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A Complete Characterization of Maximal Symmetric Difference-Free families on $\{1, \ldots n\}$.

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A Complete Characterization of Maximal Symmetric Difference-Free Families on

$\{1, \ldots n\}$

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by

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ABSTRACT

A Complete Characterization of Maximal Symmetric Difference Free Families on $\{1, \ldots n\}$

by

Travis Gerarde Buck

Prior work in the field of set theory has looked at the properties of union-free families. This thesis investigates families based on a different set operation, the symmetric difference. It provides a complete characterization of maximal symmetric difference-free families of subsets of $\{1, \ldots n\}$

DEDICATION

I would like to dedicate this to my family and everyone who encouraged me to study math as I was growing up.

ACKNOWLEDGMENTS

I would like to thank Dr. Anant Godbole for his support and especially patience in helping me prepare this thesis. I would also like to thank Dr. Teresa Haynes for encouraging me to attend the Graph Theory conference in Boone. Without her and Dr. Godbole, I might still be looking for a topic. I would like to thank Dr. Gardner, not only for being on my committee, but also for reminding me of all the bureaucratic checkpoints I had to pass. Hopefully, I can see you coming without having to worry that I've forgotten something. Finally, I'd like to thank all of the faculty and staff in the Department of Mathematics, you make everything so interesting that it's hard to focus on one thing.

CONTENTS

1 INTRODUCTION

This thesis will provide a complete characterization of maximal symmetric difference-free families of subsets of $\{1 \dots n\}$. The work presented here was inspired by an invited one-hour presentation given by Professor Dwight Duffus of Emory University at the 2005 Graph Theory Conference in Boone, NC. In this talk, Duffus introduced the union closed Frankl conjecture that states that in any union closed family of subsets of $\{1, 2, \dots, n\}$ there exists an element a that belongs to at least half the sets in the collection[1]. Union-free families were defined by Frankl and Furedi to be those families where there does not exist A, B, C, D in the family such that $A \cup B = C \cup D$.[2]

We adopted an approach and a definition slightly different than those used by the above authors, in particular delta free families were not defined analogously with union free families. For this thesis, if a family of sets is delta free, then for any A, B in the family, $A \Delta B$ is not in the family. The symmetric difference operation, denoted by Δ , of two sets constructs a set that contains all the elements that are in exactly one of the sets. For example, given the sets $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, their symmetric difference, or $A \Delta B$ would be $\{1, 4\}$. Because both A and B contain the elements 2 and 3, neither of these elements can be in the symmetric difference; because 1 and 4 are each in only one of the sets, they are in the symmetric difference. Another way to describe the symmetric difference of sets is the union of the sets minus their intersection, i.e. $A \Delta B = (A \cup B) \setminus (A \cap B)$.

One application of constructing Δ-free families of sets would be to let each set

[1] uiuc

[2] FF

represent a person and each element of the set represent a task which that person can do. Let us also assume that each person can only perform one task at a time. If the family is Δ-free, then for any three people we pick from it, each person is useful in his or her own right for completing a task and isn't simply back-up for the other two people. For example, as a Δ-free family we could have the sets $\{1, 2\}, \{1, 3\}$, and $\{1, 2, 3\}$. The symmetric difference of the first two sets is $\{2, 3\}$, that of the last two is $\{2\}$, and that of the first and last is $\{3\}$. None of these symmetric differences are in the family and each person is useful for completing a task. For example, while the first person does task 1, the second can do task 3, and the third person can do task 2. If the first person does task 2, the second can do task 1 and the third task 3 or vice versa. If we were limited to the family being union-free, we would not be able to have $\{1, 2, 3\}$ in the same family as $\{1, 2\}$ and $\{1, 3\}$, since the union of $\{1, 2\}$ and $\{1, 3\}$ is $\{1, 2, 3\}$. Considering the same definition of sets and elements, union-free would represent avoiding redundancy if all possible tasks were performed simultaneously by each person. In the above example, the person represented by $\{1, 2, 3\}$ would not be needed since $\{1, 2\}$ and $\{1, 3\}$ can perform all the possible tasks that $\{1, 2, 3\}$ can perform. Another application of Δ-free families would be to let each set represent a person and each element represent a language that person speaks. If the family is Δ-free, then there is no person who can translate for two other people all the languages those two do not share and yet not understand the languages those two other people do share.

