The Excess Method: A Multiwinner Approval Voting Procedure to Allocate Wasted Votes

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Abstract

In using approval voting to elect multiple winners to a committee or council, it is desirable that *excess votes*—approvals beyond those that a candidate needs to win a seat—not be wasted. The *excess method* does this by sequentially allocating excess votes to a voter's as-yet-unelected approved candidates, based on the Jefferson method of apportionment. It is monotonic—approving of a candidate never hurts and may help him or her get elected—computationally easy, and less manipulable than related methods. In parliamentary systems with party lists, the excess method is equivalent to the Jefferson method and thus ensures the approximate proportional representation of political parties. As a method for achieving proportional representation (PR) on a committee or council, we compare it to other PR methods proposed by Hare, Andrae, and Droop for preferential voting systems, and by Phragmén for approval voting. Because voters can vote for multiple candidates or parties, the excess method is likely to abet coalitions that cross ideological and party lines and to foster greater consensus in voting bodies.

1. Introduction

In electing a voting body, the *proportional representation* (PR) of different interests is usually an important goal: Preferences and opinions should be represented in the body roughly proportional to their presence in the electorate. But instead of restricting voters to voting for one candidate or party, we propose that they be given greater flexibility in expressing their preferences by being able to vote for multiple candidates or parties, using approval voting.[1]

We begin by focusing on the election of individual candidates to a committee or council, who need not be affiliated with a political party. Later we turn to the election of members of political parties in proportion to their numbers of supporters in the electorate.

Instead of simply summing the approval votes that each candidate receives in order to determine winners, we propose to modify a well-known apportionment method, proposed independently by Thomas Jefferson and Viktor d'Hondt, to depreciate the votes of voters who have succeeded in getting one or more of their approved candidates already elected (Brill et al., 2018; Brams et al., 2018).[2] This tends to ensure that voters with different backgrounds or interests receive a proportional share of seats on a committee or council.[3]

[1] Under approval voting, voters do not rank candidates or parties but simply express their approval for those they like or find acceptable; where voters draw the line between acceptable and nonacceptable we assume is a subjective judgment they make.

[2] On a committee or council, diversity may be more important than PR. As a case in point, one would not want all members of a university grievance committee to come from the same department or be of the same gender. Ratliff (2014) and Bredereck et al. (2018) analyze constraints of this kind.

[3] Apportionment methods can be used for other purposes. For example, in the United States, they are used to determine the number of seats in the House of Representatives to which each state, based on its population, is entitled.

The main modification of approval voting we analyze, which builds on the Jefferson method, seeks to prevent voters, insofar as possible, from wasting their votes. More specifically, the *excess method* largely eliminates the need for voters to anticipate the voting behavior of other voters and vote strategically (in particular, so as not to waste votes on shoo-ins, who would win without a voter's support). Instead, it takes the excess approvals that the most popular candidates receive, after they have been elected, and transfers them to a voter's other approved but less popular candidates, who in later rounds deserve, and may benefit from, receiving this additional support.

This is not to say that the excess method is invulnerable to manipulation through the misrepresentation of approvals (e.g., Lackner and Skowron, 2018). Sometimes voters can achieve a preferred outcome by withholding approvals from shoo-ins and concentrating their support on the remaining candidates. This form of strategic voting is called *subset manipulation*, because a voter approves of only a subset of his or her originally approved candidates on the manipulated ballot.

Recently, Peters (2018) proved an important impossibility result: If an approval-based multiwinner voting procedure satisfies a modest degree of PR, it is necessarily susceptible to subset manipulations, dashing hope of finding a rule that is not manipulable—that is, one that is strategyproof.[4]

Nevertheless, some rules are less manipulable than others, which is the key insight that motivates the excess method. The excess method not only provides voters with the flexibility of voting for multiple candidates, using an approval ballot, but it also

[4] For a broader analysis of the manipulability of voting systems, see Taylor (2005).

uses votes that would otherwise be wasted on candidates who do not need them to help voters' other approved candidates get elected.

The paper proceeds as follows. In section 2 we describe the Jefferson apportionment method—the most commonly used of the so-called divisor apportionment methods—which we modify to incorporate excess approvals that are not needed by a candidate to win (the same modification can be applied to other divisor methods).

In section 3, we illustrate the excess method with two simple examples and discuss different ways of allocating excess votes. We show that allocating excess votes to other approved but yet-to-be elected candidates respects voter sovereignty and voter equality, abetting the election of less popular but deserving candidates. We also illustrate its vulnerability to subset manipulation.

In section 4, we define more formally the excess method. We illustrate in detail its calculations with an example in which more than two candidates are elected.

In section 5, we compare the excess method with other multiwinner voting rules that are based on similar ideas—specifically, single transferable vote (STV) and a method developed by Lars Edvard Phragmén (Janson, 2016). We give reasons why we think the excess methods is preferable to other PR systems.

In section 6, we show how the excess method, when applied to the election of political parties, is equivalent to the Jefferson method, electing members of parties approximately proportional to the numbers of approvals, some depreciated, that the parties receive. We also illustrate how it can be manipulated through a form of subset manipulation. We argue, however, that there will not generally be sufficient information to make strategic calculations feasible.

In section 7, we conclude that the excess method, by passing on excess votes to as-yet-unelected approved candidates, helps to ensure that a voters' approvals are not wasted but, generally speaking, used effectively. Applied to political parties, there is no excess, because additional approvals are simply used to elect more party members. But allowing voters to vote for more than one party in a party-list system is likely to encourage coordination among political parties with overlapping interests, fostering consensus rather than partisan bickering.

