Axiomatizations of the proportional Shapley value

Manfred Besner

HFT Stuttgart, University of Applied Sciences, Department of Geomatics, Computer Science and Mathematics

19 July 2017

Axiomatizations of the proportional Shapley value

Manfred Besner*

September 26, 2018

Abstract

We present new axiomatic characterizations of the proportional Shapley value, a weighted TU-value with the worths of the singletons as weights. The presented characterizations are proportional counterparts to the famous characterizations of the Shapley value by Shapley (1953b) and Young (1985a). We introduce two new axioms, called proportionality and player splitting respectively. Each of them makes a main difference between the proportional Shapley value and the Shapley value. If the stand-alone worths are plausible weights, the proportional Shapley value is a convincing alternative to the Shapley value, for example in cost allocation. Especially the player splitting property, which states that the players' payoffs do not change if another player splits into two new players who have the same impact to the game as the original player, justifies the use of the proportional Shapley value in many economic situations.

Keywords Cost allocation · Dividends · Proportional Shapley value · (Weighted) Shapley value · Proportionality · Player splitting

1. Introduction

In contrast to Thomas (1969, 1974), who asserts that all cost allocation methods are arbitrary and no one allocation scheme can be defended against all others, we have on the one hand a large group of economists which prefers traditional cost accounting practices. On the other hand, there exists a small group of economists and academics which prefers cost allocation based on solutions to cooperative games with transferable utility dominated by the Shapley value, e. g. Shubik (1962), Spinetto (1975), Roth and Verrecchia (1979), Young (1985a), Young (1985b) and Leng and Parlar (2009). Moriarity (1975) states

> "A proposal for a new allocation procedure can be justified only on the basis of the advantages of the proposed method over existing methods."

*M. Besner
Department of Geomatics, Computer Science and Mathematics,
HFT Stuttgart, University of Applied Sciences, Schellingstr. 24
D-70174 Stuttgart, Germany
Tel.: +49 (0)711 8926 2791
E-mail: manfred.besner@hft-stuttgart.de

An empirical study by Barton (1988) shows a dramatic preference for the proportional solution, called Moriarity's method (Moriarity, 1975), also known as proportional rule, compared to the nucleolus or the Shapley value. Banker (1981) uses an axiomatic approach. He further analyzed some shortcomings of the Shapley value in cost allocation, especially the additivity axiom is considered questionable. It renders the allocation sensitive to the way cost centers are used or organized. Banker shows in an example that the allocations can differ significantly if two cost centers are merged and considered as a single entry. His own proposal for an axiomatization in the context of cost allocation contains a splitting axiom instead of additivity. It turns out that the unique value of his axiomatization is identical with the proportional rule (Moriarity, 1975).

In contrast, some other authors stress the disadvantages of the proportional rule and suggest the Shapley value. For example, Amer et al. (2007) criticize the restricted domain of the proportional rule, the lack of additivity, a doubly discriminatory level and that it does not take into account most of the marginal contributions.

The last point of criticism is avoided by the proper Shapley values (Brink et al., 2015) or the proportional value, developed by Ortmann (2000) and Feldman (1999) simultaneously. Feldman also suggests his value to cost allocation, gives a short overview of proportional cost allocation and points to Gangolly (1981) who introduced a new cost allocation scheme, denoted as "Independent Cost Proportional Scheme (ICPS)". In this scheme, Gangolly used a (proportional) weighted Shapley value for each given coalition function $\overline{v}$, where the weights are the worths of the singletons $\overline{v}(\{i\})$ for every player i. Yet a general formalization as a TU-value and an axiomatic characterization were still missing. Independently there was a "rediscovery" of this value by Besner (2016) and Béal et al. (2018). Both denote their non-linear TU-value "proportional Shapley value" and give an axiomatization by efficiency and proportional balanced contributions, in spirit to the axiomatization of the weighted Shapley values with weighted balanced contributions (Myerson, 1980; Hart and Mas-Colell, 1989). In addition, they point out that the proportional Shapley value inherits many of the properties of the weighted Shapley values. Besner (2016) offers some extensions of the proportional Shapley value, for example to graphs and level structures. Béal et al. (2018) introduce a potential and give some comparable axiomatizations to the Shapley value and economic applications.

The aim of this paper is to establish the proportional Shapley value as an application-relevant allocation scheme, where there are asymmetries that are included exclusively in the underlying game and not in exogenously given weights. Two new axioms make a main difference to the Shapley value. The first one, called proportionality, is a proportional counterpart to symmetry: the payoffs to two weakly dependent players, i. e., the marginal contribution of one of these players to any coalition which contains only one of both players is only his singleton worth, are proportional to the singleton worths of each other. Nowak and Radzik (1995) give a similar axiom for the weighted Shapley values, called ω-mutual dependence. The second axiom, called player splitting, is related to Banker (1981). If a player splits into two new players, the payoff to unconcerned players does not change under the condition that the new players contribute together the same to the game as the original player. Radzik (2012) presents a similar idea in the opposite direction by his amalgamating payoffs axiom: players who build a partnership (Kalai and Samet, 1987) amalgamate to a new player. Our two new axioms enable axiomatic characterizations which are proportional counterparts to the famous characterizations of the Shapley value by Shapley (1953b) and Young (1985a).

The paper is organized as follows. Section 2 contains some preliminaries. As the main results, we offer in Section 3 an axiomatization which is close to the axiomatization of the Shapley value by Shapley (1953b), in Section 4 we present an axiomatization which is close to the axiomatization by Young (1985a) and in Section 6 we introduce the player splitting property which leads to two axiomatizations as corollaries. In between, in Section 5 examples illustrate the proportional Shapley value in the context of cost allocation, motivate the player splitting property and show an inconsistency of the Shapley value in this case. Section 7 gives a short conclusion. All the proofs, related lemmas, and the logical independence of the axioms used for characterization are relegated to the appendix (Section 8).

2. Preliminaries

We denote by $\mathbb{R}$ the real numbers and by $\mathbb{Q}$ the rational numbers. Let $\mathfrak{U}$ an infinite set, the universe of all players. We denote by $\mathcal{N}$ the set of all finite subsets of $\mathfrak{U}$. A cooperative game with transferable utility (**TU-game**) is a pair (N, v) consisting of a set of players $N \in \mathcal{N}$ and a **coalition function** $v \in \{f: 2^N \to \mathbb{R} \,|\, f(\emptyset) = 0\}$ where 2^N is the power set of N. We refer to a TU-game also only by v. The subsets $S \subseteq N$ are called **coalitions** and $v(S)$ is called the **worth** of coalition S.

Let $N \in \mathcal{N}$. The set of all TU-games with player set N is denoted by $\mathcal{G}^N$, if the worths of the singletons must all be positive real numbers or must all be negative real numbers we denote this set by $\mathcal{G}_0^N := \{v \in \mathcal{G}^N : v(\{i\}) > 0 \text{ for all } i \in N \text{ or } v(\{i\}) < 0 \text{ for all } i \in N\}$, if the worths of the singletons must all be positive rational numbers or must all be negative rational numbers we mark this set by $\mathcal{G}_{0_\mathbb{Q}}^N := \{v \in \mathcal{G}_0^N : v(\{i\}) \in \mathbb{Q} \text{ for all } i \in N\}$.

A **TU-value** on $\mathcal{G}^N$ (respectively on subdomains of $\mathcal{G}^N$) is an operator φ, that assigns to any $v \in \mathcal{G}^N$ (respectively v is an element of a subdomain of $\mathcal{G}^N$) a payoff vector $\varphi(N, v) \in \mathbb{R}^N$ (or $\varphi(v)$ for short), with the meaning that $\varphi_i(v)$ is the payoff to player i in the TU-game v.