Besides trying to find a non-trival application, most of the early efforts were put towards trying to find bounds for symmetric difference-free families, as well as exploring the limitations for the potential members of such families. The first important observation came regarding the empty set, $\emptyset$. The second key observation dealt with the form of sets that could be guaranteed to be part of a Δ-free family if no other sets were present.

Lemma 2.1 *If a family F contains the empty set, F cannot be Δ-free.*

Proof: Assume F is Δ-free and contains the empty set. Since $A\Delta\emptyset = A$ for any A in F we reach a contradiction, so F cannot be Δ-free and still contain the empty set. Similarly, if F is Δ-free, then for any $A \in F, A\Delta A = \emptyset$ so $\emptyset$ cannot be $\in F$.$\blacksquare$

Early exploration also showed that sets consisting of only odd members from the power set were Δ-free. For example, if $n = 3$, then the family $\{1\}, \{2\}, \{3\}, \{1, 2, 3\}$ could be constructed. Since this family doesn't contain $\{1, 2\}, \{1, 3\}$, or $\{2, 3\}$ it is Δ-free. Similarly, if $n = 4$, the family $\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$ can be constucted and this family is Δ-free as well. This led to the second lemma.

Lemma 2.2 *Any family F that contains only sets of odd cardinality is Δ-free.*

Proof: If the cardinality of the union of two sets of odd cardinality is even, then the cardinality of the intersection of those two sets is also even, so the cardinality of their symmetric difference would be even. If the cardinality of the union of two sets of odd cardinality is odd, then the cardinality of the intersection of those two sets would be odd and, again, the cardinality of their symmetric difference would be even. In either case, the cardinality of the symmetric difference of two odd sets is even. The family F consists of only sets with odd cardinalty, so for any $A, B \in F, A \Delta B$ is even, so $A \Delta B \notin F$ since F consists only of sets of odd cardinality. Therefore, any family F that consists of only sets of odd cardinality is Δ-free. ■

This lemma is interesting and useful because, of the 2^n elements of the power set of $\{1, \ldots n\}$ exactly half, or 2^{n-1} of the elements are of odd cardinality. Since, from Lemma 2.2, we know that we can create a Δ-free family by selecting all of these odd sets, we know that a maximal Δ-free family of subsets of $\{1, \ldots n\}$ has to have at least 2^{n-1} members since we can construct a Δ-free family with 2^{n-1} members. With this, we have established a lower bound.

A Δ-free family can, however, contain sets with even cardinality. For example, for $n = 3$, the family $\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}$ contains sets with both even and odd cardinality. The family is Δ-free as it does not contain $\{2\}, \{3\}$, or $\{2,3\}$. Being Δ-free, the family definitely cannot contain the empty set. It is worth noting that this family has 2^{3-1} i.e. $2^2 = 4$ members which is half the members of the power set. In fact, since half (2^{n-1}) of the members of the power set of $\{1, \ldots n\}$ are odd, in order to have more than 2^{n-1} members in a family we would need to have at least one set of even cardinality which leads to our third lemma.

Lemma 3.1 *A maximal Δ-free family that contains at least one even set contains at least half the even sets and half the odd sets. The cardinality of this family is 2^{n-1}.*

Proof: Let F be a maximal Δ-free family on $\{1, \ldots n\}$ that contains at least one even set, call it F_e. We know that the cardinality of F is at least 2^{n-1}. Let O be any of the 2^{n-1} odd members of the power set. $F_e \Delta O = O_\Delta$ is an odd set that is distinct from O. If $O \in F$, then $O_\Delta \notin F$ since F is Δ-free. So, if F contains an even set, each odd set in F rules out another odd set as a potential member of F. Therefore, if F contains an even set, it can contain at most half, or 2^{2-2}, of the odd sets from the power set.

