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Legendrian circular helix links

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Abstract

Examples are given of legendrian links in the manifold of cooriented contact elements of the plane, or equivalently, in the 1-jet space of the circle which are not equivalent via an isotopy of contact diffeomorphisms. These examples have generalizations to linked legendrian spheres in contact manifolds diffeomorphic to $\mathbb{R}^n \times S^{n-1}$. These links are distinguished by applying the theory of generating functions to contact manifolds.

1. *Introduction*

A contact 3-manifold is a smooth manifold with a field of tangent 2-planes satisfying a non-degeneracy condition. This non-degeneracy implies the field of hyperplanes has no intregral surfaces. However, there are many integral curves, known as legendrian curves. A basic problem in contact topology is to classify legendrian curves up to contact isotopies of the ambient manifold. Legendrian knots are legendrian submanifolds diffeomorphic to S^1 and a legendrian link is a collection of disjoint legendrian knots. Background on legendrian knots can be found, for example, in [**B**], [**E**], [**A**].

Motivated by [**A**], this paper focuses on examples of standard links in the manifold of cooriented contact elements, a contact manifold diffeomorphic to $\mathbb{R}^2 \times S^1$. The components of the links will consist of the following legendrian knots in $\mathbb{R}^2 \times S^1, S^1 := \mathbb{R}/2\pi\mathbb{Z}$:

$$G(1) := \{(\cos\theta, \sin\theta, \theta) : \theta \in S^1\},$$
$$G(2) := \{(2\cos\theta, 2\sin\theta, \theta) : \theta \in S^1\},$$
$$R(2) := \{(2\cos\theta, 2\sin\theta, \theta + \pi) : \theta \in S^1\}.$$

It is often convenient to visualize these knots as quotients of right-handed helices in $\mathbb{R}^3$ of radii $1, 2$, respectively. $R(2)$ and $G(2)$ differ only by a translation in the S^1 coordinate. The links

$$G(1) \coprod G(2), \quad G(1) \coprod R(2)$$

are topologically the same: there exists an isotopy of $\mathbb{R}^2 \times S^1$ that takes $G(2)$ to $R(2)$ and at the same time returns $G(1)$ to $G(1)$. The legendrian knots $G(2)$ and $R(2)$ are equivalent via an isotopy of contact transformations. However,

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THEOREM. *The legendrian links $G(1) \coprod G(2)$, $G(1) \coprod R(2)$ are not equivalent via an isotopy of contact diffeomorphisms of the manifold of cooriented contact elements.*

This result has a slightly more general formulation in the contact manifold $\mathscr{J}^1(S^1)$, the 1-jet space of real-valued functions on a circle, with its standard contact structure. The 1-jet of a smooth function $f\colon S^1 \to \mathbb{R}$,

$$\Lambda_f := \left\{\left(q, \frac{df}{dq}(q), f(q)\right)\right\} \subset \mathscr{J}^1(S^1),$$

is a legendrian knot. Links are naturally formed by considering tuples of functions.

THEOREM′. *Consider $f, g, h\colon S^1 \to \mathbb{R}$ where $f(q) < g(q) < h(q)$, for all $q \in S^1$. Then the legendrian links*

$$\Lambda_g \coprod \Lambda_h, \quad \Lambda_g \coprod \Lambda_f$$

are topologically equivalent but not equivalent via an isotopy of contact transformations of $\mathscr{J}^1(S^1)$.

The theorems above are proved in Section 5 as an application of generating functions in contact topology. The proof that the above links are different is reminiscent of Viterbo's proof of the symplectic camel theorem, [**V**].

In a more 'twisted' version of the manifold of cooriented contact elements, $\mathbb{R}^2 \times S^1$ with an alternate contact structure, there exist non-equivalent legendrian links whose knot components complete more turns of either a right-handed or left-handed helix before closing; see Section 7. The above results also have generalizations to higher dimensions. There exist legendrian links in a contact manifold diffeomorphic to $\mathbb{R}^n \times S^{n-1}$ which are topologically equivalent but not equivalent via an isotopy consisting of contact transformations. Here, each component of the link is diffeomorphic to S^{n-1}; see Section 8. Limitations of this technique to the study of other links are discussed in Section 9.

2. *Background*

A contact structure on a $(2n+1)$-manifold M is a completely non-integrable tangent hyperplane field, ξ. The complete non-integrability of ξ can be expressed by the inequality $\alpha \wedge (d\alpha)^n \neq 0$ where ξ is locally described by $\xi = \ker \alpha$. For simplicity, throughout this section, it will be assumed that ξ is transversally orientable so that it can be described as the kernel of some 1-form α. A diffeomorphism $\kappa\colon (M_0, \xi_0) \to (M_1, \xi_1)$ is a *contactomorphism* or *contact diffeomorphism* if $\kappa_*(\xi_0) = \xi_1$. A *contact isotopy* is a smooth 1-parameter family of contactomorphisms, $\kappa_t\colon (M, \xi) \to (M, \xi)$, $t \in [0, 1]$, such that $\kappa_0 = id$. An n-dimensional submanifold $\mathscr{L}$ in (M^{2n+1}, ξ) is *legendrian* if it is tangent to ξ: $T\mathscr{L} \subset \xi$. For more background on contact geometry, see [**A-G**], [**E**], [**M-S**].

For a fixed contact 3-manifold (M, ξ), let $\mathscr{L}\mathscr{E}(S^1)$ denote the space of embeddings of S^1 into M with legendrian images. Then consider

$$\coprod_1^k \mathscr{L}\mathscr{E}(S^1) := \{(\ {}_1, \ldots, \ {}_k)\colon \ {}_i \in \mathscr{L}\mathscr{E}(S^1), \quad \mathrm{Im}\ {}_i \cap \mathrm{Im}\ {}_j = \emptyset, \quad i \neq j\}.$$

A *legendrian knot*, $\mathscr{K}$, is the image of $\ \in \mathscr{L}\mathscr{E}(S^1)$. A *legendrian link*, $(\mathscr{K}_1 \coprod \ldots \coprod \mathscr{K}_k)$, is the image of $\Phi \in \coprod_1^k \mathscr{L}\mathscr{E}(S^1)$. Two lengendrian links, $(\mathscr{K}_1 \amalg \ldots \amalg \mathscr{K}_k)$, $(\mathscr{L}_1 \amalg \ldots \amalg \mathscr{L}_k)$, are *equivalent* if there exists a contact isotopy $\kappa_t : M \to M$, $t \in [0,1]$, such that $\kappa_1(\mathscr{K}_i) = \mathscr{L}_i$, $i = 1, \ldots, k$. The notation $(\mathscr{K}_1 \amalg \ldots \amalg \mathscr{K}_k) \simeq (\mathscr{L}_1 \amalg \ldots \amalg \mathscr{L}_k)$ will be used to denote equivalent legendrian links.

The following three central results in contact topology hold for contact manifolds of all dimensions. The first is the contact analogue of the symplectic 'Moser stability'. Proofs can be found in [**A-G**], [**M-S**].