Let $v \in \mathcal{G}^N$ and $S \subseteq N$. We denote by (S, v) the **restriction** of (N, v) to the player set S. The **Harsanyi dividends** $\Delta_v(S)$ (Harsanyi, 1959) are defined inductively by

$$\Delta_v(S) = \begin{cases} v(S) - \sum_{R \subsetneq S} \Delta_v(R), & \text{if } |S| \geq 1, \text{ and} \\ 0 & \text{if } S = \emptyset. \end{cases} \tag{1}$$

Another well-known formula of the dividends is given for all $S \subseteq N$, $S \neq \emptyset$, by

$$\Delta_v(S) = \sum_{R \subseteq S} (-1)^{|S|-|R|} v(R). \tag{2}$$

The **marginal contribution** $MC_i^v(S)$ of player $i \in N$ to $S \subseteq N \backslash \{i\}$ is given by $MC_i^v(S) := v(S \cup \{i\}) - v(S)$. We call a coalition $S \subseteq N$ **active** in v if $\Delta_v(S) \neq 0$. Player $i \in N$ is a **dummy player** in v if $v(S \cup \{i\}) = v(S) + v(\{i\})$, $i \notin S$, $S \subseteq N$, or equivalent as a well-known fact, if $\Delta_v(S \cup \{i\}) = 0$ for all $S \subseteq N \backslash \{i\}$, $S \neq \emptyset$. If in addition $v(\{i\}) = 0$, then i is called a **null player**; players $i, j \in N$, $i \neq j$, are called **symmetric** in v, if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all coalitions $S \subseteq N \backslash \{i, j\}$.

In many applications, the assumption of symmetry of the players is not realistic, e. g. if the bargaining power or the amount of used venture capital are different. Shapley (1953a) introduced for this case the (positive) weighted Shapley values[1]. We denote by $\mathbb{R}^N_{++} := \{x \in \mathbb{R}^N : x_i > 0 \text{ for all } i \in N\}$ the set of all vectors $x \in \mathbb{R}^N$ where all coordinates x_i are positive. For all $N \in \mathcal{N}$, $v \in \mathcal{G}^N$, and a vector $\omega \in \mathbb{R}^N_{++}$ of positive weights ω_i for all $i \in N$, the (positive) **weighted Shapley value** Sh^ω (Shapley, 1953a) is defined by

$$Sh_i^\omega(v) = \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{\omega_i}{\sum_{j \in S} \omega_j} \Delta_v(S) \quad \text{for all } i \in N.$$

The next value distributes the dividends equally among all players in a coalition: for all $N \in \mathcal{N}$, $v \in \mathcal{G}^N$, the **Shapley value** Sh (Shapley, 1953b) is defined by

$$Sh_i(v) = \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{\Delta_v(S)}{|S|} \quad \text{for all } i \in N.$$

We see that the Shapley value is a weighted Shapley value where all weights are equal. The following value distributes the dividends proportionally to the singleton worths among all players in a coalition. For all $N \in \mathcal{N}$, $v \in \mathcal{G}_0^N$, the **proportional Shapley value** Sh^p (Gangolly, 1981; Besner, 2016; Béal et al., 2018) is given by

$$Sh_i^p(v) = \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_v(S) \quad \text{for all } i \in N. \tag{3}$$

Let φ be a TU-value on $\mathcal{G}^N$. We refer to some standard axioms. The first one assures that the sum of all payoffs equals the worth of the grand coalition: distributing less than $v(N)$ is not efficient and distributing more than $v(N)$ is not feasible.

Efficiency, E. For all $N \in \mathcal{N}$, $v \in \mathcal{G}^N$, we have $\sum_{i \in N} \varphi_i(v) = v(N)$.

The second axiom requires that a player whose marginal contribution is null with respect to every coalition receives a null payoff.

Null player. For all $N \in \mathcal{N}$, $v \in \mathcal{G}^N$, and $i \in N$ such that i is a null player in v, we have $\varphi_i(v) = 0$.

The third axiom is a stronger version of the null player property. It states that a player who only contributes his own worth to every coalition is getting a payoff of his singleton worth.

Dummy, D. For all $N \in \mathcal{N}$, $v \in \mathcal{G}^N$, and $i \in N$ such that i is a dummy player in v, we have $\varphi_i(v) = v(\{i\})$.

The next axiom states that a player's payoff depends only on her marginal contributions.

Marginality, M. For all $N \in \mathcal{N}$, $v, v' \in \mathcal{G}^N$, and $i \in N$ such that $MC_i^v(S) = MC_i^{v'}(S)$ for all $S \subseteq N \backslash \{i\}$, we have $\varphi_i(v) = \varphi_i(v')$.

[1]We desist from possibly null weights as by Shapley (1953a) or Kalai and Samet (1987).

Additivity requires that a TU-value applied to the sum of two TU-games gives the same payoff vector as the sum of the two payoff vectors obtained when applying the TU-value to each of the two TU-games.

Additivity, A. For all $N \in \mathcal{N}$, $v, v' \in \mathcal{G}^N$, we have $\varphi(v) + \varphi(v') = \varphi(v + v')$.

The symmetry axiom assures that equals should be treated equally.

Symmetry, S. For all $N \in \mathcal{N}$, $v \in \mathcal{G}^N$, $i, j \in N$ such that i and j are symmetric in v, we have $\varphi_i(v) = \varphi_j(v)$.

In the case of using a subdomain in the following sections, an axiom is required to hold whenever a game belongs to the subdomain.

3. A characterization similar to Shapley

Nowak and Radzik (1995) have used for axiomatizations of the weighted Shapley values, also in the spirit of Shapley (1953b) and Young (1985a), an axiom called ω-mutual dependence. There are, in contrast to our following definition, the singleton worths of dependent players zero.

Definition 3.1. *Let $v \in \mathcal{G}^N$. Two players $i, j \in N$, $i \neq j$, are called **weakly dependent** in v if $v(S \cup \{k\}) = v(S) + v(\{k\})$, $k \in \{i, j\}$, for all $S \subseteq N \backslash \{i, j\}$.*

This definition has the interpretation that a player is only interested to join a coalition which contents weakly dependent players if all weakly dependent players are in the joined coalition. So all weakly dependent players are in mutual dependence.

Lemma 3.2. *Players $i, j \in N$ are weakly dependent in $v \in \mathcal{G}^N$, iff $\Delta_v(S \cup \{k\}) = 0$, $k \in \{i, j\}$, for all $S \subseteq N \backslash \{i, j\}$, $S \neq \emptyset$.*

With Definition 3.1 for weakly dependent players, we obtain a proportional property, comparable to symmetry: the payoffs to two weakly dependent players are in the same proportion as their singleton weights.

Proportionality, P. For all $N \in \mathcal{N}$, $v \in \mathcal{G}_0^N$, $i, j \in N$ such that i and j are weakly dependent in v, we have

$$\frac{\varphi_i(v)}{v(\{i\})} = \frac{\varphi_j(v)}{v(\{j\})}.$$

In Shapley (1953b) are formulated desirable properties for a TU-value by his well-known three axioms, which can be represented by the four axioms of efficiency, null player, symmetry and additivity, where the null player property can be replaced by the dummy player property. But it is not appropriate to claim additivity in the case of a proportional value because additivity is not even satisfied in the two-player case (see Huettner (2015), page 282). So we use an axiom of additivity where in each game the stand-alone worths of all players are in the same proportion.

Weak additivity, WA. For all $N \in \mathcal{N}$, $v, w \in \mathcal{G}^N$, $w(\{i\}) = c \cdot v(\{i\})$ for all $i \in N$, $c > 0$, we have

$$\varphi(v) + \varphi(w) = \varphi(v + w).$$

It follows a characterization close to the original by Shapley (1953b).

Theorem 3.3. *Let $N \in \mathcal{N}$ and $v \in \mathcal{G}_0^N$. Sh^p is the unique TU-value that satisfies **E**, **D**, **P**, and **WA**.*

4. A characterization similar to Young

One of the most elegant characterizations of the Shapley value is suggested by Young (1985a). In addition to efficiency and symmetry, Young used marginality[2]. To characterize the proportional Shapley value we weaken marginality:

Weak marginality, WM. For all $N \in \mathcal{N}$, $v, w \in \mathcal{G}^N$, $w(\{j\}) = v(\{j\})$ for all $j \in N$, and $i \in N$ such that $MC_i^v(S) = MC_i^w(S)$ for all $S \subseteq N \backslash \{i\}$, we have

$$\varphi_i(v) = \varphi_i(w).$$

By this axiom, if the stand-alone worths of all players are given, the payoff to a player depends only on his own marginal contributions. Chun (1989) introduced the coalitional strategic equivalence axiom and Chun (1991) offers another appealing axiomatization of the Shapley value, using efficiency, symmetry and coalitional strategic equivalence, itself a generalization of strategic equivalence from von Neumann and Morgenstern (1944). But by Casajus and Huettner (2008) is shown that coalitional strategic equivalence and marginality are equivalent. Our next axiom weakens coalitional strategic equivalence.

