Catastrophic Risks with Finite or Infinite States

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ABSTRACT

Catastrophic risks are rare events with major consequences, such as market crashes, catastrophic climate change, asteroids or the extinction of a species. We show that classic expected utility theory based on Von Neumann axioms is insensitive to rare events no matter how catastrophic. Its insensitivity emerges from a requirement of continuity (e.g. Arrow's Monotone Continuity Axiom, and its relatives as defined by De Groot, Hernstein and Milnor) that anticipate average responses to extreme events*. This leads to countably additive measures and `expected utility' that are insensitive to extreme risks. In a new axiomatic extension, the author (Chichilnisky 1996, 2000, 2002) requires equal treatment of rare and frequent events, deriving the new decision criterion the axioms imply.*

These are expected utility combined with purely finitely additive measures that focus on catastrophes, and explain the presistent observations of distributions with "fat tails" in earth sciences and financial markets. Continuity is based on the `topology of fear' introduced in Chichilnisky (2009), and is linked to Debreu's 1953 work on Adam Smith's Invisible Hand. The balance between the classic and the new axioms tests the limits of non- parametric estimation in Hilbert spaces, Chichilnisky (2008).. extending the foundations of probability & statistics (Chichilnisky 2009 and 2010) to include "black swans" or rare events, and finite as well as infinite state spaces.

Keywords: *catastrophic risks, choice under uncertainty, black swans, green economics, incompleteness of mathematics, axiom of choice*

JEL Codes: *D70, D81, Q01, Q50, Q54, Q58, C1, C14*

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INTRODUCTION

This article is an investigation on the way that economics deals with catastrophic risks such as market crashes, catastrophic climate change and the extinction of a species. Catastrophic risks are rare events with major consequences. They include the 2008-9 global financial crisis that is a one-in-a-hundred-years event with momentous consequences for global markets, and more generally any rare event that threatens human survival - such as a large asteroid impact (Posner, 2004; United Nations, 2000; Chichilnisky and Eisenberger, 2010). The article shows how to rewrite the foundations of probability and statistics - the way we observe the universe around us -- so as to systematically incorporate outliers" and "fat tails" in our formal discourse, and how to develop a decision theory that agrees with the recent neurological experiments on how the human brain reacts to fear - both of which are denied in conventional Von Neumann Morgenstern theory of decision under uncertainty.

This article summarizes how Von Neumann's classic theory of choice under uncertainty evaluates catastrophic risks, and identifies why it is insensitive to rare events. The classic theory was developed axiomatically by Von Neumann and Morgenstern, Hernstein and Milnor, Villegas, Arrow, De Groot and others[1]. The point of this article is to show how the classic axioms evaluate catastrophic risks - to explain its insensitivity to rare events no matter how catastrophic they may be. Posner (2004); Tversky and Wakker (1995) and Chichilnisky (2000); (2007), leading therefore to a `disconnect' between decisions involving classic economic theory, and the way green economics and many human societies react to potential catastrophes such as global warming and widespread biodiversity extinction. Without throwing the baby with the bathtub water, the article explains a recent sharp generalization of the classic theory of decision making under uncertainty that requires *equal treatment of frequent and rare events* and is sensitive to catastrophic events, explicitly incorporating outliers and explaining the common observation of "heavy tails" as part of the standard statistical treatment, Chichilnisky (1977); (1996a); (1996b); (2002); (2006a); (2006b); (2009a); (2009b); (2010); Chichilnisky and Wu (2006), Chanel and Chichilnisky (2007). The recent treatment of catastrophic risks presented here is linked to the early work of Gerard Debreu (1954) on Adam Smith's invisible hand, the two welfare theorems of economics, and to the work of Kurt Godel (1940) on the Axiom of Choice and the "incompleteness" of mathematics. We also illustrate the connection between rare events and the foundation of statistic analysis in econometrics (Chichilnisky, 2006), and recent neuro-economic evidence of how humans choose when they face catastrophes and under conditions of fear (Chichilnisky, 1977; Le Doux, 1996; Chanel and Chichilnisky, 2007). In recent work the foundations of probability and statistics are enlarged to includes "black swans" and the results are developed to include finite as well as infinite samples.

