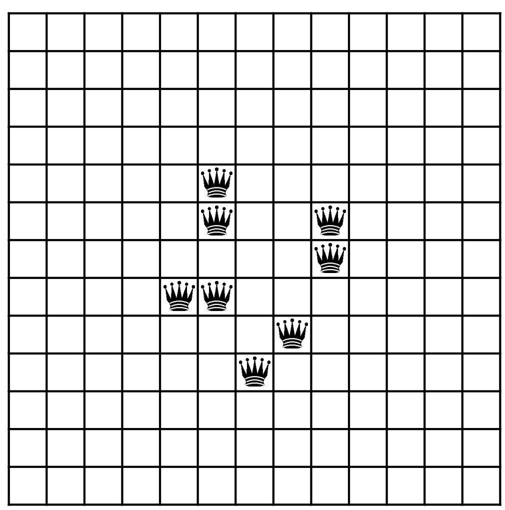

Figure 14: c) A Paired Dominating Set for $n = 13$ of 8 Queens

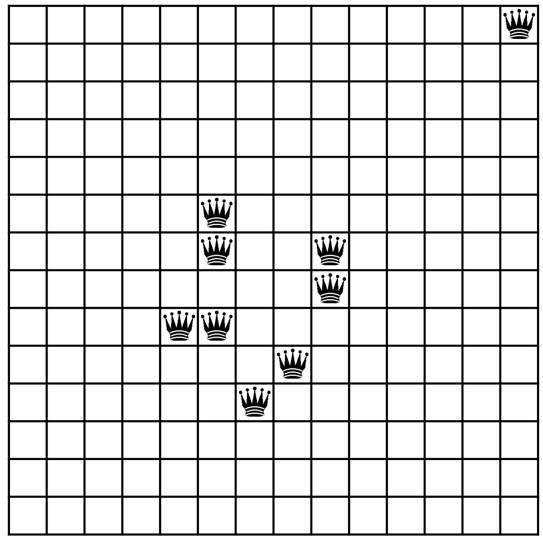

Figure 15: d) A Total Dominating Set for $n = 14$ of 9 Queens

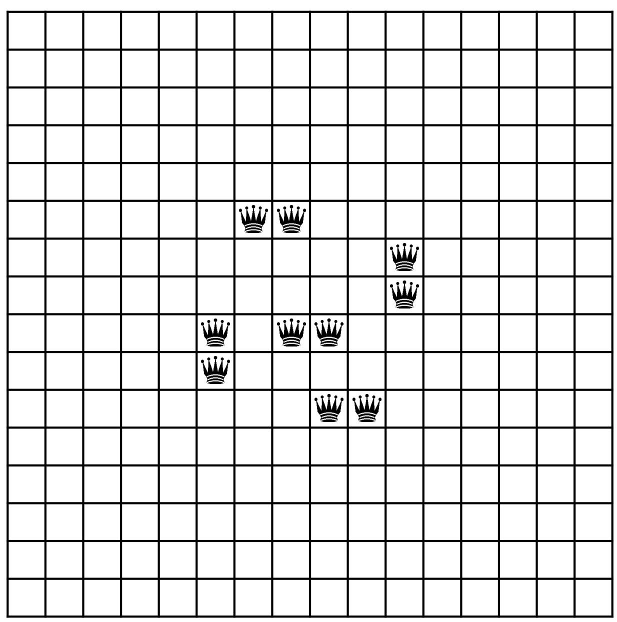

Figure 16: e) A Paired Dominating Set for $n = 16$ of 10 Queens

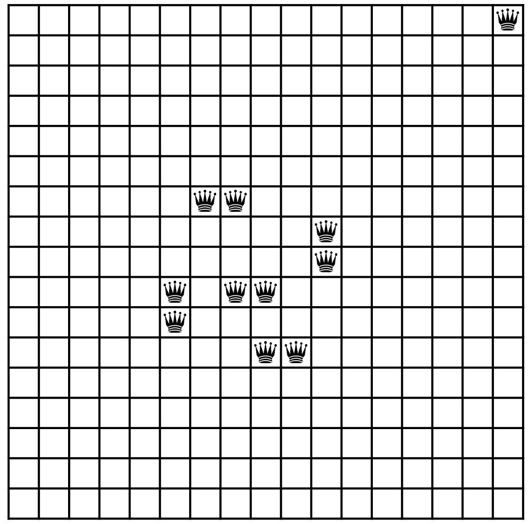

Figure 17: f) A Total Dominating Set for $n = 17$ of 11 Queens

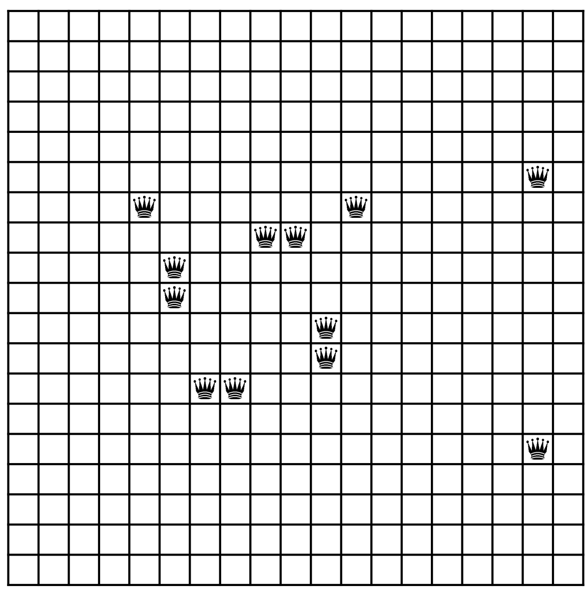

Figure 18: g) A Paired Dominating Set for $n = 19$ of 12 Queens

