A note on approximating bond returns allowing for both yield change and time passage

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A NOTE ON APPROXIMATING BOND RETURNS ALLOWING FOR BOTH YIELD CHANGE AND TIME PASSAGE

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A number of papers through the years have addressed the price-yield relationship, the approximation of bond returns and the associated components of price sensitivity. Typically, the research has been focused around the concept of duration and convexity to explain the price sensitivity of a bond to changes in its yield. Fixed income portfolio managers, however, are also interested in what happens to bond prices over a certain investment horizon, i.e. how time passage affect bond returns together with yield changes.

Chance and Jordan [1996] examines this in a very neat way by a second order Taylor series expansion around the current market yield y_0 and time to maturity t_{k0} of the price of a bond. In doing so, they are able to analyze the total return of the bond and attribute it to the five factors at hand plus an error term. The factors are the first order effects from duration and time passage, the second order effects from convexity and a squared time passage component and, finally, the cross component effect from the interaction between duration and time passage. Another advantage with this approach is that it is easy to generate returns for bonds along the yield curve for a certain investment horizon. This is very useful for creating scenarios for portfolio optimization purposes, what-if analysis or break-even calculations.

Due to the fact that the price-yield relationship is non-linear, the conventional approximation models give close results for yield changes of up to ± 100 basis points. Beyond that point, errors tend to increase significantly. Barber [1995] addresses this issue by expanding the natural logarithm of the price of a bond in a Taylor series around y_0. The relationship between the logarithm of the price of a bond and its yield is much less non-linear (or much more linear whichever way is preferred), which means that the Barber model is able to give close approximations of bond returns even when the yield range goes beyond ± 100 basis points. In fact, even as far as ± 300 basis points. Such a wide yield range would be very helpful when analyzing different yield curve scenarios, but the model does unfortunately not account for the passage of time.

In this paper, an extension to the Barber [1995] model is developed in order to draw from his close approximations over a relatively wide yield range, but also to include the time passage effect on bond returns, in the spirit of Chance and Jordan [1996], to make the model suitable for scenario analysis which allows for yield changes over a certain investment horizon.

The model

Let the price of a bond be given by

$$P = \sum_{k=1}^{K} \frac{CF_k}{(1+y)^{t_k}} \tag{1}$$

where CF_k is the k^{th} cash flow of the bond, k=1,2,...,K, y is the yield to maturity of the bond and t_k is the time to maturity of the k^{th} cash flow of the bond. The modified duration of the bond is given by

$$Mdur = -\frac{1}{P_0}\frac{\partial P}{\partial y} \tag{2}$$

and the convexity of the bond is given by

$$Cvex = \frac{1}{P_0}\frac{\partial^2 P}{\partial y^2} \tag{3}$$

Expanding the natural logarithm of the price of the bond in a Taylor series around y_0 and t_{k0} results in:

$$\begin{aligned} lnP \approx\ & lnP_0 + \frac{1}{P_0}\frac{\partial P}{\partial y}\Delta y + \frac{1}{P_0}\frac{\partial P}{\partial t_k}\Delta t_k \\ & + \frac{1}{2}\left[\frac{1}{P_0}\frac{\partial^2 P}{\partial y^2} - \frac{1}{P_0^2}\left(\frac{\partial P}{\partial y}\right)^2\right](\Delta y)^2 \\ & + \frac{1}{2}\left[\frac{1}{P_0}\frac{\partial^2 P}{\partial t_k^2} - \frac{1}{P_0^2}\left(\frac{\partial P}{\partial t_k}\right)^2\right](\Delta t_k)^2 \\ & + \left[\frac{1}{P_0}\frac{\partial^2 P}{\partial y \partial t_k} - \frac{1}{P_0^2}\frac{\partial P}{\partial y}\frac{\partial P}{\partial t_k}\right]\Delta y \Delta t_k \end{aligned} \tag{4}$$

Define bond theta as $\theta = \ln(1+y)$, which is the return increment resulting from pure time passage and is related to the first derivative of the bond price with respect to time to maturity t_k. According to (4), five partial derivatives are needed in the Taylor expansion. The five derivatives are the following:

$$i) \qquad \frac{\partial P}{\partial y} = -MdurP_0$$

$$ii) \qquad \frac{\partial^2 P}{\partial y^2} = Cvex\ P_0$$

$$iii) \qquad \frac{\partial P}{\partial t_k} = -\theta P_0$$

$$iv) \qquad \frac{\partial^2 P}{\partial t_k^2} = \theta^2 P_0$$

$$v) \qquad \frac{\partial^2 P}{\partial y \partial t_k} = P_0[Mdur\theta - (1+y)^{-1}]$$