(2·1) Gray stability. *Let M be a manifold without boundary. Suppose ξ_t, $t \in [0,1]$, is a smooth family of contact structures such that $(d/dt)\,\xi_t = 0$ on a closed submanifold Q and on the complement of C where C is a compact set. Then there exists an isotopy κ_t of M such that $(\kappa_t)_* \xi_t = \xi_0$ and $\kappa_t = id$ on Q and on the complement of C, for all $t \in [0,1]$.*

One consequence of Gray stability is the legendrian neighbourhood theorem which implies that legendrian links do not have any local invariants. For details, see [**A-G**].

(2·2) Legendrian neighbourhoods. *If $\mathscr{L}_0 \subset M_0$ and $\mathscr{L}_1 \subset M_1$ are diffeomorphic, closed, legendrian submanifolds then there exist neighbourhoods U_0, U_1 of $\mathscr{L}_0, \mathscr{L}_1$ and a contact diffeomorphism $\kappa: (U_1, \mathscr{L}_1) \to (U_0, \mathscr{L}_0)$.*

The next result implies that the equivalence of legendrian knots or links can be reduced to the study of paths in $\coprod_1^k \mathscr{L}\mathscr{E}(S^1)$. For the reader's convenience, a proof is included.

(2·3) Legendrian isotopy extension theorem. *Let L be a closed lengendrian submanifold of (M, ξ) and let $j_t : L \to M$, $t \in [0,1]$, be an isotopy ($j_0 = id$) such that $j_t(L)$ is legendrian. Then there exists a contact isotopy $\kappa_t : (M, \xi) \to (M, \xi)$ such that $\kappa_t|_L = j_t$.*

Proof. First it will be useful to review the concepts of a Reeb vector field and a contact hamiltonian. Suppose $\xi = \ker \alpha$. Given the contact form α, there exists a unique vector field $R = R_\alpha : M \to TM$ such that

$$\alpha(R) = 1, \quad i_R\, d\alpha = 0. \tag{2·3·1}$$

The vector field is called the *Reeb* vector field determined by α. Given any function $h: M \to \mathbb{R}$, there exists a unique vector field $X = X_h : M \to \mathbb{R}$ which satisfies

$$\alpha(X) = h, \quad i_X\, d\alpha = dh(R)\,\alpha - dh. \tag{2·3·2}$$

To see this, suppose h is given. Since $d\alpha|_\xi$ is non-degenerate, there exists a unique vector field $Z : M \to TM$ such that $Z \in \xi$ and $i_Z\, d\alpha|_\xi = dh|_\xi$. The vector field $X_h \coloneqq Z + hR$ is as required. Note that if X is integrable, the characterizations of X imply

$$\mathscr{L}_X \alpha = d(i_X \alpha) + i_X\, d\alpha = dh + (dh(R)\,\alpha - dh) = dh(R)\,\alpha$$

and thus X integrates to a contact isotopy.

Now suppose $j_t : L \to M$ is a legendrian isotopy. Consider the vector field X_t defined along $j_t(L)$ by

$$X_t \circ j_t = \frac{d}{dt} j_t.$$

Then consider $h_t = \alpha(X_t) : j_t(L) \to \mathbb{R}$. The idea is to extend h_t to a compactly supported function $\hat{h}_t : M \to \mathbb{R}$ so that for all $p \in j_t(L)$,

$$\alpha(X_t) = \hat{h}_t, \quad i_{X_t}\, d\alpha = d\hat{h}_t(R)\,\alpha - d\hat{h}_t.$$

Then the vector field $\hat{X}_t$ uniquely defined by $\hat{h}_t$ as in (2·3·2) will be integrable and extend X_t. First suppose $\hat{h}_t$ is chosen so that for all $p \in j_t(L)$, $\hat{h}_t(p) = h_t(p)$. This guarantees that on $j_t(L)$, $\alpha(X_t) = \hat{h}_t$. This also implies that for $v \in T(j_t(L))$,

$$d\alpha(X_t, v) = X_t\,\alpha(v) - v\alpha(X_t) - \alpha([X_t, v]) = -v\alpha(X_t) = -dh_t(v) = -d\hat{h}_t(v).$$

If more generally, it is required that $\hat{h}_t$ is chosen so that $d\hat{h}_t(p)\,(v) = -d\alpha(p)\,(X_t, v)$ for all $v \in \xi_p$, $p \in j_t(L)$, then it is easy to check that $i_{X_t} d\alpha(v) = d\hat{h}_t(R)\,\alpha(v) - d\hat{h}_t(v)$ for all $v \in T_pM$, $p \in j_t(L)$ and thus $\hat{X}_t$ will extend X_t as desired. ∎

3. *Standard links in a solid torus*

There are two standard contact manifolds which are diffeomorphic to the open, solid 3-torus. First consider the *manifold of cooriented contact elements of the plane*, $ST^*\mathbb{R}^1$. A point in this manifold consists of a point $p \in \mathbb{R}^2$ and a line in the tangent space at this point, $\ell \subset T_p(\mathbb{R}^2)$, together with a choice of one of the two half-planes into which ℓ divides the tangent plane. Standard coordinates on this manifold consist of the cartesian coordinates (x, y) of a point in the plane and the angular coordinate $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of the cooriented normal vector to ℓ. With respect to these coordinates, the standard contact structure, ξ_1, is globally defined by

$$\xi_1 = \ker(\cos\theta\, dx + \sin\theta\, dy). \qquad (3·1)$$

A second standard contact manifold diffeomorphic to $\mathbb{R}^2 \times S^1$ is the 1-*jet space of real-valued functions on the circle*, $\mathscr{J}^1(S^1) = \{(q, p, z) : q \in \mathbb{R}/2\pi\mathbb{Z}\}$, with its standard contact structure,

$$\eta = \ker(dz - p\,dq). \qquad (3·2)$$

In fact, $ST^*\mathbb{R}^2$ and $\mathscr{J}^1(S^1)$ are contactly equivalent via the hodograph transformation. This contactomorphism $\tau : (ST^*\mathbb{R}^2, \xi_1) \to (\mathscr{J}^1(S^1), \eta)$ is given by

$$\tau(x, y, \theta) = (\theta, -\sin\theta x + \cos\theta y, \cos\theta x + \sin\theta y). \qquad (3·3)$$

Thus results about the contact topology of $(ST^*\mathbb{R}^2, \xi_1)$ can be translated into results about the contact topology of $(\mathscr{J}^1(S^1), \eta)$ and vice versa. In this paper, it is often easier to visualize legendrians in $ST^*\mathbb{R}^2$ but to calculate in $\mathscr{J}^1(S^1)$.