2. The Jefferson Apportionment Method

The development and use of apportionment methods has a rich history. It is vividly recounted, mostly in the U.S. case, by Balinski and Young (1982/2001), wherein its best-known application has been to the apportionment of members of the House of Representatives according to the populations of the states. In the case of most parliamentary democracies in Europe and elsewhere, these methods have been applied to the apportionment of parliamentary seats to political parties according to the numbers of votes they receive in an election (Pukelsheim, 2014/2018).

Other jurisdictions in the United States[5] and parliaments around the world[6] use divisor methods of apportionment, of which there are exactly five that lead to *stable apportionments* (Balinski and Young, 1982/2001): No transfer of a seat from one state or

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[5] This includes ten states that use multimember districts, which vary in how winners are chosen in them (https://ballotpedia.org/State_legislative_chambers_that_use_multi-member_districts).

[6] The Webster method is used in four Scandinavian countries, whereas the Jefferson method is used in eight other countries. None of the other three divisor methods is currently used, except for the Hill method for the U.S. House of Representatives (see note 9). The nondivisor Hamilton method, also called "largest remainders" and not applicable to the sequential allocation of seats, is used in nine countries (Blais and Massicotte, 2002; Cox, 1997). For a review of apportionment methods, see Edelman (2006a), who proposed a nondivisor method that is mathematized in Edelman (2006b). In section 6, we will say more about why we favor the Jefferson method for allocating seats to political parties.

party to another can produce less disparity in apportionment, where "disparity" is measured in five different ways (there are other ways of measuring disparity, but they do not produce the stable apportionments of a divisor method). One of the five ways of defining disparity yields the Jefferson method, which we describe next.[7]

Although originally devised for allocating seats to states based on their populations, or to parties based on the votes they receive, apportionment methods can also be used to allocate seats to individual candidates based on approval voting. This was first pointed out by the Danish polymath Thorvald N. Thiele (1895); see the survey by Janson (2016).[8] As noted in section 1, voters approve a subset of candidates or parties (Brams and Fishburn, 1983/2007; Laslier and Sanver, 2010), obviating the need to rank them (e.g., as under single transferable vote, discussed in section 5).

We start with the election of individual candidates to a committee or council; later, in section 6, we describe the application of the Jefferson method to political parties. It works by iteratively depreciating the value of a voter's approvals as more and more of his or her approved candidates are elected. The *sequential versions* of these methods, which are the standard ones, give one seat on each round to the candidate, i, who maximizes a deservingness function, $d(i)$.

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[7] Balinski and Young (1982/2001) argue that the Jefferson method is preferable to the most prominent alternative divisor method, due to Daniel Webster and André Saint-Laguë, used to apportion seats to political parties, because it somewhat favors larger parties and thereby inhibits the fractionalization of party systems.

[8] What we call the Jefferson method is often referred to in the literature as sequential proportional approval voting (SPAV) or reweighted approval voting (RAV); a nonsequential version of proportional approval voting is referred to as PAV (Aziz et al., 2017). These authors show that PAV, but not SPAV, satisfies "justified representation" and "extended justified representation." For more on the history of apportionment, see Brill et al. (2018), who discuss, in particular, the contributions of Thiele and another Scandinavian mathematician, Lars Edvard Phragmén—some of whose work we turn to later—as does Janson (2016).

Let β denote the set of ballots, or multiset (ballots can appear multiple times in the set), and let $B(i) \subseteq \beta$ denote the set of ballots that include an approval vote for candidate i. On any round, let $r(b)$ denote the number of candidates on ballot b who are already elected. The *deservingness scores* of the sequential version of Jefferson (J) are defined as follows:[9]

$$d_J(i) = \sum_{b \in B(i)} \frac{1}{r(b)+1}.$$

On any round, each approval ballot supporting unelected candidate i is depreciated by an amount (in the denominator) that reflects the number of already elected candidates on the ballot. The greater the number of already elected candidates on a round, $r(b)$, the more the ballots that include candidate i are depreciated, rendering i less deserving of receiving the next seat to be filled.

On the 1st round, no candidate has yet received a seat, so $r(b) = 0$ and the Jefferson fraction for every ballot b is $1/(0+1) = 1$. Thus, the first candidate elected will be the candidate who obtains the most approvals, or the approval-vote winner. To illustrate the method beyond the 1st round, consider the following example, which is used in Brams et al. (2018) to show that the Jefferson method of apportionment may give a different outcome from the Webster method, its main competitor.

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[9] The deservingness functions of the four other divisor methods are defined similarly. For these other methods, the *denominators* on the right-hand side of $d_J(i)$ become $r(b)+1/2$ for Webster, $[r(b)(r(b)+1)]^{1/2}$ for Hill or "equal proportions," $2r(b)[r(b)+1]/[2r(b)+1]$ for Dean or "harmonic mean," and $r(b)$ for Adams or "smallest divisors" (Balinski and Young, 1982/2001; Pukelsheim, 2014/2018).

Example 1. 2 of 4 candidates $\{A, B, C, D\}$ are to be elected. The numbers of voters who approve of different subsets of candidates are

5: *AB* 2: *A* 3: *AC* 2: *BC* 4: *D*.

A, B, *C*, and *D* receive, respectively, a total of 10, 7, 5, and 4 approvals, so *A* is the candidate elected first.

On the 2nd round, *B*'s ballots (the 5 supporting *AB*, for which $r(b) = 1$, and the 2 supporting *BC*, for which $r(b) = 0$) are counted as follows: 1/2 for each of the 5 *AB* ballots (because *A* has already been elected), and 1 for each of the 2 *BC* ballots. Thus, on the 2nd round, *B*'s deservingness score is

$$B\text{: } (5)(1/2) + (2)(1) = 4\ 1/2.$$

Similarly, on the 2nd round the deservingness scores of *C* and *D* are

$$C\text{: } (3)(1/2) + (2)(1) = 3\ 1/2; \qquad D\text{: } (4)(1) = 4,$$

so the 2nd-round winner is *B*. Thus, Jefferson elects *AB*, whereas Webster would elect *AD*, for reasons we explain next.

