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# Winning an Independence Achievement Game.

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Winning an Independence Achievement Game

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by

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ABSTRACT

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The game ”Generalized Kayles (or Independence Achievement)” is played by two players A and B on an arbitrary graph G. The players alternate removing a vertex and its neighbors from G, the winner being the last player with a nonempty set from which to choose. In this thesis, we present winning strategies for some paths.

# DEDICATION

I dedicate this thesis to my family and all of my friends for all their support and patience. I also dedicate this to my girlfriend Elizabeth Tramel. Thank You Everyone and I Love You!!!

## ACKNOWLEDGMENTS

I would like to thank all of my teachers and professors for getting me to this point. I would especially like to thank my advisors Dr. Teresa Haynes and Dr. Michael Henning for all of their advice, patience, and support during the preparation of this thesis. Thank you and I will always be grateful for the help.

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# 1 Introduction

## 1.1 Graph Theory

For notation and graph theory terminology we in general follow [3] or [5]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ of order $n$ and edge set $E$, and let $v$ be a vertex in $V$. The edge $e = \{u, v\}$ is said to join the vertices $u$ and $v$. If $e = \{u, v\}$ is an edge of a graph $G$, then we say that $u$ and $v$ are **adjacent** vertices. The **order** of $G$, denoted $n$, is the cardinality of its vertex set. For example, in Figure 1, the order of the graph would be $n = 4$ and, the vertex $d$ is adjacent to $b$ and $c$. For any vertex $v \in V$, the **open neighborhood of** $v$ is the set $N(v) = \{u \in V \mid uv \in E\}$, and its **closed neighborhood** is the set $N[v] = N(v) \cup \{v\}$. For instance, in Figure 1, the open and closed neighborhoods of $a$ are as follows.

$$N(a) = \{b, c\}$$

$$N[a] = \{a, b, c\}$$

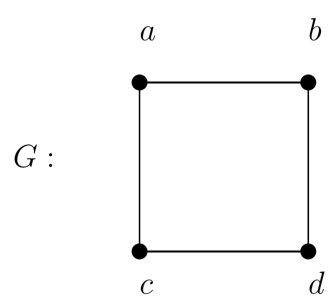


Figure 1: Example 1.

The **degree** of a vertex $v$ is the number of edges incident with $v$, denoted *deg* $v$. A vertex with degree 1 is called an **end-vertex**, while a vertex adjacent to an *end-vertex* is called a **support vertex**. Notice, in Figure 2 there are two **end-vertices**, namely $h$ and $p$. There are also two **support vertices**, namely $i$ and $o$.

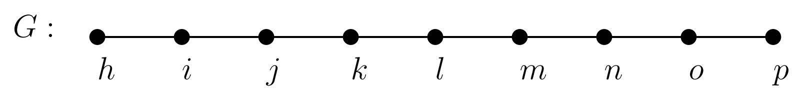


Figure 2: Example 2.

Two vertices that are not adjacent in a graph $G$ are said to be **independent**. A vertex $v$ in a graph $G$ is said to **dominate** itself and each of its neighbors, that is, $v$ dominates the vertices in its *closed neighborhood.* A set $S$ of vertices of $G$ is a **dominating set** of $G$ if every vertex of $G$ is dominated by at least one vertex of $S$. A set $S$ of vertices in a graph $G$ is called an **independent dominating set** of $G$ if $S$ is both an independent and a dominating set of $G$.

In graph theory, much time is spent studying properties of certain families of graphs. (Families are collections of graphs which may vary in order and size, but all have the same basic structure.) There are several different families of graphs that we consider, and we will briefly describe a few of them.

1) Let $u$ and $v$ be (not necessarily distinct) vertices of a graph $G$. A $u-v$ **path** of $G$ is a finite, alternating sequence

$$u = u_0, e_1, u_1, e_2, ..., u_{k-1}, e_k, u_k = v$$

of vertices and edges, beginning with vertex $u$ and ending with vertex $v$ such that $e_i = u_{i-1}u_i$ for $i = 1, 2, ..., k$, and no vertex is repeated. The number $k$ is called the **length** of the path, and a trivial path is one which contains no edges, that is, $k = 0$. A path on $n$ vertices is denoted $P_n$. An example of the path $P_9$ is shown in Figure 2.