Weak coalitional strategic equivalence, WCSE. For all $N \in \mathcal{N}$, $v, w \in \mathcal{G}^N$ such that for any coalition $R \subseteq N$, $|R| \geq 2$, $c \in \mathbb{R}$, and all coalitions $S \subseteq N$,

$$v(S) = \begin{cases} w(S) + c, & \text{if } S \supseteq R, \\ w(S), & \text{if } S \not\supseteq R, \end{cases} \tag{4}$$

we have $\varphi_i(v) = \varphi_i(w)$ for all $i \in N \backslash R$.

If the members of a coalition R are improving their cooperation and take this improvement into all supersets of R, the payoff to all non-members of R does not change. The same applies to all players outside of R if the cooperation within R has got worse and this trend has continued in all supersets of R by the same amount. Unlike our axiom, in coalitional strategic equivalence from Chun, there are also admitted singletons for the coalition R, in strategic equivalence from Neumann and Morgenstern only singletons.

We show, analog to Casajus and Huettner (2008):

Proposition 4.1. ***WM*** *is equivalent to* ***WCSE****.*

We obtain a proportional counterpart to the axiomatization by Young (1985a).

Theorem 4.2. *Let $N \in \mathcal{N}$ and $v \in \mathcal{G}_0^N$. Sh^p is the unique TU-value that satisfies **E**, **P** and **WM/WCSE** (equivalent by Proposition 4.1).*

[2]Originally Young used an axiom called strong monotonicity. Chun (1989) named the essential part of this axiom for the proof of the uniqueness marginality.

5. Examples

This section demonstrates the advantages of using the proportional Shapley value in cost allocation. It confirms the proportionality property and illustrates the player splitting property of the proportional Shapley value, presented in Section 6.

5.1. City 1

Assume that three districts of a city, district A, B, and C, wish to get a motorway ring. In Fig. 1 the lengths of the motorway sections are given in kilometers. The offer of the

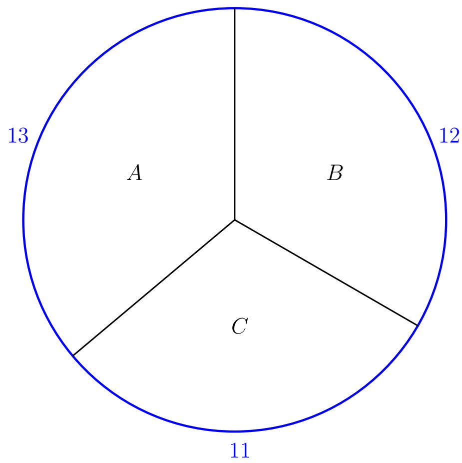

Figure 1: City with three districts and a motorway ring (lengths in km).

road-building company with the most favorable prices (in millions of monetary units) is given by

$$p(\ell) = \begin{cases} 100\ell, & \text{for } 0 \leq \ell < 20, \\ 100\ell - 100, & \text{for } 20 \leq \ell < 32, \\ 100\ell - 200, & \text{for } 32 \leq \ell, \end{cases}$$

where ℓ is the length in kilometers. To share the building costs, we can establish a cost game v on $N = \{1, 2, 3\}$ with players $1 := A$, $2 := B$, $3 := C$, where the worth of a coalition $S \subseteq N$ is the cost of the coalition S (in millions of monetary units). We get

$$v(\{1\}) = 1300, \quad v(2) = 1200, \quad v(3) = 1100,$$
$$v(\{1,2\}) = 2400, \quad v(\{1,3\}) = 2300, \quad v(\{2,3\}) = 2200,$$
$$v(\{1,2,3\}) = 3400.$$

The three districts have the problem, how to share the costs in the game v.

5.2. City 2

We modify city 1 to city 2 (Fig. 2).

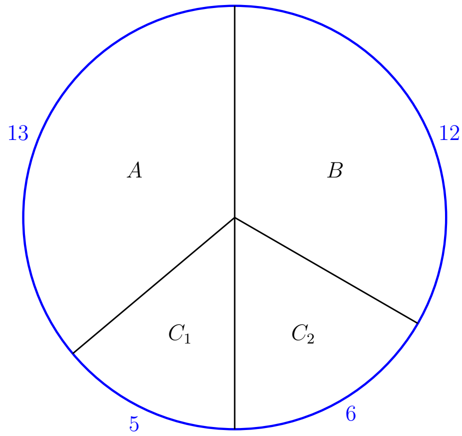

Figure 2: City with four districts and a motorway ring (lengths in km).

District C is split in two districts C_1 and C_2. We get a new coalition function v' on $N = \{1, 2, 3_1, 3_2\}$ with players $1 := A$, $2 := B$, $3_1 := C_1$, $3_2 := C_2$, given by

$$\begin{array}{lll} v'(\{1\}) = 1300, & v'(2) = 1200, & v'(3_1) = 500, \\ v'(3_2) = 600, & v'(\{1,2\}) = 2400, & v'(\{1,3_1\}) = 1800, \\ v'(\{1,3_2\}) = 1900, & v'(\{2,3_1\}) = 1700, & v'(\{2,3_2\}) = 1800, \\ v'(\{3_1,3_2\}) = 1100, & v'(\{1,2,3_1\}) = 2900, & v'(\{1,2,3_2\}) = 3000, \\ v'(\{1,3_1,3_2\}) = 2300, & v'(\{2,3_1,3_2\}) = 2200, & v'(\{1,2,3_1,3_2\}) = 3400. \end{array}$$

Here is a special kind of "dependency" between player 3_1 and 3_2:

- The sum of the singleton worths from player 3_1 and 3_2 in v' is equal to the worth of player 3 in the game v.

- The marginal contributions of player 3_1 or player 3_2 to any coalition which does not contain the respective other player are only the singleton worths of these players. So the players 3_1 and 3_2 are weakly dependent in v'.

- All coalitions which conclude both players 3_1 and 3_2 have the same worth in v' as the related coalitions in v which content the player 3.

- Coalitions which are the same in v and v' have the same worth in v and v'.

Overall, there is no effect on the other players by splitting player 3 into two new players 3_1 and 3_2. Hence, we call a value consistent for splitting in a game v if in a corresponding game v', fulfilling the same conditions as here (we will formulate these conditions in Definition 6.1), the payoff to not split players does not change.

We get with the Shapley value in the game v

$$Sh_1(v) = 1233.33,\ Sh_2(v) = 1133.33,\ Sh_3(v) = 1033.33.$$

In the game v' we obtain

$$Sh_1(v') = 1241.67,\ Sh_2(v') = 1141.67,\ Sh_{3_1}(v') = 458.33,\ Sh_{3_2}(v') = 558.33.$$

The total cost saving of cooperating is 200 million in each game. District C saves in the game v 66.67 million (one-third of the total saving) and district C_1 and district C_2 save together in the game v' 83,34 million (42% of the total saving), although district C owns only 31% of the length of the motorway. So there is, additional to the inconsistency, also a discriminatory level of players which have a greater share of costs.

On the contrary, we get with the proportional Shapley value in the game v

$$Sh_1^p(v) = 1229.94,\ Sh_2^p(v) = 1133.16,\ Sh_3^p(v) = 1036.90$$

and in the game v'

$$Sh_1^p(v') = 1229.94,\ Sh_2^p(v') = 1133.16,\ Sh_{3_1}^p(v') = 471.32,\ Sh_{3_2}^p(v') = 565.58.$$

Districts A and B have in both games the same costs and district C saves the same as districts C_1 and C_2 together, 32% of the total cost saving. The proportional Shapley value is then consistent for splitting in our sense, what we will show in general in the following subsection. In addition, this result confirms the proportionality property of the proportional Shapley value which gives the key difference in theorem 3.3 and theorem 4.2 to the related axiomatizations of the Shapley value: the players 3_1 and 3_2 are weakly dependent in v' and we have

$$\frac{Sh_{3_1}^p(v')}{v'(\{3_1\})} = \frac{Sh_{3_2}^p(v')}{v'(\{3_2\})}.$$

6. Player Splitting

In many applications, it is not desired that the players' payoffs change if another player splits into several new players, which together have only the same input to the game as the original splitting player, like in subsection 5.2 in our examples. We define a corresponding game to a TU-game where a player of the original game is "split" in two new players:

Definition 6.1. *Let $N, N^j \in \mathcal{N}$, $(N,v) \in \mathcal{G}^N$, $(N^j, v^j) \in \mathcal{G}^{N^j}$, $j \in N$, $k,\ell \in N^j$, $k,\ell \notin N$, $N^j = (N\backslash\{j\}) \cup \{k,\ell\}$. The game (N^j, v^j) is called a* ***corresponding split player game***[3] *to (N,v) if for all $S \subseteq N\backslash\{j\}$*

- $v^j(\{k\}) + v^j(\{\ell\}) = v(\{j\})$,
- $v^j(S \cup \{i\}) = v(S) + v^j(\{i\})$, $i \in \{k,\ell\}$,
- $v^j(S \cup \{k,l\}) = v(S \cup \{j\})$, *and*
- $v^j(S) = v(S)$.