Note that Δt_k in a strict mathematical sense must be ≤ 0, due to the fact that time passage only can decrease the time to maturity of the bond. For practical purposes, though, the signs of the derivatives iii) and v) will be changed[1] because it is more convenient to think of an investment horizon of, for example, three month as 0.25 years rather than -0.25 years. With that in mind, substituting the above derivatives into (4), changing the signs of iii) and v), and rearranging gives the following:

$$R_c = ln\left(\frac{P}{P_0}\right) \approx \theta\Delta t - Mdur\Delta y$$

$$+\frac{1}{2}[Cvex - Mdur^2](\Delta y)^2$$

$$+(1+y)^{-1}\Delta y\Delta t \qquad (5)$$

This is the continuously compounded return approximation, which can be annualized according to:

$$R_a = \frac{P}{P_0} - 1 \approx e^{R_c} - 1 \qquad (6)$$

Note that equation (6) also can be used to work out the approximate new price P at the horizon, given changes in y and t_k.

$$P \approx P_0\, e^{R_c} \qquad (7)$$

It is important to account for any interim coupon payments within the horizon to get a meaningful price. This kind of adjustment, however, is not needed when calculating the return approximations.

Comparison of true returns and approximate model returns

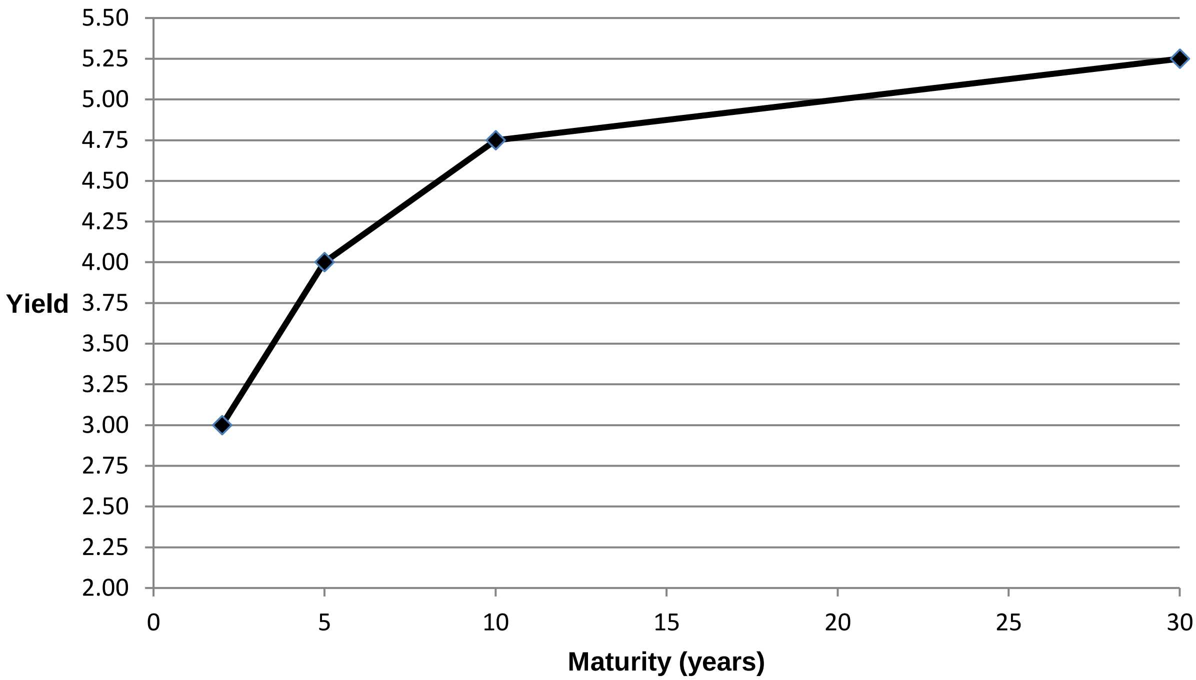
In this section some hypothetical bonds and their returns are examined over different time horizons and over a rather wide yield range in order to see how the logarithmic model (5) compares to the true returns of

[1] Derivative iv), i.e. the second order derivative of lnP with respect to t_k in the Taylor expansion cancels out and does not appear in the approximation formula (5).

these bonds. The conventional model of Chance and Jordan [1996] will also be included in this analysis to see if the logarithmic approach can improve upon the return approximations of their model as the yield range widens.

