

Bryn Mawr College
Scholarship, Research, and Creative Work at Bryn Mawr
College

Bryn Mawr College Dissertations and Theses

2017

A Product Structure on Generating Family Cohomology for Legendrian Submanifolds

Ziva Myer
Bryn Mawr College

Follow this and additional works at: <https://repository.brynmawr.edu/dissertations>

 Part of the [Mathematics Commons](#)

Custom Citation

Myer, Ziva. "A Product Structure on Generating Family Cohomology for Legendrian Submanifolds." PhD diss., Bryn Mawr College, 2017.

This paper is posted at Scholarship, Research, and Creative Work at Bryn Mawr College. <https://repository.brynmawr.edu/dissertations/160>

For more information, please contact repository@brynmawr.edu.

A Product Structure on
Generating Family Cohomology
for Legendrian Submanifolds

by

Ziva Myer

April 2017

Submitted to the Faculty of Bryn Mawr College
in partial fulfillment of the requirements for
the degree of Doctor of Philosophy
in the Department of Mathematics

© Ziva Myer, 2017. All rights reserved.

To my mother, Lori-Nan Kaye

Contents

- Abstract v

- Acknowledgments vii

- 1 Introduction 1**

- 2 Background 7**
 - 2.1 Morse Theory Basics 7
 - 2.2 Generating Family Background 8
 - 2.3 Smooth Manifolds with Corners 10

- 3 A Flow Line Approach to Generating Family Cohomology 17**
 - 3.1 Setup of $GH^*(F)$ 17
 - 3.2 Independence with respect to Metric 23
 - 3.3 Invariance with respect to F Equivalences 31
 - 3.4 $GH^*(F)$ as a Legendrian Invariant 34

- 4 Extended Difference Functions 37**

- 5 Moduli Space of Gradient Flow Trees 47**
 - 5.1 Unbroken flow trees 49

5.2	Compactification by broken flow trees	54
6	Product Structure	67
7	Invariance with respect to Equivalences of F	71
7.1	Stabilization	71
7.2	Fiber Preserving Diffeomorphism	77
8	Invariance under Legendrian isotopy	81
8.1	Continuation isomorphisms on $GH^*(F)$	82
8.2	Continuation flow trees	83
9	Future Study	93
9.1	A_∞ Background	93
9.2	Setup of A_∞ Algebra	94
	References	98
	Vita	101

Abstract

One way to obtain invariants of some Legendrian submanifolds in the standard contact manifolds of 1-jet spaces, J^1M , is through the Morse theoretic technique of generating families. This dissertation extends the effective but not complete invariant of generating family cohomology by giving it a product μ_2 . To define the product, moduli spaces of gradient flow trees are constructed and shown to live as the 0-stratum of a compact smooth manifold with corners. These spaces consist of intersecting gradient trajectories of functions whose critical points correspond to Reeb chords of the Legendrian. This dissertation lays the foundation for an A_∞ algebra which will show, in particular, that μ_2 is associative and thus gives generating family cohomology a ring structure.

Acknowledgments

I am filled with so much gratitude for each and every person that has supported me during my journey through graduate school. More people have impacted my growth as a mathematician than I can even begin to thank here, so know that even if we just shared a conversation or coffee or if you supplied a few words of encouragement along the way, I am grateful.

This dissertation would not exist without the affirming guidance of my advisor, Lisa Traynor. I have learned so much from her, and not just mathematically. She has been so generous with her time, help, and advice, and I almost always leave her office feeling more optimistic. She helps me believe in myself, and I hope to one day become a fraction of the professor and person she is.

My other Bryn Mawr professors, Paul Melvin, Helen Grundman, and Djordje Milećević, have also inspired me to be a better mathematician. I've learned so much math from each of them and they have each given me advice and support that I will take with me on my journey. My undergraduate advisor at New College of Florida, Eirini Poimenidou, encouraged me to apply to Bryn Mawr, and I am so thankful that she set me on a great path. She also encouraged me to apply to the Enhancing Diversity in Graduate Education (EDGE) program, and I have been so inspired and gained many mentors from this accomplished group of women math warriors.

My past and current office mates have been such a strong support network for me, from pep talks to chocolate sharing to personality games. I will especially cherish the years spent with Kathryn Bryant, Danielle Smiley, Samantha Pezzimenti, and Hannah Schwartz. I am so lucky to have gone through graduate school with such wonderful, supportive, brilliant women.

My research has benefited immensely from many talks with mathematicians in my field. I particularly want to thank Joshua Sabloff, Katrin Wehrheim, Dan Rutherford, and Jim Stasheff for taking time to share feedback and ideas that have ended up in this dissertation.

I feel so fortunate to have a supportive family. To my father, John Myer, for always being ready with a corny dad joke. To Tom Schvehla for modeling how to accept life's turns with an astounding amount of grace. To my mother, Lori-Nan Kaye, especially, thank you for always being there for me. Life has thrown so much at you, and you refuse to let anything stop you from being the loving, caring person you are. You have taught me how to be so strong and resilient, and for that I am so grateful.

To my friends, near and far, old and new, too many to mention, but all in my heart: Thank you for believing in me and loving me.

I am thankful for my cats, Growler and Racetrack, for all the cuddles that got me through stressful times. And last but certainly not least, to the person who brought these cats into my life, Nick Mazzuca. You have been the most supportive partner for the last five years. You are my rock, a never-wavering source of strength. You have kept my life together during this hectic year, and I appreciate everything you do for me.

Chapter 1

Introduction

A classic contact manifold is the 1-jet space of a smooth manifold M , $J^1(M^n) = T^*(M) \times \mathbb{R}$, equipped with the contact structure $\xi = \ker(dz - \lambda)$, where λ is the canonical Liouville 1-form. An important class of submanifolds in any contact manifold are the Legendrian submanifolds: Λ^n such that $T_p\Lambda \subset \xi$, for all $p \in \Lambda$. Given a smooth function $f : M \rightarrow \mathbb{R}$, the 1-jet of f , $j^1(f) = \{x, \frac{\partial f}{\partial x}(x), f(x)\}$, is a Legendrian submanifold of $J^1(M)$. Not all Legendrian submanifolds arise as the 1-jet of a function. Generating families are a Morse theoretical tool that encode a larger class of Legendrian submanifolds through considering functions defined on a vector bundle over M : if a Legendrian $\Lambda \subset J^1(M)$ has a generating family $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$, then $\Lambda = \{(x, \frac{\partial F}{\partial x}(x, e), F(x, e)) \mid \frac{\partial F}{\partial e}(x, e) = 0\}$. More background on generating families is given in Chapter 2.

In recent years, invariant cohomology groups have been defined for some Legendrian submanifolds in $J^1(M^n)$ through the different techniques of pseudoholomorphic curves and of generating families; see, for example, [4, 7, 12, 21, 35]. In both of these types of constructions, the cohomology groups have an underlying cochain complex generated by the Reeb chords of the Legendrian.

For a Legendrian Λ in J^1M with a generating family F , the Reeb chords are in bijective correspondence with the positively valued critical points of a “difference function” w associated to F . Thus by considering a cochain complex generated by the positively valued critical points of w and a coboundary map ∂ defined using the positive gradient flow of w , one can define generating family cohomology $GH^*(F)$. Sometimes a Legendrian Λ can have multiple, non-equivalent generating families: one then obtains an invariant of Λ by considering the set $\{GH^*(F)\}$ for all generating families of Λ .

Generating family cohomology is an effective but not complete invariant, so a natural problem is to build further invariant algebraic structures on $GH^*(F)$. Towards this goal, we define a product structure:

Theorem 1. *Given a Legendrian $\Lambda \subset J^1M$ with a generating family $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$, there exists a map*

$$\mu_2 : GH^i(F) \otimes GH^j(F) \rightarrow GH^{i+j}(F).$$

Moreover, this map descends to the equivalence class $[F]$ with respect to the operations of stabilization and fiber-preserving diffeomorphism. Further, if $\{\Lambda^t\}_{t \in [0,1]}$ is a Legendrian isotopy and Λ^0 has a generating family, then for F^0 and F^1 given by a lift to a path of generating families $\{F^t\}_{t \in [0,1]}$ for $\{\Lambda^t\}_{t \in [0,1]}$ guaranteed by Proposition 8, the following diagram commutes:

$$\begin{array}{ccc} GH^i(F^0) \otimes GH^j(F^0) & \xrightarrow{\mu_2^0} & GH^{i+j}(F^0) \\ \cong \downarrow & & \downarrow \cong \\ GH^i(F^1) \otimes GH^j(F^1) & \xrightarrow{\mu_2^1} & GH^{i+j}(F^1) \end{array} \quad (1)$$

where the vertical isomorphisms are induced by continuation maps constructed in Chapter 8.

This theorem appears in parts throughout this dissertation as Corollary 2, Corollary 3,

Corollary 4, and Theorem 19.

The product μ_2 is defined through a count of points in a 0-dimensional moduli space of gradient flow trees. Namely, from F one constructs the difference function w and then three different stabilizations of w , denoted by $w_{1,2,3}, w_{2,3,3}, w_{1,3,3}$. For critical points p_1, p_2, p_0 of w , which correspond to Reeb chords of Λ , and a “perturbation” parameter s , one constructs a moduli space $\mathcal{M}(p_1, p_2; p_0|s)$ of “ s ”-intersecting gradient flow trajectories of $w_{1,2,3}, w_{2,3,3}, w_{1,3,3}$; see Figure 1. For generic choices of the s parameter, $\mathcal{M}(p_1, p_2; p_0|s)$ will be a manifold; for appropriate indices of p_i , $\mathcal{M}(p_1, p_2; p_0)$ will be 0-dimensional, and then m_2 is defined by a count of trees. The boundary of the compactification of a 1-dimensional $\mathcal{M}(p_1, p_2; p_0|s)$ shows that m_2 is a cochain map and thus descends to $GH^*(F)$.

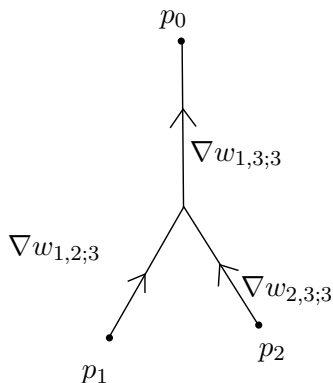


Figure 1: An element in $\mathcal{M}(p_1, p_2; p_0|0)$ is a tree with three intersecting half-infinite trajectories that follow different quadratic stabilizations of the difference function w and limit to critical points given by the p_i 's at their infinite ends.

Defining μ_2 is part of a larger project in progress to define A_∞ structures for Legendrian/Lagrangian submanifolds with generating families. This was inspired in part by Fukaya's A_∞ category of Lagrangian submanifolds in a symplectic manifold, an extension of Floer homology, [15]. In a toy model of Fukaya's construction, one gets an A_∞ category extending the Morse cohomology of a manifold M by studying gradient flow trees of Morse functions on M [14] [23]. In Fukaya's full construction, gradient flow trees are replaced by

pseudoholomorphic curves. Rather than using pseudoholomorphic curves to capture geometric information, our approach builds off of the toy model to build an A_∞ category using gradient flow trees from generating families; this involves extending the tree construction from functions on M to functions defined on vector bundles over M . There are a number of analytic challenges in this approach, including the fact that the geometric information is recorded in the subcomplex of the Morse cochain complex consisting of positive valued critical points and that standard generic perturbations of functions used for transversality arguments in Fukaya's work are no longer possible since these perturbations destroy the correspondence of critical points with the geometric information of Reeb chords.

There are a number of interesting differences between this generating family construction and analogous pseudoholomorphic curve constructions. Pseudoholomorphic curve constructions have built a DGA [7–9], whose homology is a strong invariant algebraic structure for Legendrian submanifolds of arbitrary dimensions, by using infinite-dimensional analysis of PDEs. In low dimensions, combinatorial methods have been used to extract invariant algebraic structures similar to those that we are interested in from the DGA [1, 4, 6, 26, 27]. Our work gives a different approach: It is a non-combinatorial, chain level construction for Legendrians with generating families in $J^1(M)$, where M is \mathbb{R}^n or a closed n -manifold, for arbitrary n . Our approach differs from pseudoholomorphic curve constructions by only using finite-dimensional analysis.

The outline of this dissertation is as follows: In Chapter 2 we give some background on Morse Theory, generating families, and smooth manifolds with corners. In Chapter 3 we construct cohomology groups for generating families using gradient flow lines; this differs from previous constructions of such groups using the relative singular cohomology of sublevel sets. Chapter 4 constructs the functions and metrics used in our gradient flow trees. The construction of spaces of trees themselves occurs in Chapter 5, along with compactifications

of these spaces. The spaces of trees are used to construct a chain-level product in Chapter 6 and compactification results show that the product descends to cohomology. In Chapter 7, we show that the product is invariant under the generating family operations of stabilization and fiber-preserving diffeomorphism. Chapter 8 constructs a cochain homotopy that shows that the product is invariant under Legendrian isotopy. Chapter 9 lays the foundation for a larger A_∞ structure to be constructed in the future.

Chapter 2

Background

In this chapter, we give some basic background on Morse Theory, generating families, and manifolds with corners.

2.1 Morse Theory Basics

To set notation, we recall a few facts from Morse Theory; see, for example, [19, 24, 25, 32, 38] for more details. Let X be closed manifold and let $f : X \rightarrow \mathbb{R}$ be a Morse function, i.e., a smooth function with nondegenerate critical points. We will relax the condition that X is closed in future sections to using a function with taming properties outside a compact set.

Given a critical point $p \in \text{Crit}(f)$, the **Morse index** $\text{ind}_f(p) \in \mathbb{Z}^{\geq 0}$ is the dimension of the negative eigenspace of the Hessian $D^2 f(p)$. The Morse Lemma states that, for a Morse function $f : X \rightarrow \mathbb{R}$, we may find local coordinates $\phi : \mathbb{R}^n \supset B \rightarrow X$ so that, in a neighborhood of a critical point p ,

$$\phi^* f(x_1, \dots, x_n) = f(p) - \left(x_1^2 + \dots + x_{\text{ind}(p)}^2 \right) + \left(x_{\text{ind}(p)+1}^2 + \dots + x_n^2 \right).$$

To study how a Morse function gives topological information, we pick an auxiliary Riemannian metric g and study flow lines of $\nabla_g f$. For the purposes of this dissertation, we use *positive* gradient flow. Let $\psi : \mathbb{R} \times X \rightarrow X$ denote the flow of this vector field and define the **stable and unstable manifolds** for $p \in \text{Crit}(f)$ as

$$W_p^-(f) = \{x \in X \mid \lim_{t \rightarrow -\infty} \psi_t(x) = p\} \quad W_p^+(f) = \{x \in X \mid \lim_{t \rightarrow \infty} \psi_t(x) = p\}.$$

These are smooth manifolds. Since we are using positive gradient flow, $W_p^-(f)$ is of dimension $\text{coind}(p)$ while $W_p^+(f)$ is of dimension $\text{ind}(p)$. The pair (f, g) is called **Morse-Smale** if $W_p^-(f) \pitchfork W_p^+(f)$, for all $p \in \text{Crit}(f)$.

2.2 Generating Family Background

In this subsection, we discuss the background necessary for working with generating families for Legendrian submanifolds. The germ of the idea comes from the following observation: given a function $f : M \rightarrow \mathbb{R}$, the 1-jet of f is a Legendrian submanifold of J^1M . Generating families extend this construction to “non-graphical” Lagrangians and Legendrians by expanding the domain to, for example, the trivial vector bundle $M \times \mathbb{R}^N$ for some potentially large N . We will denote the fiber coordinates by $e = (e_1, \dots, e_N)$. In this paper, M will either be \mathbb{R}^n or a closed manifold. What follows are bare-bones definitions so as to set notation; see for example [34–36] for more details.

Suppose that we have a smooth function $F : M^n \times \mathbb{R}^N \rightarrow \mathbb{R}$ such that $\mathbf{0}$ is a regular value of the map $\partial_e F : M \times \mathbb{R}^N \rightarrow \mathbb{R}^N$. We define the **fiber critical set** of F to be the n -dimensional submanifold $\Sigma_F = (\partial_\eta F)^{-1}(\mathbf{0})$. Define immersions $j_F : \Sigma_F \rightarrow J^1M$ in local coordinates by:

$$j_F(x, e) = (x, \partial_x F(x, e), F(x, e)).$$

The image Λ of j_F is an immersed Legendrian submanifold. We say that F **generates** Λ , or that F is a **generating family (of functions)** for Λ .

Two functions $F_i : M \times \mathbb{R}^{N_i} \rightarrow \mathbb{R}$, $i = 0, 1$, are **equivalent** (denoted $F_0 \sim F_1$) if they can be made equal by applying fiber-preserving diffeomorphisms and stabilizations to each; these operations are defined as follows:

1. Given a function $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$, let $Q : \mathbb{R}^K \rightarrow \mathbb{R}$ be a non-degenerate quadratic function. If we define $F \oplus Q : M \times \mathbb{R}^N \times \mathbb{R}^K \rightarrow \mathbb{R}$ by $F \oplus Q(x, e, e') = F(x, e) + Q(e')$, then $F \oplus Q$ is a (dimension K) **stabilization** of f .
2. Given a function $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$, suppose $\Phi : M \times \mathbb{R}^N \rightarrow M \times \mathbb{R}^N$ is a fiber-preserving diffeomorphism, i.e., $\Phi(x, e) = (x, \phi_x(e))$ for a smooth family of diffeomorphisms ϕ_x . Then $F \circ \Phi$ is said to be obtained from F by a **fiber-preserving diffeomorphism**.

Given a function F , denote by $[F]$ its equivalence class with respect to these two operations.

It is easy to see that if $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a generating family for a Legendrian Λ , then any $F' \in [F]$ will also be a generating family for Λ . While a Legendrian submanifold with a generating family will always have an infinite number of generating families, the set of equivalence classes may be more tractable.

Having a generating family to work with instead of a Legendrian submanifold allows us to use concepts from Morse homology. As the domain of our functions are non-compact, we impose the following ‘‘tameness’’ property on our generating families:

Definition 1. A function $f : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ is **linear-at-infinity** if f can be written as

$$f(x, e) = f^c(x, e) + A(e),$$

where f^c has compact support and A is a non-zero linear function, that is $A(e) = A(e_1, \dots, e_N) = c_1 e_1 + \dots + c_N e_N$ with $c_i \in \mathbb{R}$ not all zero.

Remark 1. As M is a closed manifold or \mathbb{R}^n , we may assume that the compact set in Definition 1 is of the form $K_M \times K_E \subset M \times \mathbb{R}^N$ where $K_M = M$ if M is closed or K_M and K_E are compact Euclidean subsets.

This convention is particularly convenient for producing compact Legendrians when $M = \mathbb{R}^n$, as seen in [12, 21]. The definition of linear-at-infinity is not preserved under stabilization. However, we have:

Lemma 1 ([31]). *If F is the stabilization of a linear-at-infinity generating family, then F is equivalent to a linear-at-infinity generating family.*

2.3 Smooth Manifolds with Corners

Many proofs of theorems in Chapter 5 use concepts from differential topology applied to smooth manifolds with corners, which we review in this section.

Definition 2. A **smooth manifold with corners** of dimension $n \in \mathbb{N}$ is a second-countable, Hausdorff space X equipped with a maximal atlas of charts $\{\phi : X \supset U \rightarrow V \subset [0, \infty)^n\}$ whose transition maps are smooth. The ℓ -**stratum** X_ℓ is the set of points $x \in X$ such that $\phi(x)$ has ℓ components equal to 0.

Lemma 2. *If X and Y are smooth manifolds with corners of dimensions m_1 and m_2 , respectively, then $X \times Y$ is a smooth manifold with corners of dimension $m_1 + m_2$ and*

$$(X \times Y)_i = \bigsqcup_{j+k=i} X_j \times Y_k.$$

Proof. Let $(x, y) \in X \times Y$. Then there are open sets $U_1 \subset X$ about x and $U_2 \subset Y$ about y , and charts $\phi_1 : U_1 \rightarrow V_1 \subset [0, \infty)^{m_1}$ and $\phi_2 : U_2 \rightarrow V_2 \subset [0, \infty)^{m_2}$ for V_1, V_2 open. Then

$V_1 \times V_2$ is an open set in $[0, \infty)^{m_1+m_2}$ and $\phi_1 \times \phi_2 : U_1 \times U_2 \rightarrow V_1 \times V_2$ is a local chart for (x, y) .

If $(x, y) \in (X \times Y)_i$ then $\phi_1 \times \phi_2(x, y) = (\phi_1(x), \phi_2(y))$ has i components equal to 0 which happens if and only if $\phi_1(x)$ has j components equal to 0, $\phi_2(y)$ has k components equal to 0, and $j + k = i$. \square

There is a natural way to understand smooth maps and derivatives on manifolds with corners, similar to the treatment of manifolds with boundary in Chapter 2 of [16]. For our purposes, understanding maps from a smooth manifold with corners to a smooth manifold without boundary or corners will suffice. Consider a smooth map $f : U \subset [0, \infty)^n \rightarrow \mathbb{R}^\ell$ for some n, ℓ . If $u \in U$ has no coordinates equal to 0, then df_u is our usual notion of derivative. However, if u has some 0 coordinates, the smoothness of f implies that we may extend f to a smooth map \tilde{f} on a neighborhood of u in \mathbb{R}^n . We define df_u to be the usual derivative $d\tilde{f}_u : \mathbb{R}^n \rightarrow \mathbb{R}^\ell$. This will not depend on the local extension of f .

With this observation, given a manifold with corners X , we may define the tangent space $T_x X$ at $x \in X$ to be the image of the derivative of any local parametrization about x . Given a map defined on a manifold with corners X , let $\partial_i f : X_i \rightarrow Y$ denote the restriction of f to the stratum X_i . Then $T_x X_i$ is a linear subspace of $T_x X$ of codimension i and $d(\partial_i f)_x = df_x|_{T_x X_i}$.

We will make use of the following natural extension of classic differential topology theorems of manifolds with boundary to manifolds with corners. We must impose extra transversality conditions on the strata to achieve these results.

The following is a known result (see, for example, [28]). For the reader's convenience, we give a proof.

Theorem 2. *Let f be a smooth map of a manifold X with corners onto a boundaryless manifold Y , and suppose $f : X \rightarrow Y$ and $\partial_i f : X_i \rightarrow Y$ are transversal to a boundaryless*

submanifold $Z \subset Y$ for all strata X_i of X . Then the preimage $f^{-1}(Z)$ is a manifold with corners with i -stratum $f^{-1}(Z)_i = f^{-1}(Z) \cap X_i$ and the codimension of $f^{-1}(Z)$ in X equals the codimension of Z in Y .

Proof. We generalize the proof in Section 2.1 of [16] for the analogous statement with manifolds with boundary (a special case of the above theorem). From [16], we know that $f^{-1}(Z) \cap X_0$ is a smooth manifold without boundary and $f^{-1}(Z) \cap (X_0 \cup X_1)$ is a smooth manifold with boundary equal to $f^{-1}(Z) \cap X_1$. We proceed inductively by assuming $f^{-1}(Z) \cap (\cup_{k=0}^{i-1} X_k)$ is a smooth manifold with corners whose k -stratum is equal to $f^{-1}(Z) \cap X_k$. Let $x \in f^{-1}(Z) \cap X_i$. We show that, near x , $f^{-1}(Z)$ is a manifold with corners.

As usual, if $\text{codim}(Z) = \ell$, we may express Z near $f(x)$ as the zero-set of ℓ linearly-independent functions; that is, there is a submersion ϕ of a neighborhood of $f(x)$ in Y onto \mathbb{R}^ℓ so that $Z = \phi^{-1}(0)$ in this neighborhood. Thus there is a neighborhood of $x \in X$ in which $f^{-1}(Z)$ is $(\phi \circ f)^{-1}(0)$. Since $x \in X_i$, around x in this neighborhood, we may pick a parametrization $h : \mathbb{R}^{n-i} \times [0, \infty)^i \supset U \rightarrow X$ near x so that $g = \phi \circ f \circ h$ is defined. We will show that $g^{-1}(0)$ is a manifold with corners near $u \in U$ for u such that $h(u) = x$.

Since $f \pitchfork Z$ and ϕ is a submersion, g is regular at u . The smoothness of g means that we may extend it to a smooth function \tilde{g} on a neighborhood of u in \mathbb{R}^n and regularity is preserved since $d\tilde{g}_u = dg_u$ (as explained in the paragraphs before this theorem). Thus, $\tilde{g}^{-1}(0)$ is a boundaryless manifold S of codimension ℓ in some neighborhood of $u \in \mathbb{R}^n$.

