A Path Integral Approach to Business Cycle Models with Large Number of Agents

Pierre Gosselin and Aïleen Lotz and Marc Wambst

Université Grenoble Alpes, Institut Fourier, UMR 5582 CNRS-UJF, BP 74, 38402 Saint Martin d'Hères, France., Cerca Trova, BP 114, 38001 Grenoble Cedex 1, France., IRMA, UMR 7501 CNRS, Université de Strasbourg, France.

October 2018

A Path Integral Approach to Business Cycle Models with Large Number of Agents

Pierre Gosselin* Aïleen Lotz† Marc Wambst‡

October 2018

Abstract

This paper presents an analytical treatment of economic systems with an arbitrary number of agents that keeps track of the systems' interactions and agents' complexity. This formalism does not seek to aggregate agents. It rather replaces the standard optimization approach by a probabilistic description of both the entire system and agents' behaviors. This is done in two distinct steps.

A first step considers an interacting system involving an arbitrary number of agents, where each agent's utility function is subject to unpredictable shocks. In such a setting, individual optimization problems need not be resolved. Each agent is described by a time-dependent probability distribution centered around his utility optimum. The entire system of agents is thus defined by a composite probability depending on time, agents' interactions and forward-looking behaviors. This dynamic system is described by a path integral formalism in an abstract space – the space of the agents' actions – and is very similar to a statistical physics or quantum mechanics system. We show that this description, applied to the space of agents' actions, reduces to the usual optimization results in simple cases.

Compared to a standard optimization, such a description markedly eases the treatment of systems with small number of agents. It becomes however useless for a large number of agents. In a second step therefore, we show that for a large number of agents, the previous description is equivalent to a more compact description in terms of field theory. This yields an analytical though approximate treatment of the system. This field theory does not model the aggregation of a microeconomic system in the usual sense. It rather describes an environment of a large number of interacting agents. From this description, various phases or equilibria may be retrieved, along with individual agents' behaviors and their interactions with the environment.

For illustrative purposes, this paper studies a Business Cycle model with a large number of agents.

Key words: path integrals, statistical field theory, business cycle, budget constraint, multi-agent model, interacting agents.

JEL Classification: C02,C60, E00, E1.

*Pierre Gosselin : Université Grenoble Alpes, Institut Fourier, UMR 5582 CNRS-UJF, BP 74, 38402 Saint Martin d'Hères, France. E-Mail: pierre.gosselin@univ-grenoble-alpes.fr

†Aïleen Lotz: Cerca Trova, BP 114, 38001 Grenoble Cedex 1, France. E-mail: a.lotz@erc-cercatrova.eu

‡Marc Wambst : IRMA, UMR 7501 CNRS, Université de Strasbourg, France. E-Mail: wambst@math.unistra.fr

Introduction

In many instances, representative agent models have proven unrealistic, lacking both the collective and emerging effects stemming from agents' interactions. Complex systems, Networks, Agent Based Systems or Econophysics are among the various paths that have been explored to remedy these pitfalls. However Agent Based and Networks Models rely on numerical simulations and may lack microeconomic foundations. Econophysics builds on statistical facts and empirical aggregate rules to derive macroeconomic laws. These laws are prone, like ad-hoc macroeconomics, to the Lucas critique (see Lucas 1976). The gap between microeconomic foundations and multi-agent systems remains.

The present paper attempts to fill this gap by adapting statistical physics methods to describe multiple interacting agents. It is an introduction to the method developped in (Gosselin, Lotz, Wambst 2017), illustrated by a basic economic application to a Business Cycle model. Our setup models individual, i.e. micro interactions in a context of statistical uncertainty, to recover a global, macroeconomic description of the system. This approach allows an analytical treatment of a broad class of economic models with an arbitrary number of agents, while keeping track of the system's interactions and complexity at the individual level. It is at the crossroads of statistical physics and economics: it preserves the microeconomic concepts of standard economic models to describe fully or partly rational agents, while enabling the study of the transition from individual to collective scale given by statistical physics.

Are microeconomic concepts still relevant at the statistical system - macro - level? Some microscopic features are known to fade away at large scales, whereas others become predominant. The relevance - or irrelevance in the physical sense - of some micro interactions when moving to a macro scale could indirectly shed some light on the change of scale in economics. Our work is a first attempt to address these questions.

Translating standard economic models into statistical ones requires a statistical field formalism. Such a formalism partly differs from those used for physical systems. The field formalism presented in this paper keeps track of the individual behaviors shaping the field theoretic description, as well as the results at the macro scale. The field description, in turn, describes the impact of the macroeconomic environment on individual behaviors.

The statistical approach of economic systems presented here is a two-step process. In a first step, the usual model of optimizing agents is replaced by a probabilistic description of the system. In an interacting system involving an arbitrary number of agents, each agent is described by an intertemporal utility function depending on an arbitrary number of variables. However each agent's utility function is subject to unpredictable shocks. In such a setting, individual optimization problems are discarded. Each agent is described by a time-dependent probability distribution centered around this agent's utility optimum. Unpredictable shocks deviate each agent from his optimal action, depending on each individual shock variance. When these variances are null, standard optimization results are recovered. This so to speak blurred behavior can be justified by the inherent complexity of agents: each period, their goals and behavior can be modified by some internal, unobservable and individual shocks.

This setup is a path integral formalism in the abstract space of agents' actions. It is actually very similar to the statistical physics or quantum mechanics systems. This description is a good approximation of standard descriptions and allows to solve otherwise intractable problems. Compared to standard optimization techniques, such a description markedly eases the treatment of systems with a small number of agents. Working with a probability distribution is often easier than solving optimization equations. This approach is thus consistent and useful in itself. It provides an alternative to the standard modeling in the case of a small number of interacting agents. The average dynamics recovered is close and at times identical to the standard approach. It also allows to study the set of agents' dynamics and its fluctuations under some external shocks.

This formalism, useful for small sets, becomes intractable for a large number of agents. It is nonetheless conveniently and classicaly modified using methods of statistical field theory (Kleinert 1989), into another and more efficient description directly grounded on our initial path integral formalism. In a second step, therefore, the individual agents' description is replaced by a model of field theory that replicates the properties of the system when N, the number of agents, is large. This modeling, although approximate, is compact enough to allow an analytical treatment of the system. A double transformation is thus performed with respect to the usual optimization models. The optimization problem is first replaced by a statistical system of N agents, that is then itself replaced by a specific field theory with a large number of degrees of freedom.

This field theory does not represent an aggregation of microeconomic systems in the usual sense. It rather describes an environment of an infinite number of interacting agents, from which various phases or equilibria may be retrieved, as well as the behaviors of the agents, and the way they are influenced by, or interact, with their environment. This is the so-called "phase transition" of field theory: the configuration of the ground state represents an equilibrium for the whole set of agents, and shapes interactions and individual dynamics. Depending on the parameters of the system, the form of the ground state may change drastically the description at the individual level. It is thus possible to compare the particular features of the macro state of a system and those of the individual level. As such, it may confirm or invalidate some aspects of the representative agent models.

By several aspects, our work is related to the Multi-Agents System economic literature, notably Agent Based Models (see Gaffard Napoletano 2012) and Economic Networks (Jackson 2010). Both rely on numerical simulation of Multi-Agents System, but are often concerned with different types of model. Agent Based Models deal with general macroeconomics models, whereas Network Models rather deal with lower scale models, such as Contract Theory, Behavior Diffusion, Information Sharing or Learning. In both type of settings, agents are typically defined by, and follow, various set of rules. These rules allow for equilibria and dynamics that would otherwise remain inaccessible to the representative agent setup.

The Agent-Based approach is similar to ours in that it does not seek to aggregate all agents, but considers the interacting system in itself. It is however highly numerical, model-dependent, and relies on microeconomic relations, such as ad-hoc reaction functions, that may be too simplistic. On the contrary, Statistical Field Theory accounts for the transition between scales. Macroeconomic patterns do not emerge from the sole dynamics of a large set of agents, but are grounded on particular behaviors and interactions structures. Describing these structures in terms of Field Theory allows the study the emergence of a particular phase at the macro scale, and in turn its impact at the individual level.

Econophysics is closer to our approach (for a review, see Chakraborti, Muni Toke, Patriarca and Abergel (2011a) and (2011b) and references therein). It often considers the set of agents as a statistical system. Moreover, Kleinert (2009) has already used path integrals to model the stock prices' dynamics. However, Econophysics does not fully apply the potentiality of Field Theory to economic systems and rather focuses on empirical laws. But the absence of microfoundations casts some doubts on the robustness of these observed empirical laws. Our approach, in contrast, keeps track of usual microeconomics concepts such as utility functions, expectations, forward looking behaviors. It includes these behaviors in the analytical treatment of Multi-Agents Systems by translating the main characteristics of a system of optimizing agents in terms of a statistical system.

To sum up, the advantages of statistical field theories are threefold. They allow, at least approximatively, to deal analytically with systems with large degrees of freedom, without reducing them to mere aggregates. They reveal features otherwise hidden in an aggregate context. Actually, they allow switching from micro to macro description, and vice-versa, and to interpret one scale in the light of the other. Moreover, and relevantly for economic systems, these model may exhibit phase transition. Depending on the parameters of the model, the system may experience structural changes in behaviors, at the individual and collective scale. In that, they allow to consider the question of multiple equilibria in economics.

The first section presents a probabilistic formalism for a system with N identical economic agents, interacting through mutual constraints. Section two introduces and discusses the associated field formalism for a large number of agents. In section three, we present an application of this formalism to a business cycle model. Section four concludes.

1 A probabilistic description of economic agents in interaction

This section presents a probabilistic formalism for a system with N identical economic agents in interaction. Agents are described by intertemporal utility functions, but do not optimize these utilities. Instead, each agent chooses for his action a path randomly distributed around the optimal path. The agent's behavior can be described as a weight that is an exponential of the intertemporal utility, that concentrates the probability around the optimal path. This feature models some internal uncertainty as well as non-measurable shocks. Gathering all agents, it yields a probabilistic description of the system in terms of a probabilistic weight. This weight includes utility functions and internalizes forward-looking behaviors, such as intertemporal budget

constraint and interactions among agents. These interactions may for instance arise through constraints, since revenue flows depend on other agents demand. The probabilist description then allows to compute the transition functions of the system, and in turn compute the probability for a system to evolve from an initial state to a final state within a given time span. They have the form of Euclidean path integrals.

For the sake of clarity, the description in terms of probabilistic representation is first explained discarding constraints. This modelization, however suitable for simple models, would be a limitation for most economic models in which constraints are relevant. So that these constraints will be considered in a second step. In a third step, we will show that the interactions between agents are best described in terms of mutual constraints. We end the section by discussing the transition functions associated with a system for a large number of economic agents, and the transition to the field formalism.

1.1 Principles

To keep track of the agents' main microeconomic features, several conditions must be satisfied. First, optimization equations should, at least in some basic cases and in average, be recovered. Second, this probabilistic description should account for the agents' individual characteristics, such as constraints, interactions and forward-looking behavior.

The probabilistic description presented here involves a probability density for the state of the system at each period t. In a system composed of N agents, each defined by a vector of action $X_i(t)$.

Notation 1 *We denote $X(t)$ the concatenation $(X_1(t), ..., X_N(t))$ of the action vectors.*

Notation 2 *We denote $X = (X(0), ..., X(T))$ the concatenation of these vectors over the entire timespan, where T is the time horizon.*

We define a probability density $P(X(t))$ for the set of actions $X(t)$ that describes the state of the system at time t. Consider first the intertemporal utility of an agent i:

$$U_t^{(i)} = \sum_{n \geqslant 0} \beta^n u_{t+n}^{(i)} \left(X_i(t+n), X(t+n-1) \right)$$

where $u_{t+n}^{(i)}$ is the instantaneous utility at time $t+n$. In the optimization setup, $X_i(t+n)$ is the agent i control variable, the variables $(X_j(t+n-1))_{j \neq i}$ are the actions of the other agents, and $(X_j(t+n-1))$ are the actions of the set of all agents.

The above utility can encompass any quantity optimized, such as the production or utility functions of consumer/producer models. It may also describe the interaction of several substructures within an individual agent (see Gosselin Lotz Wambst 2018, and previous formulations in 2013, 2015), or the motion mechanisms (decision and control) in the neurosciences literature.

We assume that agent i has no information about others (see Gosselin Lotz Wambst 2018). Their actions are perceived as random shocks by agent i. Rather than optimizing $U_t^{(i)}$ on $X_i(t)$, we postulate that agent i will choose an action $X_i(t)$ and a plan, updated every period, $X_i(t+n)$, $n > 0$, for its future actions. This plan follows a conditional probabilistic law proportional to:

$$\exp\left(U_t^{(i)}\right) = \exp\left(\sum_{n \geqslant 0} \frac{\beta^n}{\sigma_i^2} u_{t+n}^{(i)} \left(X_i(t+n), X(t+n-1) \right) \right) \tag{1}$$

This is a probabilistic law for $X_i(t)$ and the plan $X_i(t+n)$ when $n > 0$. It is conditional on the action variables $X(t+n-1)$, perceived as exogenous by agent i. The uncertainty about agent i behavior, or the variability of agents actions, is denoted σ_i^2.

Remark that, for a usual convex utility with a maximum, the closest the choices of the $X_i(t+n)$ to $U_t^{(i)}$ optimum, the higher the probability associated to $X_i(t+n)$. When $\sigma_i^2 \to 0$, the agent's action is optimal. Our choice of utility is therefore coherent with a probability peaked around the optimization optimum. It is thus different from the usual description in terms of optimal path of actions, but encompasses this approach in average.

This probabilistic description is simplified for non-strategic agents with no information about others. The agent considers the variables $X_j(t+n)$ as random noises and integrate them out. The probability for $X_i(t)$ and $X_i(t+n)$, $n>0$ will then be:

$$\int \exp\left(U_t^{(i)}\right) \exp\left(-\frac{X_j^2(s)}{\sigma_j^2}\right) \prod_{j\neq i} \prod_{s\geqslant t} dX_j(s)$$

Here, $\exp\left(-\frac{X_j^2(s)}{\sigma_j^2}\right)$ is the subjective weight attributed by i to the $X_j(s)$. In general if no information is available to agent i, we can assume that $\sigma_j^2 \to \infty$ and $\exp\left(-\frac{X_j^2(s)}{\sigma_j^2}\right) \to \delta(X_j(s))$, where $\delta(X_j(s))$ is the Dirac delta function, i.e. a function that is peaked on 0, and null everywhere else. As a consequence, as long as no further information is available, other agents may be considered as random perturbations : agent i set their future actions to 0, discarding them from his planning.

When there are no constraint and no inertia in $u_t^{(i)}$, i.e. when $u_t^{(i)}$ solely depends on $X_i(t)$ and other agents' previous actions $(X_j(t-1))_{j\neq i}$, periods are independent. Actually, action $X_i(s)$ at times $s>t$ are independent of $X_i(t)$. Consequently, $\exp\left(U_t^{(i)}\right)$ is a product of independent terms of the kind $\exp\left(\beta^n u_{t+n}^{(i)}(X_i(t+n), X(t+n-1))\right)$. The $X_i(s)$ can thus be integrated out, and the probability associated to the action $X_i(t)$ is then:

$$\int\left(\int \exp\left(U_t^{(i)}\right) \exp\left(-\frac{X_j^2(s)}{\sigma_j^2}\right) \prod_{j\neq i} \prod_{s\geqslant t} dX_j(s)\right) \prod_{s>t} dX_i(s) \propto \exp\left(\frac{u_t^{(i)}(X_i(t), X(t-1))}{\sigma_i^2}\right)$$

or in term of conditional probabilities:

$$P(X_i(t) \mid (X(t-1))) = \frac{\exp\left(\frac{u_t^{(i)}(X_i(t), X(t-1))}{\sigma_i^2}\right)}{\mathcal{N}_i} \tag{2}$$

where the normalization factor is defined by:

$$\mathcal{N}_i = \int \exp\left(u_t^{(i)}(X_i(t), X(t-1))\right) dX_i(t) \tag{3}$$

From now on, the normalization factor will be skipped, and reintroduced if needed. Formula (2) shows that each agent is described by his instantaneous utility. The lack of information induces a short sighted behavior: in absence of any period overlap, i.e. without any constraint, the behavior of agent i is described by a random distribution peaked around the optimum of $u_t^{(i)}(X_i(t), X(t-1))$ which models exactly the optimal behavior of an agent influenced by individual random shocks.

As a consequence, gathering all N agents, the full system $X(t)$, is described by a probability weight at each time t:

$$\prod_i \exp\left(\frac{u_t^{(i)}(X_i(t), X(t-1))}{\sigma_i^2}\right)$$

for any $(X_k(t-1))$. Assuming that $\sigma_i^2 = \sigma^2$ for each agent, a particular path for the whole system X, is defined by the probabililty, up to the normalization factors:

$$P(X) = \prod_t \prod_i \exp\left(\frac{u_t^{(i)}(X_i(t), X(t-1))}{\sigma^2}\right) \tag{4}$$

1.2 Introducing constraints

Let us now consider the introduction of constraints, in an exact way for simple cases, or as first approximation in the general case. To do so, we have two distinguish two types of constraints.

1.2.1 Instantaneous constraints

We define an instantaneous constraint as a dynamic identity between the control variables of the system. A standard example is the dynamic for capital accumulation of a single producer/consumer:

$$K_i(t+1) - (1-\delta) K_i(t) = Y_i(t) - C_i(t) + \epsilon_i(t) \tag{5}$$

where C_t is the consumption, δ the depreciation rate, $\epsilon_i(t)$ a gaussian random shock centered around 0 and of variance η^2, and $Y_t = F(K_t)$ the revenue. The function F may depend on other variables, such as technology, that themselves depend on the environment provided by other agents. We can generalize equation (5) for an arbitrary action variable vector $X_i(t)$:

$$X_i(t+1) - X_i(t) - H(X_i(t), X(t-1)) = \epsilon_i(t)$$

for some function H. The inclusion of this constraint in our probabilistic description is straightforward. If we assume that $\epsilon_i(t)$ is independent from any of the variables, the density of probability for the system (2) is modified by the adjunction of a gaussian term:

$$P(X_i(t) \mid (X(t-1))) = \exp\left(\frac{u_t^{(i)}(X_i(t), X(t-1))}{\sigma^2}\right) \exp\left(\frac{-(X_i(t+1) - X_i(t) - H(X_i(t), X(t-1)))^2}{\eta^2}\right) \tag{6}$$

Summing over agents and periods yields a statistical weight for a path X of the system:

$$P(X) = \exp\left(\sum_{i,t} \frac{u_t^{(i)}(X_i(t), X(t-1))}{\sigma^2}\right) \exp\left(-\sum_{i,t} \frac{(X_i(t+1) - X_i(t) - H(X_i(t), X(t-1)))^2}{\eta^2}\right) \tag{7}$$

In continuous time, an integral replaces the sum over t and formula (15) becomes:

$$P(X) = \exp\left(\sum_i \int \frac{u_t^{(i)}(X_i(t), X(t-1))}{\sigma^2} dt\right) \exp\left(-\frac{1}{\eta^2} \sum_i \int_0^T \left(\frac{d}{dt} X_i(t) - H(X_i(t))\right)^2 dt\right) \tag{8}$$

1.2.2 Intertemporal constraints

We will first consider an economic agent optimizing a quadratic utility under some budget constraint. We will then extend the result to N agents with quadratic utilities under linear arbitrary constraints. Finally, we will consider the general case of arbitrary utility.

Consider the quadratic utility of an agent whose action vector $X_i(t)$ is his sole consumption. His utility reduces to:

$$u_t^{(i)}(X_i(t), X(t-1)) = u(C_i(t)) \tag{9}$$

His current account intertemporal constraint is of the form:

$$C_i(t) = B_i(t) + Y_i(t) - B_i(t+1) \tag{10}$$

where $Y_i(t)$ is first considered as an exogenous random variable, such as revenue in standard optimization models. The state variable $B_i(t)$ represents the usual Treasury Bond. Both the interest rate r and the discount factor β are discarded here for the sake of simplicity. They can be reintroduced when needed (see Gosselin Lotz Wambst 2017). We do this explicitly for the interest rate in our example in section 3.

If we were to keep the state variable $B_i(t)$ in our description, we could consider (10) as an instantaneous constraint similar to (5). However, we will rather replace the state variable $B_i(t)$ and describe the system in terms of the usual control variable $C_i(t)$. This is in line with the usual models with intertemporal constraint such as:

$$\sum_{t \geqslant 0} Y_i(t) - \sum_{t \geqslant 0} C_i(t) = 0$$

The usual consumption smoothing is imposed through the Euler equation. However, in our formalism, both these elements will appear in a probabilistic form.

Successive periods are interconnected through the constraint. When $C_i(t)$ is replaced by the state variable $B_i(t)$, the intertemporal probability weight (1) becomes:

$$\begin{aligned} & \exp\left(u(C_i(t)) + \sum_{n>0} u(C_i(t+n))\right) \qquad (11) \\ = \quad & \exp\left(u(B_i(t) + Y_i(t) - B_i(t+1)) + \sum_{n>0} u(B_i(t+n) + Y_i(t+n) - B_i(t+n+1))\right) \end{aligned}$$

This measures the probability for a choice $C_i(t)$ and $C_i(t+n)$, $n = 1...T$. Alternately it is the probability for the state variable B_i to follow a path $B_i(t+n)$, $n \geqslant 0$ starting from $B(t)_i$. The time horizon T represents the expected remaining duration at time t of the interaction process'. It should depend decreasingly on t, but, for the sake of simplicity, it is assumed to follow a random poisson process. As a consequence, the mean expected duration will be a constant written T, irrespective of t. Integrating over the $B_i(t+n)$ with $n \geqslant 2$, yields a transition probability between $B_i(t)$ and $B_i(t+1)$ written $P(B_i(t), B_i(t+1))$, the probability to reach $B_i(t+1)$ given $B_i(t)$. It is equal to:

$$P(B_i(t), B_i(t+1)) = \int \prod_{i=2}^{T} dB_{t+i} \exp\left(u(B_i(t) + Y_i(t) - C_i(t+1)) + \sum_{n>0} u(B_i(t+n) + Y_i(t+n) - C_i(t+n+1))\right)$$

Computing $P(B_i(t), B_i(t+1))$ rather than the transition function for $C_i(t)$ does not change our approach. It merely requires that it be applied to the state variable $B_i(t)$ rather than to the control variable $C_i(t)$. In this case, due to the overlapping nature of state variables, the probability transition $P(B_i(t), B_i(t+1))$ now measures a probability involving two successive periods, so that the probability for the path $C_i(t+n)$, $n \geqslant 0$ has to be rebuilt from the data $P(B_i(t), B_i(t+1))$.

Consider a quadratic utility function of the form $u(C_i(t)) = -\alpha\left(C_i(t) - \bar{C}\right)^2$ with objective $\bar{C}$ or, should it be non quadratic, its second order approximation. Rescale it for the sake of simplicity as $-\alpha\left(C_i(t) - \bar{C}\right)^2 \to -C_i^2(t)$. The constant $\bar{C}$ can be reintroduced at the end of the computation. We assume for now that $\sigma^2 = 1$. The transition probability between two consecutive state variables thus becomes:

$$\begin{aligned} P(B_i(t), B_i(t+1)) \quad &= \quad \int \prod_{i=2}^{T} dB_i(t+n) \exp\left(u(C_i(t)) + \sum_{n>0} u(C_i(t+n))\right) \\ &= \quad \int \prod_{i=2}^{T} dB_i(t+n) \exp\left(-\left(C_i(t) - \bar{C}\right)^2 - \sum_{i>0} \left(C_i(t+n) - \bar{C}\right)^2\right) \end{aligned}$$

The successive integrals can be performed using the budget constraint (10). We find:

$$P(B_i(t), B_i(t+1)) = \exp\left(-\left(B_i(t) + Y_i(t) - B_i(t+1) - \bar{C}\right)^2 - \frac{1}{T}\left(B_i(t+1) + \sum_{n>0}\left(Y_i(t+n) - \bar{C}\right)\right)^2\right)$$

where the transversality condition $B_i(t) \to 0$ as $t \to T$ has been imposed. Recall that the number of periods T is the expected mean process duration. Appendix 1 shows how to recover the transition probability for $C_i(t)$ by integrating over $B_i(t)$ and $B_i(t+1)$, under the condition that the revenue $Y_i(t+n)$ is centered on $\bar{Y}$ with variance θ^2. Defining the centered variables:

$$\hat{C}_i(t) = C_i(t) - \bar{Y}$$

and:

$$\hat{Y}_i(t) = C_i(t) - \bar{Y}$$

the statistical weight for the consumption path C_i becomes in first approximation for T large:

$$P\left(\hat{C}_i(t)\right) = \exp\left(-\sum_{t=1}^{T}\left(\hat{C}_i(t) - \hat{C}_i(t+1)\right)^2 - \frac{\left(\sum_{t=1}^{T}\hat{Y}_i(t) - \sum_{t=1}^{T}\hat{C}_i(t)\right)^2}{\theta^2}\right) \qquad (12)$$

Equation (12) modifies (4) when agents are facing some constraints.

We can generalize (12) for any system with a constraint similar to (10), by coming back to the notation $X_i(t)$ for a general action variable and by assuming that individual utilities have a quadratic approximation around some reference value X_0:

$$u_t^{(i)}(X_i(t), X(t-1)) \simeq \text{Constant} + \left(u_t^{(i)}\right)''(X_0)(X_i(t) - X_0)^2$$

Assuming an intertemporal constraint of the form:

$$\sum_{0\leqslant t\leqslant T}\hat{Y}_i(t)=\sum_{0\leqslant t\leqslant T}X_i(t) \tag{13}$$

for some exogenous flow variable $\hat{Y}_{T+i}$, the individual weight for an individual path X_i becomes (after normalizing $\left(u_t^{(i)}\right)''(X_0)=-1$, and reintroducing the variance σ^2) :

$$P(X_i)=\exp\left(-\sum_{0\leqslant t\leqslant T}\frac{(X_i(t)-X_i(t+1))^2}{\sigma^2}-\frac{\left(\sum_{0\leqslant t\leqslant T}\hat{Y}_i(t)-\sum_{0\leqslant t\leqslant T}X_i(t)\right)^2}{\theta^2}\right) \tag{14}$$

This yields the global weight for a path X of the system when an intertemporal constraint is considered:

$$P(X)=\exp\left(-\sum_{i,0\leqslant t\leqslant T}\frac{(X_i(t)-X_i(t+1))^2}{\sigma^2}-\sum_{i,0\leqslant t\leqslant T}\frac{\left(\sum_{0\leqslant t\leqslant T}\hat{Y}_i(t)-\sum_{0\leqslant t\leqslant T}X_i(t)\right)^2}{\theta^2}\right) \tag{15}$$

If we consider a continuous time, an integral replaces the sum over t and formula (15) becomes:

$$P(X)=\exp\left(-\sum_i\left(\frac{1}{\sigma^2}\int_0^T\left(\frac{d}{dt}X_i(t)\right)^2dt+\frac{\left(\int_0^T dt\hat{Y}_i(t)-\int_0^T dtX_i(t)\right)^2}{\theta^2}\right)\right) \tag{16}$$

Equations (15) and (16) describe the statistical weight associated to a path for a system with intertemporal constraint. Gosselin, Lotz and Wambst (2018) shows that non-quadratic corrections to the utility can be considered by adding terms of the form $V_1(X_i(t))$ to the weights (15) and (16). In continuous time, this yields:

$$P(X)=\exp\left(-\sum_i\left(\int_0^T\left(\frac{1}{\sigma^2}\int_0^T\left(\frac{d}{dt}X_i(t)\right)^2+V_1(X_i(t))\right)dt+\frac{\left(\int_0^T dt\hat{Y}_i(t)-\int_0^T dtX_i(t)\right)^2}{\theta^2}\right)\right) \tag{17}$$

Ultimately, we can directly generalize (17) when the revenue $\hat{Y}_i(t)$ is itself a function of the variables of the system. This will be the case in section 3, when considering a Business Cycle model. Assuming the form $\hat{Y}_i(t)=F(X_i(t))$, yields:

$$P(X)=\exp\left(-\sum_i\left(\int_0^T\left(\frac{1}{\sigma^2}\int_0^T\left(\frac{d}{dt}X_i(t)\right)^2+V_1(X_i(t))\right)dt+\frac{\left(\int_0^T dtF(X_i(t))-\int_0^T dtX_i(t)\right)^2}{\theta^2}\right)\right) \tag{18}$$

1.2.3 Interdependent constraints

The above computations were performed under the assumption that the constraint included some exogenous, i.e. totally independent from other agents, variable $Y_i(t)$**.** However for a system of N agents, constraints are more likely imposed on agents by the entire set of interacting agents. For example the variable $Y_i(t)$ in the constraint (10) represented the agent's revenue. In a context of N interacting agents, this variable depends on others' activity. In our simple model (9), it is on their consumption. In a system of consumer/producer, the others' consumption generates the flow of revenue $Y_i(t)$. In other word, agent i revenue $Y_i(t)$ depends on other agents' consumptions $C_j(t)$ - or possibly $C_j(t-1)$ if we assume a lag between agents actions and their effect. More generally, for a system with a large number of agents, the revenue $Y_i(t)$, may depend on endogenous variables that can still be considered as exogenous in agent i's perspective. Thus, our benchmark hypothesis in this section will be that agents are too numerous to be manipulated by a single agent.

The previous procedures developed for the constraint of a single agent remain valid and can be generalized directly. Again, we impose a constraint of the form (13) in continuous time:

$$\int_0^T Y_i(t)\,dt=\int_0^T X_i(t)\,dt$$

for each agent. When the individual agent considers $Y_i(t)$ as exogenous, (16) applies. But, if $Y_i(t)$ depends endogenously on other agents, (16) must be modified accordingly. Assume for example that $Y_i(t)=\sum_j\alpha_j^iX_j(t)$ for the

i-th agent. Appendix 1 shows that under some assumptions about θ^2 and σ^2, the last term in (16) for the i-th agent can be replaced by:

$$\frac{\left(\int dtY_i(t) - \int dtX_i(t)\right)^2}{\theta^2} = \frac{1}{\theta^2}\int\int X_i(s) X_i(t) dsdt + \frac{1}{\theta^2}\sum_{i,j}\int\int V_2(X_i(s), X_j(t)) dsdt \tag{19}$$

for some constant ν depending on the system, and with:

$$V_2(X_i(s), X_j(t)) = \left(\sum_k \alpha_j^k \alpha_i^k - 2\alpha_j^i\right) X_i(s) X_j(t) \tag{20}$$

The term:

$$\frac{1}{\theta^2}\int\int X_i(s) X_i(t) dsdt \tag{21}$$

depends only on the individual agent i. It is irrelevant in modeling the interactions between agents. Moreover, (Gosselin, Lotz, Wambst 2017) shows that it can often be approximated by a term proportional to $\int X_i^2(t) dsdt$. It can thus be included in the contribution $V_1(X_i(t))$ of (17). As a result, equation (20) transcribes the constraints in some non-local interactions between agents. Each agent's constraint is shaped by the environment other agents create. Equation (20) also accounts, when necessary, for some non-linear constraints $V(X_i(s), X_j(t))$, where V is an arbitrary function.

This discussion can be generalized straightforwardly to constraints involving up to k agents. In that case, any interdependent intertemporal constraint, or any interaction between k agents is modeled by:

$$\sum_{k\geqslant 2}\sum_{i_1,...,i_k}\int_0^T\int_0^T \frac{V_k(X_{i_1}(s_1), ..., X_{i_k}(s_1))}{\theta^2} ds_1...ds_k \tag{22}$$

The functions V_k depend on the particular interactions to model. Section 3 details an example involving technology and capital. Gathering the results of (17), (8) and (22) leads to the global statistical weight for the set of agents in the continuous time version, including both instantaneous constraint (8), individual intertemporal constraint (17), and intertemporal constraint (22):

$$P(X) = \exp(-A_1 - A_2) \tag{23}$$

with:

$$\begin{aligned} A_1 &= \sum_i \int_0^T \left(\frac{1}{\sigma^2}\int_0^T \left(\frac{d}{dt}X_i(t)\right)^2 + V_1(X_i(t))\right) dt + \frac{1}{\eta^2}\sum_i \int_0^T \left(\frac{d}{dt}X_i(t) - H(X_i(t))\right)^2 dt \\ A_2 &= \sum_{k\geqslant 2}\sum_{i_1,...,i_k}\int_0^T\int_0^T \frac{V_k(X_{i_1}(s_1), ..., X_{i_k}(s_1))}{\theta^2} ds_1...ds_k \end{aligned}$$

Contribution A_1 is the individual part of the statistical weight. It depends on individual agents. It includes a utility with possible individual intertemporal constraint and instantaneous constraint. For the intertemporal part, we have chosen (17). Contribution A_2 models the agents' interactions through a potential depending on several agents.

Remark that the term:

$$\frac{\left(\int_0^T ds\hat{Y}_i(s) - \int_0^T dsX_i(s)\right)^2}{\theta^2}$$

present in (17), has desappeared. It has been decomposed into an individual part, included in $V_1\left(X_s^{(i)}\right)$, and an interaction part, included in A_2.

Finally, a slight generalization of (23) will later prove useful. Assuming the N agents have different lifespan T_1,..., T_1, we define $P_{T_1,...,T_1}(X)$ the probability for a path with variable individual lifespan by:

$$P_{T_1,...,T_1}(X) = \exp\left(-A_1' - A_2'\right) \tag{24}$$

with:

$$\begin{aligned} A_1' &= \sum_i \int_0^{T_i} \left(\frac{1}{\sigma^2}\int_0^{T_i} \left(\frac{d}{dt}X_i(t)\right)^2 + V_1(X_i(t))\right) dt + \frac{1}{\eta^2}\sum_i \int_0^{T_i} \left(\frac{d}{dt}X_i(t) - H(X_i(t))\right)^2 dt \\ A_2' &= \sum_{k\geqslant 2}\sum_{i_1,...,i_k}\int_0^{T_i}\int_0^{T_j} \frac{V_k(X_{i_1}(s_1), ..., X_{i_k}(s_1))}{\theta^2} ds_1...ds_k \end{aligned}$$

1.3 Probability transition functions

Our formalism replaces the optimization problem with a probabilistic approach. It thus allows to compute the probability transition functions (or transition functions in short) for the system between an initial and a final state. To do so, we first define the paths with initial state $\underline{X}$ and final state $\overline{X}$ as the set of paths X such that $X(0) = \underline{X}$ and $X(T) = \overline{X}$. In formula (23), $P(X)$ is the probability density for a given path X. We then define the probability of transition between an initial state $\underline{X}$, and a final state $\overline{X}$ of the system, as a sum of (23) over all paths. This probability is computed as a multiple integral:

$$P_T\left(\underline{X}, \overline{X}\right) = \int ... \int P(X) \prod_i \prod_t dX_i(t) \tag{25}$$

The integrand $\prod_i \prod_t dX_i(t)$ can be understood as the sum over the paths $X(t)$ between $\underline{X} = X(0)$ and $\overline{X} = X(T)$. A compact notation for this Path integral is $\prod_i \prod_t dX_i(t) \equiv \prod_i \mathcal{D}X_i$ (see Peskin, Schroeder, 1995). Similarly, using (24) we can define:

$$P_{T_1,...,T_1}\left(\underline{X}, \overline{X}\right) = \int ... \int P_{T_1,...,T_1}(X) \prod_i \prod_t dX_i(t) \tag{26}$$

for agents with variable lifespan.

However, the integrals in (25) and (26) are difficult to compute, particularly when the number of agents is large. Moreover, the inclusion of non-local terms in (23) is another source of complexity. Techniques such as perturbation expansions of the potential term in terms of Feynman graphs exist and may be used in some case. Nevertheless, for a large number of agents another method exists, based on Statistical Field Theory. This formalism will consider a set of an infinite number of agents and compute (25) for any number N of agents among this set. More precisely, we will rather compute the Laplace transform of (26), defined by:

$$G_\alpha\left(\underline{X}, \overline{X}\right) = \int_0^T \exp\left(-\alpha\left(T_1 + ...T_N\right)\right) P_{T_1,...,T_1}\left(\underline{X}, \overline{X}\right) dT_1...dT_N \tag{27}$$

Once (27) computed, the function $P_T\left(\underline{X}, \overline{X}\right)$ can be recovered, either analytically, through an inverse Laplace transform, either numerically. However, the function $G_\alpha\left(\underline{X}, \overline{X}\right)$ has an interest in itself. It represents a transition probability for a variable lifespan between agents: the timespans T_i are assumed to be a Poisson random process with mean $1/\alpha$. As a consequence, $G_\alpha\left(\underline{X}, \overline{X}\right)$ is the transition of probability for a system of agents with variable lifespan T.

2 A field theoretic formulation for interactions between large number agents

When the present formalism is applied to a large number of agents, transition functions can be computed as the so-called correlation functions of a field theory (see Kleinert 1989) whose action is directly derived from individual agents' statistical weight defined in section 1. Starting from the expression (23) defining $P(X)$ for a system, a functional of an abstract quantity, or "field", is built, that will both keep the collective aspects of the system, and allow to compute the transition functions of individual agents (27) defined in section 1. Field theory allows to inspect the phases of the system, phases that describe the background in which individual agents evolve. Given the parameters of the system, several phases may exist: the system may experience phase transition, switching from one type of dynamic to another.

We can now explain how to associate a field representation to (23). The idea is the following. For a large number of agents, the system described by (23), involves a large number of variables $X(t)$ that are dificult to handle. We consider the space H of complex functions defined on the space of a single agent's actions. The space H describes the collective behavior of the system. Each function Ψ of H encodes a particular state of the system. Then, to each function Ψ of H, we associate a statistical weight, i.e. a probability describing the state encoded in Ψ. This probability is written $\exp(-S(\Psi))$, where $S(\Psi)$ is a functional, i.e. a function of the function Ψ. The form of $S(\Psi)$ is derived directly from the form of (23).

This description does not represent an aggregation to a single representative agents. It keeps the information about individual agents among the whole system, and will allow to compute the transition functions for several agents among the system. Moreover, it makes possible to find some collective features of the system as a whole.

The method presented here is an adaptation of tools in statistical field theory described in (Kleinert 1989). It relies on building a field action, starting from the probabilistic description (23) in the following successive steps.

Replacing the action variable by a field Rather than describing agents by a set of action variables $X(t)$, we consider a (complex valued) function $\Psi(x)$ where the vector x belongs to the same space as the $X_i(t)$. For agents described by their consumption $C_i(t)$ and a stock of individual capital $K_i(t)$, the field will be a function $\Psi(x) = \Psi(c,k)$.

This function is an abstract encoding of the distribution of consumption and capital among the whole set of agents. It is not a distribution of probability for these values. Only a functional $S(\Psi)$ of this field will give some information about this distribution.

Translating the individual part of $P(X)$ in terms of field The individual part A_1 of (23) is the weight depending only on individual agents, excluding their mutual interactions. Recall that:

$$A_1 = \sum_i \int_0^T \left(\frac{1}{\sigma^2} \int_0^T \left(\frac{d}{ds} X_i(s) \right)^2 + V_1\left(X_s^{(i)} \right) \right) ds + \frac{1}{\eta^2} \sum_i \int_0^T \left(\frac{d}{dt} X_i(t) - H\left(X_i(t) \right) \right)^2 dt$$

Under some conditions on σ^2 (see Gosselin, Lotz, Wambst 2017), we can associate to A_1 the following functional:

$$S_0(\Psi) = \int \left(\Psi^\dagger(x) \left(-\sigma^2 \nabla^2 + V_1(x) + \alpha \right) \Psi(x) \right) dx - \sum_i \int_0^T \Psi^\dagger(x) \left(\eta^2 \nabla^2 + \nabla . H(x) \right) \Psi(x)$$

where α is the parameter arising in the Laplace transform described in the first section, formula (27), and where $\Psi^\dagger(x)$ denotes the complex conjugate of $\Psi(x)$. The operator ∇ is the gradient operator, a vector whose i-th coordinate is the first derivative $\frac{\partial}{\partial x_i}$: $\nabla = \left(\frac{\partial}{\partial x_i} \right)$. The operator ∇^2 denotes the Laplacian:

$$\nabla^2 = \sum_i \frac{\partial^2}{\partial x_i^2}$$

where the sum runs over the coordinates x_i of the vector x. Applying this to our previous example, where $\Psi(x) = \Psi(c,k)$, we get $\nabla^2 = \frac{\partial^2}{\partial c^2} + \frac{\partial^2}{\partial k^2}$.

Adding the interaction terms of $P(X)$ The last part of $P(X)$ describes specifically the interaction between the different agents. We will call it A_2.

$$A_2 = \sum_{k \geqslant 2} \sum_{i_1, \ldots, i_k} \int_0^T \int_0^T \frac{V_k\left(X_{s_1}^{(i_1)}, \ldots, X_{s_k}^{(i_k)} \right)}{\theta^2} ds_1 \ldots ds_k$$

In terms of field, it is translated into a functional:

$$S_I(\Psi) = \frac{1}{\theta^2} \sum_{k \geqslant 2} \int \Psi(x_1) \ldots \Psi(x_k) V_k(x_1 \ldots x_k) \Psi^\dagger(x_1) \ldots \Psi^\dagger(x_k) dx_1 \ldots dx_k$$

Adding $S_I(\Psi)$ to $S_0(\Psi)$ yields the field action:

$$\begin{aligned} S(\Psi) &= S_0(\Psi) + \int \left(\Psi^\dagger(x) \Psi^\dagger(y) (V_2(x,y)) \Psi(x) \Psi(y) \right) dxdy \\ &= \int \left(\Psi^\dagger(x) \left(-\sigma^2 \nabla^2 + V_1(x) + \alpha \right) \Psi(x) \right) dx - \sum_i \int_0^T \Psi^\dagger(x) \left(\eta^2 \nabla^2 + \nabla . H(x) \right) \Psi(x) \\ &\quad + \frac{1}{\theta^2} \sum_{k \geqslant 2} \sum_{k \geqslant 2} \int \Psi(x_1) \ldots \Psi(x_k) V_k(x_1 \ldots x_k) \Psi^\dagger(x_1) \ldots \Psi^\dagger(x_k) dx_1 \ldots dx_k \end{aligned}$$

Adding source fields to the action The above functional $S(\Psi)$ gathers all the necessary information. But to compute the transition functions associated to the system described by (23), it has to be supplemented with so-called source fields.

Consider a complex function $J(x)$ in $\mathcal{H}$, and add to $S(\Psi)$ the quadratic terms:

$$\int \left(J(x) \Psi^\dagger(x) + J^\dagger(x) \Psi(x) \right) dx$$

The system can be described by an action with source:

$$\begin{aligned} S\left(\Psi, J\left(x\right)\right) &= \int\Big(\Psi^{\dagger}\left(x\right)\big(-\sigma^{2}\nabla^{2}+V_{1}\left(x\right)+\alpha\big)\Psi\left(x\right)\Big)dx-\sum_{i}\int_{0}^{T}\Psi^{\dagger}\left(x\right)\big(\eta^{2}\nabla^{2}+\nabla.H\left(x\right)\big)\Psi\left(x\right) \\ &+\frac{1}{\theta^{2}}\sum_{k\geqslant 2}\sum_{i_{1},...i_{k}}\Psi\left(x_{i_{1}}\right)...\Psi\left(x_{i_{k}}\right)V_{k}\left(x_{i_{1}}...x_{i_{k}}\right)\Psi^{\dagger}\left(x_{i_{1}}\right)...\Psi^{\dagger}\left(x_{i_{k}}\right)dx_{i_{1}}...dx_{i_{k}} \\ &+\int\Big(J\left(x\right)\Psi^{\dagger}\left(x\right)+J^{\dagger}\left(x\right)\Psi\left(x\right)\Big)dx \end{aligned}$$

The successive derivatives of $S\left(\Psi, J\left(x\right)\right)$ with respect to $J\left(x\right)$ and $J^{\dagger}\left(x\right)$ will allow to recover the transition functions.

Computing the transition functions via path integrals over $\Psi\left(x\right)$ Once the action $S\left(\Psi, J\left(x\right)\right)$ derived from (23), we can compute the transition functions defined in (27). To do so, we first need to introduce two notations.

Notation 3 *We denote $X^{[k]}(t)$ the vector $(X_1(t),...,X_k(t))$ of k action vectors. Agents being identical, any set of k agents among the entire set of N agents is equivalent.*

Notation 4 *We denote $\underline{X}^{[k]}$ and $\overline{X}^{[k]}$ any initial and final conditions for this set of agents.*

Then, a result of statistical field theory (see Kleinert 1989) is the following:

The transition probability $G_{\alpha}\left(\underline{X}^{[k]}, \overline{X}^{[k]}\right)$ defined in (27) for k agents between the initial state $\underline{X}^{[k]}$ and the final state $\overline{X}^{[k]}$ for the system defined by the probability weight (23) is:

$$\begin{aligned} & G_{\alpha}\left(\underline{X}^{[k]}, \overline{X}^{[k]}\right) \qquad (28) \\ = & \left[\left(\frac{\delta}{\delta J\left(x_{i_1}\right)}\frac{\delta}{\delta J^{\dagger}\left(y_{i_1}\right)}\right)...\left(\frac{\delta}{\delta J\left(x_{i_N}\right)}\frac{\delta}{\delta J^{\dagger}\left(y_{i_N}\right)}\right)\exp\left(\int\Big(\Psi^{\dagger}\left(x\right)\big(-\sigma^{2}\nabla^{2}+V_{1}\left(x\right)+\alpha\big)\Psi\left(x\right)\Big)dx\right.\right. \\ & -\sum_{i}\int_{0}^{T}\Psi^{\dagger}\left(x\right)\big(\eta^{2}\nabla^{2}+\nabla.H\left(x\right)\big)\Psi\left(x\right) \\ & \left.\left.+\frac{1}{\theta^{2}}\sum_{k\geqslant 2}\sum_{i_{1},...i_{k}}\Psi\left(x_{i_{1}}\right)...\Psi\left(x_{i_{k}}\right)V_{k}\left(x_{i_{1}}...x_{i_{k}}\right)\Psi^{\dagger}\left(x_{i_{1}}\right)...\Psi^{\dagger}\left(x_{i_{k}}\right)+J\left(x\right)\Psi^{\dagger}\left(x\right)+J^{\dagger}\left(x\right)\Psi\left(x\right)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}\right]_{J=J^{\dagger}=0} \end{aligned}$$

As before, the notation $\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}$ denotes an integration over the space of functions $\Psi\left(x\right)$ and $\Psi^{\dagger}\left(x\right)$, that is integral in an infinite dimensional space. Actually, these integrals are formal, and solely computed in simple cases. The form of $S\left(\Psi\right)$ is often sufficient to derive good qualitative insights about the results. In terms of field theory, formula (28) means that the transition functions are the correlation functions of the field theory with action $S\left(\Psi\right)$.

