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Uphill & Downhill Domination in Graphs and Related Graph Parameters.

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Uphill & Downhill Domination in Graphs

and Related Graph Parameters

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by

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ABSTRACT

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Placing degree constraints on the vertices of a path allows the definitions of uphill and downhill paths. Specifically, we say that a path $\pi = v_1, v_2, \ldots v_{k+1}$ is a *downhill path* if for every i, $1 \leq i \leq k$, $\deg(v_i) \geq \deg(v_{i+1})$. Conversely, a path $\pi = u_1, u_2, \ldots u_{k+1}$ is an *uphill path* if for every i, $1 \leq i \leq k$, $\deg(u_i) \leq \deg(u_{i+1})$. We investigate graphical parameters related to downhill and uphill paths in graphs. For example, a downhill path set is a set $\mathcal{P}$ of vertex disjoint downhill paths such that every vertex $v \in V$ belongs to at least one path in $\mathcal{P}$, and the downhill path number is the minimum cardinality of a downhill path set of G. For another example, the downhill domination number of a graph G is defined to be the minimum cardinality of a set S of vertices such that every vertex in V lies on a downhill path from some vertex in S. The uphill domination number is defined as expected. We determine relationships among these invariants and other graphical parameters related to downhill and uphill paths. We also give a polynomial time algorithm to find a minimum downhill dominating set and a minimum uphill dominating set for any graph.

DEDICATION

I wish to dedicate this thesis to dad, who has helped me more than he could ever know, my sister, who has always allowed me to be her role model, and William, who has made my life brighter by merely being a part of it.

ACKNOWLEDGMENTS

I would like to acknowledge my advisors over the last few years, namely Dr. Bob Gardner, Dr. Teresa Haynes, and Dr. Anant Godbole, who have all helped shape me into what I am today, and who will continue to shape my future. Thanks, as well, to the leadership of the Honors College, who let me experience the awesomeness that is the Honors in Discipline program. Additionally, I would like to pay acknowledgments to the Mathematics & Statistics and Physics families at ETSU, who have made my four baccalaureate years a truly fantastic experience.

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Last but not least, I have had some wonderful friends over my relatively few years, but none compare to those I have obtained over the last four years. Thanks for the laughs and the good times, guys. Here's to many more "Matt's birthdays".

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1 Background

The main objective of this thesis is to investigate the parameters related to the downhill domination number of a graph. The understanding of fundamentals of graph theory is essential to the study of the downhill domination number and related topics.

1.1 General Graphical Definitions

We start with the definition of a graph and related terms. We generally follow the terminology of [5]. A graph $G = (V, E)$ consists of a nonempty set $V(G)$, or simply V, and a set $E(G)$, or simply E, of unordered pairs $\{x, y\}$ for x,$y \in V$. We say that each element of V is a *vertex* and that each element of E is an *edge*. Any two vertices who share an edge are considered to be *adjacent* to one another. The number of vertices, or the cardinality of V, is called the *order* of G, denoted $|V|$. Likewise, $|E|$ is called the *size* of G.

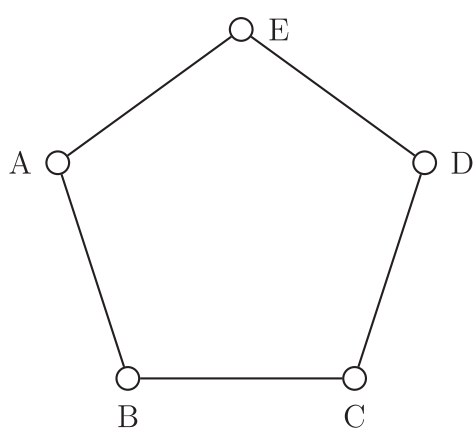

Figure 1: The cycle graph, C_5.

In the example in Figure 1, the cycle graph C_5 has vertex set $V = \{A, B, C, D, E\}$, so the order of C_5 is 5. The edge set can be given as $E = \{AB, BC, CD, DE, EA\}$, so the size of C_5 is 5.

The *open neighborhood*, $N(v)$, of a vertex v is the set of all vertices adjacent to v, and the *closed neighborhood*, $N[v]$, of a vertex v is the open neighborhood of v taken together with v. It follows that $N(v) = \{u \in V | uv \in E\}$ and $N[v] = N(v) \cup v$. Again, in Figure 1, $N(A) = \{B, E\}$ and $N[A] = \{A, B, E\}$. One may also take the neighborhood of a set of vertices; for a set $S \subseteq V$, the *open neighborhood* $N(S) = \bigcup_{v \in S} N(v)$, and the *closed neighborhood* $N[S] = N(S) \cup S$. The number of vertices adjacent to a vertex v is the *degree* of v. The *minimum* and *maximum* degrees of vertices across $V(G)$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. In reference to Figure 1, $deg(A) = 2$, $\delta(G) = 2$, and $\Delta(G) = 2$. A graph for which every vertex has degree k is called *k-regular*. The graph in Figure 1 is 2-regular; the complete graph on n vertices, K_n, which is formed by taking all possible edges among n vertices, is also regular.

A *u-v path* P in a graph G is a sequence of vertices in G, starting with u and ending at v, such that consecutive vertices in P are adjacent, and no vertex is repeated.

If we let G be the graph in Figure 2, one path in G is $P = (A, B, D, E, F)$. A *cycle* C in a graph G is a sequence of vertices in G which starts at a vertex u and ends at vertex u while allowing no other vertex repetition; in other words, a cycle is a "closed path". An example of a cycle in Figure 2 is the cycle $C = (A, B, D, E, C, A)$.

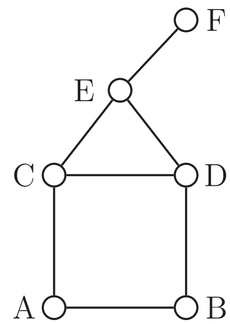

Figure 2: The "House" graph

Two vertices u and v in a graph G are *connected* if G contains a u-v path. If every pair of vertices in G are connected, then G is said to be a *connected graph.* A *tree* is a connected graph which contains no cycles. Four examples of tree graphs can be seen in Figure 3. A *path* graph is a tree graph which has only two leaves (vertices of degree 1). A path on n vertices is denoted P_n.

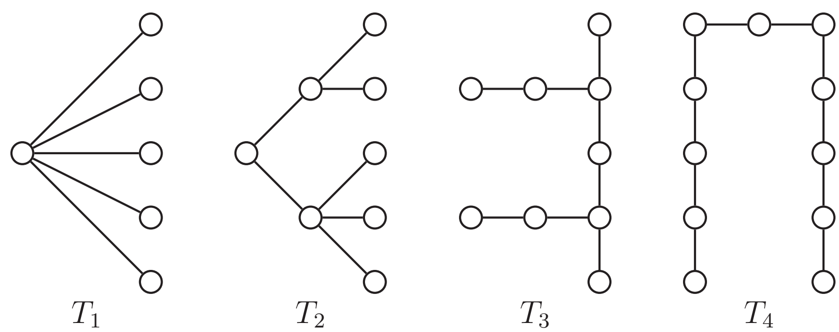

Figure 3: Examples of Trees

A set of vertices $S \subseteq V$ is said to be an *independent* set of vertices if no two vertices in S are adjacent. The maximum number of vertices in an independent set of vertices of G is the *vertex independence number*, or the *independence number*, of

G, denoted $\beta_0(G)$. The set $I = \{A, D, F\}$ forms an independent set of vertices in the House graph of Figure 2; I is also an independent set of vertices that is as large as possible for the House graph, so $\beta_0(House) = 3$. Another example of independent sets in graphs can be seen in the *complete bipartite* graph, denoted $K_{r,s}$, a graph formed by taking all possible edges between an independent set of size r and an independent set of size s, where $r < s$.