The Jefferson fractions that determine deservingness scores decrease as the number of approved candidates already elected increases. From the formula for deservingness scores $d_j(i)$, these fractions, $1/[r(b) + 1]$, decrease according to the sequence

$$1, 1/2, 1/3, 1/4, 1/5, \ldots.$$ [10]

Example 1 illustrates that after one of a voter's approved candidates is elected, the voter's approvals for a second candidate are cut by a factor of 2, and then more and more as additional approved candidates are elected. The same sequence can be used as the basis for a nonsequential method of committee election (Thiele, 1895; Janson, 2016), but it is not applicable to the excess method, because the excess is determined round-by-round.

3. The Excess Method Applied to Jefferson

We next illustrate the application of the excess method to apportionments given by the Jefferson method. We focus on two simple examples—to convey the intuition behind the excess method—and later, in section 4, we provide a formal definition of the method and illustrate it with an example that requires more than two rounds to determine the winners.

As in Example 1, we begin with a voter profile, which specifies the number of voters who approve of different subsets of candidates. From this profile, we calculate the approval scores of all the candidates, just as in the 1st round of the Jefferson method. The winner of the 1st round is the candidate with the highest approval score.

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[10] The comparable sequence of fractions for Webster is

$$1, 1/3, 1/5, 1/7, 1/9, \ldots,$$

which more depreciate the deservingness contributions of voters who have had approved candidates already elected than does Jefferson. Thus, Webster more helps candidates like D, whose supporters approve of no other candidates, whereas Jefferson tends to favor candidates (e.g., B) whose voters have already had an approved candidate elected (i.e., A).

Define the *excess* of the winner as the number of votes, beyond a tie, by which he or she wins. That is, the excess is the difference in approval scores between the candidate with the most approvals and the candidate with the second-most approvals. (If two or more candidates tie for the maximal approval score, the excess is 0.) For example, if candidate *A* gets 10 approvals, candidate *B* gets 7 approvals, and all other candidates receive less than 7 approvals, then *A* would win with an excess of 10 – 7 = 3 votes.[11]

The excess thus equals the number of approvals that could be taken away from the winner without changing the winning candidate (up to a tie). If the excess is 3, as in our example, this means that up to 3 voters approving of *A* could safely change their ballots by omitting *A*; this would not hurt *A*, who would still be elected first, but it could potentially help other candidates, approved by *A* voters, get elected later. Put differently, the excess equals the number of voters who can safely engage in subset manipulation—by transferring their votes to other approved candidates—without changing the current winner.

The idea underlying the excess method is that the method itself carries out a form of subset manipulation—on behalf of the voters who approve of multiple candidates—without requiring those voters to make strategic choices. When there is a positive excess $e > 0$, then the excess method chooses e of the voters who approve of the winner and treats them, in future rounds, as if they did not approve of the winner.

[11] We could define *A*'s excess to be 10 – (7+1) = 2—giving *A* a win with 8 votes rather than a tie with 7 votes against *B* (with 7 votes)—but instead we make 7 the baseline for determining the excess. The difference between tying and winning will be inconsequential in most applications of the excess method, whose primary purpose is to take into account the magnitude of a candidate's win—beyond tying in our definition—on each round that a new candidate is elected. This is analogous to a Vickrey, or 2nd price, auction, in which the highest bidder wins and pays exactly (not more than) the 2nd-highest bid.

We assume the excess is distributed to *all* candidates, other than *A*, whom *A* voters approve of; this distribution is proportional to the sizes of the voter subsets approving of *A*. We illustrate this calculation in the next example.

Example 2. 2 of 3 candidates {*A*, *B*, *C*} are to be elected, whose voter profile is

20: *AB* 5: *AC* 8: *C*.

The approval scores of *A*, *B*, and *C*, respectively, are 25, 20, and 13 votes. Therefore, *A* is chosen on the 1st round with an excess of 25 – 20 = 5 votes, which go to the *AB* and *AC* voters in the ratio of 20:5 = 4:1.

This means that 4/5 of the excess is allocated to the *AB* voters and 1/5 to the *AC* voters. Hence, we treat (4/5)(5) = 4 *AB* voters as if they had voted only for *B*, and (1/5)(5) = 1 *AC* voter as if he or she had voted only for *C*. Thereby the *AB* and *AC* voters, in proportion to their numbers, pass on their excess support for *A* to their other approved candidates.

Allocating the excess in this way—after subtracting 4 excess votes from the 20 *AB* voters and 1 excess vote from the 5 *AC* voters, who provided the excess—we have the following updated voter profile:

16: *AB* 4: *AC* 4: *B* 9: *C*.

Observe that the 5 excess *A* voters have become 4 new *B* voters and 1 new *C* voter; the latter voter, combined with the original 8 *C* voters, gives a total of 9 *C* voters.

We next calculate 2nd-round deservingness scores exactly as under the Jefferson method, but now with respect to the updated profile. This yields the following deservingness scores for *B* and *C*:

B: (16)(1/2) + (4)(1) = 12; C: (4)(1/2) + (9)(1) = 11.

Hence, B is chosen on the 2nd round, yielding *AB* as the outcome under *excess Jefferson*.[12]

Under *standard Jefferson* (i.e., without allocation of the excess), it is easy to show that the outcome would be *AC*. In effect, because the excess method gives a big boost to *B* (from the 20 *AB* voters) but only a small boost to *C* (from the 5 *AC* voters) on the 2nd round, *B* beats *C*, even though *C* has the undiluted support of 8 *C* voters on this round.

