Sustainability in Growth Models

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We study the relation between sustainability and national income in a neoclassical growth model with one product, which is used both as consumption good and investment good, and one natural resource, which is used in production. We analyse the possibilities for an indicator of sustainability, looking in particular at two indicators: the change in real national wealth and the ratio between sustainable constant consumption and actual consumption. It appears that both indicators can only be computed if the sustainable path of the economy is first computed, ànd that they must be computed for the whole future path of the economy, so that it is not sufficient to compute them for a single time period. For official statistics this means that sustainability indicators can only be computed by means of an economic model, and cannot be measured with actual data only.

Keywords: environment, sustainability, economic growth

1. Introduction

Time plays an important role in the environment. The consequences of environmental degradation processes are often noticeable after many years. In other words, environmental problems are often long-term problems. Therefore society has to weigh the structural, long-run consequences of economic growth for the environment. The theory of economic growth, which focusses on long-run economic development, is suited for analyzing these welfare consequences of economic growth for the environment. Such growth models have been used to study two subjects:

- 'green national income': the relation between welfare and national income
- 'sustainable national income': the relation between sustainability and national income

A survey of the literature on the first subject is given in De Boer, Brouwer en Zeelenberg (1995). In this report we survey the most important aspects of the literature on the second subject, i.e. whether there exists a measure that indicates whether the economy is sustainable. Our aim is an analysis of concepts, and not an analysis of the transition to sustainability. Therefore we use a limited economic model, with two primary factors of production, no technical change, and only one aspect of the environment, namely natural resources.

Sustainability is often defined in the literature as non-declining utility or consumption. This definition of sustainable development is closely linked to Hicks' definition of income as the amount one can consume during a period of time with welfare at the end of this period not lower than that at the start; see for example Pearce and

Atkinson (1995, pp. 167-8). This non-declining consumption is made possible by an non-declining capital stock, so that sustainability is only possible if the total capital stock (both fixed capital and environmental capital), or, taking into account population growth, per capita capital stock, is always constant or increasing. For example, Solow (1993) writes

> ”The appropriate policy is to generate an economically equivalent amount of net investment, enough to maintain society's broadly defined stock of capital intact.”

A sustainable national income can then be defined as the amount of goods and services that can be consumed instead of having to be invested in a certain period, while allowing a non-declining capital stock and thereby guaranteeing a future consumption level that is as least as high as the present level. Sustainable national income is thus the maximum level of consumption that can be maintained indefinitely. Pearce and Atkinson (1995) have used this criterion of non-declining capital to determine whether a country is on a sustainable path at a given point in time.

In this report we will investigate whether this criterion of non-declining capital stock can be justified in a neoclassical growth model. In section 2 we present the model; in section 3 we analyse the growth paths of the model; in section 4 we introduce sustainability in the model and look at the above criterion; in section 5 we describe the relation between national income and welfare; and in section 6 we give a summary.

2. Growth model with a natural resource

We analyse a closed economy with one final good, two inputs, capital and labour, and one natural resource, such as petroleum or fish. We use here only one aspect of the environment, namely a natural resource. Other aspects of the environment can be analysed in a similar way; see Vellinga en Withagen (1996) for a more general description of the environment. The production of the single good has two uses: consumption and investment. The revenue of the production process, expressed in units of the final good, is indicated by $F(K_t, L_t, R_t)$ where K is the capital stock, L the labour force, and R the intermediate consumption of the natural resource. By using the revenue function we hold implicit the optimal allocation of capital and labour over the two sectors, resource extraction and final-good production, which makes the analysis easier. Although this is theoretically not entirely correct, we will call F the production function; in De Boer et al (1995, appendix A) we show the relation with the more traditional model where the two sectors are made explicit. So, the production possiblities are represented by the production function:

$$Q_t = F(K_t, L_t, R_t), \tag{1}$$

where Q is output; we assume that F is linear homogeneous and concave. Technical change is assumed away. We assume a constant labour supply and full employment:

$$L_t = \overline{L}. \tag{2}$$

Output can be used for consumption, investment and extraction of the resource:

$$Q_t = C_t + I_t + G(R_t, S_t), \tag{3}$$

where C is consumption, I gross investment in fixed capital, and G the extraction costs of the resource, expressed in units of the final good. The initial capital stock is taken as given:

$$K_0 = \overline{K}. \tag{4}$$

We assume that the decay of the capital stock, caused by technical and economic obsolescence, is proportional to the existing stock, so that the change in the capital stock is

$$\dot{K}_t = I_t - \delta K_t, \tag{5}$$

where δ is the rate of decay.