For these standard contact manifolds, it is convenient to describe lengendrian curves from their wave fronts. In general, if a contact manifold M is a legendrian fibration over a manifold $X, \pi: M \to X$, then the *wave front* of a legendrian submanifold $\mathscr{L} \subset M$ is $\pi(\mathscr{L}) \subset X$. For example, the projection $\pi : ST^*\mathbb{R}^2 \to \mathbb{R}^2$ and the projection $\pi : \mathscr{J}^1(S^1) \to S^1 \times \mathbb{R}$ onto the (q, z)-coordinates are both legendrian fibrations. Generic wave fronts of legendrians in $ST^*\mathbb{R}^2$ will be cooriented immersed curves with semi-cubic cusp singularities. The cooriented wave front in $\mathbb{R}^2$ uniquely defines the legendrian in $ST^*\mathbb{R}^2$. For more details, see [**A**]. Wave fronts of legendrian knots in $\mathscr{J}^1(S^1)$ will include immersed graphs of multi-valued functions with non-vertical tangents and semi-cubic cusp singularities. Again, the wave front in $S^1 \times \mathbb{R}$ uniquely defines the legendrian in $\mathscr{J}^1(S^1)$. The links studied in this paper have easily described wave fronts.

Definition 3·4. For $r \geqslant 0, C^{\pm}(r) \subset ST^*\mathbb{R}^2$ are legendrian knots:

$$C^+(r) := \{(r\cos\theta, r\sin\theta, \theta) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\},$$
$$C^-(r) := \{(r\cos\theta, r\sin\theta, \theta + \pi) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}.$$

The wave front of $C^+(r)$ is the circle of radius r with outward pointing normals while the wave front of $C^-(r)$ is the circle with inward pointing normals. The $\pm$ sign of these legendrians will be referred to as the *charge* of the legendrian.

Definition 3·5. The graph of any smooth function $f: S^1 \to \mathbb{R}$ has a lift to the legendrian submanifold

$$\Lambda_f := \left\{ (q, p, z) : p = \frac{df}{dq}(q), \quad z = f(q) \right\}.$$

Remark 3·6. Under the hodograph transformation, $\tau(C^\pm(r)) = \Lambda_{\pm r}$ where $\Lambda_{\pm r}$ are the legendrians associated to the constant functions of value $\pm r$,

$$\Lambda_r = \{(q, 0, r) : q \in S^1\}, \quad r \in \mathbb{R}. \quad \diamond$$

In fact, any two legendrians knots in $\mathscr{J}^1(S^1)$ which have graph wave fronts are equivalent.

LEMMA 3·7. *For any smooth functions $f, g: S^1 \to \mathbb{R}$, the legendrians $\Lambda_f, \Lambda_g \subset \mathscr{J}^1(S^1)$ are equivalent.*

Proof. The contact isotopy $\kappa_t: \mathscr{J}^1(S^1) \to \mathscr{J}^1(S^1)$,

$$\kappa_t(q, p, z) = (q, p + t(g' - f')(q), z + t(g - f)(q)), \quad t \in [0, 1],$$

satisfies $\kappa_1(\Lambda_f) = \Lambda_g$. ▌

COROLLARY 3·8. *For $r \geqslant 0$, $C^+(1) \simeq C^+(r) \simeq C^-(r)$.*

Proof. Because of Remark 3·6, this statement is an immediate corollary of Lemma 3·7. However, for later purposes, it will be helpful to keep in mind the following isotopy of $ST^*\mathbb{R}^2$. Consider the path in $\mathscr{L}\mathscr{E}(S^1)$, ℓ_t: $C^+(r) \to ST^*\mathbb{R}^2$, $t \in [-1, 1]$, $\ell_t(x, y, \theta) = (tx, ty, \theta)$. Since $\operatorname{Im} \ell_1 = C^+(r)$ and $\operatorname{Im} \ell_{-1} = C^-(r)$, Lemma 2·3 implies $C^-(r) \simeq C^+(r)$. ▌

Consider links which are pairs of these lengendrian knots. For example, consider the four legendrian links

$$C^\pm(1) \coprod C^\pm(2), \quad C^\pm(1) \coprod C^\mp(2). \tag{3·9}$$

Each of the four links $C^\pm(1) \coprod C^\pm(2)$, $C^\pm(1) \coprod C^\mp(2)$ differ by translations in the θ coordinate and thus are pairwise topologically equivalent. The radii 1, 2 were chosen for convenience and some of these legendrian links are clearly equivalent.

PROPOSITION 3·10. *For $0 \leqslant r_1 < r_2$,*

(i) $$\begin{cases} C^+(1) \coprod C^+(2) \simeq C^+(r_1) \coprod C^+(r_2); \\ C^-(1) \coprod C^-(2) \simeq C^-(r_1) \coprod C^-(r_2). \end{cases}$$

(ii) $$\begin{cases} C^+(1) \coprod C^+(2) \simeq C^-(1) \coprod C^+(2); \\ C^+(1) \coprod C^-(2) \simeq C^-(1) \coprod C^-(2). \end{cases}$$

(iii) $$C^+(1) \coprod C^-(2) \simeq C^+(2) \coprod C^-(1) \simeq C^+(2) \coprod C^+(1).$$

Proof. Slight modifications of the proof of Corollary 3·8 prove all statements. ▌

However, not all of the legendrian links in (3·9) are equivalent. Notice that in

Proposition 3·10(iii), to change the charge on the outermost wavefront, the strands of the link were interchanged. The main result of this paper is that it is impossible to make this charge change on the outer wave front without swapping the strands.

THEOREM 3·11. $C^+(1)\coprod C^-(2) \not\simeq C^+(1)\coprod C^+(2)$.

This theorem will be proved in Section 5. Assuming Theorem 3·11, there are a number of other links that can be immediately distinguished.

COROLLARY 3·12.

(i) $$C^-(1)\coprod C^-(2) \not\simeq C^-(1)\coprod C^+(2);$$

(ii) $$C^+(1)\coprod C^+(2) \not\simeq C^+(2)\coprod C^+(1);$$

(iii) $$C^-(1)\coprod C^+(1) \not\simeq C^+(1)\coprod C^-(1).$$

Proof. Statement (i) follows from Theorem 3·11 since by Proposition 3·10,

$$C^-(1)\coprod C^-(2) \simeq C^+(1)\coprod C^-(2), \quad C^-(1)\coprod C^+(2) \simeq C^+(1)\coprod C^+(2).$$

To prove (ii), note that by Proposition 3·10,

$$C^+(1)\coprod C^+(2) \simeq C^-(1)\coprod C^+(2) \simeq C^-(2)\coprod C^+(1)$$

which by Theorem 3·11 is not equivalent to $C^+(2)\coprod C^+(1)$. Statement (iii) follows from Theorem 3·11 since $C^+(1)\coprod C^-(1) \simeq C^+(1)\coprod C^-(2)$ and, by Proposition 3·10,

$$C^-(1)\coprod C^+(1) \simeq C^-(1)\coprod C^+(2) \simeq C^+(1)\coprod C^+(2).$$

Theorem 3·11 can be generalized to the following result about $\mathcal{J}^1(S^1)$.