The inductive hypothesis implies that $S \cap (\mathbb{R}^{n-i+1} \times [0, \infty)^{i-1})$ is a manifold with corners containing $(i-1)$ strata. To extend this to one with an extra stratum, we show that $S \cap (\mathbb{R}^{n-i} \times [0, \infty)^i)$ is a manifold with corners of i strata. For $1 \leq j \leq n$, let $\pi_j : S \rightarrow \mathbb{R}$ be the j th coordinate function on \mathbb{R}^n , restricted to S . Then

$$S \cap (\mathbb{R}^{n-i} \times [0, \infty)^i) = \{s \in S \cap (\mathbb{R}^{n-i+1} \times [0, \infty)^{i-1}) \mid \pi_j(s) \geq 0 \forall j, n-i+1 \leq j \leq n\}.$$

By the inductive hypothesis, we know that the collection of $s \in S \cap (\mathbb{R}^{n-i} \times [0, \infty)^i)$ with $\pi_j(s) = 0$ for at most $i - 1$ choices of $j \in \{n - i + 1, \dots, n\}$ form a smooth manifold with corners of $i - 1$ strata. To complete the proof, we show that, given such an s , 0 is a regular value of the remaining coordinate map, which we call π . Suppose not. Then there exists $s \in S$ with $\pi_j(s) = 0$ for all $j \in \{n - i + 1, \dots, n\}$ and $d(\pi)_s \equiv 0$. This means that the coordinate of every vector in $T_s S$ given by π is 0, so $T_s S \subset \mathbb{R}^{n-1}$. By the Preimage Image Theorem for smooth manifolds, $T_s S = \ker(dg_s : \mathbb{R}^n \rightarrow \mathbb{R}^\ell)$. Thus any map onto \mathbb{R}^ℓ on the restriction to its i th stratum will have dimension $n - 1 - i - \ell$. The hypothesis that $\partial_k f \pitchfork Z$ for all k means that $d(\partial_k g)_s : \mathbb{R}^{n-k} \rightarrow \mathbb{R}^\ell$ is onto, and so has kernel with dimension $n - k - \ell$. In particular, $d(\partial_i g)_s : \mathbb{R}^{n-i} \rightarrow \mathbb{R}^\ell$ is onto, so the kernel has dimension $n - i - \ell \neq n - 1 - i - \ell$, giving a contradiction. \square

The following is a generalization of the Transversality Theorem for manifolds with boundary (see, for example Section 2.3 in [16]) to manifolds with corners.

Theorem 3. *Suppose $F : X \times S \rightarrow Y$ is a smooth map, where X is a manifold with corners and S and Y are boundaryless manifolds. Let Z be a boundaryless submanifold of Y . Suppose $F : X \times S \rightarrow Y$ is transversal to Z and $\partial_i F : X_i \times S \rightarrow Y$ is transversal to Z for all i -strata X_i of X . Then for almost every $s \in S$, f_s and $\partial_i f_s$ are transversal to Z for each i -stratum X_i , where $f_s(x) = F(x, s)$ and $\partial_i f_s = f_s|_{X_i}$.*

Proof. Again, we follow the proof of the analogous statement for manifolds with boundary in [16]. From the Preimage Theorem 2, $W = F^{-1}(Z)$ is a submanifold with corners of $X \times S$ whose i -stratum is $W_i = W \cap (X \times S)_i$. Let $\pi : X \times S \rightarrow S$ be the projection to S . Then we have the following restriction maps: $\pi_W : W \rightarrow S$ and $\pi_i : W_i \rightarrow S$. From Sard's Theorem, we know that almost every $s \in S$ is a regular value of each of these maps. We claim that if $s \in S$ is regular for π_W then $f_s \pitchfork Z$ and for each i , if $s \in S$ is regular for π_i , then $\partial_i f_s \pitchfork Z$. Each of these claims is proved by a similar argument found in [16].

We sketch a proof of the claim for an arbitrary i . We wish to show that $\partial_i f_s \pitchfork Z$, so let $x \in X_i$ and suppose $f_s(x) = z \in Z$. Since $\partial_i F : X_i \times S \rightarrow Y$ is transversal to Z and $F(x, s) = z$,

$$\text{Im} \left(d(\partial_i F)_{(x,s)} \right) + T_z Z = T_z Y.$$

Equivalently, given arbitrary $a \in T_z Y$, there exists $b \in T(X_i \times S)_{(x,s)}$ so that

$$d(\partial_i F)_{(x,s)}(b) - a \in T_z Z.$$

To show that $\partial_i f_s \pitchfork Z$, we wish to find $v \in T_x X_i$ so that $d(\partial_i f_s)_x(v) - a \in T_z Z$. Since $T(X_i \times S)_{(x,s)} = T_x X_i \times T_s S$, we may write $b = (w, e)$ for $w \in T_x X_i$ and $e \in T_s S$.

If $e = 0$, we are done, for we claim that $d(\partial_i F)_{(x,s)}(c, 0) = d(\partial_i f_s)_x(c)$ for any $c \in T_x X_i$. Identifying X_i with $X_i \times \{s\}$, we may express

$$\partial_i f_s = \partial_i F|_{X_i \times \{s\}} = \partial_i F \circ \iota,$$

where $\iota : X_i \cong X_i \times \{s\} \rightarrow X_i \times S$ is the natural inclusion map. Thus,

$$d(\partial_i f_s)_x = d\partial_i F_{(x,s)} \circ d\iota_x.$$

The fact that $T_{(x,s)}(X_i \times \{s\}) = T_x X_i \times \{0\}$ proves the claim.

To finish the proof, we use the assumption that s is a regular value for $\pi_i : W_i \rightarrow S$. Since $d(\pi_i)_{(x,s)}$ is onto, there exists $(u, e) \in T_{(x,s)} W_i$ for $e \in T_s S$ as above. Since $W_i = F^{-1}(Z) \cap (X \times S)_i$, we know that $d(\partial_i F)_{(x,s)}(u, e) \in T_z Z$. Thus, $v = w - u \in T_x X_i$ is such

that

$$\begin{aligned} & d(\partial_i f_s)_x(v) - a \\ &= d(\partial_i f_s)_x(w - u) - a \\ &= d(\partial_i F)_{(x,s)}(w - u, 0) - a \\ &= [d(\partial_i F)_{(x,s)}(w, e) - a] - d(\partial_i F)_{(x,s)}(u, e) \in T_z Z. \end{aligned}$$

□

Chapter 3

A Flow Line Approach to Generating Family Cohomology

We use gradient flow lines of a “difference function” associated to a generating family to define the generating family cohomology invariants of Legendrian submanifolds; this setup differs from past formulations of cohomology for generating families using the relative singular cohomology of sublevel sets as in [12, 21, 35], for example.

3.1 Setup of $GH^*(F)$

Suppose that $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a generating family for a Legendrian $\Lambda \subset J^1M$. The **difference function**, $w : M \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$, is defined to be:

$$w(x, e, e') = F(x, e) - F(x, e'). \quad (2)$$

The reason to work with the difference function is that its critical points capture information about the **Reeb chords** of Λ . Reeb chords are segments of the Reeb vector field

with endpoints on Λ . The **Reeb vector field** R_α of a contact structure locally given by $\ker(\alpha)$ is the unique vector field satisfying $d\alpha(R_\alpha, \cdot) \equiv 0$ and $\alpha(R_\alpha) \equiv 1$. For the contact form we use, $\alpha = dz - \lambda$ on $J^1(M)$, $R_\alpha = \frac{\partial}{\partial z}$. For our purposes, then, Reeb chords are segments $\gamma : [a, b] \rightarrow J^1M$ in the z -direction whose endpoints lie on Λ . Note that Reeb chords are in one-to-one correspondence with double points of the projection of Λ to an immersed Lagrangian submanifold of T^*M . Let $\ell(\gamma) > 0$ be the **length of the Reeb chord** γ .

Proposition 4 ([12,30]). *The critical points of the difference function w are of two types:*

1. *For each Reeb chord γ of Λ , there are two critical points (x, e, e') and (x, e', e) of w with nonzero critical values $\pm\ell(\gamma)$.*

2. *The set*

$$\{(x, e, e) : (x, e) \in \Sigma_F\}$$

is a critical submanifold of w with critical value 0.

For generic F , these critical points and submanifolds are non-degenerate, and the critical submanifold has index N .

We will work with the critical points of w of Type 1 that have positive critical value.

Definition 3. Given $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ and associated difference function $w : M \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$, let $\text{Crit}_+(w)$ be the set of critical points of w with positive critical value. Then define $C(F) := \langle \text{Crit}_+(w) \rangle_{\mathbb{Z}_2}$ to be the vector space generated over \mathbb{Z}_2 by elements in $\text{Crit}_+(w)$. Equip $C(F)$ with the following grading on generators:

$$|p| = \text{ind } w(p) - N.$$

Remark 2. The shift in index occurs so that the groups are invariant when F undergoes a stabilization operation. Note that previous formulations of $GH^*(F)$ use a shift of $N + 1$ rather than N to produce an isomorphism with linearized contact homology [12]. We use a shift of N , however, to guarantee that our product map has the standard degree.

The positive-valued critical point, and in fact all critical points of w , are contained in a compact subset of $M \times \mathbb{R}^{2N}$.

Lemma 3. *Suppose that $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a linear-at-infinity generating family that agrees with a non-zero linear function outside $K_M \times K_E$, for compact sets $K_M \subset M$ and $K_E \subset \mathbb{R}^N$ as in Remark 1. Then every critical point of the associated difference function $w : M \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ is contained in $K_M \times K_E \times K_E$.*

Proof. First consider the critical points of w . By assumption

$$F(x, e) = F^c(x, e) + A(e),$$

where $F^c = 0$ if $x \notin K_M$ or $x \notin K_E$, and A is a nonzero linear function. Thus we see that for all $(x, e_1, e_2) \in M \times \mathbb{R}^N \times \mathbb{R}^N$,

$$w(x, e_1, e_2) = F^c(x, e_1) - F^c(x, e_2) + A(e_1) - A(e_2).$$

We want to show that if (x, e_1, e_2) is a critical point of w , then $x \in K_M$, $e_1 \in K_E$, and $e_2 \in K_E$. Suppose for a contradiction that $x \notin K_M$. Then we see that $w(x, e_1, e_2)$ agrees with the linear function $A(e_1) - A(e_2)$, and thus (x, e_1, e_2) cannot be a critical point. If $e_1 \notin K_E$, $w(x, e_1, e_2) = -F_0(x, e_2) + A(e_1) - A(e_2)$, and thus the $\frac{\partial w}{\partial e_1}(x, e_1, e_2) \neq 0$, showing that (x, e_1, e_2) is not critical for w . A similar argument shows that if $e_2 \notin K_E$, $\frac{\partial w}{\partial e_2}(x, e_1, e_2) \neq 0$. Thus if (x, e_1, e_2) is critical for w , then $x \in K_M$, $e_1 \in K_E$, and $e_2 \in K_E$.

□

It is easy to check that if F is linear-at-infinity, then the associated difference function w is no longer linear-at-infinity. However, we have:

Lemma 4 ([12]). *If F is linear-at-infinity, then the associated difference function w is equivalent to a linear-at-infinity function.*

Since Reeb chords of a Legendrian with a generating family F are in bijection with positive-valued critical points of the difference function w of F , the idea behind generating family cohomology is to study the Morse cohomology of the set $\{w > 0\}$. To do this, we first equip the domain of the w with a Riemannian metric.

Definition 4. Let $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a linear-at-infinity generating family that agrees with a non-zero linear function outside $K_M \times K_E$, for compact sets $K_M \subset M$ and $K_E \subset \mathbb{R}^N$ as in Remark 1. Let F have difference function $w : M \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$. Let the set of **compatible metrics**, \mathcal{G}_F , denote the set of Riemannian metrics g_w on $M \times \mathbb{R}^N \times \mathbb{R}^N$ so that for all $p \in \text{Crit}_+ w$, there is a neighborhood U of p and a parametrization $\phi : B^{\text{ind } p} \times B^{(n+2N)-\text{ind } p} \rightarrow U$ with $\phi(0) = p$ such that

1. $\phi^* w = w(p) + \frac{1}{2}(x_1^2 + \cdots + x_{(n+2N)-\text{ind } p}^2) - \frac{1}{2}(x_{(n+2N)-\text{ind } p+1}^2 + \cdots + x_{n+2N}^2)$,
2. $\phi^* g_w = dx_1 \otimes dx_1 + \cdots + dx_{n+2N} \otimes dx_{n+2N}$.

In addition, metrics in \mathcal{G}_F will satisfy

3. Outside $K_M \times K_E \times K_E$, g_w is the standard Euclidean metric, and
4. For every pair of critical points $p, q \in \text{Crit}_+ w$, the unstable and stable manifolds of p and q have a transverse intersection.

Remark 3. 1. Conditions (1), (2), and (3) are standard Morse theoretic conditions so that we may understand gradient flow near critical points and outside the compact set. In particular, these assumptions allow us to use results of [37] in Chapter 5. While condition (2) is not generic, the gradient flow of a pair satisfying conditions (1), (3), and (4) is topologically conjugate to one satisfying all four; see [37, Remark 3.6] or [11].

2. It is possible to find metrics satisfying (1) - (4). Start with a metric satisfying (1) - (3) and then perform L^2 -small perturbations of the metric on annuli around critical points to get the additional Smale condition (4), see [2].

To compute cohomology groups on $C(F)$, we will equip $C(F)$ with a codifferential $\delta : C^*(F) \rightarrow C^{*+1}(F)$ defined by a count of isolated gradient flow lines, modulo reparametrization in time.

In particular, if $p, q \in C(F)$ and $g \in \mathcal{G}_F$, let

$$\mathcal{M}(p, q) := \left(\{ \gamma : \mathbb{R} \rightarrow M \times \mathbb{R}^N \times \mathbb{R}^N \mid \dot{\gamma} = \nabla_g w, \lim_{t \rightarrow -\infty} \gamma(t) = p, \lim_{t \rightarrow \infty} \gamma(t) = q \} \right) / \mathbb{R},$$

where \mathbb{R} denotes the action of translation in the t variable.

Theorem 5. $\mathcal{M}(p, q)$ is a smooth manifold of dimension $|q| - |p| - 1$.

Theorem 5 follows from the natural identification

$$\mathcal{M}(p, q) \cong W_p^-(w) \cap W_q^+(w) \cap w^{-1}(c),$$

for $c \in \mathbb{R}$ a regular value in $(w(p), w(q))$. The Morse-Smale assumption shows that this is a smooth manifold of the above dimension.

Definition 5. Define the map $\delta : C^*(F) \rightarrow C^{*+1}(F)$ by

$$\delta(p) = \sum_{q \in C^{|p|+1}(F)} \#_{\mathbb{Z}_2} \mathcal{M}(p, q) \cdot q.$$

Remark 4. The map δ is well defined: while $C(F)$ is generated by only positively-valued critical points of w , δ counts flow lines of the positive gradient flow, so w will increase in value along trajectories.

In fact, δ is a codifferential:

Lemma 5. *The map $\delta : C^*(F) \rightarrow C^{*+1}(F)$ satisfies $\delta^2 = 0$.*

Proof. Given our taming condition of requiring F to be linear-at-infinity, the relevant flow lines are contained in a compact set; see Lemma 3 and note that outside this compact set, the gradient is constant. Thus, the above is shown by the standard argument for functions with a compact domain— If $\mathcal{M}(p, q)$ is a 1-dimensional manifold, it may be compactified with the addition of once-broken flow-lines, which make up its boundary. As the boundary of a 1-dimensional manifold contains an even number of points, the result follows. For more details, see, for example, [32]. \square

Definition 6. The **generating family cohomology** $GH^*(F)$ of the generating family F is defined to be the cohomology of $C(F)$ with respect to the codifferential δ :

$$GH^*(F) = H^*(C(F), \delta).$$

Remark 5. Note that we built the usual grading shift into the definition of the index of $C(F)$ rather than into the definition of the cohomology as was done in past papers on $GH^*(F)$ such as in [12, 21, 35], for example.

3.2 Independence of $GH^*(F)$ with respect to metric choice:

A first continuation argument

Crucial to the above construction of $GH^*(F)$ is the metric $g \in \mathcal{G}_F$ used to produce a gradient flow of w . In this section, we show that $GH^*(F)$ is independent of the metric used in its construction. While different metrics affect the codifferential δ of $GH^*(F)$, a continuation argument shows that the cohomology is invariant up to isomorphism under generic change of metric. Continuation arguments will be used multiple times in this work, so we provide a detailed exposition for the following proposition:

Proposition 6. *Up to isomorphism, $GH^*(F)$ does not depend on the metric $g \in \mathcal{G}_F$ used in the construction.*

Proof. To show that $GH^*(F)$ does not depend on the metric, we use the idea of continuation maps in Floer homology from [10], [20]; see also [19] for an informal exposition. This technique will be used multiple times to show different notions of invariance; as this is the first use, we will provide the details here for later reference.

The strategy of the proof is to define a continuation cochain map Φ_Γ from a path of functions and metrics Γ (Lemma 6). Given a homotopy between two such paths Γ and $\widehat{\Gamma}$, we construct a cochain homotopy K between the two continuation maps of the two paths in Lemma 7. We then show that in Lemma 8 that the continuation map of the concatenation of any two paths $\Phi_{\Gamma_2 * \Gamma_1}$ is cochain homotopic to the composition of the two continuation maps from the paths $\Phi_{\Gamma_2} \circ \Phi_{\Gamma_1}$. Last, we show that the continuation map of a constant path of a Morse-Smale pair is the identity map (Lemma 9). With all of these pieces, suppose that we have an arbitrary admissible path of functions and metrics. Concatenating the path with its reverse is homotopic to the constant path, showing that the continuation map is an isomorphism on cohomology.

To construct a continuation map, we use a path $\Gamma = \{(w^t, g^t) \mid t \in [0, 1]\}$ of difference functions and metrics. For this proof, we may set $w^t = w$ for all $t \in [0, 1]$. Given two metrics $g^0, g^1 \in \mathcal{G}_F$, construct a path g^t in the space of Riemannian metrics on $M \times \mathbb{R}^{2N}$ that are standard outside $K_M \times K_E \times K_E$. This space is contractible because we can use a straight-line homotopy of the metrics on the non-standard part to contract to any given metric in this set.

Given Γ , we construct a continuation map which we will denote by Φ_Γ , with

$$\Phi_\Gamma : C^*(F^0) \rightarrow C^*(F^1),$$

where (in this case) $F^0 = F^1 = F$, the generating family that produced w . Given $\epsilon > 0$ such that $\frac{\epsilon}{4} < \rho$, where ρ is the least positive critical value of w , the continuation maps count isolated gradient flow lines of the vector field $\nabla_G W$ on $(M \times \mathbb{R}^{2N}) \times I$ with

$$\begin{aligned} W(p, t) &= w^t(p) + \epsilon \left((1/2)t^2 - (1/4)t^4 \right) \\ G_{(p,t)} &= g_p^t + dt^2. \end{aligned} \tag{3}$$

We say that the path Γ is **admissible** if the unstable and stable manifolds of $\nabla_G W$ intersect transversely. Note that this does not mean that each (w^t, g^t) is Morse-Smale.

This vector field has the following property: when projected to I , there's a critical point of index 0 at $t = 0$ and one of index 1 at $t = 1$ with none in between. The vector field flows smoothly from 0 to 1. Let $\text{Crit}_+^k(W)$ denote the set of critical points of W with critical value greater than $\frac{\epsilon}{4}$. Then we have that

$$\text{Crit}_+^k(W) = \left(\text{Crit}_+^k(w^0) \times \{0\} \right) \cup \left(\text{Crit}_+^{k-1}(w^1) \times \{1\} \right),$$

where the superscript denotes the Morse index.

For $p \in \text{Crit}_+(w^0)$ and $q \in \text{Crit}_+(w^1)$, consider the space $\mathcal{M}_\Gamma((p, 0), (q, 1))$ of flow lines of $\nabla_G W$ from $(p, 0)$ to $(q, 1)$, modulo translation reparametrization. If Γ is admissible, usual Morse Theory arguments will show that $\mathcal{M}_\Gamma((p, 0), (q, 1))$ is a manifold of dimension $\text{ind}_W(q, 1) - \text{ind}_W(p, 0) - 1 = \text{ind}_{w^1}(q) + 1 - \text{ind}_{w^0}(p) + 1 = \text{ind}_{w^1}(q) - \text{ind}_{w^0}(p)$.

Thus, we can define the map Φ_Γ on generators in $C^*(F^0)$ by

$$\Phi_\Gamma(p) = \sum_{q \in C^{|p|}(F^1)} \#_{\mathbb{Z}_2} \mathcal{M}_\Gamma((p, 0), (q, 1)) \cdot q.$$

We wish to show that the continuation map Φ_Γ gives an isomorphism on $GH^*(F)$. To show this, we must prove that Φ_Γ is a cochain map, that a homotopy of Γ induces a cochain homotopy, that concatenating paths gives a cochain homotopy, and that the constant path gives the identity. These facts together show that our continuation map will induce an isomorphism on cohomology [20].

Lemma 6. *For the path Γ , Φ_Γ is a cochain map, i.e., the following diagram commutes:*

$$\begin{array}{ccc} C^k(F^0) & \xrightarrow{\Phi_\Gamma} & C^k(F^1) \\ \downarrow \delta^0 & & \downarrow \delta^1 \\ C^{k+1}(F^0) & \xrightarrow{\Phi_\Gamma} & C^{k+1}(F^1). \end{array}$$

Thus, Φ_Γ descends to cohomology.

Proof. Consider a 1-dimensional Moduli Space $\mathcal{M}_\Gamma((p, 0), (q, 1))$ of flow lines along the vector field $\nabla_G W$ defined above (3) for $(p, 0) \in \text{Crit}_+(W)$ to $(q, 1) \in \text{Crit}_+(W)$. Since this space is one-dimensional, we have that $\text{ind}_W(q, 1) - \text{ind}_W(p, 0) = 2$, i.e., $\text{ind}_w(q) - \text{ind}_w(p) = 1$. If Γ is admissible, the usual Morse Theory compactification argument implies that $\mathcal{M}_\Gamma((p, 0), (q, 1))$ has a compactification to a compact 1-manifold with boundary consisting of once-broken flow lines. Since these flow lines may only break at points in $(M \times \mathbb{R}^{2N}) \times \{0\}$

or $(M \times \mathbb{R}^{2N}) \times \{1\}$, we have the following expression for the boundary:

$$\begin{aligned} \partial \overline{\mathcal{M}}_\Gamma((p, 0), (q, 1)) &= \bigcup_{q' \in \text{Crit}_+^{\text{ind}(p)}(w^1)} \mathcal{M}_\Gamma((p, 0), (q', 1)) \times \mathcal{M}_\Gamma((q', 1), (q, 1)) \\ &\cup \bigcup_{p' \in \text{Crit}_+^{\text{ind}(p)+1}(w^0)} \mathcal{M}_\Gamma((p, 0), (p', 0)) \times \mathcal{M}_\Gamma((p', 0), (q, 1)). \end{aligned}$$

Thus,

$$\begin{aligned} 0 &= \sum_{q' \in \text{Crit}_+^{\text{ind}(p)}(w^1)} \#_{\mathbb{Z}_2} \mathcal{M}_\Gamma((p, 0), (q', 1)) \cdot \#_{\mathbb{Z}_2} \mathcal{M}_\Gamma((q', 1), (q, 1)) \\ &+ \sum_{p' \in \text{Crit}_+^{\text{ind}(p)+1}(w^0)} \#_{\mathbb{Z}_2} \mathcal{M}_\Gamma((p, 0), (p', 0)) \cdot \#_{\mathbb{Z}_2} \mathcal{M}_\Gamma((p', 0), (q, 1)) \end{aligned}$$

The dynamics of $\nabla_G W$ imply that flows between critical points at fixed time $t = 0$ or $t = 1$ are completely contained in $M \times \mathbb{R}^{2N} \times \{t\}$. Thus, we have the following natural identifications of the following manifolds:

$$\mathcal{M}_\Gamma((q', 1), (q, 1)) \cong \mathcal{M}_{w^1}(q', q)$$

$$\mathcal{M}_\Gamma((p, 0), (p', 0)) \cong \mathcal{M}_{w^0}(p, p').$$

So we have that

$$\begin{aligned} 0 &= \sum_{q' \in \text{Crit}_+^{\text{ind}(p)}(w^1)} \#_{\mathbb{Z}_2} \mathcal{M}_\Gamma((p, 0), (q', 1)) \cdot \#_{\mathbb{Z}_2} \mathcal{M}_{w^1}(q', q) \\ &+ \sum_{p' \in \text{Crit}_+^{\text{ind}(p)+1}(w^0)} \#_{\mathbb{Z}_2} \mathcal{M}_{w^0}(p, p') \cdot \#_{\mathbb{Z}_2} \mathcal{M}_\Gamma((p', 0), (q, 1)). \end{aligned}$$

The result that $0 = \delta^1 \circ \Phi_\Gamma + \Phi_\Gamma \circ \delta^0$ follows. \square

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\delta^0} & C^{k-1}(F^0) & \xrightarrow{\delta^0} & C^k(F^0) & \xrightarrow{\delta^0} & C^{k+1}(F^0) & \xrightarrow{\delta^0} & \cdots \\
& & \searrow^{K^{k-1}} & & \searrow^{K^k} & & \searrow^{K^{k+1}} & & \\
& & \Phi_{\widehat{\Gamma}} \downarrow \downarrow \Phi_{\Gamma} & & \Phi_{\widehat{\Gamma}} \downarrow \downarrow \Phi_{\Gamma} & & \Phi_{\widehat{\Gamma}} \downarrow \downarrow \Phi_{\Gamma} & & \\
& & \swarrow & & \swarrow & & \swarrow & & \\
\cdots & \xrightarrow{\delta^1} & C^{k-1}(F^1) & \xrightarrow{\delta^1} & C^k(F^1) & \xrightarrow{\delta^1} & C^{k+1}(F^1) & \xrightarrow{\delta^1} & \cdots
\end{array}$$

Figure 2: A cochain homotopy K between Φ_{Γ} and $\Phi_{\widehat{\Gamma}}$ is a sequence of maps that makes the above diagram commute.

The construction so far depended on the path Γ and it is necessary to show that Φ_{Γ}^* , the induced map on cohomology, does not depend on the path chosen in a given homotopy class of paths. For the current proof we will take a homotopy of the path of metrics on $M \times \mathbb{R}^{2N}$ that are standard outside $K_M \times K_E \times K_E$. Since this space is contractible through straight-line homotopies, the induced isomorphism will be canonical, showing that $GH^*(F)$ is independent of the metric chosen in the construction.