2) A **cycle** on $n$ vertices, denoted $C_n$ is a path that starts and ends at the same vertex An example of the cycle $C_4$ is shown in Figure 1.

3) A graph is said to be **complete** if every pair of its vertices are adjacent. A complete graph is denoted $K_n$. An example of the complete graph $K_4$ is shown in Figure 3.

$G:$

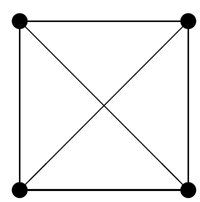

Figure 3: A complete graph on four vertices.

4) For this family, we use the definition and terminology from [5]. The $n$-cube or **hypercube**, denoted $Q_n$, can be considered to be the graph whose vertices are labeled by binary $n$-tuples and such that two vertices are adjacent if and only if their corresponding $n$-tuples differ at precisely one coordinate. The hypercube $Q_3$ is shown in Figure 4.

$G$ :

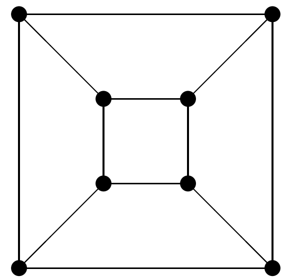

Figure 4: Hypercube, $Q_3$.

5) A **bipartite graph** is a graph with the property that the $V(G)$ can be partitioned into two subsets $V_1$ and $V_2$ such that every element of $E(G)$ joins a vertex of $V_1$ to a vertex of $V_2$. A **complete bipartite graph** $G$ is a bipartite graph having the added property that for all $u \in V_1$ and $v \in V_2$, then $uv \in E(G)$. If $|V_1| = r$ and $|V_2| = s$, then the complete bipartite graph is denoted $K_{r,s}$. An example of the complete bipartite graph $K_{2,3}$ is shown in Figure 5.

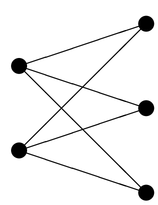

Figure 5: A complete bipartite graph $K_{2,3}$.

Now we have explained some basic definitions and graphs that will be helpful in understanding the later material. In the next sections, additional definitions will be given as needed.

## 1.2 Independent Achievement Game

The game we considered consists of two players A and B, who alternate moves on a graph $G$. On each move Player A or B selects a vertex that is not already selected. Once a vertex is selected no others in its *closed neighborhood* can be selected, i.e., the selected vertices must form an independent set. For example, in Figure 6 below if the vertex $s$ is selected, then the vertices $r$ and $t$ can no longer be selected.

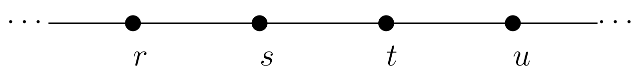


Figure 6: Example 3.

The object of the game is to be the player to make the last move. The last player to select a vertex wins. Thus the player selecting a vertex that completes an independent dominating set (of selected vertices) wins the game. We assume that each player makes the best move possible.

First let's consider playing the game on a complete graph. It is clear from Figure 7 below that Player A will always win on this family of graphs. Once any vertex is selected, all others are in its closed neighborhood. Therefore there is only one move for any complete graph.

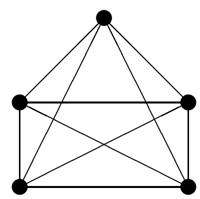

Figure 7: A complete graph on five vertices.

Next let's consider playing on a *hypercube* $Q_n$. Once Player A selects the first vertex, then Player B will always select the antipodal vertex of the one selected. For example, in Figure 8, if Player A was to select the vertex $j$, then Player B would select the vertex $p$. Basically Player B uses a reflective strategy. Since there is always an even number of moves when playing on a *hypercube*, Player B will always win.

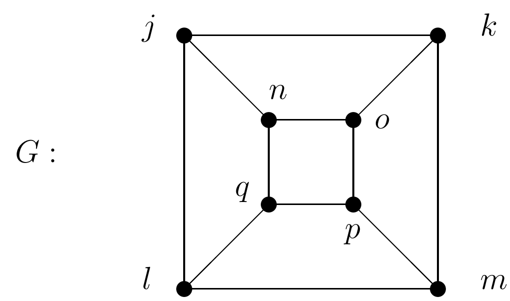


Figure 8: Hypercube, $Q_3$.