It should be observed that players k,ℓ are weakly dependent in the game v^j.

Banker (1981) notes, that an allocation scheme should not be sensitive to the way cost centers are used or organized. For this kind of games the Shapley value is not the right choice, because it does not satisfy the following axiom:

[3] In the case of using a subdomain (e. g. $\mathcal{G}_0^N$ or $\mathcal{G}_{0_{\mathbb{Q}}}^N$) for the TU-game (N,v) we require that the corresponding split player game (N^j, v^j) is defined on the related subdomain (e. g. $\mathcal{G}_0^{N^j}$ or $\mathcal{G}_{0_{\mathbb{Q}}}^{N^j}$ respectively).

Player splitting, PS. For all $N \in \mathcal{N}$, $(N, v) \in \mathcal{G}^N$, $j \in N$, and a corresponding split player game $(N^j, v^j) \in \mathcal{G}^{N^j}$ to (N, v), we have

$$\varphi_i(N, v) = \varphi_i(N^j, v^j) \text{ for all } i \in N \backslash \{j\}.$$

A suitable application of this axiom would be, that a player, who was participating in an online game full-time, now is participating part-time under cover-names. The sum of her time in part-time activities is equal to the original time in full-time activities. She participates with the same productivity in all original coalitions, but now under all her cover-names at the same time in total. In all other coalitions, she has only a pro forma membership. This means that her marginal contribution to these coalitions is only her singleton worth or, more specifically, all part-time players are weakly dependent. In such a situation the payoff to the other players should not change.

Remark 6.2. *Let $N \in \mathcal{N}$, $(N, v) \in \mathcal{G}^N$, $j \in N$, and $(N^j, v^j) \in \mathcal{G}^{N^j}$ a corresponding split player game to (N, v). If φ is a TU-value that satisfies **E** and **PS**, then we have*

$$\varphi_j(N, v) = \varphi_k(N^j, v^j) + \varphi_\ell(N^j, v^j).$$

This remark is related to the amalgamating payoffs axiom by Radzik (2012) which is there used to characterize the weighted Shapley values.

Remark 6.3. *In Definition 6.1 the game (N, v) could also be considered as a **corresponding merged player game** to (N^j, v^j) and so **PS** as a **player merging** axiom. But we expressly point out that player $j \in N$ and players $k, \ell \in N^j$ can be completely independent apart from the given properties in Definition 6.1 unlike the amalgamating payoffs property by Radzik (2012).*

It transpires that splitting fits best with the proportional Shapley value.

Proposition 6.4. *Sh^p satisfies **PS**.*

The following lemma shows dependence on symmetry for efficient values which satisfy player splitting.

Lemma 6.5. *Let $N \in \mathcal{N}$ and $v \in \mathcal{G}_0^N$. If a TU-value φ satisfies **E** and **PS**, then φ satisfies also **S**.*

We have another interesting lemma which uses Lemma 6.5 in the proof.

Lemma 6.6. *Let $N \in \mathcal{N}$ and $v \in \mathcal{G}_{0_\mathbb{Q}}^N$. If a TU-value φ satisfies **E** and **PS**, then φ satisfies also **P**.*

Thus, we obtain if the worths of all singletons are positive rational numbers, similar to Young (1985a), the following corollary.

Corollary 6.7. *Let $N \in \mathcal{N}$ and $v \in \mathcal{G}_{0_\mathbb{Q}}^N$. Sh^p is the unique TU-value that satisfies **E**, **PS**, and **WM/WCSE** (equivalent by Proposition 4.1).*

The proof follows immediately by Proposition 6.4 and Lemma 6.6 from Theorem 4.2. We have another characterization with player splitting, similar to Shapley (1953b).

Corollary 6.8. *Let $N \in \mathcal{N}$ and $v \in \mathcal{G}^N_{0_{\mathbb{Q}}}$. Sh^p is the unique TU-value that satisfies **E**, **D**, **PS**, and **WA**.*

The proof follows obviously by Proposition 6.4 and Lemma 6.6 from Theorem 3.3.

Remark 6.9. *Lemma 6.6 holds for $v \in \mathcal{G}^N_0$ if we require continuity of the TU-value in $v(\{i\})$ for all $v \in \mathcal{G}^N_0$ and all $i \in N$ in an additional axiom. So also Corollary 6.7 and Corollary 6.8 are valid for $v \in \mathcal{G}^N_0$ if there is in each case an additional continuity axiom.*

7. Conclusion

In this paper, we have shown, that in games where the stand-alone worths of the singletons are reasonable weights, the proportional Shapley value is a powerful tool to share benefits of cooperation. The value convinces due to its applicable axioms, comparable with the Shapley value. It is especially suitable for games where we want to ensure that the payoffs to uninvolved players are not changing if another player is splitting into two new players which together have the same input in the game as the single player before. The Shapley value is not appropriate for such games and this could be one of the main reasons why there is significant resistance to the use of the Shapley value in cost allocation in practice.

Acknowledgements

We would like to thank André Casajus, Winfried Hochstättler, Jörg Homberger, Frank Huettner, Hans Peters and especially an anonymous referee for their helpful comments and suggestions.

8. Appendix

8.1. Additional lemmas and a remark, used in the proofs

Remark 8.1. *We can consider the collection of all TU-games $v \in \mathcal{G}^N, N \in \mathcal{N}$, as a vector space $\mathbb{R}^{2^N-1}$. Each game v is represented by a vector $\overrightarrow{v} \in \mathbb{R}^{2^N-1}$ where the entries in the $2^{|N|-1}$ coordinates of the $2^{|N|-1}$ coalitions $S \subseteq N$, $S \neq \emptyset$, get the worth $v(S)$ of the respective coalition S. Hence there exists for every game v a vector $\overrightarrow{\Delta_v} \in \mathbb{R}^{2^N-1}$, which corresponds to the vector $\overrightarrow{v}$, where the entries of the coordinates get the dividends of the respective coalitions. By statement (1) we obtain with $v, v_1, v_2 \in \mathcal{G}^N$, and for all $S \subseteq N$,*

$$\begin{aligned}
& \overrightarrow{\Delta_v} = \overrightarrow{\Delta_{v_1}} + \overrightarrow{\Delta_{v_2}} \\
\Leftrightarrow\quad & \Delta_v(S) = \Delta_{v_1}(S) + \Delta_{v_2}(S) \\
\Leftrightarrow\quad & v(S) - \sum_{R \subsetneq S} \Delta_v(R) = v_1(S) - \sum_{R \subsetneq S} \Delta_{v_1}(R) + v_2(S) - \sum_{R \subsetneq S} \Delta_{v_2}(R) \\
\Leftrightarrow\quad & v = v_1 + v_2.
\end{aligned}$$

Lemma 8.2. *Statement (4) in **WCSE** can be replaced equivalently by*

$$\Delta_v(S) = \begin{cases} \Delta_w(R) + c, & \text{if } S = R, \\ \Delta_w(S), & \text{otherwise.} \end{cases}$$

Proof. Let the notation and the preconditions as in **WCSE**. By (2), if $v(S) = w(S)$ for all $S \not\supseteq R$, we have $\Delta_v(S) = \Delta_w(S)$ for all such S and vice versa. Hence, by (1), $v(R) = w(R) + c$ is equivalent to $\Delta_v(R) = \Delta_w(R) + c$. By induction on the size $s := |S|$ we show now $v(S) = w(S) + c \Leftrightarrow \Delta_v(S) = \Delta_w(S)$ for all proper supersets $S \supsetneq R$.