Choose another path (\widehat{g}^t) with $g^0 = \widehat{g}^0$ and $g^1 = \widehat{g}^1$ (i.e., change the path but not the endpoints), and suppose there is a generic homotopy between these two paths. We wish to say that the corresponding continuation maps are the “same” on cohomology, through the notion of a cochain homotopy (see Figure 2).

Lemma 7. *Given admissible paths Γ and $\widehat{\Gamma}$ from (w^0, g^0) to (w^1, g^1) , a fixed endpoint homotopy from the path Γ to $\widehat{\Gamma}$ induces a cochain homotopy*

$$K : C^*(F^0) \rightarrow C^{*-1}(F^1)$$

between the maps Φ_{Γ} and $\Phi_{\widehat{\Gamma}}$; see Figure 2.

Proof. The image of the path homotopy between Γ and $\widehat{\Gamma}$ traces out the shape of a digon D , a smooth two-dimensional manifold with corners consisting of two vertices, two edges in between them, and one face. The vertices correspond to fixed endpoints of the paths in the

homotopy while the edges correspond to the two homotopic paths. For every $d \in D$, the homotopy gives a pair (w^d, g^d) , where, in this case, $w^d = w$ and g^d is a metric on $M \times \mathbb{R}^{2N}$ that is standard outside the nonlinear-support compact set $K_M \times K_E \times K_E$.

Let h be a metric on D such that the edges of D have length one. Let $f : D \rightarrow \mathbb{R}$ be a nonnegative function on the digon that has an index 0 critical point at one vertex d_0 with critical value 0, an index 2 critical point at the other vertex d_1 with critical value $\frac{\epsilon}{4}$ for $\epsilon > 0$ as in the equation (3) of $\nabla_G W$, and no other critical points. Lastly suppose $\nabla_h f$ is tangent to the edges of the digon and agrees with the standard gradient of $\epsilon \left((1/2)t^2 - (1/4)t^4 \right)$ on each edge.

To get a cochain homotopy, we will consider certain flow lines of the function $W^D : (M \times \mathbb{R}^{2N}) \times D \rightarrow \mathbb{R}$ and metric G^D defined by

$$\begin{aligned} W^D(p, d) &= w^d(p) + f(d) \\ G^D_{(p,d)} &= g_p^d + h_d \end{aligned}$$

Denote the critical points of W^D with critical value greater than $\frac{\epsilon}{4}$ by $\text{Crit}_+(W^D)$. Since the only critical points of the digon occur at the vertices d_0 and d_1 ,

$$\text{Crit}_+^k(W^D) = \left(\text{Crit}_+^k(w^0) \times \{d_0\} \right) \cup \left(\text{Crit}_+^{k-2}(w^1) \times \{d_1\} \right).$$

The homotopy is admissible if the stable and unstable manifolds of $\nabla_{G^D} W^D$ have a transverse intersection. Given an admissible homotopy, the space $\mathcal{M}_D((p, d_0), (q, d_1))$ of gradient flow lines modulo reparametrization is a manifold of dimension

$$\text{ind}_{W^D}(q, 1) - \text{ind}_{W^D}(p, 0) - 1 = \text{ind}_{w^1}(q) + 2 - \text{ind}_{w^0}(p) + 1 = \text{ind}_{w^1}(q) - \text{ind}_{w^0}(p) + 1.$$

Thus, we may define a map $K_D : C^*(F^0) \rightarrow C^{*-1}(F^1)$ by counting isolated flow lines of $\nabla_{G^D} W^D$:

$$K_D(p) = \sum_{q \in C^{|p|-1}(F^1)} \#_{\mathbb{Z}_2} \mathcal{M}_D((p, d_0), (q, d_1)) \cdot q.$$

To show that K_D gives a cochain homotopy between Φ_Γ and $\Phi_{\widehat{\Gamma}}$, we use the usual argument that if $\mathcal{M}_D((p, 0), (q, 1))$ is one-dimensional, then it has a compactification to a compact one-dimensional manifold with boundary. The boundary contains the usual once-broken flow lines and has additional flow lines from the boundary of the digon, ∂D .

$$\begin{aligned} \partial \overline{\mathcal{M}_D}((p, 0), (q, 1)) &= \bigcup_{q' \in \text{Crit}_+^{\text{ind}(p)-1}(w^1)} \mathcal{M}((p, 0), (q', 1)) \times \mathcal{M}((q', 1), (q, 1)) \\ &\cup \bigcup_{p' \in \text{Crit}_+^{\text{ind}(p)+1}(w^0)} \mathcal{M}((p, 0), (p', 0)) \times \mathcal{M}((p', 0), (q, 1)) \\ &\cup \mathcal{M}_\Gamma((p, 0), (q, 1)) \cup \mathcal{M}_{\widehat{\Gamma}}((p, 0), (q, 1)) \end{aligned}$$

Thus, $K_D \delta^0 + \delta^1 K_D = \Phi_\Gamma - \Phi_{\widehat{\Gamma}}$.

□

Lemma 8. *Given admissible paths Γ_1, Γ_2 with $\Gamma_1(1) = \Gamma_2(0)$, there is a concatenation cochain homotopy between $\Phi_{\Gamma_2} \circ \Phi_{\Gamma_1}$ and $\Phi_{\Gamma_2 * \Gamma_1}$.*

Proof. This proof is similar to the proof of Lemma 7. An admissible homotopy between Γ_1 followed by Γ_2 with their concatenation $\Gamma_2 * \Gamma_1$ may be represented by a triangle T with vertices r_i representing the pair (w^i, g^i) for $i \in \{0, 1, 2\}$. For every $r \in T$, this homotopy gives a pair (w^r, g^r) with $w^r = w$ and g^r a metric that is standard outside $K_M \times K_E \times K_E$.

Equip T with a metric h that gives each edge of T length one. Let $f : T \rightarrow \mathbb{R}$ be a

nonnegative function with an index i critical point at vertex r_i with critical value $i\frac{\epsilon}{4}$ and no other critical points. Lastly suppose $\nabla_h f$ is tangent to the edges of the triangle and agrees with the standard gradient of $\epsilon((1/2)t^2 - (1/4)t^4)$ on each edge.

The remainder of the proof follows as in Lemma 7 by analyzing spaces of flow lines from (p, r_0) to (q, r_2) for $p \in \text{Crit}_+(w^0)$ and $q \in \text{Crit}_+(w^2)$. \square

Lastly, since concatenating a path with its reverse is homotopic to the constant path, we need:

Lemma 9. *Given a constant path $\Gamma = (w^t, g^t)$ with $w^t = w$ and $g^t = g \in \mathcal{G}_F$ for $t \in [0, 1]$, $\Phi_\Gamma = id_{C(F)}$.*

Proof. The fact that every point of Γ is a Morse-Smale pair makes this case different than just fixing the path of functions. Given $p \in \text{Crit}_+(w)$, we claim there is an isolated flow line from $(p, 0)$ to $(p, 1)$ along $\nabla_G W$, for W, G as in 3. In fact, for all $t \in [0, 1]$,

$$\nabla_G W(p, t) = (0, \nabla (\epsilon((1/2)t^2 - (1/4)t^4))),$$

and the result follows. \square

Lemma 10. *For an admissible path Γ , the map Φ_Γ induces an isomorphism $GH^*(F^0) \rightarrow GH^*(F^1)$.*

\square

3.3 Invariance of $GH^*(F)$ with respect to stabilization and fiber-preserving diffeomorphism

We show that the generating family (co)homology descends to equivalence classes of generating families, as defined in Section 2.2.

Proposition 7. *If $F_0 \sim F_1$, then $GH^*(F_0) \simeq GH^*(F_1)$.*

This follows from Lemmas 11 and 12.

Lemma 11. *If $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ is altered by a positive or negative stabilization resulting in $\widehat{F} : M \times \mathbb{R}^N \times \mathbb{R}^K \rightarrow \mathbb{R}$ then $GH^*(\widehat{F}) \cong GH^*(F)$.*

Proof. Given a generating family $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$, define $F^\pm : M \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ where $F^\pm(x, e, e') = F(x, e) \pm (e')^2$. It suffices to show $GH^*(F^\pm) \cong GH^*(F)$.

We will denote and express the difference functions from F^\pm by

$$\begin{aligned} w^\pm : M \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} &\rightarrow \mathbb{R} \\ (x, e_1, e'_1, e_2, e'_2) &\mapsto F^\pm(x, e_1, e'_1) - F^\pm(x, e_2, e'_2) \\ &= F(x, e_1) \pm (e'_1)^2 - F(x, e_2) \mp (e'_2)^2 \end{aligned}$$

Given $p = (x, e_1, e_2) \in \text{Crit}_+(w)$ there is a corresponding critical point $p' = (x, e_1, 0, e_2, 0) \in \text{Crit}_+(w^\pm)$ with the same critical value and $\text{ind}_{w^\pm}(p') = \text{ind}_w(p) + 1$. This gives a bijection between the generators of $C(F)$ and $C(F^\pm)$ and this bijection preserves grading: $|p'| = \text{ind}_{w^\pm}(p') - (N + 1) = \text{ind}_w(p) - N = |p|$. This is precisely why the grading on $C(F)$ depends on the dimension of the fiber of F .

We claim that there is also a correspondence of gradient flow lines, but to show this we must choose a metric from \mathcal{G}_{F^\pm} . We claim that, if $g \in \mathcal{G}_F$, then $g' = g + g_0 \in \mathcal{G}_{F^\pm}$, where

g_0 is the standard Euclidean metric on the two extra \mathbb{R} coordinates of $M \times \mathbb{R}^{2(N+1)}$. The only condition from Definition 4 that is not immediate is the Smale condition (4).

To check the Smale condition for (w^\pm, g') , we first show that the gradient flow of this pair splits. We may write the stabilized difference function as

$$w^\pm(x, e_1, e'_1, e_2, e'_2) = w(x, e_1, e_2) \pm Q^\pm(e'_1, e'_2),$$

for $Q^\pm : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $Q^\pm(e'_1, e'_2) = \pm(e'_1)^2 \mp (e'_2)^2$, so

$$d(w^\pm)_{(x, e_1, e'_1, e_2, e'_2)} = dw_{(x, e_1, e_2)} + dQ^\pm_{(e'_1, e'_2)}.$$

We then claim that $\nabla_{g'} w^\pm = (\nabla_g w, \nabla_{g_0} Q^\pm)$; the details of a similar proof may be found in Lemma 15. In particular, the unstable and stable manifolds split, and since $g \in \mathcal{G}_F$, this reduces to checking the Smale condition for (Q^\pm, g_0) . But the only critical point of Q^\pm is $0 = (0, 0)$, and the only point in $T_0 W_0^-(Q^\pm) \cap T_0 W_0^+(Q^\pm)$ is 0 , and the result holds; see Prop 11 for a similar argument with more details.

To show that $GH^*(F) \cong GH^*(F^\pm)$, we show that, with the metric chosen, $\mathcal{M}(p, q) \cong \mathcal{M}(p', q')$, where $p', q' \in \text{Crit}_+(w^\pm)$ are the images of $p, q \in \text{Crit}_+(w)$ under the bijection described earlier in this proof. Since we showed in the previous subsection that the construction of $GH^*(F^\pm)$ does not depend on the metric chosen from \mathcal{G}_{F^\pm} , the result will follow. In fact, since $\mathcal{M}(p', q') \cong (W_{p'}^-(w^\pm) \cap W_{q'}^+(w^\pm)) / \mathbb{R}$, the fact that $T_0 W_0^-(Q^\pm) \cap T_0 W_0^+(Q^\pm) = \{(0, 0)\}$ gives a diffeomorphism. \square

Lemma 12. *If $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ is altered by fiber-preserving diffeomorphism that is an isometry outside $K_M \times K_E \times K_E$ resulting in $\widehat{F} : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ then $GH^*(\widehat{F}) \cong GH^*(F)$.*

This result will follow from:

Lemma 13. *Suppose g is a Riemannian metric on X , $f : X \rightarrow \mathbb{R}$ and $\Phi : X \rightarrow \widehat{X}$ is a diffeomorphism. If V is the gradient vector field of f with respect to g , then Φ_*V is the gradient vector field of $(\Phi^{-1})^*f$ with respect to the pullback metric $(\Phi^{-1})^*g$.*

Proof. Given that the vector field V is such that for all $x \in X$, $g_x(V_x, u) = df_x(u)$ for all $u \in T_xX$, we wish to show that the vector field Φ_*V satisfies $((\Phi^{-1})^*g)_{\widehat{x}}((\Phi_*V)_{\widehat{x}}, \widehat{u}) = d(f \circ \Phi^{-1})_{\widehat{x}}(\widehat{u})$ for all $\widehat{x} \in \widehat{X}$ and $\widehat{u} \in T_{\widehat{x}}\widehat{X}$.

Let $\widehat{x} \in \widehat{X}$ and $\widehat{u} \in T_{\widehat{x}}\widehat{X}$. Since Φ is a diffeomorphism, $\widehat{x} = \Phi(x)$ for some $x \in X$ and Φ_* gives an isomorphism between T_xX and $T_{\widehat{x}}\widehat{X}$, so $\widehat{u} = \Phi_*u$ for some $u \in T_xX$. Thus

$$\begin{aligned}
& ((\Phi^{-1})^*g)_{\widehat{x}}((\Phi_*V)_{\widehat{x}}, \widehat{u}) = \\
& ((\Phi^{-1})^*g)_{\Phi(x)}((\Phi_*V)_{\Phi(x)}, \Phi_*u) = \\
& g_x(\Phi_*^{-1}((\Phi_*V)_x), \Phi_*^{-1}(\Phi_*u)) = \\
& g_x(V_x, u) = df_x(u) = \\
& df_x \circ d\Phi_{\Phi(x)}^{-1}(\Phi_*u) = \\
& d(f \circ \Phi^{-1})_{\Phi(x)}(\Phi_*u) = \\
& d(f \circ \Phi^{-1})_{\Phi(x)}(\widehat{u}),
\end{aligned}$$

as desired. □

Lemma 13 will give bijections of trajectories on the chain level that shows $GH^*(F) = GH^*(\widehat{F})$ as long as $(\Phi^{-1})^*g \in \mathcal{G}_{\widehat{F}}$. Since Φ is an isometry outside $K_M \times K_E \times K_E$, if g is Euclidean outside this set, so is $(\Phi^{-1})^*g$. Lemma 13 induces a diffeomorphism between the stable and unstable manifolds from the flows of before and after the fiber-preserving diffeomorphism, and the Smale condition holds.

Given a Legendrian submanifold $\Lambda \subset J^1M$, let:

$$\mathcal{F}^{\text{lin}}(\Lambda) = \{F : F \text{ is a linear-at-infinity generating family for } \Lambda\}.$$

On the level of equivalence, we will be interested in equivalence classes of generating families that contain linear-at-infinity representatives.

When Λ is a Legendrian unknot in the standard contact \mathbb{R}^3 with maximal Thurston-Bennequin invariant, all elements of $\mathcal{F}^{\text{lin}}(\Lambda)$ are equivalent; see [21].* In general, the set $\mathcal{F}^{\text{lin}}(\Lambda)$ is not well understood, though see [5, 12, 17] for some results.

3.4 $GH^*(F)$ as a Legendrian Invariant

To form an invariant of a Legendrian submanifold Λ with a generating family, it is important to know that the existence of a linear-at-infinity generating family persists under Legendrian isotopy. A proof of the following proposition can be given using Chekanov’s “composition formula” [3]; see, for example, [21].

Proposition 8 (Persistence of Legendrian Generating Families). *Suppose M is compact. For $t \in [0, 1]$, let $\Lambda^t \subset J^1M$ be an isotopy of Legendrian submanifolds. If Λ^0 has a linear-at-infinity generating family F , then there exists a smooth path of generating families $F^t : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ for Λ^t so that F^0 is a stabilization of F and $F^t = F^0$ outside a compact set.*

Remark 6. We will often be considering generating families for compact Legendrians in $J^1(\mathbb{R}^n)$. The above persistence will still apply since these Legendrians can be thought of as living in J^1S^n , and the linear-at-infinity condition allows the generating families to be defined on $S^n \times \mathbb{R}^N$.

*In [21], the focus was on generating families that are linear-quadratic-at-infinity. Lemma 1, however, can be used to show that linear-quadratic-at-infinity functions are equivalent to linear-at-infinity ones.

Proposition 9. *If $\Lambda^t \subset J^1M$ is an isotopy of Legendrian submanifolds for $t \in [0, 1]$, then for the path $F^t \in \mathcal{F}^{\text{lin}}(\Lambda)$ as in Proposition 8 there exists an isomorphism $\Phi^* : GH^*(F^0) \rightarrow GH^*(F^1)$.*

Proof. These isomorphisms may be constructed using a continuation argument as in Proposition 6. Given a contact isotopy and generating family, let $F^t : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a smooth path of generating families as in Proposition 8. Given $g^0 \in \mathcal{G}_{F^0}$ and $g^1 \in \mathcal{G}_{F^1}$, construct a path of metrics g^t for $t \in [0, 1]$ on $M \times \mathbb{R}^{2N}$ so that g^t is standard outside the nonlinear support compact set $K_M^t \times K_E^t \times K_E^t$. These sets vary smoothly due to the smoothness of the path F^t . The rest of the proof proceeds as in Proposition 6. □

The above proof gives an isomorphism between $GH^*(F^0)$ and $GH^*(F^1)$ that arise as a lifted path of generating families from a Legendrian isotopy. This isomorphism is independent for paths in the homotopy class of the given path F^t , but given any two generating families of isotopic Legendrians, there need not be a path between them[†].

In other words, since it may not be the case that all elements in $\mathcal{F}^{\text{lin}}(\Lambda)$ are equivalent, the generating family homology of a linear-at-infinity generating family F , is not itself an invariant of the generated Legendrian Λ . By Corollary 9, however, we do have:

Proposition 10 ([21, 35]). *For a compact Legendrian submanifold $\Lambda \subset J^1M$, the set of all generating family cohomology groups*

$$\mathcal{GH}^k(\Lambda) = \{GH^k([F]) : F \in \mathcal{F}^{\text{lin}}(\Lambda)\},$$

is invariant under Legendrian isotopy.

[†]See [29] for some results on homotopy spaces of generating families for Legendrian submanifolds.

Chapter 4

Extended Difference Functions

As seen in the previous chapter, gradient flow lines from a single difference function are used to construct generating family cohomology groups. To form gradient flow trees from a generating family, we will need intersecting gradient trajectories, so we use Sabloff's idea of using multiple "extended difference functions," sketched by Henry and Rutherford in [18]. In this chapter, we define these functions and corresponding metrics which will give an identification of gradient flow lines of these spaces with those of the original difference functions w .

Definition 7. Suppose $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a generating family for Λ . Let $P_3 = M \times \mathbb{R}^{3N}$. For each $1 \leq i < j \leq 3$ and $k \in \{1, 2, 3\} - \{i, j\}$, the **extended difference functions** $w_{i,j;3} : P_3 \rightarrow \mathbb{R}$ are defined as

$$w_{i,j;3}(x, e_1, e_2, e_3) = F(x, e_i) - F(x, e_j) + \begin{cases} e_k^2, & k < i \text{ or } k > j \\ -e_k^2, & i < k < j \end{cases}.$$

The set of positively-valued critical points of $w_{i,j;3}$ will be denoted by $\text{Crit}_+(w_{i,j;3}) \subset P_3$.

Remark 7. 1. The number 3 in the notation of the extended difference functions $w_{i,j;3}$ is a bit superfluous at this stage, but it will be useful in the future to have generalizable notation, see Chapter 9.

2. Each e_i is an N -dimensional vector, i.e., $e_i = (e_{i1}, e_{i2}, \dots, e_{iN})$ and $e_i^2 := \|e_i\|^2 = e_{i1}^2 + \dots + e_{iN}^2$. This is especially important to remember in dimension calculations.

3. The three extended difference functions on P_3 are

$$w_{1,2;3}(x, e_1, e_2, e_3) = F(x, e_1) - F(x, e_2) + e_3^2,$$

$$w_{2,3;3}(x, e_1, e_2, e_3) = F(x, e_2) - F(x, e_3) + e_1^2,$$

$$w_{1,3;3}(x, e_1, e_2, e_3) = F(x, e_1) - F(x, e_3) - e_2^2.$$

Each extended difference function may be written in the following form (see Definition 7):

$$w_{i,j;3}(x, e_1, e_2, e_3) = w(x, e_i, e_j) + Q(e_k),$$

where $k \in \{1, 2, 3\}$ is such that $k \neq i, j$ and $Q : \mathbb{R}^N \rightarrow \mathbb{R}$ is the quadratic form

$$Q(e_k) = Q(e_{k1}, \dots, e_{kN}) = \pm \left(\sum_{\ell=1}^N e_{k\ell}^2 \right).$$

Fix a point $p = (x, e_1, e_2, e_3) \in P_3$. Then for any combination of $\{i, j, k\} = \{1, 2, 3\}$,

$$T_p P_3 = T_{(x, e_i, e_j)}(M \times \mathbb{R}^N \times \mathbb{R}^N) \times T_{e_k} \mathbb{R}^N$$

and we have that

$$d(w_{i,j;3})_p = dw_{(x, e_i, e_j)} + dQ_{e_k}.$$

In fact, we can consider a larger class of extended difference functions of the form

$$w_{i,j;3}(x, e_1, e_2, e_3) = w(x, e_i, e_j) + \tilde{Q}_{i,j;3}(e_k),$$

where $\tilde{Q}_{i,j;3} : \mathbb{R}^N \rightarrow \mathbb{R}$ is a function with exactly one critical point $0_{\tilde{Q}}$ with value 0 of index 0 for $(i, j) = (1, 2), (2, 3)$ and index N for $(i, j) = (1, 3)$. We would also want $\tilde{Q}_{i,j;3}(e_k) = \pm e_k^2$ outside a compact set (with the sign corresponding to the sign of the quadratic it is generalizing). We will see in Section 7.2 the benefit of generalizing the extended difference functions in this way. For now, we use the notation in Definition 7 and will often refer to $0_{\tilde{Q}}$ as 0.

Even though we will be working with multiple functions, we will form the product on $C(F)$ as in Definition 3. While the sets of positively-valued critical points of the extended difference functions $w_{i,j;3}$ are different, there is a natural way to identify them each with positively-valued critical points of the original difference function w .

Lemma 14. *For $1 \leq i < j \leq 3$, there are bijections:*

$$\iota_{i,j;3} : \text{Crit}_+(w) \rightarrow \text{Crit}_+(w_{i,j;3})$$

which preserve critical value. In addition, we have the following index relation:

$$|p| = \text{ind } w(p) - N = \text{ind } w_{i,j;3}(\iota_{i,j;3}(p)) - (j - i)N. \quad (4)$$

Proof. The bijections are defined as follows:

$$\begin{aligned}
\iota_{1,2;3} &: \text{Crit}_+(w) \rightarrow \text{Crit}_+(w_{1,2;3}), \\
&(x, e, e') \mapsto (x, e, e', 0) \\
\iota_{2,3;3} &: \text{Crit}_+(w) \rightarrow \text{Crit}_+(w_{2,3;3}), \\
&(x, e, e') \mapsto (x, 0, e, e') \\
\iota_{1,3;3} &: \text{Crit}_+(w) \rightarrow \text{Crit}_+(w_{1,3;3}), \\
&(x, e, e') \mapsto (x, e, 0, e').
\end{aligned} \tag{5}$$

If (x, e, e') is a generator of $C^\ell(F)$ then $(x, e, e') \in \text{Crit}_+ w$ with Morse index $\ell + N$. From the definition of the extended difference functions in Definition 7, we see immediately that $\iota_{i,j;3}(x, e, e') \in \text{Crit}_+ w_{i,j;3}$. The index of $\iota_{1,2;3}(x, e, e')$ and $\iota_{2,3;3}(x, e, e')$ remains $\ell + N$, while there are N extra subtracted quadratic terms in the extended difference function $w_{1,3;3}$ so $\iota_{1,3;3}(x, e, e')$ has index $\ell + N + N = \ell + 2N$. Since we have added or subtracted terms that are just $0^2 = 0$, the critical values will not change. \square

Remark 8. Since every critical point p of w of positive critical value will correspond to a Reeb chord of the Legendrian Λ generated by F , the same is true of critical points $\iota_{i,j;3}(p)$ of $w_{i,j;3}$. The positive critical value of a critical point p (resp. $\iota_{i,j;3}(p)$) will be the length of the corresponding Reeb chord. By an abuse of notation, we will often use p to denote both p and $\iota_{i,j;3}(p)$.

Definition 8. By Definition 1, if $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a linear-at-infinity generating family, then we may write $F(x, e) = F^c(x, e) + A(e)$ where $F^c : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ is compactly supported on $K_M \times K_E \subseteq M \times \mathbb{R}^N$. We assume that $0 \in K_E$; if not, enlarge the compact set so that

this is true. We call the compact set

$$K := K_M \times K_E \times K_E \times K_E \subset P_3 \quad (6)$$

the **non-linear support** of $w_{i,j;3}$.

Remark 9. Suppose that $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a linear-at-infinity generating family that agrees with a non-zero linear function outside $K_M \times K_E$, for compact sets $K_M \subset M$ and $K_E \subset \mathbb{R}^N$. Then every critical point of an extended difference function $w_{i,j;3} : M \times \mathbb{R}^{3N} \rightarrow \mathbb{R}$ is of the form (x, e_1, e_2, e_3) where $x \in K_M$, $e_i, e_j \in K_E$, and $e_\ell = 0$ for $\ell \neq i, j$. Thus, every point in $\text{Crit}_+(w_{i,j;3})$ is contained in the set K ; see Lemma 3.