Similarly, let E be any of the 2^{n-1} even members of the power set. If $E \in F$, then we know that $E \neq \emptyset$. If $E = F_e$ then $E \Delta F_e = \emptyset \notin F$, so let us just consider if E is different from F_e. $E \Delta F_e$ is even so each even set in F rules out another even set as a potential member of F. So, again, if F contains an even set, it can contain at most half, or 2^{n-2} of the even members of the power set, including the original even set. Since a Δ-free family on $\{1, \ldots n\}$ that contains at least one even set can have at most 2^{n-2} of the odd sets from the power set and 2^{n-2} of the even sets from the power set, it can contain at most $2^{n-2} + 2^{n-2} = 2^{n-1}$ sets. Since a maximal family would have the maximum number of sets possible, we know that it will contain this maximum of 2^{n-1} sets. $\blacksquare$

Since we know that a maximal Δ-free family contains at least 2^{n-1} elements from our lower bound and at most $2n - 1$ elements from the above result, we know that a maximal Δ-free family contains exactly 2^{n-1} sets and is either all the odd sets or is half the even sets and half the odd sets. This is a very interesting result because this means we must pick exactly half the possible elements from the power set when constructing a maximal Δ-free family. The difficult question is "Which half?". It was decided that we would try to find all of the maximal Δ-free families for $n = 3, n = 4$, and $n = 5$ in order to see if some pattern was discernable.

A brute force analysis by hand of the 4 element subsets of the power set of $\{1, 2, 3\}$ yielded the following Δ-free families:

$\{1\}, \{2\}, \{3\}, \{1, 2, 3\}$

This is the family with all of the odd elements.

$\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}$

$\{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}$

$\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

These three are essentially isomorphic to each other.

$\{1\}, \{2\}, \{1,3\}, \{2,3\}$

$\{1\}, \{3\}, \{1,2\}, \{2,3\}$

$\{2\}, \{3\}, \{1,2\}, \{1,3\}$

These three are essentially isomorphic to each other.

There are $7 = 2^3 - 1$ families. Checking the results for $n = 4$ and $n = 5$ were deemed to be too time consuming to perform by hand, so a program was constructed for Maple that would check all the subsets of appropriate size in order to see if they were Δ-free. Even though the empty set could be discounted, it was still a very processor-intensive, brute force investigation with a large number of comparisons required at each step. In fact, checking with this procedure for $n = 5$ proved to be too much for the memory of the computers available. Fortunately, the computer was able to sift through the options and find the Δ-free families for $n = 4$, and a pattern was detected.

The Δ-free families for $n = 4$ are:

$\{1\}, \{2\}, \{3\}, \{4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}$

This is the family with all of the odd elements.

$\{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}$

There are three more families isomorphic to this one.

$\{1\}, \{2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{1,3,4\}, \{2,3,4\}$

There are five more families isomorphic to this one.

$\{1\}, \{2\}, \{3\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,3,4\}$

There are three more families isomorphic to this one.

There are a total of $15 = 2^4 - 1$ families. The pattern that emerged was based on what symbols were not represented as singletons. Let S be the set of singletons in F, so let S^C be the set of elements not represented by singletons in family F. We can construct a family by taking from the power set all of the odd sets with an even intersection with S^C (which includes the singletons) and all of the even sets with an odd intersection with S^C. Consider $n = 5$. If we know that $\{1\}$ and $\{2\}$ are in F but $\{3\}, \{4\}$, and $\{5\}$ are not, then S^C is $\{3,4,5\}$. Based on this, the one-element sets in F are $\{1\}$ and $\{2\}$ since they have an evem, albeit empty, intersection with S^C. The two-element sets are $\{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}$, and $\{2,5\}$, while the three-element sets are $\{1,3,4\}, \{1,3,5\}, \{2,3,4\}$, and $\{2,3,5\}$. The four-element sets are $\{1,3,4,5\}$ and $\{2,3,4,5\}$ and there are no five-element sets in F in this case. This particular family is Δ-free, but the question arises of whether this method of construction works in general.

Theorem 3.2 *Given a set S^C which consists of a subset of $\{1, \dots n\}$, we can construct a maximal Δ-free family F on $\{1, \dots n\}$ by taking all the odd members of the power set with an even intersection with S^C and the even members of the power set with an odd intersection with S^C.*

Proof: Let F be a family on $\{1, \ldots n\}$ that consists of the odd members of the power set whose intersection with a subset S^C of $\{1, \ldots n\}$ is even combined with the even members of the power set whose intersection with S^C is odd. Let A and B be $\in F$.