Initialization: Let $S \supsetneq R$ and $s = |R| + 1$. R is the only proper subset of S where there is a difference of the related dividends in both coalition functions and we obtain

$$\begin{aligned} v(S) = w(S) + c \underset{(1)}{\Leftrightarrow} \quad v(S) - \sum_{T \subsetneq S} \Delta_v(T) &= w(S) + c - \sum_{\substack{T \subsetneq S, \\ T \neq R}} \Delta_w(T) - (\Delta_w(R) + c) \\ \underset{(1)}{\Leftrightarrow} \quad \Delta_v(S) &= \Delta_w(S). \end{aligned}$$

Induction step: Assume equivalence holds for $s' = s - 1$, $|R| + 1 \leq s' \leq n - 1$ (IH). Then, by (IH), R is again the only proper subset of S with not equal related dividends in v and w. By (1), we get $v(S) = w(S) + c \Leftrightarrow \Delta_v(S) = \Delta_w(S)$ as before and Lemma 8.2 is shown. □

Lemma 8.3. (Casajus and Huettner, 2008). *If $i \in N$ and $v, w \in \mathcal{G}^N$, then $MC_i^v(S) = MC_i^w(S)$ for all $S \subseteq N \backslash \{i\}$ iff $\Delta_v(S \cup \{i\}) = \Delta_w(S \cup \{i\})$ for all $S \subseteq N \backslash \{i\}$.*

8.2. Proofs

8.2.1. Proof of Lemma 3.2

Let $i, j \in N$ and $v \in \mathcal{G}^N$. If $S = \emptyset$, we have $v(S \cup \{k\}) = v(S) + v(\{k\})$. We show by induction on the size $s := |S|$ of all coalitions $S \subseteq N \backslash \{i, j\}$, $S \neq \emptyset$,

$$v(S \cup \{k\}) = v(S) + v(\{k\}) \quad \Leftrightarrow \quad \Delta_v(S \cup \{k\}) = 0.$$

Initialization: Let $s = 1$. For $k \in \{i, j\}$ we have

$$\begin{aligned} v(S \cup \{k\}) &= v(S) + v(\{k\}) \\ \underset{(1)}{\Leftrightarrow} \quad \Delta_v(S \cup \{k\}) + \Delta_v(S) + \Delta_v(\{k\}) &= \Delta_v(S) + \Delta_v(\{k\}) \\ \Leftrightarrow \quad \Delta_v(S \cup \{k\}) &= 0. \end{aligned}$$

Induction step: Assume that equivalence and equality in the first and last line of the system above hold for all coalitions S' with $s' \geq 1$ (IH) and let $s = s' + 1$ and $k \in \{i, j\}$. We get

$$\begin{aligned} v(S \cup \{k\}) &= v(S) + v(\{k\}) \\ \underset{(1)}{\Leftrightarrow} \quad \Delta_v(S \cup \{k\}) + \sum_{R \subsetneq (S \cup \{k\})} \Delta_v(R) &= \sum_{R \subseteq S} \Delta_v(R) + \Delta_v(\{k\}) \\ \underset{(IH)}{\Leftrightarrow} \quad \Delta_v(S \cup \{k\}) + \Delta_v(\{k\}) + \sum_{R \subseteq S} \Delta_v(R) &= \sum_{R \subseteq S} \Delta_v(R) + \Delta_v(\{k\}) \\ \Leftrightarrow \quad \Delta_v(S \cup \{k\}) &= 0. \end{aligned}$$

Thus equivalence is shown. □

8.2.2. Proof of Theorem 3.3

I. Existence: By Béal et al. (2018), Sh^p satisfies **E** and **D**.

- **P**: Let $v \in \mathcal{G}_0^N$ and $i, j \in N$ such that i and j are weakly dependent in v. We have

$$\begin{aligned} Sh_i^p(v) &= \sum_{\substack{S \subseteq N,\\ S \ni i}} \frac{v(\{i\})}{\sum_{k\in S} v(\{k\})}\Delta_v(S) \underset{Lem.3.2}{=} v(\{i\}) + \sum_{\substack{S\subseteq N,\\ \{i,j\}\subseteq S}} \frac{v(\{i\})}{\sum_{k\in S} v(\{k\})}\Delta_v(S) \\ &= \frac{v(\{i\})}{v(\{j\})}v(\{j\}) + \frac{v(\{i\})}{v(\{j\})} \sum_{\substack{S\subseteq N,\\ \{i,j\}\subseteq S}} \frac{v(\{j\})}{\sum_{k\in S} v(\{k\})}\Delta_v(S) \\ &\underset{Lem.3.2}{=} \frac{v(\{i\})}{v(\{j\})} \sum_{\substack{S\subseteq N,\\ S\ni j}} \frac{v(\{j\})}{\sum_{k\in S} v(\{k\})}\Delta_v(S) = \frac{v(\{i\})}{v(\{j\})} Sh_j^p(v). \end{aligned}$$

- **WA**: Let $v, w \in \mathcal{G}_0^N$ with $w(\{i\}) = c \cdot v(\{i\})$ for all $i \in N$, $c > 0$. We have

$$\begin{aligned} Sh_i^p(v) + Sh_i^p(w) &\underset{(3)}{=} \sum_{\substack{S\subseteq N,\\ S\ni i}} \frac{v(\{i\})}{\sum_{j\in S} v(\{j\})}\Delta_v(S) + \sum_{\substack{S\subseteq N,\\ S\ni i}} \frac{w(\{i\})}{\sum_{j\in S} w(\{j\})}\Delta_w(S) \\ &= \sum_{\substack{S\subseteq N,\\ S\ni i}} \frac{v(\{i\})}{\sum_{j\in S} v(\{j\})}\Delta_v(S) + \sum_{\substack{S\subseteq N,\\ S\ni i}} \frac{c\cdot v(\{i\})}{\sum_{j\in S} c\cdot v(\{j\})}\Delta_w(S) \\ &= \sum_{\substack{S\subseteq N,\\ S\ni i}} \frac{v(\{i\})}{\sum_{j\in S} v(\{j\})}\left[\Delta_v(S) + \Delta_w(S)\right] \\ &= \sum_{\substack{S\subseteq N,\\ S\ni i}} \frac{(1+c)v(\{i\})}{\sum_{j\in S}(1+c) v(\{j\})}\left[\Delta_v(S) + \Delta_w(S)\right] \\ &= \sum_{\substack{S\subseteq N,\\ S\ni i}} \frac{v(\{i\}) + w(\{i\})}{\sum_{j\in S}\left[v(\{j\}) + w(\{j\})\right]}\Delta_{v+w}(S) = Sh_i^p(v+w). \end{aligned}$$

II. Uniqueness: Let $N \in \mathcal{N}$, $n := |N|$, $v \in \mathcal{G}_0^N$, and φ a TU-value which satisfies all axioms of Theorem 3.3. To prove uniqueness, we will show that φ equals Sh^p.

For $n = 1$, φ equals Sh^p by **E**.

Let now $n \geq 2$. For each coalition $S \subseteq N$, $S \neq \emptyset$, we define, corresponding to remark 8.1, a TU-game $v_S \in \mathcal{G}_0^N$ through a vector $\overrightarrow{v_S} \in \mathbb{R}^{2^n-1}$ by assigning the coordinates of the related vector $\overrightarrow{\Delta_{v_S}} \in \mathbb{R}^{2^n-1}$ in the entry of a coalition $R \subseteq N$, $R \neq \emptyset$, the dividend

$$\Delta_{v_S}(R) := \begin{cases} \dfrac{v(\{j\})}{2^n-1}, \text{ if } R = \{j\} \text{ for all } j \in N, \\ \Delta_v(S), \text{ if } R = S, |S| \geq 2, \\ 0, \text{ otherwise.} \end{cases}$$

Thus each vector $\overrightarrow{v_S} \in \mathbb{R}^{2^n-1}$ gets in the coordinates of coalitions $R \subseteq N$, $R \neq \emptyset$, the entry

$$v_S(R) = \begin{cases} \Delta_v(S) + \sum_{j\in R} \dfrac{v(\{j\})}{2^n-1}, \text{ if } R \supseteq S, |S| \geq 2, \\ \sum_{j\in R} \dfrac{v(\{j\})}{2^n-1}, \text{ otherwise.} \end{cases} \tag{5}$$