CLASSIC AXIOMS OF CHOICE UNDER UNCERTAINTY

Over half a century ago, Von Neumann and Morgenstern (1944); Hernstein and Milnor (1953), and Arrow (1963) introduced classic axioms to explain how we choose under conditions of uncertainty, see also Villegas (1964) and De Groot (2004) among others. The main consequence is to predict that we rank our choices under uncertainty (also called `lotteries') according to their `expected utility,' namely by optimizing a function defined on lotteries $c(t)$ by

$$\int_R u(c(t))d\mu(t),$$

more on this below. The practical importance of the classic theory is underscored by the fact that US Congress requires its use in cost benefit analysis of any major budgetary decision. Decisions based on "expected utility" have been elevated to the status of "rational decisions," and rational behavior under uncertainty has became identified with optimizing expected utility. Yet for almost as many years, experimental and empirical evidence has questioned the validity of the expected utility model. Well known examples of diverging experimental evidence include the Allais Paradox (Allais, 1988) and practical `puzzles' in finance such as the `Equity Premium Puzzle' and the `Risk Free Premium Puzzle' Mehra and Prescott (1985); (Mehra, 2003); Rietz (1988); Weil (1989) showing that the returns from equity and bonds are orders of magnitude different in practice from what expected utility analysis would predict. Early theoretical developments of this critical line of thought include Kahneman and Tversky (1979); Machina (1982); (1989); Tversky (1995); leading more recently an entire new field of decision analysis, *Behavioral Economics,* which questions rational behavior interpreted as expected utility optimization.

This article focuses on one aspect of the classic theory that is shown to be critical to the disconnect between classic theory and experimental reality: how it deals with *catastrophic risks* and the issues that arise as well as the links that these issues have with econometrics, with the classic work on Adam Smith's invisible hand, with the foundation of statistics and non - parametric estimation, with Godel's fundamental theorems on the incompleteness of Mathematics, with the foundations of statistics and how it treats `outliers' and `heavy tails', as well as with the experimental evidence of how we choose when we face rare events that inspire fear.

In the following we summarize the motivation and provide the mathematical context. We then provide new axioms for decision under uncertainty that update the classic axioms in a way that coincides with Von Neumann's axioms in the absence of catastrophic events - therefore it represents a true extension of the existing theory. A representation theorem identifies new types of probability distributions that rank lotteries in situations involving catastrophic events; these combine expected utility (which averages risk) with extremal reactions to risk and are shown to arise from convex combinations of `countably additive' and `finitely additive' measures. An intuitive 'rule of thumb' of the new type of rankings is *to optimize expected returns while minimizing losses in a catastrophe*. This is a natural criterion of choice under uncertainty -- but it is shown to be inconsistent with expected utility.

THE TOPOLOGY OF FEAR

`Expected utility' analysis anticipates average responses to average risks, where the weights are the probability of the events. This seems fair enough. But it also anticipates average responses to extreme risks. The latter is more questionable. Through Magnetic Resonance Imagining (MRI) Le Doux (1996) has shown that humans often use a different part of the brain to deal with fear: we use the ***amygdala*** rather than the ***frontal cortex*** to make decisions on events that elicit fear. The amygdala is a primitive part of the brain that handles extreme emotions, and as such provides very simplified responses to risk. It causes *extremal responses to extreme risks*. When we feel fear, rather than averaging the evidence using probabilities, as we do with expected utility, we have extreme and simplified responses, for example a zero/one response such as "flight or fight". This finding is the key to understanding how Von Neuman's (VN) theory underrepresents catastrophic risks, and to update the theory to represent more accurately how we behave when we face fear. The key can be found in the topology we use to evaluate `nearby events', the foundation of statistical measurements. The reason is simple. Topology is by definition the way we formalize the concept of "nearby."