The hypothetical yield curve is shown in Exhibit 1 and consists of par bonds with maturities of 2, 5, 10 and 30 years, with coupons and yields of 3, 4, 4.75 and 5.25% respectively.

EXHIBIT 1 ■ Hypothetical Par Yield Curve



The yields are shocked by 100 and 300 basis points respectively, over 30 and 360 days, to see how close the model can get to the true returns. To simplify for the reader to replicate the results below, the scaling of the parameters that goes into the approximation formula (5) are shown here for the 3% 2 year par bond, assuming a 100 basis point yield rise over 30 days: Mdur=1.9135, Cvex=5.5458, Theta=0.0296, Yield=0.03, Δy=0.01 and Δt=0.0833. Exhibits 2 through 5 shows the results of the above scenarios. Clearly a 300 basis points yield shift in 30 days might be a bit over the top, but the main purpose is to test the accuracy of the model approximations. As can be seen in Exhibit 2, the logarithmic model gets pretty close to the true returns, with an error of 0.0073% for the 30 year bond. For the same bond the conventional model has an error of 0.1411%, which is several times larger, but still low for practical purposes. Exhibit 3 shows what happens when the horizon goes from 30 to 360 days. The errors increase for both models, but are still relatively low at 0.0121% and 0.1587% respectively. So lengthening the time horizon does not seem to feed the errors too badly.

EXHIBIT 2 ■ Return approximations for 100 bp rise in yield over 30 days

	2y	5y	10y	30y
Coupon	3.00%	4.00%	4.75%	5.25%
Yield	3.00%	4.00%	4.75%	5.25%
Price	100.00	100.00	100.00	100.00
Mdur	1.91	4.45	7.82	14.94
Cvex	5.55	25.01	76.39	335.35
Theta	2.96%	3.92%	4.64%	5.12%
New Yield	4.00%	5.00%	5.75%	6.25%
New Price	98.44	96.06	92.98	87.03
Approx Price	98.44	96.06	92.98	87.04
Price error	0.00	0.00	0.00	0.01
True Return	-1.5650%	-3.9397%	-7.0162%	-12.9655%
Logarithmic Model Return	-1.5645%	-3.9392%	-7.0154%	-12.9582%
Return error	0.0005%	0.0005%	0.0008%	0.0073%
Conventional Model Return	-1.5629%	-3.9338%	-6.9976%	-12.8244%
Return error	0.0021%	0.0059%	0.0186%	0.1411%

EXHIBIT 3 ■ Return approximations for 100 bp rise in yield over 360 days

	2y	5y	10y	30y
Coupon	3.00%	4.00%	4.75%	5.25%
Yield	3.00%	4.00%	4.75%	5.25%
Price	100.00	100.00	100.00	100.00
Mdur	1.91	4.45	7.82	14.94
Cvex	5.55	25.01	76.39	335.35
Theta	2.96%	3.92%	4.64%	5.12%
New Yield	4.00%	5.00%	5.75%	6.25%
New Price	101.98	100.39	97.80	91.93
Approx Price	101.99	100.39	97.80	91.94
Price error	0.00	0.00	0.00	0.01
True Return	1.9833%	0.3864%	-2.2020%	-8.0699%
Logarithmic Model Return	1.9877%	0.3907%	-2.1978%	-8.0579%
Return error	0.0043%	0.0042%	0.0042%	0.0121%
Conventional Model Return	1.9732%	0.3916%	-2.1698%	-7.9112%
Return error	-0.0102%	0.0052%	0.0322%	0.1587%

Exhibit 4 and 5 shows the result of a 300 basis point yield rise over 30 and 360 days respectively. As expected the errors are much larger now, but the logarithmic model still remains comfortably below one percent at 0.1677% for the 30 year bond, whereas the conventional model now has an error of 3.2802% for the same bond. Again, increasing the time horizon from 30 to 360 days only accounts for a smaller part of the error, whereas going from a yield shift of 100 to 300 basis points accounts for the larger part of the error increase. The errors for the 30 year bond in the 360 day scenario are 0.2069% and 3.5247%, respectively, for the logarithmic and conventional models.