As announced, the formulation (28) shows how the transition of the agents, i.e. their dynamical and stochastic properties, take place in a surrounding. We do not compute the dynamic of the whole system. We rather derive agent's behaviors from the global properties of a substratum, i.e. the global action for the field $\Psi\left(x\right)$.

Remark that this change in formulation has to be related to the introduction of a variable number k of agents in (28). In the previous section, the system was described by a fixed number of agents. Here, the focus being on the environment, we can compute the transition functions for an arbitrary number of agents in this environment.

2.1 Non trivial vacuum, phase transition and Green function

In practice, it is often not necessary to compute $2k$ derivatives in (28) to compute the k agents transition functions. It is generally enough to know the transition for one agent $G_{\alpha}\left(\underline{X}^{[1]}, \overline{X}^{[1]}\right)$. From there, the transition functions $G_{\alpha}\left(\underline{X}^{[k]}, \overline{X}^{[k]}\right)$ for several agents can be deduced by techniques such as Feynman graphs (see article complet for a detailed treatment). The one agent transition functions are themselves computed through graph expansion of the interaction term $S_I\left(\Psi\right)$. However, some simplifications may arise and the graph expansion can be avoided in first approximation.

We proceed in the following way. We first look for a field minimizing the action $S(\Psi)$, that is a field Ψ_0 solution of $\frac{\delta}{\delta\Psi}S(\Psi)=0$. If such a non null solution does exist, the system is said to have a non-trivial vacuum for $S(\Psi_0)$. Then, let $\Psi=\Psi_0+\delta\Psi$ and expand $S(\Psi)$ to the second order in $\delta\Psi$. That is:

$$\begin{aligned} S(\Psi) &\simeq S(\Psi_0)+\int \delta\Psi^{\dagger}(x)\left(-\sigma^2\nabla^2+V_1(x)+\alpha\right)\delta\Psi(x)\,dx \\ &\quad+\int \delta\Psi^{\dagger}(x)\,V(\Psi_0,x,y)\,\delta\Psi(y)\,dxdy \end{aligned}$$

with:

$$V(\Psi_0,x)=\frac{\sigma^2}{\theta^2}\sum_{k\geqslant 2}\sum_{l_1,l_2}\int\left[\prod_{x_i,i\neq l_1}\Psi_0(x_i)\right]V_k(x_{i_1}...x_{i_k})\left[\prod_{x_i,i\neq l_2}\Psi^{\dagger}(x_i)\right]\delta(x-x_{l_1})\,\delta(y-x_{l_2})\,dx_1...dx_k$$

and $\delta(x-x_{l_1})$ is the Dirac function. It is then a classical computation to show that in first approximation the one agent transition function is determined by the quadratic part in $\delta\Psi$ and satisfies the differential equation:

$$\left(-\sigma^2\nabla^2+\alpha+V_1(x)+V(\Psi_0,x,y)\right)G_{\alpha}(x,y)=\delta(x-y) \tag{29}$$

in that case, the transition function describing the system can be computed at least approximatively numerically. A consequence of this set up is the notion of phase transition. For some values of the parameters, the only vacuum of the theory may be $\Psi_0=0$. In that case, $V(\Psi_0,x,y)=0$, so that the transition function is in first approximation given by the solution of:

$$\left(-\sigma^2\nabla^2+\alpha+V_1(x)\right)G_{\alpha}(x,y)=\delta(x-y) \tag{30}$$

On the other hand, if for another range of parameters $\Psi_0\neq 0$, then the transition function is computed by (29) and we say that the system experiences a phase transition. The qualitative properties of the system in the phase $\Psi_0\neq 0$ differs from those in the phase $\Psi_0=0$. Probabilities of transition and average values of quantities may differ from one phase to another.

3 Application: revisiting a standard business cycle model

In this section, we present an application of our formalism to a standard Business Cycle model. The usual assumptions of the standard model are maintained (see Romer 1996), but agents now interact through technology. In such a model, we show that a non-trivial vacuum may appear. For some values of the parameters, the equilibrium may experience a discontinuous shift. The different phases of the system induce different individual behaviors. In the following, we will present the model, compute the effect of the agents' interactions on individual dynamics for each phase, and provide an interpretation of the results.

3.1 Description

3.1.1 The model

We consider a system with a large number of identical consumer/producer agents. Each agent i consumes at time t a quantity $C_i(t)$, has a stock of capital $K_i(t)$ and a technology $A_i(t)$. The saving variable $B_i(t)$ is equal to the stock of capital $K_i(t)$ used in the production function, as usually assumed in standard Business Cycle models.

On the consumer side, we consider a utility function of the standard form (Romer 1996, Obstfeld Rogoff 1996):

$$u(C_i(t))=\frac{C_i(t)^{1-\theta}-1}{1-\theta}$$

where the coefficient θ measures the relative risk aversion, i.e. the inverse of the elasticity of substitution between consumption at different dates. A quadratic approximation of $u(C_i(t))$ can be found by an expansion around some minimal value $\hat{C}$ for the consumption:

$$u(C_i(t))=\left(C_i(t)-\hat{C}\right)-\frac{\theta}{2}\left(C_i(t)-\hat{C}\right)^2 \tag{31}$$

We can thus rewrite:

$$u(C_i(t))=\left(C_i(t)-\hat{C}\right)-\frac{\theta}{2}\left(C_i(t)-\hat{C}\right)^2=-\frac{\theta}{2}\left(C_i(t)-\tilde{C}\right)^2+\frac{1}{2\theta} \tag{32}$$

with $\tilde{C} = \hat{C} + \frac{1}{\theta}$. The constant $\tilde{C}$. We assume $\tilde{C} \gg 1$. As usual, this constant ensures decreasing marginal utility. The quadratic approximation (32) will be the utility used in the sequel.

As consumers, agents each optimize their intertemporal utility function. Written in continuous time:

$$U(C) = \int_0^T u(C_i(t))\, dt$$

Since the discount factor β does not alter this section main arguments, we set it to 1. Under the usual budget constraint:

$$C_i(t) = r_i(t) B_i(t) + Y_i(t) - \frac{d}{dt} B_i(t) \tag{33}$$

where $r_i(t)$ is the i-th agent or sector interest rate. In continuous time, integrating (33) over the entire periods yields the overall budget constraint :

$$\int_0^T (Y_i(t) - C_i(t)) \exp\left(-\int r_i(t)\, dt\right) dt = 0$$

On the production side, assuming some uncertainty in the capital accumulation process yields a dynamic equation for capital:

$$\dot{K}_i(t) = Y_i(t) - C_i(t) - \delta(K_i(t)) + \varepsilon(t) \tag{34}$$

where $\varepsilon(t)$ is a random term of variance ν^2 and $\delta(K_i(t))$ describes the depreciation of $K_i(t)$.

We endogenize the production $Y_i(t)$ and treat it as a function of capital: $Y_i(t) = A_i(t) F_i(K_i(t))$. From this relation we can deduce the form of the interest rate faced by each sector:

$$r_i(t) = A_i(t) F_i'(K_i(t)) + r_c$$

That includes an exogenous (or minimal) interest rate r_c, plus some individual determinants depending on the capital depreciation, rates of return, environment and technology of each sector. That is, r_i is defined by the marginal productivity in the sector plus some collective effect r_c. Remark that usually, $r_c = -\delta$, but we assume that other determinants allow to consider r_c as an independent variable. It is always possible to set $r_c = -\delta$ if needed.

To complete the model, the dynamics of technology should be modeled. We assume that $A_i(t)$ is a stochatic process with specific fetures. Its dynamics includes an intrinsic part that fluctuates around a technology growth path. Besides, we assume that technology and capital stock influence each other. Part of the technology random process will thus describe technology's interaction with capital. Since the dynamics of $A_i(t)$ is probabilistic, we will provide its precise description in the next section.

3.1.2 Probabilistic description

Let us now apply the method presented in section 1. Three variables describe our model. The variables K and C are standard control variables. As such, we must give them a statistical weight describing their dynamics. The third variable, technology, does not qualify as a control variable. It could be treated as an exogenous parameter. However since our formalism aims at studying interactions between variables, and explore the consequences of these interactions, we will treat technology as a variable of the system interacting with capital. Consequently, we will give it a statistical weight. So that the probability describing the system can be decomposed into several statistical weights, respectively due to consumption, capital and technology.

Statistical weight of consumption The first weight corresponds to the consumption behavior of agents with utility (31) under an intertemporal budget constraint (33). Appendix 3 shows that the exogenous interest rate in the constraint modifies the statistical weight (16) associated to consumption under constraint in:

$$\exp\left(-\sum_t \frac{1}{\varpi^2}\left(C_i(t) - \bar{C} - \frac{(C_i(t+1) - \bar{C})}{(1+r)}\right)^2 + \sum_t C_0\right) \exp\left(-\frac{\left(\int_0^T (Y_i(t) - C_i(t)) \exp\left(-\int r_i(t)\, dt\right) dt\right)^2}{\theta^2}\right)$$

where $\varpi^2 = \frac{2\sigma^2}{\theta}$ measures the uncertainty in consumption behavior among agents. For a $\varpi^2 \ll 1$, the weight is peaked around the usual optimal Euler equation in continuous time. The parameter σ^2 is the uncertainty in consumption behavior used in (16). Remark that $C_0 \equiv \frac{1}{2\theta\sigma^2}$, and that the sum $\sum_t C_0 = TC_0$ measures the agents' relative risk aversion, cumulated over the entire timespan, to change consumption.

Appendix 5 shows that the part due to the overall intertemporal constraint:

$$-\frac{\left(\int_0^T (Y_i(t) - C_i(t)) \exp\left(-\int r_i(t)\, dt\right) dt\right)^2}{\theta^2}$$

can be neglected in first approximation. In continuous time, this leaves us with:

$$\exp\left(-\int dt\frac{\left(\dot{C}_i\left(t\right)-r\left(C_i\left(t\right)-\bar{C}\right)\right)^2}{\varpi^2}+C_0\int dt\right) \tag{35}$$

Statistical weight of capital The second weight models the capital dynamics. Equation (34) shows that $\varepsilon_i\left(t\right)=\dot{K}_i\left(t\right)-\left(Y_i\left(t\right)-C_i\left(t\right)-\delta\left(K_i\left(t\right)\right)\right)$ is a gaussian variable with variance ν^2. The associated statistical weight is thus gaussian and writes:

$$\exp\left(-\int dt\frac{\left(\dot{K}_i\left(t\right)-\left(AF_i\left(K_i\left(t\right)\right)-C_i\left(t\right)-\delta\left(K_i\left(t\right)\right)\right)\right)^2}{\nu^2}\right) \tag{36}$$

Since $\varepsilon_i\left(t\right)$ is independent from consumption, this weight will be multiplied by (35).

Statistical weight of technology The third weight accounts for technology. Recall that this is a particular variable in our setting: its dynamics can be seen as intrinsic, or resulting from capital interaction. This reflects on its weight.

Statistical weight of intrinsic technology We first consider the contribution inherent to the technology itself. We denote $\langle A\rangle$ the system's average technology, to be computed later, but phase dependent. Let us also denote A_0 an exogenous level of technology and $\bar{A}$ is the optimal technology level for the agent in the system, with $\bar{A}=\varkappa\langle A\rangle+A_0$, and $\varkappa<1$. We choose the technology contribution to be of the following form:

$$\exp\left(-\int dt\left(\frac{\left(\dot{A}_i\left(t\right)-gA_i\left(t\right)\right)^2}{\lambda^2}+\left(A_i\left(t\right)-\bar{A}\right)^2\right)\right) \tag{37}$$

The first term $\frac{\left(\dot{A}_t-gA_t\right)^2}{\lambda^2}$ models the agent's technology endowment as fluctuating around a technology growth path g. Actually, the distribution is centered around the paths solutions of $\left(\dot{A}_i\left(t\right)-gA_i\left(t\right)\right)=0$. In the sequel, we will set $g=0$ to simplify, so that $A_i\left(t\right)$ can be seen as a detrended variable, but the growth factor g can be reintroduced if needed. We consider $\lambda^2\gg 1$, which means that the level of technology can adapt relatively quickly to $\bar{A}$. The second term, $\left(A_i\left(t\right)-\bar{A}\right)^2$, is the difference between the agent's technology and agents' potential level of technology in the system. So that, in the absence any other forces, the agent should be driven towards this optimal level of technology. As a consequence (37) models an individual technology that is both driven by individual factors, and a collective level of technology.

Statistical weight of capital-technology interaction We have shown in section 1 how interactions can be modeled by adding a potential involving several agents (see (22)). Here, we model the impact of capital on technology by introducing an additional term in (37) such as:

$$\exp\left(-\gamma\int\int_{t_j<t_i}\sum_j A_i\left(t_i\right)H\left(K_i\left(t_i\right),K_j\left(t_j\right)\right)K_j\left(t_j\right)dt_jdt_i\right) \tag{38}$$

This term describes the value added accumulated in the different sectors by capital stocks. The function H is any positive functions and represents the impact of sector i on sector j. We assume reciprocal interactions and assume that H is symmetric: $H\left(K_i\left(t_i\right),K_j\left(t_j\right)\right)=H\left(K_j\left(t_j\right),K_i\left(t\right)\right)$. The constant γ measures the magnitude of these interactions. To check that (38) represents the impact of capital stock on technology, notice that the agent's technology $A_i\left(t_i\right)$ is multiplied by a weighted sum of other agents' past capital stocks. This weight thus models the interaction between technology and the global capital stock.

This interaction weight would however be incomplete, since the various sectors' technology may also, in turn, accelerate the dynamics of $K_i\left(t\right)$. Considering the interaction between capital and technology is reciprocal is equivalent to adding the term:

$$\exp\left(-\gamma\int\int_{t_j<t_i}\sum_j A_i\left(t_i\right)H\left(K_i\left(t_i\right),K_j\left(t_j\right)\right)K_j\left(t_j\right)dt_idt_j\right) \tag{39}$$

Here, inverting $t_j < t_i$ accounts for the reversal of roles: it is the past technology that impacts capital stock. Consequently, the statistical weight for technology and its interaction with capital stock is:

$$\exp\left(-\int dt\left(\frac{\left(\dot{A}_i(t)-gA_i(t)\right)^2}{\lambda^2}+\left(A_i(t)-\bar{A}\right)^2\right)\right. \tag{40}$$
$$\left.-\gamma\int\int\sum_j A_i(t_i)\,H\left(K_i(t_i),K_j(t_j)\right)K_j(t_j)\,dt_j dt_i\right)$$

The second term of (40) can be better understood if we note that it is an approximation of the quadratic term for $|\gamma|\ll 1$:

$$+\frac{\gamma}{\sqrt{|\gamma|}}\int\sum_i\left(A_i(t_i)-\frac{\sqrt{|\gamma|}}{2}\int\sum_j H\left(K_i(t_i),K_j(t_j)\right)K_j(t_j)\,dt_j\right)^2 dt_i \tag{41}$$

Actually, the expansion of (40) yields the second term of (40), plus a quadratic term in $A_i(t_i)$ and a quadratic term in $K_j(t_j)$. The quadratic term in $K_j(t_j)$ is of magnitude $\left(\sqrt{|\gamma|}\right)^3$ and can be neglected. The quadratic term in $A_i(t_i)$ is of magnitude $\sqrt{|\gamma|}A_i^2(t_i)$ which is negligible with respect to $\left(A_i(t)-\bar{A}\right)^2$.

Equation (41) shows for $\gamma<0$, the interaction is attractive: the higher capital stock, the higher the technology. Interactions between capital and technology increase the likelihood for paths satisfying:

$$A_i(t_i)-\frac{\sqrt{|\gamma|}}{2}\int\sum_j H\left(K_i(t_i),K_j(t_j)\right)K_j(t_j)\,dt_j=0$$

On the contrary, for $\gamma>0$ the interaction is repulsive: interactions between capital and technology increase the likelihood of paths satisfying:

$$A_i(t_i)-\frac{\sqrt{|\gamma|}}{2}\int\sum_j H\left(K_i(t_i),K_j(t_j)\right)K_j(t_j)\,dt_j\to\infty$$

Depending on society's stock of capital, we can define a certain threshold $\tilde{A}$ of required technology:

$$\tilde{A}=\frac{\sqrt{|\gamma|}}{2}\int\sum_j H\left(K_i(t_i),K_j(t_j)\right)K_j(t_j)\,dt_j$$

Agents with technology endowment higher than this threshold have an advantage and are thus driven on a technology growth path. Agents below this threshold $\tilde{A}$ will be evicted. We will study both cases $\gamma<0$ and $\gamma>0$ later on.

Overall statistical weight We can now gather the three contributions (35) (36) and (40). The overall statistical weight writes:

$$\exp\left(-\int dt\left(\frac{\left(\dot{C}_i(t)+r\left(C_i(t)-\bar{C}\right)\right)^2}{\varpi^2}\right)+C_0\int dt\right. \tag{42}$$
$$\left.-\sum_i\int dt\left(\frac{\left(\dot{K}_i(t)-\left(A_i(t)F_i\left(K_i(t)\right)-C_i(t)-\delta\left(K_i(t)\right)\right)\right)^2+\varsigma^2C_i^2(t)}{\nu^2}\right)\right)$$
$$\times\exp\left(-\sum_i\int dt_i\left(\frac{\left(\dot{A}_i(t_i)\right)^2}{\lambda^2}+\left(A_i(t_i)-\bar{A}\right)^2\right)-\gamma\int\int\sum_{i,j}A_j(t)\,H\left(K_j(t_j),K_i(t_i)\right)K_i(t_i)\,dt_i dt_j\right)$$

3.1.3 Field theoretic description

Now that the model has been described in terms of probabilities, we can transcribe it in terms of field, and apply the method presented in section 2. We introduce a field $\Psi(K,C,A)$ depending on the relevant variables of the system, consumption, capital and technology. Appendix 6 presents the field theoretic formulation of the system given the

above assumptions. Choosing the usual linear depreciation function $\delta(K) = \delta K$ and a "distance function" of the form $H(K_2, K_1) = 1$ yields the field formulation of the system:

$$S(\Psi) = \int \Psi^{\dagger}(K, C, A)\, \mathcal{O}\Psi(K, C, A) \tag{43}$$
$$+\frac{\gamma}{2}\int \Psi^{\dagger}(K_1, C_1, A_1)\, \Psi^{\dagger}(K_2, C_2, A_2)(A_2K_1 + A_1K_2)\, \Psi(K_1, C_1, A_1)\, \Psi(K_2, C_2, A_2)$$

where the second order differential operator $\mathcal{O}$ is defined by:

$$\begin{aligned}\mathcal{O} &= -\varpi^2 \frac{\partial^2}{\partial C^2} - \frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2} - \nu^2 \frac{\partial^2}{\partial K^2} \\ &+ \left(A - \bar{A}\right)^2 - 2\left(C - AF(K) + \delta K\right)\frac{\partial}{\partial K} + 2\left(AF'(K) + r_c\right)\left(C - \bar{C}\right)\frac{\partial}{\partial C} + \varsigma^2\left(C - \bar{C}\right)^2 \\ &+ \left(\alpha + 2AF'(K) + (r_c - \delta) - C_0\right)\end{aligned}$$

The quadratic part of the action $S(\Psi)$, namely $\int \Psi^{\dagger}(K, C, A)\, \mathcal{O}\Psi(K, C, A)$, describes the individual behavior of the agents. The quartic part of the action represents the interaction between technology and capital stocks among agents.

3.2 Results

We have described the model in terms of Field Theory. We can now search for non-trivial phases in the system, and study their properties. These emerging phases will then allow us to compute the transition functions first without, then with interactions.

3.2.1 Phases of the system

Once the field action $S(\Psi)$ found, the minima that define the various phases of the system can be computed. These phases correspond to the system different economic global equilibria. The trivial phase, i.e. $\Psi_1(K, C, A) = 0$, is an analogous to a system linearized around its equilibrium. On the contrary, non-trivial phases reveal the emergence of other equilibria. In each of these phases, the variables' average values and the agents' transition probability functions can be computed and studied.

The inspection of the non-trivial phases associated to the field action (43) of our model and the computation of the variables' average values in each phase are performed in Appendix 6. The production function, assumed to be Cobb Douglas, $F(K) = AK^{\varepsilon}$ with $\varepsilon < 1$, can be approximated by a Taylor expansion above a minimal stock of capital $\bar{K}$:

$$AK^{\varepsilon} \simeq A\bar{K}^{\varepsilon}\left(1 + \varepsilon\left(\frac{K - \bar{K}}{\bar{K}}\right) - \frac{\varepsilon(1 - \varepsilon)}{2}\left(\frac{K - \bar{K}}{\bar{K}}\right)^2\right)$$

The results show that a non-trivial minimum $\Psi_1(K, C, A)$ for $S(\Psi)$ exists, provided some conditions on the parameters are fulfilled. The minimum $\Psi_1(K, C, A)$ is a product of several gaussian functions in the variables K, C, A whose precise form is not necessary to the discussion (see Appendix 6 for a details).

This non-trivial minimum of $S(\Psi)$ exists when:

$$\begin{aligned}\gamma &> 0 \\ A_0 &> \left(1 + \sqrt{2}\right)\bar{C} \\ \lambda &> > 1\end{aligned} \tag{44}$$

and:

$$1 \gg \frac{A_0\varepsilon}{(1 - \varkappa)\bar{K}^{1-\varepsilon}} - \delta > 0 \tag{45}$$

Given that $\varepsilon < 1$, these conditions are jointly satisfied for A_0 and $\bar{K}$ relatively large, and when the return of capital exceeds the depreciation value (equation (45)). An additional condition on the relative risk aversion C_0 exists (see Appendix 6 for details). Qualitatively, for an intermediate range of values for C_0, the non-trivial phase is possible and stable. We give below some interpretation for these conditions.

The equilibrium values of the variables in both phases and the global patterns of the system are computed using quadratic expansions of $S(\Psi)$ around the minima $\Psi_1(K, C, A) = 0$ and $\Psi_1(K, C, A) \neq 0$ respectively. This expansion

in turn allows to find the average values for K, C and A in each phase (see appendix 6). For $\Psi_1(K,C,A)=0$ and neglecting the interaction terms, the average values for the relevant variables are:

$$\begin{aligned}\langle A\rangle_0 &= \bar{A}=\frac{A_0}{(1-\varkappa)}\\ \langle C\rangle_0 &= \bar{C}+\sqrt{\frac{2}{\pi}}\varpi\\ \langle K\rangle_0 &= \frac{\frac{A_0}{1-\varkappa}\bar{K}^{\varepsilon}(1-\varepsilon)-x}{Y}\end{aligned}$$

with:

$$\begin{aligned}Y &= \delta-\frac{A_0}{(1-\varkappa)}\varepsilon\bar{K}^{\varepsilon-1}\\ K_1' &= -\frac{2}{\pi}\left|\bar{C}+\sqrt{\frac{2}{\pi}}\varpi-A\bar{K}^{\varepsilon}(1-\varepsilon)\right|-\frac{\sqrt{2\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}}{\sqrt{\pi}}\nu\\ x &= \bar{C}+\sqrt{\frac{2}{\pi}}\varpi-K_1'\end{aligned}$$

For the phase with $\Psi_1(K,C,A)\neq 0$, one finds:

$$\begin{aligned}\langle A\rangle_1 &= \frac{A_0}{(1-\varkappa)}-\frac{1}{2}\frac{K^{\varepsilon}A_0(1-\varepsilon)-\left(\bar{C}+\sqrt{\frac{2}{\pi}}\varpi-K_1'\right)Y(1-\varkappa)}{Y^2(1-\varkappa)^3}\gamma\eta \qquad (46)\\ \langle C\rangle_1 &= \bar{C}+\sqrt{\frac{2}{\pi}}\varpi-\frac{\varpi^2\left(\frac{A_0}{1-\chi}\right)}{2\left(\varsigma^2\varpi^2+\left(AF'(K)+r_c\right)^2\right)\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\gamma\eta\end{aligned}$$

$$\begin{aligned}\langle K\rangle_1 &= \frac{\frac{A_0}{1-\varkappa}\bar{K}^{\varepsilon}(1-\varepsilon)-x}{Y}-\frac{1}{2}K^{\varepsilon}\frac{\left(K^{\varepsilon}A_0(1-\varepsilon)-Y(1-\varkappa)x\right)\left(\varepsilon x-K\delta(1-\varepsilon)\right)}{Y^4K(1-\varkappa)^3}\gamma\eta \qquad (47)\\ &\quad-\frac{\bar{K}^{\varepsilon}(1-\varepsilon)(1-Y)K_1'}{2Y^3}\gamma\eta-\frac{\nu^2\frac{A_0}{1-\chi}}{2Y^2}\gamma\eta-\frac{\varpi^2\left(\frac{A_0}{1-\chi}\right)}{2\left(\varsigma^2\varpi^2+\left(AF'(K)+r_c\right)^2\right)Y^2}\gamma\eta\end{aligned}$$

The parameter $\eta\gamma$ depends on the parameters of the system and can be estimated as:

$$\eta\gamma\ll 1$$

In the non-trivial phase, both, the average level of consumption $\langle C\rangle_1$ and the average level of technology $\langle A\rangle_1$ are lower than in the trivial phase: $\langle C\rangle_1<\langle C\rangle_0$, $\langle A\rangle_1<\langle A\rangle_0$. Indeed, the non-trivial phase emerges under high relative risk aversion, and hence low level of consumptions. The effect on average capital stock $\langle K\rangle_1$ is mitigated: since the technology level has decreased, a higher level of capital stock may be required to reach average levels of consumption. Equation (47) shows that when capital or/and consumption volatility, ν and ϖ respectively, are high, capital stock is lower in the non-trivial phase. Uncertainty hinders accumulation. But when these volatilities are low, capital stock is higher than in the trivial phase.

One can compute the average production level in both phases. This is done in Appendix 6. It appears, that in our range of approximations:

$$\langle Y\rangle_1<\langle Y\rangle_0$$

Let us now discuss the conditions (44) and (45) in which this non trivial phase should appear. When $\gamma<0$, capital and technology are mutually enhanced. This prevents the non trivial phase to appear. A high level of consumption and capital can be reached. But when $\gamma>0$, the interaction between capital and technology is selective. In such a setting, the initial endowment in technology is crucial. Agents endowed with a level of technology above a certain threshold are favored. On the contrary, for agents poorly endowed, this level acts as a ceiling. In average, our results show that overall, the society experiences lower technology, lower production and lower consumption.

Finally, the non trivial phase corresponds to a large minimal technology and intermediate values of C_0. Actually, risk aversion has to be large enough to reduce capital accumulation. Still, this reduced capital accumulation must be sufficient to reach optimal consumption. To do so, technology must be relatively high, and compensate for a lower stock of capital. This equilibrium must also be sustainable. Thus, a large C_0 is outside the limits of our model.

3.2.2 Transition functions without interaction

In the above, we had first translated the classical business model in terms of probabilistic description, then translated it into statistical field. Now that the various phases of the system of the statistical field have been described, we can turn back to and examine the transition functions of the system. In other terms, having found the collective levels, we can now turn back to the individual level, that may include several agents and their interactions. The transition functions will now allow us to describe the individual agents' dynamics in each phase. We will then deduce the equilibrium values of these dynamics, as well as the dynamic equations of their average paths.

Transition functions In the following, we will denote the phase $\iota = 0$ the phase with minimum $\Psi_1 = 0$, the trivial phase. The phase $\iota = 1$ will correspond to $\Psi_1 \neq 0$, the non-trivial phase.

Given a phase ι of the system and neglecting the expression last term, i.e. the quartic interaction term in (43), the probability of transition for an agent from a state (C', K', A') to a state (C, K, A) during a timespan t, is equal to:

$$\begin{aligned} G\left(C,K,A,C',K',A',t\right) &= \frac{1}{\sqrt{\left(2\pi\right)^3 \frac{\Omega^2\varpi^2}{\lambda^2}t^3}} \exp\left(-\frac{\left(\left(C-\bar{C}_\iota\right)-\left(C'-\bar{C}_\iota\right)(1+\beta t)\right)^2}{2\varpi^2 t}\right) \\ &\times \exp\left(-\frac{\left(\left(K-\bar{K}+\frac{\delta\bar{K}+\bar{C}_\iota}{\alpha}\right)-\left(\left(K'-\bar{K}+\frac{\delta\bar{K}+\bar{C}_\iota}{\alpha}\right)(1-\alpha t)-\left(C'-\bar{C}_\iota\right)t+A'\bar{K}^\varepsilon t\right)\right)^2}{2\Omega^2 t}\right) \\ &\times \exp\left(-\frac{\lambda^2\left(A-A'\right)^2}{2t}-\frac{\left(\frac{A+A'}{2}-\bar{A}_i\right)^2}{2}t-m_\iota t\right) \end{aligned} \tag{48}$$

where (C', K', A') is the initial state of the system and (C, K, A) the final state for a process of duration t.

Parameters Two sorts of parameters appear in equation (48). Some parameters, such as Ω^2, α and β, do not depend on the phase of the system. They are:

$$\Omega^2 = \frac{\varpi^2}{\lambda^2}\left(\nu^2+\frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2}+\frac{3\varpi^2}{2\left(\beta^2-\alpha^2\right)}\right)$$

$$\begin{aligned} \alpha &= \delta-\frac{A+A'}{2}F'\left(\frac{K+K'}{2}\right) \\ \beta &= \frac{A+A'}{2}F'\left(\frac{K+K'}{2}\right)+r_c-\delta \end{aligned} \tag{49}$$

The parameter Ω^2 is a global variance of the system which mixes the variances of the variables C, K and A. This parameter is independent from the phase, since the phases do not affect volatilities but average values in this particular model. For $r_c = \delta$, the parameter β is the average rate of return of capital for a process starting from (C', K', A') and reaching (C, K, A). Finally, the parameter α measures the spread between marginal productivity and capital depreciation. These two variables do not depend on the phase, but merely on the producer capital and technology levels.

Other parameters in equation (48) are phase-dependent. These parameters are the average values of technology and capital in a given phase. They are, for the first phase:

$$\begin{aligned} \bar{A}_0 &= \left\langle A\right\rangle_0 = \frac{A_0}{1-\chi} \\ \bar{C}_0 &= \left\langle C\right\rangle_0 = \bar{C}+2\varpi \\ m_0 &= 0 \end{aligned}$$

and for the second phase:

$$\begin{aligned} \bar{A}_1 &= A_0+\chi\left\langle A\right\rangle_1, \bar{C}_1 = \Gamma_1 \\ m_1 &= \bar{A}_1^2-\left(\left(1-\varkappa\right)\bar{A}_1+\varkappa\Gamma_3\right)^2+\frac{\left(\left(\left(1-\varkappa\right)\bar{A}_1+\varkappa\Gamma_3\right)^2-2\bar{C}\left(\left(1-\varkappa\right)\bar{A}_1+\varkappa\bar{A}_1\right)-\bar{C}_1^2\right)}{\left(1-\varepsilon\right)\bar{K}^\varepsilon} \end{aligned}$$

where the average values $\langle A \rangle_\iota$ have been defined in (46).

Because some parameters are phase-dependent, the agent's transition probability depends on the phase of the system.

The parameter m_ι is a measure of the system's inertia. It is null in the trivial phase, but positive in the non-trivial phase. The fact that it should appear as an exponential term $\exp(-m_\iota t)$ in (48) is relevant. Transitions are quickly dampened through the interaction process in the non-trivial phase, and transition probabilities are reduced. The strong inertia of the system keeps agents closer to their initial values than in the trivial phase.

Note also that multi-agents interactions do not appear directly in (48), but only through parameters. These parameters encode the collective effects of the interactions and their impact at the individual level.

Average paths and classical dynamics Formula (48) is valid for small t (see Appendix 6 for larger t). However, it is sufficient to find the agent's average path. This is straightforward: for a gaussian weight, the average path is found by setting the exponent in (48) to 0. This expresses the fact that we select the most likely path, which is the average path. This yields the relations between the initial and final points:

$$\begin{aligned} 0 &= \left(\left(C-\bar{C}_\iota\right)-\left(C'-\bar{C}_\iota\right)(1+(\alpha+\beta)\,t)\right) \\ 0 &= \left(\left(K-\bar{K}+\frac{\delta\bar{K}+\bar{C}_\iota}{\alpha}\right)-\left(\left(K'-\bar{K}+\frac{\delta\bar{K}+\bar{C}_\iota}{\alpha}\right)(1-\alpha t)-\left(C'-\bar{C}_\iota\right)t+A'\bar{K}^\varepsilon t\right)\right) \\ 0 &= \frac{\lambda^2\left(A-A'\right)^2}{2t}+\frac{\left(\frac{A+A'}{2}-\bar{A}_i\right)^2}{2}t \end{aligned} \tag{50}$$

The treatment of these equations is usual. The equilibrium values are first found by setting both initial and final values equal to the equilibrium:

$$\left(C',K',A'\right)=(C,K,A)=(C_e,K_e,A_e)$$

One finds directly:

$$\begin{aligned} K_e &= \frac{(1-\varepsilon)\,\bar{A}_i\bar{K}^\varepsilon-\bar{C}_i}{\delta-\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon} \\ C_e &= \bar{C}_i \\ A_e &= \bar{A}_i \end{aligned}$$

Then, replacing these values in (50) yields directly the relations for the most likely path, or equivalently, the average path:

$$\begin{aligned} \left(C-\bar{C}_i\right) &= \left(C'-\bar{C}_i\right)+\left(C'-\bar{C}_i\right)(\alpha+\beta)\,t \\ \left(K-K_e\right) &= \left(K'-K_e\right)-\alpha\left(K'-K_e\right)t-\left(C'-\bar{C}_i\right)t \\ \lambda^2\left(A-A'\right) &= -\frac{\frac{A+A'}{2}-\bar{A}_i}{2}t \end{aligned}$$

In the limit of small t, and using (132) and $\frac{K+K'}{2}\to K$ leads to a differential equation for the average path:

$$\begin{aligned} \frac{d}{dt}\left(C\left(t\right)-\bar{C}_e\right) &= \left(C\left(t\right)-\bar{C}_e\right)\left(AF'\left(K\left(t\right)\right)+r_c-\delta\right) \\ \frac{d}{dt}\left(K\left(t\right)-K_e\right) &= \left(AF'\left(K\left(t\right)\right)-\delta\right)\left(K\left(t\right)-K_e\right)-\left(C\left(t\right)-\bar{C}_i\right) \\ \lambda^2\frac{d\left(A-\bar{A}_i\right)}{dt} &= -\frac{\left(A-\bar{A}_i\right)}{2} \end{aligned} \tag{51}$$

The above equations are in fact those of a simplified model of capital accumulation: the standard approach is recovered in average. The first equation is the usual Euler equation with interest rate. The second and third equations describe the dynamics of capital and technology respectively. The fixed point $\bar{A}_i$ depends both on the system and the system's interactions, as seen in (110). Linearizing the dynamics around the fixed point leads to a usual saddle path dynamics, with two eigenvalues:

$$\frac{1}{2}r_c-\delta\pm\frac{1}{2}\sqrt{r_c^2-4\bar{A}_i}+\bar{A}_i\varepsilon F'K_e$$

Usually, for $r_c=\delta$, the first eigenvalue is negative and the second positive. When the system moves along the unstable equilibrium, the linear approximation breaks down. Actually, for large values of $(K\left(t\right)-K_e)$, marginal productivity falls below the depreciation rate, and capital accumulation stops. We will not dwell on this point since once the phases of the system have been found, and the average dynamics equations have been written, interpretations are standard.

3.2.3 Corrections due to the interaction term

We go on studying the individual dynamics in the background created by system as a whole. To take into account the individual interactions and their impact on (48), we have to turn back to the field theoretic formulation and to find the modification of the transition functions due to the quartic interaction term in (43):

$$I = \frac{\gamma}{2}\int \Psi^{\dagger}\left(K_1, C_1, A_1\right)\Psi^{\dagger}\left(K_2, C_2, A_2\right)\left(A_2K_1 + A_1K_2\right)\Psi\left(K_1, C_1, A_1\right)\Psi\left(K_2, C_2, A_2\right) \tag{52}$$

Appendix 6 computes this correction. We come back to the definition of the probability transitions, and add the contribution of (52) that was discarded while computing (48). We show that the Green function is modified at the first order in γ as:

$$\bar{G}\left(C, K, A, C', K', A', t\right) = G\left(C, K, A, C', K', A', t\right) \times \exp\left(-\gamma V\left(C, K, A, C', K', A', t\right)\right)$$

where the function $V\left(C, K, A, C', K', A', t\right)$ depends on the initial and final states:

$$V\left(C, K, A, C', K', A', t\right) = \left(2t^2AK + \frac{t^3}{12}\left(A - A'\right)\left(C - C'\right) + \frac{t^3}{2}\left(A - A'\right)\left(K - K'\right) - \frac{1}{12}\bar{K}^{\varepsilon}t^3\left(A - A'\right)^2\right)$$
$$+A\left(\frac{1}{3}t^3\left(C - C'\right) + t^2\left(K - K'\right) - \frac{1}{3}\bar{K}^{\varepsilon}t^3\left(A - A'\right)\right) + \gamma t^2\left(A - A'\right)K$$

It can be shown that the trajectories correspond to an average path with initial conditions $\left(C\left(0\right), K\left(0\right), A\left(0\right)\right)$, and that for small interaction timepans, $\left(\dot{C}\left(0\right), \dot{K}\left(0\right), \dot{A}\left(0\right)\right)$ is modified at the first order in γ (see Appendix 6). Defining $\delta C\left(t\right)$, $\delta K\left(t\right)$ and $\delta A\left(t\right)$ the respective consumption, capital and technology deviations from their average paths (51) due to the interactions, we can write:

$$\begin{aligned}
\delta C\left(t\right) &= 0 \\
\delta K\left(t\right) &= \gamma\left(\frac{7ct^5}{720\bar{A}^2}C\left(0\right) + \frac{ct^4}{48\bar{A}^2}K\left(0\right) + \left(\frac{bt^3}{6K^{\varepsilon}\bar{A}^2} + \frac{K^{\varepsilon}ct^5}{90}\right)A\left(0\right)\right) \\
&\quad +\gamma\left(-\frac{7ct^6}{1440\bar{A}^2}\dot{C}\left(0\right) + \frac{ct^5}{60\bar{A}^2}\dot{K}\left(0\right) + \left(\frac{bt^4}{24K^{\varepsilon}\bar{A}^2} + \frac{3K^{\varepsilon}ct^6}{160\bar{A}^2}\right)\dot{A}\left(0\right)\right) \\
\delta A\left(t\right) &= \gamma\left(\frac{ct^3}{6K^{\varepsilon}\bar{A}^2}K\left(0\right)\right) + \gamma\left(\frac{ct^4}{24}\dot{K}\left(0\right) + t\dot{A}\left(0\right)\right)
\end{aligned} \tag{53}$$

where:

$$\begin{aligned}
b &= 2\left(\nu^2 + \frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2} + \frac{3\varpi^2}{2\left(2\alpha + \beta\right)\beta}\right) \\
c &= \frac{2}{\lambda^2}
\end{aligned}$$

The interpretation of (53) is the following. From (53) we find directly the impact of the agent's initial state on the deviations $\delta K\left(t\right)$ and $\delta A\left(t\right)$:

$$\begin{aligned}
\frac{\partial\left(\delta K\left(t\right)\right)}{\partial K\left(0\right)} &= \frac{ct^4}{48\bar{A}^2} > 0 \\
\frac{\partial\left(\delta K\left(t\right)\right)}{\partial A\left(0\right)} &= \frac{bt^3}{6K^{\varepsilon}\bar{A}^2} + \frac{K^{\varepsilon}ct^5}{90} > 0 \\
\frac{\partial\left(\delta A\left(t\right)\right)}{\partial K\left(0\right)} &= \frac{ct^3}{6K^{\varepsilon}\bar{A}^2} > 0
\end{aligned} \tag{54}$$

the interaction between individual capital and technology stocks produces a synergy effect, both stocks increase faster. This effect is proportional to the initial values of the stocks. The higher these initial individual values, the faster both stocks increase. Moreover, the polynomial time dependency of the elasticities shows that the accumulation dynamics is faster than a linear process. Remark that this synergy effect is not contradictory with the eviction effect described in the non-trivial phase. Actually, the results presented here are only valid at the individual level. In other words, the individual dynamics in a given phase does not detect the collective mechanisms of interactions. The latters are only detectable when analyzing the phase, and are indeed hidden in apparently exogenous parameters shaping the agents environment.

The accumulation dynamics is however dampened by fluctuations in technology stocks, measured by $\bar{A}^2\lambda^2$. The higher these fluctuations, the slower the accumulation process. The initial direction of the system, given by the terms proportional to $\dot{X}(0)$ amplify this synergy effect. Actually, we can also compute from (53) the impact of the agent's initial momentum on the deviations $\delta K(t)$ and $\delta A(t)$:

$$\begin{aligned}
\frac{\partial(\delta K(t))}{\partial \dot{K}(0)} &= \gamma \frac{ct^5}{60\bar{A}^2} > 0 \\
\frac{\partial(\delta K(t))}{\partial \dot{A}(0)} &= \gamma \frac{bt^4}{24K^{\varepsilon}\bar{A}^2} + \frac{3K^{\varepsilon}ct^6}{160\bar{A}^2} > 0 \\
\frac{\partial(\delta A(t))}{\partial \dot{A}(0)} &= \gamma t > 0 \\
\frac{\partial(\delta A(t))}{\partial \dot{K}(0)} &= \gamma \frac{ct^4}{24} > 0
\end{aligned}$$

a system that had started initially to accumulate both capital and technology stocks will accelerate faster compared to a system that was at first in a constant equilibrium.

The effect of the consumption initial value is ambiguous. Technology improvement, measured by the dynamics of A, increases productivity, and rates of return. It is thus optimal for agents to increase their savings and reduce their consumption. Capital stock is positively correlated to $C(0)$, as shown by its elasticity with respect to $C(0)$:

$$\frac{\partial(\delta K(t))}{\partial C(0)} = \gamma \frac{7ct^5}{720\bar{A}^2} > 0$$

In other word, a high level of initial consumption is an indicator of wealth. The agents interaction induce an accumulation process that favors capital expenditures. Since consumption elasticity with respect to consumption initial value $\frac{\partial(\delta C(t))}{\partial C(0)}$ is null, any increase of wealth is spent on capital stock**.**

Besides, any initial increase in the consumption rate impairs capital accumulation, since:

$$\frac{\partial(\delta K(t))}{\partial \dot{C}(0)} = -\gamma \frac{7ct^6}{1440\bar{A}^2} < 0$$

An initial increase in consumption will be smoothed over the entire timespan, and will eventually dampen the accumulation process.

3.2.4 Two agents transition functions and interaction

The field formalism presented here also allows the study of interaction between individual agents. Consider the simplest example of a two agents dynamics. Discarding interactions, the probability of transition between an initial state:

$$\left((K_1, C_1, A_1)_i, (K_2, C_2, A_2)_i\right)$$

and a final state:

$$\left((K_1, C_1, A_1)_f, (K_2, C_2, A_2)_f\right)$$

is simply the product of the transition probabilities (48) for each agent:

$$\begin{aligned}
& G\left((K_1, C_1, A_1)_i, (K_2, C_2, A_2)_i, (K_1, C_1, A_1)_f, (K_2, C_2, A_2)_f\right) \\
\equiv\ & G\left((K_1, C_1, A_1)_i, (K_2, C_2, A_2)_i, t\right) G\left((K_1, C_1, A_1)_f, (K_2, C_2, A_2)_f, t\right)
\end{aligned}$$

since they are considered to be independent. The global interaction effect, the impact of the entire system on each agent is included in the phase of the system, through the parameters of the transition probabilities.

When the interaction term is included, the transition probability has to be modified. In Appendix 6 compute the correction for the transition probability for two agents. We consider an initial state $\left((K_1, C_1, A_1)_i, (K_2, C_2, A_2)_i\right)$ and a final state $\left((K_1, C_1, A_1)_f, (K_2, C_2, A_2)_f\right)$.

In presence of the interaction term, appendix shows that:

$$I = \frac{\gamma}{2} \int \Psi^{\dagger}(K_1, C_1, A_1)\, \Psi^{\dagger}(K_2, C_2, A_2)(A_2K_1 + A_1K_2)\, \Psi(K_1, C_1, A_1)\, \Psi(K_2, C_2, A_2)$$

the transition probability is modified in the following way:

$$\begin{aligned} & G_I\left(\left(K_1,C_1,A_1,t\right)_i,\left(K_2,C_2,A_2,t\right)_i,\left(K_1,C_1,A_1,t\right)_f,\left(K_2,C_2,A_2,t\right)_f\right) \\ \equiv\ & G\left(\left(K_1,C_1,A_1\right)_i,\left(K_2,C_2,A_2\right)_i,\left(K_1,C_1,A_1\right)_f,\left(K_2,C_2,A_2\right)_f\right)\exp\left(-V_I\right) \end{aligned}$$

where:

$$V_I=\gamma t^2\left(\left\langle A_1\right\rangle\left\langle K_2\right\rangle+\left\langle K_1\right\rangle\left\langle A_2\right\rangle\right)+\frac{\gamma t^3}{24}\left(\left\langle A_1\right\rangle\Delta C_2+\Delta C_1\left\langle A_2\right\rangle-K^{\varepsilon}\left(\left\langle A_1\right\rangle\Delta A_2+\Delta A_1\left\langle A_2\right\rangle\right)\right)$$

with:

$$\begin{aligned} \left\langle X_j\right\rangle &= \frac{\left(X_j\right)_i+\left(X_j\right)_f}{2} \\ \Delta X_j &= \left(X_j\right)_f-\left(X_j\right)_i \end{aligned}$$

for any variable $X=C$, K, or A and agent $j=1$ or 2. The quantity $\left\langle X_j\right\rangle$ computes the average value of X for agent j along the path, and ΔX_j, the variation of X along this path.