We have $\overrightarrow{\Delta_v} = \sum_{\substack{S\subseteq N,\\ S\neq\emptyset}} \overrightarrow{\Delta_{v_S}}$ and so, by remark 8.1, $v = \sum_{\substack{S\subseteq N,\\ S\neq\emptyset}} v_S$.
By **D** we obtain

$$\varphi_i(v_S) = \begin{cases} v_S(\{i\}) = \dfrac{v(\{i\})}{2^n-1} \text{ for all } i \in N \text{ and } |S| = 1, \text{ and} \\ v_S(\{i\}) = \dfrac{v(\{i\})}{2^n-1} \text{ for all } i \in N,\, i \notin S,\, |S| \geq 2. \end{cases} \tag{6}$$

By Lemma 3.2, all players $i \in S$, $|S| \geq 2$, are pairwise weakly dependent in v_S. We get for an arbitrary $i \in S$, $|S| \geq 2$, and by $v_S(N) \underset{(5)}{=} \Delta_v(S) + \sum_{j\in N} \dfrac{v(\{j\})}{2^n-1}$

$$\begin{aligned} \textstyle\sum_{j\in S} \varphi_j(v_S) &\underset{\mathbf{(P)}}{=} \sum_{j\in S} \frac{v_S(\{j\})}{v_S(\{i\})}\varphi_i(v_S) = \sum_{j\in S} \frac{v(\{j\})}{v(\{i\})}\varphi_i(v_S) \\ &\underset{\mathbf{(E)}}{=} v_S(N) - \sum_{j\in N\setminus S} \varphi_j(v_S) \underset{(6)}{=} \Delta_v(S) + \sum_{j\in S} \frac{v(\{j\})}{2^n-1} \\ \Leftrightarrow \quad \varphi_i(v_S) &= \frac{v(\{i\})}{\sum_{j\in S} v(\{j\})}\Delta_v(S) + \frac{v(\{i\})}{2^n-1}. \end{aligned} \tag{7}$$

So we have by (3), (6) and (7) for all $S \subseteq N$, $S \neq \emptyset$,

$$\varphi_i(v_S) = Sh_i^p(v_S) \text{ for all } i \in N.$$

Sh^p and φ satisfy **WA**. It follows

$$\varphi_i(v) = Sh_i^p(v) \text{ for all } i \in N$$

and uniqueness is shown. □

8.2.3. Proof of Proposition 4.1

⇒: We show **WM** implies **WCSE**: Let v and w two TU-games satisfying the hypotheses of **WCSE**, i.e. for a coalition $R \subseteq N$, $|R| \geq 2$, $c \in \mathbb{R}$, we have

$$v(S) = \begin{cases} w(S) + c, & \text{if } S \supseteq R, \\ w(S), & \text{if } S \not\supseteq R. \end{cases}$$

Let φ be a TU-value which obeys **WM**. By Lemma 8.2, we have

$$\Delta_v(S) = \begin{cases} \Delta_w(R) + c, & \text{if } S = R, \\ \Delta_w(S), & \text{otherwise.} \end{cases}$$

Thus, we have $\Delta_v(S \cup \{i\}) = \Delta_w(S \cup \{i\})$ for all $i \in N\backslash R$ and $S \subseteq N\backslash\{i\}$. By Lemma 8.3 follows $MC_i^v(S) = MC_i^w(S)$ for all $S \subseteq N\backslash\{i\}$. So we can use **WM** and get $\varphi_i(v) = \varphi_i(w)$ for all $i \in N\backslash R$ and **WCSE** is satisfied.
⇐: We show **WCSE** implies **WM**: Let $N \in \mathcal{N}$, $i \in N$, $v, w \in \mathcal{G}^N$ two coalition functions satisfying the hypothesis of **WM**, i.e. $MC_i^v(S) = MC_i^w(S)$ for all $S \subseteq N\backslash\{i\}$ and

$w(\{k\}) = v(\{k\})$ for all $k \in N$ and φ a value satisfying **WCSE**. Then, by Lemma 8.3, we have $\Delta_v(T) = \Delta_w(T)$ for all $T \subseteq N$, $T \ni i$. Let $\mathcal{R} = \{R_j \subseteq N : \Delta_v(R_j) \neq \Delta_w(R_j)\}$ an indexed set of all subsets of N with different dividends in v and w, $1 \leq j \leq |\mathcal{R}|$. We inductively define a sequence of coalition functions w_j, $0 \leq j \leq |\mathcal{R}|$, by $w_j := w$ if $j = 0$, and, if $1 \leq j \leq |\mathcal{R}|$,

$$\Delta_{w_j}(S) := \begin{cases} \Delta_{w_{j-1}}(R_j) + \left[\Delta_v(R_j) - \Delta_{w_{j-1}}(R_j)\right], & \text{if } S = R_j, \\ \Delta_{w_{j-1}}(S), & \text{if } S \subseteq N,\ S \neq R_j. \end{cases}$$

Then we have $w_{|\mathcal{R}|} = v$ and, by Lemma 8.2 and **WCSE,** we get $\varphi_i(w_j) = \varphi_i(w_{j-1})$ for all j, $1 \leq j \leq |\mathcal{R}|$, and therefore $\varphi_i(v) = \varphi_i(w)$ and **WM** is satisfied. □

8.2.4. Proof of Theorem 4.2

I. Existence: By Theorem 3.3, Sh^p satisfies **E** and **P**.

- **WCSE**: By Lemma 8.2, we have for v and a coalition R from **WCSE**

$$\Delta_v(S) = \begin{cases} \Delta_w(R) + c, & \text{if } S = R, \\ \Delta_w(S), & \text{otherwise.} \end{cases}$$

Thus, we obtain for all $i \in N \backslash R$

$$Sh_i^p(v) = \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_v(S) = \sum_{\substack{S \subseteq N, \\ S \ni i}} \frac{w(\{i\})}{\sum_{j \in S} w(\{j\})} \Delta_w(S) = Sh_i^p(w).$$

II. Uniqueness: Let $N \in \mathcal{N}$, $n := |N|$, $v \in \mathcal{G}_0^N$, and φ a TU-value which satisfies all axioms of Theorem 4.2. We will show that φ satisfies eq. (3).

For $n = 1$, eq. (3) is satisfied by **E**.

Let $n \geq 2$. We use an induction on the size $r := |\{R \subseteq N : R \text{ is active in } v \text{ and } |R| \geq 2\}|$.

Initialization: Let $r = 0$. By Lemma 3.2, all players $i, j \in N$ are pairwise weakly dependent in v. We get for an arbitrary $i \in N$

$$\sum_{j \in N} \varphi_j(v) \underset{(\mathbf{P})}{=} \sum_{j \in N} \frac{v(\{j\})}{v(\{i\})} \varphi_i(v) \underset{(\mathbf{E})}{=} v(N).$$

With $v(N) = \sum_{j \in N} v(\{j\})$ follows $\varphi_i(v) = v(\{i\})$ and eq. (3) holds to φ if r = 0.

Induction step: Assume that eq. (3) holds to φ if $r \geq 0$, r arbitrary (IH), and let exactly $r + 1$ coalitions $Q_k \subseteq N$, $|Q_k| \geq 2$, $1 \leq k \leq r + 1$, active in v. Let Q be the intersection of all such coalitions Q_k

$$Q = \bigcap_{1 \leq k \leq r+1} Q_k.$$

We distinguish two cases: (a) $i \in N \backslash Q$ and (b) $i \in Q$.