EXHIBIT 4 ■ Return approximations for 300 bp rise in yield over 30 days

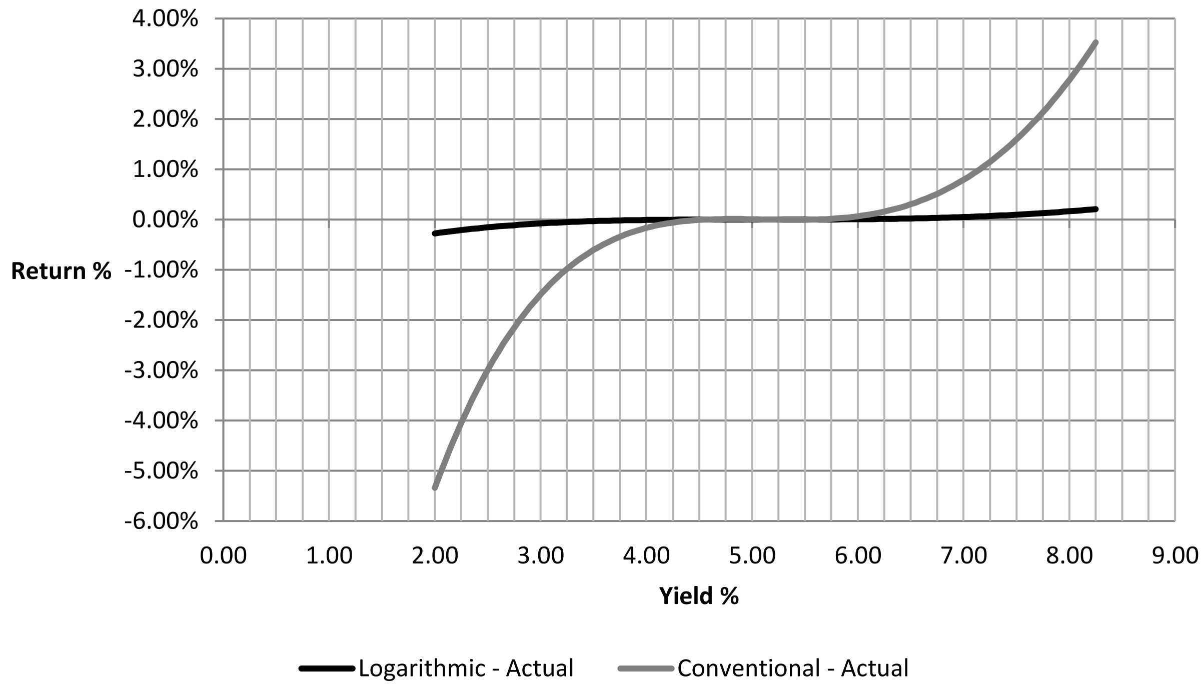
	2y	5y	10y	30y
Coupon	3.00%	4.00%	4.75%	5.25%
Yield	3.00%	4.00%	4.75%	5.25%
Price	100.00	100.00	100.00	100.00
Mdur	1.91	4.45	7.82	14.94
Cvex	5.55	25.01	76.39	335.35
Theta	2.96%	3.92%	4.64%	5.12%
New Yield	6.00%	7.00%	7.75%	8.25%
New Price	94.96	88.20	80.14	67.45
Approx Price	94.96	88.20	80.14	67.62
Price error	0.00	0.01	0.01	0.17
True Return	-5.0400%	-11.8047%	-19.8622%	-32.5475%
Logarithmic Model Return	-5.0354%	-11.7984%	-19.8568%	-32.3798%
Return error	0.0046%	0.0063%	0.0054%	0.1677%
Conventional Model Return	-5.0156%	-11.7058%	-19.4761%	-29.2673%
Return error	0.0244%	0.0989%	0.3860%	3.2802%