Due to the interaction, the two agents' transition probabilities are now entangled. The average trajectory for one agent is modified by the other agent's path (see Appendix 6). We write $\delta X_{2\to1}\left(t\right)$ the correction of agent 1's trajectory due to agent 2 for $X=C$, K, or A, and $\delta X_{1\to2}\left(t\right)$ the correction of agent 2's trajectory due to agent 1. Appendix 6 shows that:

$$\begin{aligned} \delta K_{2\to1}\left(t\right) &= \gamma bt\left\langle A_2\right\rangle \\ \delta A_{2\to1}\left(t\right) &= \gamma\left(\frac{1}{6}c\frac{\Delta C_2}{2}-\frac{1}{6}cK^{\varepsilon}\frac{\Delta A_2}{2}\right)t^3+\gamma ct\left\langle K_2\right\rangle \\ \delta K_{1\to2}\left(t\right) &= \gamma bt\left\langle A_1\right\rangle \\ \delta A_{1\to2}\left(t\right) &= \gamma\left(\frac{1}{6}c\frac{\Delta C_1}{2}-\frac{1}{6}cK^{\varepsilon}\frac{\Delta A_1}{2}\right)t^3+\gamma ct\left\langle K_1\right\rangle \end{aligned} \tag{55}$$

with:

$$\begin{aligned} \bar{X}_i &= \frac{X_i\left(0\right)+X_i\left(t\right)}{2} \\ \frac{\Delta X_i}{2} &= \frac{X_i\left(t\right)-X_i\left(0\right)}{2} \end{aligned}$$

Formula (55) allows to find the dependency of an agent behavior on other agent's path. The elasticities are:

$$\frac{\partial\left(\delta K_1\left(t\right)\right)}{\partial\bar{A}_2\left(0\right)}=\frac{\partial\left(\delta A_1\left(t\right)\right)}{\partial\bar{K}_2\left(t\right)}=t>0\text{ (and other elasticities with respect to }\bar{X}_2\text{ are null)}$$

which means that the average technology of agent 2 impacts positively the accumulation of capital for agent 1, and that the accumulated stock of agent 2 accelerates the technology improvement for agent 1. Agent 2 participates to the environment of agent 1, and both its capital and technology stocks influence the other agents.

These elasticities are proportional to the interaction timespan: the longer agents interact, the higher the final stocks. The elasticities with respect to the initial direction of agent's 2 path may seem counterintuitive.

$$\begin{aligned} \frac{\partial\left(\delta A_1\left(t\right)\right)}{\partial\Delta A_2\left(0\right)} &= -\frac{1}{6}cK^{\varepsilon}t^3<0 \\ \frac{\partial\left(\delta A_1\left(t\right)\right)}{\partial C_2\left(t\right)} &= \frac{1}{6}ct^3>0 \end{aligned}$$

Technology stock is negatively correlated to other agents' accumulation rate. This is the consequence of the acceleration of accumulation process for both agents and our choice of representation of a path as function of the average value of the path $\bar{X}_2$: since the dynamic follows an accelerating pattern, its representative curve is below the average $\bar{X}_2$ most of the time. As a consequence, the accumulated stock are below the linear approximation in $\bar{X}_2$. The term proportional to $\frac{\Delta X_2}{2}$ is thus a correction to this linear approximation.

3.3 Synthesis and Discussion

The application of our formalism to a basic Business Cycle model has shown the implications of introducing multiple agents interacting through technology and capital stocks. This has allowed us to inspect setups not accessible to the usual representative agents models. It has also allowed to detect collective effects due to large number of agents, such as the appearance of multiple phases or macro equilibria. In turn, we have seen how these phases impact the individual agents' dynamics. These individual dynamics are formally identical to those used in representative agents models. It is at this point however that some major major differences with standard models appear:

In our formalism, the individual agents' dynamics are derived from a collective background. They emerge from the model, but cannot be imposed as a defining point of the model. This translates into several features.

First, the individual dynamics parameters are not exogenous. As mentioned above, they depend on the global system. A change of phase in the entire system induces a structural break that actually modifies the parameters of the agent's dynamic equations.

Second, the fact that individual behaviors emerge from the system extends to any subset of agents. Their dynamics and interactions, too, can and should be deduced from the collective background (see section 3.2.4). This allows a straightforward and detailed analysis of agents' interactions, while preserving the agents' heterogeneity.

As a result, and third, individual agents cannot be considered as representative agents. Section (3.2.3) demonstrates that individual features do not aggregate to produce similar effects at the macro level. The synergy effect in equation (54) shows that an agent may experience a virtuous circle between his capital and technology, even in the non-trivial phase characterized at a macro level by an eviction effect and a lower production. These two macro features are present at the individual level, but only as a hidden externality that shapes the agent's environment through seemingly exogenous parameters.

To conclude, the representative agent paradigm cannot detect some macro features from the description of particular agents. Some conclusions at the individual level do not agregate.

4 Conclusion

This paper has presented an analytical treatment of economic systems with an arbitrary number of agents that keeps track of the systems' interactions and agents' complexity. As significant results, we have shown that a field theory formalism may reveal some emerging equilibria, and studied the influence of these equilibria on the agent's individual dynamics. This method can be applied to various economic models.

In this paper we have, for the sake of clarity, deliberately set aside some matters developped in (Gosselin, Lotz, Wambst 2017), such as strategic behaviors and heterogeneity among agents, in information, goals, or actions. However, our formalism extends to such cases. Social interactions and economic networks could also be included. These subjects are under current research.

Ultimately, our formalism should shed some lights on the matter of aggregation. Indeed, despite the fact that field theory does not deal with aggregates, it allows to recover macroeconomic quantities through averages. A natural question would be to find even partial relations between these macroeconomic quantities.

References

Abergel F, Chakraborti A, Muni Toke I and Patriarca M (2011) Econophysics review: I. Empirical facts, Quantitative Finance, Vol. 11, No. 7, 991-1012.

Abergel F, Chakraborti A, Muni Toke I and Patriarca M (2011) Econophysics review: II. Agent-based models, Quantitative Finance, Vol. 11, No. 7, 1013-1041.

Lucas, Robert (1976) "Econometric Policy Evaluation: A Critique". In Brunner, K.; Meltzer, A. The Phillips Curve and Labor Markets. Carnegie-Rochester Conference Series on Public Policy. 1. New York: American Elsevier. pp. 19–46. ISBN 0-444-11007-0.

Lotz A (2011) An Economic Approach to the Self: the Dual Agent, Preprint, 2011. https://mpra.ub.uni-muenchen.de/50771/1/MPRA_paper_50771.pdf

Gosselin P and Lotz A (2012) A dynamic model of interactions between conscious and unconscious, Preprint, 2012. https://mpra.ub.uni-muenchen.de/36697/1/MPRA_paper_36697.pdf.

Gosselin P, Lotz A, and Wambst M. (2013) On Apparent Irrational Behavior : Interacting Structures and the Mind, Preprint, 2013. https://hal.archives-ouvertes.fr/hal-00851309/document

Gosselin P, Lotz A, and Wambst M (2015) From Rationality to Irrationality : Dynamic Interacting Structures, IF PREPUB. 2015. https://hal.archives-ouvertes.fr/hal-01122078/document

Gosselin P, Lotz A and Wambst M (2017) A Path Integral Approach to Interacting Economic Systems with Multiple Heterogeneous Agents. IF_PREPUB. 2017. hal-01549586v2

Kleinert H (1989) Gauge fields in condensed matter Vol. I , Superflow and vortex lines, Disorder Fields, Phase Transitions, Vol. II, Stresses and defects, Differential Geometry, Crystal Melting, World Scientific, Singapore 1989.

Gaffard J-L and Napoletano M Editors: Agent-based models and economic policy. Ofce 2012.

Jackson M (2010) Social and Economic Networks, Princeton University Press 2010.

Kleinert H (2009) Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets 5th edition, World Scientific, Singapore 2009.

Hamilton J. D. (1994) Time series analysis. Princeton University Press, 1994.

Zinn-Justin J (1993) Quantum Field Theory and Critical Phenomena, 2nd edition, Oxford Science Publications, 1993.

Peskin ME, Schroeder DV (1995), An introduction to Quantum Field Theory. Addison-Wesley Publishing Company 1995.

Romer, David. Advanced Macroeconomics. New York: McGraw-Hill Companies, 1996

Maurice Obstfeld & Kenneth S. Rogoff, 1996. "Foundations of International Macroeconomics," MIT Press Books, The MIT Press.

Appendix 1

We show the derivation of (12). We will remove the agent's index i since the argument deals with one individual agent.

If $Y(t+n)$ is centered on $\bar{Y}$ with variance σ^2, $\sum_{n>0} Y(t+n)$ will be centered on $\bar{Y}$ with variance $T\sigma^2$, and the integration over $Y(t+n)$ yields:

$$\begin{aligned}
&\int \prod dY(t+n) \exp\left(-\frac{1}{T}\left(B(t+1)+\sum_{n>0}\left(Y(t+n)-\bar{C}\right)\right)^2-\frac{1}{\sigma^2}\sum_{n=1}^{T}\left(Y(t+n)-\bar{Y}\right)^2\right)\\
= &\int \prod dY'(t+n) \exp\left(-\frac{1}{T}\left(B(t+1)+\sum_{n=1}^{T}\left(Y'(t+n)-\left(\bar{C}-\bar{Y}\right)\right)\right)^2-\frac{1}{\sigma^2}\sum_{n=1}^{T}\left(Y'(t+n)\right)^2\right)
\end{aligned}$$

with $Y'(t+n)=Y(t+n)-\bar{Y}$. The exponential rewrites:

$$\begin{aligned}
&\exp\left(-\frac{1}{T}\left(B(t+1)+\sum_{n=1}^{T}\left(Y'(t+n)-\left(\bar{C}-\bar{Y}\right)\right)\right)^2-\frac{1}{\sigma^2}\sum_{n=1}^{T}\left(Y'(t+n)\right)^2\right)\\
= &\exp\left(-\frac{\left(B(t+1)-T\left(\bar{C}-\bar{Y}\right)\right)^2}{T}-\frac{2\left(B(t+1)-T\left(\bar{C}-\bar{Y}\right)\right)}{T}\sum_{n=1}^{T}Y'(t+n)-\left(\frac{1}{\sigma^2}+\frac{1}{T}\right)\sum_{n=1}^{T}\left(Y'(t+n)\right)^2\right)
\end{aligned}$$

and the integration over the $Y'(t+n)$ leads to a weight:

$$\begin{aligned}
&\exp\left(-\frac{\left(B(t+1)-T\left(\bar{C}-\bar{Y}\right)\right)^2}{T}-\frac{2\left(B(t+1)-T\left(\bar{C}-\bar{Y}\right)\right)}{T}\sum_{n=1}^{T}Y'(t+n)-\left(\frac{1}{\sigma^2}+\frac{1}{T}\right)\sum_{n=1}^{T}\left(Y'(t+n)\right)^2\right)\\
= &\exp\left(-\frac{\left(B(t+1)-T\left(\bar{C}-\bar{Y}\right)\right)^2}{T}+\frac{\sigma^2\left(B(t+1)-T\left(\bar{C}-\bar{Y}\right)\right)^2}{T\left(\sigma^2+T\right)}\right)\\
= &\exp\left(-\frac{1}{T+\sigma^2}\left(B(t+1)-T\left(\bar{C}-\bar{Y}\right)\right)^2\right) \qquad (56)
\end{aligned}$$

We can now write $B(t+1)$ as a function of past variables:

$$B(t+1)=\sum_{n\leqslant 0}Y(t+n)-\sum_{n\leqslant 0}C(t+n) \qquad (57)$$

and, along with the expression $B_t+Y(t)-B(t+1)-\bar{C}=C(t)-\bar{C}$, write the global weight (56) as:

$$\begin{aligned}
&\exp\left(-\left(C(t)-\bar{C}\right)^2-\frac{1}{T+\sigma^2}\left(\sum_{n\leqslant 0}Y(t+n)-\sum_{n\leqslant 0}C(t+n)-T\left(\bar{C}-\bar{Y}\right)\right)^2\right) \qquad (58)\\
\simeq &\exp\left(-\left(C(t)-\bar{C}\right)^2-\frac{1}{T}\left(\sum_{n\leqslant 0}Y(t+n)-\sum_{n\leqslant 0}C(t+n)-T\left(\bar{C}-\bar{Y}\right)\right)^2\right)
\end{aligned}$$

for a large enough time scale, so that $T\gg\sigma^2$. Recall that terms in the exponential depend only on past variables, and do not modify the statistical weight. This statistical weigh can thus be written:

$$\begin{aligned}
&\exp\left(-\left(\frac{T+1}{T}\right)\left(C(t)-\frac{T}{T+1}\bar{C}-\frac{1}{T+1}\left(\sum_{n\leqslant 0}Y(t+n)-\sum_{n<0}C(t+n)-T\left(\bar{C}-\bar{Y}\right)\right)\right)^2\right)\\
= &\exp\left(-\left(\frac{T+1}{T}\right)\left(C(t)-\frac{1}{T+1}\left(\sum_{n\leqslant 0}Y(t+n)-\sum_{n<0}C(t+n)+T\bar{Y}\right)\right)^2\right)\\
= &\exp\left(-\left(\frac{T+1}{T}\right)\left(C(t)-\bar{Y}-\frac{1}{T+1}\left(\sum_{n\leqslant 0}\left(Y(t+n)-\bar{Y}\right)-\sum_{n<0}\left(C(t+n)-\bar{Y}\right)\right)\right)^2\right)
\end{aligned}$$

For $T \gg 1$, the last expression reduces to:

$$\exp\left(-\left(C(t)-\bar{Y}-\frac{1}{T}\left(\sum_{n\leqslant 0}\left(Y(t+n)-\bar{Y}\right)-\sum_{n<0}\left(C(t+n)-\bar{Y}\right)\right)\right)^2\right)$$

and defining $\hat{C}(t)=C(t)-\bar{Y}$, we are left with:

$$\exp\left(-\left(\hat{C}(t)-\frac{1}{T}\left(\sum_{n\leqslant 0}\hat{Y}(t+n)-\sum_{n<0}\hat{C}(t+n)\right)\right)^2\right) \tag{59}$$

To define the global statistical weight for the system over all periods, we can now sum the exponentiated terms in (59) to obtain the weight:

$$\exp\left(-\sum_{t=1}^{T}\left(\hat{C}(t)-\frac{1}{T}\left(\sum_{n\leqslant 0}\hat{Y}(t+n)-\sum_{n<0}\hat{C}(t+n)\right)\right)^2\right)$$

This weight describes the variables:

$$X(t)=\hat{C}(t)-\frac{1}{T}\left(\sum_{n\leqslant 0}\hat{Y}(t+n)-\sum_{n<0}\hat{C}(t+n)\right)$$

as gaussian and independent. It can now be written differently by remarking that:

$$\begin{aligned}
& X(t)-X(t+1)\\
= \;& \hat{C}(t)-\frac{1}{T}\left(\sum_{n\leqslant 0}\hat{Y}(t+n)-\sum_{n<0}\hat{C}(t+n)\right)-\left(\hat{C}(t+1)-\frac{1}{T}\left(\sum_{n\leqslant 0}\hat{Y}(t+1+n)-\sum_{n<0}\hat{C}(t+1+n)\right)\right)\\
= \;& \hat{C}(t)-\hat{C}(t+1)+\frac{1}{T}\left(\hat{Y}(t+1)-\hat{C}(t)\right)\\
= \;& \frac{T-1}{T}\hat{C}(t)-\hat{C}(t+1)+\frac{1}{T}\hat{Y}(t+1)
\end{aligned}$$

It then allows computing the density probability of:

$$\frac{T-1}{T}\hat{C}(t)-\hat{C}(t+1)+\frac{1}{T}\hat{Y}(t+1)$$

by writing:

$$\begin{aligned}
& \int \exp\left(-X^2(t)-X^2(t+1)\right)\\
& \qquad\qquad \times\delta\left(X(t)-X(t+1)-\left(\frac{T-1}{T}\hat{C}(t)-\hat{C}(t+1)+\frac{1}{T}\hat{Y}(t+1)\right)\right)dX(t)\,dX(t+1)\\
= \;& \int \exp\left(-X^2(t)-\left(X(t)-\left(\frac{T-1}{T}\hat{C}(t)-\hat{C}(t+1)+\frac{1}{T}\hat{Y}(t+1)\right)\right)^2\right)dX(t)\\
= \;& \exp\left(-\frac{\left(\frac{T-1}{T}\hat{C}(t)-\hat{C}(t+1)+\frac{1}{T}\hat{Y}(t+1)\right)^2}{2}\right)
\end{aligned}$$

For T large, $\frac{T-1}{T}\hat{C}(t)\simeq\hat{C}(t)$. This describes a brownian type process for $\hat{C}(t)$, with variance 2. This brownian motion is constrained to $X_T=0$ through the constraint:

$$X_T=\hat{C}(T)-\left(\sum_{n\leqslant 0}\hat{Y}(T+n)-\sum_{n<0}\hat{C}(T+n)\right)=\sum_{n\leqslant 0}\hat{Y}(T+n)-\sum_{n\leqslant 0}\hat{C}(T+n)=0$$

so that the overall weight becomes, in the $\hat{C}(t)$ representation and in first approximation for T large:

$$\begin{aligned}
& \exp\left(-\sum\left(\hat{C}(t)-\hat{C}(t+1)+\frac{1}{T}\hat{Y}(t+1)\right)^2-\left(\sum_{n\leqslant 0}\hat{Y}(T+n)-\sum_{n\leqslant 0}\hat{C}(T+n)\right)\right) \\
& \times\delta\left(\sum_{n\leqslant 0}\hat{Y}(T+n)-\sum_{n\leqslant 0}\hat{C}(T+n)\right) \\
\simeq\ & \exp\left(-\sum^{2}\left(\hat{C}(t)-\hat{C}(t+1)+\frac{1}{T}\hat{Y}(t+1)\right)-\frac{\left(\sum_{n\leqslant 0}\hat{Y}(T+n)-\sum_{n\leqslant 0}\hat{C}(T+n)\right)^2}{\sigma^2}\right) \\
\simeq\ & \exp\left(-\sum\left(\hat{C}(t)-\hat{C}(t+1)\right)^2-\frac{\left(\sum_{n\leqslant 0}\hat{Y}(T+n)-\sum_{n\leqslant 0}\hat{C}(T+n)\right)^2}{\sigma^2}\right) \qquad (60)
\end{aligned}$$

with $\sigma^2 \ll 1$. The second line allows for small deviations from the overall constraint.

Formula (60) is straightforward to generalize for agents with varying horizon. Actually, if the time horizon at time t is $T-t$, the statistical weight (59) at time t becomes:

$$\exp\left(-\left(\hat{C}(t)-\frac{1}{T-t}\left(\sum_{n\leqslant 0}\hat{Y}(t+n)-\sum_{n<0}\hat{C}(t+n)\right)\right)^2\right) \qquad (61)$$

As before, defining the variables

$$X(t)=\left(\hat{C}(t)-\frac{1}{T-t}\left(\sum_{n\leqslant 0}\hat{Y}(t+n)-\sum_{n<0}\hat{C}(t+n)\right)\right)$$

The expression (61) shows that the $X(t)$ are gaussian and independent. Computing $X(t)-\left(\frac{T-t-1}{T-t}\right)X(t+1)$ yields:

$$\begin{aligned}
& X(t)-\left(\frac{T-t-1}{T-t}\right)X(t+1) \\
=\ & \hat{C}(t)-\frac{1}{T-t}\left(\sum_{n\leqslant 0}\hat{Y}(t+n)-\sum_{n<0}\hat{C}(t+n)\right) \\
& -\left(\frac{T-t-1}{T-t}\right)\left(\hat{C}(t+1)-\frac{1}{T-t-1}\left(\sum_{n\leqslant 0}\hat{Y}(t+1+n)-\sum_{n<0}\hat{C}(t+1+n)\right)\right) \\
=\ & \hat{C}(t)-\left(\frac{T-t-1}{T-t}\right)\hat{C}(t+1)+\frac{1}{T-t}\left(\hat{Y}(t+1)-\hat{C}(t)\right) \\
=\ & \left(\frac{T-t-1}{T-t}\right)\left(\hat{C}(t)-\hat{C}(t+1)+\frac{1}{T-t}\hat{Y}(t+1)\right)
\end{aligned}$$

which implies that the probability density for $\hat{C}(t)-\hat{C}(t+1)+\frac{1}{T-t}\hat{Y}(t+1)$ can be computed by:

$$\begin{aligned}
& \int\exp\left(-X^2(t)-X^2(t+1)\right) \\
& \times\delta\left(\left(\frac{T-t}{T-t-1}\right)X_t-X(t+1)-\left(\hat{C}(t)-\hat{C}(t+1)+\frac{1}{T-t}\hat{Y}(t+1)\right)\right)dX_t dX(t+1) \\
=\ & \int\exp\left(-X_t^2-\left(\frac{T-t}{T-t-1}\right)^2\left(X_t-\left(\frac{T-t-1}{T-t}\right)\left(\hat{C}(t)-\hat{C}(t+1)+\frac{1}{T-t}\hat{Y}(t+1)\right)\right)^2\right)dX_t \\
=\ & \exp\left(-\frac{\left(\hat{C}(t)-\hat{C}(t+1)+\frac{1}{T-t}\hat{Y}(t+1)\right)^2}{\left(\frac{T-t}{T-t-1}\right)^2\left(1+\left(\frac{T-t}{T-t-1}\right)^2\right)}\right)
\end{aligned}$$

This describes a brownian type process for $\hat{C}(t)$ of variance $\left(\frac{T-t}{T-t-1}\right)^2\left(1+\left(\frac{T-t}{T-t-1}\right)^2\right)$. For T large, we have $\left(\frac{T-t}{T-t-1}\right)^2\left(1+\left(\frac{T-t}{T-t-1}\right)^2\right) \simeq 2$ and one recovers the brownian motion, for $t \ll T$. This brownian motion is constrained to $X_T = 0$ through the constraint:

$$X(T) = \hat{C}(T) - \left(\sum_{n\leqslant 0}\hat{Y}(T+n) - \sum_{n<0}\hat{C}(T+n)\right) = \sum_{n\leqslant 0}\hat{Y}(T+n) - \sum_{n\leqslant 0}\hat{C}(T+n) = 0$$

so that the overall weight in the $\hat{C}(t)$ representation becomes in first approximation for T large:

$$\begin{aligned}
&\exp\left(-\sum\left(\hat{C}(t) - \hat{C}(t+1) + \frac{\hat{Y}(t+1)}{T-t}\right)^2 - \left(\sum_{n\leqslant 0}\hat{Y}(T+n) - \sum_{n\leqslant 0}\hat{C}(T+n)\right)\right)\\
&\times\delta\left(\left(\sum_{n\leqslant 0}\hat{Y}(T+n) - \sum_{n\leqslant 0}\hat{C}(T+n)\right)\right)\\
\simeq\ &\exp\left(-\sum\left(\hat{C}(t) - \hat{C}(t+1) + \frac{\hat{Y}(t+1)}{T-t}\right)^2 - \frac{\left(\sum_{n\leqslant 0}\hat{Y}(T+n) - \sum_{n\leqslant 0}\hat{C}(T+n)\right)^2}{\bar{\sigma}^2}\right) \qquad (62)
\end{aligned}$$

The second line allows for small deviations from the overall constraint. Under a strictly binding constraint, $\bar{\sigma}^2 \ll 1$.

In the continuous version, we can replace the sum over t by an integral. Formulas (60) and (62) become:

$$\begin{aligned}
\exp\left(U^{eff}\right) &\equiv \exp\left(\int dt U^{eff}\left(\hat{C}(t)\right)\right)\\
&= \exp\left(-\int dt\left(\frac{d}{dt}\hat{C}(t) + \frac{1}{T-t}\hat{Y}_{t+1}\right)^2 - \frac{\left(\int dt\hat{Y}_t - \int dt\hat{C}(t)\right)^2}{\bar{\sigma}^2}\right)
\end{aligned}$$

The first term means (discarding the constraint) that $\frac{d}{dt}\hat{C}(t) + \frac{1}{T-t}\hat{Y}(t+1)$ is gaussian of variance 1. Thus, if $\hat{Y}(t+1)$ is considered purely random, one can consider that $\frac{d}{dt}\hat{C}(t)$ is gaussian with variance $1 + Var\left(\frac{1}{T-t}\hat{Y}(t+1)\right) = \frac{1}{\beta} > 1$ (for T large, we can consider β as constant) and replace the first term of U^{eff} by $\beta\left(\frac{d}{dt}\hat{C}(t)\right)^2$ to obtain:

$$\exp U^{eff} = \exp\left(-\beta\int dt\left(\frac{d}{dt}\hat{C}(t)\right)^2 - \frac{\left(\int dt\hat{Y}(t) - \int dt\hat{C}(t)\right)^2}{\bar{\sigma}^2}\right) \qquad (63)$$

As a consequence of (60), (62) and (63), the introduction of a constraint is equivalent to the introduction of non-local interaction terms. The non-local terms may, in some cases, be approximated by some terms in the derivatives of $C(t)$ (see Gosselin, Lotz Wambst 2018).

Appendix 2

We consider some constraints within the context of non quadratic utilities. To do so, we start with a simple example and consider the budgent constraint (10) for a single agent:

$$C(t) = B(t) + Y(t) - B(t+1) \qquad (64)$$

At time t, the agent's statistical weight has the general form (11):

$$\int\prod_{n>1}\exp\left(U(B(t) + Y(t) - B(t+1)) + \sum_{n>0}U(B(t+n) + Y(t+n) - B(t+n+1))\right)dB(t+n) \qquad (65)$$

Performing the following change of variables for $n > 1$:

$$\begin{aligned}
B(t+n) &\rightarrow B(t+n) - \sum_{m\geqslant n}Y(t+m)\\
B(t+n) + Y(t+n) - B(t+n) &\rightarrow B(t+n) - B(t+1+n)
\end{aligned}$$

the statistical weight (65) become:

$$\int \prod_{n>1} \exp(U(B(t)+Y(t)-B(t+1))+U(B(t+1)-B(t+2)+\sum_{n\geqslant 1} Y(t+n))+\sum_{n>1} U(B(t+n)-B(t+1+m)))dB(t+n) \quad (66)$$

Except for the case of a quadratic utility function, the successive integrals

$$\int \prod_{n>1} \exp\left(U\left(B\left(t+1\right)-B\left(t+2\right)+\sum_{n\geqslant 1} Y\left(t+n\right)\right)+\sum_{n>1} U\left(B\left(t+n\right)-B\left(t+1+n\right)\right)\right) dB\left(t+n\right) \quad (67)$$

arising in (66) cannot be computed exactly, . However, we can still define a function $\check{U}\left(B\left(t+1\right)\right)$ resulting from the convolution integrals (67):

$$\begin{aligned} & \exp\left(\check{U}\left(B\left(t+1\right)+\sum_{n\geqslant 1} Y\left(t+n\right)\right)\right) \quad (68)\\ = & \int \exp\left(U\left(B\left(t+1\right)-B\left(t+2\right)+\sum_{n\geqslant 1} Y\left(t+n\right)\right)+\sum_{n>1} U\left(B\left(t+n\right)-B\left(t+n+1\right)\right)\right) \prod_{n>1} dB\left(t+n\right) \end{aligned}$$

The function $\check{U}$ can be approximatively computed - we will comment on that later in the paragraph - however its precise form is not needed here. Instead, we use the general formula (68) to write (66) as:

$$\exp\left(U\left(B\left(t\right)+Y\left(t\right)-B\left(t+1\right)\right)+\check{U}\left(B\left(t+1\right)+\sum_{n\geqslant 1} Y\left(t+n\right)\right)\right) \quad (69)$$

Here again (see the first paragraph of this section), we can get rid of the variables $Y\left(t+n\right)$ by considering them to be gaussian random variables centered on $\bar{Y}$ for $n \geqslant 1$. The transition probability for $B\left(t\right)$ is obtained by integrating (69) over the variables $Y\left(t+n\right)$:

$$\int \prod dY\left(t+n\right) \exp\left(U\left(B\left(t\right)+Y\left(t\right)-B\left(t+1\right)\right)+\check{U}\left(B\left(t+1\right)+\sum_{n\geqslant 1} Y\left(t+n\right)\right)-\frac{1}{\sigma^2}\sum_{n>0}\left(Y\left(t+n\right)-\bar{Y}\right)^2\right) \quad (70)$$

This expression can be simplified. Actually, in the gaussian integrals:

$$\int \prod dY\left(t+n\right) \exp\left(\check{U}\left(B\left(t+1\right)+\sum_{n\geqslant 1} Y\left(t+n\right)\right)-\frac{1}{\sigma^2}\sum_{n>0}\left(Y\left(t+n\right)-\bar{Y}\right)^2\right) \quad (71)$$

the variable $\sum_{n\geqslant 1} Y\left(t+n\right)$ has mean $T\bar{Y}$ and variance $T\sigma^2$. As a consequence, if we assume T large enough so that $\sqrt{T} \gg \sigma$, then

$$\sum_{n\geqslant 1} Y\left(t+n\right) \simeq T\bar{Y} \pm \sqrt{T}\sigma \simeq T\bar{Y}$$

in first approximation. This allows to simplify (71):

$$\begin{aligned} & \int \prod dY\left(t+n\right) \exp\left(\check{U}\left(B\left(t+1\right)+\sum_{n\geqslant 1} Y\left(t+n\right)\right)-\frac{1}{\sigma^2}\sum_{n>0}\left(Y\left(t+n\right)-\bar{Y}\right)^2\right)\\ \simeq & \exp\left(\check{U}\left(B\left(t+1\right)+T\bar{Y}\right)\right) \end{aligned}$$

so that, using the constraint (64) to write $B\left(t+1\right)$ as a function of the past variables:

$$\begin{aligned} B\left(t+1\right)+\sum_{n\geqslant 1} Y\left(t+n\right) &= \sum_{n\leqslant 0} Y\left(t+n\right)-\sum_{n\leqslant 0} C\left(t+n\right)+\sum_{n\geqslant 1} Y\left(t+n\right)\\ &\simeq \sum_{n\leqslant 0} Y\left(t+n\right)-\sum_{n\leqslant 0} C\left(t+n\right)+T\bar{Y} \end{aligned}$$

the weight (70) results in:

$$\int \prod dY(t+n) \exp\left(U\left(B(t)+Y(t)-B(t+1)\right)+\check{U}\left(B(t+1)+\sum_{n\geqslant 1} Y(t+1)\right)-\frac{\sum_{n>0}\left(Y(t+n)-\bar{Y}\right)^2}{\sigma^2}\right) \quad (72)$$

$$\simeq \exp\left(U\left(B\left(t\right)+Y\left(t\right)-B\left(t+1\right)\right)+\check{U}\left(\sum_{n\leqslant 0} Y(t+n)-\sum_{n\leqslant 0} C\left(t+n\right)+T\bar{Y}\right)\right)$$

We can consider that the term:

$$\sum_{n\leqslant 0} Y\left(t+n\right)-\sum_{n\leqslant 0} C\left(t+n\right)+T\bar{Y}$$

has relatively small fluctuations with respect to its average $T\bar{Y}$, we can approximate $\check{U}$ by its second order expansion:

$$\check{U}\left(\sum_{n\leqslant 0} Y\left(t+n\right)-\sum_{n\leqslant 0} C\left(t+n\right)+T\bar{Y}\right)\simeq C-\gamma\left(\sum_{n\leqslant 0} Y\left(t+n\right)-\sum_{n\leqslant 0} C\left(t+n\right)+\bar{Y}\right)^2$$

the values of C and γ depending on (68). Then, up to the irrelevant constant C, (72) simplifies to the second order approximation:

$$\exp\left(U\left(B\left(t\right)+Y\left(t\right)-B\left(t+1\right)\right)-\gamma\left(\sum_{n\leqslant 0} Y\left(t+n\right)-\sum_{n\leqslant 0} C\left(t+n\right)+\bar{Y}\right)^2\right) \quad (73)$$

$$= \exp\left(U\left(C\left(t\right)\right)-\gamma\left(\sum_{n\leqslant 0} Y\left(t+n\right)-\sum_{n\leqslant 0} C\left(t+n\right)+\bar{Y}\right)^2\right)$$

a result similar to the first example of this section. The constraint can be introduced as a quadratic and non local contribution to the utility $U\left(C_t\right)$. This result is not surprising. The constraint being imposed on the whole path of the system, the inclusion of its intertemporal quadratic expansion enforces the constraint on average, as needed. The result is similar to (58), except that the quadratic utility has been replaced by a more general function. In first approximation, one can thus share $U\left(C\left(t\right)\right)$ in a qudratic approximation plus some perturbative terms $V\left(C\left(t\right)\right)$. Then proceeding with the quadratic term as we did in Appendix 1 to define the statistical weight, one recovers a formula similar to (62) for the consumption path:

$$\exp\left(-\sum\left(\left(\hat{C}\left(t\right)-\hat{C}\left(t+1\right)\right)^2+V\left(C\left(t\right)\right)\right)-\frac{\left(\sum_{n\leqslant 0}\hat{Y}\left(T+n\right)-\sum_{n\leqslant 0}\hat{C}\left(T+n\right)\right)^2}{\bar{\sigma}^2}\right)$$

where $\frac{\hat{Y}(t+1)}{T-t}$ has been neglected for $T\gg 1$.

Let us close this section by quickly discussing the form of the function $\check{U}$ defined by (68):

$$\exp\left(\check{U}\left(B\left(t+1\right)+\sum_{n\geqslant 1} Y\left(t+n\right)\right)\right) \quad (74)$$

$$= \int \exp\left(U\left(B\left(t+1\right)-B\left(t+2\right)+\sum_{n\geqslant 1} Y\left(t+n\right)\right)+\sum_{n>1} U\left(B\left(t+n\right)-B_{t+n+1}\right)\right)\prod_{n>1} dB\left(t+n\right)$$

These integrals can be approximatively computed with the saddle path approximation technique developed in the first and second sections. The saddle path result is not exact for a non quadratic utility, but constitutes a sufficient approximation for us. The saddle path (74) for the function inside the exponential can be written as a difference equation $B\left(t+n\right)$ with $n>1$:

$$U'\left(B\left(t+n\right)-B\left(t+n+1\right)\right)-U'\left(B\left(t+n-1\right)-B\left(t+n\right)\right)=0 \text{ for } n>2$$

and:

$$U'\left(B\left(t+1\right)-B\left(t+2\right)\right)-U\left(B\left(t+1\right)-B\left(t+2\right)+\sum_{m\geqslant 1} Y\left(t+m\right)\right) \text{ for } n=2$$

Once the saddle path $\bar{B}\left(t+n\right)$ is found, it can be introduced in (74) to yield:

$$\exp\left(\check{U}\left(B\left(t+1\right)+\sum_{n\geqslant 1} Y\left(t+n\right)\right)\right)=\exp\left(U\left(B\left(t+1\right)-\bar{B}_{t+2}+\sum_{n\geqslant 1} Y\left(t+n\right)\right)+\sum_{n>1} U\left(\bar{B}\left(t+n\right)-\bar{B}\left(t+1+n\right)\right)\right)$$

and a first approximation for $\check{U}$ is thus:

$$\check{U}\left(B\left(t+1\right)+\sum_{n\geqslant 1}Y\left(t+n\right)\right)=U\left(B\left(t+1\right)-\bar{B}\left(t+2\right)+\sum_{n\geqslant 1}Y\left(t+n\right)\right)+\sum_{n>1}U\left(\bar{B}\left(t+n\right)-\bar{B}\left(t+1+n\right)\right) \quad (75)$$

Some corrections to the saddle path can be included if we expand the RHS to the second order around the saddle point by letting

$$B\left(t+n\right)=\bar{B}\left(t+n+1\right)+\delta B\left(t+n\right)$$

and then integrate over $\delta B\left(t+n\right)$:

$$\begin{aligned}
&\exp\left(\check{U}\left(B\left(t+1\right)+\sum_{n\geqslant 1}Y\left(t+n\right)\right)\right) \quad (76)\\
=\ &\exp\left(U\left(B\left(t+1\right)-\bar{B}\left(t+2\right)+\sum_{n\geqslant 1}Y\left(t+n\right)\right)+\sum_{n>1}U\left(\bar{B}\left(t+n\right)-\bar{B}\left(t+n+1\right)\right)\right)\\
&\times\int\exp\left(U''\left(B\left(t+1\right)-\bar{B}\left(t+2\right)+\sum_{n\geqslant 1}Y\left(t+n\right)\right)\left(\delta B\left(t+2\right)\right)^{2}\right.\\
&\qquad\left.+\sum_{n>1}U''\left(\bar{B}\left(t+n\right)-\bar{B}\left(t+n+1\right)\right)\left(\delta B\left(t+n\right)-\delta B\left(t+n\right)\right)^{2}\right)\prod_{n>1}d\delta B\left(t+n\right)
\end{aligned}$$

The log of the integrals in (76) will yield some corrections to (75), but we will not inspect further the precise form of these corrections.

Appendix 3

When some discount rate is introduced, we go back to the initial individual agent formulation and modify it accordingly. Recall that the transition probabilities between two consecutive state variables of the system are defined by

(**??**) with a discount rate β added:

$$P\left(B_i\left(t\right),B_i\left(t+1\right)\right)=\int\prod_{n=2}^{T}dB\left(t+n\right)\exp\left(U\left(C\left(t\right)\right)+\sum_{n>0}\beta^{n}U\left(C\left(t+n\right)\right)\right)$$

but now, the constraint rewrites:

$$B\left(t+1\right)=\left(1+r\right)\left(B\left(t\right)+Y\left(t\right)-C\left(t\right)\right)$$

or equivalently:

$$C\left(t\right)=B\left(t\right)+Y\left(t\right)-\frac{B\left(t+1\right)}{\left(1+r\right)}$$

Then, the integral over the $B\left(t+n\right)$ is similar to the previous one, since one can change the variables: $\frac{B(t+n)}{(1+r)^{n}}\rightarrow B\left(t+n\right)$ for $n>1$. We assume that the uncertainty about future periods increases with a factor $\left(1+r\right)$.

$$
\begin{aligned}
&= \int \prod_{n=2}^{T} dB\left(t+n\right) \exp\left(-\left(C\left(t\right)-\bar{C}\right)^{2} - \sum_{n>0}\left(C\left(t+n\right)-\bar{C}\right)^{2} \right) \\
&= \int \prod_{n=2}^{T} dB\left(t+n\right) \exp\left(-\left(B\left(t\right)+Y\left(t\right)-\frac{B\left(t+1\right)}{\left(1+r\right)}-\bar{C}\right)^{2} \right. \\
&\qquad \left. -\sum_{n>0}\left(\frac{\beta}{\left(1+r\right)}\right)^{n}\left(B\left(t+n\right)+Y\left(t+n\right)-\frac{B\left(t+n+1\right)}{\left(1+r\right)}-\bar{C}\right)^{2} \right) \\
&= \int \prod_{n=2}^{T} \left(1+r\right)^{n} dB\left(t+n\right)' \exp\left(-\left(B\left(t\right)+Y\left(t\right)-\frac{B\left(t+1\right)}{\left(1+r\right)}-\bar{C}\right)^{2} \right. \\
&\qquad \left. -\sum_{n>0}\left(1+r\right)^{n}\beta^{n}\left(B'\left(t+n\right)-B'\left(t+n+1\right)+\frac{Y\left(t+n\right)-\bar{C}}{\left(1+r\right)^{n}}\right)^{2} \right) \\
&= \left(\prod_{n=2}^{T}\left(1+r\right)^{n}\right) \exp\left(-\left(B\left(t\right)+Y\left(t\right)-\frac{B\left(t+1\right)}{\left(1+r\right)}-\bar{C}\right)^{2} \right. \\
&\qquad \left. -\frac{1}{\sum_{n>0}\left(\beta\left(1+r\right)\right)^{-n}}\left(\frac{B\left(t+1\right)}{\left(1+r\right)}+\sum_{n>0}\frac{Y\left(t+n\right)-\bar{C}}{\left(1+r\right)^{n}}\right)^{2} \right) \\
&= \left(\prod_{n=2}^{T}\left(1+r\right)^{n}\right) \exp\left(-\left(B\left(t\right)+Y\left(t\right)-\frac{B\left(t+1\right)}{\left(1+r\right)}-\bar{C}\right)^{2} - s\left(T\right)\left(\frac{B\left(t+1\right)}{\left(1+r\right)}+\sum_{n>0}\frac{Y\left(t+n\right)-\bar{C}}{\left(1+r\right)^{n}}\right)^{2} \right)
\end{aligned}
$$

where the sum has been performed up to T where T is the time horizon defined previously and $T \gg 1$, and $s^{-1}\left(T\right)=\sum_{n>0}\left(\beta\left(1+r\right)^{2}\right)^{-n}$. Since $\beta<1$:

$$
s^{-1}\left(T\right) > \sum_{n>0}\left(\left(1+r\right)\right)^{-n} \simeq \frac{1}{r}
$$

for $r \ll 1$. As a consequence, $s\left(T\right)<r$. For $\beta=0$, $s^{-1}\left(T\right)=\frac{1}{r}$.

The factor $\prod_{n=2}^{T}\left(1+r\right)^{n}$ can be included in the normalization factor, as explained before, and then we are left with:

$$
\begin{aligned}
P\left(B_i\left(t\right),B_i\left(t+1\right)\right) &= \int \prod_{n=2}^{T} dB\left(t+n\right) \exp\left(U\left(C\left(t\right)\right)+\sum_{n>0}U\left(C\left(t+n\right)\right)\right) \qquad (77) \\
&= \exp\left(-\left(B\left(t\right)+Y\left(t\right)-\frac{B\left(t+1\right)}{\left(1+r\right)}-\bar{C}\right)^{2} - s\left(T\right)\left(\frac{B\left(t+1\right)}{\left(1+r\right)}+\sum_{n>0}\frac{Y\left(t+n\right)-\bar{C}}{\left(1+r\right)^{n}}\right)^{2} \right)
\end{aligned}
$$

which is similar to (56), except the $\frac{1}{(1+r)}$ factor in front of $B\left(t+1\right)$ and the $\left(1+r\right)^{n}$ multiplying $\left(Y\left(t+n\right)-\bar{C}\right)$. One also replaces T by $\frac{1}{s(T)}$. Then the previous analysis following (56) applies, except that, writing $B\left(t+1\right)$ as a function of the past is now:

$$
\frac{B\left(t+1\right)}{1+r} = \sum_{n\leqslant 0}\frac{Y\left(t+n\right)}{\left(1+r\right)^{n}} - \sum_{n\leqslant 0}\frac{C\left(t+n\right)}{\left(1+r\right)^{n}} \qquad (78)
$$

with $B\left(t\right)\to 0$, $t\to T$ to impose the transversality condition. The number of periods, T, is itself unknown, but as said before T is the expected mean process duration.