Next let's consider playing on a *complete bipartite* $K_{r,s}$.

**Theorem 1** *Player A wins on a $K_{r,s}$ if and only if one of $r$ or $s$ is odd.*

**Proof.** Note that if Player A selects a vertex from a partite set, then the vertices of the other partite are eliminated from play (because they are in the neighborhood of the selected vertex).

Hence by selecting a vertex Player A determines the number of moves in the game. If one of $r$ or $s$ is odd, then Player A will select a vertex in the odd partite set. Since there is an odd number of moves, Player A will win. On the other hand if both $r$ and $s$ are even, no matter from which set A chooses, the number of moves in the game is even, resulting in a win for Player B.□

Notice in Figure 9, if Player A selects the vertex $a$, then Player B must select from either vertex $b$ or $c$. Thus leaving the final vertex for Player A.

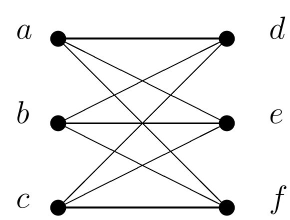


Figure 9: A complete bipartite graph $K_{3,3}$.

This game was first introduced in [1]. The authors focused on the graphs containing no cycle of length seven or smaller. They presented a characterization of the collection of graphs in which every maximal independent set of vertices is maximum. They also stated that for an arbitrary graph a winning strategy is "very difficult" to determine.

The game was also mentioned in [2]. The authors refer to the game as "Generalized Kayles". They stated that the problem of whether the first player could force a win is PSPACE-complete [6]. Their results were focused on characterizing the parity graphs of girth greater than five in such a way as to show that those graphs can be recognized in polynomial time.

In the next chapter, we use the terminology "results in a subgraph ..." to indicate the subgraph induced by the vertices available to be chosen after a given move.

# 2 Results

In this chapter, I present results that I obtained for paths. First we will focus on all paths that are odd in length. It turns out there is a strategy that will work for all odd paths.

**Theorem 2** *Player A wins on all odd paths* $P_{2k+1}$, $k \geq 0$.

**Proof.** Let the vertices be labeled $v_1, v_2, ..., v_{2k+1}$, $k \geq 0$. Player A will select the center vertex, namely vertex $v_{k+1}$. Once A has selected the center vertex, he mirrors Player B's remaining moves until A wins. More precisely, for $i = 1, 2, ..., k$, if B chooses vertex $v_{k+1+i}$ (respectively, $v_{k+1-i}$) then A selects vertex $v_{k+1-i}$ (respectively, $v_{k+1+i}$). Since there will always be an even number of moves after A selects the center vertex, Player A will win on any odd path. $\square$

For example, consider the following path $P_9$.

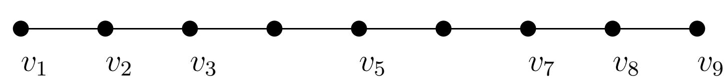


Figure 10: A path on nine vertices.

Notice that after Player A selects the vertex $v_5$, Player B must select from $v_1, v_2, v_3$, $v_7, v_8$, or $v_9$. It is easy to see that Player A will win by following the strategy given in Theorem 2.

One might think that since all of the paths odd in length were classified easily, it would be just as easy to give winning strategies for the remaining paths. However, the classification of even paths is much more difficult. As of yet we have been unable to determine a strategy for the paths even in length. The $P_2$ is trivial since it will always be won by Player A with the first move. Hence we consider paths of even order at least four. Let $P_{2k}$ be labeled $v_1, v_2, ..., v_{2k}$.

