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ON LITTLEWOOD-PALEY FUNCTIONS

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ABSTRACT. We prove that, for a compactly supported L^q function Φ with vanishing integral on $\mathbf{R}^n$, the corresponding square function operator S_Φ is bounded on L^p for $|1/p - 1/2| < \min\{(q-1)/2, 1/2\}$.

1. INTRODUCTION

Let $n \geq 1$ and $\mathbf{R}^n$ denote the n-dimensional Euclidean space. For a function $\Phi \in L^1(\mathbf{R}^n)$ which satisfies

$$\int_{\mathbf{R}^n} \Phi(x)dx = 0, \tag{1.1}$$

we define the square function operator S_Φ by

$$(S_\Phi\ f)(x) = \left(\int_0^\infty |\Phi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2}, \tag{1.2}$$

where $\Phi_t(x) = t^{-n}\Phi(x/t)$. The operator S_Φ is often called a square function or a Littlewood-Paley function. Such operators have long played important roles in harmonic analysis. The main problem under investigation concerns the boundedness of these operators on various L^p spaces.

It has been well known that, if the function Φ is sufficiently nice (in terms of decaying and smoothness properties), the corresponding operator S_Φ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$. The following result is due to Benedek, Calderón and Panzone ([2]):

Theorem A. *Suppose that Φ satisfies (1.1) and for some positive α,*

$$|\Phi(x)| \leq C(1+|x|)^{-n-\alpha}, \int_{\mathbf{R}^n} |\Phi(x-y) - \Phi(x)|dx \leq C|y|^\alpha. \tag{1.3}$$

Then S_Φ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.

Examples of functions satisfying (1.3) include the Schwartz functions, as well as the following which arise from the Poisson kernel on $\mathbf{R}^n$:

$$\Phi_0(x) = \frac{\partial}{\partial t}\left(\frac{t}{(|x|^2+t^2)^{(n+1)/2}} \right)\Bigg|_{t=1}$$

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and

$$\Phi_j(x) = \frac{\partial}{\partial x_j}\left(\frac{1}{(|x|^2+1)^{(n+1)/2}}\right).$$

When Φ is given by

$$\Phi(x) = |x|^{-n+1}\Omega(x)\chi_{[0,1]}(|x|), \tag{1.4}$$

where Ω is homogeneous of degree 0 and has mean value zero on $\mathbf{S}^{n-1}$, then S_Φ becomes the Marcinkiewicz integral operator ([13]). See also [1], [14], [16] and the extensive list of references given in the survey [6].

In [12] S. Sato proved that, among other things, the conclusion of Theorem A is still true if the smoothness condition (1.3) is eliminated (for L^2 this had been known earlier; see [5], [10]).

In this paper we shall study the L^p boundedness of S_Φ without imposing conditions (1.2) or (1.3) on Φ, or the assumption that Φ be given as in (1.4). The following is a known result:

Theorem B. *Suppose that Φ satisfies (1.1) and is compactly supported.*

(i) *If $\Phi \in L^q(\mathbf{R}^n)$ for some $q \geq 2$, then S_Φ is a bounded operator on $L^p(\mathbf{R}^n, w)$ for $p > q'$ and $w \in A_{p/q'}$.*

(ii) *If $\Phi \in L^2(\mathbf{R}^n)$, then S_Φ is a bounded operator on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

In the above statement, $L^p(\mathbf{R}^n, w)$ represents the weighted L^p space with weight w (the definition of the weight class A_p can be found in [11], [4] or [14]). When $w \equiv 1$ we write $L^p(\mathbf{R}^n, w)$ as $L^p(\mathbf{R}^n)$.

Part (i) of Theorem B is due to S. Sato (Theorem 3 in [12]). Part (ii) follows from (i) by using duality and interpolation (with $w \equiv 1$).

Theorem B (ii) covers the cases $\Phi \in L^q(\mathbf{R}^n)$, $q > 2$ as well because of the compact support assumption. However, the approach used in [12] does not appear to work when $q < 2$ (see also [9], page 241). The main purpose of the present paper is to establish the following theorem dealing with the case where $\Phi \in L^q(\mathbf{R}^n)$ for $q < 2$.

Theorem C. *Suppose that Φ is a compactly supported function satisfying (1.1). If $\Phi \in L^q(\mathbf{R}^n)$ for some $q \in (1, 2]$, then S_Φ is a bounded operator on $L^p(\mathbf{R}^n)$ for $|1/p - 1/2| < (q-1)/2$.*

Remarks 1. (i) The range of p given by $|1/p - 1/2| < (q-1)/2$ is the same as $2/q < p < 2/(2-q)$, which becomes $(1, \infty)$ when $q = 2$.