The extraction costs of the resource, G, are a function of the stock and the extraction. Indicating the natural growth of the stock S of resource by $N(S)$ we can write the extraction costs as $G(R, S)$; for a non-renewable resource the natural growth is equal to 0. Exploration for the resource is left out from the analysis. The change in the natural resource stock is then

$$\dot{S}_t = N(S_t) - R_t, \tag{6}$$

where a dot above a variable indicates the derivative with respect to time, e.g. $\dot{S}_t = \mathrm{d}S_t/\mathrm{d}t$.

As social welfare function we choose a function of future consumption:

$$V_t = V\left(\int_t^\infty \upsilon(C_\tau, \tau)\,\mathrm{d}\tau\right), \tag{7}$$

where τ indicates future time ($t \leq \tau \leq \infty$), V is a monotonously increasing function, and υ a concave function of consumption. Special cases of the social welfare function (7) are the *present-value welfare function*, where[1] $\upsilon(C_\tau, \tau) = e^{-\rho\tau}U(C_\tau)$, the *iso-elastic welfare function*[2] $V_t = (\int_t^\infty e^{-\rho\tau}C_\tau^{1-\eta}\,\mathrm{d}\tau)^{-1/(1-\eta)}$, and the *maximin welfare function*, where $V_t = \min_{\tau\geq t} C_\tau$. The maximin function is a special case of the iso-elastic function, namely for $\eta \to \infty$ and $\rho = 0$, which is easily checked by computing the limit $(V_t/\min_{\tau\geq t} C_\tau)^{1-\eta}$.

The optimal growth path of the economy is now the path that gives maximum social welfare, given the technology, the labour force and the initial capital stock; this problem is known as the optimal growth model of Ramsey. So maximum welfare is obtained by maximizing the social welfare function under the restrictions (1)-(6).

[1]The term $e^{-\rho}$ is the *discount factor*, and ρ the *rate of time preference*.

[2]The term η equals minus the *elasticity of marginal utility* and also minus the inverse of the *intertemporal elasticity of substitution*: for $U(C) = C^{1-\eta}$ there holds $\eta = -CU''(C)/U'(C)$ and $1/\eta = -\lim_{t\to s} \partial\log(C_s/C_t)/\partial\log[U'(C_s)/U'(C_t)]$.

Without loss of generality we can take $t = 0$, so that the maximization problem is given by

$$\max V_0 = V\left(\int_0^\infty \upsilon(C_t, t)\,\mathrm{d}\,t\right), \tag{8}$$

under the restrictions

$$\dot{K}_t = F(K_t, \overline{L}, R_t) - C_t - \delta K_t - G(R_t, S_t) \tag{9}$$

and

$$\dot{S}_t = N(S_t) - R_t. \tag{10}$$

Note that the solution of this problem is consumption as a function of time. The optimization problem (8-10) can be solved by means of optimal control techniques; see Appendix A.1.

3. Growth paths

3.1. Introduction

The form of the optimal growth path of the previous section has been studied by Stiglitz (1973), Solow (1973), Dasgupta and Heal (1973, 1979) and Pezzey en Withagen (1995). Their results can be summarised as follows.

3.2. Essential and non-essential resources

First we must make a distinction between essential and non-essential resources. A resource is *essential* if there exists a growth path on which consumption does not tend to zero, and *non-essential* if on every growth path, consumption tends to zero. We consider only production functions with constant elasticity of substitution, so-called CES-production functions:

$$Q_t = F(K_t, L_t, R_t) = \left[\alpha\overline{L}^{(\sigma-1)/\sigma} + \beta K_t^{(\sigma-1)/\sigma} + \gamma R_t^{(\sigma-1)/\sigma}\right]^{\sigma/(\sigma-1)}, \tag{11}$$

where

$$\alpha > 0, \quad \beta > 0, \quad \gamma > 0, \quad \alpha + \beta + \gamma = 1, \quad \text{and} \quad \sigma > 0. \tag{12}$$