It is worth noting that for independent sets, a maximal set is a set for which no additional vertex in the graph may be added to the set and preserve the independence property of the set; a maximum independent set is a set of maximum possible cardinality, that is, a set having maximum cardinality among all independent sets of G.

1.2 Graph Operations

Just as binary operations are defined for numbers, binary operations can also be defined for sets and graphs. Let A and B be two sets. The *Cartesian product* of A and B, denoted $A \times B$, is the set of all ordered pairs (a, b), such that $a \in A$ and $b \in B$, or, more succinctly, $A \times B = \{(a, b) | a \in A, b \in B\}$. The Cartesian product G of two graphs G_1 and G_2, denoted $G_1 \Box G_2$, has vertex set $V(G) = V(G_1) \times V(G_2)$, and two distinct vertices (a, b) and (c, d) of $G_1 \Box G_2$ are adjacent if either:

1. $a = c$ and $bd \in E(G_2)$, or

2. $b = d$ and $ac \in E(G_1)$.

The easiest way to think of or draw a graph Cartesian product is to imagine placing a copy of G_2 at each vertex of G_1 and joining appropriate corresponding vertices. An example of a Cartesian product can be seen in Figure 4.

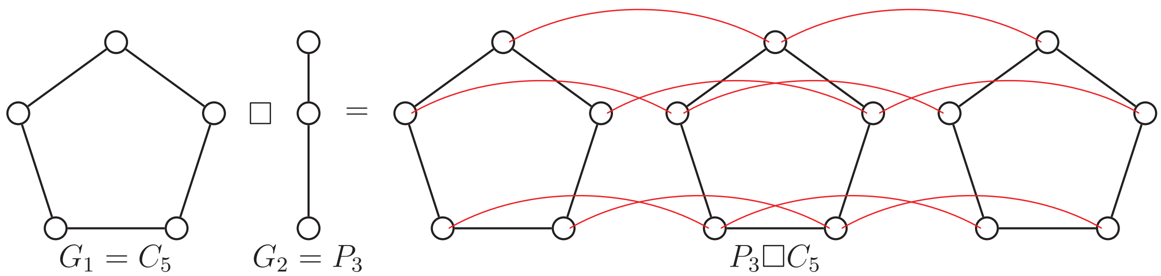

Figure 4: An Example of a Cartesian Product Between Two Graphs

The *union* $G = G_1 + G_2$ of two graphs G_1 and G_2 has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$. The union of k disjoint copies of a graph G is denoted kG. Figure 5 is an example of a graph union.

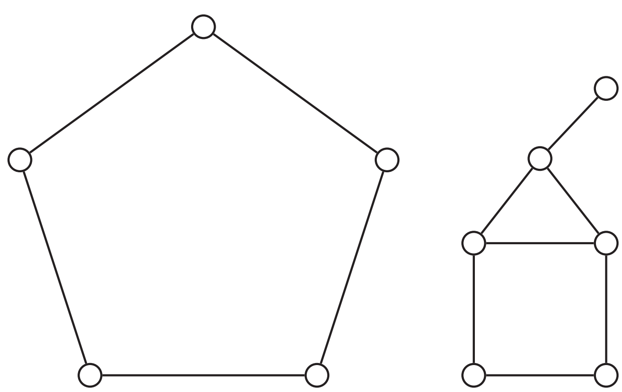

Figure 5: The union $C_5 + House$

The *join* $G = G_1 \vee G_2$ of two graphs G_1 and G_2 has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$. The graph in Figure 6 is an example of a join of two graphs.

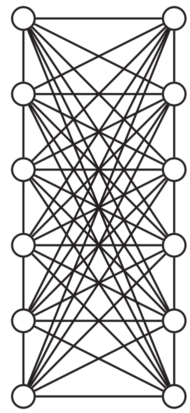

Figure 6: The join $P_6 \vee P_6$

One useful way of obtaining a new graph from an existing graph is by taking the *complement* of the graph. The *complement* of a graph G, denoted $\overline{G}$, is the graph with vertex set $V(G)$ such that two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. One interesting note is that some graphs are self complementary. The graph in Figure 1 is one such graph.

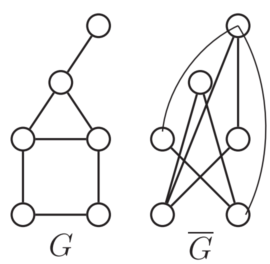

Figure 7: The House Graph and its Complement

1.3 Domination in Graphs

As defined in [5], a set $S \subseteq V(G)$ is called a *dominating set* of G if every vertex $v \in V(G)$ is either an element of S or is adjacent to an element of S, that is, if $N[S] = V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. Similarly, we can define an *independent dominating set* of G to be a subset of V which is a dominating set of G and an independent set. The *independent domination number* $i(G)$ is the minimum cardinality of an independent dominating set of G. Just as with an independent set of vertices, it is worth noting that we make a distinction between a *minimal* dominating set and a *minimum* dominating set. A minimal dominating set is a dominating set such that the removal of any vertex from the set violates the dominating property of the set. A minimum dominating set of G is a dominating set of smallest possible cardinality among all dominating sets of G. Additionally, any minimum dominating set is a minimal dominating set but the reverse is not necessarily true. Further we may refer to a minimum dominating (respectively, independent dominating) set as a γ-set (i-set), since $\gamma(G)$ ($i(G)$) is the cardinality of such a set. For more details on domination, see [5].

1.4 Digraphs

We may also define what is called a *digraph*, which consists of a finite, nonempty set of vertices together with a set of ordered pairs of distinct vertices, called *arcs* or *directed edges*. Just as with graphs, the vertex set of a digraph D is denoted $V(D)$

and the arc set of D is denoted $E(D)$. Figure 8 is an example of a digraph; notice that arcs in a digraph are associated with a direction from one vertex to another.

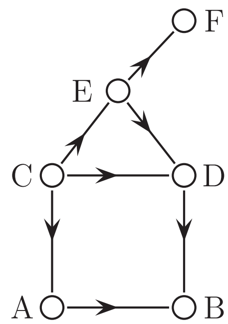

Figure 8: The Directed House Graph

Most of the terminology for graphs also holds for digraphs, with some small differences. If an arc $a = (u, v)$ is an arc of D, then we say that vertex u is *adjacent to* vertex v, while vertex v is *adjacent from* vertex u. For any vertex v in $V(D)$, the *out-degree* of v, or $od(v)$, is the number of vertices adjacent to v. Likewise, the *in-degree* of v, or $id(v)$, is the number of vertices adjacent from v. Finally, the *degree* of a vertex v in D is given by $deg(v) = od(v) + id(v)$. In Figure 8, for example, vertex D has in-degree 2, out-degree 1, and degree 3.

1.5 Downhill Domination, Uphill Domination, Covers and Path Sets

A path $\pi = v_1, v_2, \ldots v_{k+1}$ in a graph $G = (V, E)$ is a *downhill path* if for every i, $1 \leq i \leq k$, $deg(v_i) \geq deg(v_{i+1})$. We may also define uphill paths similarly, namely, a path $\pi = u_1, u_2, \ldots u_{k+1}$ in a graph $G = (V, E)$ is an *uphill path* if for every i,

$1 \leq i \leq k$, $deg(v_i) \leq deg(u_{i+1})$. We should observe that although the definition of a downhill (uphill) path is given in terms of the degrees of the vertices on the path, a similar definition can be given in terms of any function that assigns weights to the vertices of a graph, as is done in surveying when assigning elevations to the points of a topographic map, or in thermal imaging, in which the values assigned to the points in an image are a measure of their heat content.