THEOREM 3·13. *Let* $f_1, g_1, f_2, g_2 : S^1 \to \mathbb{R}$ *be functions satisfying*

$$f_1(q) < g_1(q), \quad f_2(q) < g_2(q), \quad \forall q \in S^1.$$

If $\Lambda_{f_i}, \Lambda_{g_i} \subset \mathcal{J}^1(S^1)$ *denote the associated legendrian knots then*

$$\Lambda_{f_1}\coprod \Lambda_{g_1} \simeq \Lambda_{f_2}\coprod \Lambda_{g_2} \quad \textit{and} \quad \Lambda_{f_1}\coprod \Lambda_{g_1} \not\simeq \Lambda_{g_2}\coprod \Lambda_{f_2}.$$

The pointwise ordering on the functions guarantees the knot components of the link are disjoint. Theorem 3·13 will be proved in Section 5 after the machinery of generating functions is developed in Section 4. A more 'twisted' version of Theorem 3·11 is described in Section 7 and higher dimensional generalizations of Theorems 3·11 and 3·13 are discussed in Section 8.

4. *Legendrian generating functions*

Consider the 1-jet space $\mathcal{J}^1(Z)$ of a closed n-manifold Z with its standard contact structure $\eta = \ker\alpha$ where $\alpha := dz - pdq$. The 1-jet of a smooth function, $f: Z \to \mathbb{R}$, is a closed, legendrian submanifold. More generally, if $F: Z\times\mathbb{R}^k \to \mathbb{R}$ has fibre derivatives $\partial F/\partial x$ transverse to 0 then Λ, described in local coordinates as

$$\Lambda := \left\{\left(q, \frac{\partial F}{\partial q}(q,x), F(q,x)\right) : \frac{\partial F}{\partial x}(q,x) = 0\right\},$$

is an immersed legendrian submanifold of $\mathcal{J}^1(Z)$ and F is called a *generating function* for Λ. A function $F: Z\times\mathbb{R}^k \to \mathbb{R}$ is said to be *quadratic at infinity* if outside a compact set, $F(q,x) \equiv Q(x)+C$ where Q is a non-degenerate quadratic function and C is a constant. Critical points of a generating function correspond to points where Λ intersects the set $\{p=0\}$.

The following existence theorem is proved by Chaperon in [**C**]. An alternate proof which uses a symplectization procedure is given in the appendix to this paper.

(4·1) *Let Z be a closed manifold and let $\Lambda_0 \subset \mathscr{J}^1(Z)$ denote the 1-jet of the zero function. If $\kappa_t, t\in[0,1]$, is a contact isotopy of $\mathscr{J}^1(Z)$ then there exists a smooth 1-parameter family of quadratic at infinity generating functions $F_t: Z\times\mathbb{R}^k \to \mathbb{R}$ for $\Lambda_t := \kappa_t(\Lambda_0)$.*

Following ideas of Viterbo, [**V**], there are two natural invariants associated to these quadratic at infinity functions. First, for a non-critical value b of $F: Z\times\mathbb{R}^k\to\mathbb{R}$, consider

$$F^b := \{(q,x) : F(q,x) \leqslant b\}.$$

If F is quadratic at infinity then for $0 \ll b < c, F^b$ is a deformation retract of F^c and F^{-c} is a deformation retract of F^{-b}. If $F^{\pm\infty}$ denotes $F^{\pm c}$ for $c \gg 0$, a direct calculation shows that the relative homology groups are isomorphic to the homology groups of Z:

$$\tau : H_*(Z) \xrightarrow{\simeq} H_{*+\iota}(F^{+\infty}, F^{-\infty}),$$

where ι is the index of the quadratic $Q, F\equiv Q+C$ outside a compact set. Also notice that for all b, there is a homomorphism $i_* : H_*(F^b, F^{-\infty}) \to H_*(F^{+\infty}, F^{-\infty})$ induced by inclusion. Given a quadratic at infinity function F, define

$$\left.\begin{aligned} c_+(F) &= \inf\{b : \tau(\mu_n) \in i_*(H_{n+\iota}(F^b, F^{-\infty}))\} \\ c_-(F) &= \inf\{b : \tau(\mu_0) \in i_*(H_\iota(F^b, F^{-\infty}))\}, \end{aligned}\right\} \qquad (4{\cdot}2)$$

where μ_n, μ_0 are generators of $H_n(Z)$, $H_0(Z)$, $\dim Z = n$, respectively. The following proposition is proved by Viterbo in [**V**]. A proof is given for the reader's convenience.

(4·3) *If F_t is a smooth family of quadratic at infinity functions, $F_t: Z\times\mathbb{R}^k\to\mathbb{R}$, then $c_\pm : [0,1]\to\mathbb{R}$ defined by $c_\pm(t) = c_\pm(F_t)$ are continuous, piecewise smooth functions.*

Proof. Fix $t_0 \in [0,1]$. Given $\epsilon > 0$, suppose b is a non-critical value of F_{t_0} with $|c_\pm(t) - b| \geqslant \epsilon > 0$. Choose ϵ_b so that $[b-\epsilon_b, b+\epsilon_b]$ contains no critical values of F_{t_0}. By applying fiber-preserving diffeomorphisms, it can be assumed $d/dt F_t = 0$ outside a compact set. Thus there exists $\delta > 0$ such that $\|F_t - F_{t_0}\| < \epsilon_b$ for all $t \in (t_0 - \delta, t_0+\delta) \cap [0,1]$. By choosing δ_b, $0 < \delta_b < \delta$, it can be assumed that $[b-\epsilon_b, b+\epsilon_b]$ contains no critical values of F_t for all $t\in(t_0-\delta_b, t_0+\delta_b) \cap [0,1]$ and thus the inclusions $F_t^{b-\epsilon_b} \subset F_{t_0}^b \subset F_t^{b+\epsilon_b}$ induce an isomorphism $H_*(F_{t_0}^b, F_{t_0}^{-\infty}) \simeq H_*(F_t^b, F_t^{-\infty})$. The result then follows from the definition of $c_\pm(F_t)$. ∎

In general it may be difficult to calculate $c_\pm(F)$. However, for 1-jets of constant functions, $\Lambda_r := \Lambda_f$ where $f\equiv r$, these numbers can be easily calculated.

Lemma 4·4. *If F is a quadratic at infinity generating function for Λ_r, $c_\pm(F) = r$.*

Proof. If $F: Z\times\mathbb{R}^k\to\mathbb{R}$ is a quadratic at infinity generating function for Λ_r, then

$$\Lambda_r = \{(q,0,r)\} = \left\{\left(q, \frac{\partial F}{\partial q}(q,x), F(q,x)\right) : \frac{\partial F}{\partial x}(q,x) = 0\right\}$$

and it follows that r is the only critical value of F. ∎

5. *Proof of Theorems* 3·11 *and* 3·13

By Remark 3·6, Theorem 3·11 is an immediate corollary of Theorem 3·13. Let $f_1, g_1, f_2, g_2 : S^1 \to \mathbb{R}$ be functions satisfying $f_1(q) < g_1(q), f_2(q) < g_2(q)$, for all $q \in S^1$. A slight modification of the proof of Lemma 3·7 proves that $\Lambda_{f_1} \coprod \Lambda_{g_1} \simeq \Lambda_{f_2} \coprod \Lambda_{g_2}$. Thus, to complete the proof of Theorem 3·13, it suffices to prove that

$$\Lambda_0 \coprod \Lambda_1 \not\simeq \Lambda_0 \coprod \Lambda_{-1},$$

where Λ_r is the 1-jet of the constant function $f \equiv r$.