**Lemma 3** *If the path $P_{2k}$ is won by Player B, then the path $P_{2k+2}$ will be won by Player A.*

**Proof.** Suppose we have a path $P_{2k+2}$ and Player B wins on the path $P_{2k}$. Player A will select the vertex $v_1$. This will result in Player B selecting first on a path $P_{2k}$. Since the path $P_{2k}$ is won by the second player, Player A will win on a path $P_{2k+2}$.□

**Theorem 4** *Player B wins on the $P_4$.*

**Proof.** No matter which vertex Player A selects to begin the game, there is only one more move left with Player B selecting. Therefore Player B wins on a $P_4$. □

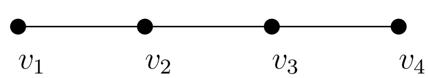


Figure 11: A path on four vertices.

When we considered the $P_6$, we found that there are only two distinct winning strategies for Player A.

**Theorem 5** *Player A wins on the $P_6$ and there are exactly two strategies for winning.*

**Proof.** (Strategy 1)

By Lemma 3, Player A can guarantee a win by selecting vertex $v_1$.

(Strategy 2)

Player A can guarantee a win by selecting vertex $v_3$. Notice this results in two disjoint paths, each with only one move. Therefore when Player B selects from one of the paths, he leaves one move for Player A. Thus A wins on a $P_6$ with this strategy.

By investigation of all possibilities, it is a simple exercise to show that these are the only two strategies.□

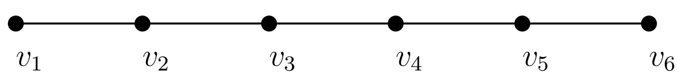


Figure 12: A path on six vertices.

The next path we considered was similar to the $P_4$, in the fact that Player B will win no matter what Player A does, so Player A cannot guarantee a win.

**Theorem 6** *Player B wins on a $P_8$.*

**Proof.** By symmetry, Player A has only four distinct choices to begin the game. Thus we must consider four different selections of Player A's first move.

**Case 1.** Suppose A selects the vertex $v_1$. This results in Player B selecting first on a $P_6$. From the previous strategies for a $P_6$ we know that B would be able to guarantee a win. Therefore Player B will win on a $P_8$ if Player A begins with vertex $v_1$.

**Case 2.** Suppose Player A selects the vertex $v_2$. This results in a $P_5$, with Player B making the next move. This would guarantee a win for B according to Theorem 2. Therefore Player B will win on a $P_8$ if Player A begins with vertex $v_2$.

**Case 3.** Suppose Player A selects the vertex $v_3$. This results in two disjoint paths a $P_1$ and a $P_4$. Thus there are a total of 3 moves remaining. Regardless of which vertex B selects, there will be exactly 2 moves remaining. Therefore Player B will win on a $P_8$ if Player A begins with vertex $v_3$.

**Case 4.** Suppose Player A selects the vertex $v_4$. This results once again in two disjoint paths a $P_2$ and a $P_3$. Player B will next select an *end-vertex* of the $P_3$, namely the vertex $v_6$ or $v_8$. This will leave a $P_1$ and a $P_2$. Therefore there are only 2 moves remaining with Player A selecting next. Thus Player B will win on a $P_8$ if Player A begins with vertex $v_4$.□

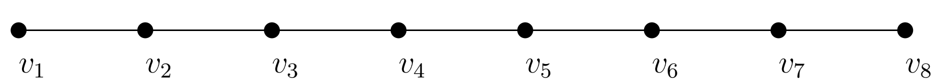


Figure 13: A path on eight vertices.

The next path we considered was similar to the $P_6$, in the fact that Player A has two distinct strategies to guarantee a win.

**Theorem 7** *Player A wins on the* $P_{10}$ *and there are at least two strategies for winning.*

**Proof.** (Strategy 1)

Player A can guarantee a win by selecting vertex $v_1$. Notice this results in a $P_8$ with Player B now selecting. By Lemma 3, A wins on a $P_{10}$ with this strategy.

(Strategy 2)

Player A can guarantee a win by selecting vertex $v_3$. Notice this results in two disjoint paths, a $P_1$ and $P_6$. Player B cannot select $v_1$ because this would result in Player A selecting first on a $P_6$, and from Theorem 5 it is clear A would win. Therefore Player B must select from the resulting $P_6$. Again from symmetry, we know that Player B has three choices for his next move. This means we must consider three cases:

**Case 1.** Suppose Player B selects the vertex $v_5$. This results in two disjoint paths, a $P_1$ and a $P_4$. From the strategy in Theorem 6 Case 3, the first player selecting on this configuration wins. Since it is Player A's turn, he would win on a $P_{10}$ with this strategy.