As in Remark 7, if we use the more general form of the extended difference functions, we choose K so that $0_{\tilde{Q}} \in K_E$.

Definition 9. Similarly, given $Q : \mathbb{R}^N \rightarrow \mathbb{R}$ as in Remark 3 after Definition 7, let \mathcal{G}_Q denote the set of Riemannian metrics g_Q on \mathbb{R}^N such that g_Q is the standard Euclidean metric outside K_E and in a neighborhood U of the origin 0 (the only critical point of Q).

Given $g_w \in \mathcal{G}_F$ and $g_Q \in \mathcal{G}_Q$, we define the following three “split” metrics $g_{i,j;3}$ pointwise on P_3 :

$$\begin{aligned} (g_{1,2;3})_{(x,e_1,e_2,e_3)} &= (g_w)_{(x,e_1,e_2)} + (g_Q)_{e_3} \\ (g_{2,3;3})_{(x,e_1,e_2,e_3)} &= (g_w)_{(x,e_2,e_3)} + (g_Q)_{e_1} \\ (g_{1,3;3})_{(x,e_1,e_2,e_3)} &= (g_w)_{(x,e_1,e_3)} + (g_Q)_{e_2}. \end{aligned}$$

The metrics $g_{i,j;3}$ from Definition 9 produce gradient vector fields of the extended difference functions that we may express in terms of gradient vector fields of the original difference function, w . To see this, first note that each extended difference function may be

written in the following form (see Definition 7):

$$w_{i,j;3}(x, e_1, e_2, e_3) = w(x, e_i, e_j) + Q(e_k),$$

where $k \in \{1, 2, 3\}$ is such that $k \neq i, j$ and $Q : \mathbb{R}^N \rightarrow \mathbb{R}$ is the quadratic form

$$Q(e_k) = Q(e_{k1}, \dots, e_{kN}) = \pm \left(\sum_{\ell=1}^N e_{k\ell}^2 \right).$$

Fix a point $p = (x, e_1, e_2, e_3) \in P_3$. Then for any combination of $\{i, j, k\} = \{1, 2, 3\}$,

$$T_p P_3 = T_{(x, e_i, e_j)}(M \times \mathbb{R}^N \times \mathbb{R}^N) \times T_{e_k} \mathbb{R}^N$$

and we have that

$$d(w_{i,j;3})_p = dw_{(x, e_i, e_j)} + dQ_{e_k}.$$

Putting this all together, we have the following Lemma:

Lemma 15. *Given $g_w \in \mathcal{G}_F$ and $g_Q \in \mathcal{G}_Q$, let $g_{i,j;3}$ as in Definition 9. Then, up to a reordering of coordinates, we have the following split of gradient vector fields:*

$$\nabla_{g_{i,j;3}} w_{i,j;3} = (\nabla_{g_w} w, \nabla_{g_Q} Q).$$

Proof. Fix $g_w \in \mathcal{G}_F$ and $g_Q \in \mathcal{G}_Q$. By definition, $\nabla w_{i,j;3} = \nabla_{g_{i,j;3}} w_{i,j;3}$ is a vector field so that for all $p = (x, e_1, e_2, e_3) \in P_3$, $g_p(\nabla w_{i,j;3}, \cdot) = d(w_{i,j;3})_p(\cdot) = \left(dw_{(x, e_i, e_j)} + dQ_{e_k} \right) (\cdot)$.

Let $v = (v_w, v_e) \in T_p P_3 = T_{(x, e_i, e_j)}(M \times \mathbb{R}^N \times \mathbb{R}^N) \times T_{e_k} \mathbb{R}^N$. We check:

$$\begin{aligned}
& g_p \left((\nabla_{g_w} w, \nabla_{g_Q} Q), (v_w, v_e) \right) \\
&= (g_w)_{(x, e_i, e_j)}(\nabla w, v_w) + (g_Q)_{e_k}(\nabla Q, v_e) \\
&= dw_{(x, e_i, e_j)}(v_w) + dQ_{e_k}(v_e) \\
&= d_p(w_{i,j;3})(v_w, v_e).
\end{aligned}$$

which, due the positive definiteness of metrics and the resulting uniqueness of gradient vector fields, implies the result. \square

The preceding Lemma shows why we chose metrics as in Definition 9. We must check, however, that such a choice of metric yields Morse-Smale pairs.

Proposition 11. *Given $g_w \in \mathcal{G}_F$ and $g_Q \in \mathcal{G}_Q$, each $(w_{i,j;3}, g_{i,j;3})$ satisfies the Smale condition on positively-valued critical points: That is, for every pair of critical points $p, q \in \text{Crit}_+ w_{i,j;3}$, the unstable and stable manifolds of p and q have a transverse intersection.*

Proof. Fix $p' = \iota_{i,j;3}(p)$ and $q' = \iota_{i,j;3}(q)$ for $p, q \in C(F)$, and suppose $a = (x, e_1, e_2, e_3) \in W_{p'}^-(w_{i,j;3}) \cap W_{q'}^+(w_{i,j;3})$.

Lemma 15 implies that the flow Ψ of $\nabla_{g_{i,j;3}} w_{i,j;3}$ on P_3 may be expressed as $\Psi = (\Psi_w, \Psi_Q)$, where Ψ_w is the flow of $\nabla_{g_w} w$ and Ψ_Q is the flow of $\nabla_{g_Q} Q$. Thus, $W_{p'}^-(w_{i,j;3}) = W_p^-(w) \times W_0^-(Q)$ and $W_{q'}^+(w_{i,j;3}) = W_q^+(w) \times W_0^+(Q)$.

Thus, we have that

$$\begin{aligned}
& T_a W_{p'}^-(w_{i,j;3}) + T_a W_{q'}^+(w_{i,j;3}) \\
&= T_a(W_p^-(w) \times W_0^-(Q)) + T_a(W_q^+(w) \times W_0^+(Q)) \\
&= \left(T_{(x,e_i,e_j)} W_p^-(w) \times T_{e_k} W_0^-(Q) \right) + \left(T_{(x,e_i,e_j)} W_q^+(w) \times T_{e_k} W_0^+(Q) \right) \\
&= \left(T_{(x,e_i,e_j)} W_p^-(w) + T_{(x,e_i,e_j)} W_q^+(w) \right) \times (T_{e_k} W_0^-(Q) + T_{e_k} W_0^+(Q)) \\
&= T_{(x,e_i,e_j)}(M \times \mathbb{R}^N \times \mathbb{R}^N) \times (T_{e_k} W_0^-(Q) + T_{e_k} W_0^+(Q)),
\end{aligned}$$

where the first term in the final equivalence is our assumption that (w, g) satisfies the Smale condition on $\text{Crit}_+(w)$. For the second term, note that $e_k \in W_0^-(Q) \cap W_0^+(Q)$ implies that $e_k = 0$. Since $T_0 W_0^-(Q) + T_0 W_0^+(Q) = T_0 \mathbb{R}^N = \mathbb{R}^N$, we have that

$$T_a W_{p'}^-(w_{i,j;3}) + T_a W_{q'}^+(w_{i,j;3}) = T_{(x,e_i,e_j)}(M \times \mathbb{R}^N \times \mathbb{R}^N) \times T_{e_k} \mathbb{R}^N = T_a P_3,$$

as desired. □

We can now define trajectory spaces of the extended difference functions.

Definition 10. For $p_-, p_+ \in \text{Crit}_+(w)$, the **unbroken infinite Morse trajectory spaces** between p_- and p_+ is

$$\begin{aligned}
\mathcal{M}_{i,j;3}(p_-, p_+) &:= \{ \gamma : (-\infty, \infty) \rightarrow P_3 \mid \dot{\gamma} = \nabla_{g_{i,j;3}} w_{i,j;3}, \\
&\quad \lim_{t \rightarrow -\infty} \gamma(t) = \iota_{i,j;3}(p_-), \lim_{t \rightarrow \infty} \gamma(t) = \iota_{i,j;3}(p_+) \} / \mathbb{R},
\end{aligned}$$

where $/\mathbb{R}$ denotes quotienting by the action of \mathbb{R} that takes $\gamma(t)$ to $\gamma(t+a)$ for $a \in \mathbb{R}$.

Given the correspondence of the (positively-valued) critical points of w and the extended difference functions $w_{i,j;3}$, we would like there to also be a correspondence of gradient flow

lines. This is where we see the benefit of choosing our metrics $g_{i,j;3}$ to be “split” as in Definition 9.

Proposition 12. *For appropriate choice of metrics, there are bijections*

$$\mathcal{M}(p, q) \leftrightarrow \mathcal{M}_{i,j;3}(p, q)$$

for each $1 \leq i < j \leq 3$.

Proof. Express $\gamma : \mathbb{R} \rightarrow M \times \mathbb{R}^N \times \mathbb{R}^N \in \mathcal{M}(p, q)$ as $\gamma(t) = (a(t), b_1(t), b_2(t))$ for $a : \mathbb{R} \rightarrow M$ and $b_1, b_2 : \mathbb{R} \rightarrow \mathbb{R}^N$. Define paths $\gamma_{i,j;3} : \mathbb{R} \rightarrow P_3$ by

$$\gamma_{1,2;3}(t) = (a(t), b_1(t), b_2(t), 0)$$

$$\gamma_{2,3;3}(t) = (a(t), 0, b_1(t), b_2(t))$$

$$\gamma_{1,3;3}(t) = (a(t), b_1(t), 0, b_2(t)).$$

We claim that $\gamma_{i,j;3} \in \mathcal{M}_{i,j;3}(p, q)$ and that this identification defines a bijection (that is, up to reparametrization, all trajectories in $\mathcal{M}_{i,j;3}(p, q)$ are of this form).

Lemma 15 implies that gradient trajectories of $w_{i,j;3}$ may be written in terms of a gradient trajectory of w and one of $Q : \mathbb{R}^N \rightarrow \mathbb{R}$. Since $Q(e_k) = \pm \left(\sum_{\ell=1}^N e_{k\ell}^2 \right)$, 0 is its only critical point, and hence the constant trajectory at 0 is the only gradient trajectory of Q . □

Chapter 5

Moduli Space of Gradient Flow Trees

We will study positive gradient flow lines of the extended difference functions $w_{i,j;3}$ with respect to metrics as in Definition 9. Gradient flow lines are well-studied objects in Morse Theory, and we will work with moduli spaces of intersecting flow lines, which we will refer to as **gradient flow trees**. Understanding the structure of these spaces will play an integral role in defining our product. In particular, to define products with correct properties, we will need that our moduli spaces are smooth manifolds with certain compactification properties.

A gradient flow tree is made of three half-infinite gradient trajectories, one from each extended difference function, that limit to critical points at their infinite ends and intersect at their finite ends. To achieve transversality of this intersection, we consider trees that “almost” intersect at their finite ends, up to a small fixed vector at each finite end.

To compactify the space of flow trees, we use results from [37] that give a smooth manifold with corners structure to spaces of broken, half-infinite gradient trajectories of a Morse-Smale pair (f, g) , where f is a function on closed manifold. There are differences in

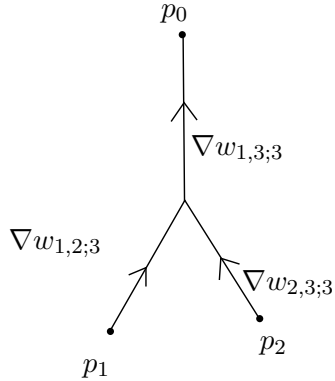


Figure 3: A gradient flow tree with three intersecting half-infinite trajectories.

our setup: Our functions are Morse-Bott rather than Morse and defined on a noncompact space. However, the positive-valued critical points are isolated and contained in a compact set. This set is not boundaryless but we show that the trajectories in our trees are contained in an open subset of this compact set and hence do not approach the boundary. Thus, the space of flow trees sits in a larger, compact space of flow trees with broken branches.

To be able to prove this, we build a few choices into our construction, explained in the following remark:

Remark 10. Lemma 9 implies that there are only a finite number of critical points with positive critical value since such points are isolated. Thus, we know that there exists a smallest positive critical value

$$\rho := \rho_F = \min\{w(p) \mid p \in \text{Crit}_+(w)\}. \quad (7)$$

To prove certain results in Chapter 5 (see Lemma 17), we need to use this fact to build a couple of choices into our construction:

1. We shrink the fiber coordinates in the following way: We apply a fiber-preserving diffeomorphism to P_3 that is the identity outside of K and so that every point $y =$

$(x, e_1, e_2, e_3) \in K \subset P_3$ is such that $e_1^2 + e_2^2 + e_3^2 < \rho$. In Chapter 7 we will see that the product is invariant under fiber preserving diffeomorphism, so this choice will not affect the outcome.

2. By Lemma 9, each $w_{i,j;3}|_K$ is uniformly continuous. In particular, for ρ as above, there exists $\delta = \min\{\delta_{1,2;3}, \delta_{2,3;3}, \delta_{1,3;3}\} > 0$ such that for all $y_1, y_2 \in K$, $|y_1 - y_2| < \delta_{i,j;3}$ implies that $|w_{i,j;3}(y_1) - w_{i,j;3}(y_2)| < \frac{\rho}{4}$.

5.1 Unbroken flow trees

Definition 11. The **unbroken half-infinite Morse trajectory spaces** to/from a critical point $p \in \text{Crit}_+(w)$ are defined as:

$$\mathcal{M}_{i,j;3}(P_3, p) := \{\gamma : [0, \infty) \rightarrow P_3 \mid \dot{\gamma} = \nabla w_{i,j;3}, \lim_{t \rightarrow \infty} \gamma(t) = \iota_{i,j;3}(p)\} \text{ and}$$

$$\mathcal{M}_{i,j;3}(p, P_3) := \{\gamma : (-\infty, 0] \rightarrow P_3 \mid \dot{\gamma} = \nabla w_{i,j;3}, \lim_{t \rightarrow -\infty} \gamma(t) = \iota_{i,i;3}(p)\}.$$

Remark 11. 1. The sets in Definition 11 inherit smooth structures from unstable and stable manifolds:

$$\mathcal{M}_{i,j;3}(P_3, p) \cong W_p^+(w_{i,j;3}),$$

$$\mathcal{M}_{i,j;3}(p, P_3) \cong W_p^-(w_{i,j;3}).$$

2. In contrast to the infinite gradient trajectory spaces in the previous chapters, quotienting by reparametrization is not needed for half-infinite trajectory spaces because the image of the trajectory changes under reparametrization.

Definition 12. Define **evaluation maps** on the half-infinite trajectory spaces to record

the finite endpoint:

$$\text{ev}_{i,j;3}^- : \mathcal{M}_{i,j;3}(P_3, p) \rightarrow P_3, \quad \text{ev}_{i,j;3}^+ : \mathcal{M}_{i,j;3}(p, P_3) \rightarrow P_3$$

are given by $\text{ev}_{i,j;3}^-(\gamma) := \gamma(0)$ and $\text{ev}_{i,j;3}^+(\gamma) := \gamma(0)$.

Next, we define the perturbation ball S and maps $E_{i,j;3}$ that perturb the evaluation at endpoints map by vectors in S . The perturbation ball S and the subsequent maps are defined in slightly different ways depending on whether the manifold M in $P_3 = M \times \mathbb{R}^{3N}$ is a Euclidean space or a closed manifold.

Definition 13. If $M = \mathbb{R}^n$, define the **perturbation ball** $S \subset P_3$ to be an open ϵ -ball centered at 0 in $P_3 = \mathbb{R}^{n+3N}$. We will denote such a ball as $B^{n+3N}(\epsilon)$ or just $B(\epsilon)$ if the dimension is clear.

If M is a closed manifold, then we know $M \subset \mathbb{R}^m$ for some $m \in \mathbb{N}$. Every $B^m(\epsilon)$ defines a space $M^\epsilon \subset \mathbb{R}^m$, the open set of points in \mathbb{R}^m of distance less than ϵ to M . By the ϵ -Neighborhood Theorem (see, for example [16]), if ϵ is small enough, there is a well defined submersion $\pi_M : M^\epsilon \rightarrow M$ that takes a point in M^ϵ to the unique closest point in M and is the identity when restricted to M . We can extend this map to get a submersion $\pi : M^\epsilon \times \mathbb{R}^{3N} \rightarrow P_3$ defined by

$$\pi(x, e_1, e_2, e_3) = (\pi_M(x), e_1, e_2, e_3).$$

For $M \subset \mathbb{R}^m$ compact, the **perturbation ball** is

$$S := B^{m+3N}(\epsilon) \subset B^m(\epsilon) \times B^{3N}(\epsilon) \subset \mathbb{R}^m \times \mathbb{R}^{3N}.$$

Remark 12. For δ as in Remark 10, we choose the size of the perturbation ball S so that

for all $s \in S$, $|s| < \delta$. This choice is used in Lemma 17.

Definition 14. For $M = \mathbb{R}^n$, define **perturbed evaluation maps** $E_{i,j;3}$ as follows:

$$\begin{aligned}
E_{1,2;3} &: \mathcal{M}_{1,2;3}(p_1, P_3) \times S \rightarrow P_3 \\
&(\gamma, s) \mapsto \text{ev}_{1,2;3}^+(\gamma) + s = \gamma(0) + s, \\
E_{2,3;3} &: \mathcal{M}_{2,3;3}(p_2, P_3) \times S \rightarrow P_3 \\
&(\gamma, s) \mapsto \text{ev}_{2,3;3}^+(\gamma) + s = \gamma(0) + s, \\
E_{1,3;3} &: \mathcal{M}_{1,3;3}(P_3, p_0) \times S \rightarrow P_3 \\
&(\gamma, s) \mapsto \text{ev}_{1,3;3}^-(\gamma) + s = \gamma(0) + s.
\end{aligned}$$

If M is a closed manifold, we define $E_{i,j;3}$ using the map $\pi : M^\epsilon \times \mathbb{R}^{3N} \rightarrow P_3$ defined in Definition 13:

$$\begin{aligned}
E_{1,2;3} &: \mathcal{M}_{1,2;3}(p_1, P_3) \times S \rightarrow P_3 \\
&(\gamma, s) \mapsto \pi \left(\text{ev}_{1,2;3}^+(\gamma) + s \right) = \pi (\gamma(0) + s), \\
E_{2,3;3} &: \mathcal{M}_{2,3;3}(p_2, P_3) \times S \rightarrow P_3 \\
&(\gamma, s) \mapsto \pi \left(\text{ev}_{2,3;3}^+(\gamma) + s \right) = \pi (\gamma(0) + s), \\
E_{1,3;3} &: \mathcal{M}_{1,3;3}(P_3, p_0) \times S \rightarrow P_3 \\
&(\gamma, s) \mapsto \pi \left(\text{ev}_{1,3;3}^-(\gamma) + s \right) = \pi (\gamma(0) + s).
\end{aligned}$$

Remark 13. The $E_{i,j;3}$ maps are well-defined: This is clear when $M = \mathbb{R}^n$, and for closed $M \subset \mathbb{R}^m$, the evaluation maps have outputs in $P_3 = M \times \mathbb{R}^{3N} \subset M^\epsilon \times \mathbb{R}^{3N} \subset \mathbb{R}^{m+3N}$.

Adding an element $s \in S$ to this output will give a point within distance ϵ of the endpoint. This is a valid input for the map π , which we use to get a corresponding point P_3 .

The following definitions set notation for the statement and proof of Theorem 13.

Definition 15. For $p_1, p_2, p_0 \in \text{Crit}_+(w)$, let

$$X := \mathcal{M}_{1,2,3}(p_1, P_3) \times \mathcal{M}_{2,3,3}(p_2, P_3) \times \mathcal{M}_{1,3,3}(P_3, p_0),$$

and define the following **triple perturbed evaluation map**:

$$\begin{aligned} E : X \times (S \times S \times S) &\rightarrow P_3 \times P_3 \times P_3 \\ ((\gamma_1, \gamma_2, \gamma_3), (s_1, s_2, s_3)) &\mapsto (E_{1,2,3}(\gamma_1, s_1), E_{2,3,3}(\gamma_2, s_2), E_{1,3,3}(\gamma_3, s_3)). \end{aligned}$$

For a given $s = (s_1, s_2, s_3) \in S^3$, let $E_s = E|_{X \times \{s\}} : X \rightarrow (P_3)^3$ be the restriction of E to s .

Denote the diagonal of $(P_3)^3$ by Δ^3 , i.e. $\Delta^3 = \{(y, y, y) \mid y \in P_3\}$. The following theorem shows that, for almost every choice of perturbation $s \in S^3$, there is manifold structure on flow trees with a midpoint perturbation by s .

Theorem 13. *For almost every $s = (s_1, s_2, s_3) \in S^3 = S \times S \times S$, $E_s^{-1}(\Delta^3) \subset X$ is a smooth manifold of dimension $|p_0| - |p_1| - |p_2|$.*

Proof. We show that, for almost every $s \in S^3$, $E_s \pitchfork \Delta^3$. Thus, by the Transversality Theorem (see, for example Section 2.3 in [16]), $E_s^{-1}(\Delta^3)$ is a smooth manifold and the codimension of $E_s^{-1}(\Delta^3)$ in X equals to codimension of Δ^3 in $(P_3)^3$.

For $M = \mathbb{R}^n$, the map $E_{i,j,3}$ restricted to γ is the translation $s \mapsto \gamma(0) + s$ and for M compact, $E_{i,j,3}$ restricted to γ is this translation composed with the submersion π (see Definitions 13 and 14). Thus, fixing a triple of trajectories $\hat{\gamma} := (\gamma_1, \gamma_2, \gamma_3) \in X$, the map

$E_{\tilde{\gamma}} : S \times S \times S \rightarrow P_3 \times P_3 \times P_3$ is a product of submersions of the ball S , so the whole map E is transversal to any submanifold of $(P_3)^3$.

Since the hypothesis of the Transversality Theorem is satisfied, for almost every $s \in S \times S \times S$, $E_s^{-1}(\Delta^3)$ is a smooth manifold whose codimension in X equals the codimension of Δ^3 in $(P_3)^3$. From this, Remark 11, and Equation 4, we can calculate:

$$\begin{aligned}
& \dim(E_s^{-1}(\Delta^3)) \\
&= \dim(X) - (\dim((P_3)^3) - \dim(\Delta^3)) \\
&= \dim(\mathcal{M}_{1,2;3}(p_1, P_3)) + \dim(\mathcal{M}_{2,3;3}(p_2, P_3)) + \dim(\mathcal{M}_{1,3;3}(P_3, p_0)) - 2(n + 3N) \\
&= \dim(W_{p_1}^-(w_{1,2;3})) + \dim(W_{p_2}^-(w_{2,3;3})) + \dim(W_{p_0}^+(w_{1,3;3})) - 2(n + 3N) \\
&= (n + 3N) - \text{ind}_{w_{1,2;3}}(p_1) + (n + 3N) - \text{ind}_{w_{2,3;3}}(p_2) + \text{ind}_{w_{1,3;3}}(p_0) - 2(n + 3N) \\
&= -(|p_1| + N) - (|p_2| + N) + (|p_0| + 2N) \\
&= |p_0| - |p_1| - |p_2|.
\end{aligned}$$

□

Definition 16. We denote the manifold $E_s^{-1}(\Delta^3)$ from Theorem 13 by $\mathcal{M}(p_1, p_2; p_0|s)$.

We may describe $\mathcal{M}(p_1, p_2; p_0|s)$ in the following way:

Given a generating family $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$, pick metrics $g_{i,j;3}$ as in Definition 9. Let S be a perturbation ball as in Definition 13 and form the $E_{i,j;3}$ maps as in Definition 14. Theorem 13 implies that we can choose $s = (s_1, s_2, s_3) \in S \times S \times S$ so that the following

set is a smooth manifold.

$$\mathcal{M}(p_1, p_2; p_0|s) = \left\{ (\gamma_1, \gamma_2, \gamma_3) \left[\begin{array}{l} \gamma_1 : (-\infty, 0] \rightarrow M \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N, \\ \gamma_2 : (-\infty, 0] \rightarrow M \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N, \\ \gamma_3 : [0, \infty) \rightarrow M \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N, \\ \frac{d\gamma_1}{dt} = \nabla_{g_{1,2;3}} w_{1,2;3}, \quad \frac{d\gamma_2}{dt} = \nabla_{g_{2,3;3}} w_{2,3;3}, \\ \frac{d\gamma_3}{dt} = \nabla_{g_{1,3;3}} w_{1,3;3}, \\ E_{1,2;3}(\gamma_1, s_1) = E_{2,3;3}(\gamma_2, s_2) = E_{1,3;3}(\gamma_3, s_3), \\ \lim_{t \rightarrow -\infty} \gamma_1(t) = \iota_{1,2;3}(p_1), \quad \lim_{t \rightarrow -\infty} \gamma_2(t) = \iota_{2,3;3}(p_2), \\ \lim_{t \rightarrow \infty} \gamma_3(t) = \iota_{1,3;3}(p_0) \end{array} \right. \right\}.$$

5.2 Compactification by broken flow trees

The compactification of $\mathcal{M}(p_1, p_2; p_0|s)$ relies on a few technical lemmas that allow us to apply the compactifications of half-infinite Morse trajectories from [37]. In particular, we show that the images of flow trees in $\mathcal{M}(p_1, p_2; p_0|s)$ are contained in a compact set (Lemma 16) and are bounded away from the critical submanifolds of the extended difference functions (Lemma 17).

Although the gradient trajectories are in the non-compact space P_3 , the following shows that all trees will have their images in a compact subset of P_3 .