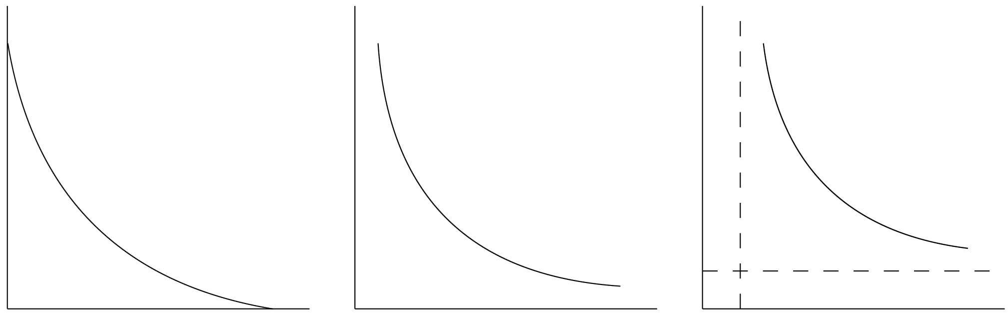
For $\sigma = 1$, (11) has the Cobb-Douglas form

$$Q_t = \overline{L}^{\alpha} K_t^{\beta} R_t^{\gamma}. \tag{13}$$

The isoquants of the CES production function are sketched in figure 1. They have asymptotes that for $\sigma > 1$ lie in the negative quadrant, for $\sigma = 1$ coincide with the axes, and for $\sigma < 1$ lie in the positive quadrant.

Figure 1 shows that for $\sigma > 1$ there holds $Q > 0$ if $R = 0$, so that the resource is not necessary in production and therefore not essential. Thus exhaustibility is not a

Figure 1. Isoquants of the CES-function



$\sigma > 1$ $\sigma = 1$ $\sigma < 1$

problem if the elasticity of substitution between fixed capital and the resource is larger than one.

For $\sigma < 1$ the average product Q/R is bounded so that the total output $\int_0^\infty Q_\tau \, d\tau$ that can be produced in the course of time, is finite. The resource is essential and exhaustibility poses a limit to economic development.

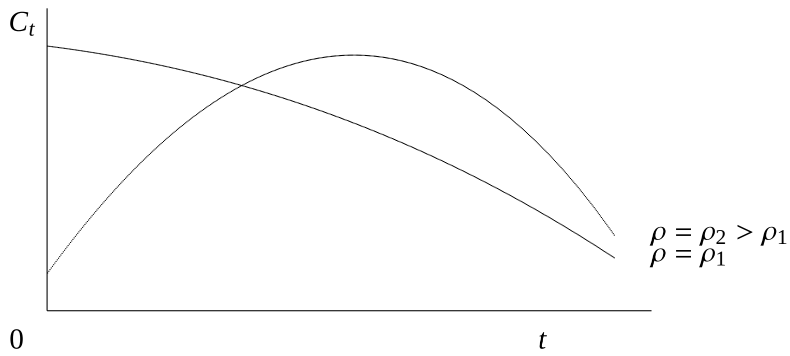
The only case that needs investigation is the Cobb-Douglas production function. On the one hand the resource is necessary in production, because $Q = 0$ if $R = 0$. On the other hand the average product of the resource is unbounded. It appears that when β is larger than γ, the resource is non-essential (Dasgupta en Heal, 1979, pp 200-3), because then there exists a growth path on which capital grows linearly and consumption is constant. Note that β equals the share of capital in output, and γ the share of the resource in output. In most economies the share of capital in national income is larger than the share of natural resources, so that for a Cobb-Douglas production structure natural resources are in general non-essential.

Therefore, whether exhaustion of a natural resource poses a problem for economic development, depends in this model on the elasticity of substitution between fixed capital and natural resources. If this elasticity is smaller than one, then in the long-run output will tend to zero, if it is larger than one, then there is no problem, and if it is equal to one, then there is no problem if the share of capital in national income is larger than the share of natural resources, a condition which is usually fulfilled.