**Case 2.** Suppose Player B selects the vertex $v_6$. This results in two disjoint paths, a $P_1$ and a $P_3$. Player A would then select one of the *end-vertices* of the $P_3$. This results in only two moves remaining with Player B selecting first. Therefore A wins on a $P_{10}$ with this strategy.

**Case 3.** Suppose Player B selects the vertex $v_7$. This results in three disjoint paths each with only one move. Since Player A is selecting and there are only three moves remaining, he wins on a $P_{10}$ with this strategy.

Thus Player A can guarantee a victory on a $P_{10}$ by using one of these two strategies. □

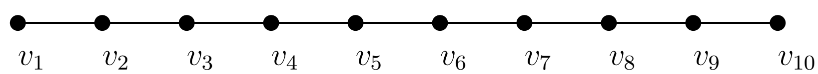


Figure 14: A path on ten vertices.

On the next path $P_{12}$, Player A once again has a definite strategy to guarantee a win.

**Theorem 8** *Player A wins on the* $P_{12}$.

**Proof.** Player A will begin by selecting vertex $v_4$. Notice this results in two disjoint paths, a $P_2$ and $P_7$. Player B cannot select $v_1$ or $v_2$ because this would result in Player A selecting first on a $P_7$, and from Theorem 2 it is clear A would win. That leaves, by symmetry, only four vertices from which Player B must select. Thus we have four cases to consider.

**Case 1.** Suppose Player B selects the vertex $v_6$. Player A would then select the vertex $v_9$ with his second move. This results in only two moves remaining. Since

Player B must select next, Player A will win on a $P_{12}$ with this strategy.

**Case 2.** Suppose Player B selects the vertex $v_7$. Player A would then select the vertex $v_{10}$ with his second move. This results in only two moves remaining. Since Player B must select next, Player A will win on a $P_{12}$ with this strategy.

**Case 3.** Suppose Player B selects the vertex $v_8$. Player A would then select the vertex $v_{11}$ with his second move. This results in only two moves remaining. Since Player B must select next, Player A will win on a $P_{12}$ with this strategy.

**Case 4.** Suppose Player B selects the vertex $v_9$. This results in three disjoint paths each with only one move. Since Player A is selecting and there is only three moves remaining, he wins on a $P_{12}$ with this strategy.

Therefore Player A will always win on the path $P_{12}$.□

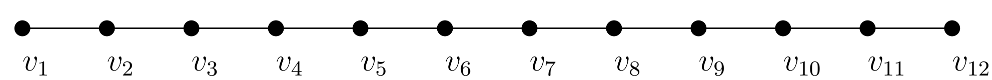


Figure 15: A path on twelve vertices.

The next path we considered was similar to both $P_4$ and $P_8$, in the fact that Player A cannot guarantee a victory.

**Theorem 9** *Player B wins on a* $P_{14}$.

**Proof.** By symmetry, Player A has seven distinct choices to begin the game. Thus we must consider seven different selections of Player A's first move.

**Case 1.** Suppose A selects the vertex $v_1$. This results in Player B selecting first on a $P_{12}$. From the previous strategy for a $P_{12}$ we know that B would be able to guarantee a win. Therefore Player B will win on a $P_{14}$ if Player A begins with vertex $v_1$.

**Case 2.** Suppose Player A selects the vertex $v_2$. This results in a $P_{11}$, with Player B making the next move. This would guarantee a win for B according to Theorem 2. Therefore Player B will win on a $P_{14}$ if Player A begins with vertex $v_2$.