A *downhill dominating set*, abbreviated DDS, is a set $S \subseteq V$ having the property that every vertex $v \in V$ lies on a downhill path originating from some vertex in S. The *downhill domination number* $\gamma_{dn}(G)$ equals the minimum cardinality of a DDS of G. A downhill dominating set S having minimum cardinality is called a $\gamma_{dn}(G)$-set.

A *downhill cover* of a graph $G = (V, E)$ is a set $\mathcal{P}$ of downhill paths such that every vertex $v \in V$ belongs to at least one path in $\mathcal{P}$. The *downhill cover number* $\alpha_{dn}(G)$ equals the minimum cardinality of a downhill cover of G. A downhill cover $\mathcal{P}$ having minimum cardinality is called an α_{dn}-set.

A *downhill path set* is a set $\mathcal{Q}$ of *vertex disjoint* downhill paths such that every vertex $v \in V$ belongs to at least one path in $\mathcal{Q}$. The *downhill path number* $\rho_{dn}(G)$ equals the minimum cardinality of a downhill path set of G. A downhill path set $\mathcal{Q}$ having minimum cardinality is called a ρ_{dn}-set. This parameter mirrors the well studied *path number* of a graph G, denoted $\rho(G)$, which equals the minimum number of vertex disjoint paths in a graph containing every vertex $v \in V$.

The analogous uphill versions of these parameters could be defined as expected, but we note that $\alpha_{dn}(G) = \alpha_{up}(G)$ and $\rho_{dn}(G) = \rho_{up}(G)$, since the reverse of every downhill path is an uphill path. On the other hand, we define the uphill domination number as it can be quite different than $\gamma_{dn}(G)$. An *uphill dominating set*, abbreviated UDS, is a set $S \subseteq V$ having the property that every vertex $v \in V$ lies on an uphill path originating from some vertex in S. The *uphill domination number* $\gamma_{up}(G)$ equals the minimum cardinality of a UDS of G. An uphill dominating set S having minimum cardinality is called a γ_{up}*-set*.

We can show that $\gamma(G)$ and $\gamma_{dn}(G)$ are incomparable in general. For instance, let G be the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ for $n \geq 6$. If n is even, then $\gamma(G) = 2 > 1 = \gamma_{dn}(G)$. On the other hand, for odd n, $\gamma(G) = 2 < \lfloor \frac{n}{2} \rfloor = \gamma_{dn}(G)$. Additionally, $i(G)$ and $\gamma_{dn}(G)$ are also incomparable. For instance, form a graph H by taking the complete bipartite graph $K_{r,s}$, for $r \geq 3$ and $s \geq r + 2$, and deleting any arbitrary edge. In this case, $i(H) = 2 < r = \gamma_{dn}(H)$. On the other hand, the graph H given in Figure 9 has $\gamma_{dn}(H) = 1 < 6 = i(H)$. For more details on domination, see [5].

We note that all four parameters, namely, $\gamma_{dn}(G)$, $\gamma_{up}(G)$, $\alpha_{dn}(G)$, and $\rho_{dn}(G)$ can be different for a graph. The graph in Figure 9 is one graph for which this is the case. For this graph G, $\{d\}$ is a γ_{dn}-set, $\{t, u\}$ is a γ_{up}-set, $\{(d, a, f, c, i, j, o, t),$ $(d, b, g, e, k, j, o, u), (n, s, r, m, h, i, j, k, l, q, w, v, p)\}$ is an α_{dn}-set, and $\{(d, f, a, c, i, h, m, r, s, n),$ $(g, b, e, k, l, q, w, v, p), (j, o, t), (u)\}$ is a ρ_{dn}-set. Thus, $\gamma_{dn}(G) = 1 < \gamma_{up}(G) = 2 < \alpha_{dn}(G) = 3 < \rho_{dn}(G) = 4$.

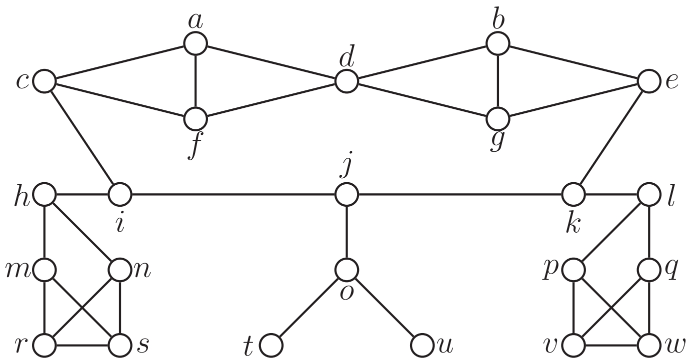

Figure 9: A Graph for Which All Four Major Parameters are Different

We define the *downhill run number*, $run(G)$, to equal the maximum length of a downhill path in G. The *elevation* $el(v)$ of a vertex v is the maximum of the number of inequalities $deg(v_i) > deg(v_{i+1})$ (called *falls*) minus the number of inequalities $deg(v_i) < deg(v_{i+1})$ (called *rises*) in a path from vertex v, and the *elevation* $el(G)$ of a graph G equals $el(G) = max\{el(v) : v \in V\}$.

Notice that if G is a k-regular graph, then every path in G is, by definition, both a downhill and an uphill path. In such cases, we speak of a *plain*. A *plain at elevation* k in a graph G is a maximal set of vertices $S \subset V$ such that (i) the subgraph $G[S]$ induced by S is connected, (ii) for every vertex $v \in S$, $el(v) = k$, and (iii) $|S| \geq 2$. A *plateau* is a plain in which no vertex has a neighbor of higher elevation, and at least one vertex has a neighbor of lower elevation. A *valley*, by contrast, is a plain in which no vertex has a neighbor of lower elevation, and at least one vertex has a neighbor of higher elevation.

A *peak* in a graph G is a vertex v whose elevation is greater than the elevation of every neighbor of v. Equivalently, a peak is a vertex v for which $deg(v) > deg(u)$ for every $u \in N(v)$. Such vertices have been called *very strong* in two papers by Hedetniemi, Hedetniemi, Hedetniemi, and Lewis [6, 7]. A graph G is said to have a *global peak* if it contains a peak vertex v such that every vertex in $V \setminus \{v\}$ lies on a downhill path from v. It should be noted that not all graphs have peaks. Regular graphs and path graphs are two such examples. Similarly, a *lowpoint* in a graph G is a vertex v whose degree is less than or equal to the degree of every neighbor of v. These are called *very weak* vertices in [6, 7].

2 Preliminary Observations

First, note that the notions of elevation and degree give the following observations concerning the elevations and degrees of vertices in a subset of V.

Observation 1. *In any graph G, all vertices in the same plain, plateau or valley have the same degree, and every pair of adjacent vertices having the same degree belong to a common plain, plateau or valley.*

Observation 2. *In any graph G, two adjacent vertices have the same elevation if and only if they have the same degree.*

Observation 3. *For any connected graph G, $el(G) = 0$ if and only if G is a regular graph.*

Observation 4. *For every connected graph G of order n,*

1. $el(G) = 0$ *if and only if G is a regular graph.*

2. $el(G) \leq run(G) \leq detour(G) \leq n - 1$.

3. $el(G) \leq \Delta(G) - 1$.

Peak numbers of graphs are also related to the downhill cover and downhill domination numbers in the following ways.

Observation 5. *For any connected graph G, $\alpha_{dn}(G) \geq |\{v \in V : v \text{ is a peak}\}|$.*

Observation 6. *If a connected graph G has a global peak, then $\gamma_{dn}(G) = 1$.*

Additionally, we can make the following preliminary observations about the downhill domination and uphill domination numbers of some basic families of graphs.