Mathematically, the VN axioms postulate nearby responses to nearby events, a form of continuity that is necessary for empirical validation and robustness of the theory. But the specific topology used in the definition of continuity by Arrow (1963); Milnor and Hernstein (1953) and De Groot (1970); (2004) means that they anticipate that *nearby is measured by averaging distances*. Under conditions of fear, however, we measure distances differently, by ***extremals***. Mathematically the difference is as follows: ***Average distance*** is defined by averaging coordinate by coordinate as in Euclidean space;[3] in continuous function spaces on the real line (such as $L_p\,(p < \infty)$, and Sobolev spaces) the average distance is defined as

$$\|f - g\|_p = (\int |(f-g)^p|\, dt)^{1/p}$$

Extremal distances are measured instead by the supremum of the difference, namely by focusing on extremes,[2] for example, using the *sup norm* of L_∞ :

$$\|f - g\|_\infty = ess \sup_R |(f-g)|$$

Changing the topology, namely the way we measure distances, changes our approach to risk: it leads to new ways to evaluate catastrophes. If we use a topology that takes extremals into consideration, our actions become more sensitive to extremal events no matter how rare they may be. We call this the *"topology of fear"* (Chichilnisky, 2009) because it focuses sharply if not uniquely on the worst that can happen in the case of catastrophes. This is the approach followed in (Chichilnisky, 1977; 1996a; 2000; 2002; 2006; 2009). The new axioms for choice under uncertainty introduced in this work agree with the classic theory when catastrophic events are absent, but otherwise they lead to a different type of statistics as well as econometrics, based on standard "gaussian" measures combined with singular measures (Chichilnisky, 2000), more on this below.

MATHEMATICS OF RISK

A few definitions are needed. A system is in one of several possible states which can be described by real numbers, the integers, a bounded segment of real numbers or a finite set of states $S = \{s_1, ... s_K\}$. Here we choose the real line for simplicity. At the end of the article we address the case of finite states. To each state $s \in R$ there is an associated outcome, so that one has $f(s) \in R^N$, $N \geq 1$. A description of probabilities across all states is called a *lottery* $x(s): R \to R^N$. The space of all lotteries L is therefore a ***function space*** L. Under uncertainty one ranks lotteries in L. Von Neumann's (VN) axioms provided a mathematical formalization of how to rank lotteries. Optimization according to such a ranking is called **`*expected utility maximization'***** and defines decision making under uncertainty. To choose among risky outcomes, we rank lotteries. A lottery x is ranked above another y if and only if there is a real valued continuous function on lotteries, W, that assigns to x a larger real number than y:

$$x \succ y \Leftrightarrow W(x) > W(y)$$

EXPECTED UTILITY

The main result from the VN axioms is a *representation theorem* that explains in practice how to choose under uncertainty:

Theorem: (VN, Hernstein and Milnor, Villegas, De Groot, Arrow) A ranking over lotteries which satisfies the VN axioms admits a representation by an *integral operator* $W : L \to R$, which has as a `kernel' a countably additive measure over the set of states, with an integrable density. This is called ***expected utility***.

The VN representation theorem proves that the ranking of lotteries is given by a function $W : L \to R$,

$$W(x) = \int_{s \in R} u(x(s)) d\mu(s)$$

where the real line R is the state space, $x : R \to R^N$ is a lottery, $u : R^N \to R^+$ is a (bounded) utility function describing the utility provided by the (certain) outcome of the lottery in each state s, $u(x)$, and where $d\mu(s)$ is a standard countably additive measure over states s in R. The optimization of an *expected utility* W is a widely used procedure for evaluating choices under uncertainty. Euler equations are used to characterize optimal solutions.

WHAT ARE CATASTROPHIC RISKS?