Lemma 16. *For a given a linear-at-infinity generating family $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$, there is a compact set K_s with $K \subseteq K_s \subseteq P_3$ such that for all p_1, p_2, p_0 and all $\Gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathcal{M}_F^s(p_1, p_2; p_0)$, $\text{Im } \Gamma \subseteq K_s$.*

Proof. As shown above in Lemma 9, the critical points $p_1, p_2, p_0 \in K$. To show that the image of every $\Gamma \in \mathcal{M}(p_1, p_2; p_0|s)$ is contained in K_s , we first work with a tree with

$s = 0$. We show that every trajectory of $\nabla w_{i,j;3}$ that leaves K cannot reenter K . This implies that every edge E in a tree, $\text{Im}(\Gamma|_E)$ can intersect ∂K at most once. To show that these edges in fact never intersect ∂K , we show that the point of intersection of the three trajectories is in K .

For these arguments, it will be useful to first analyze some properties of $\nabla w_{i,j;3}$ outside a compact set. Since F is linear-at-infinity and our metrics are chosen to be standard outside K , we know that for $(x, e_1, e_2, e_3) \notin K$,

$$\begin{aligned} \nabla w_{i,j;3}(x, e_1, e_2, e_3) &= \left(\frac{\partial F}{\partial x}(x, e_i) - \frac{\partial F}{\partial x}(x, e_j) \right) \frac{\partial}{\partial x} \\ &\quad + \left(\frac{\partial F}{\partial e_i}(x, e_i) \right) \frac{\partial}{\partial e_i} - \left(\frac{\partial F}{\partial e_j}(x, e_j) \right) \frac{\partial}{\partial e_j} \\ &\quad \pm 2e_\ell \frac{\partial}{\partial e_\ell}, \end{aligned}$$

where $\ell \neq i, j$ and the $\frac{\partial}{\partial e_\ell}$ sign is $-$ if $\ell = 2$ and $+$ else.

More specifically, suppose that outside $K_M \times K_E$, $F(x, e)$ is the nonzero linear function $A(e_1, \dots, e_N)$ and $\frac{\partial}{\partial e_k} A(e) = c_k \in \mathbb{R}^N - \{0\}$. Then we have that $\forall i, j$,

1. if $x \notin K_M$, the $\frac{\partial}{\partial x}$ component of $\nabla w_{i,j;3}(x, e_1, e_2, e_3)$ equals 0;
2. if $e_i \notin K_E$, the $\frac{\partial}{\partial e_{ik}}$ component of $\nabla w_{i,j;3}(x, e_1, e_2, e_3)$ equals c_k ;
3. if $e_j \notin K_E$, the $\frac{\partial}{\partial e_{jk}}$ component of $\nabla w_{i,j;3}(x, e_1, e_2, e_3)$ equals $-c_k$;
4. for e_ℓ , $\ell \neq i, j$, the $\frac{\partial}{\partial e_\ell}$ component of $\nabla w_{i,j;3}(x, e_1, e_2, e_3)$ is $2e_\ell$ when $\ell = 1$ or $\ell = 3$ and is $-2e_\ell$ when $\ell = 2$.

First, suppose γ is a trajectory of $\nabla w_{i,j;3}$ and there exists a $t_0 < t_1$ so that $\gamma(t_0) \in K, \gamma(t_1) \notin K$. The following argument then shows that for all $t > t_1$, $\gamma(t) \notin K$. Since $\gamma(t_1) \notin K$, $\gamma(t_1) = (x, e_1, e_2, e_3)$ where $x \notin K_M$ or $e_i \notin K_E$, for some i . From the form of $\nabla w_{i,j;3}$ outside

K , it is easy to see that for all $t > t_1$, $\gamma(t_1) \notin K$. For example, if $\gamma(t_1) = (x, e_1, e_2, e_3)$, where $e_i \notin K_E$, then since the $\frac{\partial}{\partial e_i}$ component of $\nabla w_{i,j;k}(x, e_1, \dots, e_{k+1})$ is constant or linear, it follows that for all $t > t_1$, the i^{th} component of $\gamma(t)$ will not lie in K_E .

To complete the proof, we first assume the perturbation vector $s = 0 \in S \times S \times S$ (so that the trajectories intersect) and show that for the vertex v in the interior of the tree, $\Gamma(v) \in K$; in other words, we show that the intersection point of gradient trajectories in a tree must be contained in K . Let $y \in M \times \mathbb{R}^{3N}$ denote the intersection point of gradient trajectories.

Suppose $y = (x^y, e_1^y, e_2^y, e_3^y) \notin K$. From (1), we see that $y \notin K$ can only follow from $e_i^y \notin K_E$ for some $1 \leq i \leq 3$. We complete the argument by finding contradictions to $e_i^y \notin K_E$, by cases depending on i .

Suppose $e_1^y \notin K_E$. Then by 2, we see that the $\frac{\partial}{\partial e_1}$ components of $\nabla w_{1,2,3}(x, e_1, e_2, e_3)$ and $\nabla w_{1,3,3}(x, e_1, e_2, e_3)$ both equal the same constants c_k , but the first flows from K to y and the other flows from y to K , giving a contradiction.

A similar contradiction is reached if $e_2^y \notin K_E$: The trajectories along $\nabla w_{1,2,3}$ and $\nabla w_{2,3,3}$ both flow to y , but by 2 and 3, we see the $\frac{\partial}{\partial e_2}$ components of the trajectories outside K are constant with opposite signs.

Lastly, if $e_3^y \notin K_E$, we obtain a similar contradiction as in the case $i = 1$ using 3 and the fact that $\nabla w_{2,3,3}$ flows from K to y while $\nabla w_{1,3,3}$ flows away from y back to K . Thus we must have $y \in K$.

For trajectories with a nonzero perturbation, let $y = (x^y, e_1^y, e_2^y, e_3^y)$ be the intersection point of the perturbed trajectories, i.e., if $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\} \in \mathcal{M}(p_1, p_2; p_0|s)$, then

$$y = \pi(\gamma_1(0) + s_1) = \pi(\gamma_2(0) + s_2) = \pi(\gamma_3(0) + s_3),$$

where π is the identity map or a submersion that is the identity on P_3 , see Definition 14.

We built the perturbation ball S so that $|s_i| < \delta$; see Remark 12. Either $B_y(3\delta) \subset K$ or there exists a larger compact set K_s containing K so that this is true.

Hence Γ is contained in a compact set K_s , as desired. \square

To apply results in [37], we must bound the half-infinite trajectories of the extended difference functions in our trees away from the critical submanifold of their respective function. Since the critical submanifolds have value 0, we show that the points in the image of each trajectory have positive value bounded away from 0. Since our trajectories follow a positive gradient, the trajectories that flow from positive-valued critical points p_1 and p_2 naturally are bounded away from their respective Morse-Bott submanifolds. The following lemma bounds the remaining trajectory.

Lemma 17. *Given $\Gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathcal{M}(p_1, p_2; p_0|s)$, $\gamma_3(0) > \frac{\rho}{4} > 0$, where ρ is the least positive critical value of w .*

Proof. Consider

$$\begin{aligned} y &= E_{1,2,3}(\gamma_1, s_1) = E_{2,3,3}(\gamma_2, s_2) = E_{1,3,3}(\gamma_3, s_3) \\ &= (x(y), e_1(y), e_2(y), e_3(y)) \in P_3. \end{aligned}$$

While the $E_{i,j,3}$ maps were defined in slightly different ways dependent on if the underlying manifold M was Euclidean or closed (see Definition 14), we may express them as $\pi(\gamma_k + s_k)$ for $k = 1, 2, 3$, where π is the identity or the submersion described in the definition.

Since the trees in $\mathcal{M}(p_1, p_2; p_0|s)$ are defined using the positive gradient flow of the extended difference functions, we have that $w_{1,2,3}(\gamma_1(0)) \geq w_{1,2,3}(p_1)$ and $w_{2,3,3}(\gamma_2(0)) \geq w_{2,3,3}(p_2)$.

By construction of the extended difference functions,

$$w_{1,3;3}(y) = w_{1,2;3}(y) + w_{2,3;3}(y) - e_1(y)^2 - e_2(y)^2 - e_3(y)^2. \quad (8)$$

We now make use of a couple of the choices we have built into our constructions of $\mathcal{M}(p_1, p_2; p_0 | s)$; see Remark 10. Since the perturbation terms $s_i \in S$, we have ensured that $|s_i| < \delta$ so that, using the uniform continuity of $w_{i,j;3}$ on K , we have that

$$|w_{i,j;3}(y) - w_{i,j;3}(\gamma_k(0))| < \frac{\rho}{4}. \quad (9)$$

From this and (8) we see that

$$\begin{aligned} & w_{1,3;3}(\gamma_3(0)) \\ & > w_{1,3;3}(y) - \frac{\rho}{4} \\ & = w_{1,2;3}(y) + w_{2,3;3}(y) - (e_1(y))^2 - (e_2(y))^2 - (e_3(y))^2 - \frac{\rho}{4} \\ & > w_{1,2;3}(\gamma_1(0)) - \frac{\rho}{4} + w_{2,3;3}(\gamma_2(0)) - \frac{\rho}{4} - (e_1(y))^2 - (e_2(y))^2 - (e_3(y))^2 - \frac{\rho}{4} \\ & \geq w_{1,2;3}(p_1) + w_{2,3;3}(p_2) - (e_1(y))^2 - (e_2(y))^2 - (e_3(y))^2 - \frac{3\rho}{4} \\ & > w_{1,2;3}(p_1) + w_{2,3;3}(p_2) - \rho - \frac{3\rho}{4} \\ & > 2\rho - \rho - \frac{3\rho}{4} > 0. \end{aligned}$$

□

The idea behind compactifying Morse trajectory spaces is that unbroken flow lines limit to broken ones. Topologically, spaces of multiply-broken flow lines have a manifold with corners structure. This notion has been made precise for Morse functions on closed manifolds in [37]. Lemma 16 will be used in the proof of Theorem 4 for compactification, but to get

a (noncompact) manifold with corners structure on the spaces of half-infinite trajectories, we use the result of Lemma 17 to justify restricting the space $\mathcal{M}_{1,3;3}(P_3, p_0)$ to a space of trajectories that have their finite end bounded away from $\{w_{1,3;3} = 0\}$, so that broken trajectories cannot break at the critical submanifold. With this in mind, we set up broken half-infinite trajectory spaces:

Definition 17. To ease notation, let

$$(\mathcal{U}_-, \mathcal{U}_+) = (p_1, P_3), (p_2, P_3), \text{ or } \left(\left\{ w_{1,3;3} > \frac{\rho}{8} \right\}, p_0 \right).$$

We define the ℓ -fold broken half-infinite trajectories to be

$$\overline{\mathcal{M}}_{i,j;3}(\mathcal{U}_-, \mathcal{U}_+)_{\ell} := \bigcup \mathcal{M}_{i,j;3}(\mathcal{U}_-, q_1) \times \mathcal{M}_{i,j;3}(q_1, q_2) \times \cdots \times \mathcal{M}_{i,j;3}(q_{\ell}, \mathcal{U}_+),$$

where the union is taken over sequence of critical points $q_1, \dots, q_{\ell} \in \text{Crit}_+(w)$ such that $\mathcal{M}_{i,j;3}(\mathcal{U}_-, q_1), \mathcal{M}_{i,j;3}(q_1, q_2), \dots, \mathcal{M}_{i,j;3}(q_{\ell}, \mathcal{U}_+) \neq \emptyset$.

Definition 18. The generalized Morse trajectory space is

$$\overline{\mathcal{M}}_{i,j;3}(\mathcal{U}_-, \mathcal{U}_+) := \bigcup_{\ell \in \mathbb{N}} \overline{\mathcal{M}}_{i,j;3}(\mathcal{U}_-, \mathcal{U}_+)_{\ell}.$$

We will use $\overline{\gamma} = \{\gamma_1, \dots, \gamma_{\ell}\}$ to denote an element of $\overline{\mathcal{M}}_{i,j;3}(\mathcal{U}_-, \mathcal{U}_+)$.

Remark 14. The union in Definition 18 is finite: There are only a finite number of points in $\text{Crit}_+ w$ and they live in a compact subset of P_3 (see Lemma 9). The finite ends of the generalized trajectories from p_1 and p_2 may leave the non-linear support set K , so these spaces are not necessarily contained in a compact set.

There is a natural metric on $\overline{\mathcal{M}}_{i,j;3}(\mathcal{U}_-, \mathcal{U}_+)$:

Definition 19. On $\overline{\mathcal{M}}_{i,j;3}(\mathcal{U}_-, \mathcal{U}_+)$ consider the metric $d_{\overline{\mathcal{M}}}$ which is the Hausdorff distance on the images of broken trajectories.

$$d_{\overline{\mathcal{M}}}(\overline{\gamma}, \overline{\gamma'}) := d_{\text{Haus}}(\overline{\text{im}\overline{\gamma}}, \overline{\text{im}\overline{\gamma'}}).$$

By $\overline{\text{im}\overline{\gamma}}$, we mean the closure of the union of the images of trajectories that make up the trajectory sequence $\overline{\gamma}$, i.e., we are including the critical point limits of the trajectories in the sequence. Recall that the Hausdorff distance d_{Haus} is a metric on non-empty compact subsets of a space defined by

$$d_{\text{Haus}}(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}.$$

Theorem 14. For $p_0, p_1, p_2 \in \text{Crit}_+(w)$, the half-infinite broken trajectory spaces $(\overline{\mathcal{M}}_{1,2;3}(p_1, P_3), d_{\overline{\mathcal{M}}})$, $(\overline{\mathcal{M}}_{2,3;3}(p_2, P_3), d_{\overline{\mathcal{M}}})$, and $(\overline{\mathcal{M}}_{1,3;3}(\{w_{1,3;3} > \frac{\rho}{8}\}, p_0), d_{\overline{\mathcal{M}}})$ are metric spaces that can be equipped with the structure of a smooth manifold with corners. In each case, the ℓ -stratum is $\overline{\mathcal{M}}_{i,j;3}(\mathcal{U}_-, \mathcal{U}_+)_\ell$ as in Definition 17.

Proof. While our setup differs from that in [37, Theorem 2.3], we argue that constructions in [37] suffice to claim this result. In particular, the constructions in [37] give a maximal atlas of charts and associative gluing maps to define a manifold with corners structure for Morse-Smale pairs on a closed manifold. In neighborhoods not containing critical points, there is a natural smooth structure induced by the smoothness of the gradient flow. The careful work to define the corner structure occurs in neighborhood of the critical points. Thus, while the extended difference functions are Morse-Bott and defined on a noncompact manifold, Lemmas 16 and 17 show that neighborhoods of the trajectories that occur in trees occur in an open set contained in compact set. Hence, the charts in [37] suffice to give a manifold with corners structure on the relevant trajectory spaces. In contrast

to [37, Theorem 2.3], we lose the compactness of the trajectory spaces themselves. \square

Definition 20. Extend the maps in Definition 12 to **generalized evaluation maps**

$$\text{ev}_{i,j;3}^- : \overline{\mathcal{M}}_{1,3;3} \left(\left\{ w_{1,3;3} > \frac{\rho}{8} \right\}, p_0 \right) \rightarrow P_3, \quad \text{ev}_{i,j;3}^+ : \overline{\mathcal{M}}_{i,j;3}(p, P_3) \rightarrow P_3$$

by

$$\text{ev}_{i,j;3}^-(\overline{\gamma}) = \text{ev}_{i,j;3}^-(\{\gamma_1, \dots, \gamma_\ell\}) := \gamma_1(0)$$

and

$$\text{ev}_{i,j;3}^+(\overline{\gamma}) = \text{ev}_{i,j;3}^+(\{\gamma_1, \dots, \gamma_\ell\}) := \gamma_\ell(0).$$

Remark 15. [37, Lemma 3.3] proved that the extended evaluation maps in Definition 20 are continuous with respect to the Hausdorff metric defined in 19. As shown in [37, Remark 5.5], the evaluation maps are smooth.

Definition 21. For $M = \mathbb{R}^n$, define the **generalized perturbation maps** $\overline{E}_{i,j;3}$ as follows:

$$\begin{aligned} \overline{E}_{1,2;3} : \overline{\mathcal{M}}_{1,2;3}(p_1, P_3) \times S &\rightarrow P_3 \\ (\overline{\gamma}, s) &\mapsto \text{ev}_{1,2;3}^+(\overline{\gamma}) + s = \gamma_\ell(0) + s, \\ \overline{E}_{2,3;3} : \overline{\mathcal{M}}_{2,3;3}(p_2, P_3) \times S &\rightarrow P_3 \\ (\overline{\gamma}, s) &\mapsto \text{ev}_{2,3;3}^+(\overline{\gamma}) + s = \gamma_\ell(0) + s, \\ \overline{E}_{1,3;3} : \overline{\mathcal{M}}_{1,3;3} \left(\left\{ w_{1,3;3} > \frac{\rho}{8} \right\}, p_0 \right) \times S &\rightarrow P_3 \\ (\overline{\gamma}, s) &\mapsto \text{ev}_{1,3;3}^-(\overline{\gamma}) + s = \gamma_1(0) + s. \end{aligned}$$

If M is a compact manifold, we define the **generalized perturbation maps** $\overline{E}_{i,j;3}$

using the map $\pi : M^\epsilon \times \mathbb{R}^{3N} \rightarrow P_3$ defined in Definition 13:

$$\begin{aligned} \bar{E}_{1,2,3} &: \bar{\mathcal{M}}_{1,2,3}(p_1, P_3) \times S \rightarrow P_3 \\ &(\bar{\gamma}, s) \mapsto \pi \left(\text{ev}_{1,2,3}^+(\bar{\gamma}) + s \right) = \pi (\gamma_\ell(0) + s), \\ \bar{E}_{2,3,3} &: \bar{\mathcal{M}}_{2,3,3}(p_2, P_3) \times S \rightarrow P_3 \\ &(\bar{\gamma}, s) \mapsto \pi \left(\text{ev}_{2,3,3}^+(\bar{\gamma}) + s \right) = \pi (\gamma_\ell(0) + s), \\ \bar{E}_{1,3,3} &: \bar{\mathcal{M}}_{1,3,3} \left(\left\{ w_{1,3,3} > \frac{\rho}{8} \right\}, p_0 \right) \times S \rightarrow P_3 \\ &(\bar{\gamma}, s) \mapsto \pi \left(\text{ev}_{1,3,3}^-(\bar{\gamma}) + s \right) = \pi (\gamma_1(0) + s). \end{aligned}$$

Definition 22. For $p_1, p_2, p_0 \in C(F)$, let

$$\bar{X} := \bar{\mathcal{M}}_{1,2,3}(p_1, P_3) \times \bar{\mathcal{M}}_{2,3,3}(p_2, P_3) \times \bar{\mathcal{M}}_{1,3,3} \left(\left\{ w_{1,3,3} > \frac{\rho}{8} \right\}, p_0 \right).$$

Applying Lemma 2 twice shows that the space \bar{X} is a manifold with corners whose ℓ -stratum is

$$\bar{X}_\ell = \bigsqcup_{i+j+k=\ell} \bar{\mathcal{M}}_{1,2,3}(p_1, P_3)_i \times \bar{\mathcal{M}}_{2,3,3}(p_2, P_3)_j \times \bar{\mathcal{M}}_{1,3,3} \left(\left\{ w_{1,3,3} > \frac{\rho}{8} \right\}, p_0 \right)_k.$$

Theorem 15. For almost every $s = (s_1, s_2, s_3) \in S^3 = S \times S \times S$, $\bar{E}_s^{-1}(\Delta^3) = \bar{\mathcal{M}}(p_1, p_2; p_0|s)$ is a compact manifold with corners of dimension $|p_0| - |p_1| - |p_2|$ with i -stratum $\bar{\mathcal{M}}(p_1, p_2; p_0|s)_i = \bar{X}_i \cap \bar{E}_s^{-1}(\Delta^3)$ given by trees with a total of i breaks on the tree edges. In particular, $\bar{\mathcal{M}}(p_1, p_2; p_0|s)_0 = \mathcal{M}(p_1, p_2; p_0|s)$. Thus, $\bar{\mathcal{M}}(p_1, p_2; p_0|s)$ is a compactification of $\mathcal{M}(p_1, p_2; p_0|s)$.

Proof. Following a similar strategy as in the proof of Theorem 13, we transversely cut out a smooth manifold with corners from \bar{X} . We will make use of extensions of Transversality

and Preimage Theorems for manifolds with corners; see Theorems 2 and 3.

Combining the generalized perturbation maps into one map, define:

$$\begin{aligned} \bar{E} : \bar{X} \times (S \times S \times S) &\rightarrow P_3 \times P_3 \times P_3 \\ ((\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3), (s_1, s_2, s_3)) &\mapsto (\bar{E}_{1,2;3}(\bar{\gamma}_1, s_1), \bar{E}_{2,3;3}(\bar{\gamma}_2, s_2), \bar{E}_{1,3;3}(\bar{\gamma}_3, s_3)). \end{aligned}$$

We use Theorem 3 to show that for almost every $s = (s_1, s_2, s_3) \in S \times S \times S$, $\partial_\ell \bar{E}_s = \bar{E}_s|_{\bar{X}_\ell}$ is transversal to the diagonal $\Delta^3 \subset P_3 \times P_3 \times P_3$. Thus, by Theorem 2, $\bar{E}_s^{-1}(\Delta^3)$ is a smooth submanifold with corners of \bar{X} whose ℓ -stratum $\bar{E}_s^{-1}(\Delta^3)_\ell$ is $\bar{X}_\ell \cap \bar{E}_s^{-1}(\Delta^3)$.

To use Theorem 3, we need to show that $\partial_i \bar{E}$ is transversal to Δ^3 for all strata of \bar{X} . Fix a trajectory sequence $\bar{\gamma}$ in $\bar{\mathcal{M}}_{1,2;3}(p_1, P_3)$, $\bar{\mathcal{M}}_{2,3;3}(p_2, P_3)$, or $\bar{\mathcal{M}}_{1,3;3}(\{w_{1,3;3} > \frac{\ell}{8}\}, p_0)$.

For $M = \mathbb{R}^n$, the map $\bar{E}_{i,j;3}$ restricted to $\bar{\gamma}$ is the translation $s \mapsto x + s$ where $x = \text{ev}_{i,j;3}^\pm(\bar{\gamma})$ and so is a submersion, and since $\bar{\gamma}$ was arbitrary, $E_{i,j;3}$ restricted to any strata (which is exactly the map $\partial^i \bar{E}_{i,j;3}$ of $\bar{\mathcal{M}}_{i,j;3}(p_i, P_3)$ or $\bar{\mathcal{M}}_{1,3;3}(\{w_{1,3;3} > \frac{\ell}{8}\}, p_0)$) is a submersion.

The case where M is closed is similar. For fixed $\bar{\gamma}$ as above, the map $\bar{E}_{i,j;3}$ sends s to $\pi(x + s)$ and is a composition of a translation with π , which was chosen through the ϵ -Neighborhood Theorem to be a submersion (see Definition 13). Since a restriction to one trajectory sequence is a submersion, a restriction to any stratum will be as well.

Thus, fixing a triple of trajectories $\hat{\gamma} := (\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3) \in \bar{X}$, note that, no matter which stratum this triple lives in, the map $E_{\hat{\gamma}} : S \times S \times S \rightarrow P_3 \times P_3 \times P_3$ is a product of submersions of the ball S . Thus, any restriction of E to any strata of \bar{X} is transversal to any submanifold of $(P_3)^3$, which shows that $\partial_i \bar{E} \pitchfork \Delta^3$.

Thus, for almost every $s \in S \times S \times S$, $\bar{\mathcal{M}}(p_1, p_2; p_0|s)$ will be a smooth manifold with corners whose codimension in \bar{X} equals the codimension of Δ^3 in $(P_3)^3$. The dimension follows from the exact calculation Theorem 13.

It remains to show that $\overline{\mathcal{M}}(p_1, p_2; p_0|s)$ is compact. This would be immediate if \overline{X} were compact, as Δ^3 is closed in $(P_3)^3$ and so $\overline{E_s^{-1}(\Delta^3)}$ is closed in \overline{X} . We will argue, instead, that trajectory sequences that cause \overline{X} to be noncompact do not show up in trees in $\overline{\mathcal{M}}(p_1, p_2; p_0|s)$. We know that broken half-infinite trajectory spaces for Morse-Smale pairs on a closed manifold are compact from [37, Theorem 2.3], so issues of noncompactness in

$$\overline{X} = \overline{\mathcal{M}}_{1,2;3}(p_1, P_3) \times \overline{\mathcal{M}}_{2,3;3}(p_2, P_3) \times \overline{\mathcal{M}}_{1,3;3} \left(\left\{ w_{1,3;3} > \frac{\rho}{8} \right\}, p_0 \right)$$

stem from the noncompactness of P_3 and $\{w_{1,3;3} > \frac{\rho}{8}\}$. In particular, the spaces $\overline{\mathcal{M}}_{i,j;3}(p_i, P_3)$ could contain a sequence of trajectories whose finite ends (i.e., images of $\text{ev}_{i,j;3}^+$) diverge. Similarly, there could be sequence in $\overline{\mathcal{M}}_{1,3;3}(\{w_{1,3;3} > \frac{\rho}{8}\}, p_0)$ whose limit has finite end, given by $\text{ev}_{i,j;3}^-$, in the level set $\{w_{1,3;3} = \frac{\rho}{8}\}$.

As $\overline{\mathcal{M}}(p_1, p_2; p_0|s)$ is a metric space, to show that it is compact it suffices to prove sequential compactness. Suppose $\Gamma_n = (\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3)_n$ is a sequence of trees in $\overline{\mathcal{M}}(p_1, p_2; p_0|s)$. The same proofs of Lemma 16 and 17 show that, $\forall n$, $\Gamma_n \subset K_s$ and $w_{1,3;3}((\gamma_3)_n) > \frac{\rho}{4} > \frac{\rho}{8}$.