3.3. Form of the growth path

The results on the form of the growth path have been systematically presented by Pezzey and Withagen (1995). We assume that the welfare function has the present-value form, and we first consider the case of a positive time preference (i.e. $\rho > 0$). Then it can be shown that if the production function exhibits constant returns to scale in fixed capital and the natural resource, the time path of consumption either has a peak or always falls. This also holds if the production function has the Cobb-Douglas form and exhibits decreasing returns. The possible forms of the time path of consumption

Figure 2. Time path of consumption under positive rate of time preference



are shown in in figure 2. We now consider the special case of the Cobb-Douglas production function. It can be shown that if the rate of time preference ρ is sufficiently small, the time path of consumption at first rises. Combining this with the result of the previous paragraph, this means that the time path has a peak. It can also be shown that for a sufficiently large value of the rate of time preference the time path at first and thus always falls. That the peak shifts to the right if the rate of time preference falls, has at present only been proved for the very special case of both constant returns to scale in capital and natural resource and a rate of time preference equal to the coefficient of capital in the production function (i.e. $\rho = \beta$); see Pezzey and Withagen (1995).

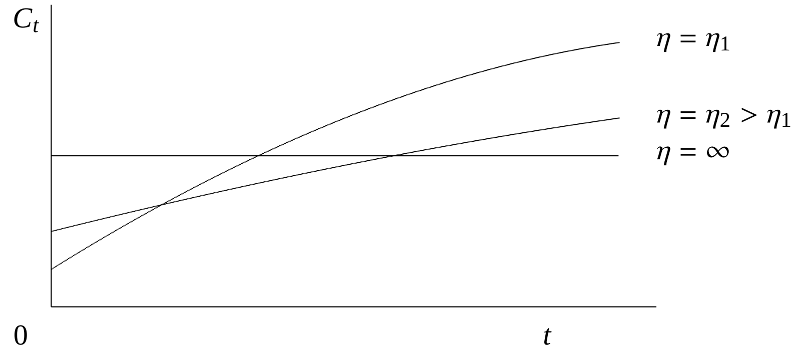
Dasgupta en Heal (1979, pp 305-8) have investigated the case where the rate of time preference equals zero, the production function has the Cobb-Douglas form, and the social welfare function has the iso-elastic form. It appears that if the elasticity of marginal utility is sufficiently large, namely if $\eta > (1 - \gamma)/(\beta - \gamma)$, permanent growth of consumption is possible. If η tends to ∞, then consumption is constant, because the iso-elastic welfare function is then equal to the maximin welfare function. If $\eta \leq (1 - \gamma)/(\beta - \gamma)$, then there is no solution of the welfare maximization problem. The possible forms of the consumption path are shown in figure 3.

4. Sustainability

4.1. What is sustainability?

We define sustainability as follows. An economy *follows a sustainable path (is sustainable)* if on the entire path social welfare does not decrease. An economy is *sustainable at a certain point in time* if given the situation at that point, it can reach a path of non-decreasing welfare. Note that this definition does not imply that sustainability is impossible if the stocks of natural resources fall below a certain level. Whether sustainability is technically feasible if these stocks keep on falling, depends on the technology, in particular the substitutability between fixed capital and natural resources and the form of the social welfare function. For the former we refer to section 3.2,

Figure 3. Consumption paths with time preference equal to 0



where it has been shown for the CES production function that if the elasticity of substitution between fixed capital and the natural resource is smaller than one, sustainable development is impossible.

4.2. Sustainability in growth models

Sustainability can be regarded as keeping intact the possibilities of the environment for later generations. We view sustainability here as 'weak sustainability', where the environment itself may deteriorate if this is compensated for by an increase in alternatives, such as fixed capital. 'Weak sustainability' is for our purposes then the same as non-decreasing welfare.

There are several ways in which sustainability can be incorporated in growth models. First we can take sustainability as the criterion of social welfare, which is then measured by the maximin welfare function

$$\min_{\tau \geq t} C_\tau. \tag{14}$$

Then the optimal path of consumption is constant:

$$C_\tau = \overline{C}_t, \quad \tau \geq t. \tag{15}$$

where $\overline{C}_t$ is the constant consumption that is maximally possible from t onwards.

Another way to incorporate sustainability is to add it as an additional restriction to the optimization problem (7) (see Pezzey, 1995, chapter 3), i.e. we add the restriction

$$\dot{C}_\tau \geq 0, \quad \tau \geq t. \tag{16}$$

This approach does not necessarily lead to constant consumption. In many cases the solution of (7) with (16) tends in the long run to a constant level of consumption. It is not entirely clear which of these two approaches is to be preferred. If society has a strong preference for sustainability, then incorporation in the social welfare function, such as in the first approach, is preferable. On the other hand, the second approach gives more general solutions.