A catastrophic risk is a small probability - or ***rare*** - event which can lead to major and widespread losses. Classic methods do not work: it has been shown empirically and theoretically (Chichilnisky, 1996a; 2000; 2009; Posner, 2004) that using VN expected utility criteria undervalues catastrophic risks and conflicts with the observed evidence of how humans evaluate such risks. Mathematically the problem is that the measure μ which emerges from the VN representation theorem presented above, is *countably additive* implying that any two lotteries $x, y \in L$ are ranked by W quite independently of the utility of the outcome in states whose probabilities are lower than some threshold level $\varepsilon > 0$ depending on x and y. This means that expected utility maximization is insensitive to small probability events, no matter how catastrophic these may be. As shown in Chichilnisky (2000) the properties of the measure arise from continuity under an L_1 norm and when one changes this to continuity under a L_∞ norm, the sup norm, one obtains in general different types of measure. Formally the problem with VN Axioms is the following

EXPECTED UTILITY IS INSENSITIVE TO RARE EVENTS

A ranking W is called *Insensitive to Rare Events* when it ranks two choices focusing on their outcomes in frequent events, and neglects their outcomes are rare events. Formally,

Definition 1: A ranking W is *Insensitive to Rare Events* when

$$W(x)$$
$$W(y)$$
$$\exists \varepsilon$$
$$0,\ \varepsilon \quad = \varepsilon\ (x, y):$$

$$W(x')$$
$$W(y')$$

for all x', y' such that

$$x' = x \text{ and } y' = y \text{ a.e. except on } A \subset R\ :\ \mu\ (A) < \varepsilon\ .$$

In words, W ranks x above y if and only if it ranks x' above y' for any pair of lotteries x' and y' which are obtained by modifying arbitrarily x and y in sets of states A with probability lower than ε. The interpretation of this property is that the ranking defined by W is ***insensitive*** to the outcomes of the lottery in rare events. Similarly,

Definition 2: A ranking W is called *Insensitive to Frequent Events* when it ranks choices focusing exclusively on rare events, and neglects frequent events:

$$W(x)$$
$$W(y)$$
$$\exists M$$
$$0,\ M = M(x, y):$$

$$W(x')$$
$$W(y')$$

for all x', y' such that

$$x' = x \text{ and } y' \text{ a.e.} = y \text{ on } A \subset R\ :\ \mu(A) \quad M \quad .$$

Proposition 1: Expected utility is Insensitive to Rare Events.
Proof : In Chichilnisky (2000) and (2009)

As defined by VN, the expected utility criterion W is therefore less well suited for evaluating catastrophic risks.

UPDATING VON NEUMANN MORGENSTERN AXIOMS

A well-defined set of axioms introduced in (Chichilnisky, 1996a; 2000) update and extend Von-Neumann Morgenstern axioms by treating symmetrically rare and frequent risks. These axioms lead to a new representation theorem and define *new types of criteria or functions* that are maximized under uncertainty. These functions are commonly used in practice but were not used in Economics, Mathematics or Statistics before. The result is quite different from expected utility maximization or any

other optimization criterion used so far. The space of lotteries is L_∞ with the sup norm.

NEW AXIOMS FOR CHOICES THAT INVOLVE CATASTROPHES

Here are the new axioms about how to choose under uncertainty, how to "rank lotteries," when catastrophes are at stake:

Axiom 1. The ranking $W : L_\infty \to R$ is sensitive to rare events.

Axioms 2. The ranking W is sensitive to frequent events

Axiom 3: The ranking W is continuous and linear

Axioms 2 and 3 are standard, they are satisfied for example by expected utility, and agree with VN axioms. But Axiom 1 ***negates*** Arrow's "Axiom of Monotone Continuity", which leads to Expected Utility. Indeed:

Theorem 1: The Monotone Continuity Axiom (Hernstein & Milnor, 1953; Arrow, 1963; De Groot, 1970) **is equivalent to "Insensitivity to Rare Events". It is the logical negation of our Axiom 1.**
Proof: In Theorem 2, "The Topology of Fear" (Chichilnisky, 1977)

Example: The Monotone Continuity Axioms of Arrow and Milnor provides **A Statistical Value of Life** (Chichilnisky, 1977). This defies the experimental evidence on how people value life (Chanel and Chichilnisky, 2007)