With these bounds, the convergence of a subsequence of Γ_n follows as in the proof of [37, Theorem 2.3] and [2, Proposition 3]. This was shown by defining a continuous reparametrization of the images of trajectories in the sequence with bounded derivatives on the complements of neighborhoods of critical points. This implies the equicontinuity of these reparametrizations, which, by the Arzelà-Ascoli Theorem, gives a convergent subsequence. \square

Geometrically, the content of Theorem 15 is that $\mathcal{M}(p_1, p_2; p_0|s)$ lives as the 0-stratum in a larger manifold with corners whose ℓ -stratum is made up of “almost” intersecting trees with a total of ℓ breaks spread over its three branches.

We apply Theorem 15 to see that a 1-dimensional $\mathcal{M}(p_1, p_2; p_0|s)$ has a natural com-

pactification through the addition of trees with once-broken edges, see Figure 4.

Corollary 1. *Given p_1, p_2, p_0 with $|p_0| - |p_1| - |p_2| = 1$, $\mathcal{M}(p_1, p_2; p_0|s)$ can be compactified to a 1-manifold $\overline{\mathcal{M}}(p_1, p_2; p_0|s)$ with boundary*

$$\begin{aligned} \partial\overline{\mathcal{M}}(p_1, p_2; p_0|s) := & \bigcup_{p'_1} \mathcal{M}_{1,2;3}(p_1, p'_1) \times \mathcal{M}(p'_1, p_2; p_0|s) \\ & \bigcup_{p'_2} \mathcal{M}_{2,3;3}(p_2, p'_2) \times \mathcal{M}(p_1, p'_2; p_0|s) \\ & \bigcup_{p'_0} \mathcal{M}(p_1, p_2; p'_0|s) \times \mathcal{M}_{1,3;3}(p'_0, p_0), \end{aligned}$$

where the unions are taken over $p'_1 \in C^{|p_1|+1}(F)$, $p'_2 \in C^{|p_2|+1}(F)$, and $p'_0 \in C^{|p_0|-1}(F)$.

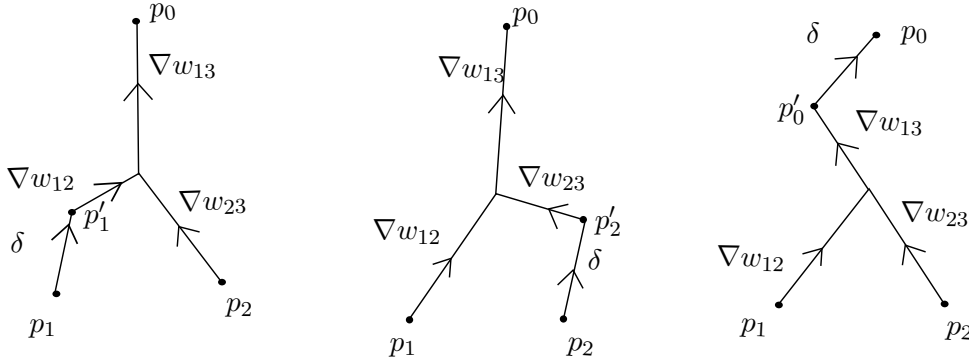


Figure 4: Elements in $\partial\mathcal{M}(p_1, p_2; p_0|s)$ for $s = (0, 0, 0)$.

Chapter 6

Product Structure

We define a map by counting isolated trees in $\mathcal{M}(p_1, p_2; p_0|s)$. Note that, for $p_1, p_2, p_0 \in \text{Crit}_+(w)$, Theorem 13 implies that isolated trees in $\mathcal{M}(p_1, p_2; p_0|s)$ satisfy

$$|p_0| = |p_1| + |p_2|$$

Definition 23. Given a generating family $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$, we define a map

$$m_2 : C^i(F) \otimes C^j(F) \rightarrow C^{i+j}(F)$$

as follows: For critical points $p_1, p_2 \in \text{Crit}_+(w)$, define

$$m_2(p_1 \otimes p_2) = \sum (\#_{\mathbb{Z}_2} \mathcal{M}(p_1, p_2; p_0|s)) \cdot p_0$$

where the sum is taken over $p_0 \in \text{Crit}_+(w)$, such that $|p_0| = |p_1| + |p_2|$. Extend the product bilinearly over the tensor product.

The following lemma shows that m_2 descends to a map on cohomology:

Lemma 18. *The map*

$$m_2 : C^i(F) \otimes C^j(F) \rightarrow C^{i+j}(F)$$

is a cochain map, i.e., the following diagram commutes:

$$\begin{array}{ccc} C(F) \otimes C(F) & \xrightarrow{m_2} & C(F) \\ \delta \otimes 1 + 1 \otimes \delta \downarrow & & \downarrow \delta \\ C(F) \otimes C(F) & \xrightarrow{m_2} & C(F). \end{array} \quad (10)$$

Proof. Consider a 1-dimensional moduli space of flow trees, $\mathcal{M}(p_1, p_2; p_0 | s)$. Theorem 13 shows that such a space occurs when

$$|p_0| = |p_1| + |p_2| + 1,$$

so let $p_1 \in C^i(F)$, $p_2 \in C^j(F)$, and $p_0 \in C^{i+j+1}(F)$.

Corollary 1 gives an expression for $\partial\mathcal{M}(p_1, p_2; p_0 | s)$. In particular, the boundary of $\mathcal{M}(p_1, p_2; p_0 | s)$ consists of isolated trees with a single broken edge. After compactification, $\mathcal{M}(p_1, p_2; p_0 | s)$ is a compact 1-manifold, so its boundary contains an even number of points. Thus, a \mathbb{Z}_2 count of both sides of the expression for $\partial\mathcal{M}(p_1, p_2; p_0 | s)$ gives us:

$$\begin{aligned} 0 &= \sum_{p'_1} \#_{\mathbb{Z}_2} \mathcal{M}_{1,2,3}(p_1, p'_1) \cdot \#_{\mathbb{Z}_2} \mathcal{M}(p'_1, p_2; p_0 | s) \\ &+ \sum_{p'_2} \#_{\mathbb{Z}_2} \mathcal{M}_{2,3,3}(p_2, p'_2) \cdot \#_{\mathbb{Z}_2} \mathcal{M}(p_1, p'_2; p_0 | s) \\ &+ \sum_{p'_0} \#_{\mathbb{Z}_2} \mathcal{M}(p_1, p_2; p'_0 | s) \cdot \#_{\mathbb{Z}_2} \mathcal{M}_{1,3,3}(p'_0, p_0). \end{aligned} \quad (11)$$

This now implies the cochain map condition

$$m_2(\delta p_1 \otimes p_2) + m_2(p_1 \otimes \delta p_2) = \delta m_2(p_1 \otimes p_2).$$

This follows since the terms on the right hand side of Equation 11 are exactly the coefficients of the three terms in the cochain map condition.

As an example, consider the term $m_2(\delta p_1 \otimes p_2)$:

$$\begin{aligned} m_2(\delta p_1 \otimes p_2) &= \sum_{p_0} \#_{\mathbb{Z}_2} \mathcal{M}(\delta p_1, p_2, p_0 \mid s) \cdot p_0 \\ &= \sum_{p_0} \#_{\mathbb{Z}_2} \mathcal{M}(\sum_{p'_1} \#_{\mathbb{Z}_2} \mathcal{M}_{1,2;3}(p_1, p'_1) \cdot p'_1, p_2, p_0 \mid s) \cdot p_0 \\ &= \sum_{p_0} (\sum_{p'_1} \#_{\mathbb{Z}_2} \mathcal{M}_{1,2;3}(p_1, p'_1) \cdot \#_{\mathbb{Z}_2} \mathcal{M}(p'_1, p_2, p_0 \mid s)) \cdot p_0. \end{aligned}$$

The other two terms follow similarly, which shows that m_2 is a cochain map, as desired. □

Corollary 2. *Given a generating family $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$, there is a product map on Generating Family Cohomology*

$$\mu_2 : GH^i(F) \otimes GH^j(F) \rightarrow GH^{i+j}(F).$$

Chapter 7

Invariance with respect to Equivalences of F

Recall from Section 2.2 that there is a notion of equivalence \sim of generating families for a Legendrian submanifold $\Lambda \subset J^1(M)$. Lemma 7 shows that $GH^*(F)$ is invariant under \sim . In this chapter, we show that the product is unchanged under \sim as well. This amounts to showing that, when \widehat{F} is obtained from F by stabilization or fiber-preserving diffeomorphism resulting in isomorphisms $GH^*(F) \rightarrow GH^*(\widehat{F})$ as in Section 3.3, the following diagram commutes:

$$\begin{array}{ccc}
 GH^*(F) \otimes GH^*(F) & \xrightarrow{\mu_2} & GH^*(F) \\
 \cong \downarrow & & \downarrow \cong \\
 GH^*(\widehat{F}) \otimes GH^*(\widehat{F}) & \xrightarrow{\widehat{\mu}_2} & GH^*(\widehat{F}).
 \end{array} \tag{12}$$

7.1 Stabilization

Given a generating family $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$, define $F^\pm : M \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ where $F^\pm(x, e, e') = F(x, e) \pm (e')^2$. To show invariance under stabilization, it suffices to show

that the diagram 12 commutes for $\widehat{F} = F^\pm$. Observe that we can write out new extended difference functions

$$w_{i,j;3}^\pm : M \times \mathbb{R}^{3N} \times \mathbb{R}^3 \rightarrow \mathbb{R}$$

from F^\pm in terms of stabilizations of F :

$$\begin{aligned} w_{1,2;3}^+(x, e_1, e'_1, e_2, e'_2, e_3, e'_3) &= F(x, e_1) + (e'_1)^2 - F(x, e_2) - (e'_2)^2 + e_3^2 + (e'_3)^2 \\ w_{2,3;3}^+(x, e_1, e'_1, e_2, e'_2, e_3, e'_3) &= F(x, e_2) + (e'_2)^2 - F(x, e_3) - (e'_3)^2 + e_1^2 + (e'_1)^2 \\ w_{1,3;3}^+(x, e_1, e'_1, e_2, e'_2, e_3, e'_3) &= F(x, e_1) + (e'_1)^2 - F(x, e_3) - (e'_3)^2 - e_2^2 - (e'_2)^2 \\ w_{1,2;3}^-(x, e_1, e'_1, e_2, e'_2, e_3, e'_3) &= F(x, e_1) - (e'_1)^2 - F(x, e_2) + (e'_2)^2 + e_3^2 + (e'_3)^2 \\ w_{2,3;3}^-(x, e_1, e'_1, e_2, e'_2, e_3, e'_3) &= F(x, e_2) - (e'_2)^2 - F(x, e_3) + (e'_3)^2 + e_1^2 + (e'_1)^2 \\ w_{1,3;3}^-(x, e_1, e'_1, e_2, e'_2, e_3, e'_3) &= F(x, e_1) - (e'_1)^2 - F(x, e_3) + (e'_3)^2 - e_2^2 - (e'_2)^2, \end{aligned}$$

which induces isomorphisms with $GH^*(F)$ by Lemma 11. We may also express these stabilized extended difference functions as

$$w_{i,j;3}^\pm(x, e_1, e'_1, e_2, e'_2, e_3, e'_3) = w_{i,j;3}(x, e_1, e_2, e_3) + Q_{i,j;3}^\pm(e'_1, e'_2, e'_3)$$

for different nondegenerate quadratic functions $Q_{i,j;3}^\pm : \mathbb{R}^3 \rightarrow \mathbb{R}$.

Remark 16. As in Section 3.3, given a generating family F and F^\pm as above, if $p \in \text{Crit}_+(w)$, then there are corresponding critical points $p^\pm \in \text{Crit}_+(w^\pm)$ with the same critical value and $|p^\pm| = |p|$, even though the Morse index changes. Note that, by construction, this correspondence passes to the extended difference functions and bijections in Lemma 14: If $p \in \text{Crit}(w_{i,j;3})$, then there is a corresponding critical point $p^\pm \in \text{Crit}(w_{i,j;3}^\pm)$ whose primed coordinates are 0. Hence, $w_{i,j;3}(p) = w_{i,j;3}^\pm(p^\pm)$. The Morse index increases by 1 if $j - i = 1$

and 2 if $j - i = 2$.

The gradient trajectories we are interested in now live in $P_3 \times \mathbb{R}^3$ rather than P_3 . To study trajectories, we equip $P_3 \times \mathbb{R}^3$ with split metrics $g'_{i,j;3} = g_{i,j;3} + g_0$ where $g_{i,j;3}$ is a metric on P_3 as in Definition 9 and g_0 is the standard Riemannian metric on \mathbb{R}^3 . Such a metric facilitates comparison of gradient trajectories of the stabilized extended difference functions to those before stabilization. It is necessary to check that such a metric satisfies the conditions in Definition 9.

Lemma 19. *If $g_{i,j;3}$ is a metric of the form in Definition 9; that is, if $g_{i,j;3} = g_w + g_Q$, then $g'_{i,j;3} = g_{w^\pm} + g_{Q'}$ for $g_{w^\pm} \in \mathcal{G}_{F^\pm}$ and $g_{Q'} \in \mathcal{G}_{Q'}$. Here, \mathcal{G}_{F^\pm} and $\mathcal{G}_{Q'}$ are the metric sets defined in Definition 9 for the stabilized generating family $F^\pm : M \times \mathbb{R}^{N+1}$ and corresponding quadratic form $Q' : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ so that $w'_{i,j;3} = w^\pm + Q'$.*

Proof. The only non-immediate condition to check is the Smale condition, but since $w^\pm(x, e_1, e'_1, e_2, e'_2) = w(x, e_1, e_2) \pm (e'_1)^2 \mp (e'_2)^2$, the techniques in the proof of Proposition 11 show that these metrics will ensure the Smale condition. \square

Remark 17. With this choice of metrics $g_{i,j;3}$, the below relations between the unstable/stable manifolds hold, where $p_i^\pm \in C(F^\pm)$ denotes the corresponding critical point to $p_i \in C(F)$, see Remark 16. Note that we are abusing notation as promised in Remark 8. The first diffeomorphism, as noted by Remark 11, is due to the fact that the Morse trajectory spaces inherit their smooth structures from the unstable and stable manifolds.

$$\begin{aligned}
\mathcal{M}_{1,2;3}^+(p_1^+, P_3 \times \mathbb{R}^3) &\cong W_{p_1^+}^-(w_{1,2;3}^+) \cong W_{p_1^-}(w_{1,2;3}) \times \mathbb{R}_{e'_1} \times \{0\}_{e'_2} \times \mathbb{R}_{e'_3} \\
\mathcal{M}_{2,3;3}^+(p_2^+, P_3 \times \mathbb{R}^3) &\cong W_{p_2^+}^-(w_{2,3;3}^+) \cong W_{p_2^-}(w_{2,3;3}) \times \mathbb{R}_{e'_1} \times \mathbb{R}_{e'_2} \times \{0\}_{e'_3} \\
\mathcal{M}_{1,3;3}^+(P_3 \times \mathbb{R}^3, p_0^+) &\cong W_{p_0^+}^+(w_{1,3;3}^+) \cong W_{p_0^+}(w_{1,3;3}) \times \{0\}_{e'_1} \times \mathbb{R}_{e'_2} \times \mathbb{R}_{e'_3} \\
\mathcal{M}_{1,2;3}^-(p_1^-, P_3 \times \mathbb{R}^3) &\cong W_{p_1^-}^-(w_{1,2;3}^-) \cong W_{p_1^-}(w_{1,2;3}) \times \{0\}_{e'_1} \times \mathbb{R}_{e'_2} \times \mathbb{R}_{e'_3} \\
\mathcal{M}_{2,3;3}^-(p_2^-, P_3 \times \mathbb{R}^3) &\cong W_{p_2^-}^-(w_{2,3;3}^-) \cong W_{p_2^-}(w_{2,3;3}) \times \mathbb{R}_{e'_1} \times \{0\}_{e'_2} \times \mathbb{R}_{e'_3} \\
\mathcal{M}_{1,3;3}^-(P_3 \times \mathbb{R}^3, p_0^-) &\cong W_{p_0^-}^+(w_{1,3;3}^-) \cong W_{p_0^+}(w_{1,3;3}) \times \mathbb{R}_{e'_1} \times \mathbb{R}_{e'_2} \times \{0\}_{e'_3}
\end{aligned}$$

Remark 17 tells us how the space $X = \mathcal{M}_{1,2;3}(p_1, P_3) \times \mathcal{M}_{2,3;3}(p_2, P_3) \times \mathcal{M}_{1,3;3}(P_3, p_0)$ in which our moduli space of flow trees lives, compares to the space

$$X^\pm = \mathcal{M}_{1,2;3}^\pm(p_1^\pm, P_3 \times \mathbb{R}^3) \times \mathcal{M}_{2,3;3}^\pm(p_2^\pm, P_3 \times \mathbb{R}^3) \times \mathcal{M}_{1,3;3}^\pm(P_3 \times \mathbb{R}^3, p_0^\pm).$$

It remains to check transversality of perturbed evaluation at endpoints maps with the diagonal $\widehat{\Delta}^3 \cong \Delta^3 \times \Delta_{\mathbb{R}^3} \subset (P_3 \times \mathbb{R}^3)^3$ persists, and that the resulting preimage, the moduli space of flow trees, is diffeomorphic to the preimage from before. Given a perturbation $s = (s_1, s_2, s_3) \in S^3$, we claim that $(s, 0) \in \widehat{S}^3$ achieves transversality, where \widehat{S} is the perturbation ball in $P_3 \times \mathbb{R}^3$ as defined in Definition 13. As we will prove in the following chapter, the product does not depend on the choice of perturbation used in its construction.

$$\begin{array}{ccc}
 & & \Delta^3 \\
 & \swarrow \text{dotted} & \downarrow \\
 X & \xrightarrow{E_s} & (P_3)^3 \\
 \downarrow & & \downarrow \\
 X^\pm & \xrightarrow{E_{(s,0)}^\pm} & (P_3 \times \mathbb{R}^3)^3 \\
 & \swarrow \text{dotted} & \uparrow \\
 & & \Delta^3
 \end{array} \tag{13}$$

Lemma 20. *Given $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ and $F^\pm : M \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$, let $p_1, p_2, p_0 \in C(F)$ have corresponding critical points $p_1^\pm, p_2^\pm, p_0^\pm \in C(F^\pm)$ (see Remark 16). Then if $E_s \pitchfork \Delta^3$ then $\widehat{E}_{(s,0)} \pitchfork \widehat{\Delta}^3$.*

Proof. Our choice of metric and perturbation reduces this question to a elementary differential topology one. We wish to show the following in \mathbb{R}^9 :

$$\left(W_0^-(Q_{1,2,3}^\pm) \times W_0^-(Q_{2,3,3}^\pm) \times W_0^+(Q_{1,3,3}^\pm) \right) \pitchfork \Delta_{\mathbb{R}^3}^3.$$

Remark 17 tells us what these stable and unstable manifolds are. Since the product of these manifolds in both the + and - case is 6-dimensional and $\Delta_{\mathbb{R}^3}^3$ is 3-dimensional, the result follows because

$$\left(W_0^-(Q_{1,2,3}^\pm) \times W_0^-(Q_{2,3,3}^\pm) \times W_0^+(Q_{1,3,3}^\pm) \right) \cap \Delta_{\mathbb{R}^3}^3 = \{0\}$$

□

With our choice of split metric and no extra perturbation, the following lemma shows that the moduli space of flow trees that define out product splits into a space of trees defined through the original generating family F and “constant” trees, that is, three constant

trajectories at $0 \in \mathbb{R}^3$.

Proposition 16. *Given $F, F^\pm, p_i \in C(F)$, and $P_i^\pm \in C(F^\pm)$, $\widehat{\mathcal{M}}(p_1^\pm, p_2^\pm; p_0^\pm | (s, 0))$ is diffeomorphic to $\mathcal{M}(p_1, p_2; p_0 | s)$.*

Proof. Our setup shows that we may split this preimage

$$\widehat{E}_{(s,0)}^{-1}(\widehat{\Delta}^3) \cong E_s^{-1}(\Delta^3) \times E_0^{-1}(\Delta_{\mathbb{R}^3}).$$

As $E_s^{-1}(\Delta^3) = \mathcal{M}(p_1, p_2; p_0 | s)$, we consider $E_0^{-1}(\Delta_{\mathbb{R}^3})$. This space consists of triples of trajectories $\{\gamma_1, \gamma_2, \gamma_3\}$ with $\gamma_1, \gamma_2 : (-\infty, 0] \rightarrow \mathbb{R}^3$ and $\gamma_3 : [0, \infty) \rightarrow \mathbb{R}^3$. The trajectory γ_1 flows from $0 \in \mathbb{R}^3$ and follows $\nabla_{g_0} Q_{1,2,3}^\pm$, and so, by Remark 17, is contained in the (e'_1, e'_3) -plane in the + case and the (e'_2, e'_3) -plane in the - case. Similarly, γ_2 flows from $0 \in \mathbb{R}^3$ and follows $\nabla_{g_0} Q_{2,3,3}^\pm$, and so is contained in the (e'_1, e'_2) -plane in the + case and the (e'_1, e'_3) -plane in the - case. Thus, in the + case, γ_1 and γ_2 intersect along the e'_1 -axis; in the - case, they intersect along the e'_3 -axis. In either case, γ_3 intersects the intersection of γ_1 and γ_2 and flows to $0 \in \mathbb{R}^3$ along $\nabla_{g_0} Q_{1,3,3}^\pm$. In both the + and - case, however, γ_3 trajectory will only intersect the e'_1 -axis (e'_3 -axis) at 0, and thus has to be the constant trajectory. This implies that both γ_1 and γ_2 never flowed away from 0, and are also constant trajectories. \square

Corollary 3. *If $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ is altered by a positive or negative stabilization resulting in $\widehat{F} : M \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ then the following diagram commutes:*

$$\begin{array}{ccc} GH^*(F) \otimes GH^*(F) & \xrightarrow{\mu_2} & GH^*(F) \\ \cong \downarrow & & \downarrow \cong \\ GH^*(F^\pm) \otimes GH^*(F^\pm) & \xrightarrow{\mu_2^\pm} & GH^*(F^\pm). \end{array} \quad (14)$$

7.2 Fiber Preserving Diffeomorphism

In this section, we analyze how the product is affected when we pre-compose our generating family $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ with a fiber-preserving diffeomorphism $\Phi : M \times \mathbb{R}^N \rightarrow M \times \mathbb{R}^N$. Recall from Subsection 2.2 that, by definition, $\Phi(x, e) = (x, \phi_x(e))$ for a smooth family of diffeomorphisms $\phi_x : \mathbb{R}^N \rightarrow \mathbb{R}^N$. As in Lemma 12, we consider diffeomorphisms ϕ that are isometries outside the compact set K_E since our setup of gradient flow uses metrics that are Euclidean outside K ; see Definitions 4 and 9.

Remark 18. We may extend Φ naturally to a diffeomorphism on $P_3 = M \times \mathbb{R}^{3N}$: Abusing notation, let $\Phi : P_3 \rightarrow P_3$ be defined as

$$(x, e_1, e_2, e_3) \mapsto (x, \phi(e_1), \phi(e_2), \phi(e_3)).$$

Lemma 21. For $\Delta^3 \subset (P_3)^3$, $\Phi^3(\Delta^3) = \Delta^3$.

Proof. This is not a hard fact, but we write out the proof to recall the space Δ^3 . We use coordinates (x, e_1, e_2, e_3) on $P_3 = M \times \mathbb{R}^{3N}$, so we have natural coordinates

$$(x_1, e_{11}, e_{21}, e_{31}, x_2, e_{12}, e_{22}, e_{32}, x_3, e_{13}, e_{23}, e_{33})$$

on P_3^3 . With these coordinates, Δ^3 is the submanifold in which $x_1 = x_2 = x_3$ and $e_{i1} = e_{i2} = e_{i3}$ for $i = 1, 2, 3$, which is preserved under Φ . \square

Lemma 13 showed that critical points and gradient trajectories correspond under diffeomorphism. This induces diffeomorphisms of the stable and unstable manifolds which gives a diffeomorphism $\tilde{X} \cong X$.

$$\begin{array}{ccccc}
X \times S^3 & \xrightarrow{E} & (P_3)^3 & \longleftarrow \supset & \Delta^3 \\
\downarrow & & \downarrow \cong & & \downarrow \\
\tilde{X} \times \tilde{S}^3 & \xrightarrow{E} & (P_3)^3 & \longleftarrow \supset & \Delta^3
\end{array} \tag{15}$$

Proposition 17. *Suppose \tilde{F} is obtained from F through fiber-preserving diffeomorphism. Let $s = (s_1, s_2, s_3) \in S \times S \times S$ and $p_i \in C(F)$ so that $\mathcal{M}(p_1, p_2; p_0|s)$ is a 0-dimensional manifold. Then, for corresponding $\tilde{p}_i \in C(\tilde{F})$, there exists an $\tilde{s} = (\tilde{s}_1, \tilde{s}_2, \tilde{s}_3) \in \tilde{S} \times \tilde{S} \times \tilde{S}$ for some $\tilde{\delta}$ ball \tilde{S} such that $\tilde{\mathcal{M}}(\tilde{p}_1, \tilde{p}_2; \tilde{p}_0 | \tilde{s})$ is in bijection with $\mathcal{M}(p_1, p_2; p_0|s)$.*

Proof. Given a tree $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\} \in \mathcal{M}(p_1, p_2; p_0|s)$, there exists a $(y, y, y) \in \Delta^3 \subset (P_3)^3$ such that $E(\Gamma) = (y, y, y)$, i.e., $E_{1,2;3}(\gamma_1) = E_{2,3;3}(\gamma_2) = E_{1,3;3}(\gamma_3) = y$.