(ii) A result relevant to the theorems mentioned above is Theorem 1 in [7]. While in general an L^q function Φ does not satisfy the pointwise decay condition imposed on its Fourier transform in Theorem 1 of [7], a modification of the proof given in [7] can yield the L^p boundedness of S_Φ for $|1/p - 1/2| < (q-1)/(2q)$ under the conditions in Theorem C. Since $q > 1$, the range of p given by $|1/p - 1/2| < (q-1)/2$ in our theorem is considerably better.

(iii) While the inequality $|1/p - 1/2| < (q-1)/2$ gives the full range $1 < p < \infty$ when $q = 2$, it would be an interesting problem to determine whether it also represents the best possible range for p when $q < 2$.

The proof of Theorem C will be given in Section 2. Section 3 contains a result on the boundedness of S_Φ when the compact support condition is replaced by some other conditions.

2. Proof of Theorem C

Lemma 2.1. *Let Ψ be a compactly supported function in $L^q(\mathbf{R}^n)$ for some $q \in (1,2]$. Then, for every p satisfying $|1/p - 1/2| < (q-1)/2$, there exists a $C_p > 0$ such that*

$$\left\| \left(\int_0^\infty |(\Psi_t * F^t)(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbf{R}^n)} \le C_p \left\| \left(\int_0^\infty |F^t(x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbf{R}^n)} \tag{2.1}$$

holds for every measurable function $F^t(x) = F(t,x)$ on $(0,\infty) \times \mathbf{R}^n$.

Proof. By duality and interpolation we may assume that $p > 2$. We shall also assume that $\mathrm{supp}(\Psi)$ is contained in $B(0,1)$, where $B(x_0,t) = \{x \in \mathbf{R}^n : |x - x_0| \le t\}$. Let T be the operator acting on functions defined on $(0,\infty) \times \mathbf{R}^n$ given by

$$T(F)(t,x) = (\Psi_t * F^t)(x), \tag{2.2}$$

where $F^t(y) = F(t,y)$ for $(t,y) \in (0,\infty) \times \mathbf{R}^n$. For $1 \le p, q < \infty$ we shall use the following notation:

$$\|F\|_{L^p(L^q(t^{-1}dt),dx)} = \left(\int_{\mathbf{R}^n} \left(\int_0^\infty |F(t,x)|^q \frac{dt}{t} \right)^{p/q} dx \right)^{1/p}. \tag{2.3}$$

Thus we have

$$\begin{aligned} \|T(F)\|_{L^1(L^1(t^{-1}dt),dx)} &= \int_{\mathbf{R}^n} \left(\int_0^\infty |(\Psi_t * F^t)(x)| t^{-1} dt \right) dx \\ &= \int_0^\infty \|\Psi_t * F^t\|_{L^1(\mathbf{R}^n)} t^{-1} dt \\ &\le \|\Psi\|_{L^1(\mathbf{R}^n)} \int_0^\infty \|F^t\|_{L^1(\mathbf{R}^n)} t^{-1} dt \\ &= \|\Psi\|_{L^1(\mathbf{R}^n)} \|F\|_{L^1(L^1(t^{-1}dt),dx)}. \end{aligned} \tag{2.4}$$

It follows from $1/2 - 1/p < (q-1)/2$ that

$$p'q/2 > 1.$$

Let $r = (p'q/2)'$. By $p > 2$ we have $r > q'$. Thus, for any F that satisfies

$$\|T(F)\|_{L^r(L^{q'}(t^{-1}dt),dx)} < \infty,$$

there exists a function $h \in L^{(r/q')'}(\mathbf{R}^n)$ such that

$$\|h\|_{L^{(r/q')'}(\mathbf{R}^n)} = 1$$

and

$$\|T(F)\|^{q'}_{L^r(L^{q'}(t^{-1}dt),dx)} = \int_{\mathbf{R}^n} \left(\int_0^\infty |(\Psi_t * F^t)(x)|^{q'} t^{-1} dt \right) h(x) dx. \tag{2.5}$$

By Hölder's inequality,

$$\begin{aligned} |\Psi_t * F^t(x)|^{q'} &= t^{-nq'} \left| \int_{\mathbf{R}^n} \Psi\left(\frac{x-y}{t} \right) F(t,y) dy \right|^{q'} \\ &\le \|\Psi\|^{q'}_{L^q(\mathbf{R}^n)} |B(0,t)|^{-1} \int_{\mathbf{R}^n} |F(t,y)|^{q'} \chi_{B(0,t)}(x-y) dy. \end{aligned} \tag{2.6}$$

