**SCHOOL *of*
GRADUATE STUDIES**
EAST TENNESSEE STATE UNIVERSITY

Electronic Theses and Dissertations

Student Works

5-2009

Decompositions of Mixed Graphs with Partial Orientations of the P_4.

Adam M. Meadows
East Tennessee State University

Follow this and additional works at: https://dc.etsu.edu/etd

🎨 Part of the Discrete Mathematics and Combinatorics Commons

Decompositions of Mixed Graphs with Partial Orientations of the P_4

A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

by

Adam Meadows

May 2009

Robert A. Beeler, Ph.D. Chair

Robert B. Gardner, Ph.D.

Teresa W. Haynes, Ph.D.

Keywords: graph theory, graph decomposition, graceful labeling, mixed graphs.

ABSTRACT

Decompositions of Mixed Graphs with Partial Orientations of the P_4

by

Adam Meadows

A decomposition $\mathscr{D}$ of a graph H by a graph G is a partition of the edge set of H such that the subgraph induced by the edges in each part of the partition is isomorphic to G. A *mixed graph* on V vertices is an ordered pair (V, C), where V is a set of vertices, $|V| = v$, and C is a set of ordered and unordered pairs, denoted (x, y) and $[x, y]$ respectively, of elements of V [8]. An ordered pair $(x, y) \in C$ is called an *arc* of (V, C) and an unordered pair $[x, y] \in C$ is called an *edge* of graph (V, C). A path on n vertices is denoted as P_n. A *partial orientation on* G is obtained by replacing each edge $[x, y] \in E(G)$ with either (x, y), (y, x), or $[x, y]$ in such a way that there are twice as many arcs as edges. The *complete mixed graph* on v vertices, denoted M_v, is the mixed graph (V, C) where for every pair of distinct vertices $v_1, v_2 \in V$, we have $\{(v_1, v_2), (v_2, v_1), [v_1, v_2]\} \subset C$. The goal of this thesis is to establish necessary and sufficient conditions for decomposition of M_v by all possible partial orientations of P_4.

DEDICATION

I would like to dedicate my thesis to my father. My father has been an inspiration to me my whole life. My father has always been there when I needed him and has always been proud of everything I have achieved or ever done. Thank you for making me the man I am today.

ACKNOWLEDGMENTS

I would like to thank Dr. Robert A. Beeler for working with me during the time of my thesis. Thank you for always being there whenever I needed help throughout this whole process. I would also like to thank my other committee members Dr. Robert B. Gardner and Dr. Teresa W. Haynes for being on my committee and taking the time to review my work and give feedback along the way. I want to thank my grandparents for being supportive of me along the way and always being there when I've needed their help. I also want to thank my former college professor, Jennifer Wilson. I would not be where I am today without the support and guidance of her along the way. She has always believed in me and been an inspiration in my life. Finally, I would like to thank my fiancé Crystal. Thank you for always being there when I needed you. You are my best friend and I love you.

CONTENTS

1 INTRODUCTION

1.1 Basic Graph Theory Definitions

A *graph G* consists of a *vertex set V(G)*, an *edge set E(G)*, and a relation that associates the edges with vertices. If a vertex is associated with an edge, it is an *endpoint* of that edge. A vertex v is *incident* to an edge e if v is an endpoint of e. An arc, $a = (x, y)$, is considered to be a directed edge from x to y. For a vertex v in a graph G, the *degree* of v denoted $deg_G(v)$ is the number of edges of G incident with v. For a vertex v in graph G, the *out degree* denoted $od(v)$ of v is the number of vertices of G to which v is adjacent, while the *in degree* of G denoted $id(v)$ of v is the number of vertices of G from which v is adjacent. We will assume that all graphs are *simple*, i.e., there are no *loops* (edges whose endpoints are equal) and no multiple edges [25]. All graphs presented here are finite unless otherwise noted. Examples are given in Figure 1.
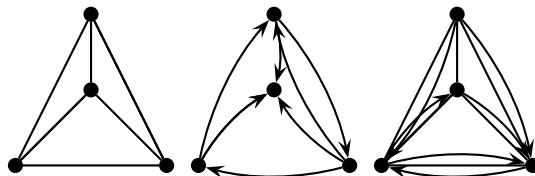
Figure 1: Simple Graph with No Arcs, with Arcs, and with Arcs and Edges.

The *order* of G, denoted $n(G) = |V(G)|$, is the number of vertices in G. The *size* of G, denoted $e(G) = |E(G)|$, is the number of edges in G. The *maximum degree* of G, denoted $\Delta(G) = max\{deg_G(v)\}$, is the largest number of edges incident to a vertex v in G. The *minimum* degree of G, denoted as $\delta(G) = min\{deg_G(v)\}$, is the smallest number of edges incident to a vertex v in G [25].

8

A *path* on n vertices is denoted P_n. The *cycle* on n vertices is denoted C_n. A *bipartite graph* is a graph whose vertices can be divided into two disjoint sets V_1 and V_2 such that every edge connects a vertex in V_1 to one in V_2; that is, V_1 and V_2 are *independent sets*. Equivalently, a *bipartite graph* is a graph that does not contain any cycles of odd length. A *complete bipartite graph* $G : (V_1 \cup V_2, E)$ is a bipartite graph such that $v_1 v_2 \in E(G)$ if and only if $v_1 \in V_1$ and $v_2 \in V_2$. The *complete bipartite graph* with partitions $|V_1| = m$ and $|V_2| = n$ is denoted $K_{m,n}$. The *wheel graph*, denoted W_n, is a graph with $n+1$ vertices, formed by connecting a single vertex to all the vertices of an $n - cycle$. The *complete graph* is a simple graph in which every pair of distinct vertices is connected by an edge. The complete graph on n vertices is denoted by K_n [8]. Examples of these graphs are given in Figure 2.

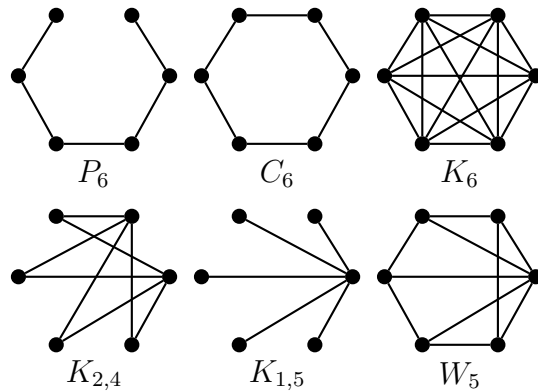Figure 2: Special Graphs on Six Vertices

A *directed graph* or *digraph*, denoted D, consists of a vertex set $V(D)$, and a set of arcs $A(D)$. If a directed path leads from x to y, then y is said to be a successor of x and reachable from x, and x is said to be a predecessor of y. An arc, $a = (x, y)$, is considered to be directed from x to y. The arc (y, x) is the arc (x, y) inverted.

9

A *mixed graph* on V vertices is an ordered pair (V, C), where V is a set of vertices, $|V| = v$, and C is a set of ordered and unordered pairs, denoted (x, y) and $[x, y]$ respectively, of elements of V [8]. An ordered pair $(x, y) \in C$ is called an *arc* of (V, C) and an unordered pair $[x, y] \in C$ is called an *edge* of graph (V, C). Figure 3 illustrates a mixed graph on four vertices.
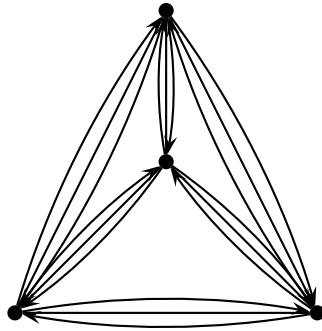
Figure 3: Mixed Complete Graph on Four Vertices, M_4

An *isomorphism* between two graphs G and H is a bijection $f : V(G) \rightarrow V(H)$ such that $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$. Two graphs are *isomorphic* if such a bijections exists.