(a) Each player $i \in N \backslash Q$ is a member of at most r active coalitions Q_k, $|Q_k| \geq 2$, and v gets at least one active coalition R_i, $|R_i| \geq 2$, $i \notin R_i$. Hence exists a coalition function

$w_i \in \mathcal{G}_0^N$ where all coalitions get the same dividend in w_i as in v, except the coalition R_i which gets the dividend $\Delta_{w_i}(R_i) = 0$, and there is existing a scalar $c \in \mathbb{R}$, $c \neq 0$, with

$$\Delta_v(S) = \begin{cases} \Delta_{w_i}(R_i) + c, & \text{if } S = R_i, \\ \Delta_{w_i}(S), & \text{otherwise.} \end{cases}$$

By Lemma 8.2 and **WCSE**, we get $\varphi_i(v) = \varphi_i(w_i)$ with $i \in N \backslash R_i$ and because there exists for all $i \in N \backslash Q$ a such R_i, we get $\varphi_i(v) = \varphi_i(w_i)$ for all $i \in N \backslash Q$. All coalition functions w_i get at most r active coalitions with at least two players and eq. (3) follows by (IH). Thus, we have

$$\varphi_i(v) = Sh_i^p(v) \text{ for all } i \in N \backslash Q. \tag{8}$$

(b) If $Q = \{i\}$, we get, by **E** of φ and Sh^p and case (a), $\varphi_i(v) = Sh_i^p(v)$. If $|Q| \geq 2$, each player $j \in Q$ is a member of all $r+1$ active coalitions $Q_k \subseteq N$, $|Q_k| \geq 2$, $1 \leq k \leq r+1$, and therefore, by Lemma 3.2, all players $j \in Q$ are weakly dependent. By **P** and **E** of φ and Sh^p, we get for an arbitrary $i \in Q$

$$\sum_{j \in Q} \varphi_j(v) \underset{(\mathbf{P})}{=} \sum_{j \in Q} \frac{v(\{j\})}{v(\{i\})} \varphi_i(v) \underset{\substack{(\mathbf{E}) \\ (8)}}{=} v(N) - \sum_{j \in N \backslash Q} Sh_j^p(v) \underset{(\mathbf{E})}{=} \sum_{j \in Q} Sh_j^p(v)$$

$$\underset{(\mathbf{P})}{=} \sum_{j \in Q} \frac{v(\{j\})}{v(\{i\})} Sh_i^p(v) \quad \Leftrightarrow \quad \varphi_i(v) = Sh_i^p(v)$$

and together with $I.$ the proof is complete. □

8.2.5. Proof of Proposition 6.4

Let $(N, v) \in \mathcal{G}_0^N$, $j \in N$, and $(N^j, v^j) \in \mathcal{G}_0^{N^j}$ a corresponding split player game to (N, v). We point out that we have for all $S \subseteq N \backslash \{j\}$, $S \neq \emptyset$, $\Delta_{v^j}(S) = \Delta_v(S)$, $\Delta_{v^j}(S \cup \{k, l\}) = \Delta_v(S \cup \{j\})$, and $\Delta_{v^j}(S \cup \{k\}) = \Delta_{v^j}(S \cup \{\ell\}) = 0$. Then we get for all $i \in N \backslash \{j\}$

$$\begin{aligned} Sh_i^p(N, v) &= \sum_{\substack{R \subseteq N, \\ R \ni i}} \frac{v(\{i\})}{\sum_{m \in R} v(\{m\})} \Delta_v(R) \\ &= \sum_{\substack{S \subseteq N \backslash \{j\}, \\ S \ni i}} \frac{v(\{i\})}{\sum_{m \in S} v(\{m\})} \Delta_v(S) + \sum_{\substack{S \subseteq N \backslash \{j\}, \\ S \ni i}} \frac{v(\{i\})}{\sum_{m \in S \cup \{j\}} v(\{m\})} \Delta_v(S \cup \{j\}) \\ &= \sum_{\substack{S \subseteq N^j \backslash \{k, \ell\}, \\ S \ni i}} \frac{v^j(\{i\})}{\sum_{m \in S} v^j(\{m\})} \Delta_{v^j}(S) \\ &\quad + \sum_{\substack{S \subseteq N^j \backslash \{k, \ell\}, \\ S \ni i}} \frac{v^j(\{i\})}{\sum_{m \in S \cup \{k, \ell\}} v^j(\{m\})} \Delta_{v^j}(S \cup \{k, \ell\}) \\ &= \sum_{\substack{R \subseteq N^j, \\ R \ni i}} \frac{v^j(\{i\})}{\sum_{m \in R} v^j(\{m\})} \Delta_{v^j}(R) = Sh_i^p(N^j, v^j). \end{aligned}$$

□

8.2.6. Proof of Lemma 6.5

Let $N = \{1, 2, ..., n\}$, $|N| \geq 2$, $v \in \mathcal{G}_0^N$, φ a TU-value which satisfies **E** and **PS** for all $v \in \mathcal{G}_0^N$, and w.l.o.g., player 1 and player 2 be symmetric in v. If we split player 1 according to **PS** into two new players, player $n+1$ and player $n+2$, $N^1 = \{2, 3, ..., n, n+1, n+2\}$, we have

$$\varphi_2(N^1, v^1) = \varphi_2(N, v), \tag{9}$$

and, if we split player 2 according to **PS** into the same players as before, player $n+1$ and player $n+2$, instead, $N^2 = \{1, 3, 4, ..., n, n+1, n+2\}$, we have

$$\varphi_1(N^2, v^2) = \varphi_1(N, v), \tag{10}$$

where we choose $v^2(\{n+1\}) := v^1(\{n+1\})$ and $v^2(\{n+2\}) := v^1(\{n+2\})$.

In the same manner we split now in the game (N^1, v^1) player 2 into two new players, player $n+3$ and player $n+4$, and analogous in the game (N^2, v^2) player 1 into the same players as before, player $n+3$ and player $n+4$, and choose $v^{2^1}(\{n+3\}) := v^{1^2}(\{n+3\})$ and $v^{2^1}(\{n+4\}) := v^{1^2}(\{n+4\})$. We have $N^{1^2} = N^{2^1} = \{3, 4, ..., n, n+1, n+2, n+3, n+4\}$ and $v^{1^2} = v^{2^1}$ and get by **E**, according to remark 6.2,

$$\varphi_{n+3}\left(N^{1^2}, v^{1^2}\right) + \varphi_{n+4}\left(N^{1^2}, v^{1^2}\right) = \varphi_2(N^1, v^1) \underset{(9)}{=} \varphi_2(N, v),$$
$$\varphi_{n+3}\left(N^{2^1}, v^{2^1}\right) + \varphi_{n+4}\left(N^{2^1}, v^{2^1}\right) = \varphi_1(N^2, v^2) \underset{(10)}{=} \varphi_1(N, v).$$

Hence, we have $\varphi_1(N, v) = \varphi_2(N, v)$ and **S** is shown. □

8.2.7. Proof of Lemma 6.6

Let $N \in \mathcal{N}$, $|N| \geq 2$, $v \in \mathcal{G}_{0_\mathbb{Q}}^N$ a TU-game, and, w.l.o.g., player $i, j \in N$ weakly dependent in v. Furthermore, let φ a TU-value which satisfies **E** and **PS** for all $v \in \mathcal{G}_{0_\mathbb{Q}}^N$ and therefore, by Lemma 6.5, also **S**. Due to $v(\{i\}), v(\{j\}) \in \mathbb{Q}\backslash\{0\}$, the worths of the singletons $v(\{k\})$, $k \in \{i, j\}$, can be written as a fraction. We distinguish two cases: (a) $v(\{k\}) > 0$ and (b) $v(\{k\}) < 0$.

(a) We have

$$v(\{k\}) = \frac{p_k}{q_k} \text{ with } p_k, q_k \in \mathbb{N}.$$

We choose a main denominator q of these two fractions by $q := q_i q_j$. With $z_i := p_i q_j$ and $z_j := p_j q_i$, we get

$$v(\{i\}) = \frac{z_i}{q} \text{ and } v(\{j\}) = \frac{z_j}{q}. \tag{11}$$

Now we define a player set N' and a coalition function v' by "splitting" each player $k \in \{i, j\}$ into z_k players k_1 to k_{z_k} such that we have $N' = (N\backslash\{i, j\}) \cup \{i_m : 1 \leq m \leq z_i\} \cup \{j_m : 1 \leq m \leq z_j\}$. Each player $k_m \in N'\backslash(N\backslash\{i, j\})$, $1 \leq m \leq z_k$, get a singleton worth $v'(\{k_m\}) = \frac{1}{q}$ for $k \in \{i, j\}$, synonymous with

$$v'(\{\ell\}) = \frac{1}{q} \text{ for all } \ell \in N'\backslash(N\backslash\{i, j\}),$$

where $|N'\backslash(N\backslash\{i,j\})| = z_i + z_j$ and $v(\{k\}) = \sum_{1\leq m\leq z_k} v'(\{k_m\})$, $k \in \{i,j\}$. We define $v'(R') := v(R)$ for all $R' = R\backslash\{i,j\}\cup N'\backslash(N\backslash\{i,j\})$, $R \subseteq N, \{i,j\} \subseteq R$, and $v'(S) := v(S)$ for all $S \subseteq N'$ with $S \subseteq N$. All other coalitions $T \subseteq N'$ are defined as not active in v'.