A REPRESENTATION THEOREM

As in the case of the prior VN axioms, the new axioms lead to a new representation theorem.
Theorem 2 (Chichilnisky 1992, 1996, 2000)
There exist criteria or functionals $\Psi : L_\infty \to R$ which satisfy all three new axioms. All such functionals are defined by a convex combination of purely and countably additive measures, with both parts present.
Formally, there exists $\nu,\ 0<\nu<1,$ a utility function $u(x):\ R \to R$ and a countably additive regular measure μ on R, represented by an L_1 density Γ, such that the ranking of lotteries $\Psi : L_\infty \to R$ is of the form

$$\Psi(x) = \nu \int u(x(s))\Gamma(s)d\mu(s) + (1-\nu)\Phi(u(x(s))). \qquad (1)$$

where Φ denotes a purely finite measure on R.

The interpretation of (1) is straightforward. The first part of Ψ is an integral operator with a countably additive kernel such as $\lambda^{-2s} \in L_1$ emphasizing the weight of frequent events in the ranking of a lottery $x \in L_\infty$. This satisfies the second axiom Axiom 2, but not the first. The second purely finitely additive part assigns positive weight to rare events which have small probability according to μ, it satisfies Axiom 1 but not the second axiom. An example of a function of this nature is Long Run Averages when the catastrophic event is in the long run future. When both parts are present, Ψ is sensitive to both rare and frequent events, it satisfies Axioms 1 and 2, both axioms are satisfied. Catastrophic risks are therefore ranked more realistically by such functionals.

When there are no catastrophic events, the second axiom is void. Therefore the second component of Ψ "collapses" to its first component, and we have

Theorem 3:

In the absence of catastrophic events, the choice function Ψ agrees with the Expected Utility criterion for evaluating lotteries. Therefore, absent catastrophes, the new theory coincides with the old.

Proof**:**

In "The Topology of Fear", (Chichilnisky, 2009)

FINITE STATES AND BLACK SWANS

Some system are in one of a finite number of states, so the state space is $S = \{s_1, ..., s_S\}$. In this case one can develop a similar theory. One of the states could be identified as a "Black Swan", a rare event with momentuous consequences. Lotteries are now vectors in R^S representing the frequencies of the events. In such systems we define *insensitivity to rare and frequent events* as follows. A ranking of lotteries $W : R^S \to R$ is called *insensitive to rare events* when $\forall f, g \in X$ $\exists \varepsilon > 0 : W(f) > W(g) \Leftrightarrow W(f') > W(g')$ when $f' = f$ and $g' = g$ except on events of measure smaller than ε. Similarly we say that $W : R^S \to R$ is insensitive to frequent events when $\forall f, g \in X$ $\exists N > 0 : W(f) > W(g) \Leftrightarrow W(f') > W(g')$ when $f' = f$ and $g' = g$ except on events of measure larger than N. The axioms for finitely many states (state space R or Z below are the new axioms:

Axiom 1. The ranking $W : R^S \to R$ is sensitive to rare events.

Axioms 2. The ranking W is sensitive to frequent events

Axiom 3: The ranking W is continuous and linear

Theorem 4: Expected utility is insensitive to rare events with finitely many states.

Proof**:** For every f, g let $\varepsilon(f,g) \in R_+$ satisfy $\varepsilon < Min_{1,2,...S}\{f_1,..., f_S, g_1,..., g_S\}$.

Then $\forall f, g \in R^S$ $\varepsilon = \varepsilon(f,g) > 0$ and it satisfies: $W(f) > W(g) \Leftrightarrow W(f') > W(g')$ when $f' = f$ and $g' = g$ except on events of measure smaller than ε because there are no events of measure less than ε.