Case 1: A and B are both odd

If A and B are both odd, then their symmetric difference would be even. A and B each have an even intersection with S^C, so the symmetric difference of their intersections with S^C would be even, as well as a subset of $A\Delta B$. Thus, $A\Delta B$ would be an even member of the power set with an even intersection with S^C. Since F does not contain even members of the power set with an even intersection with S^C, we know that $A\Delta B \notin F$.

Case 2: A and B are both even

If A and B are both even, then their symmetric difference would be even. A and B each have an odd intersection with S^C, so the symmetric difference of their intersections with S^C would be even. Since F does not contain even members of the power set with an even intersection with S^C we know that $A\Delta B \notin F$.

Case 3: WLOG A is even and B is odd

Since A is even and in F it has an odd intersection with S^C. Since B is odd and in F it has an even intersection with S^C. $A\Delta B$ is an odd member of the power set. The symmetric difference of the intersection of A with S^C and the intersection of B with S^C is a subset of $A\Delta B$ and is odd. Therefore, $A\Delta B$ is an odd member of the power set with an odd intersection with S^C. Since F does not contain odd members of the power set with an odd intersection with S^C, we know that $A\Delta B \notin F$.

In any case, for A and $B \in F, A\Delta B \notin F$ so F is Δ-free. Of the 2^{n-1} odd members of the power set, 2^{n-2} have an odd intersection with S^C and, of the 2^{n-1} even members of the power set, 2^{n-2} have an even intersection with S^C. Combined, $2^{n-2} + 2^{n-2} = 2^{n-1}$ which is the maximal size of a Δ-free family, so we have constructed a family that is Δ-free and is of maximal size. $\blacksquare$

From the above proof, we know that this method will constuct a Δ-free family of maximal size. A family consisting of all the odd members of the power set can be constructed by letting S^C be the empty set. Since every odd set has a trivially even, empty intersection with SC, that can all be included. Since every even set also has a trivially even, empty intersection with S^C, none of them can be included. The above examples of Δ-free families for $n = 3$ and $n = 4$ fit the pattern as well.

4 UNIQUENESS

We know the method outlined in Theorem 3.2 will construct maximal Δ-free families, but in order to completely characterize maximal Δ-free families, we need to make sure that those families are the only such maximal Δ-free families. For instance, for some similarly defined S^C can we find an even member if the power set with an even intersection with S^C in a maximal Δ-free family F?

Theorem 4.1 *The procedure outlined in Theorem 3.2 generates all the possible maximal Δ-free families on $\{1, \ldots n\}$.*

Proof: Let us assume that there is some maximal Δ-free family F on $\{1, \ldots n\}$ that is not generated by taking the odd sets with even intersections with S^C and the even sets with an odd intersection with S^C.

Case 1: F contains an even set with an even intersection with its corresponding S^C.

For a two element set to have an even intersection with S^C it must contain either no elements or two elements from S^C. WLOG, let the proposed two element set be $\{1, 2\}$. If $\{1, 2\} \cap S^C = \emptyset$ then $\{1\} \notin S^C$ so $\{1\} \in S$ so, by definition, $\{1\} \in F$. Similarly, $\{2\} \notin S^C$ so $\{2\} \in F$. However, $\{1\}\Delta\{2\} = \{1, 2\}$, so if $\{1, 2\} \cap S^C = \emptyset$, then $\{1, 2\} \notin F$, a contradiction to our assumption that $\{1, 2\} \in F$. If $\{1, 2\}$ contains exactly two elements from S^C, then $\{1\} \notin F$ and $\{2\} \notin F$ since they are both $\in S^C$. However, in order to be maximal, if $\{1, 2\} \in F$, then exactly one of $\{1\}$ and $\{2\}$ must be in F as well, which is a contradiction. So, if F contains a two element set with an even intersection with its corresponding S^C, it cannot be maximal.