Suppose there exists a contact isotopy $\kappa_t, t \in [0,1]$, of $\mathscr{J}^1(S^1)$ such that $\kappa_1(\Lambda_0) = \Lambda_0$, $\kappa_1(\Lambda_1) = \Lambda_{-1}$. Let $\Lambda_0(t) := \kappa_t(\Lambda_0)$, $\Lambda_1(t) := \kappa_t(\Lambda_1)$. By (4·1), for all t, $\Lambda_0(t)$, $\Lambda_1(t)$ have quadratic at infinity generating functions,

$$F_0^t : S^1 \times \mathbb{R}^N \to \mathbb{R}, \quad F_1^t : S^1 \times \mathbb{R}^M \to \mathbb{R}.$$

Consider $D^t : S^1 \times \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}$,

$$D^t(q, \eta_1, \eta_0) := F_1^t(q, \eta_1) - F_0^t(q, \eta_0).$$

D^t is a quadratic at infinity generating function for the (immersed) legendrian

$$\Delta(t) := \{(q, p_1 - p_0, z_1 - z_0) : (q, p_1, z_1) \in \Lambda_1(t), \quad (q, p_0, z_0) \in \Lambda_0(t)\}.$$

By (4·3), the function $c_+ : [0,1] \to \mathbb{R}, c_+(t) = c_+(D^t)$ is continuous. Since D^0 is a generating function for Λ_1 and D^1 is a generating function for Λ_{-1}, Lemma 4·4 implies $c_+(0) = 1$ and $c_+(1) = -1$. By the continuity of c_+, there exists t' such that $c_+(t') = 0$. Then since $c_+(t') = 0$ is a critical value of $D^{t'}$, there exists $q' \in S^1$ so that $(q', 0, 0) \in \Delta(t')$ and thus, by the above description of $\Delta(t')$, there exists $(q', p_1', z_1') = (q', p_0', z_0') \in \Lambda_1(t') \cap \Lambda_0(t')$, a contradiction. ∎

6. *Non-concentric wave fronts*

The links in $ST^*\mathbb{R}^2$ studied above have concentric, non-intersecting wave fronts. To study links with intersecting wave fronts, let $\tau_z : ST^*\mathbb{R}^2 \to ST^*\mathbb{R}^2$ denote the translation

$$\tau_z(x, y, \theta) = (x + z, y, \theta).$$

Notice that $\tau_z(C^{\pm}(r))$ has a wave front centred at $(x, y) = (z, 0)$ and thus when $0 < z \leqslant r$, the wave fronts of $\tau_{-z}(C^+(r))$ and $\tau_z(C^-(r))$ will intersect. In fact, a link whose wave front consists of intersecting circles of opposite charge is equivalent to one of the links already studied.

PROPOSITION 6·1. *For* $0 < z < r$, $0 \leqslant z_1 < r_1$,

$$\tau_{-z}(C^+(r)) \coprod \tau_z(C^-(r)) \simeq \tau_{-z}(C^+(r)) \coprod \tau_{z_1}(C^-(r_1)) \simeq C^+(1) \coprod C^-(2),$$
$$\tau_{-z}(C^-(r)) \coprod \tau_z(C^+(r)) \simeq \tau_z(C^-(r)) \coprod \tau_{-z}(C^+(r)) \simeq C^-(1) \coprod C^+(2).$$

Proof. $0 \leqslant z_1 < r_1$ implies that the wave fronts of both $\tau_{-z}(C^+(r))$ and $\tau_{z_1}(C^-(r_1))$ intersect the line $\{x = 0\}$. A proof similar to the proof of Corollary 3·8 proves that $\tau_{-z}(C^+(r)) \coprod \tau_z(C^-(r)) \simeq \tau_{-z}(C^+(r)) \coprod \tau_{z_1}(C^-(r_1))$. When $r \ll r_1, z_1 = 0$, the wave front of

$\tau_{-z}(C^+(r))$ will be completely contained in the bounded component of the complement of $\tau_{z_1}(C^-(r_1))$'s wavefront and thus $\tau_{-z}(C^+(r)) \amalg \tau_z(C^-(r)) \simeq C^+(1) \amalg C^-(2)$. A similar argument proves the second statement. ▌

COROLLARY 6·2. *For* $r > 0$, $\tau_{-r}(C^+(2r)) \amalg \tau_r(C^-(2r)) \not\simeq \tau_{-r}(C^-(2r)) \amalg \tau_r(C^+(2r))$.

7. *Higher twisting*

As a more 'twisted' version of $ST^*\mathbb{R}^2$, consider the open solid torus, $\mathbb{R}^2 \times S^1$, with the alternative contact structure ξ_n,

$$\xi_n = \ker(\cos(n\theta)\,dx + \sin(n\theta)\,dy), \quad n \in \mathbb{Z}\backslash\{0\} \tag{7·1}$$

Notice that when $n = 1$, this is precisely $ST^*\mathbb{R}^2$. In $(\mathbb{R}^2 \times S^1, \xi_n)$, $|n| > 1$, it is possible to construct legendrian knots which complete more turns of the helix before closing. When $n < 0$, these knots will be portions of a circular, left-handed helix. More precisely, for $r \geqslant 0$,

$$\left.\begin{aligned} C_n^+(r) &:= \{(r\cos(n\theta), r\sin(n\theta), \theta) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\} \\ C_n^-(r) &:= \{(r\cos(n\theta), r\sin(n\theta), \theta + \pi/n) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\} \end{aligned}\right\} \tag{7·2}$$

are legendrian knots in $(\mathbb{R}^2 \times S^1, \xi_n)$. (Notice that

$$\{(r\cos(n\theta), r\sin(n\theta), \theta + 2\pi/n) : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$$

is a different parameterization of $C_n^+(r)$.)

The circles above and in previous sections had length 2π. More generally, for $\lambda > 0$, let $S^1(\lambda)$ denote the circle of length λ:

$$S^1(\lambda) := \mathbb{R}/\lambda\mathbb{Z}, \quad \lambda > 0. \tag{7·3}$$

The 1-jet spaces of $S^1(\lambda)$, $\mathscr{J}^1(S^1(\lambda))$, can be considered as a contact manifold with the contact structure, η, again defined by

$$\eta = \ker(dz - p\,dq). \tag{7·4}$$

For $\lambda < 0$, $\mathscr{J}^1(S^1(\lambda))$ will denote the 1-jet space of $S^1(-\lambda)$ with the contact structure

$$\eta = \ker(dz + p\,dq). \tag{7·5}$$

Then for each $n \in \mathbb{Z}\backslash\{0\}$, $(\mathbb{R}^2 \times S^1, \xi_n)$ is contactly equivalent to the 1-jet space of a circle of appropriate 'size and orientation':

$$\left.\begin{aligned} &\tau_n : (\mathbb{R}^2 \times S^1, \xi_n) \to (\mathscr{J}^1(S^1(2n\pi)), \eta), \\ &\tau_n(x, y, \theta) = \begin{cases} (n\theta, -\sin(n\theta)\,x + \cos(n\theta)\,y, \cos(n\theta)\,x + \sin(n\theta)\,y), & n > 0 \\ (-n\theta, -\sin(n\theta)\,x + \cos(n\theta)\,y, \cos(n\theta)\,x + \sin(n\theta)\,y), & n < 0. \end{cases} \end{aligned}\right\} \tag{7·6}$$

Under these contact diffeomorphisms, $C_n^{\pm}(r)$ correspond to $\Lambda_{\pm r}$, the 1-jets of the constant functions $f \equiv \pm r$.