Can excess Jefferson be manipulated by a subset of voters to elect *AC* rather than *AB* (obviously, the 5 *AC* voters in Example 2 would prefer this outcome). Assume 4 of these 5 *AC* voters change their approval strategy to vote only for *C*, giving this revised voter profile:

20: *AB* 1: *AC* 12: *C*.

The vote totals of *A*, *B*, and *C* are now 21, 20, and 13 votes, respectively, so A wins the 1st seat, and there is an excess of 1 vote.

Because this excess is small, we will ignore it for now and then assess whether it would have made a difference in the outcome. Applying standard Jefferson, the deservingness scores for *B* and *C* are as follows:

B: (20)(1/2) = 10; C: (1)(1/2) + (12)(1) = 12 1/2.

Distributing the excess of 1 vote in the ratio of 20:1 to *AB* and *AC* voters would have benefited *B* by less than 1 vote and *C* by much less.

[12] Because the excess method can be applied to other divisor methods of apportionment, such as that of Webster, we refer to it here as "excess Jefferson" to distinguish it from "standard Jefferson."

Clearly, this would not have been sufficient to enable *B* to win the 2nd seat, as occurred with excess Jefferson before manipulation (*B* won by a margin of 13 – 11 = 2 votes). Consequently, subset manipulation is successful in electing *C* on the 2nd round not only for standard Jefferson but also for excess Jefferson, demonstrating that these apportionment methods are not strategyproof.

A special case arises when some voters, who approve of the 1st-round winner, approve of no other candidates. In this situation, the transfer of excess votes to these voters is fictitious, as our next example illustrates.

Example 3. 2 of 3 candidates {*A*, *B*, *C*} are to be elected, whose voter profile is

5: *A* 15: *AB* 5: *AC* 6: *C*.

The approval scores of *A*, *B*, and *C*, respectively, are 25, 15, and 11. Therefore, *A* is chosen on the 1st round with an excess of 25 – 15 = 10 votes, which go to the *A*, *AB* and *AC* voters in the proportion 5:15:5 = 1:3:1. This translates to

- (1/5)(10) = 2 excess votes to the *A* voters;
- (3/5)(10) = 6 excess votes to the *AB* voters;
- (1/5)(10) = 2 excess votes to the *AC* voters.

Because there is no other approved candidate to whom the 5 *A* voters can pass on their excess, however, the allocation of 2 votes to *A* is fictitious. Allocating the remaining excess of 6 votes of the 15 *AB* voters and 2 votes of the 5 *AC* voters in the ratio of 3:1 to *B* and *C*—after subtracting the 6 excess votes from the 15 *AB* voters and the 2 excess votes from the 5 *AC* voters, who provide the excess—we have the following updated voter profile:

9: *AB* 3: *AC* 6: *B* 8: *C*

On the 2nd round, the deservingness scores of *B* and *C* are as follows:

B: (9)(1/2) + (6)(1) = 10 1/2; *C*: (3)(1/2) + (8)(1) = 9 1/2,

Hence, *B* is the 2nd candidate elected, yielding *AB* as the outcome.

Compare this outcome with that for standard Jefferson, in which *AC* would have won because of the 6 *C* voters in the original profile, who more than counterbalance the more numerous 15 *AB*—compared with the 5 *AC*—voters. As in Example 2, the excess method gives a sufficiently big boost to *B* on the 2nd round so that he or she beats *C* for the 2nd seat.

We assumed in Example 3 that none of the excess from the 5 *A* voters goes to *B* or *C*, because the *A* voters did not indicate any other candidate of whom they approved. As an alternative assumption, we could have assumed that the *entire* excess of 10 votes (instead of the 8 votes that come from *AB* and *AC* voters) goes to *B* and *C* in the ratio 15:5 = 3:1. This would result in allocating (3/4)(10) = 7 1/2 excess votes to *B* and (1/4)(10) = 2 1/2 excess votes to *C*. One advantage of this approach is that none of the excess is wasted, including that which comes from the *A* voters.

However, there are two serious problems with this approach. First, it violates *voter sovereignty*—the 5 *A* voters offered no indication of a preference between *B* and *C*, much less that they favored *B* over *C* by a factor of 3.[13] We think voter sovereignty

[13] We could make this "support transfer" explicit by informing voters that if there are candidates whom they have *not* approved of after all their approved candidate(s) have been elected—as occurred with the *A* voters in Example 3—the excess method will attribute approval in the transfer process proportional to the approval of *A* voters who approved of *other* candidates. But simply informing voters that their unexpressed

should be respected: Voters who approved of no other candidates should not be presumed to have a preference for the transfer of their portion of the excess.

The second problem with transferring the entire excess is illustrated by assuming that the 5 *A* voters in Example 3 are instead 25 *A* voters. This modification of Example 3 increases the excess from 10 to 30 votes in the 2nd round. But now transferring 30 votes to a combined total of 20 *AB* and *AC* voters means that the excess is greater than the number of voters to which it is allocated, giving each voter an average of 30/20 = 1 1/2 votes after the transfer.

If there were another subset of, say, *D* voters who approved of no other candidates and hence did not receive any transfer of the excess, this would give the *B* and *C* voters a greater say on the 2nd-round choice than each of the *D* voters. In our opinion, this would be an excessive transfer by the excess method (pun intended), violating *voter equality*—that the excess cannot exceed the number of voters to whom it is transferred, giving its recipients more than a single vote.[14]

Because both voter sovereignty and voter equality may be violated when the entire excess is transferred, we henceforth assume that transfers do not include any excess from those voters who have had all their approved candidates elected (only *A* voters in Example 3). In section 4, we offer a more formal description of the excess method, first stating its rules and then illustrating it with an example in which more than two candidates are elected.