A four element set, like all even sets, limits us to having at most half of the other even sets and half the odd sets if we are to have a Δ-free family. If, WLOG, $\{1,2,3,4\}$ has an even intersection with S^C, then there are at least two two-element sets each with an even intersection with S^C whose symmetric difference is $\{1,2,3,4\}$. In order to be a maximal Δ-free family, for each pair of two-element sets exactly one would need to be in F. However, as shown above, neither can be in F since F cannot contain a two-element set with an even intersection with its S^C, so we cannot have a four-element set with an even intersection with its S^C in a maximal Δ-free family.
From here, we can use induction. We know we cannot have a two-element set with an even intersection with the S^C in a maximal Δ-free family. Suppose we cannot have any $2k$-element set with an even intersection with the S^C in a maximal Δ-free family. A $2(k+1)$-element set with an even intersection with the S^C is the symmetric difference of a $2k$-element set with even intersection and a 2-element set with even intersection. In order for F to be maximal, exactly one of these two sets would need to be in F. However, neither are in F by our assumption so we cannot have an even set with an even intersection with the corresponding S^C in a maximal Δ-free family.$\square$

Case 2: F contains an odd set with an odd intersection with its S^C.

We now know that in a maximal Δ-free family the only even sets possible are those with an odd intersection with the corresponding S^C and that there are 2^{n-2} such sets. The symmetric difference of an even set with an odd intersection with the S^C and an odd set with an odd intersection with the S^C is an odd set with an even intersection with the S^C. Therefore, if the family under consideration contains all the even sets with an odd intersection with the S^C and an odd set with an odd intersection with the S^C, it cannot contain an odd set with an even intersection with the S^C including the singletons we used to define S and thus S^C, which leads to a contradiction. Therefore, if a family contains an odd set with an odd intersection with its corresponding S^C, it cannot be a maximal Δ-free family. $\square$

Since a maximal Δ-free family cannot contain an even set with an even intersection with its S^C or an odd set with an odd intersection with its S^C, then our assumption that can construct a maximal Δ-free family that is not constructed by Theorem 3.2 is contradicted. Therefore, Theorem 3.2 constructs all possible maximal Δ-free familes. $\blacksquare$.

If $S^C = \{1, \dots n\}$ for a family, then we cannot have any even sets in the proposed maximal family. Also, we cannot have any one-element sets in the family, so the cardinality of a family constructed from such an S^C is less than 2^{n-1}.

Alternate proof: If the result does not hold, then for some maximal family F, there exists a corresponding, non-empty set S^C and a set A^* such that A^* is an even set such that $A^* \cap S^C$ is even. The other sets in the maximal F are either even or odd and have either an even or odd intersection with the S^C that corresponds with F. Let J be the class of odd sets from the power set with an odd intersection with S^C, K be the class of odd sets from the power set with an even intersection with S^C, L be the class of even sets from the power set with an odd intersection with S^C, and M be the class of even sets from the power set with an even intersection with S^C. The cardinality of each of these classes is 2^{n-2}. We have supposed that there is a maximal Δ-free family F that contains a set A^* from class M. The symmetric difference of any set A with a set from class M is another set from the same class as A. So, since F contains a set from class M it can contain at most half, or 2^{n-3} members from classes J, K, and L. This is because of the important fact that $C \neq D$ implies that $C\Delta E \neq D\Delta E$. Let M consist of sets M_1, M_2, etc. Note that if we are to remain Δ-free, M cannot contain the empty set and so there are at most $2^{n-2} - 1$ options for membership in F from class M. Excluding A^*, we get $2^{n-2} - 2$ sets in M. Since $A^*\Delta M_1, A^*\Delta M_2$ etc. cannot be in F if M_1, M_2 etc. are in F, we are left with the fact that F has at most $2^{n-3} - 1$ elements from M. In a similar fashion A^* causes the elimination of one set from the collection for each set in J, K, L so J, K, L can have at most 2^{n-3} members in F. Thus, F is not maximal, a contradiction. $\blacksquare$

5 CONCLUSIONS

We know that all maximal Δ-free families on $\{1, \ldots n\}$ have 2^{n-1} members. Additionally, given a set $S^C \subset \{1, \ldots n\}$ we can construct a corresponding maximal Δ-free family consisting of all the odd members of the power set with an even intersection with S^C and all the even members of the power set with an odd intersection with S^C, and this method of construction generates all possible maximal Δ-free families. Due to the isomorphisms from the arbitrary assignment of elements, there are a total of $2^n - 1$ maximal Δ-free families of $\{1, \ldots n\}$ with S^C being not allowed.

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