A third way is to define sustainability as a constant level of the services provided by the environment and the fixed capital stock to society. If technology is constant, as in the models discussed here, this level can only remain constant by substituting environmental services by capital services; for this to continue indefinitely, the elasticity of substitution has to be equal to or larger than one. If there is technical change, this may compensate a part of the degradation of the environment. In both cases consumption is held at least constant. Applications with this definition will not lead to conclusions that differ much from those with the other two definitions, but are more in line with the thought of environmental scientists, who use standards for the flows of environmental services. In this more practical approach, consumption is maximized at each point of time under the restriction that environmental standards are not not exceeded.

In the remainder of this paper we will continue with the first definition and not consider the other two approaches.

4.3. Maximin welfare maximization and Hartwick's rule

As shown above, the optimal maximin path has constant consumption. Hartwick (1977, 1978) has derived an important policy rule that leads to constant consumption. This rule says that fixed capital formation should be equal to net revenue from resource extraction:

$$\dot{K}_t = -(F_R - G_R)\dot{S}_t = (F_R - G_R)(R_t - N_t), \tag{17}$$

where $F_R - G_R$ is the shadow price of the resource, which is equal to the difference between its marginal product and its marginal extraction cost. Thus Hartwick's rule implies that real national wealth is constant, since equation (17) says that the value of the change in real national wealth equals zero, so that the Divisia quantity index of national wealth is constant.

If resource extraction is efficient, then Hotelling's rule says that the change in its rate of return equals the interest rate:

$$\frac{\dot{F}_R - \dot{G}_R - G_S}{F_R - G_R} + N_S = F_K - \delta. \tag{18}$$

The interpretation of this rule is: the right-hand side is the shadow rate of interest, i.e. net marginal product of capital; the left-hand side is the rate of return on the resource stock, and consists of three terms: the first term, $(\dot{F}_R - \dot{G}_R)/(F_R - G_R)$, is the change in the shadow price of the stock (the capital gain), the second term, $G_S/(F_R - G_R)$, is the change in the rate of return on the whole stock resulting from a change in extraction costs when an addtional unit is extracted, and the third term, N_S, is the rate of return resulting from natural growth. Equality (18) can also be derived from the first-order conditions of the optimization problem (7) (see Appendix A.2).

From (3) and (5) we have

$$\dot{C}_t = \dot{Q}_t - \dot{G}_t - \ddot{K}_t - \delta \dot{K}_t, \tag{19}$$

so that using (17) and (18) we get

$$\begin{aligned}
\dot{C}_t =& F_K \dot{K}_t + F_L \dot{L}_t + F_R \dot{R}_t - G_R \dot{R}_t - G_S \dot{S}_t - \ddot{K}_t - \delta \dot{K}_t \\
=& -(F_R - G_R)\left(\frac{\dot{F}_R - \dot{G}_R - G_S}{F_R - G_R} + N_S\right)\dot{S}_t + F_R \dot{R}_t - G_R \dot{R}_t - G_S \dot{S}_t \\
& + (\dot{F}_R - \dot{G}_R)\dot{S}_t - (F_R - G_R)(\dot{R}_t - \dot{N}_t) \\
=& -(\dot{F}_R - \dot{G}_R - G_S)\dot{S}_t - (F_R - G_R) N_S \dot{S}_t + (F_R - G_R)\dot{R}_t \\
& + (\dot{F}_R - \dot{G}_R - G_S)\dot{S}_t - (F_R - G_R) N_S \dot{S}_t + (F_R - G_R)\dot{R}_t - G_S \dot{S}_t \\
=& 0;
\end{aligned} \tag{20}$$

in other words: if all net revenue from the resource is always invested in fixed capital, then consumption is constant. Thus in this growth model sustainability can be obtained by following Hartwick's rule *for every point of time*. Therefore it is not correct, as in Pearce and Atkinson (1994), to use the change in national wealth at one point in time as the criterion of sustainability. Asheim (1994) even gives a counter example, where the economy does not follow a sustainable path, but the change in real national wealth is positive.

Note moreover that the prices that have been used in (18) are the prices on the sustainable path, which, if the economy is not actually sustainable, do not have to equal actual prices. To compute the prices under sustainability, we have to solve the optimization problem (8)-(10) using the maximin function as social welfare function.