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approval will be delegated in this way still violates their sovereignty. It's as if they were told: "Even if you don't have an opinion, trust other *A* voters who do."

[14] Voter equality cannot be violated if there are no fictitious transfers, because then the excess allocated to a voter subset can never exceed the number of voters in this subset. (In Example 3, as modified with 25 *A* voters, 25/35 = 5/7 of the 30 excess votes, or 21 3/7 votes, would be fictitiously transferred to the 25 *A* voters.)

4. Rules of the Excess Method

Although the calculations of the excess method, as illustrated by Examples 2 and 3, are more complicated than those of standard Jefferson, the application of the method is in fact quite simple, based on the following five rules:

1. Voters submit their approvals for subsets of candidates. If k candidates are to be elected, there are k rounds of voting and up to k–1 transfers after the 1st round.

2. The candidate elected on the 1st round, A, is the approval-vote winner (if there is a tie, it is broken randomly). The difference between A's approval and B's, the candidate with the second-most approvals, is the excess (which is zero if there is a tie).

3. On the 2nd round, the excess is distributed to subsets of voters who approved of A in proportion to their contributions to A's deservingness score. The excess part of the subset comprises all approved candidates other than A; the remaining part includes A.

4. This dichotomization of A-subsets, one containing and the other not containing A—plus the subsets of voters voting for other candidates—provides an updated voter profile. Applying the Jefferson method to this updated profile yields new deservingness scores for the remaining not-yet-elected candidates, which determines a winner on the 2nd round.

5. Steps 2 – 4 are repeated for the 3rd round, with the winner from the 2nd round replacing A in the determination of the deservingness scores of the 3rd-round winner. This iterative process is repeated until k candidates are elected.

To summarize the excess method, after A is elected, the excess that A did not need to tie the 2nd-place candidate goes to subsets of candidates approved of by A voters. On each subsequent round, *A* is replaced by a new winner, whose excess is passed on to the not-yet-elected candidates who approved of that winner.

In Examples 2 and 3, after the election of *A* on the 1st round, the excess on the 2nd round was distributed to other candidates supported by *A* voters (*B* and *C* in the case of *AB* and *AC* voters) in proportion to the *sizes* of voter subsets that supported *A*. These proportions determined the *contributions* of these subsets to *A*'s support.

But on later rounds, after the excesses on earlier rounds are transferred, the size and composition of voter subsets change. These are reflected in updated voter profiles, which determine a winner on each round, his or her excess, and transfers to subsets who approved of the winner to not-yet-elected candidates.

These transfers depend not only on the size of the subsets of candidates to which they are transferred but also on which members of the subset have already been elected. To illustrate, assume there is a subset of 6 *ABC* voters. On the 1st round, if *A* is elected, the contribution of *ABC* to *A*'s victory will be (6)(1) = 6, because none of the *ABC* candidates had been previously elected, and similarly for the contributions of other subsets of voters who approved of *A*.

On the 2nd round, assume that *B* wins the next seat, and the updated voter profile now includes 3 *ABC* voters. Then the contribution of these voters to *B*'s victory will be depreciated to (3)(1/2) = 1 1/2, because one of the *ABC* candidates (i.e., *A*) had been previously elected. This will be the contribution of the *ABC* voters to the excess of the 2nd winner, *B*, which will be distributed to subsets of voters who approved of *B*.

When the excess method is used to elect candidates after the 2nd round, as specified by rule 5, the excess will similarly be distributed to all subsets of voters that include the candidate just elected. Whereas updated voter profiles specify the sizes of the current voter subsets, the already elected candidates determine how much each subset is depreciated and, therefore, contributes to the new winner's victory. Both factors determine the deservingness scores of the unelected candidates and, therefore, who wins on each round.

In our next example, we illustrate all the rules of the excess method, which requires three rounds to elect three candidates. As in Examples 2 and 3, the method gives a different set of winners from those of standard Jefferson.

Example 4. 3 of 4 candidates $\{A, B, C, D\}$ are to be elected, whose voter profile is

5: *A* 20: *AB* 10: *AC* 5: *ABC* 6: *C* 12: *D*.

The approval scores of *A*, *B*, *C*, and *D* are 40, 25, 21, and 12 votes, respectively. Therefore, *A* is elected on the 1st round with an excess of 40 – 25 = 15 votes, which go to the *A*, *AB*, *AC*, and *ABC* voters in the proportion 5:20:10:5 = 1:4:2:1. This yields

- (1/8)(15) = 1 7/8 excess votes to *A* voters, which go to no other candidate and so are fictitious;
- (1/2)(15) = 7 1/2 excess votes to *AB* voters, which go to *B*;
- (1/4)(15) = 3 3/4 excess votes to *AC*, which go to *C*;
- (1/8)(15) = 1 7/8 excess votes to *ABC*, which go to *BC*.

As in Example 3, we assume that the excess votes to *A* are discarded since the *A* voters approve of nobody else. Allocating the remaining excess votes of *AB*, *AC*, and *ABC* voters to subsets B, C, and BC—after subtracting these excess votes from the 20 AB, 10 AC, and 5 ABC voters—we have the following updated voter profile (note that C's total is the excess of 3 3/4 votes plus 6 votes from the original profile):

12 1/2: AB 6 1/4: AC 3 1/8: ABC 7 1/2: B 1 7/8: BC 9 3/4: C 12: D.

On the 2nd round, the deservingness scores are

- B: (25/2)(1/2) + (25/8)(1/2) + (15/2)(1) + (15/8)(1) = 17 3/16;
- C: (25/4)(1/2) + (25/8)(1/2) + (15/8)(1) + (39/4)(1) = 16 5/16;
- D: (12)(1) = 12.