Applying **PS** (repeatedly) to v, φ and the two players $i,j \in N$ we can get the coalition function v' defined just before and, by remark 6.2, we have

$$\varphi_k(N,v) = \sum_{1\leq m\leq z_k} \varphi_{k_m}(N',v') \text{ for } k \in \{i,j\}. \tag{12}$$

All players $\ell \in N'\backslash(N\backslash\{i,j\})$ are symmetric in v'. Hence follows by **S**

$$\varphi_\ell(N',v') = \frac{\varphi_i(N,v) + \varphi_j(N,v)}{z_i + z_j} \text{ for all } \ell \in N'\backslash(N\backslash\{i,j\}).$$

We get

$$\varphi_k(N,v) \underset{(12)}{=} \sum_{1\leq m\leq z_k} \varphi_{k_m}(N',v') = \frac{z_k}{z_i + z_j}\left[\varphi_i(N,v) + \varphi_j(N,v)\right] \text{ for } k \in \{i,j\}.$$

It follows

$$\varphi_i(N,v) = \frac{z_i}{z_j}\varphi_j(N,v) \underset{(11)}{=} \frac{v(\{i\})}{v(\{j\})}\varphi_j(N,v)$$

and **P** is shown.

(b) We have

$$v(\{k\}) = \frac{p_k}{q_k} \quad \text{with} \quad (-p_k), q_k \in \mathbb{N}.$$

We choose a main denominator q of these two fractions by $q := -q_i q_j$. With $z_i := -p_i q_j$ and $z_j := -p_j q_i$, we get

$$v(\{i\}) = \frac{z_i}{q} \quad \text{and} \quad v(\{j\}) = \frac{z_j}{q}.$$

The remaining part of the proof equals the related part in case (a). □

8.3. Logical independence

Finally, we want to show the independence of the axioms used in the characterizations.

Remark 8.4. *Let $v \in \mathcal{G}_0^N$, $N \in \mathcal{N}$. The axioms in Theorem 3.3/Corollary 6.8 are logically independent:*

- ***E**: The TU-value φ defined by*

$$\varphi_i(v) = v(\{i\}) + 2 \cdot \sum_{\substack{S\subseteq N,\\ S\ni i, S\neq\{i\}}} \frac{v(\{i\})}{\sum_{j\in S} v(\{j\})}\Delta_v(S) \quad \text{for all } i \in N$$

*satisfies **D**, **P**/**PS**, and **WA** but not **E**.*

- ***D***: *The* ***proportional rule*** π *(Moriarity, 1975), given by*

$$\pi_i(v) = \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N) \ \ \text{for all } i \in N \tag{13}$$

satisfies ***E***, ***P/PS***, *and* ***WA*** *but not* ***D***.

- ***P/PS***: *Sh satisfies* ***E***, ***D***, *and* ***WA*** *but not* ***P/PS***.

- ***WA***: *The TU-value* φ *defined for all* $i \in N$ *by*

$$\varphi_i(v) = \begin{cases} v(\{i\}), & \text{if } i \text{ is a dummy player}, \\ \dfrac{v(\{i\})}{\sum_{\substack{j\in N, \\ j \text{ is no dummy}}} v(\{j\})} \Big[v(N) - \sum_{\substack{j\in N, \\ j \text{ is a dummy}}} v(\{j\}) \Big], & \text{otherwise}, \end{cases}$$

satisfies ***E***, ***D***, *and* ***P/PS*** *but not* ***WA***.

Remark 8.5. *Let* $v \in \mathcal{G}_0^N$, $N \in \mathcal{N}$. *The axioms in Theorem 4.2/Corollary 6.7 are logically independent:*

- ***E***: *The TU-value* φ *defined for all* $i \in N$ *by*

$$\varphi_i(v) = \begin{cases} 0, & \text{if } |N| = 1, \\ Sh_i^p(v), & \text{else}, \end{cases}$$

satisfies ***P/PS*** *and* ***WCSE*** *but not* ***E***.

- ***P/PS***: *Sh satisfies* ***E*** *and* ***WCSE*** *but not* ***P/PS***.

- ***WCSE***: *The proportional rule* π *(eq. (13)) satisfies* ***E*** *and* ***P/PS*** *but not* ***WCSE***.

References

Amer, R., Carreras, F., & Magaña, A. (2007). Two Main Methods for Utility Sharing in Joint Business: A Discussion. *Journal of Mathematics and Statistics 3*(1), 28–35.

Banker, R. D. (1981). Equity Considerations in Traditional Full Cost Allocation Practices: An Axiomatic Perspective. *Carnegie-Mellon University.*

Barton, T. L. (1988). Intuitive choice of cooperative sharing mechanisms for joint cost savings: Some empirical results. *Abacus, 24*(2), 162–169.

Béal, S., Ferrières, S., Rémila, E., & Solal, P. (2017). The proportional Shapley value and applications. Games and Economic Behavior..

Besner, M. (2016). Lösungskonzepte kooperativer Spiele mit Koalitionsstrukturen, Master's thesis at the Chair of Discrete Mathematics, FernUniversität in Hagen.

van den Brink, R., Levínský, R., & Zelený, M. (2015). On proper Shapley values for monotone TU-games. *International Journal of Game Theory, 44*(2), 449–471.

Casajus, A., & Huettner, F. (2008). Marginality is equivalent to coalitional strategic equivalence. Working paper.

Chun, Y. (1989). A new axiomatization of the Shapley value. *Games and Economic Behavior 1*(2), 119–130.

Feldman, B. (1999). The proportional value of a cooperative game. *Manuscript. Chicago: Scudder Kemper Investments.*

Gangolly, J. S. (1981). On joint cost allocation: Independent cost proportional scheme (ICPS) and its properties. *Journal of Accounting Research,* 299–312.

Harsanyi, J. C. (1959). A bargaining model for cooperative n-person games. In: *A. W. Tucker & R. D. Luce (Eds.), Contributions to the theory of games IV* (325–355). Princeton NJ: Princeton University Press.

Hart, S., & Mas-Colell, A. (1989). Potential, value, and consistency. *Econometrica, 57*(3) 589–614.

Kalai, E., & Samet, D. (1987). On weighted Shapley values. *International Journal of Game Theory 16*(3), 205–222.

Leng, M., & Parlar, M. (2009). Allocation of cost savings in a three-level supply chain with demand information sharing: A cooperative-game approach. *Operations Research, 57*(1), 200–213.

Moriarity, S. (1975). Another approach to allocating joint costs, *The Accounting Review, 50*(4), 791–795.

Myerson, R. B. (1980). Conference Structures and Fair Allocation Rules, *International Journal of Game Theory, Volume 9, Issue 3,* 169–182.

von Neumann, J., & Morgenstern, O. (1944). Theory of games and economic behavior. *Princeton, NJ: Princeton Univ. Press.*

Nowak, A. S., & Radzik, T. (1995). On axiomatizations of the weighted Shapley values. *Games and Economic Behavior, 8*(2), 389–405.

Ortmann, K. M. (2000). The proportional value for positive cooperative games. *Mathematical Methods of Operations Research, 51*(2), 235–248.

Radzik, T. (2012). A new look at the role of players' weights in the weighted Shapley value. *European Journal of Operational Research, 223*(2), 407–416.

Roth, A. E., & Verrecchia, R. E. (1979). The Shapley value as applied to cost allocation: a reinterpretation. *Journal of Accounting Research,* 295–303.

Shapley, L. S. (1953a). Additive and non-additive set functions. *Princeton University.*

Shapley, L. S. (1953b). A value for n-person games. H. W. Kuhn/A. W. Tucker (eds.), *Contributions to the Theory of Games,* Vol. 2, Princeton University Press, Princeton, pp. 307–317.

Shubik, M. (1962). Incentives, decentralized control, the assignment of joint costs and internal pricing. *Management science, 8*(3), 325–343.

Spinetto, R. D. (1975). Fairness in cost allocations and cooperative games. *Decision Sciences, 6*(3), 482–491.

Thomas, A. L. (1969). The Allocation Problem in Financial Accounting Theory-Studies in Accounting Research No. 3. *American Accounting Association, Evanston, Illinois.*

Thomas, A. L. (1974). The allocation problem in financial accounting theory (No. 9). *American Accounting Association.*

Young, H. P. (1985a). Monotonic solutions of Cooperative Games. *International Journal of Game Theory, 14*(2), 65–72.

Young, H. P. (1985b). Cost Allocation: Methods, Principles, Applications. *North Holland Publishing Co..*