If $Y\left(t+n\right)$ is centered on $\bar{Y}$ with variance $\left(1+r\right)^{2n}\sigma^{2}$ (we assume that the discounted variable $\frac{Y(t+n)}{(1+r)^{n}}$ has a constant variance σ^{2}), $\sum_{n>0}Y\left(t+n\right)$ centered on $\bar{Y}$ with variance $T\sigma^{2}$, integration over $Y\left(t+n\right)$ yields:

$$
\begin{aligned}
&\int \prod dY\left(t+n\right) \exp\left(-s\left(T\right)\left(\frac{B\left(t+1\right)}{\left(1+r\right)}+\sum_{n>0}\frac{Y\left(t+n\right)-\bar{C}}{\left(1+r\right)^{n}}\right)^{2} - \frac{1}{\sigma^{2}}\sum_{n=1}^{T}\left(Y\left(t+n\right)-\bar{Y}\right)^{2} \right) \\
&= \int \prod dY'\left(t+n\right) \exp\left(-s\left(T\right)\left(\frac{B\left(t+1\right)}{\left(1+r\right)}+\sum_{n>0}\frac{Y'\left(t+n\right)+\bar{Y}-\bar{C}}{\left(1+r\right)^{n}}\right)^{2} - \frac{1}{\sigma^{2}}\sum_{n=1}^{T}\left(Y\left(t+n\right)-\bar{Y}\right)^{2} \right)
\end{aligned}
$$

with $Y'(t+n) = Y(t+n) - \bar{Y}$. Neglecting the terms $Y'(t+n)Y'(t+m)$ for $n \neq m$, since they are null in expectations, the exponential rewrites (for a time horizon $T \gg 1$):

$$\begin{aligned}
& \exp\left(-s(T)\left(\frac{B(t+1)}{(1+r)} + \sum_{n>0} \frac{Y'(t+n) + \bar{Y} - \bar{C}}{(1+r)^n}\right)^2 - \frac{1}{\sigma^2}\sum_{n=1}^{T}\left(Y(t+n) - \bar{Y}\right)^2\right) \\
= \quad & \exp\left(\left(-s(T)\left(\frac{B(t+1)}{(1+r)} + \frac{1}{r}\left(\bar{Y} - \bar{C}\right)\right)^2\right)\right. \\
& \left. -2s(T)\left(\frac{B(t+1)}{(1+r)} + \frac{1}{r}\left(\bar{Y} - \bar{C}\right)\right)\sum_{n=1}^{T}\frac{Y'(t+n)}{(1+r)^n} - \sum_{n=1}^{T}\left(\frac{1}{\sigma^2} + \frac{s(T)}{(1+r)^{2n}}\right)\left(Y(t+n) - \bar{Y}\right)^2\right) \\
\simeq \quad & \exp\left(-s(T)\left(\frac{B(t+1)}{(1+r)} + \frac{1}{r}\left(\bar{Y} - \bar{C}\right)\right)^2\right. \\
& \left. -2s(T)\sum_{n=1}^{T}\left(\frac{B(t+1)}{(1+r)} + \frac{1}{r}\left(\bar{Y} - \bar{C}\right)\right)\frac{Y'(t+n)}{(1+r)^n} - \frac{1}{\sigma^2}\sum_{n=1}^{T}\left(Y(t+n) - \bar{Y}\right)^2\right)
\end{aligned}$$

for $\sigma^2 < 1$ and since $s(T) < r \ll 1$.
and the integration over the $Y'(t+n)$ leads to a weight:

$$\begin{aligned}
& \exp\left(-s(T)\left(\frac{B(t+1)}{(1+r)} + \frac{1}{r}\left(\bar{Y} - \bar{C}\right)\right)^2 + \frac{\sigma^2 s^2(T)(1+r)}{r}\left(\frac{B(t+1)}{(1+r)} + \frac{1}{r}\left(\bar{Y} - \bar{C}\right)\right)^2\right) \\
\simeq \quad & \exp\left(-s(T)\left(\frac{B(t+1)}{(1+r)} + \frac{1}{r}\left(\bar{Y} - \bar{C}\right)\right)^2\right)
\end{aligned}$$

since $s(T) \ll r$ and thus $\frac{\sigma^2 s^2(T)(1+r)}{r} \ll s(T)\sigma^2(1+r) \ll s(T)$. Using that:

$$\frac{B(t+1)}{1+r} = \sum_{n\leqslant 0}\frac{Y(t+n)}{(1+r)^n} - \sum_{n\leqslant 0}\frac{C(t+n)}{(1+r)^n}$$

the weight can be written:

$$\begin{aligned}
& \exp\left(-\left(B(t) + Y(t) - \frac{B(t+1)}{(1+r)} - \bar{C}\right)^2 - s(T)\left(\sum_{n\leqslant 0}\frac{Y(t+n)}{(1+r)^n} - \sum_{n\leqslant 0}\frac{C(t+n)}{(1+r)^n} + \frac{1}{r}\left(\bar{Y} - \bar{C}\right)\right)^2\right) \\
= \quad & \exp\left(-\left(C(t) - \bar{C}\right)^2 - s(T)\left(\sum_{n\leqslant 0}\frac{Y(t+n)}{(1+r)^n} - \sum_{n\leqslant 0}\frac{C(t+n)}{(1+r)^n} + \frac{1}{r}\left(\bar{Y} - \bar{C}\right)\right)^2\right) \\
= \quad & \exp\left(\begin{array}{c} -(1+s(T))\left(C(t) - \bar{C}\right)^2 - s(T)\left(\sum_{n\leqslant 0}\frac{Y(t+n)}{(1+r)^n} - \sum_{n<0}\frac{C(t+n)}{(1+r)^n} - \bar{C} + \frac{1}{r}\left(\bar{Y} - \bar{C}\right)\right)^2 \\ +2s(T)\left(\sum_{n\leqslant 0}\frac{Y(t+n)}{(1+r)^n} - \sum_{n<0}\frac{C(t+n)}{(1+r)^n} - \bar{C} - \frac{1}{r}\left(\bar{C} - \bar{Y}\right)\right)\left(C(t) - \bar{C}\right) \end{array}\right) \\
\simeq \quad & \exp\left(\begin{array}{c} -\left(C(t) - \bar{C}\right)^2 - s(T)\left(\sum_{n\leqslant 0}\frac{Y(t+n)}{(1+r)^n} - \sum_{n<0}\frac{C(t+n)}{(1+r)^n} - \frac{\bar{C}}{(1+r)} + \frac{1}{r}\left(\bar{Y} - \bar{C}\right)\right)^2 \\ +2s(T)\left(\sum_{n\leqslant 0}\frac{Y(t+n)}{(1+r)^n} - \sum_{n<0}\frac{C(t+n)}{(1+r)^n} - \bar{C} - \frac{1}{r}\left(\bar{C} - \bar{Y}\right)\right)\left(C(t) - \bar{C}\right) \end{array}\right)
\end{aligned}$$

as in the text, the terms in the exponential depending only of past and predetermined variables are irrelevant to the statistical weight, so that this one can be written:

$$\begin{aligned}
& \exp\left(-\left(C(t) - \bar{C} - s(T)\left(\sum_{n\leqslant 0}\frac{Y(t+n)}{(1+r)^n} - \sum_{n<0}\frac{C(t+n)}{(1+r)^n} - \bar{C} + \frac{1}{r}\left(\bar{Y} - \bar{C}\right)\right)\right)^2\right) \\
= \quad & \exp\left(-\left(\left(C(t) - \bar{C} + s(T)\left(\bar{C} - \frac{1}{r}\left(\bar{Y} - \bar{C}\right)\right)\right) - s(T)\left(\sum_{n\leqslant 0}\frac{Y(t+n)}{(1+r)^n} - \sum_{n<0}\frac{C(t+n)}{(1+r)^n}\right)\right)^2\right) \\
\simeq \quad & \exp\left(-\left(\hat{C}(t) - s(T)\left(\sum_{n\leqslant 0}\frac{\hat{Y}(t+n)}{(1+r)^n} - \sum_{n<0}\frac{\hat{C}(t+n)}{(1+r)^n}\right)\right)^2\right)
\end{aligned}$$

for $s(T) \ll r$, with $\hat{Y}(t+n) = Y(t+n) - \bar{C} + s(T)\left(\bar{C} - \frac{1}{r}\left(\bar{Y} - \bar{C}\right)\right)$ for $n \leqslant 0$ and $\hat{C}(t+n) = C(t+n) + s(T)\left(\bar{C} - \frac{1}{r}\left(\bar{Y} - \bar{C}\right)\right)$ for $n \leqslant 0$. For $\beta = 1$, $s(T) \simeq r$, and then $\hat{Y}(t+n) = Y(t+n) - \left(\bar{Y} - r\bar{C}\right)$, $\hat{C}(t+n) = C(t+n) - r\left(\bar{Y} - r\bar{C}\right)$. In first approximation, $\hat{Y}(t+n) = Y(t+n)$, $\hat{C}(t+n) = C(t+n)$.

We can now proceed as in the derivation of (60), and switch the representation to express the probabilies for the variables. This weight can be written differently. Actually, it describes the variables

$$X(t) = \hat{C}(t) - \frac{s(T)}{(1+r)}\left(\sum_{n\leqslant 0}\frac{\hat{Y}(t+n)}{(1+r)^n} - \sum_{n<0}\frac{\hat{C}(t+n)}{(1+r)^n}\right)$$

as gaussian and independent. Now, remark that at the first order in r:

$$\begin{aligned}
& (1+r)X(t) - X(t+1) \\
= \;& (1+r)\hat{C}(t) - \frac{s(T)}{(1+r)}\left(\sum_{n\leqslant 0}\frac{\hat{Y}(t+n)}{(1+r)^n} - \sum_{n<0}\frac{\hat{C}(t+n)}{(1+r)^n}\right) \\
& - \left(\hat{C}(t+1) - \frac{s(T)}{(1+r)}\left(\sum_{n\leqslant 0}\frac{\hat{Y}(t+1+n)}{(1+r)^n} - \sum_{n<0}\frac{\hat{C}(t+1+n)}{(1+r)^n}\right)\right) \\
\simeq \;& (1+r)\hat{C}(t) - \hat{C}(t+1) + s(T)\left(\frac{\hat{Y}(t+1)}{(1+r)} - \hat{C}(t)\right) \\
= \;& (1+r-s(T))\hat{C}(t) - \hat{C}(t+1) + \frac{s(T)}{(1+r)}\hat{Y}(t+1) \\
\simeq \;& (1+r)\hat{C}(t) - \hat{C}(t+1) + \frac{s(T)}{(1+r)}\hat{Y}(t+1)
\end{aligned}$$

It then allows to compute the density probability of:

$$(1+r)\hat{C}(t) - \hat{C}(t+1) + \frac{s(T)}{(1+r)}\hat{Y}(t+1)$$

by writing:

$$\begin{aligned}
& \int \exp\left(-X^2(t) - X^2(t+1)\right) \\
& \times\delta\left((1+r)X(t) - X(t+1) - \left((1+r)\hat{C}(t) - \hat{C}(t+1) + \frac{s(T)}{(1+r)}\hat{Y}(t+1)\right)\right)dX(t)\,dX(t+1) \\
= \;& \int \exp\left(-X^2(t) - \left((1+r)X(t) - \left((1+r)\hat{C}(t) - \hat{C}(t+1) + \frac{s(T)}{(1+r)}\hat{Y}(t+1)\right)\right)^2\right)dX(t) \\
= \;& \exp\left(-\frac{\left(\hat{C}(t) - \frac{\hat{C}(t+1)}{(1+r)} + \frac{s(T)}{(1+r)}\hat{Y}(t+1)\right)^2}{1+(1+r)^2}\right) \\
\simeq \;& \exp\left(-\frac{\left(\hat{C}(t) - \hat{C}(t+1)(1-r) + s(T)\hat{Y}(t+1)\right)^2}{2}\right)
\end{aligned}$$

This stochastic process is constrained to $X_T = 0$ through the constraint:

$$X(T) = \hat{C}(T) - \left(\sum_{0\leqslant n\leqslant T}\frac{\hat{Y}(n)}{(1+r)^n} - \sum_{0\leqslant n\leqslant T}\frac{\hat{C}(n)}{(1+r)^n}\right) = \sum_{0\leqslant n\leqslant T}\frac{\hat{Y}(n)}{(1+r)^n} - \sum_{0\leqslant i\leqslant T}\frac{\hat{C}(n)}{(1+r)^n} = 0$$

Given that $s(T)\hat{Y}(t+1)$ has variance $s^2(T)\sigma^2 \ll 1$, including the global constraint yields the statistical weight over all periods:

$$\begin{aligned}
& \exp\left(-\sum_t \frac{\left(\hat{C}(t) - \hat{C}(t+1)(1-r)\right)^2}{2\left(1+s^2(T)\sigma^2\right)}\right)\delta\left(\sum_{0\leqslant n\leqslant T}\frac{\hat{Y}(n)}{(1+r)^n} - \sum_{0\leqslant n\leqslant T}\frac{\hat{C}(n)}{(1+r)^n}\right) \\
\simeq \;& \exp\left(-\sum_t \frac{\left(\hat{C}(t+1) - \hat{C}(t) - r\hat{C}(t+1)\right)^2}{2} - \frac{\left(\sum_{0\leqslant n\leqslant T}\frac{\hat{Y}(n)}{(1+r)^n} - \sum_{0\leqslant n\leqslant T}\frac{\hat{C}(n)}{(1+r)^n}\right)^2}{T\sigma^2}\right)
\end{aligned}$$

Appendix 4

The results of the first section can be summed up as follows. We described a set of several individual economic agents by a stochastic process defined in a space whose dimension depends of the number of degrees of freedom, that is number of state variables, of the system. We now eplain how this description can e replaced by a field formalism that will facilitate the computations for a large number of agents. For the sake of the exposition, we choose a simplified version of the model developed previously, in its continuous time version. We start first by considering a single agent discarding the global part of the constraint. We will then introduce an arbitrary number of agents without interactions, and include the interactions ultimately. The behavior of this agent can be represented during a time span T by a probability weight for each possible path of actions. For a path $X(t)$ of actions - such as consumption, production, signals - for $t \in [0,T]$, the weight (17) is:

$$P(X) = \exp\left(-\left(\frac{1}{\sigma^2}\int_0^T \left(\left(\frac{d}{dt}X(t)\right)^2 + V_1(X(t))\right)dt\right)\right)$$

If we impose some initial and final conditions, $\underline{X}$ and $\overline{X}$ on the path, we can also write, as in the first section, the probability of transition from $\underline{X}$ to $\overline{X}$:

$$P\left(\underline{X},\overline{X},T\right) = \exp\left(-\int_{X(0)=\underline{X}}^{X(T)=\overline{X}} \left(\frac{1}{2}\left(\frac{d}{dt}X(t)\right)^2 + V_1(X(t))\right)dt\right)\mathcal{D}X(t) \tag{79}$$

where $K(X(t))$ is a "potential term" whose form depends explicitly on the agent's utility function, or any other intertemporal function the agent optimizes. It represents the probability for an agent to reach $\overline{X}$ starting from $\underline{X}$ during the time span T. It is the probability of social mobility - moving from point $\underline{X}$ to $\overline{X}$ - for an agent in the social space. Written under this form, the probability transition (79) is given by a path integral: The weight in the exponential includes a random, brownian motion, plus a potential $V_1(X(t))$ describing the individual goals as well as social/economical influences. It can be seen as an intertemporal utility whose optimization would yield the usual brownian noise plus some external determinants. As explained before, we have to compute the Laplace transform (27) of $P\left(\underline{X},\overline{X},T\right)$:

$$G_\alpha\left(\underline{X},\overline{X}\right) = \int_0^\infty \exp(-\alpha t)\, P_t\left(\underline{X},\overline{X}\right)dt \tag{80}$$

To do so, it is straightforward to check that (see Kleinert 1989 for example) $P_t\left(\underline{X},\overline{X}\right)$ satisfies a partial differential equation:

$$\frac{\partial}{\partial t}P_t\left(\underline{X},\overline{X}\right) = \left(\frac{1}{2}\nabla^2_{\underline{X}} - V_1\left(\underline{X}\right)\right)P\left(\underline{X},\overline{X},s\right)$$

and that, as a consequence, its Laplace transform $G_\alpha\left(\underline{X},\overline{X}\right)$ satisfies:

$$\left(-\frac{1}{2}\nabla^2 + \alpha + V_1\left(\underline{X}\right)\right)G_\alpha\left(\underline{X},\overline{X}\right) = \delta\left(\underline{X}-\overline{X}\right) \tag{81}$$

where $\delta\left(\underline{X}-\overline{X}\right)$ denotes the Dirac function. The solution of equation (81) is the resolvent, or the kernel, of the operator:

$$L = -\frac{1}{2}\nabla^2 + \alpha + V_1\left(\underline{X}\right)$$

Introducing N identical agents without interaction is straightforward. Without interaction, the agents are indpendent and as a consequence, the probability transition is a the product of individual probability transitions:

$$P_{t_1,\ldots,t_n}\left(\underline{X},\overline{X},T\right) = P_{t_1}\left(\underline{X_1},\overline{X_1}\right)...P_{t_N}\left(\underline{X}_N,\overline{X}_N\right)$$

and the same applies for their Laplace transform:

$$\begin{aligned} G_\alpha\left(\underline{X},\overline{X}\right) &= \int_0^\infty \exp\left(-\alpha\left(t_1+...+t_N\right)\right)P_{t_1}\left(\underline{X_1},\overline{X_1}\right)...P_{t_N}\left(\underline{X}_N,\overline{X}_N\right)dt_1...dt_n \\ &= G_\alpha\left(\underline{X_1},\overline{X_1}\right)...G_\alpha\left(\underline{X}_N,\overline{X}_N\right) \end{aligned} \tag{82}$$

From now on, in order to alleviate the notation, we will denote x_i and y_i the inital and final state for agent i (in the case of a single agent, we will simply use x and y). The transition probability for N agents will thus be written:

$$\prod_{i=1}^N G_\alpha(x_i,y_i)$$

We will write $G_\alpha\left(\left(x_i,y_i\right)_N\right)$ the probability of transition for N agents.

As explained in the first section, the field theory allows to express the product in (82) as derivatives of a single function. To do so we consider (81). We can actually infer from (81) that the determinant of the integral operator G_α whose kernel is $G_\alpha\left(\underline{X},\overline{X}\right)$ can be expressed as an infinite dimensional integral:

$$\left(\det\left(G_\alpha\right)\right)^{-1}=\int\exp\left(-\int\Psi\left(x\right)\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)\Psi^\dagger\left(x\right)dx\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger \tag{83}$$

where the integrals over $\Psi\left(x\right)$ and $\Psi^\dagger\left(x\right)$ are performed over the space of complex-valued functions of one variable x living in the same space as $\underline{X}$ and $\overline{X}$, the initial and final states of one single agent. The function $\Psi^\dagger\left(x\right)$ is the complex conjugate of $\Psi\left(x\right)$. These functions are the fields introduced in section 1.

The formula (83) is simply the generalization in infinite dimension of the gaussian integral formula:

$$\left(\det\left(M\right)\right)^{-1}=\int\exp\left(-X\left(M\right)X^\dagger\right)\mathcal{D}X\mathcal{D}X^\dagger$$

for X a vector of $\mathbb{C}^N$, $X^\dagger$ it's complex conjugate, and M an invertible linear operator on $\mathbb{C}^N$, i.e. an invertible matrix. To recover (82) from (83), one introduces the source term $J\left(x\right)\Psi^\dagger\left(x\right)+J^\dagger\left(x\right)\Psi\left(x\right)$ defined in section 2. We claim that:

$$\begin{aligned}&\frac{\int\exp\left(-\int\left(\Psi\left(x\right)\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)\Psi^\dagger\left(x\right)+J\left(x\right)\Psi^\dagger\left(x\right)+J^\dagger\left(x\right)\Psi\left(x\right)\right)dx\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger}{\int\exp\left(\left(-\int\left(\Psi\left(x\right)\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)\Psi^\dagger\left(x\right)\right)dx\right)\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger}\\ =\;&\exp\left(\int J\left(x\right)\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)^{-1}J^\dagger\left(x\right)dx\right)\\ =\;&\exp\left(\int J\left(x\right)G_\alpha\left(x,y\right)J^\dagger\left(x\right)dx\right)\end{aligned} \tag{84}$$

This results directly from (84) by changing the variable $\Psi\left(x\right)\rightarrow\Psi\left(x\right)+J\left(x\right)$ in the numerator and using (83). As a consequence, the terms in (82) can be recovered from (84). Actually, the transition function for N agents (84):

$$\prod_{i=1}^N G_\alpha\left(x_i,y_i\right) \tag{85}$$

can directly be written as:

$$\prod_{i=1}^N G_\alpha\left(x_i,y_i\right)=\left[\left(\frac{\delta}{\delta J\left(x_{i_1}\right)}\frac{\delta}{\delta J^\dagger\left(y_{i_1}\right)}\right)...\left(\frac{\delta}{\delta J\left(x_{i_N}\right)}\frac{\delta}{\delta J^\dagger\left(y_{i_N}\right)}\right)\exp\left(J\left(x\right)G_\alpha\left(x,y\right)J^\dagger\left(x\right)\right)\right]_{J=J^\dagger=0}$$

Consequently, we now have an infinite dimensional integral representation for the transition functions for N agents:

$$\begin{aligned}&\prod_{i=1}^N G_\alpha\left(x_i,y_i,\alpha\right)=\frac{1}{\int\exp\left(\left(-\int\left(\Psi\left(x\right)\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)\Psi^\dagger\left(x\right)\right)dx\right)\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger}\\ &\times\left[\left(\frac{\delta}{\delta J\left(x_{i_1}\right)}\frac{\delta}{\delta J^\dagger\left(x_{i_1}\right)}\right)...\left(\frac{\delta}{\delta J\left(x_{i_N}\right)}\frac{\delta}{\delta J^\dagger\left(x_{i_N}\right)}\right)\right]_{J=J^\dagger=0}\\ &.\int\exp\left(-\int\left(\Psi\left(x\right)\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)\Psi^\dagger\left(x\right)+J\left(x\right)\Psi^\dagger\left(x\right)+J^\dagger\left(x\right)\Psi\left(x\right)\right)dx\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger\end{aligned} \tag{86}$$

The normalization factor

$$\frac{1}{\int\exp\left(\left(-\int\left(\Psi\left(x\right)\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)\Psi^\dagger\left(x\right)\right)dx\right)\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger}\;=\det\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)^{-1}$$

is constant and will thus be - whenever possible - omitted in the formula. Thus, the transition functions associated to (79) are computed by taking the derivatives with respect to $J\left(x\right)$ and $J^\dagger\left(x\right)$ of

$$\int\exp\left(-\int\left(\Psi\left(x\right)\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)\Psi^\dagger\left(x\right)+J\left(x\right)\Psi^\dagger\left(x\right)+J^\dagger\left(x\right)\Psi\left(x\right)\right)dx\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger$$

The source term is usually implied and only reintroduced ultimately, at the end of the computations. As a consequence,

$$\int\exp\left(-\int\left(\Psi\left(x\right)\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)\Psi^\dagger\left(x\right)\right)dx\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger \tag{87}$$

will describe the same system of identical non interacting agents whose probabilistic description is (79).

We can now consider the case of interacting agents, which means including a potential term involving k agents as in (23):

$$\sum_{i_1,...,i_k} \int_0^{T_1} ... \int_0^{T_n} V_k \left(X_{s_1}^{(i_1)}, ..., X_{s_k}^{(i_k)} \right) ds_1 ... ds_k$$

Where we set $\theta = 1$ and each agent having its own lifespan T_i. We explain how to translate our probabilistic formalism in a field description similar to (86), and including the interactions.

To do so, we introduce the so called Grand Partition Function for an infinite set of interacting individual agents associated to the partition function (23):

$$\sum_N \frac{1}{N!} \prod_{i=1}^N \int \exp(-\alpha t_i) \int \mathcal{D}X_i(t) \exp\left(- \sum_i \int_{X_i(0)=x_i}^{X_i(t_i)=y_i} \left(\left(\frac{1}{2} \left(\frac{d}{dt} X_i(t) \right)^2 + V_1(X_i(t)) \right) dt \right) \right. \tag{88}$$
$$\left. - \sum_{i_1,...i_k} \int_{X_i(0)=x_i}^{X_i(s)=y_i} V_k(X_{i_1}(t_1) ... X_{i_k}(t_k)) \, dt_1 ... dt_k \right)$$

Compared to (23), two differences arise.

First, the number of agents is variale. This is in line with our description of the field formalism in section 2. We do not focus on a fixed number of agents, but rather on the interaction of several agents among a set of an infinite number of agents. The number N of agents involved in the interaction process can be variable, eventhough very large, and this is why we sum over N expressions similar to (23). The $N!$ reflects the fact that agents are identical in that context and is here to avoid redundancies in the sum over agents.

Second, the lifespan of the agents is different from one agent to another. As explained in section one we assume this lifespan to be a random Poisson process of average $\frac{1}{\alpha}$ and we take the average over this process for all agents. This the reason for the Laplace transform.

As explained above, without interactions, the transition probabilities associated to (88) can be computed with the formalism defined by (87). To include the interaction part, we will now consider the potential $V_1(X_i(t))$ as a source term. To do so, we follow the presentation of (Kleinert 1989). Starting with the simplest case of no interaction, i.e. $V_k(X_1(t_1) ... X_k(t_k)) = 0$, the function of interest to us is:

$$\sum_N \frac{1}{N!} \prod_{i=1}^N \int \exp(-\alpha t_i) \int \mathcal{D}X_i(t) \exp\left(- \sum_i \int_{X_i(0)=x_i}^{X_i(t_i)=y_i} \left(\left(\frac{1}{2} \left(\frac{d}{dt} X_i(t) \right)^2 + V_1(X_i(t)) \right) dt \right) \right) \tag{89}$$

Each of these integrals being independent from each others, the results for (89) is:

$$\sum_N \frac{1}{N!} \prod_{i=1}^N \int \exp(-\alpha t_i) \int \mathcal{D}X_i(t) \exp\left(- \sum_i \int_{X_i(0)=x_i}^{X_i(t_i)=y_i} \left(\left(\frac{1}{2} \left(\frac{d}{dt} X_i(t) \right)^2 + V_1(X_i(t)) \right) dt \right) \right) = \sum_N \frac{1}{N!} \prod_{i=1}^N G_\alpha(x_i, y_i) \tag{90}$$

which is a mixed sum over N of transition functions for N agents. Each product $\frac{1}{N!} \prod_{i=1}^N G_K(x_i, y_i, \alpha)$ computes, as needed, the transition probability from $\{x_i\}_{i=1...N}$ to $\{y_i\}_{i=1...N}$ for N ordered agents during a process of mean duration $\frac{1}{\alpha}$. Thus the sum can be seen as a generating series for these probabilities with N agents. However, between identical agents, order is irrelevant, so that the probability of transition of the system from $\{x_i\}_{i=1...N}$ to $\{y_i\}_{i=1...N}$ is the sum over the permutations with N elements of the terms on (90) rhs. Since these terms are equal, the "true" probability of transition is $\prod_{i=1}^N G_K(x_i, y_i, \alpha)$. The whole problem at stake is to recover the case with interaction (88) from the "free" case (89). This is done using the following method. Using the functional derivative with respect to $V_1(x_{i_1})$ we write:

$$\begin{aligned}
&\frac{\delta}{\delta V_1(x_{i_1})} \sum_N \frac{1}{N!} \prod_{i=1}^N \int \exp(-\alpha t_i) \int \mathcal{D}X_i(t) \exp\left(- \sum_i \int_{X_i(0)=x_i}^{X_i(t_i)=y_i} \left(\left(\frac{1}{2} \left(\frac{d}{dt} X_i(t) \right)^2 + V_1(X_i(t)) \right) dt \right) \right) \\
= \; & \sum_N \frac{1}{N!} \prod_{i=1}^N \int \exp(-\alpha t_i) \int \mathcal{D}X_i(t) \exp\left(- \sum_i \int_{X_i(0)=x_i}^{X_i(t_i)=y_i} \left(\left(\frac{1}{2} \left(\frac{d}{dt} X_i(t) \right)^2 + V_1(X_i(t)) \right) dt \right) \right) \\
& \times \left\{ - \sum_i \int_{X_{i_1}(0)=x_{i_1}}^{X_{i_1}(t_{i_1})=y_{i_1}} dt \delta(X_{i_1}(t) - x_{i_1}) \right\}
\end{aligned}$$

where $\delta\left(x_{i_1}\left(t\right)-x_{i_1}\right)$ is the delta of Dirac function. By extension, this generalizes for any function $V\left(x_{i_1}\right)$, to yield:

$$\begin{aligned}
&\int dx_{i_1} V\left(x_{i_1}\right)\frac{\delta}{\delta V_1\left(x_{i_1}\right)}\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp\left(-\alpha t_i\right)\\
&\qquad\times\int\mathcal{D}X_i\left(t\right)\exp\left(-\sum_i\int_{X_i(0)=x_i}^{X_i(t_i)=y_i}\left(\left(\frac{1}{2}\left(\frac{d}{dt}X_i\left(t\right)\right)^2+V_1\left(X_i\left(t\right)\right)\right)dt\right)\right)\\
=\quad&=\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp\left(-\alpha t_i\right)\int\mathcal{D}X_i\left(t\right)\exp\left(-\sum_i\int_{X_i(0)=x_i}^{X_i(t_i)=y_i}\left(\left(\frac{1}{2}\left(\frac{d}{dt}X_i\left(t\right)\right)^2+V_1\left(X_i\left(t\right)\right)\right)dt\right)\right)\\
&\times\left\{-\sum_i\int_{X_{i_1}(0)=x_{i_1}}^{X_{i_1}\left(t_{i_1}\right)=y_{i_1}}dtV\left(X_{i_1}\left(t\right)\right)\right\}
\end{aligned}$$

and for any function of several variables, one has similarly:

$$\begin{aligned}
&\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp\left(-\alpha t_i\right)\int\mathcal{D}X_i\left(t\right)\exp\left(-\sum_i\int_{X_i(0)=x_i}^{X_i(t_i)=y_i}\left(\left(\frac{1}{2}\left(\frac{d}{dt}X_i\left(t\right)\right)^2+V_1\left(X_i\left(t\right)\right)\right)dt\right)\right)\qquad(91)\\
&\times\sum_{i_1,...i_k}\int_{X_{i_1}(0)=x_{i_1}}^{X_{i_1}\left(t_{i_1}\right)=y_{i_1}}...\int_{X_{i_k}(0)=x_{i_k}}^{X_{i_k}\left(t_{i_k}\right)=y_{i_k}}V_k\left(X_{i_1}\left(t_k\right)...X_{i_k}\left(t_k\right)\right)dt_1...dt_k\\
=\quad&\sum_{i_1,...i_k}\left\{\left(-1\right)^k\int dx_{i_1}...dx_{i_k}V_k\left(x_{i_1}...x_{i_k}\right)\frac{\delta}{\delta V_1\left(x_{i_1}\right)}...\frac{\delta}{\delta V_1\left(x_{i_k}\right)}\right\}\\
&\times\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp\left(-\alpha t_i\right)\int\mathcal{D}X_i\left(t\right)\exp\left(-\sum_i\int_{X_i(0)=x_i}^{X_i(t_i)=y_i}\left(\left(\frac{1}{2}\left(\frac{d}{dt}X_i\left(t\right)\right)^2+V_1\left(X_i\left(t\right)\right)\right)dt\right)\right)
\end{aligned}$$

To find (88) from (89), the next step is to exponentiate (91) to express (88) as:

$$\begin{aligned}
&\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp\left(-\alpha t_i\right)\int\mathcal{D}X_i\left(t\right)\exp\left(-\sum_i\int_{X_i(0)=x_i}^{X_i(t_i)=y_i}\left(\left(\frac{1}{2}\left(\frac{d}{dt}X_i\left(t\right)\right)^2+V_1\left(X_i\left(t\right)\right)\right)dt\right.\right.\\
&\qquad\left.\left.-\sum_{i_1,...i_k}\int_{X_i(0)=x_i}^{X_i(s)=y_i}V_k\left(X_{i_1}\left(t_1\right)...X_{i_k}\left(t_k\right)\right)dt_1...dt_k\right)\right)\\
=\quad&\exp\left(-\int dx_{i_1}...dx_{i_k}V_k\left(x_{i_1}...x_{i_k}\right)\frac{\delta}{\delta V_1\left(x_{i_1}\right)}...\frac{\delta}{\delta V_1\left(x_{i_k}\right)}\right)\\
&\times\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp\left(-\alpha t_i\right)\int\mathcal{D}X_i\left(t\right)\exp\left(-\sum_i\int_{X_i(0)=x_i}^{X_i(t_i)=y_i}\left(\left(\frac{1}{2}\left(\frac{d}{dt}X_i\left(t\right)\right)^2+V_1\left(X_i\left(t\right)\right)\right)dt\right)\right)
\end{aligned}$$

In other words, using (90) one finds the partition function for the system of agents in interaction:

$$\begin{aligned}
&\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp\left(-\alpha t_i\right)\int\mathcal{D}X_i\left(t\right)\exp\left(-\sum_i\int_{X_i(0)=x_i}^{X_i(t_i)=y_i}\left(\left(\frac{1}{2}\left(\frac{d}{dt}X_i\left(t\right)\right)^2+V_1\left(X_i\left(t\right)\right)\right)dt\right.\right.\qquad(92)\\
&\qquad\left.\left.-\sum_{i_1,...i_k}\int_{X_i(0)=x_i}^{X_i(s)=y_i}V_k\left(X_{i_1}\left(t_1\right)...X_{i_k}\left(t_k\right)\right)dt_1...dt_k\right)\right)\\
=\quad&\exp\left(-\int dx_{i_1}...dx_{i_k}V_k\left(x_{i_1}...x_{i_k}\right)\frac{\delta}{\delta V_1\left(x_{i_1}\right)}...\frac{\delta}{\delta K\left(x_{i_k}\right)}\right)_{K\equiv0}\times\sum_N\frac{1}{N!}\prod_{i=1}^N G_\alpha\left(x_i,y_i\right)
\end{aligned}$$

We can now find the field formulation associated to (92). We have seen that (see (86)):

$$\begin{aligned}
&\prod_{i=1}^N G_\alpha\left(x_i,y_i,\alpha\right)=\det\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)^{-1}\times\left[\left(\frac{\delta}{\delta J\left(x_{i_1}\right)}\frac{\delta}{\delta J^\dagger\left(x_{i_1}\right)}\right)...\left(\frac{\delta}{\delta J\left(x_{i_N}\right)}\frac{\delta}{\delta J^\dagger\left(x_{i_N}\right)}\right)\right]_{J=J^\dagger=0}\\
&.\times\int\exp\left(-\int\left(\Psi\left(x\right)\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)\Psi^\dagger\left(x\right)+J\left(x\right)\Psi^\dagger\left(x\right)+J^\dagger\left(x\right)\Psi\left(x\right)\right)dx\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger
\end{aligned}$$

so that, discarding the constant factor $\det\left(-\frac{1}{2}\nabla^2+\alpha+K(x)\right)^{-1}$, one has:

$$\begin{aligned}
&\sum_N \frac{1}{N!}\prod_{i=1}^N \int \exp(-\alpha t_i)\int \mathcal{D}X_i(t)\exp\left(-\sum_i \int_{X_i(0)=x_i}^{X_i(t_i)=y_i}\left(\left(\frac{1}{2}\left(\frac{d}{dt}X_i(t)\right)^2+V_1(X_i(t))\right)dt\right)\right.\\
&\qquad\left.-\sum_{i_1,...i_k}\int_{X_i(0)=x_i}^{X_i(s)=y_i} V_k\left(X_{i_1}(t_1)...X_{i_k}(t_k)\right)dt_1...dt_k\right)\\
=\;&\exp\left(-\int dx_{i_1}...dx_{i_k}V_k\left(x_{i_1}...x_{i_k}\right)\frac{\delta}{\delta V_1(x_{i_1})}...\frac{\delta}{\delta K(x_{i_k})}\right)_{K\equiv 0}\\
&\times\left[\left(\frac{\delta}{\delta J(x_{i_1})}\frac{\delta}{\delta J^\dagger(x_{i_1})}\right)...\left(\frac{\delta}{\delta J(x_{i_N})}\frac{\delta}{\delta J^\dagger(x_{i_N})}\right)\right]_{J=J^\dagger=0}\\
&\times\int\exp\left(-\int\left(\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+K(x)\right)\Psi^\dagger(x)+J(x)\Psi^\dagger(x)+J^\dagger(x)\Psi(x)\right)dx\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger
\end{aligned}$$

and this quantity is equal to:

$$\begin{aligned}
&\left[\left(\frac{\delta}{\delta J(x_{i_1})}\frac{\delta}{\delta J^\dagger(x_{i_1})}\right)...\left(\frac{\delta}{\delta J(x_{i_N})}\frac{\delta}{\delta J^\dagger(x_{i_N})}\right)\right]_{J=J^\dagger=0}\\
&\times\int\exp\left(-\int\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+V_1(x)\right)\Psi^\dagger(x)\,dx-\int J(x)\Psi^\dagger(x)+J^\dagger(x)\Psi(x)\,dx\right.\\
&\left.-\sum_{i_1,...i_k}\int\Psi(x_{i_1})...\Psi(x_{i_k})V_k(x_{i_1}...x_{i_k})\Psi^\dagger(x_{i_1})...\Psi^\dagger(x_{i_k})\,dx_{i_1}...x_{i_k}\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger
\end{aligned}$$

In other word, the probabilitic description (88) of a large number of interacting agents is encompassed in the path integral:

$$\begin{aligned}
&\int\exp\left(-\int\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+V_1(x)\right)\Psi^\dagger(x)\,dx-\int J(x)\Psi^\dagger(x)+J^\dagger(x)\Psi(x)\,dx\right.\\
&\left.-\sum_{i_1,...i_k}\int\Psi(x_{i_1})...\Psi(x_{i_k})V_k(x_{i_1}...x_{i_k})\Psi^\dagger(x_{i_1})...\Psi^\dagger(x_{i_k})\,dx_{i_1}...x_{i_k}\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger
\end{aligned}$$

The functional:

$$\begin{aligned}
S(\Psi,J)\;=\;&\int\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+V_1(x)\right)\Psi^\dagger(x)\,dx\\
&+\sum_{i_1,...i_k}\int\Psi(x_{i_1})...\Psi(x_{i_k})V_k(x_{i_1}...x_{i_k})\Psi^\dagger(x_{i_1})...\Psi^\dagger(x_{i_k})\,dx_{i_1}...x_{i_k}+\int J(x)\Psi^\dagger(x)+J^\dagger(x)\Psi(x)\,dx
\end{aligned}$$

is a particular case of a field action functional with source as defined in section 2. It is straightforward to generalize this formula for any type of potential involving an arbitrary number of agents by introducing over k yielding an action:

$$\begin{aligned}
S(\Psi,J)\;=\;&\int\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+V_1(x)\right)\Psi^\dagger(x)\,dx\\
&+\sum_{k\geqslant 2}\sum_{i_1,...i_k}\int\Psi(x_{i_1})...\Psi(x_{i_k})V_k(x_{i_1}...x_{i_k})\Psi^\dagger(x_{i_1})...\Psi^\dagger(x_{i_k})\,dx_{i_1}...x_{i_k}+\int J(x)\Psi^\dagger(x)+J^\dagger(x)\Psi(x)\,dx
\end{aligned}$$

As a consequence:

$$\begin{aligned}
&\int\exp\left(-\int\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+V_1(x)\right)\Psi^\dagger(x)\,dx\right.\\
&\left.-\sum_{k\geqslant 2}\sum_{i_1,...i_k}\int\Psi(x_{i_1})...\Psi(x_{i_k})V_k(x_{i_1}...x_{i_k})\Psi^\dagger(x_{i_1})...\Psi^\dagger(x_{i_k})\,dx_{i_1}...x_{i_k}+\int J(x)\Psi^\dagger(x)+J^\dagger(x)\Psi(x)\,dx\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger
\end{aligned}\tag{93}$$

computes, by successive derivatives with respect to $J(x)$ and $J^\dagger(x)$, the transition functions of a system of infinite number of identical agents, with arbitrary, non local in time, interactions $V_k\left(X_{i_1}(t_1)...X_{i_k}(t_k)\right)$ involving k agents,

with k arbitrary. The constant α is the characteristic scale of the interaction process, and $\frac{1}{\alpha}$ the mean duration of the interaction process, or alternately the mean lifespan of the agents. The transition functions are given by:

$$\begin{aligned} & G_K\left(\{x_i\},\{y_i\},\alpha\right) \\ = & \left[\left(\frac{\delta}{\delta J\left(x_{i_1}\right)}\frac{\delta}{\delta J^\dagger\left(y_{i_1}\right)}\right)...\left(\frac{\delta}{\delta J\left(x_{i_N}\right)}\frac{\delta}{\delta J^\dagger\left(y_{i_N}\right)}\right)\int \exp\left(-\Psi\left(x\right)\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)\Psi^\dagger\left(x\right)\right. \\ & \left.\left.-\sum_{k\geqslant 2}\sum_{i_1,...i_k}\Psi\left(x_{i_1}\right)...\Psi\left(x_{i_k}\right)V_k\left(x_{i_1}...x_{i_k}\right)\Psi^\dagger\left(x_{i_1}\right)...\Psi^\dagger\left(x_{i_k}\right)+J\left(x\right)\Psi^\dagger\left(x\right)+J^\dagger\left(x\right)\Psi\left(x\right)\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger\right]_{J=J^\dagger=0} \end{aligned} \tag{94}$$

and $G_K\left(\{x_i\},\{y_i\},\alpha\right)$ is the probability of transition for N agents from a state $\{x_i\}$ to a state $\{y_i\}$. Remark that this formulation realizes what was announced before. The switch in formulation induces that the transition of the agents, i.e. their dynamical and stochastic properties, takes place in a surrounding. Instead of computing directly the dynamic of the system, we derive this behavior from the global properties of a substratum, the global action for the field $\Psi\left(x\right)$. By global action we denote the functional, or action:

$$S\left(\Psi\right)=\int dx\left(\Psi\left(x\right)\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)\Psi^\dagger\left(x\right)+\sum_{k\geqslant 2}^{A}\sum_{i_1,...i_k}\Psi\left(x_{i_1}\right)...\Psi\left(x_{i_k}\right)V_k\left(x_{i_1}...x_{i_k}\right)\Psi^\dagger\left(x_{i_1}\right)...\Psi^\dagger\left(x_{i_k}\right)\right)$$

Appendix 5

We start with the statistical weight associated to the intertemporal budget constraint:

$$\begin{aligned} & \exp\left(-\sum_s\frac{1}{\varpi^2}\left(C_s-\bar{C}-\frac{C_{s+1}}{\left(1+r\right)}\right)^2+\sum_s C_0\right)\exp\left(-\frac{\left(\int_0^T\left(Y_i\left(s\right)-C_i\left(s\right)\right)\exp\left(-\int r_i\left(s\right)ds\right)ds\right)^2}{\theta^2}\right) \\ \simeq & \exp\left(-\sum_s\frac{\left(C_{s+1}-C_s-rC_{s+1}\right)^2}{\varpi^2}+\sum_s C_0\right)\exp\left(-\frac{\left(\int_0^T\left(Y_i\left(s\right)-C_i\left(s\right)\right)\exp\left(-\int r_i\left(s\right)ds\right)ds\right)^2}{\theta^2}\right) \end{aligned}$$

with $C_0\equiv\frac{1}{2\sigma\varpi^2}$. Now, remark that the budgt constraint:

$$\int\left(Y_i\left(s\right)-C_s\right)\exp\left(-rs\right)ds=0$$

can be expressed as:

$$\int\left(\dot{K}_i\left(s\right)+\varepsilon\left(s\right)\right)\exp\left(-rs\right)ds=0$$

or as:

$$\int\dot{K}_i\left(s\right)\exp\left(-rs\right)ds=-\int\varepsilon\left(s\right)\exp\left(-rs\right)ds$$

The last term has variance $\frac{\nu^2}{2r}$ which implies that the overall constraint can be included in the global weight through a term:

$$\frac{2\bar{r}}{\nu^2}\left(\int\dot{K}_i\left(s\right)\exp\left(-rs\right)ds\right)^2$$

with:

$$\bar{r}=\frac{1}{\int\exp\left(-rs\right)ds}$$

$$\begin{aligned} \int\dot{K}_i\left(s\right)\exp\left(-rs\right)ds &= \left[K_i\left(s\right)\exp\left(-rs\right)\right]_0^T+\int rK_i\left(s\right)\exp\left(-rs\right)ds \\ &= -K_i\left(0\right)+\int rK_i\left(s\right)\exp\left(-rs\right)ds \end{aligned}$$

if the transversality condition is satisfied. At the lowest order in $\bar{r}$ or r, the contribution is approximated by $K_i\left(0\right)$ and can be neglected. We end up with a contribution:

$$\exp\left(-\sum_s\frac{1}{\varpi^2}\left(C_s-\bar{C}-\frac{\left(C_{s+1}\right)}{\left(1+r\right)}\right)^2+\sum_s C_0\right)$$

or in continuous time:

$$\exp\left(-\int ds\frac{\left(\dot{C}_s - rC_s + \bar{C}\right)^2}{\varpi^2} + C_0\int ds\right)$$

Appendix 6

Business cycle model, field theoretic representation

The field theoretic equivalent of (42) is obtained by the same methods we used previously. One obtains the following action for the field:

$$\begin{aligned}
S(\Psi) = \Psi^{\dagger}(K,C,A)\Bigg(&-\nabla.\left(\begin{pmatrix}\nu^2 & 0 & 0\\ 0 & \varpi^2 & 0\\ 0 & 0 & \frac{1}{\lambda^2}\end{pmatrix}\nabla + 2\begin{pmatrix}-AF(K)+C+\delta(K)\\ -\left(AF'(K)+r_c\right)C+\bar{C}\\ 0\end{pmatrix}\right)\\
&+\varsigma^2\left(C-\bar{C}\right)^2+\left(A_i-\bar{A}\right)^2+\left(-\left(AF(K)-\delta(K)\right)'-AF'(K)-r_c\right)+\alpha-C_0\Bigg)\Psi(K,C,A)\\
&+\gamma\Psi^{\dagger}(K_1,C_1,A_1)\Psi^{\dagger}(K_2,C_2,A_2)\left\{A_2H(K_1,K_2)K_1\right\}\Psi(K_1,C_1,A_1)\Psi(K_2,C_2,A_2)\\
= \int\Psi^{\dagger}(K,C,A)\Bigg\{&-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}+\varsigma^2\left(C-\bar{C}\right)^2+\left(A_i-\bar{A}\right)^2-\nu^2\frac{\partial^2}{\partial K^2}\\
&+\Bigg(-2\left(C-AF(K)+\delta(K)\right)\frac{\partial}{\partial K}+2\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)\frac{\partial}{\partial C}+2\frac{\partial}{\partial K}\left(AF(K)-\delta(K)\right)\Bigg)\\
&+2\left(AF'(K)+r_c\right)\Bigg\}\Psi(K,C,A)\\
&+\int\Psi^{\dagger}(K,C,A)\Bigg(-\left(\frac{\partial}{\partial K}\left(AF(K)-\delta(K)\right)\right)-\left(AF'(K)+r_c\right)+\alpha-C_0\Bigg)\Psi(K,C,A)\\
&+\gamma\frac{1}{2}\int\Psi^{\dagger}(K_1,C_1,A_1)\Psi^{\dagger}(K_2,C_2,A_2)\left\{A_2H(K_1,K_2)K_1+A_1H(K_2,K_1)K_2\right\}\Psi(K_1,C_1,A_1)\Psi(K_2,C_2,A_2)
\end{aligned}$$