EXHIBIT 5 ■ Return approximations for 300 bp rise in yield over 360 days

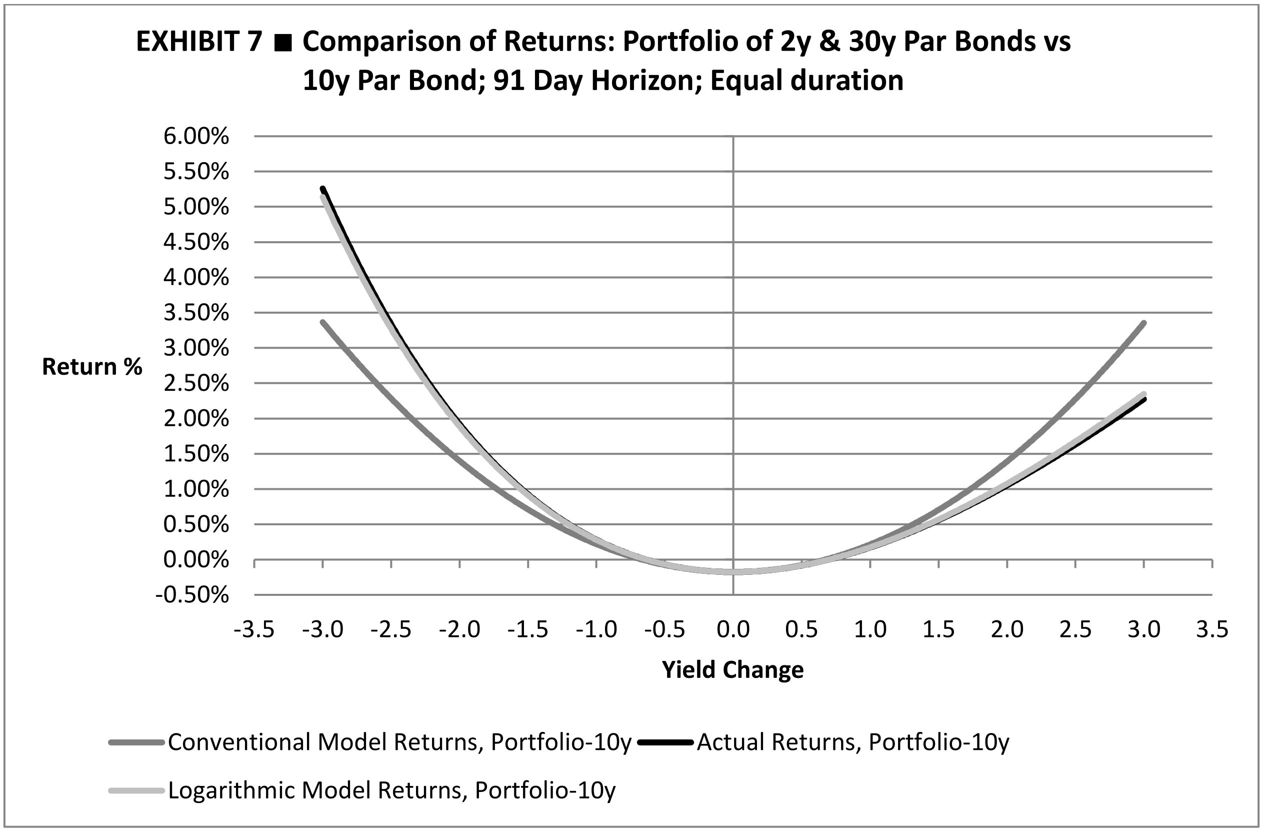
	2y	5y	10y	30y
Coupon	3.00%	4.00%	4.75%	5.25%
Yield	3.00%	4.00%	4.75%	5.25%
Price	100.00	100.00	100.00	100.00
Mdur	1.91	4.45	7.82	14.94
Cvex	5.55	25.01	76.39	335.35
Theta	2.96%	3.92%	4.64%	5.12%
New Yield	6.00%	7.00%	7.75%	8.25%
New Price	100.09	93.75	85.72	72.46
Approx Price	100.13	93.79	85.76	72.66
Price error	0.04	0.04	0.04	0.21
True Return	0.0883%	-6.2496%	-14.2760%	-27.5439%
Logarithmic Model Return	0.1315%	-6.2085%	-14.2389%	-27.3370%
Return error	0.0431%	0.0411%	0.0370%	0.2069%
Conventional Model Return	0.1713%	-5.9595%	-13.5796%	-24.0192%
Return error	0.0830%	0.2901%	0.6964%	3.5247%

Exhibit 6 shows the difference between model and actual returns for the logarithmic and conventional models over a yield range of 2.00% to 8.25% for a 5.25% 30 year par bond with a 360 day time horizon. As can be seen, the conventional model starts to diverge badly at about ± 100 basis points from the initial yield of 5.25%. The logarithmic model, on the other hand, behaves pretty well within the entire yield range with errors far below one percent.