Let M denote the Hardy-Littlewood maximal operator on $\mathbf{R}^n$. Then by (2.5) and (2.6),

$$\begin{aligned}\|T(F)\|^{q'}_{L^r(L^{q'}(t^{-1}dt),dx)} &\le \|\Psi\|^{q'}_{L^q(\mathbf{R}^n)} \int_{\mathbf{R}^n} \left(\int_0^\infty |F(t,y)|^{q'} t^{-1} dt\right)(Mh)(y)dy \\ &\le \|\Psi\|^{q'}_{L^q(\mathbf{R}^n)} \left\| \left(\int_0^\infty |F(t,y)|^{q'} t^{-1} dt\right)\right\|^{q'}_{L^{(r/q')}(\mathbf{R}^n)} \|Mh\|_{L^{(r/q')'}(\mathbf{R}^n)} \\ &\le C_{r,q} \|\Psi\|^{q'}_{L^q(\mathbf{R}^n)} \|F\|^{q'}_{L^r(L^{q'}(t^{-1}dt),dx)}. \qquad (2.7)\end{aligned}$$

Since

$$\frac{1}{2} = \frac{\theta}{q'} + \frac{(1-\theta)}{1}, \qquad \frac{1}{p} = \frac{\theta}{r} + \frac{(1-\theta)}{1}$$

hold with $\theta = q/2$, by interpolating between (2.4) and (2.7) (see, for example, [3]) we obtain

$$\|T(F)\|_{L^p(L^2(t^{-1}dt),dx)} \le C_p \|F\|_{L^p(L^2(t^{-1}dt),dx)},$$

which proves (2.1) for $|1/p - 1/2| < (q-1)/2$.

For $s > 0$ we let $s^{\pm\alpha} = \min\{s^\alpha, s^{-\alpha}\}$. □

Lemma 2.2. *Let $\Psi \in \mathcal{S}(\mathbf{R}^n)$ and $\Phi \in L^q(\mathbf{R}^n)$ for some $q \in (1, 2]$. Suppose that Φ is compactly supported and $\operatorname{supp}(\widehat{\Psi}) \subset \{1/2 < |\xi| < 2\}$. Then for every p satisfying $|1/p - 1/2| < (q-1)/2$, there exist $C_p, \alpha_p > 0$ such that*

$$\left\| \left(\int_0^\infty |\Phi_t * \Psi_{st} * f|^2 \frac{dt}{t}\right)^{1/2} \right\|_{L^p(\mathbf{R}^n)} \le C_p (s^{\pm\alpha_p}) \|f\|_{L^p(\mathbf{R}^n)} \qquad (2.8)$$

for all $f \in L^p(\mathbf{R}^n)$ and $s > 0$.

Proof. Let

$$T_s f(x) = \left(\int_0^\infty |\Phi_t * \Psi_{st} * f(x)|^2 \frac{dt}{t}\right)^{1/2}.$$

Then, by Lemma 2.1 and Theorem A, for each p satisfying $|1/p - 1/2| < (q-1)/2$,

$$\begin{aligned}\|T_s f\|_{L^p(\mathbf{R}^n)} &\le C_p \|S_\Psi f\|_{L^p(\mathbf{R}^n)} \\ &\le C_p \|f\|_{L^p(\mathbf{R}^n)}. \qquad (2.9)\end{aligned}$$

On the other hand, by Plancherel's Theorem,

$$\begin{aligned}\|T_s f\|^2_{L^2(\mathbf{R}^n)} &= \int_0^\infty \int_{\mathbf{R}^n} |\Phi_t * \Psi_{st} * f(x)|^2 dx t^{-1} dt \\ &= \int_{\mathbf{R}^n} |\hat{f}(\xi)|^2 \left(\int_0^\infty |\widehat{\Phi}(t\xi)|^2 |\widehat{\Psi}(st\xi)|^2 \frac{dt}{t}\right) d\xi. \qquad (2.10)\end{aligned}$$

It follows from Lemmas 2, 3 of [12] and (1.1) that

$$\int_{1/2}^2 |\widehat{\Phi}(t\xi)|^2 dt \le C|\xi|^{\pm 1/(2q')}. \qquad (2.11)$$

Let $\xi' = |\xi|^{-1}\xi$ for $\xi \neq 0$. By (2.10) and (2.11) we have

$$\|T_s f\|^2_{L^2(\mathbf{R}^n)} \le \int_{\mathbf{R}^n} |\hat{f}(\xi)|^2 \left(\int_{1/2}^{2} |\widehat{\Phi}(ts^{-1}\xi')|^2 \frac{dt}{t} \right) d\xi$$
$$\le C(s^{\pm 1/(2q')}) \|f\|^2_{L^2(\mathbf{R}^n)}. \tag{2.12}$$

By (2.9), (2.12) and interpolation we conclude that (2.8) holds for $|1/p - 1/2| < (q-1)/2$. □

Proof of Theorem C. It suffices to establish

$$\|S_\Phi f\|_{L^p(\mathbf{R}^n)} \le C_p \|f\|_{L^p(\mathbf{R}^n)}$$

for $f \in \mathcal{S}(\mathbf{R}^n)$ and $|1/p - 1/2| < (q-1)/2$.