As sustainability indicator one could also use the ratio $\overline{C}_t / C_t$. To compute this indicator we do not need prices on the sustainable path, but to obtain the necessary information, one has again to solve the optimization problem (8)-(10) using the maximin function as social welfare function.

5. National income and welfare

Using the Hamiltonian, we can transform the infinite-horizon optimization problem (8)-(10), into optimization problems for each point in time. Maximization of the Hamiltonian is thus equivalent to welfare maximization, so that one may view the Hamiltonian as the *instantaneous welfare indicator*. This equivalence has been used by, amongst others, Hartwick (1990) and Mäler (1991) to analyse the relation between welfare, environment and national income. Equation (A.23) in appendix A.3 shows that the Hamiltonian is equal to

$$H_t = \upsilon(C_t, t) + \upsilon_C I_t^f + \upsilon_C (F_R - G_R) I_t^h, \tag{21}$$

where $I_t^f = \dot{K}_t - \delta K_t$ is net fixed capital formation, and $I_t^h = N(S_t) - R_t$ is the net increase in the stock of the natural resource. So the Hamiltonian is equal to the sum of the utility of consumption and the real change in national wealth, evaluated at marginal utility. The first two terms in (21) correspond to the components of national income, consumption and fixed capital formation. Thus the welfare indicator (21) encompasses more than national income: the change in the resource stock has to be included as well. In this sense one may say that in order to obtain a welfare measure, one has to correct national income for the exhaustion of natural resources.

If Hartwick's rule, equation (17), is followed, then the real change in national wealth is equal to zero, so that we have

$$\overline{H}_t = \upsilon(\overline{C}_t, t) = \overline{C}_t. \tag{22}$$

Again we see that the welfare indicator is not equal to national income. Thus one might say that the quest for 'sustainable national income' has not gone far enough: not only must we determine sustainable national income, but we must also correct for investment in the sustainable situation in order to measure welfare.

6. Conclusion

Sustainability has been defined in this paper as a situation where welfare of future generations is not lower than that of the present generation. In economic terms this means that the environment may deteriorate if alternative means, such as capital goods, that may compensate for the loss of possible uses of the environment, increase; this form of sustainability is sometimes called 'weak sustainability'.

In the literature on sustainable development it is often said that the real change in national wealth, the sum of the value of capital goods and the value of the environment, can be used as an indicator for sustainability. We have investigated this claim in a simple growth model with a single natural resource and constant labour supply and constant technology. We have shown that sustainability is only correctly indicated by a non-declining real national wealth if this holds at *any* point of the growth path, ànd the indicator is evaluated at the prices on the sustainable path. So, to be able to use the indicator, one must compute the sustainable path of the economy. A similar conclusion holds for an alternative indicator, the ratio of maximum constant consumption and actual consumption. Therefore, the computation of a sustainability indicator requires a lot of information on the production possibilities of the economy; in general, this information will be hard to obtain, because it concerns a part of the production possibilities that can be observed only if the economy actually follows a sustainable path.

For official statistics we can draw the following conclusions. Because evaluation of the sustainability indicators requires computation of the sustainable path, they can only be determined by using an integrated environmental-economic model. In other words: the sustainability indicators must be computed by means of an economic model and cannot be measured by means of actual statistical data only. To which extent this still belongs to official statistics, is a matter of judgement.

Appendix A. Derivations

A.1. Welfare maximization

The optimal growth problem (8)-(10) is

$$\max V_0 = V\left(\int_0^\infty \upsilon(C_t, t)\,\mathrm{d}\,t\right), \tag{A.1}$$

under the restrictions

$$\dot{K}_t = F(K_t, \overline{L}, R_t) - C_t - \delta K_t - G(R_t, S_t) \tag{A.2}$$

and

$$\dot{S}_t = N(S_t) - R_t. \tag{A.3}$$

Because V is a monotonously increasing function, the problem (A.1)-(A.3) is equivalent to

$$\max V_0 = \int_0^\infty \upsilon(C_t, t)\,\mathrm{d}\,t, \tag{A.4}$$

under the restrictions

$$\dot{K}_t = F(K_t, \overline{L}, R_t) - C_t - \delta K_t - G(R_t, S_t) \tag{A.5}$$