In the case that $M = \mathbb{R}^m$, this means that $y = \gamma_1(0) + s_1 = \gamma_2(0) + s_2 = \gamma_3(0) + s_3$. From Lemma 13, we know that there are corresponding $\tilde{\gamma}_i$. Remark 18 gives us an element $\tilde{y} = \Phi(y) \in \Delta^3$. Thus, there is a unique way to pick \tilde{s}_i so that $\tilde{y} = \tilde{\gamma}_1(0) + \tilde{s}_1 = \tilde{\gamma}_2(0) + \tilde{s}_2 = \tilde{\gamma}_3(0) + \tilde{s}_3$.

For this $\tilde{s} = (\tilde{s}_1, \tilde{s}_2, \tilde{s}_3)$, we have that $E_{\tilde{s}} \cap \Delta^3$ and that $\tilde{s}_i \in \tilde{S}$ for each i , that is, $|\tilde{s}_i| < \tilde{\delta}$. Here, $\tilde{\delta}$ is such that for all $y_1, y_2 \in \tilde{K}$, $|y_1 - y_2| < \tilde{\delta}$ implies that $|(w_{i,j;3} \circ \Phi)(y_1) - (w_{i,j;3} \circ \Phi)(y_2)| < \rho/4$, where ρ is the least positive critical value of w , which is the same as the least positive critical value of $w \circ \Phi$. \square

Remark 19. While using Φ as in Remark 18 gives the desired bijection, $w_{i,j;3} \circ \Phi$ is not an extended difference function as defined in Definition 7. Rather,

$$(w_{i,j;3} \circ \Phi)(x, e_1, e_2, e_3) = (w \circ \Phi)(x, e_i, e_j) \pm (\phi_x(e_k))^2,$$

and $(\phi_x(e_k))^2$ is not necessarily a quadratic form. It is however, a function with only one critical point with preserved index and preserved critical value. While quadratic forms are

especially nice, such a function also suits our needs, as pointed out in Remark 7.

Corollary 4. *Suppose $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ is altered by a fiber-preserving diffeomorphism $\Phi : M \times \mathbb{R}^N \rightarrow M \times \mathbb{R}^N$, where $\Phi(x, e) = (x, \phi_x(e))$ for some diffeomorphisms $\phi_x : \mathbb{R}^N \rightarrow \mathbb{R}^N$ that are isometries outside K_E , resulting in $\tilde{F} = F \circ \Phi$. Then the following diagram commutes:*

$$\begin{array}{ccc}
 GH^*(F) \otimes GH^*(F) & \xrightarrow{\mu_2} & GH^*(F) \\
 \cong \downarrow & & \downarrow \cong \\
 GH^*(\tilde{F}) \otimes GH^*(\tilde{F}) & \xrightarrow{\tilde{\mu}_2} & GH^*(\tilde{F}).
 \end{array} \tag{16}$$

Chapter 8

Invariance under Legendrian isotopy

In this chapter, we study the product as the underlying Legendrian Λ undergoes a Legendrian isotopy. In particular, suppose we have a Legendrian isotopy Λ^t with $t \in [0, 1]$, and suppose Λ^0 has a generating family. From the Persistence of Legendrian Generating Families (see Proposition 8), there exists a smooth path of generating families F^t for Λ^t . In Section 3, we constructed a chain map that induces an isomorphism between $GH^*(F^0) \rightarrow GH^*(F^1)$ (Corollary 9). Similar isomorphisms can be constructed using extended difference functions.

Given F^0 and the resulting F^1 guaranteed by Proposition 8, we may assume by stabilization that both are functions on $M \times \mathbb{R}^N$. We wish to compare the product from F^0 with the product from F^1 . In particular, we wish to show that the following diagram commutes:

$$\begin{array}{ccc}
 GH^*(F^0) \otimes GH^*(F^0) & \xrightarrow{\mu_2^0} & GH^*(F^0) \\
 \cong \downarrow & & \downarrow \cong \\
 GH^*(F^1) \otimes GH^*(F^1) & \xrightarrow{\mu_2^1} & GH^*(F^1).
 \end{array} \tag{17}$$

While we know there exists isomorphisms $GH^*(F^0) \rightarrow GH^*(F^1)$, for the vertical isomorphisms in the diagram in 17, we construct maps that will be compatible with the product. To do this, we slightly alter the setup of the continuation map in Section 3 to produce three continuation maps using paths of extended difference functions. We then extend this idea to form a moduli space of “continuation flow trees” on $P_3 \times I$ and define a map K counting isolated spaces of such trees. Studying the compactification of a 1-dimensional space of the trees shows that the map K defines a cochain homotopy that induces the commutative diagram 17 on cohomology.

The first subsection of this section deals with the vertical isomorphisms in the above diagrams, while the second constructs “continuation trees” which will define a cochain homotopy that implies the commutativity of 17.

8.1 Continuation isomorphisms on $GH^*(F)$

Given the path of linear-at-infinity generating families from $F^0 : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ to $F^1 : M \times \mathbb{R}^N \rightarrow \mathbb{R}$, we wish to compare the product at time $t = 0$ to the one at $t = 1$. For F_0 and F_1 , we constructed continuation maps from the path of difference functions $w^t : M \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ such that $w^t(x, e_1, e_2) = F^t(x, e_1) - F^t(x, e_2)$.

To get continuation isomorphisms that are compatible with the product, we will construct them on the paths of extended difference functions for $t \in [0, 1]$, denoted $w_{i,j,3}^t : M \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ defined as usual by:

$$w_{1,2,3}^t(x, e_1, e_2, e_3) = F^t(x, e_1) - F^t(x, e_2) + e_3^2$$

$$w_{2,3,3}^t(x, e_1, e_2, e_3) = F^t(x, e_2) - F^t(x, e_3) + e_1^2$$

$$w_{1,3,3}^t(x, e_1, e_2, e_3) = F^t(x, e_1) - F^t(x, e_3) - e_2^2$$

For each $w_{i,j;3}^t$, there is the corresponding non-linear support compact set K^t , which will vary smoothly with t .

Given F^0 and the resulting F^1 , construct the resulting extended difference functions as above and pick metrics $g_{i,j;3}^0, g_{i,j;3}^1$ as in Definition 9. Then let $\Gamma_{i,j;3} = \{(w_{i,j;3}^t, g_{i,j;3}^t) \mid t \in [0, 1]\}$ be a path of the extended difference functions and metrics on P_3 that are standard outside K^t from $(w_{i,j;3}^0, g_{i,j;3}^0)$ to $(w_{i,j;3}^1, g_{i,j;3}^1)$.

For each of these three paths we have a continuation map $\Phi_{i,j;3} : C^*(F^0) \rightarrow C^*(F^1)$ defined by counting isolated flow lines of the vector field $\nabla_{G_{i,j;3}} W_{i,j;3}$ on $(M \times \mathbb{R}^{3N}) \times I$ with

$$\begin{aligned} W_{i,j;3}(p, t) &= w_{i,j;3}^t(p) + \epsilon \left((1/2)t^2 - (1/4)t^4 \right) \\ (G_{i,j;3})_{(p,t)} &= (g_{i,j;3}^t)_p + dt^2, \end{aligned} \tag{18}$$

for $\epsilon > 0$ such that $\frac{\epsilon}{4} < \rho$, where ρ is the least positive critical value of w .

As done in detail in Section 3, these maps induce isomorphisms which we will denote by $\Phi_{i,j;3}^*$, with

$$\Phi_{i,j;3}^* : GH^*(F^0) \rightarrow GH^*(F^1),$$

and the arguments in Proposition 6 show that this map does not depend on the path F^t up to homotopy class.

8.2 Continuation flow trees

To get the commutative diagram in 17, we construct a cochain homotopy by defining a moduli space of ‘‘continuation flow trees.’’ The construction will be similar to that of $\mathcal{M}(p_1, p_2; p_0|s)$. Now, our trees will live in $P_3 \times I$ rather than P_3 and we will require that the trees span I , i.e., flow along trajectories out of two critical points p_1, p_2 at $t = 0$ and

along a trajectory that limits to a critical point p_1 at $t = 1$. We will denote the moduli space of continuation flow trees by $\mathcal{M}_I(p_1, p_2; p_0 | \{s^t\})$ and describe its construction in the following paragraphs.

Each branch in a continuation tree will follow one of the vector fields $V_{i,j;3} = \nabla_{G_{i,j;3}} W_{i,j;3}$ defined in the previous subsection in 18. Recall that the path of metrics $g_{i,j;3}^t$ used to define $G_{i,j;3}$ was chosen to be admissible so that the unstable and stable manifolds from each $V_{i,j;3}$ intersect transversely. This does not guarantee the transverse intersection in the flow trees. To fix this and prove that the product does not depend on the perturbation used to achieve transversality, we add the data of a path of perturbation vectors into the construction of the continuation trees.

The perturbation balls S^0 for F^0 and S^1 for F^1 might be of different sizes; see Remark 12. There is, however, a smooth path S^t of perturbation balls connecting them. To construct a path $s^t = (s_1^t, s_2^t, s_3^t) \in (S^t)^3$, first pick endpoints $s^0 \in (S^0)^3$ so that $\mathcal{M}_{F^0}(p_1, p_2; p'_0 | s^0) \neq \emptyset$ is a smooth manifold for any choice of $p'_0 \in \text{Crit}_+(w)$ and $s^1 \in (S^1)^3$ so that $\mathcal{M}_{F^1}(p'_1, p'_2; p_0 | s^1) \neq \emptyset$ is a smooth manifold for any choices of $p'_1, p'_2 \in \text{Crit}_+(w)$. By Theorem 13, almost every choice of s^0 and s^1 will suffice for a fixed triplet of critical points so almost every choice still suffices because $\text{Crit}_+(w)$ is a finite set. Using these endpoints, construct a smooth path $s^t = (s_1^t, s_2^t, s_3^t) \in (S^t)^3$. This path may be perturbed while keeping the admissible endpoint fixed to achieve transversality as described in the following after we setup perturbed continuation evaluation maps.

To get a manifold structure and compactification results on $\mathcal{M}_I(p_1, p_2; p_0 | \{s^t\})$ we need to transversely cut it out of Morse trajectory spaces. As before in Definition 11, we have half-infinite Morse trajectory spaces, for $(p, t) \in C(F^t) \times \{t\}$ for $t = 0$ or $t = 1$.

$$\mathcal{M}_{i,j;3}((p, t), P_3 \times I) = \{\gamma : (-\infty, 0] \rightarrow P_3 \times I \mid \dot{\gamma} = V_{i,j;3}, \lim_{s \rightarrow -\infty} \gamma(s) = (p, t)\} \text{ and}$$

$$\mathcal{M}_{i,j;3}(P_3 \times I, (p, t)) = \{\gamma : [0, \infty) \rightarrow P_3 \times I \mid \dot{\gamma} = V_{i,j;3}, \lim_{s \rightarrow \infty} \gamma(s) = (p, t)\},$$

where we abuse notation and use p to denote a critical point in $\text{Crit}_+(w^t)$ and its bijective image in $\text{Crit}_+(w_{i,j;3}^t)$. Given $p_1, p_2 \in \text{Crit}_+(w^0)$ and $p_0 \in \text{Crit}_+(w^1)$, let

$$X_I = \mathcal{M}_{1,2;3}((p_1, 0), P_3 \times I) \times \mathcal{M}_{2,3;3}((p_2, 0), P_3 \times I) \times \mathcal{M}_{1,3;3}((P_3 \times I, (p_0, 1))).$$

This space has a smooth structure induced by a diffeomorphism with $W_{(p_1,0)}^-(V_{1,2;3}) \times W_{(p_2,0)}^-(V_{2,3;3}) \times W_{(p_0,1)}^+(V_{1,3;3})$.

Given a path $\{s^t\}$ through $(S^t)^3$ as described above, we have analogous perturbed evaluation maps as in Definition 14 that we will use to construct a map E_I as in Definition 15. Given a half-infinite trajectory γ in one of the above spaces, the evaluation maps $\text{ev}_{i,j;3}^\pm(\gamma)$ give a point $\gamma(0) = (\gamma(0)|_{P_3}, \gamma(0)|_I) \in P_3 \times I$ and we will perturb $\gamma(0)|_{P_3}$ by s^t in $P_3 \times \{t\}$ for $t = \gamma(0)|_I$. That is, we have the following three maps, with π representing the submersion or the identity to remain consistent with Definition 14.

$$E_{1,2;3} : \mathcal{M}_{1,2;3}((p_1, 0), P_3 \times I) \rightarrow P_3 \times I$$

$$E_{2,3;3} : \mathcal{M}_{2,3;3}((p_2, 0), P_3 \times I) \rightarrow P_3 \times I$$

$$E_{1,3;3} : \mathcal{M}_{1,3;3}((P_3 \times I, (p_0, 1))) \rightarrow P_3 \times I$$

$$\gamma \mapsto \pi \left(\gamma(0)|_{P_3} + s^{\gamma(0)|_I} \right)$$

Thus, we have the map $E : X_I \rightarrow (P_3 \times I)^3$ defined by

$$E(\gamma_1, \gamma_2, \gamma_3) = (E_{1,2;3}(\gamma_1), E_{2,3;3}(\gamma_2), E_{1,3;3}(\gamma_3)).$$

Definition 24. The moduli space of continuation trees is

$$\mathcal{M}_I(p_1, p_2; p_0 | \{s^t\}) := E^{-1}(\Delta(P_3 \times I)^3).$$

We may express this moduli space as the following set:

$$\mathcal{M}_I(p_1, p_2; p_0 | \{s^t\}) = \left\{ (\gamma_1, \gamma_2, \gamma_3) \left| \begin{array}{l} \gamma_1 : (-\infty, 0] \rightarrow P_3 \times I, \\ \gamma_2 : (-\infty, 0] \rightarrow P_3 \times I, \\ \gamma_3 : [0, \infty) \rightarrow P_3 \times I, \\ \frac{d\gamma_1}{ds} = V_{1,2;3}, \frac{d\gamma_2}{ds} = V_{2,3;3}, \frac{d\gamma_3}{ds} = V_{1,3;3}, \\ E_{1,2;3}(\gamma_1) = E_{2,3;3}(\gamma_2) = E_{1,3;3}(\gamma_3) \\ \lim_{s \rightarrow -\infty} \gamma_1(s) = (p_1, 0), \lim_{s \rightarrow -\infty} \gamma_2(s) = (p_2, 0), \\ \lim_{s \rightarrow \infty} \gamma_3(s) = (p_0, 1) \end{array} \right. \right\}.$$

Lemma 22. *There is a perturbation of the path $\{s^t\}$ so that $E \pitchfork \Delta(P_3 \times I)^3$. Then $\mathcal{M}_I(p_1, p_2; p_0 | \{s^t\})$ is a manifold of dimension $|p_0| - |p_1| - |p_2| + 1$.*

Proof. The freedom given by perturbing the path $\{s^t\}$ together with the larger class of met-

rics used to define $V_{i,j;3}$ give us room to achieve transversality. We calculate the dimension:

$$\begin{aligned}
& \dim(E^{-1}(\Delta(P_3 \times I)^3)) \\
&= \dim(X_I) - \text{codim}((\Delta(P_3 \times I)^3)) \\
&= \dim(W_{(p_1,0)}^-(V_{1,2;3})) + \dim(W_{(p_2,0)}^-(V_{2,3;3})) + \dim(W_{(p_0,1)}^+(V_{1,3;3})) - 2(n + 3N + 1) \\
&= (n + 3N + 1) - \text{ind}_{V_{1,2;3}}((p_1, 0)) + (n + 3N + 1) - \text{ind}_{V_{2,3;3}}((p_2, 0)) \\
&\quad + \text{ind}_{V_{1,3;3}}((p_0, 1)) - 2(n + 3N + 1) \\
&= \text{ind}_{w_{1,3;3}^1}(p_0) + 1 - \text{ind}_{w_{1,2;3}^0}(p_1) - \text{ind}_{w_{2,3;3}^0}(p_2) \\
&= (|p_0| + 2N) + 1 - (|p_1| + N) - (|p_2| + N) \\
&= |p_0| - |p_1| - |p_2| + 1.
\end{aligned}$$

□

As in previous arguments in this dissertation, we will need to understand the boundary of the compactification of a 1-dimensional $\mathcal{M}_I(p_1, p_2; p_0|\{s^t\})$. Rather than defining a larger manifold with corners structure as in Chapter 5, we will use similar arguments to classify possible limits of unbroken continuation trees.

To apply similar arguments, we need bounds on the continuations trees as in Lemmas 16 and 17. The compact non-linear support set from each generating family F^t gives a path of compact non-linear support sets K^t as in Definition 6. A similar argument to Lemma 16 shows that, for all $\Gamma \subset \mathcal{M}_I(p_1, p_2; p_0|\{s^t\})$, $\text{Im}(\Gamma) \subset \bigcup_{t \in I} (K^t \times \{t\}) \subset P_3 \times I$. Similarly, given $\{\rho^t\}$, the path of smallest positive critical values of w^t , we may bound the “midpoint” of any tree Γ , which occurs at a specific slice $P_3 \times \{t\}$ away from the critical submanifold of $w_{1,3;3}^t$ as in Lemma 17.

Proposition 18. *Given $p_1, p_2 \in \text{Crit}_+(w^0)$ and $p_0 \in \text{Crit}_+(w^1)$ with $|p_0| = |p_1| + |p_2|$, if*

$\{s_t\}$ is a path so that $\mathcal{M}_I(p_1, p_2; p_0|\{s^t\})$ is a 1-manifold, then it may be compactified to a 1-manifold $\overline{\mathcal{M}}_I(p_1, p_2; p_0|\{s^t\})$ with boundary

$$\begin{aligned} \partial\overline{\mathcal{M}}_I(p_1, p_2; p_0|\{s^t\}) = & \\ & \bigcup_{p'_1} \mathcal{M}_{1,2,3}((p_1, 0), (p'_1, 0)) \times \mathcal{M}_I(p'_1, p_2; p_0|\{s^t\}) \\ & \bigcup_{p'_2} \mathcal{M}_{2,3,3}((p_2, 0), (p'_2, 0)) \times \mathcal{M}_I(p_1, p'_2; p_0|\{s^t\}) \\ & \bigcup_{p'_0} \mathcal{M}_I(p_1, p_2, p'_0|\{s^t\}) \times \mathcal{M}_{1,3,3}((p'_0, 1), (p_0, 1)) \\ & \bigcup_{p''_0} \mathcal{M}_{F^0}(p_1, p_2, p''_0|s^0) \times \mathcal{M}_{1,3,3}((p''_0, 0), (p_0, 1)) \\ & \bigcup_{p''_1, p''_2} \mathcal{M}_{1,2,3}((p_1, 0), (p''_1, 1)) \times \mathcal{M}_{2,3,3}((p_2, 0), (p''_2, 1)) \times \mathcal{M}_{F^1}(p''_1, p''_2; p_0|s^1), \end{aligned}$$

where the unions are taken over $p'_1 \in C^{|p_1|+1}(F^0)$, $p'_2 \in C^{|p_2|+1}(F^0)$, $p'_0 \in C^{|p_0|-1}(F^1)$, $p''_0 \in C^{|p_0|}(F^0)$, $p''_1 \in C^{|p_1|}(F^1)$, and $p''_2 \in C^{|p_2|}(F^1)$, respectively.

Proof. Let $\mathcal{M}_I(p_1, p_2; p_0|\{s^t\})$ be of dimension 1. A similar argument as in Chapter 5 gives a compactification of this space by trees with once broken branches. By construction of our vector fields, all critical points of $V_{i,j;3}$ live in $P_3 \times \{0\}$ and $P_3 \times \{1\}$. With three branches that may break at critical points in either of these manifolds, we seemingly have six cases of broken trees that might show up in the boundary of a compactified 1-dimensional moduli space of continuation trees:

1. The branch flowing from $(p_1, 0)$ along $V_{1,2,3}$ breaks in $P_3 \times \{0\}$: This would mean that p_1 flows along $\nabla_{g_{1,2,3}} w_{1,2,3}^0$ to another critical point $p'_1 \in C(F^0)$ with $|p'_1| = |p_1| + 1$. An index calculation shows that a tree from $(p'_1, 0)$ and $(p_2, 0)$ to $(p_0, 1)$ would be

isolated.

2. In the same way, $(p_2, 0)$ could flow along $\nabla_{g_{2,3,3}} w_{2,3,3}^0$ to a point $(p'_2, 0)$ with $p'_2 \in C^{|p_2|+1}(F^0)$. Note that the indices force only one edge to break in this way at a time.
3. If the branch flowing along $V_{1,3,3}$ ending at $(p_0, 1)$ breaks at a point in $P_3 \times \{1\}$, then the trajectories form a tree from $(p_1, 0)$ and $(p_2, 0)$ to a critical point $(p'_0, 1)$, where $p'_0 \in C^{|p_0|-1}(F^1)$ and then p'_0 flows along $\nabla_{g_{1,3,3}} w_{1,3,3}^1$ to p_0 .
4. If the branch flowing along $V_{1,3,3}$ to $(p_0, 1)$ breaks at $t = 0$ at a point $(p''_0, 0)$, then we see a tree that must be contained in $P_3 \times \{0\}$. Since $\text{ind}_{V_{1,3,3}}(p''_0, 0) = \text{ind}_{V_{1,3,3}}(p_0, 1) - 1$, it must be that $p''_0 \in C^{|p_0|}(F^0)$. The tree in $P_3 \times \{0\}$ is in a moduli space $\mathcal{M}_{F^0}(p_1, p_2, p''_0 | s^0)$ of flow trees from F^0 , and our conditions on the endpoints of the path $\{s^t\}$ guarantee that this a manifold of dimension $|p''_0| - |p_1| - |p_2| = 0$. We then see a flow line from $(p''_0, 0)$ to $(p_0, 1)$, which is in the 0-dimensional continuation moduli space $\mathcal{M}_{1,3,3}((p''_0, 0), (p_0, 1))$.
5. Suppose the branch following $V_{1,2,3}$ from $(p_1, 0)$ breaks in $P_3 \times \{1\}$ at a point $(p'_1, 1)$. This would imply that $(p'_1, 1) \in \text{Crit}_+^{\text{ind}_{w_{1,2,3}^1}(p_1)+1}(V_{1,2,3})$, so $p'_1 \in C^{|p_1|}(F^1)$. Thus, this is a flow line in the isolated continuation moduli space $\mathcal{M}_{1,2,3}((p_1, 0), (p'_1, 1))$. Then we see a tree with the $V_{1,2,3}$ branch contained in $P_3 \times \{1\}$. In particular, the “midpoint” of the tree (the point in $\Delta(P_3 \times I)^3$) is a point $y \in P_3 \times \{1\}$. This means that the branch γ_2 of the tree that flows along $V_{2,3,3}$ has finite endpoint $\gamma_2(0)$ in $P_3 \times \{1\}$. Due to the ∂t component of the vector field vanishing as $t \rightarrow 1$, this cannot happen in finite time. Thus, the branch flowing along $V_{2,3,3}$ must break at a critical point $(p''_2, 1)$ with $p''_2 \in C^{|p_2|}(F^1)$. Then there is a tree from $(p'_1, 1)$ and $(p''_2, 1)$ to $(p_0, 1)$ completely contained in $P_3 \times \{1\}$. This tree lives in a moduli space $\mathcal{M}_{F^1}(p'_1, p''_2; p_0 | s^1)$, which, due to the construction of the perturbation path $\{s^t\}$, is a manifold of dimension 0.

□

Definition 25. We define a map $K : C^i(F^0) \otimes C^j(F^0) \rightarrow C^{i+j-1}(F^1)$ as follows:

Given $p_1 \in \text{Crit}_+(w^0)$ and $p_2 \in \text{Crit}_+(w^0)$, then

$$K(p_1 \otimes p_2) = \sum (\#_{\mathbb{Z}_2} \mathcal{M}_I(p_1, p_2; p_0 | \{s^t\})) \cdot p_0$$

where the sum is taken over $p_0 \in \text{Crit}_+(w^1)$ with $|p_0| = |p_1| + |p_2| - 1$. Extend the product bilinearly over the tensor product.

The following Corollary follows directly from the description of the boundary of a compactified one-dimensional continuation flow tree moduli space in Proposition 18.

Corollary 5. *The map $K : C^i(F^0) \otimes C^j(F^0) \rightarrow C^{i+j-1}(F^1)$ is a cochain homotopy, i.e.,*

$$\delta_{1,3;3} \circ K + K \circ (\delta_{1,2;3} \otimes 1 + 1 \otimes \delta_{2,3;3}) = (\Phi_{1,3;3} \circ m_2^0) + m_2^1 \circ (\Phi_{1,2;3} \otimes \Phi_{2,3;3})$$

$$\begin{array}{ccc}
 C(F^0) \otimes C(F^0) & \xrightarrow{\delta \otimes 1 + 1 \otimes \delta} & C(F^0) \otimes C(F^0) & (19) \\
 \downarrow \Phi_{1,3;3} \circ m_2^0 & \searrow K & \downarrow \Phi_{1,3;3} \circ m_2^0 \\
 C(F^1) & \xrightarrow{\delta} & C(F^1)
 \end{array}$$

The following results follow from the fact that K is a cochain homotopy:

Theorem 19. *Let $\Lambda_t \subset J^1 M$, $t \in [0, 1]$ be isotopy of Legendrian submanifolds, and suppose Λ_0 has a linear-at-infinity generating family. Then for F^0 and F^1 guaranteed by Proposition*

8, the following diagram commutes:

$$\begin{array}{ccc}
 GH^*(F^0) \otimes GH^*(F^0) & \xrightarrow{\mu_2^0} & GH^*(F^0) \\
 \cong \downarrow & & \downarrow \cong \\
 GH^*(F^1) \otimes GH^*(F^1) & \xrightarrow{\mu_2^1} & GH^*(F^1).
 \end{array} \tag{20}$$

Since paths of metrics and perturbations appeared in the construction of continuation flow trees, the resulting cochain homotopy also shows the following two results of invariance.