Proposition 7. *The following characterizations for the listed graph families hold for the downhill (uphill) domination number.*

- *For any connected k-regular graph G, $\gamma_{dn}(G) = \gamma_{up}(G) = 1$.*
- *For any m-by-n grid graph G, $\gamma_{dn}(G) = 1$, and $\gamma_{up}(G) = 4$.*
- *For any complete k partite graph $K_{n_1,\ldots,n_k}$ where for any j, $1 \leq j \leq k$ $n_j \leq n_{j+1}$, and $n_1 \neq n_k$, $\gamma_{dn}(K_{n_1,\ldots,n_k}) = n_1$ and $\gamma_{up}(K_{n_1,\ldots,n_k}) = n_k$. If $n_1 = n_k$ then $\gamma_{dn}(K_{n_1,\ldots,n_k}) = \gamma_{up}(K_{n_1,\ldots,n_k}) = 1$.*
- *For any path P_n, $\gamma_{dn}(P_n) = 1$ and $\gamma_{up}(P_n) = 2$.*

As previously mentioned, if G is a k-regular graph, then every path in G is, by definition, both a downhill and an uphill path. Hence, since graphs such as complete graphs and cycles are regular graphs, their downhill and uphill domination numbers are 1. This fact will become more useful later on for constructions and eliminating possibilities.

Additionally, the following proposition can be very useful in proving things about a DDS and a UDS.

Proposition 8. *For a graph G, a DDS must contain at least one vertex with degree $\Delta(G)$ and a UDS must contain at least one vertex with degree $\delta(G)$.*

As a simple bound on the degree of a vertex in any DDS, the following can be useful, as well.

Proposition 9. *For every vertex $v \in DDS$ of a connected graph, $deg(v) \geq 2$.*

Note that any tree graph has at least two leaves. The following relates the uphill domination number to the number of leaves in a graph.

Proposition 10. *For any graph G with order greater than or equal to 3, let S be the set of all leaf vertices in G. Then $\gamma_{up}(G) \geq |S|$.*

3 Results

This section details our main results pertaining to the new graph theoretical parameters.

3.1 Minimum Downhill and Uphill Dominating Sets

We now show that any minimal DDS (respectively, UDS) of a graph is a minimum DDS.

Theorem 11. *Every minimal DDS of a graph G is a minimum DDS of G.*

Proof. Suppose to the contrary that there exists a minimal DDS, say D, of G, such that $|D| > \gamma_{dn}(G)$. Among all γ_{dn}-sets of G, select D' to be one that has the maximum number of vertices in common with D, that is, $|D' \cap D|$ is maximized.

Since $|D'| < |D|$, there exists a vertex $u \in (D \setminus D')$. Thus u is downhill dominated by a vertex, say d' in D'. Then u and all the vertices downhill from u are downhill dominated by d'. If $d' \in D$, then $D \setminus \{u\}$ is a DDS with cardinality less than $|D|$, contradicting the minimality of D. Hence we may assume that $d' \notin D$.

Thus there exists a vertex $v \in D$ that downhill dominates d' and all of the vertices downhill from d'. Suppose $u \neq v$, then v downhill dominates u and so, again, $D \setminus \{u\}$ is a DDS, contradicting the minimality of D. If $u = v$, then since v downhill dominates d' and d' downhill dominates u, it follows that $\deg(u) = \deg(d')$. Moreover, u downhill dominates d' and the vertices downhill dominated by d'. Thus, $D'' = (D' \setminus \{d'\}) \cup \{u\}$

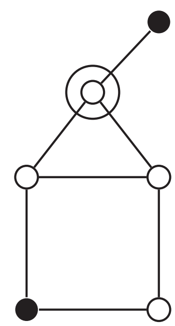

Figure 10: A graph G with $\gamma_{dn}(G) = 1 < 2 = \gamma_{up}(G)$.

is a γ_{dn}-set of G such that $|D'' \cap D| > |D' \cap D|$, contradicting our choice of D'.

□

An analogous argument shows that any minimal UDS of a graph G is a γ_{up}-set of G.

Theorem 12. *Every minimal UDS of a graph G is a minimum UDS of G.*

3.2 Incomparability of the Downhill and Uphill Domination Numbers

The parameters $\gamma_{dn}(G)$ and $\gamma_{up}(G)$ are incomparable. To see this, we note that for regular graphs $\gamma_{dn}(G) = 1 = \gamma_{up}(G)$, the House graph G in Figure 10 has $\gamma_{dn}(G) < \gamma_{up}(G)$, and the graph H in Figure 11 has $\gamma_{dn}(H) > \gamma_{up}(H)$. In these figures, the darkened vertices form a γ_{up}-set and the circled vertices form a γ_{dn}-set.

Theorem 13. *For all $u, d \in \mathbb{N}$, there exists a graph G, such that $\gamma_{up}(G) = u$ and $\gamma_{dn}(G) = d$.*

Proof. Let $u, d \in \mathbb{N}$. We prove a series of cases.

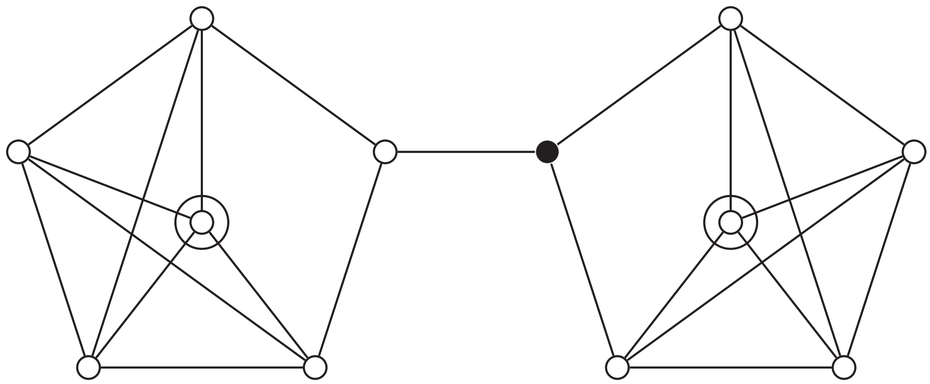

Figure 11: A graph H with $\gamma_{up}(H) < \gamma_{dn}(H)$.

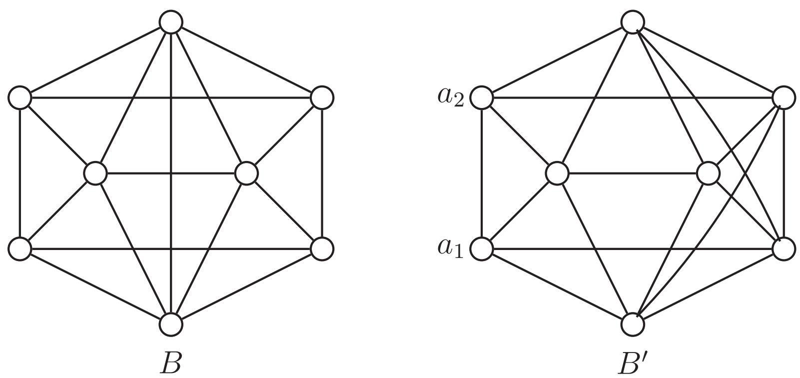

Figure 12: B and B'

Case 1. $d = 1$ *and* $u = 1$

Any connected regular graph suffices.

Case 2. $d = 1$ *and* $u \geq 2$

The star $K_{1,u}$ suffices.

Case 3. $d = 2$ *and* $u = 1$

The graph in Figure 11 suffices. Darkened vertices represent a γ_{up}-set, and circled vertices represent a γ_{dn}-set.