An alternative definition of insensitivity to rare events can be provided. For any two vectors $p, f \in R^K$ let $\langle p, f\rangle$ denote the inner product, $\langle p, f\rangle = \sum_{s=1}^{K} p_s.f_s$. If f, g are two lotteries in R^K, and $W(.)$ is an expected utility, then we say that W is insensitive to rare events if $\forall f, g$ $\exists \varepsilon > 0$:

$$W(f) > W(g) \Leftrightarrow W(f') > W(g')$$

$\forall f'$ and g' satisfying

$$(f_1 - f_1') < \varepsilon \text{ and } (g_1 - g_1') < \varepsilon$$

Under this second definition, expected utility is still insensitive to rare events. This can be seen as follows:

$$W(f) > W(g) \Rightarrow \exists x \in R_+^S \ : \ W(f) = \langle x, f\rangle > W(g) = \langle x, g\rangle \ \Leftrightarrow \tag{2}$$

$$\Leftrightarrow \ \sum_{i=1}^{S} x_i(f_i - g_i) > 0 \ \Leftrightarrow \exists \delta > 0 \ : \ \sum_{i=1}^{S} x_i(f_i - g_i) > \delta. \tag{3}$$

Now assume without loss of generality that $x_1 = \min\,(x_1,..., x_S)$, where state $s = 1$ represents a `black swan', and $(f_1 - g_1) > 0$, then (2) and (3) are equivalent to

$$\sum_{i>1}^{S} x_i(f_i - g_i) + x_1(f_1 - g_1) - \delta/2 > \delta/2$$

$$\text{so that } \forall f' \text{ and } g' \text{satisfying} \,|\, x_1[(f_1' - g_1') - (f_1 - g_1)] \,|< \delta/2$$

or, equivalently, $\forall f'$ and g' satisfying

$$(f_1 - f_1') < \delta/2x_1 \text{ and } (g_1 - g_1') < \delta/2x_1$$

$$W(f) > W(g) \Rightarrow W(f') > W(g').$$

The reciprocal is immediate.

Therefore we have shown that $\forall f, g$ $\exists \varepsilon > 0$:

$$W(f) \; > W(g) \Leftrightarrow W(f \quad)' \; > W(g')$$

$\forall f'$ and g' satisfying

$$(f_1 - f_1^{'}) < \delta / 2x_1 \text{ and } (g_1 - g_1^{'}) < \delta / 2x_1$$

where $\delta / 2x_1 = \varepsilon$ and δ satisfies (2) and (3).

Theorem 5: With finitely many states, the choice function $W(f) = Min_{s=1,\dots,K}(f_s)$ is insensitive to frequent events, and therefore violates axiom 2. It also fails to be linear, so it violates axiom 3.
Proof: The insensitivity to frequent events is immediate from the definition. Linearity is clearly violated, for example,

$$Min(0,1) + Min(1,0) = 0 + 0 = 0 \neq Min((0,1) + (1,0)) = 1.$$

Theorem 6: With finitely many states, the function $W(f) = \langle p, f \rangle + Min_s(f_s)$ satisfies axioms 1 and 2, but does not satisfy axiom 3.
Proof: The proof that W satisfies axioms 1 and 2 is identical to the case of infinitely many states. Theorem 5 showed that this function violates axiom 3. ?

Generally speaking the standard averaging function that provide the expected value of a lottery is insensitive to rare events, while 'positional' choice functions, such as the $\min, \max,$ or median, are insensitive to frequent events. Combining the two gives a choice function that satisfies both axioms 1 and 2, but linearity is lost. Intuitively, optimizing the function $W(f) = \langle p, f \rangle + Min_s(f_s)$ corresponds to maximizing the expected value of the lottery f according to a probability distribution denoted $p \in R^K$, plus minimizing the worse that can happen in case of a catastrophe.