LEMMA 7·7. *For* λ, $\mu \in \mathbb{R}\backslash\{0\}$, $\mathscr{J}^1(S^1(\lambda))$ *is contactly equivalent to* $\mathscr{J}^1(S^1(\mu))$. *For*

$n, m \in \mathbb{Z}\backslash\{0\}$, $(\mathbb{R}^2 \times S^1, \xi_n)$ and $(\mathbb{R}^2 \times S^1, \xi_m)$ are equivalent contact manifolds. Moreover, for all $n, m \in \mathbb{Z}\backslash\{0\}$ there exists a contact diffeomorphism

$$w: (\mathbb{R}^2 \times S^1, \xi_n) \to (\mathbb{R}^2 \times S^1, \xi_m) \quad \textit{such that}$$

$$w(C_n^{\pm}(r)) = \begin{cases} C_m^{\pm}\left(\dfrac{mr}{n}\right), & nm > 0 \\[2ex] C_m^{\mp}\left(-\dfrac{mr}{n}\right), & nm < 0. \end{cases}$$

Proof. For $\lambda, \mu \in \mathbb{R}^+$, consider the contact diffeomorphisms

$$s_{\lambda,\mu}: \mathscr{J}^1(S^1(\lambda)) \to \mathscr{J}^1(S^1(\mu)), \quad s_{\lambda,\mu}(q,p,z) = \left(\frac{\mu}{\lambda}q, p, \frac{\mu}{\lambda}z\right),$$

$$\rho: \mathscr{J}^1(S^1(\lambda)) \to \mathscr{J}^1(S^1(-\lambda)), \quad \rho(q,p,z) = (q, -p, -z).$$

A combination of these diffeomorphisms shows the equivalence of $\mathscr{J}^1(S^1(\lambda))$ and $\mathscr{J}^1(S^1(\mu))$, for all $\lambda, \mu \in \mathbb{R}\backslash\{0\}$. Together with the contactomorphisms τ_n, τ_m, these diffeomophisms imply $(\mathbb{R}^2 \times S^1, \xi_n)$ and $(\mathbb{R}^2 \times S^1, \xi_m)$ are contactomorphic when $n, m \in \mathbb{Z}\backslash\{0\}$. If $w := \tau_m^{-1} \circ s_{n,m} \circ \tau_n$, then it is a contactomorphism with the specified property. ▮

COROLLARY 7·8. *For any $n \in \mathbb{Z}\backslash\{0\}$, $C_n^+(1) \amalg C_n^-(1)$ and $C_n^-(1) \amalg C_n^+(1)$ are non-equivalent legendrian links in $(\mathbb{R}^2 \times S^1, \xi_n)$.*

8. *Higher dimensions*

The links studied above have natural generalizations to 'links' in $ST^*\mathbb{R}^n$ and $\mathscr{J}^1(S^{n-1})$. Each of these contact manifolds are diffeomorphic to $\mathbb{R}^n \times S^{n-1}$ and the hodograph transformation again proves that these contact manifolds are equivalent, [**A**]. Natural legendrian knots in these contact manifolds consist of legendrian embeddings of S^{n-1}: the $(n-1)$-sphere of radius r in $\mathbb{R}^n$ with outward (inward) normals defines the legendrian $C^{\pm}(r) \in ST^*\mathbb{R}^n$ and functions $f: S^{n-1} \to \mathbb{R}$ define legendrians $\Lambda_f \subset \mathscr{J}^1(S^{n-1})$. Under the hodograph transformation, $C^{\pm}(r)$ correspond to $\Lambda_{\pm r}$. It is easy to see that the links $\Lambda_0 \amalg \Lambda_1$ and $\Lambda_0 \amalg \Lambda_{-1}$ are topologically equivalent. However, the generating function argument from Section 5 again proves that these legendrian links are not contactly equivalent.

9. *Others links*

There are a number of obstacles when attempting to apply the theory of generating functions to other links in $ST^*\mathbb{R}^2$ or $\mathscr{J}^1(S^1)$. First, many legendrian knots in $\mathscr{J}^1(S^1)$ do not admit quadratic at infinity generating functions. For example, consider the transversal knot $\mathscr{T} := \{p = 1, z = 0\} \subset \mathscr{J}^1(S^1)$. There exists a C^0-perturbation τ so that $\mathscr{L} := \tau(\mathscr{T})$ is legendrian, [**E**]. It can be assumed that $\mathscr{L}$ does not intersect $\{p = 0\}$. However, if $\mathscr{L}$ admits a quadratic at infinity generating function F, by the arguments in Section 5, the existence of $c_{\pm}(F)$ implies that $\mathscr{L}$ must intersect $\{p = 0\}$. As another example, consider the (topologically trivial) legendrian knot $L^+ \subset ST^*\mathbb{R}^2 (L^- \subset ST^*\mathbb{R}^2)$ whose wave front is the 'eye' with two cusps with 'upward' ('downward') normals. Under the hodograph transformation, $L^+(L^-)$ is mapped to a legendrian in $\mathscr{J}^1(S^1)$ that does not project onto S^1. It is easy to check that such a legendrian cannot have a quadratic at infinity generating function as defined in Section 4. The following is an open question about legendrian versions of the Hopf link. It is interesting to compare this with Corollary 6·2.

Question 9·1. Assume τ_-, τ_+ are translations as in Section 6 so that the wave fronts

$$\pi(\tau_-(L^+)) \cap \pi(\tau_+(L^+)) \neq \emptyset.$$

It is easy to check that $\tau_-(L^+) \coprod \tau_+(L^+) \simeq \tau_+(L^-) \coprod \tau_-(L^-)$. Is it true that

$$\tau_-(L^+) \coprod \tau_+(L^+) \not\simeq \tau_-(L^-) \coprod \tau_+(L^-)\,?$$

Appendix: legendrian generating functions

Versions of Theorem A·1 are proved by Chaperon in [**Ca**] and in the thesis of David Théret, [**Th**]. The purpose of this appendix is to give an alternate proof, described to me by Ya. Eliashberg, which uses a symplectization procedure and Yuri Chekanov's 'formula' (cf. {**Ce**]). Background on symplectic geometry can be found in [**A-G**], [**M-S**].

Throughout this appendix, $\Lambda_0 \subset \mathscr{J}^1(Z)$, $\Lambda_0^m \subset \mathscr{J}^1(\mathbb{R}^m)$ will denote the 1-jets of the zero function.