Case 4. *$d \geq 3$ and $u = 1$*

Form the graph G by taking the cycle C_d with vertex set $\{v_1, v_2, \ldots, v_d\}$ and d copies of B' as seen in Figure 12. For the j^{th} copy of B' join the vertices labeled a_1 and a_2 to vertex v_j for $1 \leq j \leq d$. This means that each of the vertices of the B' blocks have degree 5, while the vertices of the C_d block have degree 4. Note that $\{v_i\}$ is a γ_{up}-set for any v_i in C_d, so $\gamma_{up}(G) = 1$. Also, a set S formed by taking one vertex from each of the d B' blocks is a DDS of G, so $\gamma_{dn}(G) \leq |S| = d$. To see that $\gamma_{dn}(G) \geq d$, we note that at least one vertex from each B' block must be in any DDS. Hence, $\gamma_{dn}(G) = d$.

Case 5. *$d \geq 3$ and $u = 2$*

Form the graph G by taking cycles C_d with vertex sets $\{v_1^1, v_2^1, \ldots, v_d^1\}$ and $\{v_1^2, v_2^2, \ldots, v_d^2\}$, respectively. Again take d copies of B', and now for the j^{th} copy of B' join vertex a_1 to vertex v_j^1 by an edge and vertex a_2 to vertex v_j^2 by an edge. In this case, every vertex on one of the original cycles has degree 3, while every vertex in one of the original B' blocks has degree 5. Note that $\{v_i^1, v_j^2\}$, for $1 \leq i \leq d$ and $1 \leq j \leq d$, is a γ_{up}-set, so $\gamma_{up}(G) \leq 2$. To see that one vertex cannot uphill dominate G, note that no uphill path between the two cycles exists. Hence, $\gamma_{up}(G) = 2$. Also, a set S formed by taking one vertex from each of the d B' blocks is a DDS of G, so $\gamma_{dn}(G) \leq |S| = d$.

To see that $\gamma_{dn}(G) \geq d$, we note that at least one vertex from each B' block must be in any DDS. Hence, $\gamma_{dn}(G) = d$.

Case 6. *$d \geq 3$ and $u \geq 3$*

We will give a construction for a graph $G_{u,d}$ with $\gamma_{up}(G_{u,d}) = u$ and $\gamma_{dn}(G_{u,d}) = d$. First note that the graph B in Figure 12 has four vertices of degree 4 and four vertices of degree 5. We construct the graph $G_{u,d}$ as follows:

Begin with the Cartesian product $C_u \Box C_d$ of the two cycles C_u and C_d. For each of the d copies of C_u do the following:

1. For each edge in the C_u subdivide the edge exactly twice creating vertices x and y, and then adding a copy of the graph B from Figure 12 identifying vertex x with one vertex of degree 4 in B and vertex y with another vertex of degree 4 in B. Notice that this copy of B will now have two vertices of degree 4 and six vertices of degree 5.

2. Next add u new edges to form a cycle of length $2u$ to the remaining vertices of the B subgraphs associated with the particular C_u of degree 4. Notice now that these copies of B will have all vertices of degree 5 and will form a regular subgraph.

Notice now that each vertex from the original $C_u \Box C_d$ will be of degree 4.

Now we establish γ_{up}. Note that the set S_u formed by selecting one vertex from each copy of C_d in the original $C_u \Box C_d$ will form a UDS of $G_{u,d}$. Thus, $\gamma_{up}(G_{u,d}) \leq$

$|S_u| = u$. Then for some $x \in S_u$, the set $S \backslash \{x\}$ does not dominate the copy of C_d containing x. Thus S_u is a minimal UDS of G, so by Theorem 11 S_u is a γ_{up}-set of $G_{u,d}$. Hence, $\gamma_{up}(G_{u,d}) = u$.

Finally we establish γ_{dn}. For each of the d copies of C_u in $G_{u,d}$, select one of the created subdivision vertices to form a set S_d. Notice that S_d is a DDS of $G_{u,d}$, implying that $\gamma_{dn}(G_{u,d}) \leq |S_d| = d$. Now for some $x \in S_d$, the set $S \backslash \{x\}$ does not downhill dominate the copy of the B graph identified with x, thus S_d is a minimal DDS of $G_{u,d}$. Therefore by Theorem 11 S_d is a γ_{dn}-set of $G_{u,d}$. Hence, $\gamma_{dn}(G_{u,d}) = d$. □

To illustrate Case 6 the graph G in Figure 13 is a graph for which $\gamma_{dn}(G) = 4$ and $\gamma_{up}(G) = 3$. Note the 12 individual B blocks in G and the paths between blocks.

3.3 The Downhill (Uphill) Domination and Independence Numbers of a Graph

Recall that the *independence number* of G, denoted $\beta_0(G)$, is the maximum number of vertices in an independent set of vertices of G. In order to provide a bound, we first prove a useful lemma.

Theorem 14. *Any minimal downhill (respectively, uphill) dominating set is an independent set.*

Proof. Assume S is a DDS of G. If two vertices u and v of S are adjacent, then without loss of generality there exists a downhill path from u through v to all vertices which are downhill from v. Therefore the removal of v from S creates a minimal DDS

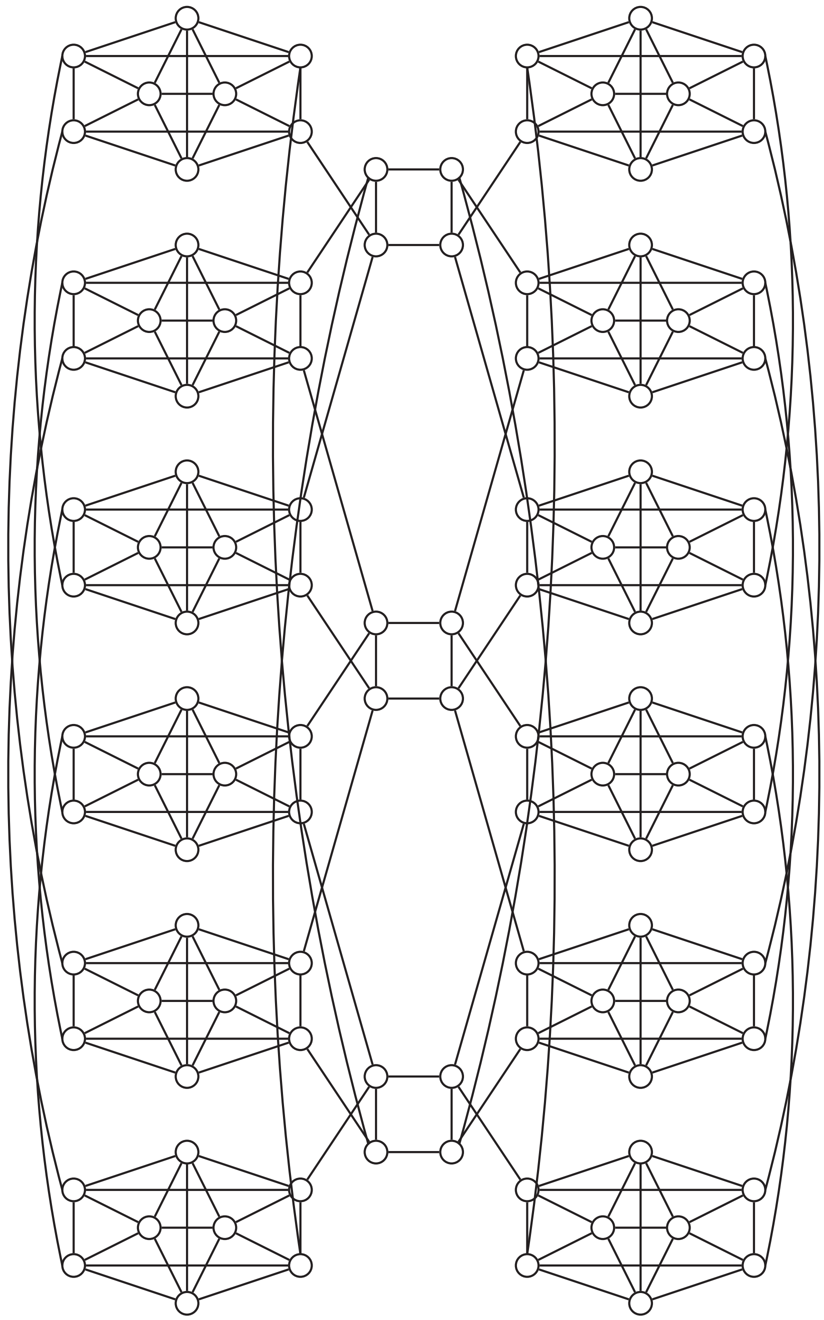

Figure 13: A Graph Constructed Using Case 6 of the Proof of Theorem 13

of G. Hence, a minimal DDS of G forms an independent set. A similar argument using uphill paths yields the result for minimal UDSs.