1.2   Decompositions

A *decomposition* $\mathscr{D}$ of a digraph H is a partition of the arc set of H. The graph H is called the *host* graph for the decomposition. For each P of the partition, the subgraph of H induced by P is called a *block* of the partition. We will be concerned with the case that all blocks are isomorphic to a single *block prototype* G. In this case we say that $\mathscr{D}$ is a G-decomposition of H and that G is a *divisor* of H. This

situation is denoted $G|H$. Analogous definitions exist for decompositions of directed
and mixed graphs. Figure 4 illustrates a P_3-decomposition of Q_3.
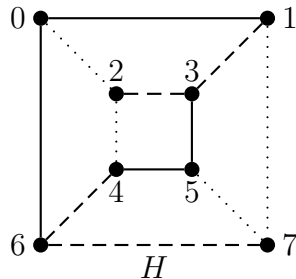
Figure 4: A P_3-Decomposition of H

If G is a simple graph then the mixed graph $M(G)$ has $V(M(G)) = V(G)$, and
$C = \{(x, y), [x, y], [y, x]\} \subseteq C$ if and only if $xy \in E(G)$. For convenience, $M(K_n) =$
M_n. Figure 5 illustrates a graph and its associated mixed graph.
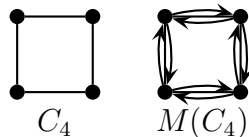
Figure 5: C_4 and $M(C_4)$

A *partial orientation of* G is obtained by replacing each edge $[x, y] \in E(G)$ with
either (x, y), (y, x), or $[x, y]$. We restrict this to partial orientations in which there are
twice as many arcs as edges. Since the mixed complete graph has twice as many arcs
as edges, $e(G) \equiv 0 \pmod 3$. The *converse* of a directed (mixed) graph G, denoted
G^c, is obtained from G by reversing the orientation on all arcs, i.e., $(x, y) \in G^c$ if and
only if $(y, x) \in G$.

11

The goal of this thesis is to establish necessary and sufficient conditions for the decomposition of M_v by all possible partial orientations of P_4.
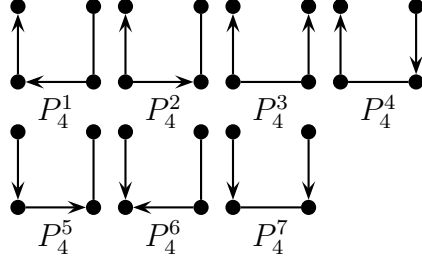
Figure 6: Partial Orientations of P_4

To aid in this, we give the following proposition.

Proposition 1.1   *i. If $G|H$ and $H|K$, then $G|K$.*

ii. $G|H$ if and only if $G^c|H^c$.

iii. $G|H$ if and only if $M(G)|M(H)$.

Proof.

i. Suppose $G|H$ and $H|K$. If $G|H$, then there is a decomposition $\mathscr{D}_1$ of a graph H into copies of G. Similarly, since $H|K$, there exists a decomposition $\mathscr{D}_2$ of K into copies of H. For each H-block in $\mathscr{D}_2$, we replace it with edge disjoint copies of G via $\mathscr{D}_1$. This gives the required G-decomposition of K.

ii. Suppose $G|H$. If $G|H$, then there is a decomposition $\mathscr{D}$ of a digraph H is a partition of the arc set of H. Then take any G-block B such that $(x, y) \in B$. Note that $(x, y) \in A(H)$. Then for each $(x, y) \in A(H)$, there exists and G^c-block B^c such that $(y, x) \in A(B^c)$. Hence, this gives the required G^c-decomposition

12

of H^c. Conversely, if $G^c|H^c$ then there is a decomposition $\mathcal{D}$ of a graph H^c into copies of G^c. Then for each $(x, y) \in A(H)$, there exists and G^c-block, B^c, such that $(y, x) \in A(B^c)$. Since $(x, y) \in A(H)$, there exists a G-block B such that $(x, y) \in A(B)$. This gives the required G-decomposition of H.

iii. If $G|H$, then there is a decomposition $\mathcal{D}$ of a graph H into copies of G. Then for each $xy \in E(H)$, there exists a G-block B such that $xy \in B$. For every

$$\{(x, y), (y, x), [x, y]\} \subset C(M(H)),$$

there exists $M(G)$-block B such that $\{(x, y), (y, x), [x, y]\} \in C(B)$. This means we have a $\mathcal{D}$ of a graph $M(H)$ into copies of $M(G)$. This then gives the required $M(G)$-decomposition of $M(H)$. Conversely, if $M(G)|M(H)$ then there is a decomposition $\mathcal{D}$ of a graph $M(H)$ into copies of $M(G)$. For every

$$\{(x, y), (y, x), [x, y]\} \subset C(M(H)),$$

there exists $M(G)$-block B such that $\{(x, y), (y, x), [x, y]\} \in M(H)$. Hence for each $xy \in E(M(H))$ there exists a $M(G)$-block B such that; $xy \in B$. This means we have a decomposition $\mathcal{D}$ of a graph H into copies of G. Hence this gives the required G-decomposition of H.

$\blacksquare$

1.3   Algebra

The constructions that we will use to prove the sufficient conditions for our results will often be algebraic in nature. To facilitate this we give relevant definitions and

13

results from algebra.

Let a, d be integers. If d *divides* a, then there exists $c \in \mathbb{Z}$ such that $cd = a$, and this situation is denoted $d|a$. If there is no such c, we say that d does not divide a, and this situation is denoted $d \nmid a$. Let $S \subseteq \mathbb{Z}^+$. Then the greatest common divisor of S, denoted $gcd(S)$, is the largest $d \in \mathbb{Z}^+$ such that $d|a$ for all $a \in S$ [2, 10, 13].

Define $\mathbb{Z}_n^* = \{x \in \mathbb{Z}_n : 0 < x \leq n/2\}$. Define $|x|_n = min\{|x|, |n - x|\}$. Let $n \in \mathbb{Z}^+$ and $S \subseteq \mathbb{Z}_n^*$ be given. The *circulant graph* $C_n(S)$ is the undirected graph with vertex set $V = \mathbb{Z}_n$ and edge set: $E = \{xy : x, y \in \mathbb{Z}_n \text{ and } |x - y|_n \in S\}$[3]. Figure 7 illustrates an example of a circulant graph.
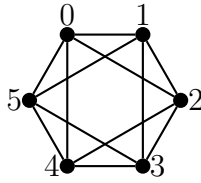
Figure 7: Circulant - $C_6(1, 2)$

Let $\mathbb{Z}_n = \{0, ..., n - 1\}$. If $a \in \mathbb{Z}_n$ and $gcd(a, n) = 1$, then for every $x \in \mathbb{Z}_n$ there exists a $y \in \mathbb{Z}_n$ such that $ya = x$. Hence a generates $\mathbb{Z}_n$. In general, if $gcd(a, n) = d$, then a will generate d disjoint cycles, each of length n/d.

Two integers v and a are said to be *congruent modulo n*, if their difference $a - b$ is an integer multiple of n. An equivalent definition is that both numbers have the same remainder when divided by n. If this is the case, it is expressed as $v \equiv a \pmod{n}$.

2   REVIEW OF LITERATURE

This chapter introduces the motivation behind the study of triple systems and mixed triple systems.

2.1   Triple Systems

Our problem originates from the study of triple systems. This originated with Kirkman's schoolgirl problem. Kirkman's schoolgirl problem is a problem in combinatorics solved by Thomas Kirkman in 1847 as Query VI in "The Lady's and Gentleman's Diary"[17].

The problem states: "Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast."

The problem can be generalized to n girls, where n is an odd multiple of three. The n girls need to be walking in triplets for $(n-1)/2$ days, with the requirement that no pair of girls walk in the same row twice. A complete solution to the general case was given by D. K. Ray-Chaudhuri and R. M. Wilson in 1969 [23].