Let $\delta(K)=\delta K$ as usually assumed. The previous expression simplifies as:

$$\begin{aligned}
S(\Psi) = &\int\Psi^{\dagger}(K,C,A)\Bigg\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\frac{\partial^2}{\partial K^2}+\left(A-\bar{A}\right)^2-2\left(C-AF(K)+\delta K\right)\frac{\partial}{\partial K}\\
&+2\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)\frac{\partial}{\partial C}+\varsigma^2\left(C-\bar{C}\right)^2\Bigg\}\Psi(K,C,A)\\
&+\int\Psi^{\dagger}(K,C,A)\left(\alpha+2AF'(K)+(r_c-\delta)-C_0\right)\Psi(K,C,A)\\
&+\gamma\frac{1}{2}\int\Psi^{\dagger}(K_1,C_1,A_1)\Psi^{\dagger}(K_2,C_2,A_2)\left\{A_2H(K_1,K_2)K_1+A_1H(K_2,K_1)K_2\right\}\Psi(K_1,C_1,A_1)\Psi(K_2,C_2,A_2)
\end{aligned}$$

If we consider that the rate $\left(AF'(K)+r_c\right)$ is slowly varying, as an interest rate, we can perform a change of variable:

$$\begin{aligned}
\Psi(K,C,A) &= \exp\left(\frac{1}{2\varpi^2}\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)^2\right)\hat{\Psi}(K,C,A)\\
\hat{\Psi}^{\dagger}(K,C,A) &= \exp\left(-\frac{1}{2\varpi^2}\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)^2\right)\hat{\Psi}(K,C,A)
\end{aligned}$$

and rewrite the action as a function of $\hat{\Psi}$:

$$\begin{aligned}
S\left(\hat{\Psi}\right) = &\int\hat{\Psi}^{\dagger}(K,C,A)\Bigg\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\frac{\partial^2}{\partial K^2}+\left(A-\bar{A}\right)^2-2\left(C-AF(K)+\delta K\right)\frac{\partial}{\partial K}\\
&+\left(\varsigma^2+\frac{\left(AF'(K)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}\right)^2+\alpha-C_0+AF'(K)-\delta\Bigg\}\hat{\Psi}(K,C,A)\\
&+\gamma\frac{1}{2}\int\hat{\Psi}^{\dagger}(K_1,C_1,A_1)\hat{\Psi}^{\dagger}(K_2,C_2,A_2)\left\{A_2H(K_1,K_2)K_1+A_1H(K_2,K_1)K_2\right\}\hat{\Psi}(K_1,C_1,A_1)\hat{\Psi}(K_2,C_2,A_2)
\end{aligned}$$

Then, a change of variable

$$\begin{aligned} K' &= C - AF(K) + \delta K \\ \frac{\partial}{\partial K} &= \left(\delta - AF'(K)\right)\frac{\partial}{\partial K'} \end{aligned}$$

associated also with the assumption that the rate $\delta - AF'(K)$ slowly varying, as well as a rescaling $\frac{\bar{C}}{(AF'(K)+r_c)} \to \bar{C}$ leads to:

$$\begin{aligned} S\left(\hat{\Psi}\right) = \int \hat{\Psi}^{\dagger}\left(K', C, A\right)\Big\{ & -\varpi^2\frac{\partial^2}{\partial C^2} - \frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2} - \nu^2\left(\delta - AF'(K)\right)^2\frac{\partial^2}{\partial K'^2} + \left(A-\bar{A}\right)^2 - 2\left(\delta - AF'(K)\right)K'\frac{\partial}{\partial K'} \\ & + \left(\varsigma^2 + \frac{\left(AF'(K)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}\right)^2 + \alpha - C_0 + AF'(K) - \delta\Big\}\hat{\Psi}\left(K', C, A\right) \\ +\gamma\frac{1}{2}\int \hat{\Psi}^{\dagger}\left(K'_1, C_1, A_1\right)\hat{\Psi}^{\dagger}\left(K'_2, C_2, A_2\right)\left\{A_2 H(K_1, K_2)K_1 + A_1 H(K_2, K_1)K_2\right\}\hat{\Psi}\left(K'_1, C_1, A_1\right)\hat{\Psi}\left(K'_2, C_2, A_2\right) \end{aligned}$$

Ultimately, one can recast the action in a tractable form through a second rescaling of the field:

$$\begin{aligned} \hat{\Psi}\left(K', C, A\right) &= \exp\left(-\frac{\left(K'\right)^2}{2\nu^2\left(\delta - AF'(K)\right)}\right)\bar{\Psi}\left(K', C, A\right) \\ \hat{\Psi}^{\dagger}\left(K', C, A\right) &= \exp\left(\frac{\left(K'\right)^2}{2\nu^2\left(\delta - AF'(K)\right)}\right)\bar{\Psi}\left(K', C, A\right) \end{aligned}$$

and thus:

$$\begin{aligned} S\left(\bar{\Psi}\right) = \int \bar{\Psi}^{\dagger}\left(K', C, A\right)\Big\{ & -\varpi^2\frac{\partial^2}{\partial C^2} - \frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2} - \nu^2\left(\delta - AF'(K)\right)^2\frac{\partial^2}{\partial K'^2} + \left(A-\bar{A}\right)^2 \\ & + \left(\varsigma^2 + \frac{\left(AF'(K)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}\right)^2 + \frac{\left(K'\right)^2}{\nu^2} + \alpha - C_0\Big\}\bar{\Psi}\left(K', C, A\right) \\ +\gamma\frac{1}{2}\int \bar{\Psi}^{\dagger}\left(K'_1, C_1, A_1\right)\bar{\Psi}^{\dagger}\left(K'_2, C_2, A_2\right)\left\{A_2 H(K_1, K_2)K_1 + A_1 H(K_2, K_1)K_2\right\}\bar{\Psi}\left(K'_1, C_1, A_1\right)\bar{\Psi}\left(K'_2, C_2, A_2\right) \end{aligned}$$

The relation between $\Psi(K, C, A)$ and $\bar{\Psi}(K', C, A)$

$$\begin{aligned} \Psi(K, C, A) &= \exp\left(\frac{1}{2\varpi^2}\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)^2\right)\exp\left(-\frac{\left(K'\right)^2}{2\nu^2\left(\delta - AF'(K)\right)}\right)\bar{\Psi}\left(K', C, A\right) \\ \Psi^{\dagger}(K, C, A) &= \exp\left(-\frac{1}{2\varpi^2}\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)^2\right)\exp\left(\frac{\left(K'\right)^2}{2\nu^2\left(\delta - AF'(K)\right)}\right)\bar{\Psi}^{\dagger}\left(K', C, A\right) \end{aligned}$$

implies at first sight that $\bar{\Psi}(K', C, A)$ and $\bar{\Psi}^{\dagger}(K', C, A)$ are not complex conjugate. This is the consequence from the fact that the operator involved in the definition of $S(\Psi)$ is not hermitian, or self adjoint in the real interpretation. This non hermiticity is itself the consequence of an asymmetry in the transition functions: due to a drift term, the transition probability between two points is not symmetric. However, one can make sense of the partition function:

$$\int \exp\left(-S\left(\bar{\Psi}\right)\right)\mathcal{D}\bar{\ominus}\mathcal{D}\bar{\Psi}^{\dagger} \tag{95}$$

and show that it computes the same partition function as:

$$\int \exp\left(-S\left(\Psi\right)\right)\mathcal{D}\ominus\mathcal{D}\Psi^{\dagger} \tag{96}$$

To do so, we first define:

$$\begin{aligned} L &= -\varpi^2\frac{\partial^2}{\partial C^2} - \frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2} - \nu^2\frac{\partial^2}{\partial K^2} + \left(A-\bar{A}\right)^2 - 2\left(C - AF(K) + \delta K\right)\frac{\partial}{\partial K} + 2\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)\frac{\partial}{\partial C} \\ &\quad + \varsigma^2\left(C-\bar{C}\right)^2 + \alpha - C_0 \\ L' &= -\varpi^2\frac{\partial^2}{\partial C^2} - \frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2} - \nu^2\left(\delta - AF'(K)\right)^2\frac{\partial^2}{\partial K'^2} + \left(A-\bar{A}\right)^2 + \left(\varsigma^2 + \frac{\left(AF'(K)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}\right)^2 \\ &\quad + \frac{\left(K'\right)^2}{\nu^2} + \alpha - C_0 \end{aligned}$$

and then:

$$\Psi^{\dagger}(K,C,A)\,L\Psi(K,C,A)=\bar{\Psi}^{\dagger}\left(K',C,A\right)L'\bar{\Psi}\left(K',C,A\right)$$

so that:

$$\begin{aligned}L' &= \exp\left(-\frac{1}{2\varpi^2}\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)^2+\frac{(K')^2}{2\nu^2\left(\delta-AF'(K)\right)}\right)\\ &\quad\times L\exp\left(\frac{1}{2\varpi^2}\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)^2-\frac{(K')^2}{2\nu^2\left(\delta-AF'(K)\right)}\right)\end{aligned}$$

Then, to make sense of the partition function (95), we will compare it to the computation of (96). To do so, recall that the partition function for $\gamma=0$, that is (96), is defined as $\det\left(L^{-1}\right)$ and that this quantities is computed via the eigenvalues of L:

$$\det L^{-1}=\prod_n dx_n dy_n\exp\left(-x_n\lambda_n x_n-y_n\lambda_n y_n\right)$$

This expression makes sense since the eigenvalues of operator L have positive real part. Let $\alpha_n=x_n+iy_n$ and $\alpha_n^{\dagger}=x_n-iy_n$, $\det L^{-1}$ rewrites:

$$\det L^{-1}=\prod_n d\alpha_n d\alpha_n^{\dagger}\exp\left(\alpha_n^{\dagger}\lambda_n\alpha_n\right)$$

This expression is real, since L is a real operator, and if λ_n is an eigenvalue of L, so is $\bar{\lambda}_n$. Consider the expansion of $\Psi(K,C,A)$:

$$\Psi(K,C,A)=\sum\alpha_n\Psi_n(K,C,A)$$

where $\Psi_n(K,C,A)$ are eigenfunctions for λ_n of L, then, define $\Psi^{\dagger}(K,C,A)$ as:

$$\Psi^{\dagger}(K,C,A)=\sum\alpha_n^{\dagger}\Psi_n^{\dagger}(K,C,A)\tag{97}$$

where $\Psi_n^{\dagger}(K,C,A)$, eigenfunctions for λ_n of the adjoint L^{+}, and $\left\langle\Psi_m^{\dagger}(K,C,A),\Psi_n(K,C,A)\right\rangle=\delta_{m,n}$.

As a consequence, the partition function rewrites:

$$\det L^{-1}=\exp\left(-S(\Psi)\right)\mathcal{D}\ominus\mathcal{D}\Psi^{\dagger}$$

This is (96), but the field $\Psi^{\dagger}$ is not the complex conjugate of $\ominus$, and has rather to be understood as given by the expansion (97). Now, focusing on (95), consider the transformed eigenfunctions:

$$\bar{\Psi}_n\left(K',C,A\right)=\exp\left(-\frac{1}{2\varpi^2}\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)^2+\frac{(K')^2}{2\nu^2\left(\delta-AF'(K)\right)}\right)\Psi_n(K,C,A)$$

that are eigenfunctions of L' for eigenvalues λ_n. Actually:

$$\begin{aligned}L'\bar{\Psi}_n\left(K',C,A\right) &= \exp\left(-\frac{1}{2\varpi^2}\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)^2+\frac{(K')^2}{2\nu^2\left(\delta-AF'(K)\right)}\right)L\Psi_n(K,C,A)\\ &= \lambda_n\bar{\Psi}_n\left(K',C,A\right)\end{aligned}\tag{98}$$

Moreover, we define:

$$\bar{\Psi}_n^{\dagger}\left(K',C,A\right)=\exp\left(\frac{1}{2\varpi^2}\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)^2-\frac{(K')^2}{2\nu^2\left(\delta-AF'(K)\right)}\right)\Psi_n^{\dagger}(K,C,A)$$

the eigenfunction of $(L')^{+}$ for λ_n. The functions $\bar{\Psi}_n(K',C,A)$ and $\bar{\Psi}_n(K',C,A)$ are orthogonal :

$$\left\langle\bar{\Psi}_m^{\dagger}(K,C,A),\bar{\Psi}_n(K,C,A)\right\rangle=\left\langle\Psi_m^{\dagger}(K,C,A),\Psi_n(K,C,A)\right\rangle=\delta_{m,n}\tag{99}$$

as a direct consequence of:

$$\begin{aligned}\left(L'\right)^{+} &= \exp\left(\frac{1}{2\varpi^2}\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)^2-\frac{(K')^2}{2\nu^2\left(\delta-AF'(K)\right)}\right)L^{+}\\ &\quad\times\exp\left(-\frac{1}{2\varpi^2}\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)^2+\frac{(K')^2}{2\nu^2\left(\delta-AF'(K)\right)}\right)\end{aligned}$$

and

$$\begin{aligned}\left(L'\right)^{+}\bar{\Psi}_n^{\dagger}\left(K',C,A\right) &= \exp\left(\frac{1}{2\varpi^2}\left(AF'\left(K\right)+r_c\right)\left(C-\bar{C}\right)^2-\frac{\left(K'\right)^2}{2\nu^2\left(\delta-AF'\left(K\right)\right)}\right)L^{+}\Psi_n^{\dagger}\left(K,C,A\right)\\ &= \lambda_n\bar{\Psi}_n^{\dagger}\left(K',C,A\right)\end{aligned}$$

since these two relations imply:

$$\begin{aligned}&\int\bar{\Psi}^{\dagger}\left(K',C,A\right)\left\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\left(\delta-AF'\left(K\right)\right)^2\frac{\partial^2}{\partial K'^2}+\left(A-\bar{A}\right)^2\right.\\ &\qquad\left.+\left(\varsigma^2+\frac{\left(AF'\left(K\right)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}\right)^2+\frac{\left(K'\right)^2}{\nu^2}+\alpha-C_0\right\}\bar{\Psi}\left(K',C,A\right)\\ =\;&\int\bar{\Psi}^{\dagger}\left(K',C,A\right)L'\bar{\Psi}\left(K',C,A\right)=\int\sum\alpha_m^{\dagger}\bar{\Psi}_m^{\dagger}\left(K',C,A\right)\lambda_n\alpha_n\bar{\Psi}_n\left(K,C,A\right)\\ =\;&\int\sum\alpha_m\lambda_n\alpha_n^{\dagger}\\ =\;&\int\sum\alpha_m^{\dagger}\Psi_m^{\dagger}\left(K,C,A\right)\lambda_n\alpha_n\Psi_n\left(K,C,A\right)\\ =\;&\int\Psi^{\dagger}\left(K,C,A\right)L\Psi\left(K,C,A\right)\end{aligned}$$

As a consequence of (98) and (99),

$$\int\mathcal{D}\ominus^{\dagger}\mathcal{D}\Psi\exp\left(-\int\Psi^{\dagger}\left(K,C,A\right)L\Psi\left(K,C,A\right)\right)$$

and

$$\int\mathcal{D}\bar{\ominus}\mathcal{D}\bar{\Psi}\exp\left(-\int\bar{\Psi}^{\dagger}\left(K',C,A\right)L'\bar{\Psi}\left(K',C,A\right)\right)$$

compute the same partition function. We can thus consider the following action:

$$\begin{aligned}S\left(\bar{\Psi}\right)=&\int\bar{\Psi}^{\dagger}\left(K',C,A\right)\left\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\left(\delta-AF'\left(K\right)\right)^2\frac{\partial^2}{\partial K'^2}+\left(A-\bar{A}\right)^2\right.\\ &\left.+\left(\varsigma^2+\frac{\left(AF'\left(K\right)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}\right)^2+\frac{\left(K'\right)^2}{\nu^2}+\alpha-C_0\right\}\bar{\Psi}\left(K',C,A\right)\\ &+\gamma\frac{1}{2}\int\bar{\Psi}^{\dagger}\left(K_1',C_1,A_1\right)\bar{\Psi}^{\dagger}\left(K_2',C_2,A_2\right)\left\{A_2H\left(K_1,K_2\right)K_1+A_1H\left(K_2,K_1\right)K_2\right\}\bar{\Psi}\left(K_1',C_1,A_1\right)\bar{\Psi}\left(K_2',C_2,A_2\right)\end{aligned}\tag{100}$$

as stated in the text.

Existence of a saddle point

We first set $H\left(K_2,K\right)=1$, and $\bar{A}=A_0+\varkappa\left\langle A\right\rangle$ with $\varkappa<1$ to simplify the computations, but any function $H\left(K_2,K\right)$ could be considered. Before considering the saddle point equation, we can note that for $\gamma<0$, no minimum can exist for $S\left(\bar{\Psi}\right)$. Actually, for $\gamma<0$, the quartic term:

$$\gamma\frac{1}{2}\int\bar{\Psi}^{\dagger}\left(K_1',C_1,A_1\right)\bar{\Psi}^{\dagger}\left(K_2',C_2,A_2\right)\left\{A_2K_1+A_1K_2\right\}\bar{\Psi}\left(K_1',C_1,A_1\right)\bar{\Psi}\left(K_2',C_2,A_2\right)$$

is negative, and if we let $\left\|\bar{\Psi}\left(K_1',C_1,A_1\right)\right\|\to\infty$, then this term dominates, so that $S\left(\bar{\Psi}\right)\to-\infty$. Thus, to inspect the possiblity of a minimum for $S\left(\bar{\Psi}\right)$ we have to consider $\gamma>0$. The saddle point equation is:

$$\begin{aligned}&\left\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\left(\delta-AF'\left(K\right)\right)^2\frac{\partial^2}{\partial K'^2}+\left(A-\bar{A}\right)^2+2\left(\bar{A}-\Gamma_3\right)\varkappa A\right.\\ &\left.+\left(\varsigma^2+\frac{\left(AF'\left(K\right)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}\right)^2+\frac{\left(K'\right)^2}{\nu^2}+\alpha-C_0\right\}\Psi_1\left(K,C,A\right)\\ &+\gamma\eta\left(\Gamma_2'A+\Gamma_3K\right)\Psi_1\left(K,C,A\right)=0\end{aligned}$$

where:

$$\begin{aligned}\int \Psi_1^{\dagger}(K_2,C_2,A_2)\,K_2\Psi_1(K_2,C_2,A_2) &= \Gamma_2'\\ \int \Psi_1^{\dagger}(K_2,C_2,A_2)\,A_2\Psi_1(K_2,C_2,A_2) &= \Gamma_3\\ g^{-1}(K) &= K-\frac{A}{\delta}F(K)\end{aligned}$$

The term $2\left(\bar{A}-\Gamma_3\right)\varkappa A$ comes from variation of $\bar{A}=A_0+\varkappa\int\bar{\Psi}^{\dagger}(K',C,A)\,A\bar{\Psi}(K',C,A)$. We impose $\|\Psi_1(K,C,A)\|=1$, so that $\Psi(K,C,A)=\sqrt{\eta}\Psi_1(K,C,A)$. We can also replace K with K' through the relation:

$$K'=C-AF(K)+\delta K\equiv\delta g^{-1}(K)+C$$

$$\begin{aligned}&\left\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\left(\delta-AF'(K)\right)^2\frac{\partial^2}{\partial K'^2}+\left(A-\bar{A}\right)^2+2\left(\bar{A}-\Gamma_3\right)\varkappa A\right.\\ &\left.+\left(\varsigma^2+\frac{\left(AF'(K)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}\right)^2+\frac{\left(K'\right)^2}{\nu^2}+\alpha-C_0\right\}\Psi_1(K,C,A)\\ &+\gamma\eta\left(\Gamma_2'A+\Gamma_3\left(g\left(\frac{K'-C}{\delta}\right)\right)\right)\Psi_1(K,C,A)=0\end{aligned}\tag{101}$$

For the usual form $F(K)=K^{\varepsilon}$, with $\varepsilon<1$,

$$g^{-1}(K)=K\left(1-\frac{A}{\delta K^{1-\varepsilon}}\right)$$

and above a minimal level $\bar{K}$:

$$\begin{aligned}\frac{AK}{\delta K^{1-\varepsilon}} &\simeq \frac{A\bar{K}^{\varepsilon}}{\delta}\left(1+\varepsilon\left(\frac{K-\bar{K}}{\bar{K}}\right)-\frac{\varepsilon(1-\varepsilon)}{2}\left(\frac{K-\bar{K}}{\bar{K}}\right)^2\right)\\ &\simeq \frac{A\bar{K}^{\varepsilon}}{\delta}\left(1+\varepsilon\left(\frac{K-\bar{K}}{\bar{K}}\right)\right)\end{aligned}$$

and:

$$g^{-1}(K)\simeq K-\frac{A\bar{K}^{\varepsilon}}{\delta}\left(1+\varepsilon\left(\frac{K-\bar{K}}{\bar{K}}\right)\right)$$

so that:

$$K=g\left(\frac{K'-C}{\delta}\right)=\frac{(K'-C)+A\bar{K}^{\varepsilon}(1-\varepsilon)-\frac{A}{2}\varepsilon(1-\varepsilon)\bar{K}^{\varepsilon-2}}{\delta-A\varepsilon\bar{K}^{\varepsilon-1}}$$

$$C+\left(\delta-A\varepsilon\bar{K}^{\varepsilon-1}\right)\left(\frac{\varepsilon A}{\delta}\right)^{\frac{1}{1-\varepsilon}}-A\bar{K}^{\varepsilon}(1-\varepsilon)<K'C-A\bar{K}^{\varepsilon}(1-\varepsilon)$$

$$A>\frac{\delta\bar{K}^{1-\varepsilon}}{\varepsilon}$$

For this particular form of production function, (101) is then:

$$\begin{aligned}&\left\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\left(\delta-AF'(K)\right)^2\frac{\partial^2}{\partial K'^2}+\left(A-\bar{A}\right)^2+2\left(\bar{A}-\Gamma_3\right)\varkappa A\right.\\ &\left.+\left(\varsigma^2+\frac{\left(AF'(K)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}\right)^2+\frac{\left(K'\right)^2}{\nu^2}+\alpha-C_0\right\}\Psi_1(K,C,A)\\ &+\gamma\eta\left(\Gamma_2'A+\Gamma_3\frac{(K'-C)+A\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta-A\varepsilon\bar{K}^{\varepsilon-1}}\right)\Psi_1(K,C,A)=0\end{aligned}\tag{102}$$

where:

$$\begin{aligned}\Gamma_1 &= \langle C\rangle,\ \Gamma_2=\langle K'\rangle,\ \Gamma_3=\langle A\rangle\\ \Gamma_2' &= \left\langle\frac{(K'-C)+A\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta-A\varepsilon\bar{K}^{\varepsilon-1}}\right\rangle\end{aligned}$$

and the brackets denotes the expectation of the quantities in the state Ψ_1. The potential terms in (102):

$$\begin{aligned}
&\left(\varsigma^2+\frac{\left(AF'(K)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}\right)^2+\frac{\left(K'\right)^2}{\nu^2}+\left(A-\bar{A}\right)^2+2\left(\bar{A}-\Gamma_3\right)\varkappa A+\gamma\eta\left(\Gamma_2'A+\Gamma_3\frac{\left(K'-C\right)+A\bar{K}^{\varepsilon}\left(1-\varepsilon\right)}{\delta-A\varepsilon\bar{K}^{\varepsilon-1}}\right)\\
\simeq\ &\left(\varsigma^2+\frac{\left(AF'(K)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}\right)^2+\frac{\left(K'\right)^2}{\nu^2}+\left(A-\bar{A}\right)^2+2\left(\bar{A}-\Gamma_3\right)\varkappa A+\gamma\eta\left(\Gamma_2'A+\Gamma_3\frac{\left(K'-C\right)+A\bar{K}^{\varepsilon}\left(1-\varepsilon\right)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}\right)
\end{aligned}$$

with

$$\Gamma_2'=\left\langle\frac{\left(K'-C\right)+A\bar{K}^{\varepsilon}\left(1-\varepsilon\right)}{\delta-A\varepsilon\bar{K}^{\varepsilon-1}}\right\rangle\simeq\frac{\left(\Gamma_2-\Gamma_1\right)+\bar{K}^{\varepsilon}\left(1-\varepsilon\right)\Gamma_3}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}$$

and the potential terms are then:

$$\begin{aligned}
V &= \left(\varsigma^2+\frac{\left(AF'(K)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}\right)^2+\frac{\left(K'\right)^2}{\nu^2}+\left(A-\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)\right)^2\\
&\quad+\gamma\eta\left(\frac{\left(\Gamma_2-\Gamma_1\right)+\bar{K}^{\varepsilon}\left(1-\varepsilon\right)\Gamma_3}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}A+\Gamma_3\frac{\left(K'-C\right)+A\bar{K}^{\varepsilon}\left(1-\varepsilon\right)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}\right)+\bar{A}^2-\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)^2
\end{aligned}$$

and can be written in a compact form:

$$\begin{aligned}
V &= \left({}^t\left(X-\hat{X}\right)\Omega\left(X-\hat{X}\right)+\left({}^t\Gamma\right)MX\right) \qquad (103)\\
&= \left({}^t\left(X-\hat{X}+\frac{1}{2}\Omega^{-1}M\Gamma\right)\right)\Omega\left(X-\hat{X}+\frac{1}{2}\Omega^{-1}M\Gamma\right)-\frac{1}{4}\left({}^t\Gamma M\Omega^{-1}M\Gamma\right)+\left({}^t\Gamma\right)M\hat{X}\\
&\quad+\bar{A}^2-\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)^2
\end{aligned}$$

with:

$$\begin{aligned}
X &= \begin{pmatrix}C\\K'\\A\end{pmatrix},\hat{X}=\begin{pmatrix}\bar{C}\\0\\\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\end{pmatrix},\Gamma=\begin{pmatrix}\Gamma_1\\\Gamma_2\\\Gamma_3\end{pmatrix},\Omega=\begin{pmatrix}\left(\varsigma^2+\frac{\left(\Gamma_3F'(K)+r_c\right)^2}{\varpi^2}\right)&0&0\\0&\frac{1}{\nu^2}&0\\0&0&1\end{pmatrix}\\
M &= \frac{\gamma\eta}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}\begin{pmatrix}0&0&-1\\0&0&1\\-1&1&\bar{K}^{\varepsilon}\left(1-\varepsilon\right)\end{pmatrix}
\end{aligned}$$

From these equations we can identify the mean values of the variables. Given that we anticipate a gaussian form for $\Psi_1(K,C,A)$, we have to implement several constraint. Actully, we assume that $C>0$, $A>0$, so that the distribution for these variables has to be cut off for the egative values. The variable K' on its side is also constrained by the model and our assumptions. First, given that:

$$K=\frac{\left(K'-C\right)+A\bar{K}^{\varepsilon}\left(1-\varepsilon\right)}{\delta-A\varepsilon\bar{K}^{\varepsilon-1}}$$

The variable K has to be positive, so that:

$$K'<C-A\bar{K}^{\varepsilon}\left(1-\varepsilon\right)$$

Moreover, we have assumed that:

$$\delta-\varepsilon A\left(K\right)^{\varepsilon-1}<0$$

which translates in:

$$\delta-\varepsilon A\left(\frac{\left(K'-C\right)+A\bar{K}^{\varepsilon}\left(1-\varepsilon\right)}{\delta-A\varepsilon\bar{K}^{\varepsilon-1}}\right)^{\varepsilon-1}<0$$

or equivalently:

$$C+\left(\delta-A\varepsilon\bar{K}^{\varepsilon-1}\right)\left(\frac{\varepsilon A}{\delta}\right)^{\frac{1}{1-\varepsilon}}-A\bar{K}^{\varepsilon}\left(1-\varepsilon\right)<K'$$

As a consequence, the domain for is bounded K':

$$C+\left(\delta-A\varepsilon\bar{K}^{\varepsilon-1}\right)\left(\frac{\varepsilon A}{\delta}\right)^{\frac{1}{1-\varepsilon}}-A\bar{K}^{\varepsilon}\left(1-\varepsilon\right)<K'<C-A\bar{K}^{\varepsilon}\left(1-\varepsilon\right)$$

We will see later that the existence a non trivial phase is possible if $A \gg 0$. As a consequence, we can assume that $C - A\bar{K}^{\varepsilon}(1-\varepsilon) < 0$ and

$$\left|\delta - A\varepsilon\bar{K}^{\varepsilon-1}\right| \left(\frac{\varepsilon A}{\delta}\right)^{\frac{1}{1-\varepsilon}}$$

wich means that a good approximation for the domain in question is:

$$K' < C - A\bar{K}^{\varepsilon}(1-\varepsilon)$$

Given these conditions on the variables, the equations for the average values can thus be written as:

$$\begin{aligned} \Gamma &= \hat{X} - \frac{1}{2}\Omega^{-1}M\Gamma + \hat{X}_1 \\ \Gamma &= \left(1 + \frac{1}{2}\Omega^{-1}M\right)^{-1}\left(\hat{X} + \hat{X}_1\right) \end{aligned} \tag{104}$$

with $\hat{X}_1 = \begin{pmatrix} C_1 \\ K_1' \\ A_1 \end{pmatrix}$.The variables C_1 and A_1 express the shift due to the cut off for the negative values of C and A.

$$C_1 = \frac{\int_0^{\infty} X_1 \exp\left(-\frac{\left(X-\hat{X}+\frac{1}{2}\Omega^{-1}M\Gamma\right)_1^2}{2\varpi^2}\right) dX_1}{\int_0^{\infty} \exp\left(-\frac{\left(X-\hat{X}+\frac{1}{2}\Omega^{-1}M\Gamma\right)_1^2}{2\varpi^2}\right) dX_1} = \sqrt{\frac{2}{\pi}}\varpi \frac{\exp\left(-\frac{\left(\hat{X}-\frac{1}{2}\Omega^{-1}M\Gamma\right)_1^2}{2\varpi^2}\right)}{1 - \operatorname{erf}\left(\frac{\left(\hat{X}-\frac{1}{2}\Omega^{-1}M\Gamma\right)_1}{\sqrt{2}\varpi}\right)} \tag{105}$$

$$A_1 = \frac{\int_0^{\infty} X_2 \exp\left(-\frac{\lambda\left(X-\hat{X}+\frac{1}{2}\Omega^{-1}M\Gamma\right)_2^2}{2}\right) dX_2}{\int_0^{\infty} \exp\left(-\frac{\lambda\left(X-\hat{X}+\frac{1}{2}\Omega^{-1}M\Gamma\right)_2^2}{2}\right) dX_2} = \frac{2}{\sqrt{\pi\lambda}} \frac{\exp\left(-\frac{\lambda\left(\hat{X}-\frac{1}{2}\Omega^{-1}M\Gamma\right)_3^2}{2}\right)}{1 - \operatorname{erf}\left(\frac{\sqrt{\lambda}\left(\hat{X}-\frac{1}{2}\Omega^{-1}M\Gamma\right)_3}{\sqrt{2}}\right)} \tag{106}$$

whereas K_1' expresses the shift due to the superior bound for K':

$$\begin{aligned} K_1' &= \frac{\int_{-\infty}^{C-A\bar{K}^{\varepsilon}(1-\varepsilon)} X_3 \exp\left(-\frac{\left(X-\hat{X}+\frac{1}{2}\Omega^{-1}M\Gamma\right)_2^2}{2\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|\nu^2}\right) dX_3}{\int_{-\infty}^{C-A\bar{K}^{\varepsilon}(1-\varepsilon)} \exp\left(-\frac{\left(X-\hat{X}+\frac{1}{2}\Omega^{-1}M\Gamma\right)_2^2}{2\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|\nu^2}\right) dX_3} \\ &= -\frac{\sqrt{\frac{2}{\pi}}\sqrt{\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\nu \exp\left(-\frac{\left(C-A\bar{K}^{\varepsilon}(1-\varepsilon)-\left(\hat{X}-\frac{1}{2}\Omega^{-1}M\Gamma\right)_2\right)^2}{2\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|\nu^2}\right)}{\left(\operatorname{erf}\left(\frac{\left(C-A\bar{K}^{\varepsilon}(1-\varepsilon)-\left(\hat{X}-\frac{1}{2}\Omega^{-1}M\Gamma\right)_2\right)}{\sqrt{2}}\right)+1\right)} \end{aligned}$$

and this last expression can be approximated by:

$$K_1' \simeq -2\nu^2\left|\delta - A\varepsilon\bar{K}^{\varepsilon-1}\right| \frac{\exp\left(-\frac{\left(C-A\bar{K}^{\varepsilon}(1-\varepsilon)-\left(\hat{X}-\frac{1}{2}\Omega^{-1}M\Gamma\right)_2\right)^2}{2\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|\nu^2}\right)}{2 - \exp\left(-1.9\left(\frac{\left|C-A\bar{K}^{\varepsilon}(1-\varepsilon)\right|}{\sqrt{2\nu^2\left|\delta-A\varepsilon\bar{K}^{\varepsilon-1}\right|}}\right)^{1.3}\right)} \tag{107}$$

For a saddle point equation written as:

$$\begin{aligned} 0 = \Bigg\{ &-\varpi^2\frac{\partial^2}{\partial C^2} - \frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2} - \nu^2\left(\delta - AF'(K)\right)^2\frac{\partial^2}{\partial K'^2} + {}^t\left(X - \hat{X} + \frac{1}{2}\Omega^{-1}M\Gamma\right)\Omega\left(X - \hat{X} + \frac{1}{2}\Omega^{-1}M\Gamma\right) \\ &-\frac{1}{4}\left({}^t\Gamma M\Omega^{-1}M\Gamma\right) + \left({}^t\Gamma\right)M\hat{X} + \left(\bar{A}^2 - \left((1-\varkappa)\bar{A} + \varkappa\Gamma_3\right)^2\right) + \alpha - C_0\Bigg\}\Psi_1(K,C,A) \end{aligned} \tag{108}$$

Given that the σ_i have been considered relatively small for capital and technology, we can neglect the term proportional to the exponential in (104) for these two variables. We will keep this additional contribution for C only which shift $\hat{C}$ by $\frac{2}{\sqrt{2\pi}}\varpi$ for small $\hat{C}$.

The constant term in (103) is thus given by:

$$\begin{aligned}
-\frac{1}{4}\left({}^t\Gamma M\Omega^{-1}M\Gamma\right)+\left({}^t\Gamma\right)M\hat{X} &= -\frac{1}{4}\left({}^t\Gamma M\Omega^{-1}M\Gamma\right)+\left({}^t\Gamma\right)M\left(1+\frac{1}{2}\Omega^{-1}M\right)\Gamma \\
&= \left({}^t\Gamma\right)\left(M+\frac{1}{4}M\Omega^{-1}M\right)\Gamma \\
&= \left({}^t\hat{X}+\hat{X}_1\right)\left(1+\frac{1}{2}M\Omega^{-1}\right)^{-1}\left(M+\frac{1}{4}M\Omega^{-1}M\right)\left(1+\frac{1}{2}\Omega^{-1}M\right)^{-1}\left(\hat{X}+\hat{X}_1\right)
\end{aligned}$$

Equation (104) can be expressed as an equation for Γ_3:

$$\begin{aligned}
\Gamma_3 &= 2\frac{2\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3+A_1\right)+\left(\left(\bar{C}+C_1\right)-K_1'\right)+\alpha\gamma\eta}{4-\left(a+b\right)\alpha^2\left(\gamma\eta\right)^2+2\alpha\beta\gamma\eta} \\
&= 2\frac{2\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3+A_1\right)\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)^2+\left(\left(\bar{C}+C_1\right)-K_1'\right)\gamma\eta\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)}{4\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)^2-\left(a+b\right)(\gamma\eta)^2\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)+2\gamma\eta\bar{K}^{\varepsilon}\left(1-\varepsilon\right)} \\
&= 2\frac{2\left(\left(1-\varkappa\right)A_0+\left(2-\varkappa\right)\varkappa\Gamma_3+A_1\right)\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)^2+\left(\left(\bar{C}+C_1\right)-K_1'\right)\gamma\eta\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)}{4\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)^2-\left(a+b\right)\left(\gamma\eta\right)^2\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)+2\gamma\eta\bar{K}^{\varepsilon}\left(1-\varepsilon\right)}
\end{aligned} \tag{109}$$

where:

$$a+b=\frac{\varpi^2}{\varsigma^2\varpi^2+\left(AF'\left(K\right)+r_c\right)^2}+\nu^2\ll 1$$

and:

$$\alpha=\frac{1}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}},\beta=\frac{\bar{K}^{\varepsilon}\left(1-\varepsilon\right)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}$$

At the first order in $\gamma\eta$, the equation rewrites:

$$\Gamma_3=2\frac{2\left(\left(1-\varkappa\right)A_0+\left(2-\varkappa\right)\varkappa\Gamma_3+A_1\right)\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)^2+\left(\left(\bar{C}+C_1\right)-K_1'\right)\gamma\eta\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)}{4\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)^2+2\gamma\eta\bar{K}^{\varepsilon}\left(1-\varepsilon\right)}$$

For $\gamma\eta\ll 1$, we check that $\Gamma_3\simeq\frac{A_0}{1-\varkappa}$, in that case, using (106):

$$A_1=\frac{2}{\sqrt{\lambda}}\frac{\exp\left(-\frac{\lambda\left(\hat{X}-\frac{1}{2}\Omega^{-1}M\Gamma\right)_3^2}{2}\right)}{2-\operatorname{erf}\left(\frac{\sqrt{\lambda}\left(\hat{X}-\frac{1}{2}\Omega^{-1}M\Gamma\right)_3}{\sqrt{2}}\right)}\simeq\frac{2}{\sqrt{\lambda}}\exp\left(-\frac{\lambda\left(\frac{A_0}{1-\varkappa}\right)^2}{2}\right)\ll 1$$

and A_1 can be neglected, whivh leads to:

$$\Gamma_3=2\frac{2\left(\left(1-\varkappa\right)A_0+\left(2-\varkappa\right)\varkappa\Gamma_3\right)\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)^2+\left(\left(\bar{C}+C_1\right)-K_1'\right)\gamma\eta\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)}{4\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)^2+2\gamma\eta\bar{K}^{\varepsilon}\left(1-\varepsilon\right)}$$

We assume that $\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)<0$, so that the marginal productivity exceeds the depreciation rate of capital, we see below that the solution for $\gamma\eta$ is of first order in $\delta-\frac{A_0}{(1-\varkappa)}\varepsilon\bar{K}^{\varepsilon-1}$, which allows for a first order solution in $\gamma\eta$ to be considered:

$$\Gamma_3=\frac{A_0}{(1-\varkappa)}-\frac{1}{2}\frac{K^{\varepsilon}A_0\left(1-\varepsilon\right)-\left(\left(\bar{C}+C_1\right)-K_1'\right)\left(\delta-\frac{A_0}{(1-\varkappa)}\varepsilon\bar{K}^{\varepsilon-1}\right)(1-\varkappa)}{\left(\delta-\frac{A_0}{(1-\varkappa)}\varepsilon\bar{K}^{\varepsilon-1}\right)^2(1-\varkappa)^3}\gamma\eta \tag{110}$$

The quantities Γ_1 and Γ_2 can be expressed in terms of Γ_3:

$$\begin{aligned}
\Gamma_1 &\simeq \frac{4\left(\bar{C}+C_1\right)-\left(\bar{C}+C_1\right)b\alpha^2\left(\gamma\eta\right)^2+2a\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)\alpha\gamma\eta+2\left(\bar{C}+C_1\right)\alpha\beta\gamma\eta+\frac{\alpha^2\varpi^2(\gamma\eta)^2K_1'}{\left(\varpi^2\varsigma^2+(\Gamma_3F'(K)+r_c)^2\right)}}{4-\left(a+b\right)\alpha^2\left(\gamma\eta\right)^2+2\alpha\beta\gamma\eta} \\
&= \frac{4\left(\bar{C}+C_1\right)-\left(\bar{C}+C_1\right)b\alpha^2\left(\gamma\eta\right)^2+2a\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)\alpha\gamma\eta+2\left(\bar{C}+C_1\right)\alpha\beta\gamma\eta+\frac{\alpha^2\varpi^2(\gamma\eta)^2K_1'}{\left(\varpi^2\varsigma^2+(\Gamma_3F'(K)+r_c)^2\right)}}{2\left(2\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)+\left(\bar{C}+C_1\right)\alpha\gamma\eta\right)}\Gamma_3 \\
\Gamma_2 &= -\frac{\left(\gamma\eta\right)^2b\alpha^2\left(\bar{C}+C_1\right)+2b\alpha\gamma\eta\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)-\left(4+2K^{\varepsilon}\left(1-\varepsilon\right)\alpha\gamma\eta-\frac{(\alpha\gamma\eta)^2}{\left(\varsigma^2+\frac{(\Gamma_3F'(K)+r_c)^2}{\varpi^2}\right)}\right)K_1'}{4-\left(a+b\right)\alpha^2\left(\gamma\eta\right)^2+2\alpha\beta\gamma\eta} \\
&= -\frac{\left(\gamma\eta\right)^2b\alpha^2\left(\bar{C}+C_1\right)+2b\alpha\gamma\eta\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)-\left(4+2K^{\varepsilon}\left(1-\varepsilon\right)\alpha\gamma\eta-\frac{(\alpha\gamma\eta)^2}{\left(\varsigma^2+\frac{(\Gamma_3F'(K)+r_c)^2}{\varpi^2}\right)}\right)K_1'}{2\left(2\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)+\left(\bar{C}+C_1\right)\alpha\gamma\eta\right)}\Gamma_3
\end{aligned}$$

For $\gamma\eta\neq 0$, one has at the first order in $\gamma\eta$:

$$\Gamma_1=\bar{C}+C_1-\frac{\varpi^2\left(\frac{A_0}{1-\chi}\right)}{2\left(\varsigma^2\varpi^2+\left(AF'\left(K\right)+r_c\right)^2\right)\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\gamma\eta$$

C_1 can be found by writing (105) as:

$$\begin{aligned}
C_1 &= \sqrt{\frac{2}{\pi}}\varpi\frac{\exp\left(-\frac{\left(\hat{X}-\frac{1}{2}\Omega^{-1}M\Gamma\right)_1^2}{2\varpi^2}\right)}{1-\operatorname{erf}\left(\frac{\left(\hat{X}-\frac{1}{2}\Omega^{-1}M\Gamma\right)_1}{\sqrt{2}\varpi}\right)}=\sqrt{\frac{2}{\pi}}\varpi\frac{\exp\left(-\frac{\left(\Gamma_1-C_1\right)_1^2}{2\varpi^2}\right)}{1-\operatorname{erf}\left(\frac{\left(\Gamma_1-C_1\right)_1}{\sqrt{2}\varpi}\right)} \\
&= \sqrt{\frac{2}{\pi}}\varpi\frac{\exp\left(-\frac{\left(\bar{C}\right)_1^2}{2\varpi^2}\right)}{1-\operatorname{erf}\left(\frac{\left(\bar{C}\right)_1}{\sqrt{2}\varpi}\right)}
\end{aligned}$$

where we used (104). For $\bar{C}\ll 1$, one has:

$$C_1\simeq\sqrt{\frac{2}{\pi}}\varpi$$

The value of Γ_2 is computed in the same way:

$$\Gamma_2=-\frac{2b\alpha\gamma\eta\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)-\left(4+2K^{\varepsilon}\left(1-\varepsilon\right)\alpha\gamma\eta\right)K_1'}{4+2\alpha\beta\gamma\eta}$$

At the zeroth order in $\gamma\eta$, it reduces to:

$$\Gamma_2=K_1'$$

and the parameter K_1' is found by considering (107):

$$\begin{aligned}
K_1' &= -\frac{\sqrt{\frac{2}{\pi}}\sqrt{\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\nu\exp\left(-\frac{\left(C-A\bar{K}^{\varepsilon}(1-\varepsilon)-\left(\hat{X}-\frac{1}{2}\Omega^{-1}M\Gamma\right)_2\right)^2}{2\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|\nu^2}\right)}{\left(\operatorname{erf}\left(\frac{\left(C-A\bar{K}^{\varepsilon}(1-\varepsilon)-\left(\hat{X}-\frac{1}{2}\Omega^{-1}M\Gamma\right)_2\right)}{\sqrt{2\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|\nu^2}}\right)+1\right)} \\
&\simeq -\frac{\sqrt{\frac{2}{\pi}}\sqrt{\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\nu\exp\left(-\frac{\left(C-A\bar{K}^{\varepsilon}(1-\varepsilon)\right)^2}{2\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|\nu^2}\right)}{\left(\operatorname{erf}\left(\frac{\left(C-A\bar{K}^{\varepsilon}(1-\varepsilon)-\left(\hat{X}-\frac{1}{2}\Omega^{-1}M\Gamma\right)_2\right)}{\sqrt{2}}\right)+1\right)} \\
&\simeq -\frac{2}{\pi}\left|\bar{C}+\sqrt{\frac{2}{\pi}}\varpi-A\bar{K}^{\varepsilon}\left(1-\varepsilon\right)\right|-\frac{\sqrt{2\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}}{\sqrt{\pi}}\nu
\end{aligned}$$

for $\frac{\left|C-A\bar{K}^{\varepsilon}(1-\varepsilon)\right|}{\sqrt{\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\nu}<1$ and using (104).