□

Corollary 15. *For any graph G, $\gamma_{dn}(G) \leq \beta_0(G)$ and $\gamma_{up}(G) \leq \beta_0(G)$.*

To see the sharpness of Corollary 15 consider K_n, for which $\gamma_{dn}(K_n) = \beta_0(K_n) = 1$, and the complete bipartite graph $K_{r,s}$, for $r \neq s$. This provides an example for which $\gamma_{up}(K_{r,s}) = \beta_0(K_{r,s}) = max\{r, s\}$.

Now we present a construction for graphs with $\gamma_{dn}(G) = a$ and $\beta_0(G) = b$ for all values of a and b, where $a \leq b$. This implies that the difference in $\gamma_{dn}(G)$ and $\beta_0(G)$ may be arbitrarily large. Likewise, we can provide an additional construction for graphs with $\gamma_{up}(G) = a$ and $\beta_0(G) = b$ for all values of a and b, where $a \leq b$.

Theorem 16. *Given positive integers a and b such that $a \leq b$,*

1. *there exists a graph G for which $\gamma_d(G) = a$ and $\beta_0(G) = b$, and*

2. *there exists a graph H for which $\gamma_u(H) = a$ and $\beta_0(G) = b$.*

Proof. Let G be the join $\overline{K}_a + bK_a$. (See Figure 14 for an example where $a = 2$ and $b = 4$). Then, $\gamma_d(G) = a$ and $\beta_0(G) = b$.

For $a = 1$, let H be the cycle C_{2b}. For $a > 1$, begin with a cycle $C_{2b-a} = v_1, v_2, ..., v_{2b-a}, v_1$. Then for each v_i, $1 \leq i \leq a-1$, add a vertex v_i' and edge v_iv_i'. (See Figure 15 for an example where $a = 3$ and $b = 4$). The set $\{v_i', v_a \,|\, 1 \leq i \leq a-1\}$ is a γ_d-set of H. Also the set $\{v_i' | 1 \leq i \leq a-1\}$ unioned with $\lceil \frac{2b-2a+1}{2} \rceil$ independent vertices from the path $v_a, v_a+1, ..., v_{2b-a}$ forms a maximum independent set. □

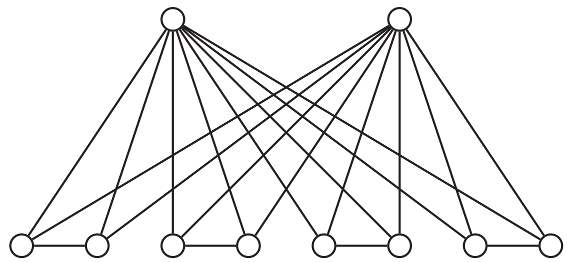

Figure 14: The construction $\overline{K}_2 \vee 4K_2$. Note that $\gamma_{dn}(G) = 2$ and $\beta_0(G) = 4$.

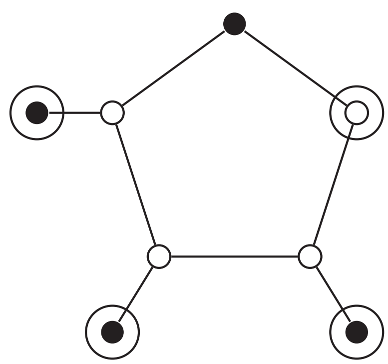

Figure 15: A graph G having $\beta_0(G) = 4$ (circled) and $\gamma_{up}(G) = 4$ (bolded).

3.4 Path Domination Numbers and Graph Operations

Given the constructibility of graphs, it is beneficial to investigate the preservations of graph invariants over graph operations. The downhill domination numbers of unions and Cartesian products of graphs can be found as given by the following results.

Observation 17. *For any two graphs G and H, $\gamma_{dn}(G \cup H) = \gamma_{dn}(G) + \gamma_{dn}(H)$.*

Theorem 18. *For any two graphs G and H, $\gamma_{dn}(G \Box H) = \gamma_{dn}(G)\gamma_{dn}(H)$.*

Proof. We first show that $\gamma_{dn}(G \Box H) \leq \gamma_{dn}(G)\gamma_{dn}(H)$. Let S_1 and S_2 be a DDS of G and a DDS of H, respectively. Let $S = \{(u, v)|u \in S_1 \text{ and } v \in S_2\}$ be a set of vertices in $G \Box H$. In order to show that S is a set of downhill dominating vertices of $G \Box H$, it suffices to show that every element in $V(G \Box H)\backslash S$ is downhill from a vertex in S. Let (x, y) be an arbitrary vertex of $V(G \Box H)\backslash S$. By symmetry, we consider two cases:

Case 1: $x \in S_1$ and $y \notin S_2$. Then x is a downhill dominating vertex in G and y is downhill from some downhill dominating vertex, say $v \in S_2$, in H. Thus $deg_{G \Box H}((x, y)) = deg_G(x) + deg_H(y) \leq deg_G(x) + deg_H(v) = deg_{G \Box H}((x, v))$. Thus (x, y) is downhill from (x, v) in S.

Case 2: $x \notin S_1$ and $y \notin S_2$. Thus x is downhill from some downhill dominating vertex, say $u \in S_1$ and y is downhill from some downhill dominating vertex, say $v \in S_2$. As before, $deg_{G \Box H}((x, y)) = deg_G(x) + deg_H(y) \leq deg_G(u) + deg_H(v) = deg_{G \Box H}((u, v))$. Thus (x, y) is downhill from $(u, v) \in S_1$. It follows that S is a set of peaks of $G \Box H$, so $\gamma_{dn}(G \Box H) \leq |S| = |S_1||S_2|$.

To complete the proof, we show that $\gamma_{dn}(G \Box H) \geq \gamma_{dn}(G)\gamma_{dn}(H)$. Let S be defined as above, and let S^* be a DDS of H. Assume, for the purposes of a contradiction, that $|S^*| < |S| = \gamma_{dn}(G)\gamma_{dn}(H)$. Thus there exists a vertex $(u, v) \in S\backslash S^*$, implying that (u, v) is downhill from some vertex $(x, y) \in S^*$. It follows that u is downhill from x in G or v is downhill from y in H. Thus there exists a DDS of G S_1^* containing x or a DDS of H S_2^* containing y. Since this is true for every $(x, y) \in S^*$ and $|S^*| < |S_1||S_2|$, then either $|S_1^*| < |S_1|$ or $|S_2^*| < |S_2|$, thus providing a DDS of smaller cardinality than either S_1 or S_2, giving a contradiction.

Hence, $\gamma_{dn}(G \Box H) = \gamma_{dn}(G)\gamma_{dn}(H)$.

$\Box$

Once again, we can invoke a similar argument to give the uphill domination number of the Cartesian product $G \Box H$ of two graphs G and H.