Let $\eta \in C^\infty(\mathbf{R})$ such that $\operatorname{supp}(\eta) \subset (1/4, 4)$ and

$$\int_0^\infty \eta(s) \frac{ds}{s} = 2. \tag{2.13}$$

Define the Schwartz function Ψ on $\mathbf{R}^n$ by

$$\widehat{\Psi}(\xi) = \eta(|\xi|^2)$$

for $\xi \in \mathbf{R}^n$. Then by (2.13) we have

$$\int_0^\infty \widehat{\Psi}(s\xi) \frac{ds}{s} = 1$$

and

$$\Phi_t * f = \int_0^\infty (\Phi_t * \Psi_{st} * f) \frac{ds}{s} \tag{2.14}$$

for all $f \in \mathcal{S}(\mathbf{R}^n)$ and $t > 0$. By Minkowski's inequality we have

$$|S_\Phi f(x)| \le \int_0^\infty \left(\int_0^\infty |\Phi_t * \Psi_{st} * f(x)|^2 \frac{dt}{t} \right)^{1/2} \frac{ds}{s}.$$

By Lemma 2.2, for every p satisfying $|1/p - 1/2| < (q-1)/2$,

$$\|S_\Phi f\|_{L^p(\mathbf{R}^n)} \le C_p \|f\|_{L^p(\mathbf{R}^n)} \int_0^\infty s^{\pm \alpha_p} \frac{ds}{s}$$
$$= 2\alpha_p^{-1} C_p \|f\|_{L^p(\mathbf{R}^n)}. \tag{2.15}$$

The proof of Theorem C is now complete. □

3. Further results

For Φ's which are not necessarily compactly supported, one can easily deduce the following result from Theorem 2 of [12] by using duality and interpolation:

Theorem 3.1. *Let $\Phi \in L^1(\mathbf{R}^n)$ and satisfy (1.1). Suppose that*

(i) $\displaystyle\int_{|x|<1} |\Phi(x)|^{1+\varepsilon} dx + \int_{|x|>1} |\Phi(x)||x|^\varepsilon dx < \infty$ *for some $\varepsilon > 0$;*

(ii) $|\Phi(x)| \le h(|x|)\Omega(x')$, *where*

(ii.a) h is non-negative, non-increasing and satisfies $\displaystyle\int_0^\infty h(r) r^{n-1} dr < \infty$;

(ii.b) $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q \ge 2$.

Then S_Φ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.

Using the above theorem one can see that the condition (1.4) in Theorem A is redundant, as observed in [12].

Below we shall show that the requirement "$q \geq 2$" in (ii.b) of the above theorem can be lowered to $q > 1$ without affecting the validity of the claim.

Theorem 3.2. *If the condition (ii.b) in Theorem 3.1 is replaced by the weaker condition (ii.b)′: $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$, while all other conditions in Theorem 3.1 remain unchanged, then S_Φ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

Proof. By Lemmas 1–3 in [12], we see that (2.11) still holds (note that $q > 1$ is needed when applying Lemma 2 of [12]). Thus, it suffices to show that (2.1) holds for all $p \in (1, \infty)$.

For each $y' \in \mathbf{S}^{n-1}$ and $x \in \mathbf{R}^n$, let

$$M_{y'} f(x) = \sup_{s>0} \left(\frac{1}{s} \int_0^s |f(x - sy')| ds \right).$$

For $t > 0$ and $x \in \mathbf{R}^n$,

$$\begin{aligned} |\Phi_t * f(x)| &\leq \int_{\mathbf{S}^{n-1}} |\Omega(y')| \left(\int_0^\infty |f(x - try')| h(r) r^{n-1} dr \right) d\sigma(y') \\ &\leq \int_{\mathbf{S}^{n-1}} |\Omega(y')| \left(\sum_{j=-\infty}^{\infty} 2^{n(j+1)} h(2^j) M_{y'} f(x) \right) d\sigma(y') \\ &\leq 2^{2n} \left(\int_0^\infty h(r) r^{n-1} dr \right) \int_{\mathbf{S}^{n-1}} |\Omega(y')| M_{y'} f(x) d\sigma(y'). \qquad (3.1) \end{aligned}$$

By (3.1) and the uniform boundedness of the operators $M_{y'}$ on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$, we have

$$\| \sup_{t>0} |\Phi_t * f| \|_{L^p(\mathbf{R}^n)} \leq C_p \|\Omega\|_1 \|f\|_{L^p(\mathbf{R}^n)} \qquad (3.2)$$

for $1 < p < \infty$. It follows from (3.2) and the proof of the lemma on p. 544 of [8] (after some trivial modifications) that (2.1) holds for $1 < p < \infty$. ☐

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