At the first order in $\gamma\eta$, one finds:

$$\Gamma_2 = K_1' - \frac{\nu^2 \frac{A_0}{1-\chi}\left(\delta - \Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right) + \bar{K}^{\varepsilon}(1-\varepsilon)\left(1-\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)\right)K_1'}{2\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)^2}\gamma\eta$$

These expressions allow to find the expectation value of K in the state Ψ_1:

$$\begin{aligned}\langle K\rangle &= \left\langle \frac{(K'-C)+A\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta - A\varepsilon\bar{K}^{\varepsilon-1}}\right\rangle \\ &\simeq \frac{(\Gamma_2-\Gamma_1)+\Gamma_3\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}} \\ &= \frac{\left(\Gamma_2-\left(\bar{C}+2\varpi\right)\right)+\Gamma_3\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}\end{aligned}$$

which expands, at first order in $\gamma\eta$, as:

$$\begin{aligned}\langle K\rangle = - \quad & \frac{x - K^{\varepsilon}\frac{A_0}{1-\varkappa}(1-\varepsilon)}{Y} - \frac{1}{2}K^{\varepsilon}\left(K^{\varepsilon}A_0(1-\varepsilon) - Y(1-\varkappa)x\right)\frac{\varepsilon x - K\delta(1-\varepsilon)}{Y^4K(1-\varkappa)^3}z\gamma\eta \\ & -\frac{\bar{K}^{\varepsilon}(1-\varepsilon)(1-Y)K_1'}{2Y^3}\gamma\eta - \frac{\nu^2\frac{A_0}{1-\chi}}{2Y^2}\gamma\eta - \frac{\varpi^2\left(\frac{A_0}{1-\chi}\right)}{2\left(\varsigma^2\varpi^2+\left(AF'(K)+r_c\right)^2\right)Y^2}\gamma\eta\end{aligned}$$

where:

$$\begin{aligned}Y &= \left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right) \\ x &= \bar{C}+\sqrt{\frac{2}{\pi}}\varpi - K_1'\end{aligned}$$

One can also compare the average production $\langle Y\rangle = \langle A\rangle\langle K\rangle$ in both phases. For $\gamma\eta = 0$:

$$\langle Y\rangle_0 = \frac{A_0}{1-\varkappa}\left(\frac{x - K^{\varepsilon}\frac{A_0}{1-\varkappa}(1-\varepsilon)}{\left|\delta - K^{\varepsilon-1}\varepsilon\frac{A_0}{1-\varkappa}\right|}\right)^{\varepsilon}$$

For $\gamma\eta = 0$, neglecting the terms proportional to ν^2 and ϖ^2, one finds:

$$\begin{aligned}\langle Y\rangle_1 \quad < \quad & \frac{A_0}{1-\varkappa}\left(\frac{x - K^{\varepsilon}\frac{A_0}{1-\varkappa}(1-\varepsilon)}{\left|\delta - K^{\varepsilon-1}\varepsilon\frac{A_0}{1-\varkappa}\right|}\right)^{\varepsilon} \\ & -\frac{\left(K^{\varepsilon}A_0(1-\varepsilon) - Yx(1-\varkappa)\right)}{2Y^2(1-\varkappa)^3}\left(1-\frac{\bar{r}}{\bar{r}-\delta}\left(\varepsilon+\frac{(1-\varepsilon)\bar{K}}{\langle K\rangle}\right)\right)\left(\frac{x - K^{\varepsilon}\frac{A_0}{1-\varkappa}(1-\varepsilon)}{\left|\delta - K^{\varepsilon-1}\varepsilon\frac{A_0}{1-\varkappa}\right|}\right)^{\varepsilon}\gamma\eta\end{aligned}$$

with:

$$\bar{r} = \frac{\varepsilon K^{\varepsilon}A_0}{1-\varkappa}$$

For a minimal stock of capital lower than the average one, one has $\frac{\bar{K}}{\langle K\rangle} \ll 1$ and $\bar{r} \gg \delta$ since for the minimal capital stock, the marginal productivity exceeds the depreciation rate to allow accumulation. As a consequence:

$$1-\frac{\bar{r}}{\bar{r}-\delta}\left(\varepsilon+\frac{(1-\varepsilon)\bar{K}}{\langle K\rangle}\right) \simeq 1-\varepsilon\left(1+\frac{\delta}{\bar{r}}\right) > 0$$

for usual values of ε, $\varepsilon \lesssim 0.3$. As a consequence $\langle Y\rangle_1 < \langle Y\rangle_0$ in most cases.

For the rest of the section, we redefine $\bar{C} + \sqrt{\frac{2}{\pi}}\varpi - K_1' \to \bar{C}$. The previous results lead then to the quadratic term in (102):

$$\begin{aligned}& \left({}^t\hat{X}\right)\left(1+\frac{1}{2}M\Omega^{-1}\right)^{-1}\left(M+\frac{1}{4}M\Omega^{-1}M\right)\left(1+\frac{1}{2}\Omega^{-1}M\right)^{-1}\hat{X} \\ = \quad & -\frac{4\gamma\eta\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)\left(2\left(\frac{\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}+4\right)\left((1-\varkappa)\bar{A}+\varkappa\Gamma_3\right)\bar{C}+\left(\frac{\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}+3\right)\bar{C}^2\right)}{4\left(\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)+2\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)\right)^2} \\ & +\frac{4\gamma\eta\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)\left(\frac{\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}+4\right)\left((1-\varkappa)\bar{A}+\varkappa\Gamma_3\right)^2}{4\left(\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)+2\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)\right)^2}\end{aligned}$$

Gathering the equation for Γ_3, and the condition for the cancellation of the denominator leads to the system:

$$\begin{aligned}\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)+2\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right) &= 0\\ \Gamma_3 &= \frac{2\left((1-\varkappa)A_0+(2-\varkappa)\varkappa\Gamma_3\right)\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)^2+\bar{C}\gamma\eta\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)}{2\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)^2-(a+b)(\gamma\eta)^2\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)+\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)}\end{aligned}$$

$$\frac{2\delta+x\bar{K}^{\varepsilon}(1-\varepsilon)}{2\varepsilon\bar{K}^{\varepsilon-1}}=2\frac{2\left((1-\varkappa)A_0+(2-\varkappa)\varkappa\frac{2\delta+x\bar{K}^{\varepsilon}(1-\varepsilon)}{2\varepsilon\bar{K}^{\varepsilon-1}}\right)\left(x\frac{\bar{K}^{\varepsilon}(1-\varepsilon)}{2}\right)^2-\bar{C}x^2\frac{\bar{K}^{\varepsilon}(1-\varepsilon)}{2}}{4\left(x\frac{\bar{K}^{\varepsilon}(1-\varepsilon)}{2}\right)^2+(a+b)x^3\frac{\bar{K}^{\varepsilon}(1-\varepsilon)}{2}+2x\bar{K}^{\varepsilon}(1-\varepsilon)}$$

which has no solution, and thus the potential is defined for all $\gamma\eta$. The solution Γ_3 for $\gamma\eta\to\infty$ has the asymptotic form:

$$\Gamma_3=c\gamma\eta$$

with constant c satisfying:

$$c=\frac{(2-\varkappa)\varkappa\varepsilon\bar{K}^{\varepsilon-1}c^2}{(a+b)}$$

so that:

$$\Gamma_3=\frac{(a+b)\bar{K}^{1-\varepsilon}}{(2-\varkappa)\varkappa\varepsilon}\gamma\eta$$

The term $\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}$ cancels at $\gamma\eta\gg 1$, for:

$$\gamma\eta\to\frac{\delta(2-\varkappa)\varkappa}{a+b}\gg 1$$

$$\Gamma_3-\to\frac{\delta}{\varepsilon\bar{K}^{\varepsilon-1}}$$

and our approximations are no more valid above these values. For $\gamma\eta\to\frac{\delta(2-\varkappa)\varkappa}{a+b}$, one finds:

$$\begin{aligned}&-\frac{4\gamma\eta\left(2\left(\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)\right)\left((1-\varkappa)\bar{A}+\varkappa\Gamma_3\right)\bar{C}+\left(\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)\right)\bar{C}^2-\left(\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)\right)\left((1-\varkappa)\bar{A}+\varkappa\Gamma_3\right)^2\right)}{4\left(\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)\right)^2}\\ =\;&\frac{\left(\left((1-\varkappa)\bar{A}+\varkappa\Gamma_3\right)^2-2\bar{C}\left((1-\varkappa)\bar{A}+\varkappa\Gamma_3\right)-\bar{C}^2\right)}{(1-\varepsilon)K^{\varepsilon}}\end{aligned}$$

If $\left((1-\varkappa)\bar{A}+\varkappa\Gamma_3\right)^2-2C\left((1-\varkappa)\bar{A}+\varkappa\Gamma_3\right)-C^2>0$, i.e. for:

$$(1-\varkappa)A_0+(2-\varkappa)\varkappa\Gamma_3>\left(1+\sqrt{2}\right)\bar{C}$$

that is:

$$A_0>\frac{\left(1+\sqrt{2}\right)\bar{C}-\frac{(2-\varkappa)\varkappa\delta}{\varepsilon}\bar{K}^{1-\varepsilon}}{1-\varkappa}$$

then:

$$\begin{aligned}0 &= \left({}^t\hat{X}\right)\left(1+\frac{1}{2}M\Omega^{-1}\right)^{-1}\left(M+\frac{1}{4}M\Omega^{-1}M\right)\left(1+\frac{1}{2}\Omega^{-1}M\right)^{-1}\hat{X}(0)\\ &< \left({}^t\hat{X}\right)\left(1+\frac{1}{2}M\Omega^{-1}\right)^{-1}\left(M+\frac{1}{4}M\Omega^{-1}M\right)\left(1+\frac{1}{2}\Omega^{-1}M\right)^{-1}\hat{X}\left(\frac{\delta(2-\varkappa)\varkappa}{a+b}\right)\end{aligned}$$

Equation (**??**) has solutions of the type:

$$\begin{aligned}\Psi_{n_1,n_2,n_3} &= H_{n_1}\left((C-\Gamma_1)\right)\exp\left(-\frac{\left(\sqrt{\frac{\varsigma^2}{\varpi^2}+\frac{(\hat{r}+r_c)^2}{\varpi^4}}\right)(C-\Gamma_1)^2}{2}\right)\\ &\quad\times H_{n_2}(A-\Gamma_3)\exp\left(-\frac{\lambda(A-\Gamma_3)^2}{2}\right)H_{n_3}\left(K'-\Gamma_2\right)\exp\left(-\frac{(K'-\Gamma_2)^2}{2\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|\nu^2}\right)\end{aligned}$$

with $\hat{r}$ defined by:

$$\hat{r}+r_c \simeq \left\langle \varepsilon A\left(\frac{(K'-C)+A\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta - A\varepsilon\bar{K}^{\varepsilon-1}}\right)^{\varepsilon-1}\right\rangle + r_c \simeq \varepsilon\Gamma_3\left(\frac{(\Gamma_2-\Gamma_1)+\bar{K}^{\varepsilon}(1-\varepsilon)\Gamma_3}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}\right)^{\varepsilon-1}+r_c$$

if the integers n_1, n_2, n_3 and C_0 satisfy the compatibility condition:

$$\begin{aligned} & \alpha - C_0 + \left({}^t\Gamma\right)\left(M+\frac{1}{4}M\Omega^{-1}M\right)\Gamma + \bar{A}^2 - \left((1-\varkappa)\bar{A}+\varkappa\Gamma_3\right)^2 \\ = \quad & -(2n_1+1)\sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2} - \frac{2n_2+1}{\lambda} - (2n_3+1)\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right| \end{aligned}$$

A minimum for the action may thus exist for

$$C_0 > \alpha + Min_{\eta}\left(\left({}^t\Gamma\right)\left(M+\frac{1}{4}M\Omega^{-1}M\right)\Gamma\right) + \sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2} + \frac{1}{\lambda} + \left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right| + \left(\bar{A}^2 - \left((1-\varkappa)\bar{A}+\varkappa\Gamma_3\right)^2\right)$$

Since the minimum for $\left(\left({}^t\Gamma\right)\left(M+\frac{1}{4}M\Omega^{-1}M\right)\Gamma\right)$ is 0 for $\gamma\eta=0$, and since for $\gamma\eta=0$, $\Gamma_3=\frac{A_0}{(1-\varkappa)}$, $\bar{A}=\frac{A_0}{(1-\varkappa)}$, so that:

$$\bar{A}^2 - \left((1-\varkappa)\bar{A}+\varkappa\Gamma_3\right)^2 = 0$$

and the condition reduces to:

$$C_0 > \alpha + \sqrt{\varsigma^2\varpi^2+(\hat{r}(0)+r_c)^2} + \frac{1}{\lambda} + \left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right| \tag{111}$$

In that case, the compatibility fixes the value for $\gamma\eta$. For $n_1=n_2=n_3=0$ it is:

$$\begin{aligned} 0 \quad & = \alpha - C_0 - \frac{4\gamma\eta\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)\left(2\left(\frac{\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}+4\right)\bar{A}\bar{C}+\left(\frac{\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}+3\right)\bar{C}^2-\left(\frac{\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}+4\right)\bar{A}^2\right)}{4\left(\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)+2\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)\right)^2} \\ & \quad + \sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2} + \frac{1}{\lambda} + \left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right| + \left(\bar{A}^2-\left((1-\varkappa)\bar{A}+\varkappa\Gamma_3\right)^2\right) \\ & \equiv \quad g(\eta) \end{aligned} \tag{112}$$

for $(a+b)\ll 1$. If we find a solution to (112) with $\eta\neq 0$, then (**??**) will have a solution with $\gamma\eta\neq 0$. To inspect (112), we compare the case $\gamma\eta=0$ and the case $\gamma\eta\to\frac{\delta(2-\varkappa)\varkappa}{a+b}$. For $\gamma\eta=0$:

$$g(0) = \alpha - C_0 + \sqrt{\varsigma^2\varpi^2 + \left(\varepsilon\left(\frac{A_0}{(1-\varkappa)}\right)^{\varepsilon}\left(\frac{\bar{K}^{\varepsilon}(1-\varepsilon)-\bar{C}}{\delta-\frac{A_0}{(1-\varkappa)}\varepsilon\bar{K}^{\varepsilon-1}}\right)^{\varepsilon-1}+r_c\right)^2\varpi^2} + \frac{1}{\lambda} + \left|\delta-\frac{A_0\varepsilon\bar{K}^{\varepsilon-1}}{(1-\varkappa)}\right|$$

If one has:

$$\left|\delta-\frac{A_0\varepsilon\bar{K}^{\varepsilon-1}}{(1-\varkappa)}\right| \ll 1$$

then:

$$\frac{\bar{K}^{\varepsilon}(1-\varepsilon)-\bar{C}}{\delta-\frac{A_0}{(1-\varkappa)}\varepsilon\bar{K}^{\varepsilon-1}} \gg 1$$

which implies:

$$\varepsilon\left(\frac{A_0}{(1-\varkappa)}\right)^{\varepsilon}\left(\frac{\bar{K}^{\varepsilon}(1-\varepsilon)-\bar{C}}{\delta-\frac{A_0}{(1-\varkappa)}\varepsilon\bar{K}^{\varepsilon-1}}\right)^{\varepsilon-1} \ll 1$$

so that:

$$g(0) = \alpha - C_0 + \sqrt{\varsigma^2\varpi^2+(r_c)^2\varpi^2} + \frac{1}{\lambda}$$

Now consider $\gamma\eta\to\frac{\delta(2-\varkappa)\varkappa}{a+b}$. In that case:

$$\begin{aligned} \bar{A}^2-\left((1-\varkappa)\bar{A}+\varkappa\Gamma_3\right)^2 \quad & = \quad (A_0+\varkappa\Gamma_3)^2-\left((1-\varkappa)A_0+(2-\varkappa)\varkappa\Gamma_3\right)^2 \\ & = \quad \left(A_0+\varkappa\frac{\delta}{\varepsilon\bar{K}^{\varepsilon-1}}\right)^2-\left((1-\varkappa)A_0+(2-\varkappa)\varkappa\frac{\delta}{\varepsilon\bar{K}^{\varepsilon-1}}\right)^2 \\ & = \quad -(1-\varkappa)K\frac{\varkappa\left(\delta-K^{\varepsilon-1}\varepsilon\frac{A_0}{(1-\varkappa)}\right)\left((2-\varkappa)A_0+(3-\varkappa)\varkappa\frac{\delta}{\varepsilon\bar{K}^{\varepsilon-1}}\right)}{\varepsilon K^{\varepsilon}} \end{aligned}$$

and this is positive given our assumptions. Moreover, given the same assumptions:

$$\begin{aligned}
&\frac{\left(\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)^2-2\bar{C}\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)-\bar{C}^2\right)}{\left(1-\varepsilon\right)K^{\varepsilon}}\\
=&\frac{\left(\left(\left(1-\varkappa\right)\left(A_0+\varkappa\frac{\delta\bar{K}^{1-\varepsilon}}{\varepsilon}\right)+\varkappa\frac{\delta\bar{K}^{1-\varepsilon}}{\varepsilon}\right)^2-2\bar{C}\left(\left(1-\varkappa\right)\left(A_0+\varkappa\frac{\delta\bar{K}^{1-\varepsilon}}{\varepsilon}\right)+\varkappa\frac{\delta\bar{K}^{1-\varepsilon}}{\varepsilon}\right)-\bar{C}^2\right)}{\left(1-\varepsilon\right)K^{\varepsilon}}>0
\end{aligned}$$

and as a consequence:

$$U=\bar{A}^2-\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)^2+\frac{\left(\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)^2-2\bar{C}\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)-\bar{C}^2\right)}{\left(1-\varepsilon\right)K^{\varepsilon}}>0$$

$$\begin{aligned}
\hat{r}+r_c &\simeq \left\langle\varepsilon A\left(\frac{\left(K'-C\right)+A\bar{K}^{\varepsilon}\left(1-\varepsilon\right)}{\delta-A\varepsilon\bar{K}^{\varepsilon-1}}\right)^{\varepsilon-1}\right\rangle+r_c\\
&\simeq \varepsilon\Gamma_3\left(\frac{\left(\Gamma_2-\Gamma_1\right)+\bar{K}^{\varepsilon}\left(1-\varepsilon\right)\Gamma_3}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}\right)^{\varepsilon-1}+r_c\rightarrow r_c
\end{aligned}$$

since $\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\rightarrow 0$ and $\varepsilon-1<0$. Then:

$$g\left(\frac{\delta\left(2-\varkappa\right)\varkappa}{a+b}\right)=\alpha-C_0+\sqrt{\varsigma^2\varpi^2+\left(r_c\right)^2\varpi^2}+\frac{1}{\lambda}+U$$

with $U>0$. As a consequence, $g\left(0\right)<g\left(\frac{\delta(2-\varkappa)\varkappa}{a+b}\right)$, and for all the parameters satisfying our assumptions:

$$\begin{aligned}
A_0 &> \frac{\left(1+\sqrt{2}\right)\bar{C}-\frac{(2-\varkappa)\varkappa\delta}{\varepsilon}\bar{K}^{1-\varepsilon}}{1-\varkappa}\\
\left|\delta-\frac{A_0\varepsilon\bar{K}^{\varepsilon-1}}{\left(1-\varkappa\right)}\right| &< <1
\end{aligned}$$

and all C_0 such that $g\left(\frac{\delta(2-\varkappa)\varkappa}{a+b}\right)>0>g\left(0\right)$, that is:

$$C_0\in\left]\alpha+\sqrt{\varsigma^2\varpi^2+\left(r_c\right)^2\varpi^2}+\frac{1}{\lambda},\alpha+\sqrt{\varsigma^2\varpi^2+\left(r_c\right)^2\varpi^2}+\frac{1}{\lambda}+U\right[$$

there is $\left(\gamma\eta\right)_0$ such that the equation $g\left(\left(\gamma\eta\right)_0\right)=0$. An estimation for $\gamma\eta$ can be obtained by rewriting the compatibility condition (112):

$$\begin{aligned}
0 &= \alpha-C_0-\frac{4\gamma\eta\left(2\left(\frac{\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}+4\right)\bar{A}\bar{C}+\left(\frac{\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}+3\right)\bar{C}^2-\left(\frac{\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}+4\right)\bar{A}^2\right)}{4\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)\left(\frac{\gamma\eta\bar{K}^{\varepsilon}(1-\varepsilon)}{\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)}+2\right)^2}\\
&+\sqrt{\varsigma^2\varpi^2+\left(\hat{r}+r_c\right)^2}+\frac{1}{\lambda}+\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|+\bar{A}^2-\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)^2
\end{aligned}$$

as:

$$\frac{x\left(2\left(x+4\right)\bar{A}\bar{C}+\left(x+3\right)\bar{C}^2-\left(x+4\right)\bar{A}^2\right)}{\left(x+2\right)^2}=D$$

with:

$$\begin{aligned}
x &= \frac{\gamma\eta\bar{K}^{\varepsilon}\left(1-\varepsilon\right)}{\left(\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right)}\\
D &= \bar{K}^{\varepsilon}\left(1-\varepsilon\right)\left(\alpha-C_0+\sqrt{\varsigma^2\varpi^2+\left(\hat{r}+r_c\right)^2}+\frac{1}{\lambda}+\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|+\bar{A}^2-\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)^2\right)
\end{aligned}$$

for $\varkappa\ll 1$, we can approximate $\left(\bar{A}^2-\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)^2\right)$ by its value for $\gamma\eta=0$:

$$\bar{A}^2-\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)^2=0$$

as well as $\hat{r}$ by 0. Note that $D > 0$, due to (111). The compatibility equation can be be expanded as:

$$\begin{aligned} & x\left(2\left(x+4\right)\bar{A}\bar{C}+\left(x+3\right)\bar{C}^{2}-\left(x+4\right)\bar{A}^{2}\right)-\left(x+2\right)^{2}D \\ = & \left(C^{2}+2C\bar{A}-\bar{A}^{2}-D\right)x^{2}+\left(3C^{2}+8C\bar{A}-4\bar{A}^{2}-4D\right)x-4D \end{aligned}$$

Given our assumptions $C^2 + 2C\bar{A} - \bar{A}^2 < 0$ and $3C^2 + 8C\bar{A} - 4\bar{A}^2 < 4\left(C^2 + 2C\bar{A} - \bar{A}^2\right) < 0$, so that:

$$x = \frac{-\left(3C^{2}+8C\bar{A}-4\bar{A}^{2}-4D\right)-\sqrt{\left(3C^{2}+8C\bar{A}-4\bar{A}^{2}-4D\right)^{2}+16D\left(C^{2}+2C\bar{A}-\bar{A}^{2}-D\right)}}{2\left(C^{2}+2C\bar{A}-\bar{A}^{2}-D\right)}$$

For $A_0 \gg 1$ and $D < A_0$ and thus:

$$\begin{aligned} |x| & \simeq & \frac{4D}{\left|3C^{2}+8C\bar{A}-4\bar{A}^{2}-4D\right|} \ll 1 \\ \gamma\eta & < & \frac{\left|\delta-\frac{A_{0}}{1-\chi}\varepsilon\bar{K}^{\varepsilon-1}\right|}{\bar{K}^{\varepsilon}\left(1-\varepsilon\right)}\frac{4D}{\left|3C^{2}+8C\bar{A}-4\bar{A}^{2}-4D\right|} \ll 1 \end{aligned}$$

Since:

$$C_{0}\in\left]\alpha+\sqrt{\varsigma^{2}\varpi^{2}+\left(r_{c}\right)^{2}\varpi^{2}}+\frac{1}{\lambda},\alpha+\sqrt{\varsigma^{2}\varpi^{2}+\left(r_{c}\right)^{2}\varpi^{2}}+\frac{1}{\lambda}+U\right[$$

with U defined by:

$$\begin{aligned} U = & -\left(1-\varkappa\right)K\frac{\varkappa\left(\delta-K^{\varepsilon-1}\varepsilon\frac{A_{0}}{\left(1-\varkappa\right)}\right)\left(\left(2-\varkappa\right)A_{0}+\left(3-\varkappa\right)\varkappa\frac{\delta}{\varepsilon\bar{K}^{\varepsilon-1}}\right)}{\varepsilon K^{\varepsilon}} \\ & +\frac{\left(\left((1-\varkappa)\left(A_{0}+\varkappa\frac{\delta\bar{K}^{1-\varepsilon}}{\varepsilon}\right)+\varkappa\frac{\delta\bar{K}^{1-\varepsilon}}{\varepsilon}\right)^{2}-2\bar{C}\left((1-\varkappa)\left(A_{0}+\varkappa\frac{\delta\bar{K}^{1-\varepsilon}}{\varepsilon}\right)+\varkappa\frac{\delta\bar{K}^{1-\varepsilon}}{\varepsilon}\right)-\bar{C}^{2}\right)}{\left(1-\varepsilon\right)K^{\varepsilon}} \end{aligned} \tag{113}$$

then

$$\alpha-C_{0}+\sqrt{\varsigma^{2}\varpi^{2}+\left(r_{c}\right)^{2}}+\frac{1}{\lambda}\in\left]0,U\right[$$

and:

$$\gamma\eta\in\left]0,8\frac{\left|\delta-\frac{A_{0}}{1-\chi}\varepsilon\bar{K}^{\varepsilon-1}\right|}{\bar{K}^{\varepsilon}\left(1-\varepsilon\right)}U\right[\tag{114}$$

Saddle point stability

The solution of (**??**) may thus present a non trivial minimum, as asserted before. To prove this point, we have to show that among the set of possible solutions of (**??**), the action $S(\Psi)$ is bounded from below. Moreover, the second order variation of $S(\Psi)$ around the solution with the lowest value of $S(\Psi)$ has to be positive. We write this second order variation $\delta^2 S(\Psi)$. We decompose the variation $\varphi(K,C,A)$ in three parts. The first part, $\varphi(K,C,A)$ is orthogonal to the fundamental $\Psi_1(K,C,A)$. We compute below its contribution to $\delta^2 S(\Psi)$. The second part is proportional to $\Psi_1(K,C,A)$ and corresponds to a variation of the norm η of $\sqrt{\eta}\Psi_1(K,C,A)$, and we write this variation $\delta\sqrt{\eta}\Psi_1(K,C,A)$. The variation of the action with respect to η is thus:

$$\begin{aligned} \frac{1}{2}\delta_{\sqrt{\eta}}S\left(\Psi\right) = & \left(\delta\sqrt{\eta}\right)^{2}\int\Psi_{1}^{\dagger}\left(K,C,A\right)\left\{-\varpi^{2}\frac{\partial^{2}}{\partial C^{2}}-\frac{1}{\lambda^{2}}\frac{\partial^{2}}{\partial A^{2}}-\nu^{2}\left(\delta-AF'\left(K\right)\right)^{2}\frac{\partial^{2}}{\partial K'^{2}}\right. \\ & \left.+\left(A-\bar{A}\right)^{2}+2\left(\bar{A}-\Gamma_{3}\right)\varkappa A+\left(\varsigma^{2}+\frac{\left(AF'\left(K\right)+r_{c}\right)^{2}}{\varpi^{2}}\right)\left(C-\bar{C}\right)^{2}+\frac{\left(K'\right)^{2}}{\nu^{2}}+\alpha-C_{0}\right\}\Psi_{1}\left(K,C,A\right) \\ & +3\gamma\left(\delta\sqrt{\eta}\right)^{2}\eta\int\Psi_{1}^{\dagger}\left(K,C,A\right)K\Psi_{1}\left(K,C,A\right)\int\Psi_{1}^{\dagger}\left(K,C,A\right)A\Psi_{1}\left(K,C,A\right) \end{aligned}$$

Given the saddle point equation (102), it reduces to:

$$\frac{1}{2}\delta_{\sqrt{\eta}}S\left(\Psi\right)=2\gamma\left(\delta\sqrt{\eta}\right)^{2}\eta\int\Psi_{1}^{\dagger}\left(K,C,A\right)K\Psi_{1}\left(K,C,A\right)\int\Psi_{1}^{\dagger}\left(K,C,A\right)A\Psi_{1}\left(K,C,A\right)>0 \tag{115}$$

Since this the average value of A multiplied by the average value of K in the state $\Psi_1(K,C,A)$. The third part is a combination of the variation in the direction of $\Psi_1(K,C,A)$ and of the $\varphi(K,C,A)$ variation orthogonal to $\Psi_1(K,C,A)$. The corresponding second order variation is:

$$\begin{aligned}
\frac{1}{2}\delta_{\sqrt{\eta},\varphi}S(\Psi) &= \delta\sqrt{\eta}\int\Big(\varphi^{\dagger}(K,C,A)+\varphi(K,C,A)\Big)\times \\
&\quad\left\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\left(\delta-AF'(K)\right)^2\frac{\partial^2}{\partial K'^2}+\left(A-\bar{A}\right)^2+2\left(\bar{A}-\Gamma_3\right)\varkappa A\right. \\
&\quad\left.+\left(\varsigma^2+\frac{\left(AF'(K)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}\right)^2+\frac{\left(K'\right)^2}{\nu^2}+\alpha-C_0\right\}\Psi_1(K,C,A) \\
&\quad+2\gamma\left(\delta\sqrt{\eta}\right)^2\eta\int\Big(\varphi^{\dagger}(K,C,A)+\varphi(K,C,A)\Big)K\Psi_1(K,C,A)\int\Psi_1^{\dagger}(K,C,A)A\Psi_1(K,C,A) \\
&\quad+2\gamma\left(\delta\sqrt{\eta}\right)^2\eta\int\Psi_1^{\dagger}(K,C,A)K\Psi_1(K,C,A)\int\Big(\varphi^{\dagger}(K,C,A)+\varphi(K,C,A)\Big)A\Psi_1(K,C,A) \\
&= \gamma\left(\delta\sqrt{\eta}\right)^2\eta\int\Big(\varphi^{\dagger}(K,C,A)+\varphi(K,C,A)\Big)K\Psi_1(K,C,A)\int\Psi_1^{\dagger}(K,C,A)A\Psi_1(K,C,A) \\
&\quad+\gamma\left(\delta\sqrt{\eta}\right)^2\eta\int\Psi_1^{\dagger}(K,C,A)K\Psi_1(K,C,A)\int\Big(\varphi^{\dagger}(K,C,A)+\varphi(K,C,A)\Big)A\Psi_1(K,C,A)
\end{aligned} \tag{116}$$

where the saddle point equation (102) has been used in the last equation.We will show below that such a contibution is neglible with respect to (115) or with respect variations involving $\varphi(K,C,A)$ only. As a consequence, the second order variation involving a variation in the direction of $\Psi_1(K,C,A)$ is positive. We can now turn to the part involving only variations $\varphi(K,C,A)$ orthogonal to $\Psi_1(K,C,A)$.

$$\begin{aligned}
\frac{1}{2}\delta^2S(\Psi) &= \int\varphi^{\dagger}(K,C,A)\left\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\left(\delta-AF'(K)\right)^2\frac{\partial^2}{\partial K'^2}+(A-\Gamma_3)^2\right. \\
&\quad\left.+\left(\varsigma^2+\frac{\left(\hat{r}+r_c\right)^2}{\varpi^2}\right)(C-\Gamma_1)^2+\frac{\left(K'-\Gamma_2\right)^2}{\nu^2}\right. \\
&\quad\left.+\alpha+\left({}^t\Gamma\right)\left(M+\frac{1}{4}M\Omega^{-1}M\right)\Gamma-C_0+\varkappa\left((2-\varkappa)A_0+\varkappa(3-\varkappa)\Gamma_3\right)\left(A_0-\Gamma_3(1-\varkappa)\right)\right\}\varphi(K,C,A) \\
&\quad-2(1-\varkappa)\varkappa\left(\int\Big(\varphi(K,C,A)+\varphi^{\dagger}(K,C,A)\Big)A\Psi_1(K,C,A)\right)^2 \\
&\quad+\gamma\eta\int\Big(\varphi(K,C,A)+\varphi^{\dagger}(K,C,A)\Big)K\Psi_1(K,C,A)\int\Big(\varphi(K,C,A)+\varphi^{\dagger}(K,C,A)\Big)A\Psi_1(K,C,A)
\end{aligned} \tag{117}$$

Where $\sqrt{\eta}\Psi_1(K,C,A)$ is the fundamental previously computed for $n_1=n_2=n_3=0$, $\left({}^t\Gamma\right)$, and $\left({}^t\Gamma\right)\left(M+\frac{1}{4}M\Omega^{-1}M\right)\Gamma$ is evaluated for this state. The perturbation $\varphi(K,C,A)$ orthogonal to this fundamental state $n_1=n_2=n_3=0$, and normalized to 1.

Given the compatibility condition,

$$0=\alpha-C_0+\left({}^t\Gamma\right)\left(M+\frac{1}{4}M\Omega^{-1}M\right)\Gamma+\sqrt{\varsigma^2\varpi^2+\left(\hat{r}+r_c\right)^2}+\frac{1}{\sqrt{\lambda}}+\left|\delta-A\varepsilon\bar{K}^{\varepsilon-1}\right|+\left(\bar{A}^2-\left((1-\varkappa)\bar{A}+\varkappa\Gamma_3\right)^2\right)$$

and the variation becomes:

$$\begin{aligned}
\frac{1}{2}\delta^2S(\Psi) &> \int\varphi^{\dagger}(K,C,A)\left\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\left(\delta-AF'(K)\right)^2\frac{\partial^2}{\partial K'^2}+(A-\Gamma_3)^2\right. \\
&\quad\left.+\left(\varsigma^2+\frac{\left(\hat{r}+r_c\right)^2}{\varpi^2}\right)(C-\Gamma_1)^2+\frac{\left(K'-\Gamma_2\right)^2}{\nu^2}-\left(\sqrt{\varsigma^2\varpi^2+\left(\hat{r}+r_c\right)^2}+\frac{1}{\sqrt{\lambda}}+\left|\delta-A\varepsilon\bar{K}^{\varepsilon-1}\right|\right)\right\}\varphi(K,C,A) \\
&\quad+\gamma\eta\int\Big(\varphi(K,C,A)+\varphi^{\dagger}(K,C,A)\Big)K\Psi_1(K,C,A)\int\Big(\varphi(K,C,A)+\varphi^{\dagger}(K,C,A)\Big)A\Psi_1(K,C,A)
\end{aligned}$$

The first part of $\frac{1}{2}\delta^2S(\Psi)$ is positive given the definition of the operator, only the last part can be negative. Given that:

$$\begin{aligned}
&\gamma\eta\int\Big(\varphi(K,C,A)+\varphi^{\dagger}(K,C,A)\Big)K\Psi_1(K,C,A)\int\Big(\varphi(K,C,A)+\varphi^{\dagger}(K,C,A)\Big)A\Psi_1(K,C,A) \\
\simeq\ &\gamma\eta\int\Big(\varphi(K,C,A)+\varphi^{\dagger}(K,C,A)\Big)\frac{(K'-C)+A\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}\Psi_1(K,C,A)\int\Big(\varphi(K,C,A)+\varphi^{\dagger}(K,C,A)\Big)A\Psi_1(K,C,A)
\end{aligned}$$

in first order approximation in $\Gamma_3 \varepsilon \bar{K}^{\varepsilon-1}$. This expression is non null for the components $(n_1, n_2, n_3) = (1,0,0)$, $(0,1,0)$ or $(0,0,1)$ of $\varphi(K,C,A)$, to inspect the sign of $\frac{1}{2}\delta^2 S(\Psi)$, we can restrict to:

$$\varphi = a_1 \Psi_{(1,0,0)} + a_2 \Psi_{(0,1,0)} + a_3 \Psi_{(0,0,1)} \equiv \sum_{i=1}^{3} a_i \Psi_i$$

with

$$|a_1|^2 + |a_2|^2 + |a_3|^2 = 1$$

For each variable X_i,

$$\begin{aligned}
& \int \Big(\varphi(K,C,A) + \varphi^\dagger(K,C,A)\Big) X_i \Psi_1(K,C,A) \\
= & \int \Big(\varphi(K,C,A) + \varphi^\dagger(K,C,A)\Big)(X_i - \Gamma_i)\Psi_1(K,C,A) + \Gamma_i \int \Big(\varphi(K,C,A) + \varphi^\dagger(K,C,A)\Big)\Psi_1(K,C,A) \\
= & (a_i + a_i^*) \int \Psi_i(K,C,A)(X_i - \Gamma_i)\Psi_1(K,C,A)
\end{aligned}$$

Now,

$$(X_i - \Gamma_i) = \frac{A_i^\dagger + A_i}{\sqrt{2\omega_i}}$$

where are the annihilation/creation operators, and with:

$$\begin{aligned}
\omega_1 &= \sqrt{\frac{\varsigma^2}{\varpi^2} + \frac{(\hat{r} + r_c)^2}{\varpi^4}} \\
\omega_2 &= \frac{1}{|\delta - AF'(K)|\,\nu} \\
\omega_3 &= \lambda
\end{aligned}$$

As a consequence:

$$\int \Big(\varphi(K,C,A) + \varphi^\dagger(K,C,A)\Big) X_i \Psi_1(K,C,A) = \frac{(a_i + a_i^*)}{\sqrt{2\omega_i}} \tag{118}$$

and this allows to estimate the various contributions in (117). The term proportional to $\varkappa$ can be computed as:

$$-2(1-\varkappa)\varkappa \left(\int \Big(\varphi(K,C,A) + \varphi^\dagger(K,C,A)\Big) A \Psi_1(K,C,A)\right)^2 = -2(1-\varkappa)\varkappa \left(\frac{a_3 + a_3^*}{\sqrt{2\lambda}}\right)^2$$

and

$$\begin{aligned}
& 8 \frac{\left|\delta - \frac{A_0}{1-\chi}\varepsilon \bar{K}^{\varepsilon-1}\right|}{\bar{K}^\varepsilon (1-\varepsilon)} \left|\alpha - C_0 + \sqrt{\varsigma^2 \varpi^2 + (r_c)^2} + \frac{1}{\lambda}\right| \\
& - \frac{4\left(K^\varepsilon A_0 (1-\varepsilon) - CY(1-\varkappa)\right)}{(1-\varkappa)\left|\delta - \frac{A_0}{1-\chi}\varepsilon \bar{K}^{\varepsilon-1}\right| \bar{K}^\varepsilon (1-\varepsilon)} \left|\alpha - C_0 + \sqrt{\varsigma^2 \varpi^2 + (r_c)^2} + \frac{1}{\sqrt{\lambda}}\right|
\end{aligned}$$

Other terms can be estimated in the same way:

$$\begin{aligned}
& \gamma\eta \int \Big(\varphi(K,C,A) + \varphi^\dagger(K,C,A)\Big) K \Psi_1(K,C,A) \int \Big(\varphi(K,C,A) + \varphi^\dagger(K,C,A)\Big) A \Psi_1(K,C,A) \\
= & \gamma\eta \int \Big(\varphi(K,C,A) + \varphi^\dagger(K,C,A)\Big) \frac{(K' - C) + A\bar{K}^\varepsilon(1-\varepsilon)}{\delta - \Gamma_3 \varepsilon \bar{K}^{\varepsilon-1}} \Psi_1(K,C,A) \int \Big(\varphi(K,C,A) + \varphi^\dagger(K,C,A)\Big) A \Psi_1(K,C,A) \\
= & \gamma\eta \frac{\left(\frac{(a_2 + a_2^*)}{\sqrt{2\omega_2}} - \frac{(a_1 + a_1^*)}{\sqrt{2\omega_1}}\right) + \frac{(a_3 + a_3^*)}{\sqrt{2\omega_3}} \bar{K}^\varepsilon (1-\varepsilon)}{\delta - \Gamma_3 \varepsilon \bar{K}^{\varepsilon-1}} \frac{(a_3 + a_3^*)}{\sqrt{2\omega_3}} \\
= & \gamma\eta \frac{\left(\frac{(a_2 + a_2^*)\sqrt{|\delta - AF'(K)|\nu}}{2\sqrt{\lambda}} - \frac{(a_1 + a_1^*)\varpi}{2\sqrt{\lambda\sqrt{\varsigma^2\varpi^2 + (\hat{r} + r_c)^2}}}\right)(a_3 + a_3^*) + \frac{(a_3 + a_3^*)^2}{2\lambda} \bar{K}^\varepsilon (1-\varepsilon)}{\delta - \Gamma_3 \varepsilon \bar{K}^{\varepsilon-1}}
\end{aligned}$$

Moreover, for:

$$\varphi = a_1 \Psi_{(1,0,0)} + a_2 \Psi_{(0,1,0)} + a_3 \Psi_{(0,0,1)} \equiv \sum_{i=1}^{3} a_i \Psi_i$$

the first part of $\frac{1}{2}\delta^2 S(\Psi)$ is equal to:

$$\begin{aligned}
&\int \varphi^{\dagger}(K,C,A)\left\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\left(\delta-AF'(K)\right)^2\frac{\partial^2}{\partial K'^2}+(A-\Gamma_3)^2\right.\\
&\left.+\left(\varsigma^2+\frac{(\hat{r}+r_c)^2}{\varpi^2}\right)(C-\Gamma_1)^2+\frac{(K'-\Gamma_2)^2}{\nu^2}-\left(\sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2}+\frac{1}{\lambda}+\left|\delta-A\varepsilon\bar{K}^{\varepsilon-1}\right|\right)\right\}\varphi(K,C,A)\\
=\;& 2|a_1|^2\sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2\varpi^2}+\frac{2|a_3|^2}{\lambda}+2\left|\delta-A\varepsilon\bar{K}^{\varepsilon-1}\right||a_2|^2
\end{aligned}$$

and then:

$$\begin{aligned}
\frac{1}{2}\delta^2 S(\Psi) \;>\;& 2|a_1|^2\sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2}+\frac{2|a_3|^2}{\lambda}+2\left|\delta-A\varepsilon\bar{K}^{\varepsilon-1}\right||a_2|^2+\gamma\eta\frac{\frac{\left(a_3+a_3^*\right)^2}{2\lambda}\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}\\
&+\gamma\eta\frac{\left(\frac{\left(a_2+a_2^*\right)\sqrt{\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\nu}{2\sqrt{\lambda}}-\frac{\left(a_1+a_1^*\right)\varpi}{2\sqrt{\lambda\sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2}}}\right)\left(a_3+a_3^*\right)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}-(1-\varkappa)\varkappa\frac{\left(a_3+a_3^*\right)^2}{\lambda}
\end{aligned}$$

Given that:

$$\begin{aligned}
\left|\left(a_2+a_2^*\right)\left(a_3+a_3^*\right)\right| &< 4|a_2||a_3|<2\left(|a_3|^2+|a_2|^2\right)\\
\left|\left(a_1+a_1^*\right)\left(a_3+a_3^*\right)\right| &< 2\left(|a_3|^2+|a_1|^2\right)
\end{aligned}$$

$$\begin{aligned}
\left|\left(a_2+a_2^*\right)\left(a_3+a_3^*\right)\right| &< 4|a_2||a_3|<2\left(|a_3|^2+|a_2|^2\right)\\
\left|\left(a_1+a_1^*\right)\left(a_3+a_3^*\right)\right| &< 2\left(|a_3|^2+|a_1|^2\right)
\end{aligned}$$

one has:

$$\begin{aligned}
&\gamma\eta\left|\left(\frac{\left(a_2+a_2^*\right)\sqrt{\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\nu}{2\sqrt{\lambda}}-\frac{\left(a_1+a_1^*\right)\varpi}{2\sqrt{\lambda\sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2}}}\right)\left(a_3+a_3^*\right)\right.\\
&\left.+\gamma\eta\frac{\frac{\left(a_3+a_3^*\right)^2}{2\lambda}\bar{K}^{\varepsilon}(1-\varepsilon)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}-(1-\varkappa)\varkappa\frac{\left(a_3+a_3^*\right)^2}{\lambda}\right|\\
<\;& \frac{\varpi\left(\gamma\eta\left(|a_3|^2+|a_2|^2\right)\frac{\sqrt{\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\nu}{\sqrt{\lambda}}+\gamma\eta\left(|a_3|^2+|a_1|^2\right)\right)}{\sqrt{\lambda\sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2}}}+\gamma\eta\frac{2|a_3|^2\bar{K}^{\varepsilon}(1-\varepsilon)}{\lambda\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}+4(1-\varkappa)\varkappa\frac{|a_3|^2}{\lambda}\\
<\;& \gamma\eta|a_1|^2\frac{\varpi}{\sqrt{\lambda\sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2}}}+\gamma\eta|a_2|^2\frac{\sqrt{\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\nu}{\sqrt{\lambda}}\\
&+\left(\gamma\eta\left(\frac{\sqrt{\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\nu}{2\sqrt{\lambda}}+\frac{\varpi}{2\sqrt{\lambda\sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2}}}+\frac{2\bar{K}^{\varepsilon}(1-\varepsilon)}{\lambda\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\right)+\frac{1}{\lambda}\right)|a_3|^2
\end{aligned}$$

As a consequence:

$$\begin{aligned}\frac{1}{2}\delta^2 S(\Psi) &> \left(2\sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2}-\gamma\eta\frac{\varpi}{\sqrt{\lambda\sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2}}}\right)|a_1|^2 \\ &+\left(2\left|\delta-A\varepsilon\bar{K}^{\varepsilon-1}\right|-\gamma\eta|a_2|^2\frac{\sqrt{\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\nu}{\sqrt{\lambda}}\right)|a_2|^2 \\ &+\left(\frac{1}{\lambda}-\gamma\eta\left(\frac{\sqrt{\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\nu}{2\sqrt{\lambda}}+\frac{\varpi}{2\sqrt{\lambda\sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2}}}+\frac{2\bar{K}^{\varepsilon}(1-\varepsilon)}{\lambda\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\right)\right)|a_3|^2\end{aligned}$$

Since we found that:

$$\gamma\eta<\frac{\left|\delta-\frac{A_0}{1-\chi}\varepsilon\bar{K}^{\varepsilon-1}\right|}{\bar{K}^{\varepsilon}(1-\varepsilon)}\frac{4D}{\left|3C^2+8C\bar{A}-4\bar{A}^2-4D\right|}$$

For $A_0\gg 1$, and for C_0 such that $D<A_0$, there is a large range of parameters such that the contributions proportional to $|a_i|^2$, $i=1,2,3$ are positive, and thus $\frac{1}{2}\delta^2 S(\Psi)>0$.