Thus, B is elected on the 2nd round with a small excess of 17 3/16 – 16 5/16 = 14/16 = 7/8. Ignoring this excess for now, we update the forgoing voter profile by omitting the AB and B voters, whose approved candidates have already been elected:

6 1/4: AC 3 1/8: ABC 1 7/8: BC 9 3/4: C 12: D.

If we had not ignored the excess, it would have been distributed entirely to the four subsets of voters that include B (AB, ABC, B, BC), who was just elected, in the proportion given in the first bulleted deservingness score above.[15] This will slightly help C, compared with D. Even without this help, however, C's deservingness score is greater than D's by 14 1/3 – 12 = 2 1/3:

[15] More precisely, the 7/8 excess would have been distributed to the four subsets that include B in the proportion (25/2)(1/2) : (25/8)(1/2) : (15/1)(1) : (15/8)(1) = 100:25:120:130, giving C a bit higher deservingness score—and victory margin over D—than we show in the text next. Note that the allocation of excess votes to the AB and B voters is fictitious, because these two subsets each approved only of candidates who have already been elected.

- C: (25/4)(1/2) + (25/8)(1/3) + (15/8)(1/2) + (39/4)(1) = 14 1/3;
- D: (12)(1) = 12.

Hence, *C* wins the 3rd seat, making the outcome *ABC*.

Under standard Jefferson, the outcome would have been *ACD* rather than *ABC*, with *C* rather than *B* winning on the 2nd round. More significantly, *D* would have won on the 3rd round, rendering the set of winners—not just the order in which they are elected—different.

In effect, the excess methods boosts *B* on the 2nd round and enables *C* to win on the 3rd round, each of whom benefits from *A*'s excess on the 1st round. *D* receives no such benefit, because his 12 supporters (20% of the total) are all bullet voters.

We hesitate to say that the excess method "improves" on standard Jefferson, because an argument can be made that the 20% whom only *D* represents deserve to have some representation in a 3-member voting body. Instead, we emphasize that the excess method takes into account that *A* is a shoo-in, obtaining approval from 40 voters (2/3 of the 60 voters).

Because *A* voters can rest assured that *A*'s excess will not be wasted but will be put to good use to help elect their other approved candidates, they have little incentive to strategize.[16] To be sure, the excess method is not strategyproof, as we illustrated in Example 2, but it certainly attenuates the need to vote strategically.

[16] If *A* did not have a big excess (say, only 1 more vote than *B*), then *B* and *C* would not be significantly helped, enabling *D* to win a seat in a modified Example 4.

5. Comparison with Other Methods

It is instructive to compare standard and excess Jefferson with other methods that have been proposed to elect multiple winners to a committee or a council. The most well-known method is the Hare system of single transferable vote (STV). STV was developed independently in the 1850s by two prominent figures—Thomas Hare of England and Carl Christoffer Georg Andrae of Denmark—and it was praised by such distinguished political theorists as John Stuart Mill (1861/1991), who placed it "among the greatest improvements yet made in the theory and practice of government."

STV has been widely used in public and private elections to elect (i) single winners to offices, such as a president or a prime minister, and (ii) multiple winners to voting bodies. While there are several variations of STV, all assume that voters rank candidates; if more than one candidate is to be elected, the candidates who reach a "quota"—the most common being the Droop quota—are chosen.[17]

More specifically, STV progressively eliminates candidates who receive the fewest first-choice votes and transfers their votes to voters' next-lower-ranked candidates, with these eliminations and transfers culminating when some candidate achieves a quota. When more than one candidate is to be elected, STV on each round selects the candidate with the most votes and transfers so-called surplus votes to the next-lower-ranked candidates. *Surplus votes* are similar in spirit to excess votes, except that they are defined with respect to a fixed quota instead of the difference in votes of the two biggest vote-getters—which changes from round to round—under the excess method.

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[17] If there are v voters and s seats to be filled, the Droop quota is $v/(s+1)$, rounded down to the nearest integer. It is the smallest number that guarantees that no more than s candidates can achieve the quota.

Compared with the excess method, in which voters can approve of a subset of candidates but do not rank them, STV has several practical and theoretical drawbacks:

1. Voters often have difficulty ranking more than a few candidates, whereas expressing approval for multiple candidates is relatively easy.

2. STV is *nonmonotonic*—a voter may hurt a candidate by ranking him or her higher—just the opposite of what one would expect. But expressing approval for a candidate never hurts and may help him or her get elected.[18]

3. STV is vulnerable to the *no-show paradox*—voters may benefit from not voting—whereas not voting can never benefit voters in an approval-voting election.

4. STV may eliminate candidates who otherwise could win if they had remained in the race, whereas the excess method, which never eliminates unelected candidates, ensures that all candidates remain viable.

Interestingly, the quota-based surplus approach motivating STV can also be applied to approval ballots, using a method proposed by Swedish mathematician Lars Edvard Phragmén (Janson, 2016, section 18.5).[19] It employs the Hare quota, which is similar to the Droop quota.[20]

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[18] This is true for individual candidates under the excess method: A voter can never hurt a candidate by approving of him or her. As we showed in section 3, however, approving of a subset of multiple candidates may lead to fewer of them being elected than approving of a smaller subset, because of subset manipulation (Peters, 2018). This strategy, however, is risky in the absence of complete information about the voting behavior of other voters that would enable one to determine an optimal manipulative strategy—not to mention taking into account possible best responses of other voters and whether they lead to an equilibrium outcome.

[19] This method, called "Phragmén's first method" by Janson (2016), is different from Phragmén's sequential method that is analyzed by Brill et al. (2017).

[20] If there are v voters and s seats to be filled, the Hare quota is v/s.