We can show that Kirkman's Schoolgirl problem is equivalent to the decomposition of K_{15} into copies of K_3. Let the fifteen girls be represented by the vertices of K_{15}. As any two girls can walk in a row, there is an edge between any two vertices. The "three abreast rows" mean K_3-blocks. Further, "no two shall walk abreast" means that the edges of these triangles are disjoint [17]. The solution to this generalization is a Steiner Triple System: $S(2, 3, 6t + 3)$ with parallelism (that is, one in which each

of the $6t + 3$ elements occurs exactly once in each block of $3 - element$ sets). A solution to Kirkman's Schoolgirl Problem is given in Table 1.

Table 1: A solution to Kirkman's schoolgirl problem

Sun	Mon	Tue	Wed	Thur	Fri	Sat
1,6,11	1,2,5	2,3,6	5,6,9	3,5,11	5,7,13	11,13,4
2,7,12	3,4,7	4,5,8	7,8,11	4,6,12	6,8,14	12,14,5
3,8,13	8,9,12	9,10,13	12,13,1	7,9,15	9,11,2	12,2,8
4,9,14	10,11,14	11,12,15	14,15,3	8,10,01	10,12,3	1,3,9
5,10,15	13,15,6	14,1,7	2,4,10	13,14,2	15,1,4	6,7,10

Next, this problem was worked on by Jacob Steiner[24], who was unaware at the time of the earlier work by Kirkman. In combinatorial mathematics, a *Steiner system* is a type of block design. A Steiner system with parameters l, m, n, written $STS(l, m, n)$, is an n-element set S together with a set of $m - element$ subsets of S (called blocks) with the property that each $l - element$ subset of S is contained in exactly one block. A Steiner system with parameters ℓ, m, n is often called simply $STS(\ell, m, n)$. An $STS(2, 3, n)$ is called a *Steiner triple system*, and its blocks are called triples. The number of triples is $n(n - 1)/6$.

A *Steiner triple system* of order v, $STS(v)$, is a decomposition of the complete graph on v vertices, K_v, into $3 - cycles$.

Steiner proved that $v \equiv 1, 3 \pmod 6$ is necessary[24] and Reiss[21] showed that this was also sufficient. Reiss and Steiner were both unaware that the problem had been posed and solved by Kirkman in 1847 [17].

A *Mendelsohn triple system*, denoted $MTS(v)$, is a decomposition of D_v into

copies of A, where A is the directed graph with $V(D_v) = \{a, b, c\}$,

$$A(A) = \{(a, b), (b, c), (c, a)\}$$

. This graph is also called a $3 - circuit$. A *Mendelsohn triple system* of order v is equivalent to an A-decomposition of D_v and exists if and only if $v \equiv 0, 1 \pmod 3, v \neq 6$.

Directed triple system denoted $DTS(v)$ is decomposition of D_v into copies of B where $V(B) = \{a, b, c\}$, $A(B) = \{(a, b), (c, b), (c, a)\}$. A *Directed triple system* of order v is equivalent to an B-decomposition of D_v and exists if and only if $v \equiv 0, 1 \pmod 3$. A *Directed Triple System* exists if and only if $v = 0, 1 \pmod 3$ [8]. Figure 8 illustrates an example of these graphs.

Triple systems were further expanded in Hartman and Mendelsohn's "Last of the Triple systems". Figure 9 illustrates the thirteen connected digraphs on three vertices by Hartman and Mendelsohn.
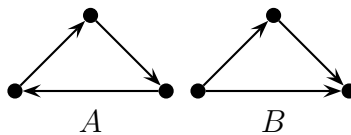
Figure 8: Mendelsohn and Directed Triple System

Inspired by Hartman and Mendelsohn, Robert Gardner [8] gave necessary and sufficient conditions for the existence of new triple systems which are given by presenting decompositions of the complete mixed graph into partial orientations of K_3. We will denote T_i by the ordered triple $(a, b, c)_i$. Figure 10 gives examples of these
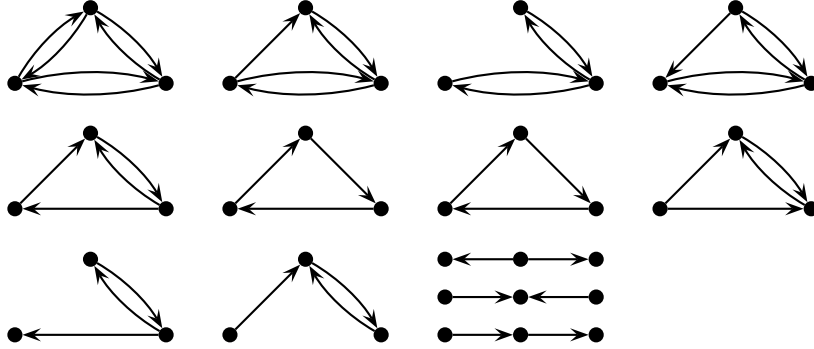
Figure 9: Digraphs From The "Last of the Triple Systems"

graphs. This is decomposition is called a T_i- *triple system* of order v where $i = 1, 2$ or 3 [8]. The necessary and sufficient conditions are stated in Proposition 2.1.

Proposition 2.1 *[8]*

  i. *A T_1-triple system of order v exists if and only if $v \equiv 1 \pmod 2$.*

  ii. *A T_2-triple system of order v exists if and only if $v \equiv 1 \pmod 2$.*

  iii. *A T_3-triple system of order v exists if and only if $v \equiv 1 \pmod 2$, $v \notin \{3, 5\}$.*

  A decomposition of M_v into copies of T_i is a T_i *Mixed Triple System* of order v [8].
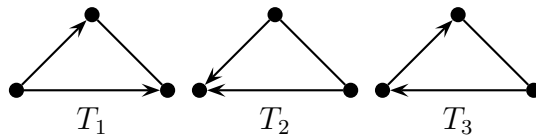
Figure 10: Mixed Triple Systems

18

2.2   Ringel's Conjecture

It has be conjectured by Ringel that any tree T size m will decompose K_{2m+1}[**?**].

Ringel's Conjecture was the primary motivation for Rosa to introduce valuations. Alexander Rosa developed several valuations on graphs. The most influential are β - valuations [22]. These valuations were popularized by Golomb under the name *graceful labelings* [9].

A *graceful labeling* of a graph with n vertices and e edges is a labeling of its vertices with distinct integers between 0 and e inclusive, such that each edge is uniquely identified by the positive, or absolute difference, between its endpoints. A graceful labeling will induce a cyclic decomposition of the complete graph [9]. Figure 11 illustrates a graph with a graceful labeling.
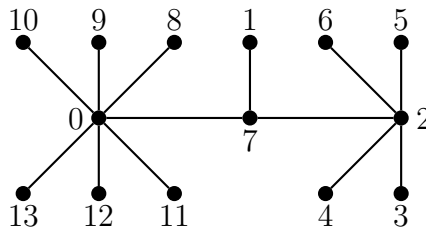
Figure 11: A Gracefully Labeled Graph

An outline of results for graceful labelings and related labelings are given in Gallian's [7] "Dynamic Survey of Graph Labeling. " This is summarized in Proposition 2.2.

Proposition 2.2 *The following graphs are known to have a graceful labeling:*

(i) *Trees with at most four endpoints [15, 16, 22, 26].*

19

(ii) Trees in which, at any two vertices u and v, there is a P_k with u and v as its endpoints, where $k \leq 5$ [14, 26].

(iii) Trees of order at most 27 [1].

(iv) Complete bipartite graphs [9, 22].

(v) Cycles of length n where $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$ [18].

(vi) W_n (i.e., Wheel Graphs) [6][5].

(vii) The n-dimensional hypercube, Q_n [19, 20].