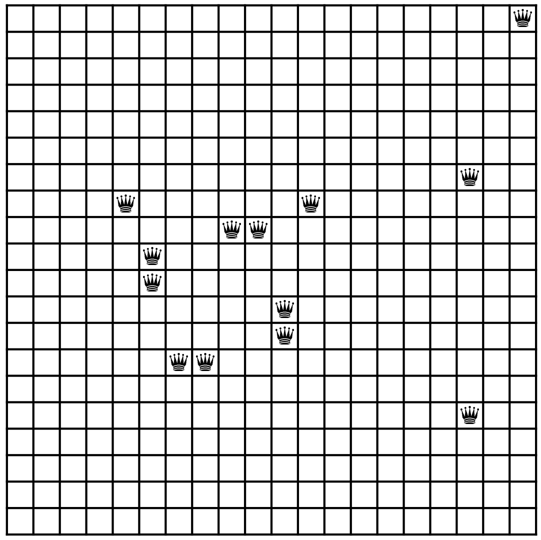

Figure 19: h) A Total Dominating Set for $n = 20$ of 13 Queens

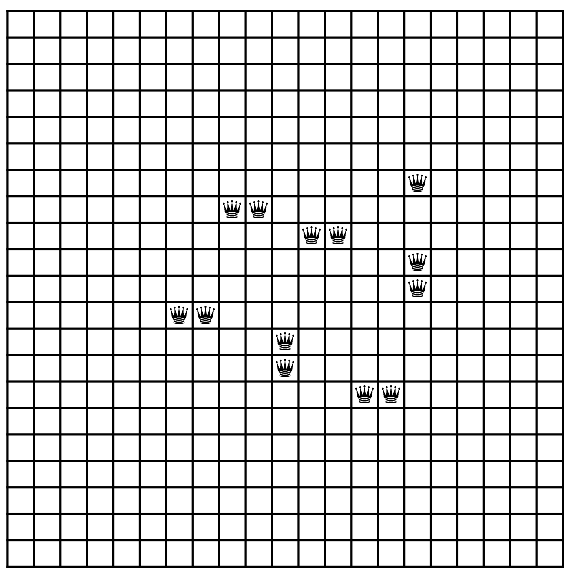

Figure 20: i) A Total Dominating Set for $n = 21$ of 13 Queens

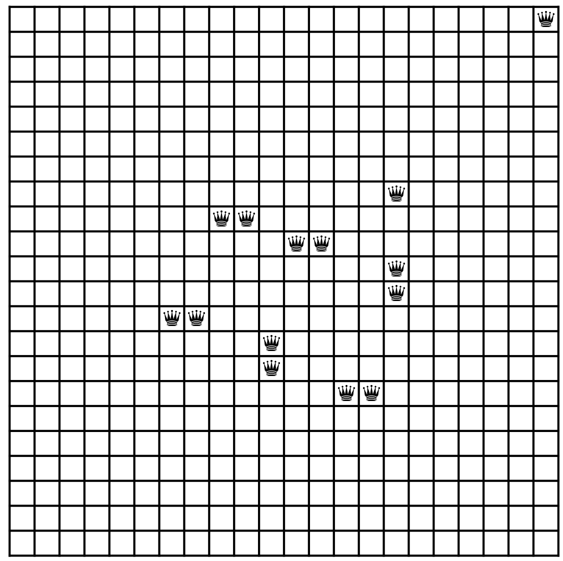

Figure 21: j) A Total Dominating Set for $n = 22$ of 14 Queens

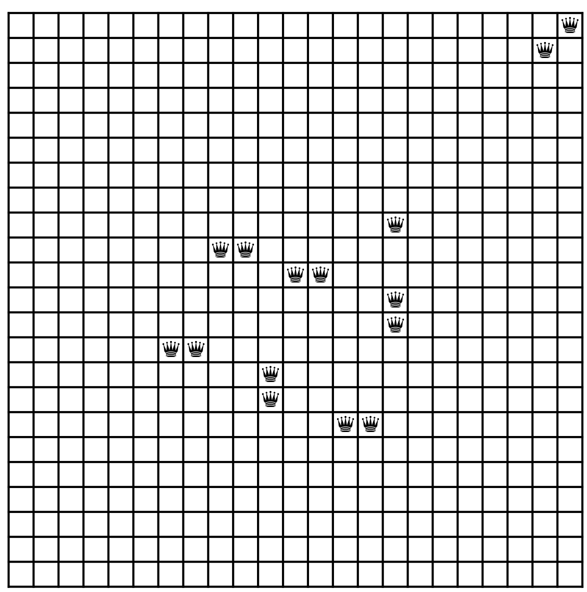

Figure 22: k) A Total Dominating Set for $n = 23$ of 15 Queens

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