Phragmén's method successively elects candidates with the greatest voting weights, updating these weights after each round. Initially, every voter has a voting weight of 1. The voting weight of a candidate is the sum of the voting weights of all voters approving of this candidate.

Let A be the candidate with the greatest voting weight; on the 1st round, this candidate is the approval-vote winner. Consequently, A is elected on the 1st round, and the voting weights are updated as follows: If A's vote total v_A exceeds the quota $q = v/s$ (i.e., if $v_A > q$), then the voting weight of each voter approving of A is multiplied by the factor $f_A = (v_A - q)/v_A$.[21] On the other hand, if $v_A \leq q$, then the voting weight of each voter approving of A is set equal to 0.

To illustrate Phragmén's method, consider Example 4 (see section 4), wherein 3 of the 4 candidates are to be elected, and q=58/3 = 19 1/3:

5: *A* 20: *AB* 10: *AC* 5: *ABC* 6: *C* 12: *D*.

The approval-vote winner, A, has a vote total of 40 votes, which exceeds q. Therefore, A is elected on the 1st round, and the voting weights of all voters approving of A are multiplied by f_A = (40 – 58/3)/40 = 31/60. This gives updated voting weights of (5)(31/60) = 31/12 for the A voters, (20)(31/60) = 31/3 for the AB voters, (10)(31/60) = 31/6 for the AC voters, and (5)(31/60) = 31/12 for the ABC voters. The voting weights on the 2nd round for candidates other than A sum the votes of their bullet voters and the forgoing weights that include these candidates:

- B: 31/3 + 31/12 = 12 11/12;

[21] This factor is chosen such that the total voting weight goes down by q. Proportionally reducing voting weights in this way is known as the *weighted inclusive Gregory method* (Janson, 2016).

- C: 6 + 31/6 + 31/12 = 13 3/4;
- D: 12.

Thus, C is elected on the 2nd round. Since the voting weight of C does not exceed the quota, the voting weights of all voters approving of C are updated to 0. On the 3rd round, the vote total of B is 31/3 = 10 1/3, and the vote total of D is 12, making D the 3rd candidate elected. Summarizing, Phragmén's method elects *ACD*.[22]

Phragmén's method depends heavily on the quota. By contrast, standard and excess Jefferson do not depreciate votes according to an exogenously defined quota; the "quota," as it were, is endogenous—defined immediately after one candidate obtains more votes than his or her competitors. In addition, Phragmén's method, unlike excess Jefferson, does not take into account the margin by which a candidate wins and incorporate this information into transfers to unelected candidates, whose election may—and we believe should—be influenced by these margins.

There is another crucial difference between Phragmén's method and standard and excess Jefferson: Phragmén's method requires that one know the number of candidates to be elected, because q depends on s, the number of seats to be filled. This can lead to anomalous situations wherein candidates who are elected for a given number of seats, s, are no longer elected when s is increased (leaving everything else unchanged).[23] This

[22] This is different from the outcome under the excess method, which we showed in section 4 is *ABC*. While Phragmén's method gives the same outcome as standard Jefferson in this example, voter profiles for which this is not the case can easily be constructed.

[23] In technical terms, quota-based methods like STV or Phragmén's are not *house monotonic*: Increasing the house size (for a fixed voter profile) does not guarantee that candidates elected before the increase remain elected afterwards.

cannot happen under standard and excess Jefferson, which do not depend on s and therefore q.

In section 6, we apply the excess method to the election of political parties, each of which may win multiple seats. We also illustrate how it can be manipulated, though doing so is probably well-nigh impossible for reasons that we will discuss.

6. Application of the Excess Method to Parties

When applying standard Jefferson to the apportionment of seats to political parties in a legislature, according to the numbers of votes they receive, voters are assumed to vote for only one party. But if they can approve of as many parties as they like, how would the apportionment of seats to parties work?

To calculate the numbers of seats that parties receive, we assume that each party nominates as many candidates, s, as will be elected to the legislature and does so in a particular order. Thus, party I nominates candidates $i_1, i_2, \ldots, i_s$; if an apportionment method allocates $k \leq s$ seats to I, they go to its top k candidates $i_1, i_2, \ldots, i_k$. We assume that a voter who votes for a party approves of all its candidates.

The following example, which appears in Brams et al. (2018) but without all the calculations made explicit, illustrates how standard Jefferson would allocate seats to parties when voters are not restricted to voting for one party but can vote for more than one:

Example 5. 2 of 6 candidates, $\{a_1, a_2, b_1, b_2, c_1, c_2\}$ from parties $\{A, B, C\}$ are to be elected. The numbers of voters who approve of different subsets parties are

7: AB 5: AC 2: B 3: C,

which translates into votes for the following sets of candidates:

$$7: a_1a_2b_1b_2 \qquad 5: a_1a_2c_1c_2 \qquad 2: b_1b_2 \qquad 3: c_1c_2.$$

Each of the three candidates of parties A, B, and C receives, respectively, 12, 9, and 8 approvals. Thus, candidate a_1 is the first candidate elected under standard Jefferson. On the 2nd round, we compare the deservingness scores for a_2 (since a_1 has already been elected from party A), b_1 (from party B), and c_1 (from party C). These scores are

$$a_2: 7(1/2) + 5(1/2) = 6; \quad b_1: 7(1/2) + 2(1) = 5\ 1/2; \quad c_1: 5(1/2) + 3(1) = 5\ 1/2,$$

so a_2, who receives the greatest score, is the 2nd candidate elected, making the winning pair a_1a_2.

In fact, excess Jefferson does not change these results. Why? Because at the conclusion of a round, another member of the party that just won a seat receives exactly the same number of votes, so there is no excess. In particular, this is true for party A at the end of the 1st round (a_2 receives the same number of votes as the winner, a_1), and this equality would hold true if there were additional candidates to be elected. In determining winners with party voting, therefore, excess Jefferson and standard Jefferson are equivalent.