GERARD DEBREU AND ADAM SMITH'S INVISIBLE HAND

Decisions involving catastrophic events, and the topology of fear that they induce, have deep mathematical roots. We show below that the decision rule involves a new type of statistics, based on a combination of regular measures (countably additive) as well as singular measures (finitely additive). In the most general cases, the construction of a singular measure is equivalent to the construction of Hahn Banach's separating hyperplanes and depends on the Axiom of Choice (Godel, 1954; Chichilnisky, 2009). Thus extreme responses to risk conjure up the `Axiom of Choice' an axiom that K. Godel has proved is independent from the rest of Mathematics (Godel, 1954). This finding is

the so called "Godel's Incompleteness Theorem," and demonstrates a profound ambiguity that is part and parcel of Logics and Mathematics, no matter how rigorous they may be. This is a critical finding in the philosophy of science and has many important practical implications.
There are also interesting implications for the foundations of statistics, or how we measure reality. The new types of probability distributions that emerge with catastrophic events are both regular and singular, exhibiting "heavy tails," and were never used before. However the "sup norm" topology of fear that is required to introduce sensitivity to rare events, is not new in economics. This topology was already used in 1953 by Gerard Debreu to prove Adam Smith's Invisible Hand Theorem (Debreu, 1954). He used the function space L_∞ with the sup norm - namely what the author has called "the topology of fear" (Chichilnisky, 2009) - to describe commodities, and to prove Adam Smith's Invisible Hand Theorem. This is the famous second welfare theorem of economics, which establishes that any optimal social distribution can emerge from a market equilibrium. The market reaches any optimal social solution, including the optimal allocation of risk bearing. Using Hahn Banach's theorem, Debreu found a separating hyperplane that represents the market prices and transforms any Pareto efficient allocation into a competitive equilibrium. This is what we now call Adam Smith's Invisible Hand.

The critical point in all this is that Debreu's theorem is correct as stated -- but is not constructible since, as Godel has shown, the construction of the market prices depends on Hahn Banach's theorem and this in turn requires the Axiom of Choice that is independent from the rest of Mathematics (Chichilnisky, 1977; 2009). The logical and mathematical tools are rigorous and deep, but the full scope and practical implications for decision theory of Debreu's 1953 results have not been understood before. His 1953 welfare theorems are correct - markets can support any socially optimal solutions - *but the market prices at stake may not be constructible*. It may not be possible to design an algorithm to compute market prices that correspond to Adam's Smith's Invisible Hand.

Similarly, we can decide on catastrophic events with criteria that treat rare and frequent events symmetrically, as in the new axioms required here. But the criteria may not always be constructible: in some cases the decisions may be ambiguous and we cannot `construct' an algorithm that decides on the solution. The constructability of the second (finitely additive) term of Ψ is equivalent to the constructability of Hahn - Banach theorem, as used by G. Debreu in proving the second welfare theorem. An interesting historical observation is that Gerard Debreu published his original 1953 (Debreu, 1954) results in the Proceedings of the US National Academy of Sciences -- yet his result contains the seed for expanding Von Neumann's theory of choice under uncertainty in a completely different direction that was meant by Von Neumann, as we saw above. In an interesting historical twist, Debreu's NAS publication shows in its header that the article was introduced to the National Academy by John Von Neumann himself.

ARROW'S MONOTONE CONTINUITY AND THE VALUE OF LIFE

The axiom that leads to the second term of Ψ - namely the author's Axiom 1 of sensitivity to rare events - is equivalent to the Axiom of Choice in Mathematics and to the existence of Ultrafilters - and is therefore independent of the rest of the axioms in Mathematics, see Chichilnisky (2009). An example that comes to mind is in computing the "value of life" - and recent experimental results by Chanel and Chichilnisky (2007) show the deep ambiguity involving choice when death is a possible outcome. This point comes out clearly in the "Monotone Continuity Axiom" in Arrow (1963), who makes this point by means of the following sharp example and comments (see, Arrow, 1963): "If one prefers 10 cents to 1 cent, and 1 cent to death, then one should prefer 10 cents and a small probability of death, to 1 cent". This example comes directly from the definition of Monotone Continuity Definition in Arrow [arrow], who calls it, "at first sight, outreageous." Arrow also points out that the choice may be different if instead of 10 cents one would consider a large sum of money such as US$1 billion. This can be seen as an ambiguity in the choice that was denied in the classical theory but appears in the new axiomatic approach to choice under uncertainty presented above. We must learn to live with ambiguity.