EXHIBIT 6 ■ Difference between Model and Actual Returns for a 5.25% 30y Par Bond over 360 Days



Using the par bonds in the examples above, Exhibit 7 shows the return profile of being long an equal duration barbell of the 2 and 30 year par bonds versus being short a bullet in the 10 year par bond over a 91 day time horizon. The barbell position has a convexity of 154.95 and the 10 year has a convexity of 76.39. The duration is 7.82 for both positions. As shown, the logarithmic model approximation of returns is very close to the actual returns of this position, whereas the conventional model starts to diverge at about ±100 basis points.



It is also interesting to see how different coupon levels change the accuracy of the model. In Exhibit 8, the return errors are shown for a 30 year bond with a coupon of zero, 1pp below par, par and 1pp above par respectively for the logarithmic and conventional models, with a time horizon of 360 days and a yield rise of 300 basis points.

Exhibit 8 ■ Return Errors for 30 Year Bond with Different Coupons for a 300 bp Rise in Yield over 360 Days

Coupon	Logarithmic	Conventional
Par + 1 %-point	0.2192%	3.3144%
Par	0.2069%	3.5247%
Par - 1%-point	0.1842%	3.8078%
Zero	0.0283%	9.4024%

As can be seen, the accuracy of the logarithmic model is far better than that of the conventional model. The logarithmic model performs better the lower the coupon relative to yield and is extremely close in the zero coupon case. The conventional model, on the other hand, seem to perform better the higher the coupon and very poorly so in the zero coupon case.

Since the relationship between the logarithm of the price of a bond and its yield gets closer to linearity the lower the coupon, there is less curvature to account for in the Taylor approximation and therefore the second order yield term contributes relatively less. This is the reason why the accuracy is so high for zero coupon bonds in the logarithmic model, since there is literally no curvature to account for at all. In the case of continuous compounding, as opposed to discrete compounding which is used in this paper, the relationship is exactly linear.

Increasing the coupon will introduce more curvature and make the model relatively more dependent on the second order yield term, which in turn will increase the approximation error. But even as coupons get higher, the relationship between the logarithm of the price of a bond and its yield is still markedly more linear than in the case of the normal price-yield curve and this is why the logarithmic model is far more accurate in general.

A closer look at the approximation formula of the logarithmic model (5), shows that the second order term with regard to squared yield changes has convexity minus modified duration squared within the brackets. This is very similar to the definition of cash flow variance, which is the excess convexity of a coupon bond over an equal duration zero coupon bond, due to the fact that the former has dispersed cash flows and the latter does not.

The cash flow variance of a bond is defined as

$$CFV = \sum_{k=1}^{K} w_k (t_k - D)^2 = C - D^2 \qquad (8)$$

where w_k is the present value weight of the k^{th} cash flow of the bond using continuous compounding, k=1,2,...,K, t_k is the time to maturity of the k^{th} cash flow of the bond, D is the duration of the bond defined as $D = \sum w_k t_k$, and C is the convexity of the bond defined as $C = \sum w_k t_k^2$.

So, in the conventional Taylor expansion, the second order yield term is related to convexity, whereas in the logarithmic case it is related to cash flow variance because most of the price-yield curvature has been stripped out to a degree equal to the square of duration.

CONCLUSIONS

By extending the logarithmic Barber [1995] model to include the passage of time, in accordance with the ideas of Chance and Jordan [1996], very tight approximations can be achieved not only for a wide yield range, but also for time horizons as long as a year. These results lend themselves very neatly to creating scenarios and calculating bond returns for portfolio optimization purposes, what-if analysis and break-even calculations. The model has proved to be far more accurate than the conventional model in all kind of scenarios.

ENDNOTES

The author wants to thank Per Von Rosen and Per-Olov Karlsson for double checking the mathematics in this paper and for useful comments. The author is, of course, solely responsible for any remaining errors in this paper.

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