**Case 3.** Suppose Player A selects the vertex $v_3$. This results in 2 disjoint paths a $P_1$ and a $P_{10}$. Player B will then select the vertex $v_6$. This results in two disjoint paths a $P_1$ and a $P_7$. Player A could not select $v_1$ because it would result in an odd path with Player B selecting. Because of symmetry Player A has four distinct choices for the next move. We must consider all four selections. **Case 3a.** Suppose Player A selects vertex $v_8$ with his second move. Player B will then select the vertex $v_{11}$ with his second move. This results in two disjoint paths a $P_1$ and a $P_2$. Thus there are only two moves remaining with Player A selecting. Therefore Player B will win on a $P_{14}$ if this strategy is used. **Case 3b.** Suppose Player A selects vertex $v_9$ with his second move. Player B will then select the vertex $v_{11}$ with his second move. This results in two disjoint paths a $P_1$ and a $P_2$. Thus there are only two moves remaining with Player A selecting. Therefore Player B will win on a $P_{14}$ if this strategy is used. **Case 3c.** Suppose Player A selects vertex $v_{10}$. Player B will then select the vertex $v_{13}$ with his second move. This results in two disjoint $P_1$'s. Thus there are only two moves remaining with Player A selecting. Therefore Player B will win on a $P_{14}$ if this strategy is used. **Case 3d.** Suppose Player A selects vertex $v_{11}$. This results in three disjoint paths each with only one move. Since Player B is selecting and there are exactly three moves remaining, he wins on a $P_{14}$ with this strategy.

**Case 4.** Suppose Player A selects the vertex $v_4$. Player B will then select the vertex $v_6$. This results in almost the same configuration as in **Case 3.** In fact, it does result in the same number of remaining moves. Therefore Player B will win with this strategy by the same argument in **Case 3.**

**Case 5.** Suppose Player A selects the vertex $v_5$. Player B will then select the vertex $v_2$. This results in a $P_8$ with Player A selecting next. From the Theorem 6 we know Player B will win with this strategy on a $P_{14}$.

**Case 6.** Suppose Player A selects the vertex $v_6$. Player B will then select the vertex $v_3$. This results in the exact configuration as in **Case 3.** Therefore from previous arguments we know Player B will win on a $P_{14}$ with this strategy.

**Case 7.** Suppose Player A selects the vertex $v_7$. Player B will then select the vertex $v_4$. This results in two disjoint paths a $P_2$ and a $P_6$. Thus, there are four distinct moves for Player A's next selection. We must consider all four choices. **Case 7a.** Suppose Player A selects the vertex $v_1$ (respectively, $v_2$) with his second move. This results in Player B selecting first on a $P_6$. From Theorem 5 we know Player B would win. **Case 7b.** Suppose Player A selects the vertex $v_9$ with his second move. This results in two disjoint paths a $P_2$ and a $P_4$. Thus there are only three moves remaining. Since Player B is next to select, he wins with this strategy on a $P_{14}$. **Case 7c.** Suppose Player A selects the vertex $v_{10}$ with his second move. This results in two disjoint paths a $P_2$ and a $P_3$. Player B would then select one of the *end-vertices* of the $P_3$ (either vertex $v_{12}$ or $v_{14}$). This results in only two moves remaining with Player A selecting first. Therefore B wins on a $P_{14}$ with this strategy. **Case 7d.** Suppose Player A selects vertex $v_{11}$ with his second move. This results in three disjoint paths each with only one move. Since Player B is selecting and there are exactly three moves remaining, he wins on a $P_{14}$ with this strategy.

Therefore Player B will always win on a $P_{14}$. □

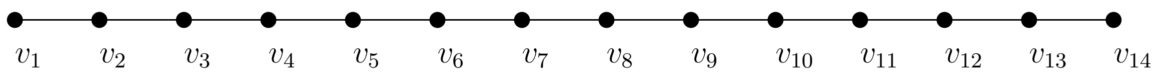


Figure 16: A path on fourteen vertices.

By Lemma 3, we have the following Corollary.

**Corollary 10** *Player A wins on* $P_{16}$.

Our results so far

| Path | Winner |
|---|---|
| $P_2$ | A |
| $P_4$ | B |
| $P_6$ | A |
| $P_8$ | B |
| $P_{10}$ | A |
| $P_{12}$ | A |
| $P_{14}$ | B |
| $P_{16}$ | A |
| $P_{18}$ | A ? |

Of course, it would be nice to settle this problem for paths. However, the problem is extremely difficult. Once the strategies are clear for all paths, then the strategies for all cycles will be known as well. Is there a clear pattern for paths in which Player B will always win? How about other families of graphs with this achievement game? Hopefully we can continue to find strategies for not only paths, but many other graphs.

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