Proposition 2.3 *[3] If $n \geq 2q + 1$, then a graceful graph of size q will decompose $C_n(1, ..., q)$.*

We will use these results for our constructions in the next chapter.

In this chapter we will give necessary and sufficient conditions for the decomposition of M_v using all possible partial orientations of the P_4.
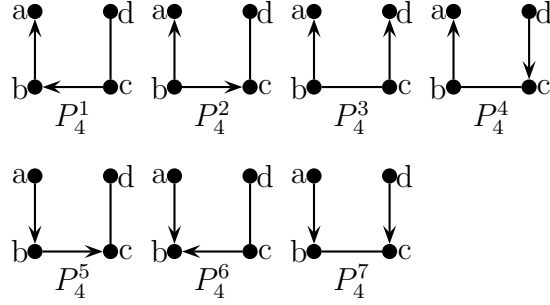
Figure 12: Labeled Partial Orientations of P_4

As P_4^5 is the converse of P_4^1, P_4^6 is the converse of P_4^2, and P_4^7 is the converse of P_4^3, we only need to find necessary and sufficient conditions of partial orientations of P_4^1, P_4^2, P_4^3, P_4^4 by Proposition 1.1. Our notation will be as follows:

i. $[a, b, c, d]_1$ denotes a P_4^1-block with $\{(b, a), (c, b), [c, d]\}$.

ii. $[a, b, c, d]_2$ denotes a P_4^2-block with $\{(b, a), (b, c), [c, d]\}$.

iii. $[a, b, c, d]_3$ denotes a P_4^3-block with $\{(b, a), (c, d), [b, c]\}$.

iv. $[a, b, c, d]_4$ denotes a P_4^4-block with $\{(b, a), (d, c), [b, c]\}$.

Theorem 3.1   *i. There exists a P_4^1-decomposition of $M(C_n)$ if and only if $n \geq 4$*

   and $n \equiv 0 \pmod 2$.

ii. There exists a P_4^2-decomposition of $M(C_n)$ if and only if $n \geq 4$.

iii. There exists a P_4^3-decomposition of $M(C_n)$ if and only if $n \geq 4$.

21

iv. *There exists a P_4^4-decomposition of $M(C_n)$ if and only if $n \geq 4$ and $n \equiv 0$ (mod 4).*

Proof. Note $n \geq 4$ is necessary in all cases [4].

i. Take the set of blocks $[2i, 2i + 1, 2i + 2, 2i + 3]_1$ and $[2i, 2i - 1, 2i - 2, 2i - 3]_1$ $i = 0, 1, 2, ..., n - 1$. This will give P_4^1-decomposition for $M(C_n)$, $n \geq 4$ and $n \equiv 0$ (mod 4). Suppose n is odd and there exists a P_4^1-decomposition of C_{2n+1}. Without loss of generality, suppose we start with the block $[0, 1, 2, 3]_1$. We then need the block $[2, 3, 4, 5]_1$ which is forced. Since the arc $(2, 3)$ cannot be in another block, we must use the set of blocks of form $[2i, 2i+1, 2i+2, 2i+3]_1$. However, this gives a contradiction since $i = n$ and as $[0, 1, 2, 3]_1$ and $[2n, 0, 1, 2]_1$ share the arc $(1, 2)$. Thus, there exists a P_4^1-decomposition of $M(C_n)$ if and only if $n \geq 4$ and $n \equiv 0$ (mod 2).

ii. Take the set of blocks $[i, i + 1, i + 2, i + 3]_2$ where $i = 0, 1, 2, ..., n - 1$. This will yield the P_4^2-decomposition of $M(C_n)$, for $n \geq 4$.

iii. Take the set of blocks $[i, i + 1, i + 2, i + 3]_3$ where $i = 0, 1, 2, ..., n - 1$. This will yield the P_4^3-decomposition of $M(C_n)$, for $n \geq 4$.

iv. Without loss of generality take the blocks $[0, 1, 2, 3]_4$ and $[4, 3, 2, 1]_4$. This forces the set of blocks $[5, 4, 3, 2]_4$ and $[3, 4, 5, 6]_4$. These in turn force the set of blocks $[4i, 4i+1, 4i+2, 4i+3]_4$, $[4i+4, 4i+3, 4i+2, 4i+1]_4$, $[4i+5, 4i+4, 4i+3, 4i+2]_4$, and $[4i + 3, 4i + 4, 4i + 5, 4i + 6]_4$. Thus, if $n \equiv 0$ (mod 4) this gives the required P_4^1-decomposition for C_n, where $n = 4t$, $i = 1, ..., t$ and $n \geq 4$. If

22

$n \equiv 1 \pmod 4$, $n = 4t + 1$, we notice that the set of blocks $[0, 1, 2, 3]_4$ and $[n-2, n-1, 0, 1]_4$ share the arc $(0, 1)$. If $n \equiv 2 \pmod 4$ we will have the set of blocks $[0, 1, 2, 3]_4$, $[n-3, n-2, n-1, 0]_4$, and $[n-1, n-2, n-3, n-4]_4$. Now we are missing the directed arcs $(1, 2), (0, n-1), (n-2, n-1)$ and the undirected edges $[0, 1], [n-1, 0]$. This then forces the block $[n-2, n-1, 0, 1]_4$, but the arc $(0, 1)$ is shared. When $n \equiv 3 \pmod 4$, $4i + 3 = n$ and $i = (n-3)/4$ we remove the block $[0, 1, 2, 3]_4$. Now we have the set of blocks $[4, 3, 2, 1]_4$, $[2, 1, 0, n-1]_4$ which share the arc $(2, 1)$. Thus, there exists a P_4^4-decomposition of C_n if and only if $n \equiv 0 \pmod 4$, $n \geq 4$.

∎

Theorem 3.2   *i. There exists a P_4^1-decomposition of $M(W_n)$ if and only if $n \geq 3$.*

ii. There does not exist a P_4^2-decomposition of $M(W_n)$ for any n.

iii. There exists a P_4^3-decomposition of $M(W_n)$ if and only if $n \geq 3$.

iv. There exists a P_4^4-decomposition of $M(W_n)$ if and only if $n \geq 3$.

Proof.   Note that $n \geq 3$ is necessary in all cases [4]. Also note that c denotes the center of the wheel and the vertices of n-cycle are denoted by elements of $\mathbb{Z}_n$.

   i. Take the set of blocks $[i, i+1, c, i+2]_1$ and $[c, i+2, i+1, i]_1$ for $i = 0, 1, 2..., n-1$. This will yield a P_4^1-decomposition of $M(W_n)$.

   ii. Note that P_4^2 has only one vertex with positive out degree and this vertex "b" has out degree two, and the vertices on the cycle of W_n have odd out degree. It

follows that a necessary condition to the existence of a P_4^2-decomposition of H is for every vertex in H to have an even out degree. Hence there does not exist a P_4^2-decomposition of $M(W_n)$.

iii. Take the set of blocks $[i, i+1, i+2, c]_3$ and $[i, c, i+1, i+2]_3$ for $i = 0, 1, 2..., n-1$. This will yield the P_4^3-decomposition of $M(W_n)$ for $n \geq 3$.

iv. Take the set of blocks $[c, i+2, i+1, i]_4$ and $[i, c, i+1, i+2]_4$ for $i = 0, 1, 2..., n-1$. This will yield the P_4^4-decomposition of $M(W_n)$ for $n \geq 3$.

$\blacksquare$

Theorem 3.3   *i. There exists a P_4^1-decomposition of $M(C_n(1, 2))$ if and only if $n \geq 4$.*

ii. There exists a P_4^2-decomposition of $M(C_n(1, 2))$ if and only if $n \geq 5$.

iii. There exists a P_4^3-decomposition of $M(C_n(1, 2))$ if and only if $n \geq 4$.

iv. There exists a P_4^4-decomposition of $M(C_n(1, 2))$ if and only if $n \geq 4$.