When voting is for individual candidates, as we showed for Example 2, voting is not strategyproof. Is this also true when voting is for parties? We show that it may be, though in a special sense.

Consider the outcome, a_1a_2 in Example 5. Assume that polls just before the election show that party A is a shoo-in to win one seat (a_1) and possibly two (a_1a_2). If you are one of the 5 AC voters and would prefer a more diverse committee comprising a_1c_1

instead of a_1a_2, you might well consider voting for just C to boost the chances of c_1 winning the 2nd seat.[24] In that case, the votes for the following subsets of candidates would be

$$7: a_1a_2b_1b_2 \qquad 4: a_1a_2c_1c_2 \qquad 2: b_1b_2 \qquad 4: c_1c_2.$$

Each of the two candidates of parties A, B, and C would receive, respectively, 11, 9, and 7 approvals. As before, candidate a_1 would be the first candidate elected. On the 2nd round, the deservingness scores for a_2, b_1, and c_1 are

$$a_2\text{: } (7)(1/2) + (4)(1/2) = 5\ 1/2; \quad b_1\text{: } (7)(1/2) + (2)(1) = 5\ 1/2; \quad c_1\text{: } (4)(1/2) + (4)(1) = 6,$$

so c_1 is the 2nd candidate elected, making the winning pair a_1c_1 and demonstrating that sincerity is not a Nash equilibrium in Example 5.

That strategic voting may be optimal is, of course, not surprising, because virtually all voting systems are vulnerable to manipulation. What complicates matters in the case of the apportionment methods is that the determination of winners, and therefore optimal strategies to produce a preferred outcome, is anything but straightforward. This makes it difficult to use information from polls or other sources to determine optimal strategic choices (Lackner and Skowron, 2018, offer experimental evidence on manipulability).

[24] This might be considered a second-order preference, expressed as $a_1c_1 > a_1a_2$; it is not reflected in the approval of AC by 5 voters, but we postulate it to show why a voter who approves of AC might choose to vote for just C. We do not have evidence that voters in a party-list system with approval voting would deliberately make such a choice, but there is evidence that there is a spoiler effect in non-approval party-list systems. For example, Kaminski (2018) shows that some voters could have done better in certain Polish parliamentary elections by voting for a different party (Kaminski, 2018). However, it is not clear that voters could have anticipated the consequences of making a different choice and thereby would have been motivated to do so.

7. Conclusions

If the depreciation calculations of standard Jefferson are simpler than excess Jefferson, why do we think the excess version is superior? In electing individual candidates to committees and councils, the excess method eliminates wasted votes, especially of shoo-ins, that may lead to different winners.

It, therefore, largely relieves voters of the need to make difficult strategic choices. More specifically, it provides them with some assurance that their excess votes will be used to help their other approved candidates, which may well encourage them to approve of additional candidates. In turn, this may encourage the formation of coalitions that cross ideological and party lines, reducing cleavages and promoting consensus in voting bodies.

The excess method does not relieve voters of the need to think carefully about whom they consider acceptable and wish to approve. This task is usually less arduous when the contestants are political parties rather than individuals, because parties are more likely to formulate platforms and promulgate their positions.

Because parties win seats in proportion to their votes and so do not accumulate excess votes in parliamentary elections, the excess method is equivalent to the Jefferson method. In applying the Jefferson method to approval votes, parties will have an incentive to coordinate their policies with ideologically similar parties to accommodate voters who like more than one (perhaps for different reasons). Promoting their common interests to voters in campaigns will facilitate the formation of both coherent and more stable governing coalitions in parliaments.

In theory, the excess method is vulnerable to subset manipulation. But in practice this does not seem to be a serious problem, because acquiring the information and making the calculations required to successfully manipulate an election is exceedingly difficult, especially when other candidates or parties may be making similar calculations and not act sincerely.

Approval voting is much less demanding of voters than having to rank candidates or political parties, as required by voting systems like STV that, in addition, are susceptible to some well-known voting paradoxes that standard and excess approval-voting methods are not. While Phragmén's method uses approval votes, like STV it postulates a fixed quota, whereas it seems more natural to make the election threshold variable—a function of the number of votes that a candidate needs to win a seat on each round.

Recall that the Jefferson method makes the criterion for getting elected on each round the deservingness scores of candidates. The excess method carries Jefferson one step further by incorporating into these scores the votes that previously elected candidates did not need in order to be elected. We think this is preferable to using an exogenous threshold, such as the Droop or Hare quota, which ignores the wasted votes of candidates.

In this paper, we analyzed one particular way of calculating and allocating excess votes. Other definitions are conceivable (and defensible); this situation is similar to that of different versions of STV, for which Tideman (1995, p. 37) noted a tradeoff between sophistication and comprehensibility.

We defined the excess of the winner on a given round as the number of votes a candidate could afford to lose and still win on that round. If the candidate were to lose more votes, he or she would not win on that round but might still be elected on a later round.

An alternative concept of excess could be based on the number of votes an elected candidate can afford to lose and still be elected on *some* round. While appealing in theory, such a definition would be much harder to operationalize. Because the excess method, as proposed in this paper, defines excess with reference to the current round, it is well-suited to scenarios in which the order in which candidates are elected matters—in particular, when being elected earlier is preferable to being elected later (Skowron et al., 2017).

The idea behind the excess method is readily applicable to other sequential multiwinner procedures that use approval ballots, including, but not limited to, other divisor apportionment methods. Indeed, for a wide range of multiwinner voting procedures, incorporating the excess into voting outcomes alleviates both the problem of wasted votes and the possibility of subset manipulation.

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