We conclude by noting that for similar arguments, the sum of (116) and (115) is positive. Actually (116) has the same form as (118) and is equal to:

$$\begin{aligned}&2\gamma\left(\delta\sqrt{\eta}\right)^2\eta\int\Psi_1^{\dagger}(K,C,A)K\Psi_1(K,C,A)\int\Psi_1^{\dagger}(K,C,A)A\Psi_1(K,C,A) \\ &+\gamma\left(\delta\sqrt{\eta}\right)^2\eta\int\Psi_1^{\dagger}(K,C,A)K\Psi_1(K,C,A)\int\left(\varphi^{\dagger}(K,C,A)+\varphi(K,C,A)\right)A\Psi_1(K,C,A) \\ &+\gamma\left(\delta\sqrt{\eta}\right)^2\eta\int\left(\varphi^{\dagger}(K,C,A)+\varphi(K,C,A)\right)K\Psi_1(K,C,A)\int\Psi_1^{\dagger}(K,C,A)A\Psi_1(K,C,A) \\ =\;&2\gamma\left(\delta\sqrt{\eta}\right)^2\eta\langle K\rangle\langle A\rangle \\ &+\gamma\left(\delta\sqrt{\eta}\right)^2\eta\left(\langle K\rangle\frac{(a_3+a_3^*)}{\sqrt{\lambda}}+\frac{\left(\frac{(a_2+a_2^*)\sqrt{\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\nu}{2}-\frac{(a_1+a_1^*)\varpi}{2\sqrt{\sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2}}}\right)(a_3+a_3^*)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}\langle A\rangle\right)\end{aligned}$$

and

$$\begin{aligned}&\left|\langle K\rangle\frac{(a_3+a_3^*)}{\sqrt{\lambda}}+\frac{\left(\frac{(a_2+a_2^*)\sqrt{\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\nu}{2}-\frac{(a_1+a_1^*)\varpi}{2\sqrt{\sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2}}}\right)(a_3+a_3^*)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}\langle A\rangle\right| \\ <\;&2\left(\frac{\langle K\rangle}{\sqrt{\lambda}}+\frac{\left(\sqrt{\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\nu+\frac{\varpi}{\sqrt{\sqrt{\varsigma^2\varpi^2+(\hat{r}+r_c)^2}}}\right)}{\left|\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}\right|}\langle A\rangle\right) \\ <\;&2\gamma\left(\delta\sqrt{\eta}\right)^2\eta\langle K\rangle\langle A\rangle\end{aligned}$$

For values of A_0 and $\bar{K}$ that are large enough.

Computation of the Green functions in both Phases

As explained in the text, to inspect the transition functions in the various phases of the system, one has to come back to the initial set of variables (K,C,A). We consider each phase separately.

In the phase $\Psi_1(K,C,A)=0$, one can directly come back to the initial action for $\Psi(K,C,A)$, and write:

$$\begin{aligned}\int \Psi^\dagger(K,C,A)\Big\{&-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\frac{\partial^2}{\partial K^2}+\left(A-\bar{A}\right)^2-2\frac{\partial}{\partial K}\left(C-AF(K)+\delta K\right)\\ &-2\frac{\partial}{\partial C}\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)+(\alpha-C_0)\Big\}\Psi(K,C,A)\\ +\frac{\gamma}{2}\int&\Big(\Psi^\dagger(K_1,C_1,A_1)A_1\Psi^\dagger(K_1,C_1,A_1)\Big)\Big(\Psi^\dagger(K_2,C_2,A_2)K_2\Psi^\dagger(K_2,C_2,A_2)\Big)\end{aligned} \tag{119}$$

We have set $\varsigma=0$, since we assumed $\varsigma\ll 1$. We first neglect the interaction term and compute the transition function associated to:

$$\begin{aligned}\int \Psi^\dagger(K,C,A)\Big\{&-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\frac{\partial^2}{\partial K^2}+\left(A-\bar{A}\right)^2-2\frac{\partial}{\partial K}\left(C-AF(K)+\delta K\right)\\ &+2\frac{\partial}{\partial C}\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)+(\alpha-C_0)\Big\}\Psi(K,C,A)\end{aligned} \tag{120}$$

with $\bar{A}=\frac{A_0}{1-\chi}$ as computed in the previous sections. The corrections due to the interactions will be inspected in the next section. The transition function is equal to the Green function for the operator:

$$\begin{aligned}L &= -\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\frac{\partial^2}{\partial K^2}+\left(A-\bar{A}\right)^2-2\left(C-AF(K)+\delta K\right)\frac{\partial}{\partial K}-2\left(AF'(K)+r_c\right)\left(C-\bar{C}\right)\frac{\partial}{\partial C}\\ &\quad+\alpha-C_0\end{aligned}$$

In the phase where $\Psi_1(K,C,A)\neq 0$, one has to proceed indirectly. Starting with action (100) (with $H(K_1,K_2)=1$) whose saddle point equation is (108):

$$\begin{aligned}S\left(\bar{\Psi}\right)=\int\bar{\Psi}^\dagger\left(K',C,A\right)\Big\{&-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\left(\delta-AF'(K)\right)^2\frac{\partial^2}{\partial K'^2}+\left(A-\bar{A}\right)^2\\ &+\left(\varsigma^2+\frac{\left(AF'(K)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}\right)^2+\frac{\left(K'\right)^2}{\nu^2}+\alpha-C_0\Big\}\bar{\Psi}\left(K',C,A\right)\\ +\gamma\frac{1}{2}\int\bar{\Psi}^\dagger\left(K'_1,C_1,A_1\right)&\bar{\Psi}^\dagger\left(K'_2,C_2,A_2\right)\left\{A_2K_1+A_1K_2\right\}\bar{\Psi}\left(K'_1,C_1,A_1\right)\bar{\Psi}\left(K'_2,C_2,A_2\right)\end{aligned} \tag{121}$$

One has to shift the field by letting: $\bar{\Psi}(K,C,A)=\Psi(K',C,A)+\Psi_1(K',C,A)$. Using that $\Psi_1(K,C,A)$ is a saddle point, it yields the following expansion in the shifted field $\Psi(K',C,A)$:

$$\begin{aligned}\hat{S}(\Psi) &= \Psi(K,C,A)\Big\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\left(\delta-AF'(K)\right)^2\frac{\partial^2}{\partial K'^2}+\left(A-\bar{A}\right)^2\\ &\quad+2\left(\bar{A}-\Gamma_3\right)\varkappa A+\left(\varsigma^2+\frac{\left(AF'(K)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}\right)^2+\frac{\left(K'\right)^2}{\nu^2}+\alpha-C_0\Big\}\Psi(K,C,A)\\ &\quad+\gamma\eta\int\Big(\Psi(K,C,A)+\Psi^\dagger(K,C,A)\Big)K\Psi_1(K,C,A)\int\Big(\Psi(K,C,A)+\Psi^\dagger(K,C,A)\Big)A\Psi_1(K,C,A)\\ &\quad+\frac{\gamma}{2}\int\Big(\Psi_1^\dagger(K,C,A)A\Psi_1^\dagger(K,C,A)\Big)\int\Big(\Psi^\dagger(K,C,A)K\Psi^\dagger(K,C,A)\Big)\\ &\quad+\frac{\gamma}{2}\int\Big(\Psi^\dagger(K,C,A)A\Psi^\dagger(K,C,A)\Big)\int\Big(\Psi_1^\dagger(K,C,A)K\Psi_1^\dagger(K,C,A)\Big)\\ &\quad+\frac{\gamma}{2}\int\Big(\Psi^\dagger(K,C,A)A\Psi(K,C,A)\Big)\int\Big(\Psi^\dagger(K,C,A)K\Psi(K,C,A)\Big)\end{aligned} \tag{122}$$

The term $2\left(\bar{A}-\Gamma_3\right)\varkappa A$ comes from the second order expansion of $\bar{A}=\int\bar{\Psi}^\dagger(K',C,A)A\bar{\Psi}(K',C,A)$, using that, as shown in the previous section (stability analysis), the product of the projections of $\left(\Psi(K,C,A)+\Psi^\dagger(K,C,A)\right)K$ and $\left(\Psi(K,C,A)+\Psi^\dagger(K,C,A)\right)A$ on Ψ_1 can be neglected. For the same reasons, the term:

$$\gamma\eta\int\Big(\Psi(K,C,A)+\Psi^\dagger(K,C,A)\Big)K\Psi_1(K,C,A)\int\Big(\Psi(K,C,A)+\Psi^\dagger(K,C,A)\Big)A\Psi_1(K,C,A)$$

can also be neglected. The last term:

$$\frac{\gamma}{2}\int\Big(\Psi^{\dagger}\left(K,C,A\right)A\Psi\left(K,C,A\right)\Big)\int\Big(\Psi^{\dagger}\left(K,C,A\right)K\Psi\left(K,C,A\right)\Big)$$

is a quartic interaction term in $\Psi\left(K,C,A\right)$. As for the other phase, we discard this term that will be considered perturbatively. Let:

$$\begin{aligned}\Gamma_3 &= \int\Psi_1^{\dagger}\left(K',C,A\right)A\Psi_1\left(K',C,A\right)\\ \Gamma_1 &= \int\Psi_1^{\dagger}\left(K',C,A\right)C\Psi_1\left(K,C,A\right)\\ \Gamma_2 &= \int\Psi_1^{\dagger}\left(K',C,A\right)K'\Psi_1\left(K,C,A\right)\end{aligned}$$

Moreover:

$$\int\Psi_1^{\dagger}\left(K,C,A\right)K\Psi_1\left(K,C,A\right)=\frac{\left(\Gamma_2-\Gamma_1\right)+\Gamma_3\bar{K}^{\varepsilon}\left(1-\varepsilon\right)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}$$

We are thus left with the following quadratic action:

$$\begin{aligned}\hat{S}\left(\Psi\right) &= \Psi\left(K,C,A\right)\Big\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\left(\delta-AF'\left(K\right)\right)^2\frac{\partial^2}{\partial K'^2}+\left(A-\bar{A}\right)^2+2\left(\bar{A}-\Gamma_3\right)\varkappa A\\ &+\left(\varsigma^2+\frac{\left(AF'\left(K\right)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}\right)^2+\frac{\left(K'\right)^2}{\nu^2}+\alpha-C_0+\gamma\eta\Gamma_3K+\gamma\eta A\frac{\left(\Gamma_2-\Gamma_1\right)+\Gamma_3\bar{K}^{\varepsilon}\left(1-\varepsilon\right)}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}\Big\}\Psi\left(K,C,A\right)\end{aligned}$$

It is shown in the previous sections that under our assumptions $\Gamma_2=0$, and that Γ_1 and Γ_3 are defined such that the previous action rewrites:

$$\begin{aligned}\hat{S}\left(\Psi\right) &= \Psi\left(K,C,A\right)\Big\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\left(\delta-AF'\left(K\right)\right)^2\frac{\partial^2}{\partial K'^2}+\left(A-\bar{A}_1\right)^2\\ &+\left(\varsigma^2+\frac{\left(AF'\left(K\right)+r_c\right)^2}{\varpi^2}\right)\left(C-\bar{C}_1\right)^2+\frac{\left(K'\right)^2}{\nu^2}+\left({}^t\Gamma\right)\left(M+\frac{1}{4}M\Omega^{-1}M\right)\Gamma+m_1+\alpha-C_0\Big\}\Psi\left(K,C,A\right)\end{aligned}$$

where:

$$\begin{aligned}\bar{A}_1 &= A_0+\chi\Gamma_3\\ \bar{C}_1 &= \Gamma_1\end{aligned}$$

have been computed previously, and:

$$m_1=\left({}^t\Gamma\right)\left(M+\frac{1}{4}M\Omega^{-1}M\right)\Gamma+\left(\bar{A}^2-\left(\left(1-\varkappa\right)\bar{A}+\varkappa\Gamma_3\right)^2\right)+\alpha-C_0$$

$$\begin{aligned}\Gamma &= \begin{pmatrix}\Gamma_1\\ \Gamma_2\\ \Gamma_3\end{pmatrix},\Omega=\begin{pmatrix}\left(\varsigma^2+\frac{\left(\Gamma_3F'(K)+r_c\right)^2}{\varpi^2}\right) & 0 & 0\\ 0 & \frac{1}{\nu^2} & 0\\ 0 & 0 & 1\end{pmatrix}\\ M &= \frac{\gamma\eta}{\delta-\Gamma_3\varepsilon\bar{K}^{\varepsilon-1}}\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 1\\ -1 & 1 & \bar{K}^{\varepsilon}\left(1-\varepsilon\right)\end{pmatrix}\end{aligned}$$

Remark that m_1 has been computed in the previous section, and $m_1>0$. More precisely:

$$m_1=\bar{A}_1^2-\left(\left(1-\varkappa\right)\bar{A}_1+\varkappa\Gamma_3\right)^2+\frac{\Big(\left(\left(1-\varkappa\right)\bar{A}_1+\varkappa\Gamma_3\right)^2-2\bar{C}\left(\left(1-\varkappa\right)\bar{A}_1+\varkappa\bar{A}_1\right)-\bar{C}_1^2\Big)}{\left(1-\varepsilon\right)\bar{K}^{\varepsilon}}$$

Now, it is possible to come back to the initial variables, through a transformation similar to the one performed in the dirst section of this Appendix (and in the reverse direction):

$$\Psi_n\left(K,C,A\right)=\exp\left(\frac{1}{2\varpi^2}\left(\left(AF'\left(K\right)+r_c\right)\left(C-\bar{C}_1\right)^2\right)-\frac{\left(K'\right)^2}{2\nu^2\left(\delta-AF'\left(K\right)\right)}\right)\bar{\Psi}_n\left(K',C,A\right)$$

$$\Psi_n^{\dagger}(K,C,A)=\exp\left(-\frac{1}{2\varpi^2}\left(\left(AF'(K)+r_c\right)\left(C-\bar{C}_1\right)^2\right)+\frac{\left(K'\right)^2}{2\nu^2\left(\delta-AF'(K)\right)}\right)\bar{\Psi}_n\left(K',C,A\right)$$

This leads to the following action:

$$\int\Psi^{\dagger}(K,C,A)\left\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\frac{\partial^2}{\partial K^2}+\left(A-\bar{A}_1\right)^2-2\frac{\partial}{\partial K}\left(C-AF(K)+\delta K\right)\right.\tag{123}$$
$$\left.+2\frac{\partial}{\partial C}\left(AF'(K)+r_c\right)\left(C-\bar{C}_1\right)+m_1+\left(\alpha-C_0\right)\right\}\Psi(K,C,A)$$

Then, the two phases (120) and (123) can be put on the same footing. Both actions have the form:

$$\int\Psi^{\dagger}(K,C,A)\left\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\frac{\partial^2}{\partial K^2}+\left(A-\bar{A}_i\right)^2-2\frac{\partial}{\partial K}\left(C-AF(K)+\delta K\right)\right.\tag{124}$$
$$\left.+2\frac{\partial}{\partial C}\left(\left(AF'(K)+r_c\right)\left(C-\bar{C}_i\right)\right)+m_i+\left(\alpha-C_0\right)\right\}\Psi(K,C,A)$$

for $i=0$ (phase with $\Psi_1=0$) or $i=0$ (phase with $\Psi_1\neq 0$). We have defined:

$$\begin{aligned}\bar{A}_0&=\frac{A_0}{1-\chi}\\ \bar{C}_0&=\bar{C}\\ m_0&=0\end{aligned}$$

The difference between manifests thus both through the different values of $\bar{A}$ and $\bar{A}_1$ and by the "mass terms" m_i In the phase (123), characterized by a non zero fundamental Ψ_1, the mass term m_1 is greater than 0, which implies reduced transitions probabilities compared to the other phases. The duration of interaction is lower than phase "0", which means a more static system. To solve explicitly for the transition function in both phases, one can use, as before, the expansion above a minimal level of capital to express the production function:

$$AF(K)=K^{\varepsilon}\simeq A\bar{K}^{\varepsilon}+\varepsilon\bar{A}_i\frac{K-\bar{K}}{\bar{K}^{1-\varepsilon}}$$

with:

$$\bar{A}_0=\bar{A}=\frac{A_0}{1-\chi},\bar{A}_1=A_0+\chi\Gamma_3$$

However, we will see below how to avoid this approximations. Neglecting the interaction term, and rescaling:

$$G\left(K,C,A,K',C',A'\right)\rightarrow\exp\left(-m_is\right)G\left(K,C,A,K',C',A'\right)$$

the Green functions for phase $i=0,1$ satisfy a similar equation:

$$\left\{-\varpi^2\frac{\partial^2}{\partial C^2}-\frac{1}{\lambda^2}\frac{\partial^2}{\partial A^2}-\nu^2\frac{\partial^2}{\partial K^2}+\left(A-\bar{A}_i\right)^2-2\frac{\partial}{\partial K}\left(C-\bar{C}-\left(A\bar{K}^{\varepsilon}+\varepsilon\bar{A}_i\frac{K-\bar{K}}{\bar{K}^{1-\varepsilon}}\right)+\delta\left(K-\bar{K}\right)+\left(\delta\bar{K}+\bar{C}\right)\right)\right.\tag{125}$$
$$\left.+2\frac{\partial}{\partial C}\left(\bar{A}_i\varepsilon\bar{K}^{\varepsilon-1}+r_c\right)\left(C-\bar{C}\right)+m_i\right\}G\left(K,C,A,K',C',A'\right)=\delta\left(\left(K,C,A\right)-\left(K',C',A'\right)\right)$$

Since $\lambda^2\gg 1$, we first neglect the term $\left(A-\bar{A}_i\right)^2$, to reintroduce it later.

in Fourier components:

$$\hat{G}=\int\exp\left(il_C\left(C-\bar{C}\right)+il_K\left(K-\bar{K}\right)+il_AA\right)G$$

and (125) becomes:

$$\left\{\varpi^2l_C^2+\frac{1}{\lambda^2}l_A^2+\nu^2l_K^2-2il_K\left(\delta\bar{K}+\bar{C}\right)-2l_K\left(\frac{\partial}{\partial l_C}-\bar{K}^{\varepsilon}\frac{\partial}{\partial l_A}+\left(\delta-\frac{\varepsilon\bar{A}_i}{\bar{K}^{1-\varepsilon}}\right)\frac{\partial}{\partial l_K}\right)\right.$$
$$\left.+2l_C\left(\bar{A}_i\varepsilon\bar{K}^{\varepsilon-1}+r_c\right)\frac{\partial}{\partial l_C}+m_i\right\}\hat{G}\left(K,C,A,K',C',A'\right)=\exp\left(-i\sum_i l_i.x_i\right)$$

We also define $\bar{G}$ as:

$$\hat{G}=\exp\left(-i\frac{\delta\bar{K}+\bar{C}}{\delta-\frac{\varepsilon\bar{A}_i}{\bar{K}^{1-\varepsilon}}}l_K\right)\bar{G}$$

and this function satisfies the equation:

$$\left\{\varpi^2 l_C^2+\frac{1}{\lambda^2}l_A^2+\nu^2 l_K^2-2l_K\left(\frac{\partial}{\partial l_C}-\bar{K}^{\varepsilon}\frac{\partial}{\partial l_A}+\left(\delta-\frac{\varepsilon\bar{A}_i}{\bar{K}^{1-\varepsilon}}\right)\frac{\partial}{\partial l_K}\right)\right. \tag{126}$$
$$\left.+2l_C\left(\bar{A}_i\varepsilon\bar{K}^{\varepsilon-1}+r_c\right)\frac{\partial}{\partial l_C}+m_i\right\}\bar{G}\left(K,C,A,K',C',A'\right)=\exp\left(-i\sum_i l_i.x_i\right)$$

whose solution is:

$$\bar{G}=\int\exp\left(-m_i s\right)\exp\left(-\frac{1}{2}\sum_{i,j}l_i H_{i,j}l_j-i\sum_i J_i l_i\right)ds$$

with H and J satisfying the equations:

$$\begin{aligned}2\hat{\Omega}-NH-H\left({}^tN\right) &= 2\frac{\partial}{\partial s}H \\ \frac{\partial}{\partial s}J &= -\frac{1}{2}NJ\end{aligned} \tag{127}$$

and where:

$$\begin{aligned}\hat{\Omega} &= \begin{pmatrix}\varpi^2 & 0 & 0\\ 0 & \nu^2 & 0\\ 0 & 0 & \frac{1}{\lambda^2}\end{pmatrix}, N=\begin{pmatrix}-2\left(\bar{A}_i\varepsilon\bar{K}^{\varepsilon-1}+r_c\right) & 0 & 0\\ -2 & -2\left(\delta-\frac{\varepsilon\bar{A}_i}{\bar{K}^{1-\varepsilon}}\right) & 2\bar{K}^{\varepsilon}\\ 0 & 0 & 0\end{pmatrix}\\ H &= \begin{pmatrix}a & b & c\\ b & d & e\\ c & e & f\end{pmatrix}\end{aligned}$$

The initial initial conditions to solve (126):

$$\begin{aligned}H\left(0\right) &= 0 \\ J\left(0\right) &= \begin{pmatrix}C'-\bar{C}\\ K'-\bar{K}\\ A'\end{pmatrix}\end{aligned} \tag{128}$$

$$\begin{aligned}J_1 &= \left(C'-\bar{C}\right)\exp\left(\left(\bar{A}_i\varepsilon\bar{K}^{\varepsilon-1}+r_c\right)s\right)\\ J_2 &= \left(K'-\bar{K}-\left[\frac{\left(C'-\bar{C}\right)}{2\bar{A}_i\varepsilon\bar{K}^{\varepsilon-1}+r_c-\delta}+\frac{\bar{K}^{\varepsilon}A'}{\delta-\frac{\varepsilon\bar{A}_i}{\bar{K}^{1-\varepsilon}}}\right]\right)\exp\left(-\left(\delta-\frac{\varepsilon\bar{A}_i}{\bar{K}^{1-\varepsilon}}\right)s\right)\\ &\quad+\frac{\left(C'-\bar{C}\right)\exp\left(\left(\bar{A}_i\varepsilon\bar{K}^{\varepsilon-1}+r_c\right)s\right)}{2\bar{A}_i\varepsilon\bar{K}^{\varepsilon-1}+r_c-\delta}+\frac{\bar{K}^{\varepsilon}A'}{\delta-\frac{\varepsilon\bar{A}_i}{\bar{K}^{1-\varepsilon}}}\\ J_3 &= \bar{J}_3=A'\end{aligned}$$

To find H, we first compute $NH+H\left({}^tN\right)$:

$$\begin{aligned}&NH+H\left({}^tN\right)\\ =&\begin{pmatrix}-2\left(\bar{A}_i\varepsilon\bar{K}^{\varepsilon-1}+r_c\right) & 0 & 0\\ -2 & -2\left(\delta-\frac{\varepsilon\bar{A}_i}{\bar{K}^{1-\varepsilon}}\right) & 2\bar{K}^{\varepsilon}\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}a & b & c\\ b & d & e\\ c & e & f\end{pmatrix}\\ &+\begin{pmatrix}a & b & c\\ b & d & e\\ c & e & f\end{pmatrix}\begin{pmatrix}-2\left(\bar{A}_i\varepsilon\bar{K}^{\varepsilon-1}+r_c\right) & -2 & 0\\ 0 & -2\left(\delta-\frac{\varepsilon\bar{A}_i}{\bar{K}^{1-\varepsilon}}\right) & 0\\ 0 & 2\bar{K}^{\varepsilon} & 0\end{pmatrix}\\ =&\begin{pmatrix}-2a\left(2r_c+2K^{\varepsilon-1}\bar{A}_i\varepsilon\right) & \begin{pmatrix}2\bar{K}^{\varepsilon}c-b\left(2r_c+2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)\\ -2a-b\left(2\delta-2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)\end{pmatrix} & -c\left(2r_c+2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)\\ \begin{pmatrix}2\bar{K}^{\varepsilon}c-b\left(2r_c+2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)\\ -2a-b\left(2\delta-2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)\end{pmatrix} & 4\bar{K}^{\varepsilon}e-4b-4d\left(\delta-\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right) & 2\bar{K}^{\varepsilon}f-2c-e\left(2\delta-2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)\\ -c\left(2r_c+2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right) & 2\bar{K}^{\varepsilon}f-2c-e\left(2\delta-2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right) & 0\end{pmatrix}\end{aligned}$$

and (127) leads to the set of differential equations:

$$\begin{aligned}
\dot{a} &= 2a\left(2r_c + 2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right) + 2\varpi^2 \\
\dot{b} &= -2\bar{K}^{\varepsilon}c + b\left(2r_c + 2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right) + 2a + b\left(2\delta - 2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right) \\
\dot{c} &= 2c\left(r_c + \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right) \\
\dot{d} &= -\left(4\bar{K}^{\varepsilon}e - 4b - 4d\left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)\right) + 2\nu^2 \\
\dot{e} &= -2\bar{K}^{\varepsilon}f + 2c + 2e\left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right) \\
\dot{f} &= \frac{2}{\lambda^2}
\end{aligned}$$

whose solution involves 6 constants of integration, and given the initial conditions, it yields

$$\begin{aligned}
a &= -\frac{\varpi^2}{2\left(r_c + \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)} + a_1\exp\left(4\left(r_c + \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)s\right) \qquad (129) \\
&= \frac{\varpi^2\left(\exp\left(4\left(r_c + \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)s\right) - 1\right)}{2\left(r_c + \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)} \\
f &= \frac{2}{\lambda^2}s \\
c &= a_3\exp\left(2\left(r_c + \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)s\right) = 0 \\
b &= a_2\exp\left(2\left(\delta + r_c\right)s\right) + \frac{\varpi^2\exp\left(4\left(r_c + \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)s\right)}{2\left(r_c + \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)\left(r_c - \delta + 2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)} + \frac{\varpi^2}{2\left(\delta + r_c\right)\left(r_c + K^{\varepsilon-1}\bar{A}_i\varepsilon\right)} \\
&= -\frac{\varpi^2\exp\left(2\left(\delta + r_c\right)s\right)}{\left(r_c - \delta + 2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)\left(\delta + r_c\right)} + \frac{\varpi^2\exp\left(4\left(r_c + \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)s\right)}{2\left(r_c + \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)\left(r_c - \delta + 2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)} \\
&\quad + \frac{\varpi^2}{2\left(\delta + r_c\right)\left(r_c + \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)} \\
e &= a_5\exp\left(2\left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)s\right) + \frac{2\bar{K}^{\varepsilon}}{\lambda^2\left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)}s + \frac{\bar{K}^{\varepsilon}}{\lambda^2\left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)^2} \\
&= -\frac{K^{\varepsilon}\left(\exp\left(2\left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)s\right) - 1\right)}{\lambda^2\left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)^2} + \frac{2\bar{K}^{\varepsilon}}{\lambda^2\left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)}s \\
d &= -\frac{\nu^2}{2\left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)} + a_4\exp\left(4\left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)s\right) \\
&\quad - \frac{2\bar{K}^{2\varepsilon}\left(\exp\left(2\left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)s\right)\right)}{\lambda^2\left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)^3} \\
&\quad + \frac{2\bar{K}^{2\varepsilon}}{\lambda^2\left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)^2}s + \frac{\bar{K}^{2\varepsilon}}{2\lambda^2\left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)^3} \\
&\quad - \frac{\varpi^2}{2\left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)\left(\delta + r_c\right)\left(r_c + \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)} \\
&\quad + \frac{\varpi^2\exp\left(4\left(r_c + \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)s\right)}{2\left(r_c + \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)\left(r_c - \delta + 2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)^2} \\
&\quad - \frac{\varpi^2\exp\left(2\left(\delta + r_c\right)s\right)}{2\left(r_c - \delta + 2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)^2\left(\delta + r_c\right)}
\end{aligned}$$

which is the result stated in the text. These expressions can be simplified given our assumptions about the parameters and for m_i relatively large, $m_i > \delta, r_c, \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon$. Equivalently it corresponds to consider $s \ll 1$. In this case, this can

be approximated by:

$$\begin{aligned}
a &= 2\varpi^2 s \\
f &= \frac{2}{\lambda^2} s, \\
c &= 0, \\
b &= 0 \\
e &= 0 \\
d &= 2\nu^2 s + \frac{4\bar{K}^{2\varepsilon} s}{\lambda^2 \left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)^2} - \frac{3\varpi^2 s}{(\delta + r_c)\left(\delta - r_c - 2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)} \\
&\simeq \frac{4\bar{K}^{2\varepsilon} s}{\lambda^2 \left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)^2}
\end{aligned}$$

so that:

$$H = \begin{pmatrix} 2\varpi^2 & 0 & 0 \\ 0 & 2\nu^2 + \frac{4\bar{K}^{2\varepsilon}}{\lambda^2\left(\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)^2} - \frac{3\varpi^2}{(\delta + r_c\left(\delta - r_c - 2\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon\right)} & 0 \\ 0 & 0 & \frac{2}{\lambda^2} \end{pmatrix} s$$

For J, the formula simplify as:

$$\begin{aligned}
J_1 &= \left(C' - \bar{C}\right)\left(1 + \left(\bar{A}_i\varepsilon\bar{K}^{\varepsilon-1} + r_c\right) s\right) \\
J_2 &= \left(K' - \bar{K} - \left[\frac{\left(C' - \bar{C}\right)}{2\bar{A}_i\varepsilon\bar{K}^{\varepsilon-1} + r_c - \delta} + \frac{A'\bar{K}^{\varepsilon}}{\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon}\right]\right)\left(1 + \left(\delta - \varepsilon\bar{A}_i\bar{K}^{\varepsilon-1}\right) s\right) \\
&\quad + \frac{\left(C' - \bar{C}\right)}{2\bar{A}_i\varepsilon\bar{K}^{\varepsilon-1} + r_c - \delta}\left(1 + \left(\bar{A}_i\varepsilon\bar{K}^{\varepsilon-1} + r_c\right) s\right) + \frac{A'\bar{K}^{\varepsilon}}{\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon} \\
&= \left(K' - \bar{K}\right)\left(1 + \left(\delta - \varepsilon\bar{A}_i\bar{K}^{\varepsilon-1}\right) s\right) - \left(C' - \bar{C}\right) s + 2A'\bar{K}^{\varepsilon} s \\
J_3 &= \bar{J}_3 = A'
\end{aligned}$$

The Green function is computed through the inverse Fourier transform:

$$\begin{aligned}
& G\left(C, K, A, C', K', A', s\right) \\
= & \int \exp\left(-il_C\left(C - \bar{C}\right) - il_K\left(K - \bar{K} + \frac{\delta\bar{K} + \bar{C}}{\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon}\right) - il_A A\right) \exp\left(-\frac{1}{2}\sum_{i,j} l_i H_{i,j} l_j - i\sum_i J_i l_i\right) dl_C dl_K dl_A \\
= & \frac{\exp\left(-\frac{1}{2}\left({}^t X H^{-1} X\right)\right)}{\sqrt{\det H}}
\end{aligned}$$

with:

$$\begin{aligned}
X &= \begin{pmatrix} \left(C - \bar{C}\right) + J_1 \\ \left(K - \bar{K} + \frac{\delta\bar{K} + \bar{C}}{\delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon}\right) + J_2 \\ A + J_3 \end{pmatrix} \\
H &= \begin{pmatrix} \varpi^2 & 0 & 0 \\ 0 & \nu^2 + \frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2} + \frac{3\varpi^2}{2(2\alpha+\beta)\beta} & 0 \\ 0 & 0 & \frac{1}{\lambda^2} \end{pmatrix}
\end{aligned}$$

As a consequence, the green function between two points (C', K', A') and (C, K, A) is:

$$\begin{aligned}
& G\left(C, K, A, C', K', A', s\right) = \frac{\exp\left(-\frac{1}{2}\left({}^t X H^{-1} X\right)\right)}{\sqrt{2\pi \det H}} \qquad (130) \\
= & \frac{\exp\left(-\frac{\left((C - \bar{C}_i) - \left(C' - \bar{C}_i\right)(1 + (\alpha+\beta)s)\right)^2}{2\varpi^2 s} - \frac{\left(\left(K - \bar{K} + \frac{\delta\bar{K} + \bar{C}_i}{\alpha}\right) - \left(\left(K' - \bar{K} + \frac{\delta\bar{K} + \bar{C}_i}{\alpha}\right)(1 - \alpha s) - \left(C' - \bar{C}_i\right)s + A'\bar{K}^{\varepsilon} s\right)\right)^2}{2\left(\nu^2 + \frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2} + \frac{3\varpi^2}{2(2\alpha+\beta)\beta}\right)s} - \frac{\lambda^2\left(A - A'\right)^2}{2s}\right)}{2\sqrt{2\pi\frac{\varpi^2}{\lambda^2}\left(\nu^2 + \frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2} + \frac{3\varpi^2}{2(2\alpha+\beta)\beta}\right)s}}
\end{aligned}$$

with:

$$\begin{aligned}
\alpha &= \delta - \bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon \\
\beta &= 2\bar{A}_i\varepsilon\bar{K}^{\varepsilon-1} + r_c - \delta \\
\delta + r_c &= 2\alpha + \beta \\
\bar{A}_i\varepsilon\bar{K}^{\varepsilon-1} + r_c &= \alpha + \beta
\end{aligned}$$

The contribution due to the potential term $\left(A - \bar{A}\right)^2$ can be reintroduced. Since the contribution of A in the exponential is gaussian, that is, $\frac{\lambda^2\left(A-A'\right)^2}{4s}$, the quadratic contribution can be introduced by adding a contribution $\frac{\left(\frac{A+A'}{2}-\bar{A}_i\right)^2}{2}s$. Reintroducing also the factor $\exp\left(-m_i s\right)$, yields:

$$\begin{aligned}
& G\left(C,K,A,C',K',A',s\right) = \frac{\exp\left(-m_i s\right)\exp\left(-\frac{1}{2}\left({}^tXH^{-1}X\right)\right)}{\sqrt{2\pi\det H}} \qquad (131)\\
= \quad & \frac{1}{2\sqrt{2\pi\frac{\varpi^2}{\lambda^2}\left(\nu^2+\frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2}+\frac{3\varpi^2}{2(2\alpha+\beta)\beta}\right)s}} \times \exp\left(-\frac{\left(\left(C-\bar{C}_i\right)-\left(C'-\bar{C}_i\right)(1+(\alpha+\beta)\,s)\right)^2}{2\varpi^2 s}\right) \\
& \times\exp\left(-\frac{\left(\left(K-\bar{K}+\frac{\delta\bar{K}+\bar{C}_i}{\alpha}\right)-\left(\left(K'-\bar{K}+\frac{\delta\bar{K}+\bar{C}_i}{\alpha}\right)(1-\alpha s)-\left(C'-\bar{C}_i\right)s+A'\bar{K}^{\varepsilon}s\right)\right)^2}{2\left(\nu^2+\frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2}+\frac{3\varpi^2}{2(2\alpha+\beta)\beta}\right)s}\right) \\
& \times\exp\left(-\frac{\lambda^2\left(A-A'\right)^2}{2s}-\frac{\left(\frac{A+A'}{2}-\bar{A}_i\right)^2}{2}s-m_i s\right)
\end{aligned}$$

Remark that we could also find an expression for $G\left(C,K,A,C',K',A',s\right)$ for all s, using (129) and (130): the inversion of the matrix H would produce an exponential weight with non quadratic exponents. We will not develop this point here.

As said before we can also avoid our approximations about the production function. Since we have considered s relatively small, the expansion of the production function could have been done between the final and initial point, for any form of production function. It amounts to replace the coefficients in the previous expression by:

$$\begin{aligned}
\alpha &= \delta - \frac{A+A'}{2}F'\left(\frac{K+K'}{2}\right) \qquad (132)\\
\beta &= 2\frac{A+A'}{2}F'\left(\frac{K+K'}{2}\right) + r_c - \delta \\
\delta + r_c &= 2\alpha + \beta \\
\frac{A+A'}{2}F'\left(\frac{K+K'}{2}\right) + r_c &= \alpha + \beta
\end{aligned}$$

Equation (131) represent a stochatic motion around an average path. The equilibrium value can be found by letting:

$$\begin{aligned}
K &= K' = K_e \\
C &= C' = C_e \\
A &= A' = A_e
\end{aligned}$$

and by setting the exponent equal to 0. One finds:

$$\begin{aligned}
K_e &= \frac{\left(1-\varepsilon\right)\bar{A}_i\bar{K}^{\varepsilon}-\bar{C}_i}{\delta-\bar{K}^{\varepsilon-1}\bar{A}_i\varepsilon} \\
C_e &= \bar{C}_i \\
A_e &= \bar{A}_i
\end{aligned}$$

and replacing these values in the exponent, and equating this one with 0, yields directly the relations:

$$\begin{aligned}
\left(C-\bar{C}_i\right) &= \left(C'-\bar{C}_i\right)+\left(C'-\bar{C}_i\right)(\alpha+\beta)\,s \\
\left(K-K_e\right) &= \left(K'-K_e\right)-\alpha\left(K'-K_e\right)s-\left(C'-\bar{C}_i\right)s \\
\lambda^2\left(A-A'\right) &= -\frac{\frac{A+A'}{2}-\bar{A}_i}{2}s
\end{aligned}$$

In the limit of small s, and using (132), as well as $\frac{K+K'}{2} \to K$ leads to a differential equation for the average path:

$$\begin{aligned}
\frac{d}{dt}\left(C\left(t\right)-\bar{C}_i\right) &= \left(C\left(t\right)-\bar{C}_i\right)\left(AF'\left(K\left(t\right)\right)+r_c-\delta\right)\\
\frac{d}{dt}\left(K\left(t\right)-K_e\right) &= \left(AF'\left(K\left(t\right)\right)-\delta\right)\left(K\left(t\right)-K_e\right)-\left(C\left(t\right)-\bar{C}_i\right)\\
\lambda^2\frac{d\left(A-\bar{A}_i\right)}{dt} &= -\frac{\left(A-\bar{A}_i\right)}{2}
\end{aligned}$$

This describes a simplified model of capital accumulation: the first equation id the usual Euler equation with interest rate. The second one is the dynamic for the capital variable. The last equation describes the dynamic for the technology level. The fixed point $\bar{A}_i$ depends on the whole system and it's interaction as seen in (110).

We end up this section by computing the Laplace transform of $G\left(C,K,A,C',K',A',s\right)$, that is, the one agent propagator of the system:

$$\begin{aligned}
& G\left(C,K,A,C',K',A',\alpha\right) \qquad (133)\\
= & \int \exp\left(-\left(m_i+\alpha-C_0\right)s\right)\times\frac{1}{2\sqrt{2\pi\frac{\varpi^2}{\lambda^2}\left(\nu^2+\frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2}+\frac{3\varpi^2}{2(2\alpha+\beta)\beta}\right)s}}\\
& \times\exp\left(-\frac{\left(\left(C-\bar{C}\right)-\left(C'-\bar{C}_i\right)\left(1+\left(\alpha+\beta\right)s\right)\right)^2}{2\varpi^2 s}\right)\\
& \times\exp\left(-\frac{\left(\left(K-\bar{K}+\frac{\delta\bar{K}+\bar{C}_i}{\alpha}\right)-\left(\left(K'-\bar{K}+\frac{\delta\bar{K}+\bar{C}_i}{\alpha}\right)\left(1-\alpha s\right)-\left(C'-\bar{C}_i\right)s+A'\bar{K}^{\varepsilon}s\right)\right)^2}{2\left(\nu^2+\frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2}+\frac{3\varpi^2}{2(2\alpha+\beta)\beta}\right)s}\right)\\
& \times\exp\left(-\frac{\lambda^2\left(A-A'\right)^2}{2s}-\frac{\left(\frac{A+A'}{2}-\bar{A}_i\right)^2}{2}s-m_i s\right)ds
\end{aligned}$$

$$C=\left(C'-\bar{C}_i\right)\left(1-\left(\alpha+\beta\right)s\right)$$

This can be found explicitly as the Laplace transform of a gaussian expression. Actually:

$$\begin{aligned}
& \int\exp\left(-m_i s\right)\frac{\exp\left(-\frac{1}{2s}\left({}^t\left(X-sY\right)H^{-1}\left(X-sY\right)\right)\right)}{\sqrt{2\pi\det Hs}}ds\\
= & \int\exp\left(-\left(m_i+\frac{1}{2}\left({}^tY\right)H^{-1}Y\right)s\right)\frac{\exp\left(-\frac{1}{2s}\left({}^tXH^{-1}X\right)\right)}{\sqrt{2\pi\det Hs}}ds\exp\left(\left({}^tX\right)H^{-1}\left(Y\right)\right)\\
= & \frac{\exp\left(-\sqrt{2m_i+\left({}^tY\right)H^{-1}Y}\sqrt{\left({}^tXH^{-1}X\right)}+\left({}^tX\right)H^{-1}\left(Y\right)\right)}{\sqrt{2m_i+\left({}^tY\right)H^{-1}Y}}
\end{aligned}$$

and as a consequence:

$$\begin{aligned}
& G\left(C,K,A,C',K',A',m_i\right)\\
= & \frac{\exp\left(-\sqrt{2m_i+\left({}^tY\right)H^{-1}Y}\sqrt{\left({}^tXH^{-1}X\right)}+\left({}^tX\right)H^{-1}\left(Y\right)\right)}{\sqrt{2m_i+\left({}^tY\right)H^{-1}Y}}\\
= & \frac{\exp\left(-\sqrt{2m_i+\frac{(\alpha+\beta)^2\left(C'-\bar{C}\right)^2}{\varpi^2}+\frac{\left(\left(K'-\bar{K}\right)\alpha+\delta\bar{K}+C'-A'\bar{K}^{\varepsilon}\right)^2}{\left(\nu^2+\frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2}+\frac{3\varpi^2}{2(2\alpha+\beta)\beta}\right)}}\sqrt{\frac{(C-C')^2}{4\varpi^2}+\frac{((K-K'))^2}{4\left(\nu^2+\frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2}+\frac{3\varpi^2}{2(2\alpha+\beta)\beta}\right)}+\frac{\lambda^2(A-A')^2}{4}}\right)}{\sqrt{2m_i+\frac{(\alpha+\beta)^2\left(C'-\bar{C}\right)^2}{\varpi^2}+\frac{\left(\left(K'-\bar{K}\right)\alpha+\left(C'-\bar{C}\right)-A'\bar{K}^{\varepsilon}\right)^2}{\left(\nu^2+\frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2}+\frac{3\varpi^2}{2(2\alpha+\beta)\beta}\right)}}}\\
& \times\exp\left(\frac{(\alpha+\beta)(C-C')\left(C'-\bar{C}\right)}{2\varpi^2}+\frac{((K-K'))\left(\left(K'-\bar{K}\right)\alpha+\delta\bar{K}+C'-A'\bar{K}^{\varepsilon}\right)}{2\left(\nu^2+\frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2}+\frac{3\varpi^2}{2(2\alpha+\beta)\beta}\right)}\right)
\end{aligned}$$

Correction to the Green function due to the interaction term

The interaction term

$$\frac{\gamma}{2}\int \Psi^{\dagger}\left(K_1,C_1,A_1\right)\Psi^{\dagger}\left(K_2,C_2,A_2\right)\left\{A_2K_1+A_1K_2\right\}\Psi\left(K_1,C_1,A_1\right)\Psi\left(K_2,C_2,A_2\right)$$

modifies the Green functions of individual agents. The correction at first order in γ is obtained by the application of the Wick theorem to the interaction term. The contractions $\underbrace{\Psi^{\dagger}\left(K_i,C_i,A_i\right)\Psi^{\dagger}\left(K_j,C_j,A_j\right)}$ being replaced by propagators $G\left(K_i,C_i,A_i,K_j,C_j,A_j,m_i\right)$. It leads to the contribution:

$$\begin{aligned}
&\delta G\left(C,K,A,C',K',A',m_i\right)\\
=\;&\gamma\int G\left(C,K,A,C_1,K_1,A_1,m_i\right)\left(A_2K_1+A_1K_2\right)G\left(C_1,K_1,A_1,C_2,K_2,A_2,m_i\right)\\
&\qquad\times G\left(C_2,K_2,A_2,C',K',A',m_i\right)d\left(C_1,K_1,A_1\right)d\left(C_2,K_2,A_2\right)
\end{aligned}$$

which is given by the following contribution.

$$\begin{aligned}
&\frac{\exp\left(-\sqrt{2m_i+\left({}^tX_1\right)H^{-1}X_1}\sqrt{\left({}^tXH^{-1}X\right)}+\left({}^tX\right)H^{-1}\left(X_1\right)\right)}{\sqrt{2m_i+\left({}^tX_1\right)H^{-1}X_1}}\\
&\times\left({}^tX_1\right)B\left(X_2\right)\frac{\exp\left(-\sqrt{2m_i+\left({}^tX_2\right)H^{-1}X_2}\sqrt{\left({}^tX_1H^{-1}X_1\right)}+\left({}^tX_1\right)H^{-1}\left(X_2\right)\right)}{\sqrt{2m_i+\left({}^tX_2\right)H^{-1}X_2}}\\
\times\quad&\frac{\exp\left(-\sqrt{2m_i+\left({}^tX'\right)H^{-1}X'}\sqrt{\left({}^tX_2H^{-1}X_2\right)}+\left({}^tX_2\right)H^{-1}\left(X'\right)\right)}{\sqrt{2m_i+\left({}^tX'\right)H^{-1}X'}}
\end{aligned}$$