Theorem 19. *For any two graphs G and H, $\gamma_{up}(G \Box H) = \gamma_{up}(G)\gamma_{up}(H)$.*

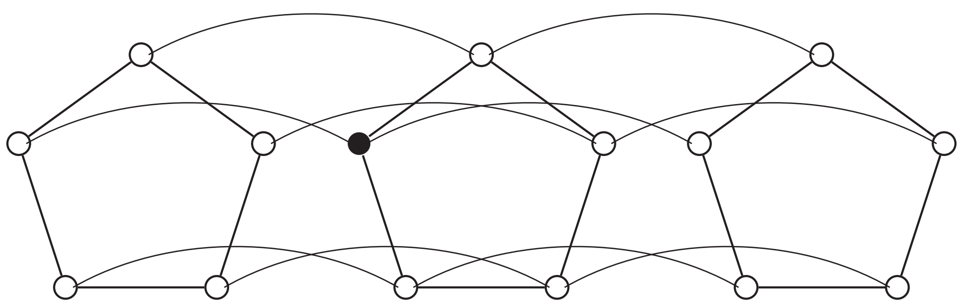

Figure 16: A DDS on $C_5 \times P_3$; $\gamma_{dn}(C_5) = 1$, $\gamma_{dn}(P_3) = 1$, and $\gamma_{dn}(C_5 \Box P_3) = 1$.

3.5 Complements of Graphs

Throughout this thesis, we have investigated the downhill and uphill domination numbers (in particular). Our investigation yielded the following result and a multitude of thoughts and open questions.

Theorem 20. *Let G be a connected graph with at least one leaf vertex, v, with support vertex u.*

$\gamma_{dn}(\overline{G}) = 1$ *if and only if there exists a vertex* $w \in V(G)$ *such that* $deg_G(w) \leq deg_G(u)$ *and* $uw \notin E(G)$. *Furthermore, if* $\gamma_{dn}(\overline{G}) \neq 1$, *then* $\gamma_{dn}(\overline{G}) = 2$.

Proof. Let G be defined as above with vertices v and u. Consider the vertex v, note that $deg_{\overline{G}}(v) = \Delta(\overline{G})$ since v is of minimum degree in G; further note that $N_{\overline{G}}[v]$ contains every vertex except u. Thus every vertex in $\overline{G}$ is downhill directly from v except u.

First, assume that $\gamma_{dn}(\overline{G}) = 1$. It follows that every vertex in $\overline{G}$ lies on some downhill path from one vertex. Thus there must exist a vertex $w' \neq u$ such that $deg_{\overline{G}}(u) \leq deg_{\overline{G}}(w')$ and $uw' \in E(\overline{G})$. Thus, $deg_G(w') \leq deg_G(u)$ and $uw' \notin E(G)$ giving the desired w vertex.

Now assume that the vertex w exists in G. Thus in $\overline{G}$, w is downhill from v and $vw \in E(\overline{G})$. Note that $uw \in E(\overline{G})$ and $deg_{\overline{G}}(w) \geq deg_{\overline{G}}(u)$, so u is downhill from w. Therefore, $\gamma_{dn}(\overline{G}) = 1$.

Suppose that $\gamma_{dn}(\overline{G}) \neq 1$. Since the set $\{v, u\}$ clearly downhill dominates $\overline{G}$ and $\gamma_{dn}(G)$ is nonzero, $\gamma_{dn}(\overline{G}) = 2$.

$\square$

Corollary 21. *If* T *is a tree and not a star, then* $\gamma_{dn}(\overline{T}) = 1$. *If* T *is a star, then* $\gamma_{dn}(\overline{T}) = 2$.

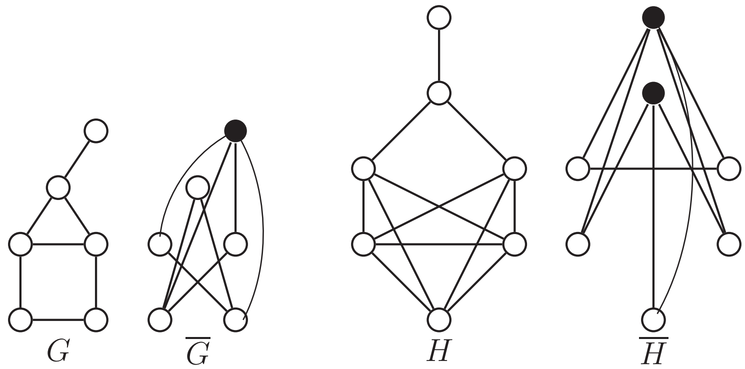

Figure 17: In G the vertex w exists, but in H the vertex w does not exist.

4 Computing $\gamma_d(G)$ and $\gamma_u(G)$

In this section, we present a polynomial algorithm for for determining $\gamma_d(G)$ and $\gamma_u(G)$ for any connected graph G. For this purpose, we first define two graphs that can be obtained from the graph G.

Definition 22. *Let G be a connected graph. The downhill edge graph $DE(G)$ is the graph formed from G by replacing each $e \in E(G)$, where $e = uv$ and $\deg(u) > \deg(v)$, by the directed arc from u to v, denoted (u, v). The remaining edges $e = uv \in E(G)$, for which $\deg(u) = \deg(v)$, are left as undirected edges. Thus, if G is not a regular graph, the edge set of $DE(G)$ consists of both directed and undirected edges.*

Notice that $DE(G)$ can be formed from a connected graph G in linear time in terms of its size m as follows:

Algorithm 1

begin

Let $G = (V, E)$ be a connected (undirected) graph;

Let $E_0 = E(G) = \{e_1, e_2, \ldots, e_m\}$, and let $e_j = u_j v_j$;

for $1 \leq j \leq m$ **do**

  if $\deg_G(u_j) > \deg_G(v_j)$ **then**
    Let $E_j = (E_{j-1} \setminus \{e_j\}) \cup \{(u_j, v_j)\}$;

  else

    if $\deg_G(u_j) < \deg_G(v_j)$ **then**
      Let $E_j = (E_{j-1} \setminus \{e_j\}) \cup \{(v_j, u_j)\}$;

    else
      Let $E_j = E_{j-1}$;

    end

  end

end

Let $DE(G) = (V(G), E_m)$;

end

We now define a new directed graph that is obtained from the graph $DE(G)$.

Definition 23. *Let G be a connected graph and $DE(G)$ be the downhill edge graph of G. The downhill representative graph of G, $DR(G)$, is the directed graph formed by contracting each undirected edge of $DE(G)$.*

Hence, $DR(G)$ can be formed from $DE(G)$ using an edge contraction algorithm that is linear in terms of the size m of G.

Let $in_0(G)$ be the number of vertices of in-degree equal to zero in $DR(G)$ and $out_0(G)$ number of vertices of out-degree equal to zero in $DR(G)$. We shall prove that $\gamma_{dn}(G) = in_0(G)$ and $\gamma_{up}(G) = out_0(G)$. Thus applying Algorithm 1 to form $DE(G)$ from a connected graph G, followed by an edge contraction algorithm to form $DR(G)$, and then counting the vertices of in-degree 0 (respectively, out-degree 0) will compute $\gamma_{dn}(G)$ (respectively, γ_{up}) in polynomial time.

We need another definition.

Definition 24. *Let G be a connected graph. For a vertex $v \in V(G)$, the regular path neighborhood (RPN) of v, denoted $A(v)$, is the set all $u \in V(G)$ such that there is a v-u path $\Pi = (v = x_1, x_2, \ldots, x_k = u)$ for which $\deg(x_i) = \deg(v)$ for $1 \le i \le k$.*

If $A(u)$ and $A(v)$ are RPNs in a graph G, we say that they are adjacent if there is at least one pair of adjacent vertices x and y in G, where $x \in A(u)$ and $y \in A(v)$. Also, if in $DE(G)$, there is a directed arc from a vertex in $A(u)$ to a vertex in $A(v)$, abusing notation, we shorten it to say that there is an arc from $A(u)$ to $A(v)$.