THE LIMITS OF ECONOMETRICS: NON PARAMETRIC ESTIMATION IN HILBERT SPACES

At the frontiers of econometrics we find non-parametric estimation - namely econometric estimation in which we make no assumptions about functional forms. Therefore we work within a general function space - in this case a Hilbert space, which is the closest there is to euclidean space in continuous time and was first introduced in economics in 1977 (Chichilnisky, 1977). The limit of this NP estimation technology is reached when one tries to remove not just the constraint on functional form but also any *a priori* determination of a bound for the sample data, using as sample space the entire real line R. In such cases, a new result identifies necessary and sufficient conditions for NP estimation, and curiously enough this relates to the same issues that underlies choices that are sensitive to rare events:

Theorem: (Chichilnisky, 2009) Axiom 1 requiring Sensitivity to Rare Events is the logical negation of

Assumption SP_4 in De Groot (1970), which compares the relative likelihood of bounded and unbounded events. Furthermore, Assumption SP_4 is both Necessary and Sufficient for ***NP*** estimation in Hilbert Spaces on a unbounded sample space R (Chichilnisky, 2009).

EXAMPLES OF THE NEW CRITERIA OF CHOICE

Example 1. Financial choices. In finance a typical example involves maximizing expected returns while minimizing the drop in a portfolio's value in case of a severe market downturn.

Example 2. Network choices In terms of network optimization a well known criterion is to maximize expected electricity throughput in the grid, while minimizing the probability of a "black out."

Both examples seem reasonable, and they agree with our three new Axioms for choice under uncertainty. However they contradict expected utility, the Monotone Continuity Axiom of Arrow (1963),

the corresponding continuity axiom in Hernstein Milnor (1953) and axiom SP_4 introduced in De Groot (1970).

CONCLUSIONS

This article summarizes the classic theory of choice under uncertainty introduced by Von Neumann and its insensitivity to rare events, even to those with major consequences such as environmental catastrophes. The cause is a continuity axiom that is essential in this theory, which averages risks and defines rationality in terms of *averaging responses* even when one is confronted with extreme risks. Through this bias against rare events our classic tools of analysis underestimate the importance of environmental catastrophes, an issue of concern for green economics. This helps explain the gap between standard economic thinking and green economics. New tools are required to properly account for environmental issues and more generally for situations involving rare events with major consequences.

The article presented new axioms of choice under uncertainty introduced in Chichilnisky (1996a); (2000); (2002) that are consistent with Von Neumann's expected utility in the absence of rare events - but require *equal treatment* for frequent and rare events and as such are quite different from expected utility. The new axioms lead to criteria of choice that seem quite natural, yet inconsistent with expected utility analysis when rare events are at stake. For example, when choosing according to the new criteria of choice one optimizes expected utility moderated by a constraint on the worst outcomes in case of a catastrophe. The article showed how the new axioms rely on statistical foundations based on a notion of continuity that the author has called elsewhere "the topology of fear" (chichilnisky, 2009) as it reproduces a type of behavior (extremal responses to extremal events) that neuro-economics has observed in situations involving fear (Le Doux, 1996). The approach leads to new types of statistical distributions that are consistent with the persistent observation of "heavy tails" and "jump diffusion" processes, both of which are inconsistent with normal distributions, and offers a systematic approach to including `outliers' into the formal discourse. Finally we showed the connection between the new criteria of choice, the frontiers of econometrics in Non Parametric Estimation in Hilbert Spaces, Godel's work on the incompleteness of Mathematics and the classic Axioms of Choice.

NOTES

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[1] The classic theory of choice under uncertainty that paralells the treatment of uncertainty in physics, was created over half a century by Von Neumann and Morgentern (1944), see also Hernstein and Milnor (1953), Arrow (1963), and is based on the statistical foundations of Villegas (1964) and De Groot (1970).

[2] The two topologies are the same in finite dimensional space - whether using extremals or averages - but they are radically different in continuous time models or with infinite horizons, which are needed to explain "rare events"

[3] $d(x,y) = (\sum_i x_i - y_i)^{1/2}$

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