Proof. Note that in all cases $C_4(1, 2)$ is isomorphic to W_3. Hence a decomposition of $C_4(1, 2)$ exists in all cases except for P_4^2. Further, for $n \geq 4$ is necessary in all cases [4]. As such, it suffices to give constructions for $n \geq 5$.

i. Take the set of blocks $[i + 2, i + 3, i + 1, i]_1$ and $[i + 1, i + 3, i + 2, i]_1$, $i = 0, 1, 2, ..., n - 1$. This will give the P_4^1-decomposition of $M(C_n(1, 2))$.

ii. Take the set of blocks $[i + 1, i, i + 2, i + 3]_2$ and $[i, i + 2, i + 1, i + 3]_2$, $i = 0, 1, 2, ..., n - 1$. This will give the P_4^2-decomposition of $M(C_n(1, 2))$.

24

iii. Take the set of blocks $[i + 3, i + 1, i + 2, i]_3$ and $[i + 4, i + 3, i + 2, i]_3$, $i = 0, 1, 2, ..., n - 1$. This will give the P_4^3-decomposition of $M(C_n(1, 2))$.

iv. Take the blocks $[i, i+2, i+3, i+1]_4$ and $[i+2, i+3, i+1, i]_4$, $i = 0, 1, 2, ..., n-1$. This will give the P_4^4-decomposition of $M(C_n(1, 2))$.

∎

Theorem 3.4   *i. There exists a P_4^1-decomposition of $M(C_n(1, 2, 3))$ if and only if $n \geq 6$.*

ii. There exists a P_4^2-decomposition of $M(C_n(1, 2, 3))$ if and only if $n \geq 7$.

iii. There exists a P_4^3-decomposition of $M(C_n(1, 2, 3))$ if and only if $n \geq 6$.

iv. There exists a P_4^4-decomposition of $M(C_n(1, 2, 3))$ if and only if $n \geq 6$.

Proof. Note that $n \geq 6$ is necessary in all cases [4].

i. Take the set of blocks $[i + 2, i + 3, i + 1, i]_1$ and $[i + 3, i + 5, i + 2, i]_1$, and $[i + 1, i + 4, i + 3, i]_1$, $i = 0, 1, 2, ..., n - 1$. This will give the P_4^1-decomposition of $M(C_n(1, 2, 3))$.

ii. Note that when $n = 6$ we have $C_6(1, 2, 3)$ is isomorphic to K_6. Note that every vertex in $C_6(1, 2)$ has five out degree arcs, this leaves only one vertex with a positive degree namely "b" which gives a contradiction by an in degree out degree argument from previous work. Hence a necessary condition for all vertices to have an even out degree. In all other cases, take the set of blocks

25

$[i+6, i+3, i+1, i]_2$, $[i+6, i+5, i+2, i]_2$, and $[i+6, i+4, i+3, i]_2$, $i = 0, 1, 2, ..., n-1$ for $n \geq 7$. This will give the P_4^2-decomposition of $M(C_n(1, 2, 3))$.

iii. Take the set of blocks $[i, i+2, i+1, i+3]_3$, $[i, i+3, i+1, i+4]_3$, and $[i+2, i+3, i, i+1]_3$, $i = 0, 1, 2, ..., n-1$. This will give the P_4^3-decomposition of $M(C_n(1, 2, 3))$.

iv. Take the blocks $[i+1, i+3, i+2, i]_4$, $[i+2, i+5, i+3, i]_4$, and $[i+3, i+4, i+1, i]_4$, $i = 0, 1, 2, ..., n-1$. This will give the P_4^4-decomposition of $M(C_n(1, 2, 3))$. ∎

Theorem 3.5 *There exists a P_4^i-decomposition for $M(C_n(1, 2, 3, 4))$ for every $n \geq 9$.*

Proof. Note $P_4^i | M(C_4)$ by Theorem 3.1, further C_4 is graceful [22]. Thus by [3] we have a C_4-decomposition of $C_n(1, 2, 3, 4))$ for every $n \geq 9$. Thus, by Proposition 1.1 there is a P_4^i-decomposition of $M(C_n(1, 2, 3, 4)$ for $n \geq 9$. ∎

Lemma 3.6 *Let $\mathcal{K} = \{W_p : n \geq 5, p = 0, p \geq 3\}$. Then there exists a $\mathcal{K}$-decomposition of $K(3k + 4p + 1, k)$, for all k.*

Proof. Treat k vertices as fixed points. The center of the wheel requires k pairs differences from $\mathbb{Z}_{2k+4p+1}$. $\mathbb{Z}_{2k+4p+1}$ has differences $\{1, 2, ..., 2p, 2p+1, ..., 2p+k\}$. Use differences $2p+1, ..., 2p+k$ to generate wheels. This leaves $C_{2k+4p+1}(1, ..., 2p)$. Note W_p has a graceful labeling [5] and $e(W_p) = 2p$, a graceful graph size q will decompose $C_n(1, ..., q)$, where $n \geq 2q + 1$ [3]. ∎

Theorem 3.7 *There exists P_4^1-decomposition of M_v for every $v \geq 4$.*

Proof. Note that $v \geq 4$ is necessary. Let $v \equiv 0 \pmod 4$. Then $v = 4p + 4$. Use a single vertex as a single fixed point and we have $M(K(4p+3, 1))$. Use $2p+1$ differences to generate the outside cycle of the wheel. We know that $W_p|C_{4p+3}(1, ..., 2p)$ for $p = 0$, $p \geq 3$. Thus, we must decompose the remaining cases $p = 1$ and $p = 2$. For $p = 1$, we have $M(C_7(1, 2))$ which by Theorem 3.3 can be decomposed by P_4^1. When $p = 2$, we have $M(C_{11}(1, 2, 3, 4))$ which by Theorem 3.5 $M(C_{11}(1, 2, 3, 4))$ can be decomposed by P_4^1.

Next, let $v \equiv 1 \pmod 4$, say $v = 4p + 1$. Note that K_{4p+1} is isomorphic to $C_{4p+1}(1, ..., 2p)$. Since $W_p|C_{4p+1}(1, ..., 2p)$ for $p = 0$ and $p \geq 3$, we are left with $p = 1$ and $p = 2$. When $p = 1$, we have $M(C_5(1, 2))$ which can be decomposed by P_4^1 by Theorem 3.3. For $p = 2$, we have $M(C_9(1, 2, 3, 4))$ has a P_4^1-decomposition by Theorem 3.5.

For $v = 6$, take the blocks $[2j + 2, 2j + 1, 2j, 2j + 3]_1$, $[i + 2, i + 4, i + 1, i]_1$, $[i + 3, i + 1, i + 2, i]_1$, $i = 0, 1, ..., n - 1$ and $j = 0, 1, 2$. This we will give the P_4^1-decomposition for M_6. For $v = 7$, we have $K_7 = C_7(1, 2, 3)$ which by Theorem 3.4 can be decomposed by P_4^1.