THEOREM A·1. *Let Z be a closed manifold. If κ_t, $t \in [0,1]$, is a compactly supported contact isotopy of $\mathscr{J}^1(Z)$ then there exists a smooth 1-parameter family of quadratic at infinity generating functions $F_t: Z \times \mathbb{R}^k \to \mathbb{R}$ for $\Lambda_t := \kappa_t(\Lambda_0)$. This means that in local coordinates*

$$\Lambda_t = \left\{\left(q, \frac{\partial F_t}{\partial q}(q,x), F_t(q,x)\right): \frac{\partial F_t}{\partial x}(q,x) = 0\right\}.$$

By the following proposition, it suffices to prove the analogue of (A·1) for the situation where $Z = \mathbb{R}^m$ and $\kappa_t, t \in [0,1]$, is an isotopy of compactly supported contactomorphisms of $\mathscr{J}^1(\mathbb{R}^m)$.

PROPOSITION A·2. *Let Z be a closed manifold. Given a contact isotopy κ_t, $t \in [0,1]$, of $\mathscr{J}^1(Z)$ there exists an embedding $e: \mathscr{J}^1(Z) \to \mathscr{J}^1(\mathbb{R}^m)$ satisfying $e(\Lambda_0) \subset \Lambda_0^m$ and a compactly supported contact isotopy κ_t^m of $\mathscr{J}^1(\mathbb{R}^m)$, $t \in [0,1]$, such that $\kappa_t^m \circ e(\Lambda_0) = e \circ \kappa_t(\Lambda_0)$. If $\kappa_t^m(\Lambda_0^m)$ has a quadratic at infinity generating function, $\kappa_t(\Lambda_0)$ has a quadratic at infinity generating function.*

Proof. There exists an embedding $j: Z \to \mathbb{R}^m$, for some m. This induces an embedding $e: \mathscr{J}^1(Z) \to \mathscr{J}^1(\mathbb{R}^m)$ so that $e(\Lambda_0) \subset \Lambda_0^m$, $e^*(\alpha^m) = \alpha$, where α, α^m are the standard contact forms on $\mathscr{J}^1(Z), \mathscr{J}^1(\mathbb{R}^m)$. Since the focus is on $\kappa_t(\Lambda_0)$, $t \in [0,1]$, by applying an argument using Gray stability, it can be assumed that κ_t is compactly supported. Let X_t be the contact vector field on $\mathscr{J}^1(Z)$ whose flow is κ_t. If $h_t := \alpha(X_t)$: $\mathscr{J}^1(\mathscr{Z}) \to \mathbb{R}$, then h_t is compactly supported and if R denotes the Reeb vector field of α (see proof of (2·3)) then

$$\mathscr{L}_{X_t}\alpha = dh_t + i_{X_t}d\alpha = dh_t(R)\,\alpha.$$

Consider a function $h_t^m: \mathscr{J}^1(\mathbb{R}^m) \to \mathbb{R}$ such that $h_t^m = (e^{-1})^* h_t$ on points of $\operatorname{Im} e$ and the derivatives of h_t^m vanish in directions normal to $\operatorname{Im} e$. If R^m denotes the Reeb vector field of α^m, then on points of $\operatorname{Im} e, dh_t^m(R^m) = dh_t(R)\circ e^{-1}: \operatorname{Im} e \to \mathbb{R}$. Thus the vector field X_t^m uniquely defined by the conditions

$$\alpha^m(X_t^m) = h_t^m, \quad dh_t^m + i_{X_t^m} d\alpha^m = dh_t^m(R^m)\,\alpha^m,$$

(see proof of (2·3)) will satisfy $X_t^m = e_*(X_t)$ on $\operatorname{Im} e$. Thus if X_t^m is integrable, it will integrate to a contact isotopy κ_t^m such that $\kappa_t^m \circ e = e \circ \kappa_t$. By choosing the function h_t^m to be zero outside a compact set of $\mathscr{J}^1(\mathbb{R}^m), X_t^m$ will be integrable and κ_t^m will be compactly supported. If $F^m: \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ is a quadratic at infinity generating function for $\kappa_t^m(\Lambda_0^m)$, then $F := (j \times id)^*(F^m): Z \times \mathbb{R}^k \to \mathbb{R}$ will be a quadratic at infinity generating function for $\kappa_t(\Lambda_0)$. ∎

The $\mathbb{R}^m$ version of (A·1) follows from a slight modification of the proof of the symplectic, $\mathbb{R}^m$ version of (A·1): if $\mathscr{L} \subset T^*(\mathbb{R}^m)$ is a lagrangian with quadratic at infinity generating function and ψ_t, $t \in [0,1]$, is a compactly supported symplectic isotopy of $T^*(\mathbb{R}^m)$, then $\psi_t(\mathscr{L})$ has a quadratic at infinity generating function. In fact, the idea is to transform all relevant contact objects in $\mathscr{J}^1(\mathbb{R}^m)$ into 'symmetric' symplectic objects in $T^*(\mathbb{R}^m \times \mathbb{R}^+)$.

Given the contact manifold $(\mathscr{J}^1(\mathbb{R}^m), \eta)$, $\eta = \ker\alpha$, $\alpha := dz - p\,dq$, let $\mathbb{R}^+ := (0, \infty)$ and consider the symplectic manifolds

$$(\mathscr{J}^1(\mathbb{R}^m) \times \mathbb{R}^+, d(t\alpha)), \quad (T^*(\mathbb{R}^m \times \mathbb{R}^+), \quad \omega := dq \wedge dp + dt \wedge dz).$$

These symplectic manifolds are equivalent:

$$\sigma: \mathscr{J}^1(\mathbb{R}^m) \times \mathbb{R}^+ \to T^*(\mathbb{R}^m \times \mathbb{R}^+), \quad \sigma(q,p,z,t) = (q,t,tp,z).$$

For a legendrian submanifold $\Lambda \subset \mathscr{J}^1(\mathbb{R}^m)$, consider the lagrangian submanifolds

$$\hat{\Lambda} \subset (\mathscr{J}^1(\mathbb{R}^m) \times \mathbb{R}^+), \quad \mathscr{L}_\Lambda \subset T^*(\mathbb{R}^m \times \mathbb{R}^+)$$
$$\hat{\Lambda} := \{(q,p,z,t) : (q,p,z) \in \Lambda\}, \quad \mathscr{L}_\Lambda := \sigma(\hat{\Lambda}). \tag{A·3}$$

For a contact difeomorphism κ isotopic to id, $\kappa^*\alpha = f\alpha$ where f is a positive function. If κ is written $\kappa(q,p,z) = (\kappa_q(q,p,z),\ \kappa_z(q,p,z))$, consider the symplectic diffeomorphism $\hat{\kappa}$ of $\mathscr{J}^1(\mathbb{R}^m) \times \mathbb{R}^+$ defined by

$$\hat{\kappa}(q,p,z,t) := \left(\kappa_q(q,p,z), \kappa_p(q,p,z), \kappa_z(q,p,z), \frac{t}{f(q,p,z)}\right), \quad \kappa^*\alpha = f\alpha.$$