Theorem 25. *If G is a connected graph, then $\gamma_{dn}(G) = in_0(G)$.*

Proof. Let G be a connected graph and D be a γ_{dn}-set of G. First note that any RPN of $DE(G)$ is contracted to a single vertex in $DR(G)$.

We prove a series of claims.

Claim A *If $A(u)$ and $A(v)$ are two distinct adjacent RPNs for vertices u and v in G, then, without loss of generality, in $DE(G)$, there is an arc from $A(u)$ to $A(v)$ and there is no arc from $A(v)$ to $A(u)$.*

Proof of Claim A. Let $A(u)$ and $A(v)$ be two distinct adjacent RPNs for vertices u and v in G. Since $A(u)$ and $A(v)$ are distinct, they are disjoint and $\deg_G(u) \neq \deg_G(v)$. Thus, without loss of generality, we may assume that $\deg_G(u) > \deg_G(v)$. Since $A(u)$ and $A(v)$ are adjacent, there exists a pair of vertices x and y such that $x \in A(u)$, $y \in A(v)$, and $xy \in E(G)$. It follows that $\deg_G(x) > \deg_G(y)$. Thus, in $DE(G)$ there is a directed edge from x to y, giving the result as desired. Clearly, since $\deg_G(x) > \deg_G(y)$ for all $x \in A(u)$ and $y \in A(v)$, there is no directed edge from $A(v)$ to $A(u)$ in $DE(G)$. *(end of proof of claim)*

Note that Claim A establishes that Algorithm 1 always assigns at most one direction (never two) to an arc in $DE(G)$.

Claim B *For a vertex $v \in V$,*

$$\sum_{u \in A(v)} in_{DE(G)}(u) = 0 \tag{1}$$

if and only if $A(v) \cap D \neq \emptyset$.

Proof of Claim B. Suppose to the contrary that there exists a vertex $v \in V(G)$ such that $A(v) \cap D \neq \emptyset$ for every γ_d-set D of G, and that in $DE(G)$

$$\sum_{u \in A(v)} in_{DE(G)}(u) \geq 1. \tag{2}$$

Since $A(v) \cap D \neq \emptyset$, there exists a vertex $v' \in A(v) \cap D$. Further, (2) implies that there is some vertex $u \in A(v)$ such that there is a vertex $w \in N_G(u)$, where $\deg_G(w) > \deg_G(u) = \deg_G(v) = \deg_G(v')$. Thus, v' does not downhill dominate w, so there exists a vertex $w' \in D \backslash \{v'\}$ (w' could be w), such that w lies on a downhill path from w'. Since u is downhill from w, there is a downhill path from w' to u. Now since $u, v' \in A(v)$, there exists a downhill path from u to v and a downhill path from v to v'. Hence, there is a downhill path from u to v', and so there is a downhill path from w' to v'. Thus, $D \backslash \{v'\}$ is a DDS of G having fewer than $\gamma_d(G)$ vertices, a contradiction. Thus, (1) holds.

Now suppose that there is a vertex v such that (1) holds in $DE(G)$, but $A(v) \cap D = \emptyset$ for some γ_d-set D of G. Then $A(v)$ contains no downhill dominating vertex and there is no directed edge into $A(v)$ in $DE(G)$. However, D downhill dominates all vertices of $A(v)$, so there is some vertex $v' \in D$ such that v lies on a downhill path from v'. Thus, $\deg_G(v') \geq \deg_G(v)$ and a v'-v downhill path exists in G. If $\deg_G(v') = \deg_G(v)$, then $v' \in A(v)$, contradicting that $A(v) \cap D = \emptyset$. Thus, $\deg_G(v') > \deg_G(v)$. But then there is some vertex $u \in A(v)$ such that there is a vertex $w \in N(u)$ with $\deg_G(w) > \deg_G(u)$, implying that there is a directed edge into $A(v)$, a contradiction. Thus, if (1) holds, then $A(v) \cap D \neq \emptyset$. *(end of proof of claim)*

Claim C *For all $u, v \in D$,*

$$A(u) \cap A(v) = \emptyset. \tag{3}$$

Proof of Claim C. Suppose to the contrary that there are two vertices $u, v \in D$ such that $A(u) \cap A(v) \neq \emptyset$, and lett $w \in A(u) \cap A(v)$. Thus, $\deg_G(w) = \deg_G(u) = \deg_G(v)$. Therefore, there is a downhill path from u to w and v to w, and so, there is a downhill path from u to v. Hence, $D \backslash \{v\}$ is a DDS of G with fewer than $\gamma_d(G)$ vertices, a contradiction. *(end of proof of claim)*

Note that by Claim B, for each vertex $v \in D$, (1) holds, and further by Claim C, for any two vertices $u, v \in D$, their RPNs are distinct. Thus in $DR(G)$ if $v \in D$, $A(v)$ contracts to a single vertex x with $in(x) = 0$, since there are no vertices with positive in-degree in $A(v)$. Further since the RPN of any vertex in D is distinct, Claim A implies that no two RPNs of vertices in D can be contracted together. Thus, $in_0(G) \geq \gamma_{dn}(G)$.

Now suppose that $in_0 > \gamma_{dn}(G)$. Then there is at least one vertex v in G which satisfies (1). But $A(v)$ contains no vertex in D, contradicting Claim B. Hence, we conclude that $in_0 \leq \gamma_{dn}(G)$, as desired. $\square$

Using an analogous argument, we show that the same result holds for the uphill domination number of a connected graph G.

Theorem 26. *If G is a connected graph, then $\gamma_{up}(G) = out_0(G)$.*

5 Open Problems

The concept of downhill and uphill paths suggest many different avenues for future research. We conclude this paper by listing a few open problems.

- Characterize the graphs for which $run(G) = diam(G)$.

- Investigate the downill/uphill domination numbers of self-complementary graphs.

- Can we determine Nordhaus-Gaddum type results for downhill/uphill domination?

- Determine bounds and properties of the downhill path numbers and downhill cover numbers of graphs.

BIBLIOGRAPHY

[1] C. Berge. *Theory of Graphs and its Applications*. Merthuen, London, 1962.

[2] F. T. Boesch, S. Chen, and J. A. M. McHugh. "On covering the points of a graph with point disjoint paths." *Graphs and Combinatorics*, 406: 202-212 (1974)

[3] F. Buckley and F. Harary. *Distance in Graphs*. Perseus Books, 1990.

[4] G. Chartrand, L. Lesniak, and P. Zhang. *Graphs and Digraphs*. CRC Press, Boca Raton, FL, 5th ed., 2004.

[5] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater. *Fundamentals of Domination in Graphs*. Marcel Dekker, New York, 1998.

[6] J. D. Hedetniemi, S. M. Hedetniemi, S. T. Hedetniemi, and T. Lewis. "Analyzing graphs by degrees." To appear in *AKCE Int. J. Graphs Comb.*

[7] J. D. Hedetniemi, S. M. Hedetniemi, S. T. Hedetniemi, and T. Lewis. "Classifying graphs by degrees." Submitted for publication, March 2012.

[8] R. Srikant, R. Sundaran, K. S. Singh, and C. P. Rangan. "Optimal path cover problem on block graphs and bipartite permutation graphs." *Theoretical Computer Science*, 115: 351-357 (1993)

[9] P. J. Slater. "Path coverings of the vertices of a tree." *Discrete Mathematics*, 25 (1): 65-74 (1979)

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