Next, let $v \equiv 2 \pmod 4$, $v = 4p + 22$. Treat seven vertices as fixed points. We know $W_p|C_{4p+15}(1, ..., 2p)$, $p = 0$, $p \geq 3$. This leaves the remaining cases $p = 1$ and $p = 2$. When $p = 1$, we have $C_{19}(1, 2)$ which can be decomposed by Theorem 3.3. For $p = 2$, we have $M(C_{23}(1, 2, 3, 4))$ and by Theorem 3.5, $M(C_{23}(1, 2, 3, 4))$ can be decomposed by P_4^1. Now we are left with $v = 6, 10, 14, 18$. By above there is a decomposition for $v = 6$. When $v = 10$, $K_{10} = W_9 \cup C_9(1, 2, 3)$. By Theorem

3.2 and Theorem 3.4, we can decompose $M(W_9)$ and $M(C_9(1, 2, 3))$ respectively. Hence there is a P_4^1-decomposition for M_{10}. When $v = 14$, we have $K_{14} = W_{13} \cup C_{13}(1, 2) \cup C_{13}(3, 4, 5)$. Then take the set of blocks $[i + 4, i + 7, i + 3, i]_1$, $[i + 2, i + 7, i + 4, i]_1$, and $[i + 7, i, i + 6, i + 1]_1$, $i = 0, 1, 2, ..., n - 1$ yields required decomposition of $C_{13}(3, 4, 5)$. By Theorem 3.2 and Theorem 3.3, we can decompose $M(W_{13})$ and $M(C_{13}(1, 2))$. Hence, there is a P_4^1-decomposition of M_{14}. When $v = 18$, we have $K_{18} = W_{17} \cup C_{17}(1, 2, 3, 4) \cup C_{17}(5, 6, 7)$. We know by Theorem 3.2 and Theorem 3.5, we can decompose $M(W_{17})$ and $M(C_{17}(1, 2, 3, 4))$. Taking the set of blocks $[i + 6, i + 11, i + 5, i]_1$, $[i + 7, i + 13, i + 6, i]_1$, and $[i + 5, i + 12, i + 7, i]_1$, $i = 0, 1, 2, ..., n - 1$ which will give the P_4^1-decomposition of $M(C_{17}(5, 6, 7))$.

Finally, let $v \equiv 3 \pmod 4$, say $v = 4p + 19$. Treat six vertices as fixed points which leaves $4p + 13$. We know $W_p | C_{4p+13}(1, ..., 2p)$ when $p = 0$, $p \geq 3$. When $p = 1$, we have $M(C_{23}(1, 2))$ which can be decomposed by Theorem 3.3. When $p = 2$, we have $M(C_{27}(1, 2, 3, 4))$ and by Theorem 3.5, $M(C_{27}(1, 2, 3, 4))$ has a P_4^1-decomposition. We are now left with $p = 7, 11, 15$. From previous work, we know the decomposition exists when $v = 7$. When $v = 11$, we have $K_{11} = C_{11}(1, 2, 3) \cup C_{11}(4, 5)$. The set of blocks $[i + 5, i + 9, i + 4, i]_1$ and $[i + 4, i + 9, i + 5, i]_1$, $i = 0, 1, 2, ..., n - 1$ will yield decomposition of $C_{11}(4, 5)$. By Theorem 3.2 and Theorem 3.4 we can decompose $M(W_{13})$ and $C_{13}(1, 2)$. Hence there exists a P_4^1-decomposition of M_{11}. When $v = 15$, we have $K_{15} = C_{15}(1, 2, 3, 4) \cup C_{15}(5, 6, 7)$. By Theorem 3.5, we can decompose $M(C_{15}(1, 2, 3, 4))$. Take the set of blocks $[i + 6, i + 11, i + 5, i]_1$, $[i + 7, i + 13, i + 6, i]_1$, and $[i + 5, i + 12, i + 7, i]_1$, $i = 0, 1, 2, ..., n - 1$ will yield P_4^1-decomposition of $M(C_{15}(5, 6, 7))$. ∎

Theorem 3.8 *There exists P_4^2-decomposition of M_v if and only if $v \equiv 1 \pmod 2$,*
$v \geq 5$.

Proof.   If $v \equiv 0 \pmod 2$, then the directed part of P_4^2 will not decompose D_v.
If $v \leq 3$, there will not be enough vertices [11]. It is known that $C_n|K_v$ [12]. If
$v = 2n + 1$. Thus, $M(C_n)|M(K_v)$ and by transitivity, $P_4^2|M_v$ when $v \geq 9$, $v \equiv 1$
(mod 2). Thus, the only cases needed to be considered are and $v = 5, 7$. When $v = 5$
we will have $K_5 = C_5 \cup C_5$. When $v = 7$, we will have $K_7 = C_7 \cup C_7 \cup C_7$.   ∎

Theorem 3.9 *There exists a P_4^3-decomposition of M_v for every $v \geq 4$.*

Proof.   Note that $v \geq 4$ is necessary. Let $v \equiv 0 \pmod 4$. Then $v = 4p + 4$. Use a
single vertex as a single fixed point and we have $M(K(4p+3, 1))$. Use the differences
$2p + 1$ to generate the outside cycle of the wheel. We know that $W_p|C_{4p+3}(1, ..., 2p)$
for $p = 0$, $p \geq 3$. Thus, we must decompose the remaining cases $p = 1$ and $p = 2$.
For $p = 1$, we have $M(C_7(1, 2))$ which by Theorem 3.3 can be decomposed by P_4^3.
When $p = 2$, we have $M(C_{11}(1, 2, 3, 4))$ which by Theorem 3.5 $C_{11}(1, 2, 3, 4)$ can be
decomposed by P_4^3.

     Next, let $v \equiv 1 \pmod 4$, say $v = 4p + 1$. Note that K_{4p+1} is isomorphic to
$C_{4p+1}(1, ..., 2p)$. Since $W_p|C_{4p+1}(1, ..., 2p)$ for $p = 0$ and $p \geq 3$, we are left with $p = 1$
and $p = 2$. When $p = 1$, we have $M(C_5(1, 2))$ which can be decomposed by P_4^3
by Theorem 3.3. For $p = 2$, we have $M(C_9(1, 2, 3, 4))$ has a P_4^3-decomposition by
Theorem 3.5.

     For $v = 6$, take the blocks $[2j + 2, 2j + 1, 2j, 2j + 3]_3$, $[i + 2, i + 4, i + 1, i]_3$,
$[i + 3, i + 1, i + 2, i]_3$, $i = 0, 1, ..., n - 1$ and $j = 0, 1, 2$. This we will give the P_4^3-

decomposition for M_6. For $v = 7$, we have $K_7 = C_7(1, 2, 3)$ which by Theorem 3.4 can be decomposed by P_4^3.

Next, let $v \equiv 2 \pmod 4$, $v = 4p + 22$. Treat seven vertices as fixed points. We know $W_p | C_{4p+15}(1, ..., 2p)$, $p = 0$, $p \geq 3$. This leaves the remaining cases $p = 1$ and $p = 2$. When $p = 1$, we have $C_{19}(1, 2)$ which can be decomposed by Theorem 3.3. For $p = 2$, we have $C_{23}(1, 2, 3, 4)$ and by Theorem 3.5, $M(C_{23}(1, 2, 3, 4))$ can be decomposed by P_4^3. Now we are left with $v = 6, 10, 14, 18$. By above there is a decomposition for $v = 6$. When $v = 10$, $K_{10} = W_9 \cup C_9(1, 2, 3)$. By Theorem 3.2 and Theorem 3.4, we can decompose $M(W_9)$ and $M(C_9(1, 2, 3))$ respectively. Hence there is a P_4^3-decomposition for M_{10}. When $v = 14$, we have $K_{14} = W_{13} \cup C_{13}(1, 2) \cup C_{13}(3, 4, 5)$. Taking the set of blocks $[i+3, i+7, i+4, i]_3$, $[i+4, i+9, i+5, i]_3$, and $[i+5, i+8, i+3, i]_3$, $i = 0, 1, 2, ..., n-1$ yields required decomposition of $M(C_{13}(3, 4, 5))$. By Theorem 3.2 and Theorem 3.3, we can decompose $M(W_{13})$ and $M(C_{13}(1, 2))$. Hence we get the P_4^3-decomposition for M_{14}. When $v = 18$, we have $K_{18} = W_{17} \cup C_{17}(1, 2, 3, 4) \cup C_{17}(5, 6, 7)$. We know by Theorem 3.2 and Theorem 3.5, $M(W_{17})$, and $M(C_{17}(1, 2, 3, 4))$ can be decomposed by P_4^3. Take the set of blocks $[i+6, i+11, i+5, i]_3$, $[i+7, i+13, i+6, i]_3$, and $[i+5, i+12, i+7, i]_3$, $i = 0, 1, 2, ..., n-1$ we have the P_4^3-decomposition $M(C_{17}(5, 6, 7))$.