Corollary 6. *The construction of μ_2 does not depend on choice of metrics from \mathcal{G}_F and \mathcal{G}_Q in Definition 9 used in the gradient vector fields.*

Corollary 7. *The construction of μ_2 does not depend on choice of perturbation s used to achieve transversality in $\mathcal{M}(p_1, p_2; p_0|s)$.*

Chapter 9

Directions of Future Study: An A_∞ Algebra on $C(F)$

Constructing μ_2 is part of a larger project to construct an A_∞ algebra for a Legendrian with a generating family. To explain this, we first give a brief introduction to A_∞ algebras and then sketch a construction for the higher order products that make up this structure.

9.1 A_∞ Background

The notion of an A_∞ algebra was introduced by Stasheff [33] to study homotopy associativity of topological spaces.

Definition 26. An A_∞ algebra over \mathbb{Z}_2 is a graded vector space V along with maps $m_k : V^{\otimes k} \rightarrow V$ of degree $2 - k$ satisfying

$$\sum_{i+j+l=k} m_{i+1+l} \circ (1^{\otimes i} \otimes m_j \otimes 1^{\otimes l}) = 0.$$

The first three relations tell us that $m_1 \circ m_1 = 0$ and m_2 descends to the cohomology

$H^*(V, m_1)$ to form an associative product there. Higher order products will not in general descend to cohomology, but we can form Massey products on cohomology modulo the images of lower order products.

There is an equivalence on A_∞ algebras:

Definition 27. An A_∞ **morphism** $\phi : (V, \mathbf{m}) \longrightarrow (W, \mathbf{n})$ is a collection of graded maps $\phi_k : V^{\otimes k} \longrightarrow W$ of degree $1 - k$ that satisfy

$$\sum_{i+j+l=k} \phi_{i+1+l} \circ (1^{\otimes i} \otimes m_j \otimes 1^{\otimes l}) = \sum_{\substack{1 \leq r \leq n \\ i_1 + \dots + i_r = n}} n_r \circ (\phi_{i_1} \otimes \dots \otimes \phi_{i_r}).$$

The morphism ϕ is an A_∞ **quasi-isomorphism** if ϕ_1 induces an isomorphism on cohomology.

For $k \geq 2$, while the m_k maps of an A_∞ algebra on V will not descend to the cohomology $H^*(V, m_1)$, we can recover an A_∞ structure on its homology using the following theorem:

Theorem 20. [22] *If (V, \mathbf{m}) is an A_∞ algebra over a field, then its cohomology $H^*(V)$ also has an A_∞ structure given by $\boldsymbol{\mu}$ such that $\mu_1 = 0$, μ_2 is induced by m_2 , and there is an A_∞ quasi-isomorphism $(H^*(V), \boldsymbol{\mu}) \longrightarrow (V, \mathbf{m})$. Further, this structure is unique up to A_∞ quasi-isomorphism.*

9.2 Setup of A_∞ Algebra

We will follow a similar procedure to define the maps $m_k : C(F)^{\otimes k} \rightarrow C(F)$ as was used to define m_2 . We define extended difference functions $w_{i,j;k+1}$ and set up gradient flow trees using these functions. One of the added complications here is that there is a space of configurations of trees that include different finite-length gradient trajectories.

Definition 28. Suppose $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a generating family for Λ . Let P_{k+1} denote the $(k+1)^{\text{th}}$ ordered fiber product $P_{k+1} = M \times \mathbb{R}^{(k+1)N}$. For each $1 \leq i < j \leq k+1$ the **extended difference function** $w_{i,j;k+1} : P_{k+1} \rightarrow \mathbb{R}$, is defined as

$$w_{i,j;k+1}(x, e_1, \dots, e_{k+1}) = F(x, e_i) - F(x, e_j) + \sum_{\ell=1}^{i-1} e_\ell^2 - \sum_{\ell=i+1}^{j-1} e_\ell^2 + \sum_{\ell=j+1}^{k+1} e_\ell^2,$$

where for $e_\ell = (e_{\ell 1}, \dots, e_{\ell N}) \in \mathbb{R}^N$, $e_\ell^2 := e_{\ell 1}^2 + \dots + e_{\ell N}^2$. The set of critical points of $w_{i,j;k+1}$ with positive critical value will be denoted by $\text{Crit}_+(w_{i,j;k+1}) \subset P_{k+1}$.

As in Lemma 14, there are bijections between the positive-valued critical points of w with those of $w_{i,j;k+1}$.

Remark 20. For $1 \leq i < j \leq k+1$, there is a bijection

$$\begin{aligned} \iota_{i,j;k+1} : \text{Crit}_+(w) &\rightarrow \text{Crit}_+(w_{i,j;k+1}) \\ (x, e, e') &\mapsto (x, 0, \dots, 0, e, 0, \dots, 0, e', 0, \dots, 0). \end{aligned} \tag{21}$$

Given $p = (x, e, e') \in \text{Crit}(w)$, we have that

$$|p| = \text{ind } w(p) - N = \text{ind } w_{i,j;k+1}(p) - (j - i - 1)N - N = \text{ind } w_{i,j;k+1}(p) - (j - i)N$$

Under this bijection the critical value of p will agree with the critical value of $\iota_{i,j;k+1}(p)$. Thus every critical point p (resp. $\iota_{i,j;k+1}(p)$) of w (resp. $w_{i,j;k+1}$), of positive critical value will correspond to a Reeb chord of the Legendrian Λ generated by F , and the positive critical value of a critical point p (resp. $\iota_{i,j;k+1}(p)$) will be the length of the corresponding Reeb chord. By an abuse of notation, we will often use p to denote both p and $\iota_{i,j;k+1}(p)$.

We will study positive gradient flow lines of the extended difference functions $w_{i,j;k+1}$ working with Riemannian metrics $g_{i,j;k+1}$ that split as in Definition 9. As in the $k = 2$

case, we will consider gradient flow lines that “intersect” to form gradient flow trees. The biggest difference in general setup is that, for $k > 2$, there are different configurations of trees involving finite interior edges. Following ideas of Fukaya, [13], we will first define a space \mathcal{T}_k of different configurations of weighted rooted planar trees with k leaves, and then define the moduli space of flow trees $\mathcal{M}_F(p_1, p_2, \dots, p_k; p_0)$ as maps of trees in \mathcal{T}_k into the ambient space P_{k+1} defined in Definition 28.

A **tree** is a one dimensional simplicial complex that is compact and simply connected. In particular, trees do not contain cycles. Recall that the valency of the vertex denotes the number of edges that connect to it.

Definition 29. For $k \geq 2$, fix a set of $k + 1$ distinct points $V_k = \{v_0, \dots, v_k\} \subset S^1 = \partial D^2$, ordered counterclockwise. The **space of weighted, rooted k -trees** \mathcal{T}_k consists of the set of trees, T , embedded in $D^2 \subset \mathbb{R}^2$ satisfying:

1. T has V_k as a subset of its vertices; all vertices in V_k have valency 1;
2. No vertex of T has valency equal to 2;
3. Each edge of T is assigned an element of $\mathbb{R}^+ \cup \{\infty\}$: each **external edge**, meaning an edge with a vertex in V_k , is assigned ∞ ; any other edge, referred to as **internal edge**, is assigned a positive real number.

Such trees form spaces sometimes called associahedra or Stasheff polytopes, after:

Theorem 21. [33] \mathcal{T}_k is homeomorphic to \mathbb{R}^{k-2} .

Since the valency of any vertex of $T \in \mathcal{T}_k$ cannot equal 2, any T will have at most $k - 2$ internal edges. Note that this gives $k - 2$ possible length parameters if we consider trees with fewer edges as having some edges of length 0.

For each $T \in \mathcal{T}_k$, $D^2 - T$ will consist of $k + 1$ connected components. Order these so that the vertex v_i is between the regions i and $i + 1$ for $1 \leq i \leq k$, and v_0 is between regions 1 and $k + 1$; see Figure 6. For $i < j$, let the edge $e_{i,j}$ be the edge between region i and region j , and denote its corresponding length parameter by $l_{i,j}$. Give the following orientation to $e_{i,j}$ with region i on its left and region j on its right:

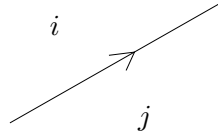


Figure 5: For $i < j$, the direction of the gradient flow of $w_{i,j;k}$.

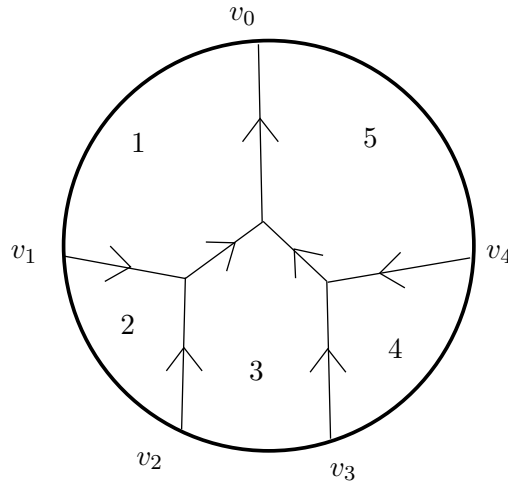


Figure 6: An example of how a tree in \mathcal{T}_4 embeds into D^2 in this way.

For a tree $T \in \mathcal{T}_k$, we will let $\text{Int } T$ denote the tree without its vertices in V_k :

$$\text{Int } T = T \cap \text{Int } D^2.$$

Given a linear-at-infinity generating family $F : M \times \mathbb{R}^N \rightarrow \mathbb{R}$ and $p_1, \dots, p_k, p_0 \in C(F)$, we form the associated unperturbed Moduli Space of Gradient Flow Trees, $\mathcal{M}(p_1, \dots, p_k; p_0)$, as follows. Let $p_1 \in \text{Crit}(w_{1,2;k+1})$, \dots , $p_k \in \text{Crit}(w_{k,k+1;k+1})$, $p_0 \in \text{Crit}(w_{1,k+1;k+1})$ be the

images of p_1, \dots, p_k, p_0 under the maps in (5). Then $\mathcal{M}(p_1, p_2, \dots, p_k; p_0) = \{\Gamma : \text{Int } T \rightarrow P_{k+1} \mid T \in \mathcal{T}_k\}$, where Γ satisfy:

1. For $1 \leq i \leq k$, after parametrizing the external edges $e_{i,i+1}$, $\Gamma|_{e_{i,i+1}} : (-\infty, 0] \rightarrow P_{k+1}$ satisfies

$$\frac{d}{dt}\Gamma|_{e_{i,i+1}} = \nabla_{g_{i,i+1}} w_{i,i+1;k+1}, \quad \lim_{t \rightarrow -\infty} \Gamma|_{e_{i,i+1}} = p_i.$$

2. After parametrizing the external edge $e_{1,k+1}$, $\Gamma|_{e_{1,k+1}} : [0, \infty) \rightarrow P_{k+1}$ satisfies

$$\frac{d}{dt}\Gamma|_{e_{1,k+1}} = \nabla_{g_{1,k+1}} w_{1,k+1;k+1}, \quad \lim_{t \rightarrow \infty} \Gamma|_{e_{1,k+1}} = p_0.$$

3. After parametrizing the internal edge $e_{i,j}$ to have length $\ell_{i,j}$,

$\Gamma|_{e_{i,j}} : [0, \ell_{i,j}] \rightarrow P_{k+1}$ satisfies

$$\frac{d}{dt}\Gamma|_{e_{i,j}} = \nabla_{g_{i,j}} w_{i,j;k+1}.$$

Future work must be done to show that such spaces, perhaps with some perturbations at some internal vertices, have the structure of a smooth manifold and may be compactified to a space with the structure of a smooth manifold with corners. Then we may define higher-order product maps $m_k : C(F)^{\otimes k} \rightarrow C(F)$ by counting isolated moduli spaces of gradient flow trees as in Chapter 6. The boundary of the compactification of a 1-dimensional space should prove the A_∞ relations, similar to how we proved that m_2 is a cochain map. Theorem 20 will transfer this structure to $GH^*(F)$. A continuation argument as in Chapter 8 will show that this structure is invariant under Legendrian isotopy up to A_∞ quasi-isomorphism.

References

- [1] F. Bourgeois and B. Chantraine. Bilinearised Legendrian contact homology and the augmentation category. Preprint available as arXiv:1210.7367, 2012.
- [2] D. Burghelea and S. Haller. On the topology and analysis of a closed one form. i (novikovs theory revisited), essays on geometry and related topics, 133–175. monogr. *Enseign. Math.*, 38.
- [3] Yu. Chekanov. Critical points of quasifunctions, and generating families of Legendrian manifolds. *Funktsional. Anal. i Prilozhen.*, 30(2):56–69, 96, 1996.
- [4] Yu. Chekanov. Differential algebra of Legendrian links. *Invent. Math.*, 150:441–483, 2002.
- [5] Yu. Chekanov and P. Pushkar. Combinatorics of Legendrian links and the Arnol’d 4-conjectures. *Russ. Math. Surv.*, 60(1):95–149, 2005.
- [6] G. Civan, J. Etnyre, P. Koprowski, J. Sabloff, and A. Walker. Product structures for Legendrian contact homology. *Math. Proc. Camb. Phil. Soc.*, 150(2):291–311, 2011.
- [7] T. Ekholm, J. Etnyre, and M. Sullivan. Legendrian contact homology in $P \times \mathbb{R}$. *Trans. Amer. Math. Soc.*, 359(7):3301–3335 (electronic), 2007.
- [8] Ya. Eliashberg. Invariants in contact topology. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, number Extra Vol. II, pages 327–338 (electronic), 1998.
- [9] Ya. Eliashberg, A. Givental, and H. Hofer. Introduction to symplectic field theory. *Geom. Funct. Anal.*, (Special Volume, Part II):560–673, 2000.
- [10] A. Floer. Symplectic fixed points and holomorphic spheres. *Comm. Math. Phys.*, 120(4):575–611, 1989.
- [11] J. M. Franks. Morse-smale flows and homotopy theory. *Topology*, 18(3):199–215, 1979.

- [12] D. Fuchs and D. Rutherford. Generating families and Legendrian contact homology in the standard contact space. *J. Topol.*, 4(1):190–226, 2011.
- [13] K. Fukaya. Morse homotopy and its Quantization.
- [14] K. Fukaya. *Morse homotopy, A-category, and Floer homologies*. Math. Sciences Research Inst., 1993.
- [15] K. Fukaya, Y-G. Oh, H. Ohta, and K. Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Parts I and II*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2009.
- [16] V. Guillemin and A. Pollack. *Differential topology*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1974.
- [17] M. B. Henry. Connections between Floer-type invariants and Morse-type invariants of Legendrian knots. *Pacific J. Math.*, 249(1):77–133, 2011.
- [18] M.B. Henry and D. Rutherford. A combinatorial DGA for Legendrian knots from generating families. Preprint available as arXiv:1106.3357, 2011.
- [19] M. Hutchings. Lecture notes on Morse homology (with an eye towards Floer theory and pseudoholomorphic curves). *available at math.berkeley.edu/~hutching/teach/276-2010/mfp.ps*, 2002.
- [20] M. Hutchings. Floer homology of families. I. *Algebr. Geom. Topol.*, 8(1):435–492, 2008.
- [21] J. Jordan and L. Traynor. Generating family invariants for Legendrian links of unknots. *Algebr. Geom. Topol.*, 6:895–933 (electronic), 2006.
- [22] T. Kadeišvili. On the theory of homology of fiber spaces. *Uspekhi Mat. Nauk*, 35(3(213)):183–188, 1980. International Topology Conference (Moscow State Univ., Moscow, 1979).
- [23] S. Mescher. Perturbed gradient flow trees and a_∞ -algebra structures on morse cochain complexes, 2016.
- [24] J. Milnor. *Morse theory*. Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.
- [25] M. Morse. *The calculus of variations in the large*, volume 18. American Mathematical Soc., 1934.
- [26] L. Ng. Computable Legendrian invariants. *Topology*, 42(1):55–82, 2003.
- [27] L. Ng, D. Rutherford, V. Shende, S. Sivek, and E. Zaslow. Augmentations are sheaves. *arXiv preprint arXiv:1502.04939*, 2015.

- [28] L.T. Nielsen. Transversality and the inverse image of a submanifold with corners. *Mathematica Scandinavica*, pages 211–221, 1982.
- [29] J. Sabloff and M. Sullivan. Families of Legendrian submanifolds via generating families. *Quantum Topol.*, To Appear.
- [30] J. Sabloff and L. Traynor. Obstructions to the existence and squeezing of Lagrangian cobordisms. *J. Topol. Anal.*, 2(2):203–232, 2010.
- [31] J. Sabloff and L. Traynor. Obstructions to Lagrangian cobordisms between Legendrian submanifolds. *Algebr. Geom. Topol.*, 13:2733–2797, 2013.
- [32] M. Schwarz. *Morse homology*, volume 111 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1993.
- [33] J. Stasheff. Homotopy associativity of H-spaces. I. *Trans. Amer. Math. Soc.*, 108(2):275–292, 1963.
- [34] D. Théret. A complete proof of Viterbo’s uniqueness theorem on generating functions. *Topology Appl.*, 96(3):249–266, 1999.
- [35] L. Traynor. Generating function polynomials for Legendrian links. *Geom. Topol.*, 5:719–760, 2001.
- [36] C. Viterbo. Symplectic topology as the geometry of generating functions. *Math. Ann.*, 292(4):685–710, 1992.
- [37] K. Wehrheim. Smooth structures on Morse trajectory spaces, featuring finite ends and associative gluing. *Geometry & Topology Monographs*, 18:369–450, 2012. Proceedings of the Freedman Fest.
- [38] E. Witten. Supersymmetry and Morse theory. *J. Differential Geom.*, 17(4):661–692 (1983), 1982.

VITA - ZIVA MYER

zmyer@brynmawr.edu \diamond sites.google.com/site/zivamyer/

Department of Mathematics, 101 N. Merion Ave \diamond Bryn Mawr, PA 19010

EDUCATION

Bryn Mawr College

Fall 2011 - Present

Ph.D., Mathematics (expected)

May 2017

Advisor: Lisa Traynor

Dissertation Title: A Product Structure on Generating Family Cohomology for Legendrian Submanifolds

New College of Florida

Fall 2007 - Spring 2011

B.A., Mathematics/Physics

Senior Thesis: An Algebraic Combinatorial Approach to Discovering Properties of Sudoku Graphs

POSITIONS

William W. Elliott Assistant Research Professor, Duke University

Fall 2017 - Summer 2020

Mildred and Carl Otto von Kienbusch Graduate Fellowship, Bryn Mawr College

Fall 2016 - Spring 2017

Teaching Assistanship, Bryn Mawr College

Fall 2015 - Spring 2016, Fall 2011 - Spring 2014

Research Fellowship, Bryn Mawr College

Fall 2014 - Spring 2015

PUBLICATIONS AND PREPRINTS

Updates at: sites.google.com/site/zivamyer/home/papers

Z. Myer, *An A_∞ algebra for a Legendrian from generating families*, In preparation.

Z. Myer, *A product structure on generating family cohomology for Legendrians*, In preparation.

PRESENTATIONS

Product structures for generating family cohomology of Legendrian submanifolds, Joint Math Meetings, Special Session on Symplectic Geometry, Moment Maps and Morse Theory, Atlanta, GA Jan. 2017

The generating family cohomology ring for Legendrian submanifolds, Joint Math Meetings, Special Session on Pure and Applied Talks by Women Math Warriors Presented by EDGE (Enhancing Diversity in Graduate Education), Atlanta, GA Jan. 2017

Product structures for generating family cohomology of Legendrian submanifolds, AMS Sectional Meeting, Special Session on low-dimensional topology, North Carolina State University, Nov. 2016

An A_∞ algebra for Legendrian submanifolds with generating families, Geometry & Topology Seminar, Massachusetts Institute of Technology, Oct. 2016

A product structure for generating family cohomology for Legendrians, Tetrahedral Geometry/Topology Seminar, Lancaster County, PA, Sept. 2016

Poster: *A_∞ Algebras for a Legendrian Submanifold from Generating Families*, Summer School on Symplectic topology, Sheaves, and Mirror symmetry, Institut de Mathématiques de Jussieu, Paris, Summer 2016

A-infinity algebras for Legendrians from generating families, Graduate Student Conference in Algebra, Geometry, and Topology, Temple University, May 2016

Product Structures for Legendrians from Generating Families, Graduate Student Topology and Geometry Conference, Indiana University, Apr. 2016

An A-infinity structure for Legendrians from generating families, Tech Topology Conference, Lightning Talk, Georgia Tech, Dec. 2015

Sheaves and Legendrian knot theory, Philadelphia Area Contact/Topology (PACT) Seminar, Series of talks with Joshua Sabloff, Nov. - Dec. 2015

Product structures for Legendrians from generating families, Graduate Student Seminar, Temple University, Nov. 2015

Distressing Math Collective

Bryn Mawr College

A weekly gathering of undergraduate students, graduate students, faculty, and alumnae.

- Contributed presentations: *The Topology of Impossible Spaces* (Mar. 2015), *Generating Families for Legendrians* (Apr. 2014), *You Cantor Ways Get What You Want – Cantor Sets, The Cantor Function, and How Calculus Lied to You* (March 2013), *Legendary Legendrians: On the Cusp of Understanding Legendrian Knots* (Sept. 2012), *Sudoku Scheming: A Combinatorial Analysis of Sudoku Graphs* (Sept. 2011)

TEACHING

Dean's Certificate in Pedagogy

Spring 2016

Certification, Bryn Mawr College

- Completed the graduate course Perspectives in Math Pedagogy for the requirements of the certificate.
- In the course, I compiled a personal teaching portfolio including teaching observations and a syllabus and course plan, lecture and non-lecture lesson plans, assessments, and exams for a linear algebra class.

Teaching Assistant

- Courses: Transitions to Higher Mathematics (Spring 2016), Abstract Algebra I (Fall 2015, Fall 2013) and (Spring 2014), Real Analysis I (Fall 2012) and II (Spring 2013), Linear Algebra (Spring 2012), Multivariable Calculus (Fall 2011)
- Responsibilities include leading four hours of problem sessions each week, grading homework, and assisting in class activities.

Substitute Lecturer

- Knot Theory and its Applications (Fall 2015) and Abstract Algebra II (Spring 2014)

Private Tutor

2012-2013

Greater Philadelphia Area

- Algebra I and II, Geometry

Math Tutor

2007 - 2011

Quantitative Resource Center, New College of Florida

PROGRAMS AND CONFERENCES ATTENDED WITH EXTERNAL FUNDING

Summer School on Symplectic topology, Sheaves, and Mirror symmetry, Institut de Mathématiques de Jussieu, Paris, Summer 2016

Georgia Topology Conference, University of Georgia, May 2016

Topological and Quantitative Aspects of Symplectic Manifolds, Columbia University, March 2016

Tech Topology Conference, Georgia Institute for Technology, Dec. 2015
Summer School in Moduli Problems in Symplectic Geometry, IHÉS, France, Summer 2015
Redbud Topology Conference, Oklahoma State University, April 2015
MSRI Graduate Summer School: Algebraic Topology, Guanajuato, Mexico, Summer 2014
Parameterized Morse Theory, Banff International Research Station, Canada, March 2014
Graduate Student Topology and Geometry Conference, Notre Dame University, April 2013
Redbud Topology Conference, University of Arkansas, March 2013
Program for Women and Mathematics, 21st Century Geometry, Institute for Advanced Study, Summer 2012
Topology Students Workshop, Georgia Institute for Technology, Summer 2012
Enhancing Diversity in Graduate Education (EDGE), Florida A & M University, Summer 2011

PROFESSIONAL SERVICE

Departmental Service

- Graduate Student Recruiting, Graduate School Fair, Joint Math Meetings, Seattle, WA 2016
- Graduate Student Representative, Mathematics Hiring Committee, BMC, Fall 2014
- Biweekly meetings with colloquium speakers, introduced the colloquium talk at least twice a semester, 2014 - present

Catalyst Conference

March 2013

- Catalyst is a conference for 7th and 8th grade girls that encourages them to take classes and pursue careers in scientific fields.
- I co-lead a workshop on *Honeycombs and Infinite Soccer Balls: Euclidean and Non-Euclidean Tessellations*.

Enhancing Diversity in Graduate Education

Summer 2015

Graduate Student Mentor

Howard University

- Responsibilities: Ongoing mentorship of fifteen women beginning Ph.D. programs in the mathematical sciences, lead problem sessions, and organized group dinners.
- Publicity: Article in the Bryn Mawr Alumnae Bulletin

SEMINAR ORGANIZER/CONTRIBUTOR

Co-organizer: Lie Group Seminar (Spring 2014)

Active participant: Philadelphia Area Contact/Topology (PACT) Seminar, Bryn Mawr College, (Fall 2011 to present)

Active participant: Philadelphia Area Topology (Contact/Hyperbolic) (PATCH) Seminar University of Pennsylvania, Temple University, and Bryn Mawr and Haverford Colleges (2012 to present)

PROFESSIONAL MEMBERSHIPS

American Mathematical Society

Association for Women in Mathematics

National Center for Faculty Development & Diversity