ψ_κ will denote the corresponding symplectic diffeomorphism of $T^*(\mathbb{R}^m \times \mathbb{R}^+)$:

$$\psi_\kappa := \sigma \circ \hat{\kappa} \circ \sigma^{-1}.$$

$\Gamma_{\psi_\kappa} := \{(x, \psi_\kappa(x))\} \subset \overline{T^*(\mathbb{R}^m \times \mathbb{R}^+)} \times T^*(\mathbb{R}^m \times \mathbb{R}^+)$ is then a lagrangian submanifold. Let τ denote the symplectic diffeomorphism

$$\tau: \overline{T^*(\mathbb{R}^m \times \mathbb{R}^+)} \times T^*(\mathbb{R}^m \times \mathbb{R}^+) \to T^*(\mathbb{R}^{2m+1} \times \mathbb{R}^+)$$
$$(q,t,p,z,Q,T,P,Z) \mapsto (p,z,Q,T,q-Q,t-T,P-p,Z-z),$$

and define

$$\tilde{\Gamma}_{\psi_\kappa} := \tau(\Gamma_{\psi_\kappa}). \tag{A·4}$$

The symplectic, $\mathbb{R}^m$ version of (A·1) follows by iterated applications of the following proposition which says that generating functions for $\mathscr{L}_\Lambda \subset T^*(\mathbb{R}^m \times \mathbb{R}^+)$ and $\tilde{\Gamma}_{\psi_\kappa} \subset T^*(\mathbb{R}^{2m+1} \times \mathbb{R}^+)$ can be 'composed' to get a generating function for $\psi_\kappa(\mathscr{L}_\Lambda) \subset T^*(\mathbb{R}^m \times \mathbb{R}^+)$. For more details or a proof of Proposition A·5, see [**Tr**, §4].

PROPOSITION A·5. *Suppose* $\mathscr{L}_\Lambda$ is a lagrangian in $T^*(\mathbb{R}^m\times\mathbb{R}^+)$ *that has a quadratic at infinity generating function* $G_1\colon \mathbb{R}^m\times\mathbb{R}^+\times\mathbb{R}^k\to\mathbb{R}$, *i.e.*

$$\mathscr{L}_\Lambda=\left\{\left(q,t,\frac{\partial G_1}{\partial q}(q,t,\eta),\frac{\partial G_1}{\partial t}(q,t,\eta)\right):\frac{\partial G_1}{\partial \eta}(q,t,\eta)=0\right\},$$

and ψ_κ *is sufficiently* C^1*-close to the identity so that* $\tilde{\Gamma}_{\psi_\kappa}\subset T^*(\mathbb{R}^{2m+1}\times\mathbb{R})$ *is the graph of an exact* 1*-form, i.e. there exists* $G_2\colon\mathbb{R}^{2m+1}\times\mathbb{R}^+\to\mathbb{R}$ *such that* $\tilde{\Gamma}_{\psi_\kappa}=\Gamma_{dG_2}$. *Then* $G_3\colon\mathbb{R}^m\times\mathbb{R}^+\times(\mathbb{R}^m\times\mathbb{R}^+\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^k)\to\mathbb{R}$, *defined by*

$$G_3(x_2,s;q,t,x_1,y,\eta)=G_1(q,t;\eta)+G_2(x_1,y,x_2,s)+x_1(x_2-q)+y(s-t),$$

is an asymptotically quadratic at infinity generating function for $\psi_\kappa(\mathscr{L}_\Lambda)=\mathscr{L}_{\kappa(\Lambda)}$ *which can be made quadratic at infinity by a fibre preserving difeomorphism.*

Thus for all $t\in[0,1]$, the lagrangian $\psi_{\kappa_t}(\mathscr{L}_{\Lambda_0})=\mathscr{L}_{\kappa_t(\Lambda_0)}\subset T^*(\mathbb{R}^m\times\mathbb{R}^+)$ has a quadratic at infinity generating function equal to zero on points corresponding to the points of $\psi_{\kappa_t}(\mathscr{L}_{\Lambda_0})$ outside a compact set. The existence of generating functions for the legendrians $\kappa_t(\Lambda_0)\subset\mathscr{J}^1(\mathbb{R}^m)$ will be a consequence of the $\mathbb{R}^+$-symmetry present in the lagrangian formed by the symplectization procedure. To describe this symmetry, consider

$$b\colon\mathbb{R}^+\times T^*(\mathbb{R}^m\times\mathbb{R}^+)\to T^*(\mathbb{R}^m\times\mathbb{R}^+),\quad b(\mu,q,t,p,z)=(q,\mu t,\mu p,z).$$

This is a *conformal symplectic action*: if for $\mu\in\mathbb{R}^+$, b_μ is defined by

$$b_\mu\colon T^*(\mathbb{R}^m\times\mathbb{R}^+)\to T^*(\mathbb{R}^m\times\mathbb{R}^+),\quad b_\mu(v)=b(\mu,v),$$

then $b_\mu^*\,\omega=\mu\omega$ and $b_{\mu_2\mu_1}=b_{\mu_2}b_{\mu_1}$. $\mathscr{L}\subset T^*(\mathbb{R}^m\times\mathbb{R}^+)$ is $\mathbb{R}^+$-*equivariant* if $b(\mathbb{R}^+\times\mathscr{L})=\mathscr{L}$.

PROPOSITION A·6. *For any legendrian* $\Lambda\subset\mathscr{J}^1(\mathbb{R}^m)$, $\mathscr{L}_\Lambda$ *is an* $\mathbb{R}^+$*-equivariant lagrangian. If* $\mathscr{L}_\Lambda$ *has a quadratic at infinity generating function* G *then*

$$F(q;x):=G(q,1;x)$$

is a quadratic at infinity generating function for the legendrian $\Lambda\subset\mathscr{J}^1(\mathbb{R}^m)$.

Proof. To see that $\mathscr{L}_\Lambda$ is an $\mathbb{R}^+$-equivariant lagrangian, consider the conformal symplectic action

$$a\colon\mathbb{R}^+\times(\mathscr{J}^1(\mathbb{R}^m)\times\mathbb{R}^+)\to\mathscr{J}^1(\mathbb{R}^m)\times\mathbb{R}^+,\quad a(\mu,q,p,z,t)=(q,p,z,\mu t).$$

It is clear that $\hat{\Lambda}$ is invariant with respect to this action. Since $b_\mu=\sigma\circ a_\mu\circ\sigma^{-1}$, $\mathscr{L}_\Lambda$ is $\mathbb{R}^+$-equivariant. Suppose $\mathscr{L}_\Lambda$ has a quadratic at infinity generating function. Since $\mathscr{L}_\Lambda\cap\{t=1\}=\{(q,1,p,z)\colon(q,p,z)\in\Lambda\}$, it follows that if $\pi\colon\mathscr{J}^1(\mathbb{R}^m)\to T^*(\mathbb{R}^m)$ denotes the projection, the lagrangian $\pi(\Lambda)$ is generated by F and thus for some constant C, $F+C$ generates Λ. By construction, Λ coincides with Λ_0 outside a compact set and thus it can be concluded that $C=0$.

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