Finally, let $v \equiv 3 \pmod 4$, say $v = 4p + 19$. Treat six vertices as fixed points and leaves $4p + 13$. We know $W_p | C_{4p+13}(1, ..., 2p)$ when $p = 0$, $p \geq 3$. When $p = 1$, we have $M(C_{23}(1, 2))$ which can be decomposed by Theorem 3.3. When $p = 2$, we have $M(C_{27}(1, 2, 3, 4))$ and by Theorem 3.5, $M(C_{27}(1, 2, 3, 4))$ has a P_4^3-decomposition. We are now left with $p = 7, 11, 15$. From previous work, we know the decomposition exists when $v = 7$. When $v = 11$, we have $K_{11} = C_{11}(1, 2, 3) \cup C_{11}(4, 5)$. The set of blocks

$[i, i+5, i+1, i+6]_3$, and $[i+1, i+5, i, i+4]_3$, $i = 0, 1, 2, ..., n-1$ will yield the P_4^3-decomposition for M_{11}. When $v = 15$, we have $K_{15} = C_{15}(1, 2, 3, 4) \cup C_{15}(5, 6, 7)$. By Theorem 3.5, we can decompose $M(C_{15}(1, 2, 3, 4))$ for P_4^3. The set of blocks $[i+6, i+11, i+5, i]_3$, $[i+7, i+13, i+6, i]_3$, and $[i+5, i+12, i+7, i]_3$ will yield the P_4^3-decomposition of $M(C_{15}(5, 6, 7))$. ■

Theorem 3.10 *There exists a P_4^4-decomposition of M_v for every $v \geq 4$.*

Proof. Note that $v \geq 4$ is necessary. Let $v \equiv 0 \pmod 4$. Then $v = 4p + 4$. Use a single vertex as a single fixed point and we have $M(K(4p+3, 1))$. Use $2p+1$ differences to generate the outside cycle of the wheel. We know that $W_p | C_{4p+3}(1, ..., 2p)$ for $p = 0$, $p \geq 3$. Thus, we must decompose the remaining cases $p = 1$ and $p = 2$. For $p = 1$, we have $M(C_7(1, 2))$ which by Theorem 3.3 can be decomposed by P_4^4. When $p = 2$, we have $M(C_{11}(1, 2, 3, 4))$ which by Theorem 3.5 $M(C_{11}(1, 2, 3, 4))$ can be decomposed by P_4^4.

Next, let $v \equiv 1 \pmod 4$, say $v = 4p + 1$. Note that K_{4p+1} is isomorphic to $C_{4p+1}(1, ..., 2p)$. Since $W_p | C_{4p+1}(1, ..., 2p)$ for $p = 0$ and $p \geq 3$, we are left with $p = 1$ and $p = 2$. When $p = 1$, we have $M(C_5(1, 2))$ which can be decomposed by P_4^4 by Theorem 3.3. For $p = 2$, we have $M(C_9(1, 2, 3, 4))$ has a P_4^4-decomposition by Theorem 3.5.

For $v = 6$, take the blocks $[2j+2, 2j+1, 2j, 2j+3]_4$, $[i+2, i+4, i+1, i]_4$, $[i+3, i+1, i+2, i]_4$, $i = 0, 1, ..., n-1$ and $j = 0, 1, 2$. This we will give the P_4^4-decomposition for M_6. For $v = 7$, we have $K_7 = C_7(1, 2, 3)$ which by Theorem 3.4 can be decomposed by P_4^4.

Next, let $v \equiv 2 \pmod 4$, $v = 4p + 22$. Treat seven vertices as fixed points. We know $W_p | C_{4p+15}(1, ..., 2p)$, $p = 0$, $p \geq 3$. This leaves the remaining cases $p = 1$ and $p = 2$. When $p = 1$, we have $C_{19}(1, 2)$ which can be decomposed by Theorem 3.3. For $p = 2$, we have $M(C_{23}(1, 2, 3, 4))$ and by Theorem 3.5, $M(C_{23}(1, 2, 3, 4))$ can be decomposed by P_4^4. Now we are left with $v = 6, 10, 14, 18$. By above there is a decomposition for $v = 6$. When $v = 10$, $K_{10} = W_9 \cup C_9(1, 2, 3)$. By Theorem 3.2 and Theorem 3.4, we can decompose $M(W_9)$ and $M(C_9(1, 2, 3))$ respectively. Hence there is a P_4^4-decomposition for M_{10}. When $v = 14$, we have $K_{14} = W_{13} \cup C_{13}(1, 2) \cup C_{13}(3, 4, 5)$. Taking the set of blocks $[i + 3, i + 7, i + 4, i]_4$, $[i + 4, i + 9, i + 5, i]_4$, and $[i + 5, i + 8, i + 3, i]_4$, $i = 0, 1, 2, ..., n - 1$ yields required decomposition of $C_{13}(3, 4, 5)$. By Theorem 3.2 and Theorem 3.3, we can decompose $M(W_{13})$ and $M(C_{13}(1, 2))$. Hence there is a P_4^4-decomposition of M_{14}. When $v = 18$, we have $K_{18} = W_{17} \cup C_{17}(1, 2, 3, 4) \cup C_{17}(5, 6, 7)$. We know by Theorem 3.2 and Theorem 3.5, how to decompose $M(W_{17})$ and $M(C_{17}(1, 2, 3, 4))$. Take the set of blocks $[i+6, i+11, i+5, i]_4$, $[i + 7, i + 13, i + 6, i]_4$, and $[i + 5, i + 12, i + 7, i]_4$, $i = 0, 1, 2, ..., n - 1$ will yield the P_4^4-decomposition of $M(C_{17}(5, 6, 7))$.

Finally, let $v \equiv 3 \pmod 4$, say $v = 4p + 19$. Treat six vertices as fixed points and leaves $4p + 13$. We know $W_p | C_{4p+13}(1, ..., 2p)$ when $p = 0$, $p \geq 3$. When $p = 1$, we have $M(C_{23}(1, 2))$ which can be decomposed by Theorem 3.3. When $p = 2$, we have $C_{27}(1, 2, 3, 4)$ and by Theorem 3.5, $M(C_{27}(1, 2, 3, 4))$ has a P_4^4-decomposition. We are now left with $p = 7, 11, 15$. From previous work, we know the decomposition exists when $v = 7$. When $v = 11$, we have $K_{11} = C_{11}(1, 2, 3) \cup C_{11}(4, 5)$. The set of blocks $[i, i+5, i+1, i+6]_4$, $[i+1, i+5, i, i+4]_4$, $i = 0, 1, 2, ..., n-1$ yields a P_4^4-decomposition

of $M(C_{11}(4,5))$. When $v = 15$ we can decompose $C_{15}(1,2,3,4)$ by Theorem 3.5. Take the set of blocks $[i+6, i+11, i+5, i]_4$, $[i+7, i+13, i+6, i]_4$, and $[i+5, i+12, i+7, i]_4$, $i = 0, 1, 2, ..., n-1$ will yield P_4^4-decomposition of $M(15(5,6,7))$. ∎

Theorem 3.11 *There exists a P_4^5-decomposition of M_v for every $v \geq 4$.*

Proof. As P_4^5 is the converse of P_4^1 and by Proposition 1.1 the same proof follows from Theorem 3.7. ∎

Theorem 3.12 *There exists a P_4^6-decomposition of M_v for every $v \geq 4$.*

Proof. As P_4^6 is the converse of P_4^6 and by Proposition 1.1 the same proof follows from Theorem 3.9. ∎