However, in this case, it is much more convenient to work with the time representation and to compute rather $\delta G\left(C,K,A,C',K',A',s\right)$. The correction at first order in γ is:

$$\begin{aligned}
&\delta G\left(C,K,A,C',K',A',s\right) \qquad (134)\\
=\;&\gamma\int G\left(C,K,A,C_1,K_1,A_1,s_1\right)\left(A_2K_1+A_1K_2\right)G\left(C_1,K_1,A_1,C_2,K_2,A_2,s_2\right)\\
&G\left(C_2,K_2,A_2,C',K',A',s-s_1-s_2\right)d\left(C_1,K_1,A_1\right)\left(C_2,K_2,A_2\right)ds_1ds_2\\
=\;&\gamma\left\langle A_2K_1+A_1K_2\right\rangle
\end{aligned}$$

with a mean taken for a stochastic process constrained to start at (C',K',A') and to end at (C,K,A). In first approximation, one can approximate $\langle A_2K_1+A_1K_2\rangle$ by its value along the average path. This one is given by the minimization of:

$$\int\left({}^t\left(\frac{d}{ds}X+MX\right)H^{-1}\left(\frac{d}{ds}X+MX\right)\right) \qquad (135)$$

for a path starting at (C',K',A',s), and ending at (C,K,A,C'). The matrices M and H are given by the exponential weight (133):

$$\begin{aligned}
\exp\Bigg(-&\frac{\left((C-C')+2\left(C'-\bar{C}\right)(\alpha+\beta)\,s\right)^2}{4\varpi^2 s}\\
&-\frac{\left((K-K')+2\left(\left(K'-\bar{K}+\frac{\delta\bar{K}+\bar{C}}{\alpha}\right)\alpha+\left(C'-\bar{C}\right)-A'\bar{K}^{\varepsilon}\right)s\right)^2}{4\left(\nu^2+\frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2}+\frac{3\varpi^2}{2(2\alpha+\beta)\beta}\right)s}-\frac{\lambda^2\left(A-A'\right)^2}{4s}\Bigg)
\end{aligned}$$

so that:

$$\begin{aligned}
M&=\begin{pmatrix}(\alpha+\beta)&0&0\\1&\alpha&-\bar{K}^{\varepsilon}\\0&0&0\end{pmatrix},{}^tM=\begin{pmatrix}(\alpha+\beta)&1&0\\0&\alpha&0\\0&-K^{\varepsilon}&0\end{pmatrix}\\
H&=\begin{pmatrix}2\varpi^2&0&0\\0&2\left(\nu^2+\frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2}+\frac{3\varpi^2}{2(2\alpha+\beta)\beta}\right)&0\\0&0&\frac{2}{\lambda^2}\end{pmatrix}=\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix}
\end{aligned}$$

$$\left(\begin{array}{ccc} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{array}\right)\left(\begin{array}{ccc} (\alpha+\beta) & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & -K^{\varepsilon} & 0 \end{array}\right)\left(\begin{array}{ccc} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{array}\right)^{-1}\left(\begin{array}{ccc} (\alpha+\beta) & 0 & 0 \\ 1 & \alpha & -\bar{K}^{\varepsilon} \\ 0 & 0 & 0 \end{array}\right)$$

The paths that minimize (135) satisfiy:

$$H^{-1}\frac{d^2}{ds^2}X + H^{-1}M\frac{d}{ds}X - \left({}^t\left(H^{-1}M\right)\right)\frac{d}{ds}X - \left({}^tM\right)H^{-1}MX = 0$$

or equivalently:

$$\frac{d^2}{ds^2}X + H\left(H^{-1}M - \left({}^t\left(H^{-1}M\right)\right)\right)\frac{d}{ds}X - H\left({}^tM\right)H^{-1}MX = 0 \tag{136}$$

We will solve (136), by looking first for a solution $\exp(Ns)$ at the first order in α and β. A straightforward computation yields:

$$\begin{aligned} N &= \left(\begin{array}{ccc} -(\alpha+\beta) & 0 & 0 \\ -1 & -\alpha & K^{\varepsilon} \\ 0 & 0 & 0 \end{array}\right) + O\left(\alpha^2\right) \\ \exp(Nu) &= \left(\begin{array}{ccc} 1-u(\beta+\alpha) & 0 & 0 \\ u\left(u\left(\alpha+\frac{\beta}{2}\right)-1\right) & 1-u\alpha & K^{\varepsilon}u\left(1-\frac{\alpha}{2}u\right) \\ 0 & 0 & 1 \end{array}\right) + O\left(\alpha^2\right) \end{aligned}$$

Then, a factorization $X = \exp(Ns)\,Y$ in the equation (136) leads to:

$$\left(\frac{d^2}{ds^2}Y + \exp(-Ns)\,H\left(H^{-1}M - \left({}^t\left(H^{-1}M\right)\right)\right)\exp(Ns)\,\frac{d}{ds}Y + 2N\frac{d}{ds}Y\right) = 0$$

That is:

$$\frac{d^2}{ds^2}Y + \left(\begin{array}{ccc} -2(\alpha+\beta) & 0 & 0 \\ -s\beta-1 & -2\alpha & (1-s\alpha)K^{\varepsilon} \\ 0 & 0 & 0 \end{array}\right)\frac{d}{ds}Y = 0$$

which is solved as:

$$\left(\frac{d}{ds}Y\right) = \exp\left(N'(s)\right)A$$

for A an initial condition and

$$N'(s) = \left(\begin{array}{ccc} 2s(\alpha+\beta) & 0 & 0 \\ \frac{1}{2}s(s\beta+2) & 2s\alpha & \frac{1}{2}K^{\varepsilon}s(s\alpha-2) \\ 0 & 0 & 0 \end{array}\right)$$

One finds given our assumptions of first order approximation:

$$\exp\left(N'(s)\right) = \left(\begin{array}{ccc} 1+2s(\alpha+\beta) & 0 & 0 \\ s\left(2s\alpha+3s\frac{\beta}{2}+1\right) & 1+2s\alpha & -K^{\varepsilon}s\left(s\frac{\alpha}{2}+1\right) \\ 0 & 0 & 1 \end{array}\right)$$

and the solutions of (136) are thus:

$$\begin{aligned} X(u) &= \exp(Nu)\left(B + \int \exp\left(N'u\right)A\right) \\ &= \exp(Nu)(B + P(u)\,A) \end{aligned}$$

where:

$$P(u) = \int \exp\left(N'u\right) = \left(\begin{array}{ccc} u(u\alpha+u\beta+1) & 0 & 0 \\ \frac{1}{6}u^2(4u\alpha+3u\beta+3) & u(u\alpha+1) & -\frac{1}{6}K^{\varepsilon}u^2(u\alpha+3) \\ 0 & 0 & u \end{array}\right)$$

For a path where $X(0)$ and $X(s)$ are fixed, the constants A and B satisfy:

$$B = X(0), A = (P(s))^{-1}(\exp(-Ns)\,X(s) - X(0))$$

To find the correction (134) in terms of initial and final points, we define:

$$\begin{aligned} \bar{X} &= \frac{X(0)+X(s)}{2} \\ \Delta X &= X(s) - X(0) \end{aligned}$$

and the solution of (136) rewrites:

$$\begin{aligned} X(u) &= \exp(Nu)\left(\left(1-(P(u))(P(s))^{-1}\right)X(0)+(P(u))(P(s))^{-1}\exp(-Ns)X(s)\right) \\ &= \exp(Nu)\left(\left(1-(P(u))(P(s))^{-1}(1-\exp(-Ns))\right)\bar{X}\right) \\ &\quad +\exp(Nu)\left((P(u))(P(s))^{-1}(\exp(-Ns)+1)-1\right)\frac{\Delta X}{2} \\ &= \exp(Nu)(P(u))(P(s))^{-1}\exp(-Ns)\Delta X \\ &\quad +\exp(Nu)\left(1-(P(u))(P(s))^{-1}(1-\exp(-Ns))\right)X(0) \end{aligned} \tag{137}$$

Some computations yield intermediate results:

$$(P(u))(P(s))^{-1} = \begin{pmatrix} u\frac{1-(s-u)(\alpha+\beta)}{s} & 0 & 0 \\ \frac{1}{6}u(s-u)\frac{2s\alpha-4u\alpha-3u\beta-3}{s} & u\frac{1-(s-u)\alpha}{s} & -\frac{1}{6}\frac{K^{\varepsilon}}{s}u(s-u)(2s\alpha-u\alpha-3) \\ 0 & 0 & \frac{1}{s}u \end{pmatrix}$$

$$\exp(Nu)(P(u))(P(s))^{-1}\exp(-Ns) = \begin{pmatrix} \frac{u}{s} & 0 & 0 \\ -\frac{1}{6}u(s-u)\frac{s\alpha+u\alpha-3}{s} & \frac{u}{s} & \frac{1}{6}uK^{\varepsilon}(s-u)\frac{s\alpha+u\alpha-3}{s} \\ 0 & 0 & \frac{u}{s} \end{pmatrix}$$

$$\exp(Nu)\left(1-(P(u))(P(s))^{-1}(1-\exp(-Ns))\right) = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2}u\alpha(s-u) & 1 & \frac{1}{2}K^{\varepsilon}u\alpha(s-u) \\ 0 & 0 & 1 \end{pmatrix}$$

and one finds for $X(u)$:

$$\begin{aligned} X(u) &= \begin{pmatrix} \frac{u}{s} & 0 & 0 \\ -\frac{1}{6}u(s-u)\frac{s\alpha+u\alpha-3}{s} & \frac{u}{s} & \frac{1}{6}uK^{\varepsilon}(s-u)\frac{s\alpha+u\alpha-3}{s} \\ 0 & 0 & \frac{u}{s} \end{pmatrix}\Delta X \\ &\quad +\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2}u\alpha(s-u) & 1 & \frac{1}{2}K^{\varepsilon}u\alpha(s-u) \\ 0 & 0 & 1 \end{pmatrix}X(0) \\ &= \begin{pmatrix} \frac{u}{s} & 0 & 0 \\ -\frac{1}{6}u(s-u)\frac{s\alpha+u\alpha-3}{s} & \frac{u}{s} & \frac{1}{6}uK^{\varepsilon}(s-u)\frac{s\alpha+u\alpha-3}{s} \\ 0 & 0 & \frac{u}{s} \end{pmatrix}X(s) \\ &\quad +\begin{pmatrix} \frac{s-u}{s} & 0 & 0 \\ -\frac{1}{6}u(s-u)\frac{2s\alpha-u\alpha+3}{s} & \frac{s-u}{s} & \frac{1}{6}K^{\varepsilon}u(s-u)\frac{2s\alpha-u\alpha+3}{s} \\ 0 & 0 & \frac{s-u}{s} \end{pmatrix}X(0) \end{aligned}$$

so that the correction to the statistical weight can be found directly. One has:

$$\gamma\int_0^s X(u)\,du = \gamma\begin{pmatrix} \frac{1}{2}s & 0 & 0 \\ \frac{1}{12}s^2 & \frac{1}{2}s & -\frac{1}{12}K^{\varepsilon}s^2 \\ 0 & 0 & \frac{1}{2}s \end{pmatrix}\Delta X + \gamma s X(0)$$

and ultimately:

$$\begin{aligned} &\gamma\langle A_2K_1+A_1K_2\rangle = \int_0^s X(u)\,duM\gamma\int_0^s X(u)\,du \\ &= \left({}^tX(0)\right)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s^2 \\ 0 & s^2 & 0 \end{pmatrix}X(0)+\left({}^t\Delta X\right)\begin{pmatrix} 0 & 0 & \frac{1}{24}s^3 \\ 0 & 0 & \frac{1}{4}s^2 \\ \frac{1}{24}s^3 & \frac{1}{4}s^2 & -\frac{1}{12}K^{\varepsilon}s^3 \end{pmatrix}\Delta X \\ &+2\left({}^t\Delta X\right)\begin{pmatrix} 0 & 0 & \frac{1}{6}s^3 \\ 0 & 0 & \frac{1}{2}s^2 \\ 0 & \frac{1}{2}s^2 & -\frac{1}{6}K^{\varepsilon}s^3 \end{pmatrix}X(0) \end{aligned}$$

This term modifies the transition functions as:

$$\begin{aligned}
&\bar{G}\left(C,K,A,C',K',A',s\right) \qquad (138)\\
=\ &\frac{\exp\left(-\frac{\left(\left(C-\bar{C}\right)-\left(C'-\bar{C}\right)\left(1-2(\alpha+\beta)s\right)\right)^2}{4\varpi^2 s}-\frac{\left(\left(K-\bar{K}+\frac{\delta\bar{K}+\bar{C}}{\alpha}\right)-\left(\left(K'-\bar{K}+\frac{\delta\bar{K}+\bar{C}}{\alpha}\right)\left(1-2\alpha s\right)-2\left(C'-\bar{C}\right)s+2A'\bar{K}^{\varepsilon}s\right)\right)^2}{4\left(\nu^2+\frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2}+\frac{3\varpi^2}{2(2\alpha+\beta)\beta}\right)s}-\frac{\lambda^2\left(A-A'\right)^2}{4s}\right)}{4\sqrt{\pi\frac{\varpi^2}{\lambda^2}\left(\nu^2+\frac{2\bar{K}^{2\varepsilon}}{\lambda^2\alpha^2}+\frac{3\varpi^2}{2(2\alpha+\beta)\beta}\right)s}}\\
&\times\exp\left(-\gamma\left\langle A_2K_1+A_1K_2\right\rangle\right)
\end{aligned}$$

We can write more precisely this correction. To do so, let us first remark that a weight of the form:

$$\left({}^t\left(\Delta X+MX\left(0\right)\right)H^{-1}\left(\Delta X+MX\left(0\right)\right)\right)+\gamma\left({}^t\Delta X\right)R_1\left(\Delta X\right)+2\gamma\left({}^t\Delta X\right)R_2X\left(0\right)+\gamma\left({}^tX\left(0\right)\right)R_3X\left(0\right)$$

with:

$$R_1=\begin{pmatrix}0&0&\frac{1}{24}s^3\\0&0&\frac{1}{4}s^2\\\frac{1}{24}s^3&\frac{1}{4}s^2&-\frac{1}{12}K^{\varepsilon}s^3\end{pmatrix},R_2=\begin{pmatrix}0&0&\frac{1}{6}s^3\\0&0&\frac{1}{2}s^2\\0&\frac{1}{2}s^2&-\frac{1}{6}K^{\varepsilon}s^3\end{pmatrix},R_3=\begin{pmatrix}0&0&0\\0&0&s^2\\0&s^2&0\end{pmatrix}$$

and the log of (**??**) be rewritten:

$$\begin{aligned}
&\left({}^t\left(\Delta X+\bar{M}X\left(0\right)\right)\left(H^{-1}+\gamma A_1\right)\left(\Delta X+\bar{M}X\left(0\right)\right)\right) \qquad (139)\\
&+\left({}^t\left(MX\left(0\right)\right)\right)\left(H^{-1}\right)\left(MX\left(0\right)\right)+\gamma\left({}^tX\left(0\right)\right)A_3X\left(0\right)\\
&-\left({}^t\left(\bar{M}X\left(0\right)\right)\right)\left(H^{-1}+\gamma A_1\right)\left(\bar{M}X\left(0\right)\right)
\end{aligned}$$

where $\bar{M}$ satisfies:

$$\left(H^{-1}+\gamma R_1\right)\bar{M}=H^{-1}M+\gamma R_2$$

that is:

$$\bar{M}=\left(1-\gamma HR_1\right)\left(M+\gamma HR_2\right)$$

In our case, it leads to:

$$\bar{M}=\begin{pmatrix}\alpha+\beta&0&\frac{1}{6}as^3\gamma\\1&\alpha-\frac{1}{8}bcs^4\gamma^2&\frac{1}{2}bs^2\gamma-K^{\varepsilon}\\-\frac{1}{4}cs^2\gamma&\frac{1}{2}cs^2\gamma&\frac{1}{4}K^{\varepsilon}cs^2\gamma-\frac{1}{6}K^{\varepsilon}cs^3\gamma\end{pmatrix}$$

To complete the computation, we rewrite the two last terms in (139) as:

$$\begin{aligned}
&\left({}^t\left(MX\left(0\right)\right)\right)\left(H^{-1}\right)\left(MX\left(0\right)\right)+\gamma\left({}^tX\left(0\right)\right)R_3X\left(0\right)-\left({}^t\left(\bar{M}X\left(0\right)\right)\right)\left(H^{-1}+\gamma R_1\right)\left(\bar{M}X\left(0\right)\right)\\
=\ &\left({}^t\left(MX\left(0\right)\right)\right)\left(H^{-1}\right)\left(MX\left(0\right)\right)+\gamma\left({}^tX\left(0\right)\right)R_3X\left(0\right)\\
&-\left({}^tX\left(0\right)\right)\left({}^tM+\gamma\left({}^tR_2\right)H\right)\left(1-\gamma\left({}^tR_1\right)H\right)\left(H^{-1}M+\gamma R_2\right)X\left(0\right)\\
=\ &\gamma\left({}^tX\left(0\right)\right)R_3X\left(0\right)-\left({}^tX\left(0\right)\right)\left(\gamma\left({}^tR_2\right)M-\gamma\left({}^tR_1\right)M+\gamma\left({}^tM\right)R_2\right)X\left(0\right)\\
=\ &\gamma\left({}^tX\left(0\right)\right)\left(R_3-\left({}^tM\right)\left(2R_2-R_1\right)\right)X\left(0\right)
\end{aligned}$$

Defining $\bar{H}^{-1}$ by:

$$\begin{aligned}
\bar{H}^{-1}&=\left(H^{-1}+\gamma R_1\right)\\
&=\begin{pmatrix}\frac{1}{as}&0&\frac{1}{12}s^3\gamma\\0&\frac{1}{bs}&\frac{1}{4}s^2\gamma\\\frac{1}{12}s^3\gamma&\frac{1}{4}s^2\gamma&-\frac{1}{6}\frac{K^{\varepsilon}cs^4\gamma-6}{cs}\end{pmatrix}
\end{aligned}$$

and:

$$\begin{aligned}
\bar{H}&=\left(H^{-1}+\gamma R_1\right)^{-1}=H-H\gamma R_1H\\
&=\begin{pmatrix}as&0&-\frac{1}{24}acs^5\gamma\\0&bs&-\frac{1}{8}bcs^4\gamma\\0&-\frac{1}{8}bcs^4\gamma&cs+\frac{1}{24}K^{\varepsilon}c^2s^5\gamma\end{pmatrix}
\end{aligned}$$

the weight including the correction (138) is:

$$\exp\left(-\left({}^t\left(\Delta X+\bar{M}X\left(0\right)\right)\bar{H}^{-1}\left(\Delta X+\bar{M}X\left(0\right)\right)\right)-\gamma\left({}^tX\left(0\right)\right)\left(R_3-\left(\left({}^tM\right)\left(2R_2-R_1\right)\right)\right)X\left(0\right)\right)$$

This result can also be studied in terms of trajectories. Actually, in (138) a term is added to the initial action. It has the form:

$$\begin{aligned} & \frac{\gamma}{2}\left({}^{t}\left(\int_0^s (X(v))\,dv\right)\right) M \left(\int_0^s (X(u))\,du\right) \\ = \quad & \gamma\left({}^{t}\left(\int_0^s (X(v))\,dv\right)\right) M \left(\int_0^v (X(u))\,du\right) \end{aligned}$$

and the correction to the dynamic equations due to the agents interaction is:

$$\frac{d^2}{ds^2}X + H\left(\left(H^{-1}M - \left({}^{t}\left(H^{-1}M\right)\right)\right)\right)\frac{d}{ds}X - \left(\left(H\left({}^{t}M\right)H^{-1}M\right)\right)X - \gamma M\left(\int_0^s (X(u))\,du\right) = 0 \tag{140}$$

We can approximate $M\left(\int_0^v (X(u))\,du\right)$ with its mean path approximation, so that (140) rewrites:

$$\frac{d^2}{ds^2}X + H\left(\left(H^{-1}M - \left({}^{t}\left(H^{-1}M\right)\right)\right)\right)\frac{d}{ds}X - \left(\left(H\left({}^{t}M\right)H^{-1}M\right) + \gamma H M_1\right)X - \gamma H M_2 X(0) = 0 \tag{141}$$

with

$$\begin{aligned} M\left(\int_0^v (X(u))\,du\right) &= \gamma\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\left(\begin{pmatrix} \frac{1}{2}s & 0 & 0 \\ \frac{1}{12}s^2 & \frac{1}{2}s & -\frac{1}{12}K^{\varepsilon}s^2 \\ 0 & 0 & \frac{1}{2}s \end{pmatrix}\Delta X + sX(0)\right) \\ &= \gamma\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}s \\ \frac{1}{12}s^2 & \frac{1}{2}s & -\frac{1}{12}K^{\varepsilon}s^2 \end{pmatrix}X(s) + \gamma\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}s \\ -\frac{1}{12}s^2 & \frac{1}{2}s & \frac{1}{12}K^{\varepsilon}s^2 \end{pmatrix}X(0) \\ &\equiv M_1X(s) + M_2X(0) \end{aligned}$$

To find the solution of (142), we first consider:

$$\frac{d^2}{ds^2}X + H\left(\left(H^{-1}M - \left({}^{t}\left(H^{-1}M\right)\right)\right)\right)\frac{d}{ds}X - \left(\left(H\left({}^{t}M\right)H^{-1}M\right) + \gamma H M_1\right)X = 0 \tag{142}$$

and proceed as for (136). We look for a solution of (142) of the form:

$$\exp(Ns)\Big(1 + \gamma\hat{N}(s)\Big)$$

so that $\hat{N}(s)$ satisfies:

$$\frac{d^2}{ds^2}\hat{N}(s) + 2N\frac{d}{ds}\hat{N}(s) + \exp(-Ns)\,H\left(H^{-1}M - \left({}^{t}\left(H^{-1}M\right)\right)\right)\exp(Ns)\frac{d}{ds}\hat{N}(s) - \exp(-Ns)\,HM_1\exp(Ns) = 0$$

whose expanded form in our order of approximation is:

$$\begin{aligned} 0 \quad = \quad & \gamma\frac{d^2}{ds^2}\hat{N}(s) + \gamma\begin{pmatrix} 0 & 0 & 0 \\ -\beta s + 1 & 0 & -K^{\varepsilon} - K^{\varepsilon}s\alpha \\ 0 & 0 & 0 \end{pmatrix}\frac{d}{ds}\hat{N}(s) \\ & -\gamma\begin{pmatrix} 0 & 0 & 0 \\ \frac{5}{12}K^{\varepsilon}cs^3 & -\frac{1}{2}K^{\varepsilon}cs^2 & \frac{1}{2}bs \\ -\frac{5}{12}cs^2 & \frac{1}{2}cs & \frac{5}{12}K^{\varepsilon}cs^2 \end{pmatrix} \end{aligned}$$

The solution $\hat{N}(s)$ is computed at the zeroth order in α and as a consequence, it satisfies:

$$\frac{d^2}{ds^2}\hat{N}(s) + \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & K^{\varepsilon} \\ 0 & 0 & 0 \end{pmatrix}\frac{d}{ds}\hat{N}(s) - \begin{pmatrix} 0 & 0 & 0 \\ \frac{5}{12}K^{\varepsilon}cs^3 & -\frac{1}{2}K^{\varepsilon}cs^2 & \frac{1}{2}bs \\ -\frac{5}{12}cs^2 & \frac{1}{2}cs & \frac{5}{12}K^{\varepsilon}cs^2 \end{pmatrix}$$

$$\hat{N}(s) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{36}K^{\varepsilon}cs^5 & -\frac{3}{48}K^{\varepsilon}cs^4 & -\frac{c}{144}K^{2\varepsilon}s^5 + \frac{1}{12}bs^3 \\ -\frac{5}{144}cs^4 & \frac{1}{12}cs^3 & \frac{5}{144}K^{\varepsilon}cs^4 \end{pmatrix}$$

The equation (142) can then be solved in the following way. We first solve

$$\frac{d^2}{ds^2}X + H\left(\left(H^{-1}M - \left({}^t\left(H^{-1}M\right)\right)\right)\right)\frac{d}{ds}X - \left(\left(H\left({}^tM\right)H^{-1}M + \gamma H N_1\right) + \gamma H M_1\right)X = 0$$

as before by setting $X = \exp(Ns)\Big(1+\gamma\hat{N}(s)\Big)Y(s)$ and $Y(s)$ satisfies:

$$\frac{d^2}{ds^2}Y + L\frac{d}{ds}Y = 0$$

with:

$$\begin{aligned}
L &= \Big(1-\gamma\hat{N}(s)\Big)\exp(-Ns)\left(H\left(H^{-1}M - \left({}^t\left(H^{-1}M\right)\right)\right)\right)\exp(Ns)\Big(1+\gamma\hat{N}(s)\Big) \\
&\quad +2\left(1-\gamma\hat{N}(s)\right)\exp(-Ns)\frac{d}{ds}\left(\exp(Ns)\Big(1+\gamma\hat{N}(s)\Big)\right) \\
&= \begin{pmatrix} 0 & 0 & 0 \\ \frac{7}{72}K^\varepsilon cs^4\gamma - s\beta + 1 & -\frac{1}{12}K^\varepsilon cs^3\gamma & -K^\varepsilon - K^\varepsilon s\alpha - \frac{7}{72}K^{2\varepsilon}cs^4\gamma \\ -\frac{1}{12}cs^3\gamma & 0 & \frac{1}{12}K^\varepsilon cs^3\gamma \end{pmatrix} \\
&\quad +2\begin{pmatrix} -\alpha-\beta & 0 & 0 \\ \frac{1}{24}K^\varepsilon cs^4\gamma - 1 & -\frac{1}{6}K^\varepsilon cs^3\gamma - \alpha & K^\varepsilon + \frac{1}{4}bs^2\gamma + \frac{1}{16}K^{2\varepsilon}cs^4\gamma \\ \frac{1}{144}cs^5\beta^2\gamma - \frac{1}{18}cs^3\gamma & \frac{1}{4}cs^2\gamma & \frac{1}{18}K^\varepsilon cs^3\gamma \end{pmatrix} \\
&= \begin{pmatrix} -2\alpha-2\beta & 0 & 0 \\ \frac{13}{72}K^\varepsilon cs^4\gamma - s\beta - 1 & -2\alpha - \frac{5}{12}K^\varepsilon cs^3\gamma & K^\varepsilon + \frac{1}{2}bs^2\gamma - K^\varepsilon s\alpha + \frac{1}{36}K^{2\varepsilon}cs^4\gamma \\ \frac{1}{72}cs^5\beta^2\gamma - \frac{7}{36}cs^3\gamma & \frac{1}{2}cs^2\gamma & \frac{7}{36}K^\varepsilon cs^3\gamma \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\int L(u)\,du &= \int_0^u \begin{pmatrix} -2\alpha-2\beta & 0 & 0 \\ \frac{13}{72}K^\varepsilon cs^4\gamma - s\beta - 1 & -2\alpha - \frac{5}{12}K^\varepsilon cs^3\gamma & K^\varepsilon + \frac{1}{2}bs^2\gamma - K^\varepsilon s\alpha + \frac{1}{36}K^{2\varepsilon}cs^4\gamma \\ \frac{1}{72}cs^5\beta^2\gamma - \frac{7}{36}cs^3\gamma & \frac{1}{2}cs^2\gamma & \frac{7}{36}K^\varepsilon cs^3\gamma \end{pmatrix} ds \\
&= \begin{pmatrix} -2u(\alpha+\beta) & 0 & 0 \\ \frac{13}{360}K^\varepsilon cu^5\gamma - \frac{1}{2}u^2\beta - u & -2u\alpha - \frac{5}{48}K^\varepsilon cu^4\gamma & K^\varepsilon u - \frac{1}{2}K^\varepsilon u^2\alpha + \frac{1}{6}bu^3\gamma + \frac{1}{180}K^{2\varepsilon}cu^5\gamma \\ \frac{1}{432}cu^6\beta^2\gamma - \frac{7}{144}cu^4\gamma & \frac{1}{6}cu^3\gamma & \frac{7}{144}K^\varepsilon cu^4\gamma \end{pmatrix}
\end{aligned}$$

As a consequence:

$$\begin{aligned}
&\exp\left(-\int L(u)\,du\right) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -2u(\alpha+\beta) & 0 & 0 \\ \frac{13K^\varepsilon cu^5\gamma}{360} - \frac{1}{2}u^2\beta - u & -2u\alpha - \frac{5K^\varepsilon cu^4\gamma}{48} & K^\varepsilon u - \frac{K^\varepsilon u^2\alpha}{2} + \frac{bu^3\gamma}{6} + \frac{K^{2\varepsilon}cu^5\gamma}{180} \\ \frac{cu^6\beta^2\gamma}{432} - \frac{7cu^4\gamma}{144} & \frac{1}{6}cu^3\gamma & \frac{7K^\varepsilon cu^4\gamma}{144} \end{pmatrix} \\
&\quad +\frac{1}{2}\begin{pmatrix} -2u(\alpha+\beta) & 0 & 0 \\ \frac{13}{360}K^\varepsilon cu^5\gamma - \frac{1}{2}u^2\beta - u & -2u\alpha - \frac{5}{48}K^\varepsilon cu^4\gamma & K^\varepsilon u - \frac{1}{2}K^\varepsilon u^2\alpha + \frac{1}{6}bu^3\gamma + \frac{1}{180}K^{2\varepsilon}cu^5\gamma \\ \frac{1}{432}cu^6\beta^2\gamma - \frac{7}{144}cu^4\gamma & \frac{1}{6}cu^3\gamma & \frac{7}{144}K^\varepsilon cu^4\gamma \end{pmatrix}^2
\end{aligned}$$

and then:

$$\begin{aligned}
Y(u) &= B + \left(\int \exp\left(-\int L(u)\,du\right)\right)A \\
&= B + \begin{pmatrix} s(s\alpha + s\beta + 1) & 0 & 0 \\ \frac{s^2\left(480s\alpha + 360s\beta - K^\varepsilon cs^4\gamma + 360\right)}{720} & s + s^2\alpha + \frac{3}{80}K^\varepsilon cs^5\gamma & -\frac{K^\varepsilon s^2}{2} - \frac{K^\varepsilon s^3\alpha}{6} - \frac{bs^4\gamma}{24} - \frac{K^{2\varepsilon}cs^6\gamma}{180} \\ -\frac{cs^5\gamma}{144} & -\frac{cs^4\gamma}{24} & s + \frac{K^\varepsilon cs^5\gamma}{144} \end{pmatrix}A
\end{aligned}$$

which leads to the solution of (142):

$$X(u) = \exp(Nu)\Big(1+\gamma\hat{N}(u)\Big)\left(B + \left(\int \exp\left(-\int L(u)\,du\right)\right)A\right)$$

Given that:

$$
\begin{aligned}
&\exp(Nu)\Big(1+\gamma\hat{N}(u)\Big)\\
=&\begin{pmatrix} 1-s(\beta+\alpha) & 0 & 0\\ s\left(s\left(\alpha+\frac{\beta}{2}\right)-1\right) & 1-s\alpha & K^{\varepsilon}s\left(1-\frac{\alpha}{2}s\right)\\ 0 & 0 & 1\end{pmatrix}\\
&\times\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\gamma\begin{pmatrix} 0 & 0 & 0\\ \frac{1}{36}K^{\varepsilon}cs^5 & -\frac{3}{48}K^{\varepsilon}cs^4 & -\frac{c}{144}K^{2\varepsilon}s^5+\frac{1}{12}bs^3\\ -\frac{5}{144}cs^4 & \frac{1}{12}cs^3 & \frac{5}{144}K^{\varepsilon}cs^4\end{pmatrix}\right)\\
=&\begin{pmatrix} 1-s\beta-s\alpha & 0 & 0\\ s^2\alpha-s+\frac{1}{2}s^2\beta-\frac{1}{144}K^{\varepsilon}cs^5\gamma & \frac{1}{48}K^{\varepsilon}cs^4\gamma-s\alpha+1 & K^{\varepsilon}s-\frac{1}{2}K^{\varepsilon}s^2\alpha+\frac{1}{12}bs^3\gamma+\frac{1}{36}K^{2\varepsilon}cs^5\gamma\\ -\frac{5}{144}cs^4\gamma & \frac{1}{12}cs^3\gamma & \frac{5}{144}K^{\varepsilon}cs^4\gamma+1\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&\exp(Nu)\Big(1+\gamma\hat{N}(u)\Big)\left(\int\exp\left(-\int L(u)\,du\right)\right)\\
=&\begin{pmatrix} s & 0 & 0\\ \frac{1}{6}s^3\alpha-\frac{1}{2}s^2-\frac{7}{1440}K^{\varepsilon}cs^6\gamma & s+\frac{1}{60}K^{\varepsilon}cs^5\gamma & \frac{1}{2}K^{\varepsilon}s^2-\frac{1}{6}K^{\varepsilon}s^3\alpha+\frac{1}{24}bs^4\gamma+\frac{3}{160}K^{2\varepsilon}cs^6\gamma\\ 0 & \frac{1}{24}cs^4\gamma & s\end{pmatrix}
\end{aligned}
$$

we have ultimately the general solution of (142):

$$
\begin{aligned}
X(u) &= \exp(Nu)\Big(1+\gamma\hat{N}(u)\Big)\left(B+\left(\int\exp\left(-\int L(u)\,du\right)\right)A\right)\\
&=\begin{pmatrix} 1-s\beta-s\alpha & 0 & 0\\ s^2\alpha-s+\frac{1}{2}s^2\beta-\frac{1}{144}K^{\varepsilon}cs^5\gamma & \frac{1}{48}K^{\varepsilon}cs^4\gamma-s\alpha+1 & K^{\varepsilon}s-\frac{1}{2}K^{\varepsilon}s^2\alpha+\frac{1}{12}bs^3\gamma+\frac{1}{36}K^{2\varepsilon}cs^5\gamma\\ -\frac{5}{144}cs^4\gamma & \frac{1}{12}cs^3\gamma & \frac{5}{144}K^{\varepsilon}cs^4\gamma+1\end{pmatrix}B\\
&+\begin{pmatrix} s & 0 & 0\\ \frac{1}{6}s^3\alpha-\frac{1}{2}s^2-\frac{7}{1440}K^{\varepsilon}cs^6\gamma & s+\frac{1}{60}K^{\varepsilon}cs^5\gamma & \frac{1}{2}K^{\varepsilon}s^2-\frac{1}{6}K^{\varepsilon}s^3\alpha+\frac{1}{24}bs^4\gamma+\frac{3}{160}K^{2\varepsilon}cs^6\gamma\\ 0 & \frac{1}{24}cs^4\gamma & s\end{pmatrix}A
\end{aligned}
$$

Then, adding the particular solution of (141):

$$
\frac{d^2}{ds^2}X+H\left(\left(H^{-1}M-\left({}^{t}\left(H^{-1}M\right)\right)\right)\right)\frac{d}{ds}X-\left(\left(H\left({}^{t}M\right)H^{-1}M\right)+\gamma HM_1\right)X-\gamma HM_2X(0)=0
$$

which is:

$$
X(s)=\gamma\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & \frac{1}{12}bs^3\\ -\frac{1}{144}cs^4 & \frac{1}{12}cs^3 & \frac{K^{\varepsilon}}{144}cs^4\end{pmatrix}X(0)
$$

we obtain the a full solution of (141):

$$
\begin{aligned}
X(s) &= \begin{pmatrix} 1-s\beta-s\alpha & 0 & 0\\ s^2\alpha-s+\frac{1}{2}s^2\beta-\frac{K^{\varepsilon}cs^5\gamma}{144} & \frac{K^{\varepsilon}cs^4\gamma}{48}-s\alpha+1 & K^{\varepsilon}s-\frac{1}{2}K^{\varepsilon}s^2\alpha+\frac{1}{12}bs^3\gamma+\frac{K^{2\varepsilon}cs^5\gamma}{36}\\ -\frac{5cs^4\gamma}{144} & \frac{1}{12}cs^3\gamma & \frac{5}{144}K^{\varepsilon}cs^4\gamma+1\end{pmatrix}B \quad (143)\\
&+\begin{pmatrix} s & 0 & 0\\ \frac{1}{6}s^3\alpha-\frac{1}{2}s^2-\frac{7K^{\varepsilon}cs^6\gamma}{1440} & s+\frac{K^{\varepsilon}cs^5\gamma}{60} & \frac{K^{\varepsilon}s^2}{2}-\frac{K^{\varepsilon}s^3\alpha}{6}+\frac{bs^4\gamma}{24}+\frac{3K^{2\varepsilon}cs^6\gamma}{160}\\ 0 & \frac{cs^4\gamma}{24} & s\end{pmatrix}A\\
&+\gamma\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & \frac{1}{12}bs^3\\ -\frac{1}{144}cs^4 & \frac{1}{12}cs^3 & \frac{K^{\varepsilon}}{144}cs^4\end{pmatrix}X(0)
\end{aligned}
$$

Several types of initial conditions are possible. The most relevant will be to chose $X(0)$ and $\dot{X}(0)$ as initial conditions, one finds A by writing:

$$
A=\dot{X}(0)-\begin{pmatrix} -\alpha-\beta & 0 & 0\\ -1 & -\alpha & K^{\varepsilon}\\ 0 & 0 & 0\end{pmatrix}X(0)
$$

Inserting this result in (143) leads ultimately to:

$$\begin{aligned} X(s) &= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2}s^2\alpha + \frac{7}{720}K^\varepsilon cs^5\gamma & \frac{1}{48}K^\varepsilon cs^4\gamma + 1 & \frac{1}{6}bs^3\gamma - \frac{1}{2}K^\varepsilon s^2\alpha + \frac{1}{90}K^{2\varepsilon}cs^5\gamma \\ 0 & \frac{1}{6}c\gamma s^3 & 1 \end{pmatrix} X(0) \\ &+ \begin{pmatrix} s & 0 & 0 \\ \frac{1}{6}s^3\alpha - \frac{1}{2}s^2 - \frac{7}{1440}K^\varepsilon cs^6\gamma & s + \frac{1}{60}K^\varepsilon cs^5\gamma & \frac{1}{2}K^\varepsilon s^2 - \frac{1}{6}K^\varepsilon s^3\alpha + \frac{1}{24}bs^4\gamma + \frac{3}{160}K^{2\varepsilon}cs^6\gamma \\ 0 & \frac{1}{24}cs^4\gamma & s \end{pmatrix} \dot{X}(0) \end{aligned}$$

Then, replacing b and c by their values, and γ by $\frac{\gamma}{A^2K^\varepsilon}$, so that γ is dimensionless, and noting that $X(0) = B$, yields the result stated in the text and the deviation to this trajectory due to the interaction term, with the same initial conditions, is thus:

$$\begin{aligned} \delta X(s) &= \gamma \begin{pmatrix} 0 & 0 & 0 \\ \frac{7}{720}K^\varepsilon cs^5 & \frac{1}{48}K^\varepsilon cs^4 & \frac{1}{6}bs^3 + \frac{1}{90}K^{2\varepsilon}cs^5 \\ 0 & \frac{1}{6}cs^3 & 0 \end{pmatrix} X(0) \\ &+ \gamma \begin{pmatrix} 0 & 0 & 0 \\ -\frac{7}{1440}K^\varepsilon cs^6 & \frac{1}{60}K^\varepsilon cs^5 & \frac{1}{24}bs^4 + \frac{3}{160}K^{2\varepsilon}cs^6 \\ 0 & \frac{1}{24}cs^4 & 0 \end{pmatrix} \dot{X}(0) \end{aligned}$$

Case 3: 2 Agents interaction via 4 points Green function

The field theoretic context allows also to study the impact of one type of agent on an other. Consider the transition functions, for two agents, without interaction. The probability of transition between $(K_1, C_1, A_1)_i$ and $(K_1, C_1, A_1)_f$ for the first one, and $(K_2, C_2, A_2)_i$ and $(K_2, C_2, A_2)_f$ for the second one, is:

$$\begin{aligned} & G\left((K_1, C_1, A_1)_i, (K_2, C_2, A_2)_i, (K_1, C_1, A_1)_f, (K_2, C_2, A_2)_f\right) \\ \equiv\ & G\left((K_1, C_1, A_1)_i, (K_2, C_2, A_2)_i, s\right) G\left((K_1, C_1, A_1)_f, (K_2, C_2, A_2)_f, s\right) \end{aligned}$$

An application of the Wick theorem to the field interaction term:

$$\frac{\gamma}{2}\int \Psi^\dagger(K_1, C_1, A_1)\Psi^\dagger(K_2, C_2, A_2)\{A_2K_1 + A_1K_2\}\Psi(K_1, C_1, A_1)\Psi(K_2, C_2, A_2)$$

leads directly to a correction, on Green function Laplace transform:

$$\begin{aligned} & \int \gamma G\left((K_1, C_1, A_1)_i, (K_1, C_1, A_1), m_i\right) G\left((K_1, C_1, A_1), (K_1, C_1, A_1)_f, m_i\right) \\ & \times \{A_2K_1 + A_1K_2\} \\ & \times G\left((K_2, C_2, A_2)_i, (K_2, C_2, A_2), m_i\right) G\left((K_2, C_2, A_2), (K_2, C_2, A_2)_f, m_i\right) \\ \equiv\ & \int \gamma G\left((X_1)_i, X_1, m_i\right) G\left(X_1, (X_1)_f, m_i\right)\left\{\left({}^tX_1\right)MX_2\right\} G\left((X_2)_i, X_2, m_i\right) G\left(X_2, (X_2)_f, m_i\right) \end{aligned}$$

and in time representation:

$$\begin{aligned} & \gamma\int G\left((X_1)_i, X_1(s_1), s_1\right) G\left(X_1(s_1), (X_1)_f, s - s_1\right) \\ & \times\left\{\left({}^tX_1(s_1)\right)MX_2(s_2)\right\} G\left((X_2)_i, X_2(s_2), s_2\right) G\left(X_2(s_2), (X_2)_f, s - s_2\right) ds_1ds_2 \\ =\ & \gamma\int\left\langle\left({}^tX_1(s_1)\right)MX_2(s_2)\right\rangle ds_1ds_2 \end{aligned}$$

where the expectation is taken for path $X_1(s_1)$ starting from $(K_1, C_1, A_1)_i$ and ending at $(K_1, C_1, A_1)_f$ and path $X_2(s_2)$ starting from $(K_2, C_2, A_2)_i$ and ending at $(K_2, C_2, A_2)_f$. Given our assumptions about the parameters, we can, as in the previous paragraph, approximate these paths by their average values to the zeroth order in the parameters:

$$X_i(u) = \begin{pmatrix} \frac{u}{s} & 0 & 0 \\ \frac{u}{2s}(s-u) & \frac{u}{s} & -\frac{u}{2s}K^\varepsilon(s-u) \\ 0 & 0 & \frac{u}{s} \end{pmatrix} X_i(s) + \begin{pmatrix} \frac{s-u}{s} & 0 & 0 \\ -\frac{1}{2}\frac{u}{s}(s-u) & \frac{s-u}{s} & K^\varepsilon\frac{u}{2s}(s-u) \\ 0 & 0 & \frac{s-u}{s} \end{pmatrix} X_i(0)$$

so that:

$$\begin{aligned}\int_0^s X_i(u)\,du &= \begin{pmatrix} \frac{1}{2}s & 0 & 0 \\ \frac{1}{12}s^2 & \frac{1}{2}s & -\frac{1}{12}K^\varepsilon s^2 \\ 0 & 0 & \frac{1}{2}s \end{pmatrix} X_i(s) + \begin{pmatrix} \frac{1}{2}s & 0 & 0 \\ -\frac{1}{12}s^2 & \frac{1}{2}s & \frac{1}{12}K^\varepsilon s^2 \\ 0 & 0 & \frac{1}{2}s \end{pmatrix} X_i(0) \\ &= \begin{pmatrix} \frac{1}{2}s & 0 & 0 \\ \frac{1}{12}s^2 & \frac{1}{2}s & -\frac{1}{12}K^\varepsilon s^2 \\ 0 & 0 & \frac{1}{2}s \end{pmatrix} \Delta X_i + sX_i(0) \\ &= \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{6}s^2 & 0 & -\frac{1}{6}K^\varepsilon s^2 \\ 0 & 0 & 0 \end{pmatrix} \frac{\Delta X_i}{2} + s\bar{X}_i\end{aligned}$$

with $\bar{X}_i = \frac{X_i(0)+X_i(s)}{2}$.

$$\begin{aligned}&\gamma \int \left\langle \left({}^tX_1(s_1)\right) MX_2(s_2) \right\rangle ds_1 ds_2 \\ = \;& \gamma \left({}^t\Delta X_2\right) \begin{pmatrix} 0 & 0 & \frac{1}{24}s^3 \\ 0 & 0 & \frac{1}{4}s^2 \\ \frac{1}{24}s^3 & \frac{1}{4}s^2 & -\frac{1}{12}K^\varepsilon s^3 \end{pmatrix} \Delta X_1 + \gamma \left({}^tX_2(0)\right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s^2 \\ 0 & s^2 & 0 \end{pmatrix} X_1(0) \\ &+\gamma \left({}^t\Delta X_2\right) \begin{pmatrix} 0 & 0 & \frac{1}{12}s^3 \\ 0 & 0 & \frac{1}{2}s^2 \\ 0 & \frac{1}{2}s^2 & -\frac{1}{12}K^\varepsilon s^3 \end{pmatrix} X_1(0) + \gamma \left({}^tX_2(0)\right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}s^2 \\ \frac{1}{12}s^3 & \frac{1}{2}s^2 & -\frac{1}{12}K^\varepsilon s^3 \end{pmatrix} \Delta X_1 \\ = \;& \gamma \left({}^t\bar{X}_2\right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s^2 \\ 0 & s^2 & 0 \end{pmatrix} \bar{X}_1 + \frac{\gamma \left({}^t\Delta X_2\right)}{2} \begin{pmatrix} 0 & 0 & \frac{1}{6}s^3 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{6}K^\varepsilon s^3 \end{pmatrix} \bar{X}_1 \\ &+\gamma \left({}^t\bar{X}_2\right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{6}s^3 & 0 & -\frac{1}{6}K^\varepsilon s^3 \end{pmatrix} \frac{\Delta X_1}{2}\end{aligned}$$

This term modifies the 4 points Green function to an interaction Green function G_γ:

$$\begin{aligned}&G_\gamma \left((K_1, C_1, A_1)_i, (K_2, C_2, A_2)_i, (K_1, C_1, A_1)_f, (K_2, C_2, A_2)_f \right) \\ \equiv \;& G \left((K_1, C_1, A_1)_i, (K_2, C_2, A_2)_i, (K_1, C_1, A_1)_f, (K_2, C_2, A_2)_f \right) \exp\left(-\gamma \int \left\langle \left({}^tX_1(s_1)\right) MX_2(s_2) \right\rangle ds_1 ds_2 \right)\end{aligned}$$

In terms of trajectory, this means that the deviation of $X_1(s)$ due to $X_2(s_2)$ is given by $\gamma \int_0^s MX_2(s_2)\,ds_2$ (and the deviation for $X_2(s)$ is $\gamma H \int_0^s MX_1(s_1)\,ds_1$). Given (144), one has:

$$\gamma HM \int_0^s X_2(u)\,du = \gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{6}cs^3 & 0 & -\frac{1}{6}cK^\varepsilon s^3 \end{pmatrix} \frac{\Delta X_2}{2} + \gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & bs \\ 0 & cs & 0 \end{pmatrix} \bar{X}_2$$