and

$$\dot{S}_t = N(S_t) - R_t. \tag{A.6}$$

To solve this problem we construct the Hamiltonian

$$H_t = \upsilon(C_t, t) + \mu_t[F(K_t, \overline{L}, R_t) - C_t - \delta K_t - G(R_t, S_t)] + \phi_t[N(S_t) - R_t], \tag{A.7}$$

where μ_t and ϕ_t are so-called co-state variables. The first-order conditions are

$$\frac{\partial H_t}{\partial C_t} = 0, \tag{A.8}$$

$$\frac{\partial H_t}{\partial R_t} = 0, \tag{A.9}$$

$$\dot{\mu}_t = -\frac{\partial H_t}{\partial K_t}, \tag{A.10}$$

$$\dot{\phi}_t = -\frac{\partial H_t}{\partial S_t}, \tag{A.11}$$

$$\lim_{t\to\infty} K_t = 0, \tag{A.12}$$

$$\lim_{t\to\infty} S_t = 0, \tag{A.13}$$

$$\lim_{t\to\infty} \mu_t \geq 0, \tag{A.14}$$

$$\lim_{t\to\infty} \phi_t \geq 0, \tag{A.15}$$

It follows from (A.8) that

$$\upsilon_C = \mu \tag{A.16}$$

and from (A.9) that

$$\phi_t = \mu_t(F_R - G_R). \tag{A.17}$$

It follows from (A.10) that

$$\dot{\mu}_t = -\mu_t(F_K - \delta) \tag{A.18}$$

and from (A.11) that

$$\dot{\phi}_t = \mu_t G_S - \phi_t N_S. \tag{A.19}$$

A.2. Efficient exploitation

Differentiating (A.9) with respect to time, we get

$$\dot{\phi}_t = \dot{\mu}_t(F_R - G_R) + \mu_t(\dot{F}_R - \dot{G}_R). \tag{A.20}$$

Substituting (A.18) and (A.19), we obtain

$$\mu_t G_S - \phi_t N_S = -\mu_t(F_K - \delta)(F_R - G_R) + \mu_t(\dot{F}_R - \dot{G}_R), \tag{A.21}$$

from which, after rearranging and using (A.17), we get Hotelling's rule

$$\frac{\dot{F}_R - \dot{G}_R - G_S}{F_R - G_R} + N_S = F_K - \delta. \tag{A.22}$$

A.3. Hamiltonian and welfare

It follows from (A.7), (A.16) and (A.17) that

$$H_t = \upsilon(C_t, t) + \upsilon_C I_t^f + \upsilon_C(F_R - G_R)I_t^h, \tag{A.23}$$

so that on the optimal path the Hamiltonian is equal to the sum of instantaneous utility and the real change in national wealth evaluated at marginal utility.

Differentiating the Hamiltonian (A.7) with respect to time we get

$$\begin{aligned}\dot{H}_t =& \upsilon_C C_t + \upsilon_C + \mu_C(F_K\dot{K}_t + F_R\dot{R}_t - \dot{C}_t - \delta\dot{K}_t G_R\dot{R}_t - G_S\dot{S}_t) \\ &+ \phi_t(N_S\dot{S}_t - \dot{R}_t) + \dot{\mu}_t\dot{K}_t + \dot{\phi}_t\dot{S}_t,\end{aligned} \tag{A.24}$$

which after substitution of (A.8)-(A.19) gives

$$\dot{H}_t = \upsilon_t. \tag{A.25}$$

Solving this differential equation we get, using (A.12) and (A.13)

$$H_t = \int_{-\infty}^{t} \upsilon_\tau(C_\tau, \tau)\,\mathrm{d}\,\tau = -\int_{t}^{\infty} \upsilon_\tau(C_\tau, \tau)\,\mathrm{d}\,\tau. \tag{A.26}$$

Thus, the Hamiltonian is an indicator of future consumption, an interpretation that becomes more clear if we consider the special case of the present-value utility function, where $\upsilon(C_\tau, \tau) = e^{-\rho\tau}U(C_\tau)$, for which we have $\upsilon_\tau = -\rho e^{-\rho\tau}U(C_\tau)$, so that

$$H_t = \rho\int_{t}^{\infty} e^{-\rho\tau}U(C_\tau)\,\mathrm{d}\,\tau; \tag{A.27}$$

i.e. the Hamiltonian is proportional to the present value of the utility of future consumption.

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