Theorem 3.13 *There exists a P_4^7-decomposition of M_v for every $v \geq 4$.*

Proof. As P_4^7 is the converse of P_4^3 and by Proposition 1.1 the same proof follows from Theorem 3.10. ∎
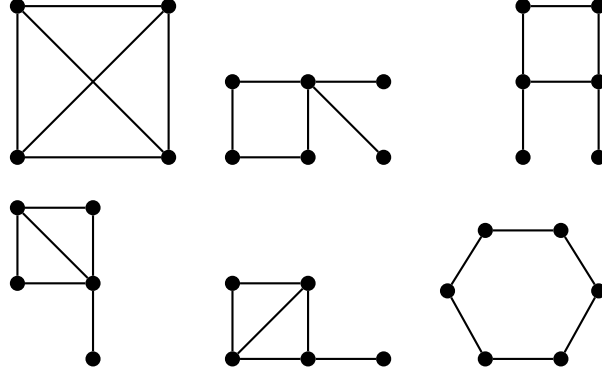
Figure 13: Future Graphs to Consider

Within this paper, we have established the necessary and sufficient conditions for decompositions of M_v and other mixed graphs using partial orientations of P_4. There are still several related problems using the partial orientations of the P_4. One open problem to look at packings and coverings. A *packing* is a set of $\{G_1, ..., G_k\}$-blocks where $G_i \subset V(H)$, $\cup_{i=1}E(G_i) \subset E(H)$, and $E(G_i) \cap E(G_j) = \emptyset$. The *leave* is $L = E(H)\backslash \cup_{i=1} E(G_i)$. A *maximal packing* of simple graph H with isomorphic copies of graph G is a set $\{G_1, G_2, ..., G_n\}$ where $G_i \cong G$ and $V(G_i) \subset V(H)$ for all i, j, $E(G_i) \cap E(G_i) = \emptyset$.

A *minimal covering* of simple graph H with isomorphic copies of graph G is a set $\{G_1, G_2, ..., G_n\}$ where $G_i \cong G$ and $V(G_i) \subset V(H)$, for every i, $H \subseteq \cup_{i=1}^n G_i$ is minimal. The graph P is called the *padding* of the covering.

Another open relation problem is looking at λM_n-designs. A λM_n-design is defined as a partition of the edges of λM_n into subgraphs called G-blocks each of which is isomorphic to G where $G = (V(G), E(G))$. An open problem to consider is taking

the partial orientations of C_6, corona graphs, and the graphs in Figure 13. Also there is hybrid decompositions where there are two or more prototypes G_1 and G_2. Let $K = \{G_1, ..., G_k\}$. There is a K-decomposition of H in there are n_i copies of G_i. Finally, an open problem to consider is looking at partial orientations of the graphs in Figure 13.

BIBLIOGRAPHY

[1] R. E. L. Aldred and Brendan D. McKay. Graceful and harmonious labellings of trees. *Bull. Inst. Combin. Appl.*, 23:69–72, 1998.

[2] George E. Andrews. *Number theory.* Dover Publications Inc., New York, 1994.

[3] Robert A. Beeler and Robert E. Jamison. Valuations, rulers, and cyclic decompositions of graphs. *Congressus Numeratium*, 183:109–127, 2006.

[4] Juraj Bosák. *Decompositions of graphs*, volume 47 of *Mathematics and its Applications (East European Series)*. Kluwer Academic Publishers Group, Dordrecht, 1990. Translated from the Slovak with a preface by Štefan Znám.

[5] Roberto W. Frucht. Nearly graceful labelings of graphs. *Sci. Ser. A Math. Sci. (N.S.)*, 5:47–59 (1995), 1992/93.

[6] Roberto W. Frucht and Joseph A. Gallian. Labeling prisms. *Ars Combin.*, 26:69–82, 1988.

[7] Joseph A. Gallian. A dynamic survey of graph labeling. *Electron. J. Combin.*, 5:Dynamic Survey 6, 2007.

[8] Robert Gardner. Triple systems from mixed graphs. *Bull. Inst. Combin. Appl.*, 27:95–100, 1999.

[9] Solomon W. Golomb. How to number a graph. In *Graph theory and computing*, pages 23–37. Academic Press, New York, 1972.

[10] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. The Clarendon Press Oxford University Press, New York, fifth edition, 1979.

[11] Alan Hartman and Eric Mendelsohn. The last of the triple systems. *Ars Combin.*, 22:25–41, 1986.

[12] Katherine Heinrich. Graph decompositions and designs. In Charles J. Colbourn and Jeffrey H. Dinitz, editors, *The CRC Handbook of Combinatorial Designs*, pages 361–366. CRC Press, Inc., Boca Raton, 1996.

[13] I.N. Herstein. *Topics in algebra*. John Wiley and Sons, Inc., New York, second edition, 1975.

[14] Pavel Hrnčiar and Alfonz Haviar. All trees of diameter five are graceful. *Discrete Math.*, 233(1-3):133–150, 2001.

[15] C. Huang, A. Kotzig, and A. Rosa. Further results on tree labellings. *Utilitas Math.*, 21:31–48, 1982.

[16] De Jun Jin, Fan Hong Meng, and Jin Gong Wang. The gracefulness of trees with diameter 4. *Acta Sci. Natur. Univ. Jilin.*, (1):17–22, 1993.

[17] T.P. Kirkman. On a problem in combinatorics. *Cambridge and Dublin Mathematics*, 2:191–204, 1847.

[18] K. M. Koh and N. Punnim. On graceful graphs: cycles with 3-consecutive chords. *Bull. Malaysian Math. Soc. (2)*, 5(1):49–64, 1982.

[19] Anton Kotzig. Decompositions of complete graphs into isomorphic cubes. *J. Combin. Theory Ser. B*, 31(3):292–296, 1981.

[20] Maryvonne Maheo. Strongly graceful graphs. *Discrete Math.*, 29(1):39–46, 1980.

[21] M. Reiss. Über eine steinersche combinatorsche aufgabe welche. *J. Reine Angew. Math.*, 56:326–344, 1859.

[22] A. Rosa. On certain valuations of the vertices of a graph. In *Theory of Graphs (Internat. Sympos., Rome, 1966)*, pages 349–355. Gordon and Breach, New York, 1967.

[23] W. W. Rouse Ball and H. S. M. Coxeter. *Mathematical recreations & essays*. University of Toronto Press, Toronto, Ont., twelfth edition, 1974.

[24] J. Steiner. Kombinatorische aufgabe. *J. Reine Angew. Math.*, 45:181–182, 1853.

[25] Douglas B. West. *Introduction to graph theory*. Prentice Hall Inc., Upper Saddle River, NJ, second edition, 2001.

[26] Shi Lin Zhao. All trees of diameter four are graceful. In *Graph theory and its applications: East and West (Jinan, 1986)*, volume 576 of *Ann. New York Acad. Sci.*, pages 700–706. New York Acad. Sci., New York, 1989.

VITA

ADAM M. MEADOWS

Education:	B.A. Mathematics with a minor in Music
	University of Virginia's College at Wise
	Wise, Virginia 2006
	M.S. Mathematical Sciences,
	East Tennessee State University
	Johnson City, Tennessee 2009
Professional Experience:	Tuition Scholar, ETSU in Johnson City,
	Johnson City, Tennessee, 2007–2007
	Graduate Assistant, Disability services, ETSU
	Johnson City, Tennessee, 2007–2008
	Graduate Assistant, Teaching Associate, ETSU
	Johnson City, Tennessee, 2008–2009
	Adjunct Professor,
	Northeast State Community College
	Blountville, Tennessee, 2008–2009
Publications:	R. Beeler and A. Meadows,
	"Decompositions of Mixed Graphs
	Using Partial Orientations of $P_3 \cup K_2$
	and $K_2 \cup K_2 \cup K_2$,"
	Proceedings of the 40th CTGC Conference,
	Pending (2009)