The weighted Shapley support levels values

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Abstract

This paper presents a new class of weighted values for level structures. The new values, called weighted Shapley support levels values, extend the weighted Shapley values to level structures and contain the Shapley levels value (Winter, 1989) as a special case. Since a level structure with only two levels coincides with a coalition structure we obtain, as a side effect, also new axiomatizations of weighted coalition structure values, presented in Levy and McLean (1989).

Keywords Cooperative game · Level structure · (Weighted) Shapley (levels) value · Weighted proportionality · Harsanyi set · Dividends

1 Introduction

Many organizations, companies, governments and so on are organized in hierarchical structures. Typically we have one entity at the apex and in the following levels each entity is splitted up in two or more subordinates which normally have a lower rank as the superior one. A similar organizational structure, in some respects, show supply chains. Effectiveness can be increased by sharing or pooling of physical objects, resources and information. Queueing problems or electricity and other networks have a related background. A central characteristic of all such organizational forms is that a cooperating unit can act as a single player to obtain cooperation benefits for the members of the unit. The question is how realized benefits should be shared and arising costs should be allocated.

To distribute profits of cooperating coalitions the use of a cooperative game seems to be a natural approach. Winter (1989) formulated a model for cooperative games with hierarchical structure, called level structure, which consists of a sequence of coalition structures (the levels). In each level the player set is partitioned into components. Winter's value (Winter, 1989) for such a model, we call it Shapley levels value, extends the Owen value (Owen, 1977), itselves an extension of the Shapley value (Shapley, 1953b). So this value satisfies extensions of the symmetry axioms which are satisfied by the Owen value.

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To treat symmetric players differently if there exist exogenous given weights for the players, Shapley (1953a) introduced the weighted Shapley values. Vidal-Puga (2012) introduced a value for coalition structures with weights given by the size of the coalitions. Gómez-Rúa and Vidal-Puga (2011) extended it to level structures with a step by step top-down proceeding. This value does not satisfy the dummy axiom. At our knowledge, the value in Gómez-Rúa and Vidal-Puga (2011) is still the only published weighted value for level structures.

A different approach took Levy and McLean (1989) and McLean (1991). They extended the weighted Shapley values in general to coalition structures. Therefore they assigned weights to the components of the coalition structure if they are regarded as players, too. Dragan (1992) called one class of these values McLean weighted coalition structure values and presented for them a formula related to that of the Owen value.

In this paper we introduce a new class[1] of weighted values for level structures which are extensions of these McLean weighted coalition structure values and thus also of the weighted Shapley values. Each value of this class can be represented by a formula with dividends (Harsanyi, 1959). The coefficients in the formulas form a dividend share system, meaning that all coefficients are non-negative and sum up to 1 for each coalition. Thus the values from this class coincide with payoff vectors from the Harsanyi set (Hammer, 1977; Vasil'ev, 1978) and inherit so all properties (adapted to level structures) of these payoff vectors. We present two axiomatizations for the values of this class which can be considered as weighted counterparts to the axiomatizations for the Shapley levels value by Winter (1989) and Khmelnitskaya and Yanovskaya (2007).

An unanimity game related to a coalition T gives a short impression how each value in our class works: each player of coalition T is supported by the weights of the largest component containing her to divide the dividend of T, within this component all players of T are supported by the weights of the next largest components containing her to divide the share of the dividend previously assigned and so on. In difference, the Shapley levels value distributes the dividends T equally among the largest components B which have a non-empty intersection with T. Then all next largest components within of such a component B which have a non-empty intersection with T divide the share of B equally and so on.

The outline of the paper is structured as follows. Section 2 contains some preliminaries, section 3 presents the axioms and section 4 gives a quick look on the Shapley levels value. As the main part we introduce in section 5 the weighted Shapley support levels values with appropriate axiomatizations. Section 6 gives a short conclusion. An appendix (section 7) provides all the proofs and some related lemmas.

2 Preliminaries

We denote by $\mathbb{R}$ the real numbers and by $\mathbb{R}_{++}$ the set of all positive real numbers. Let $\mathfrak{U}$ be a countably infinite set, the universe of all players, and denote by $\mathcal{N}$ the set of all non-empty and finite subsets of $\mathfrak{U}$. A cooperative game with transferable utility (**TU-game**) is a pair (N, v) consisting of a set of players $N \in \mathcal{N}$ and a **coalition function**

[1]This class is a special case of values for level structures proposed in Besner (2016) as f-weighted-ILS-Shapley-values. Also exists a working paper (Besner, 2017) where this class is discussed.

$v: 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, where 2^N is the power set of N. The subsets $S \subseteq N$ are called **coalitions**, $v(S)$ is the **worth** of coalition S and the set of all nonempty subsets of S is denoted by Ω^S. The set of all TU-games with player set N is denoted by $\mathbb{V}^N$.

Let $(N, v) \in \mathbb{V}^N$ and $S \subseteq N$. The **dividends** $\Delta_v(S)$ (Harsanyi, 1959) are defined inductively by

$$\Delta_v(S) := \begin{cases} v(S) - \sum_{R \subsetneq S} \Delta_v(R), \text{ if } S \in \Omega^N, \text{ and} \\ 0, \text{ if } S = \emptyset. \end{cases} \tag{1}$$

A game (N, u_T), $T \in \Omega^N$, with $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise for all $S \subseteq N$ is called an **unanimity game**. It is well-known that any coalition function v on N has a unique presentation

$$v = \sum_{T \in \Omega^N} \Delta_v(T) u_T. \tag{2}$$

The **marginal contribution** $MC_i^v(S)$ of player $i \in N$ to $S \subseteq N \backslash \{i\}$ is given by $MC_i^v(S) := v(S \cup \{i\}) - v(S)$. We call a coalition $S \subseteq N$ **active** in v if $\Delta_v(S) \neq 0$. Player $i \in N$ is called a **dummy player** in v if $v(S \cup \{i\}) = v(S) + v(\{i\})$, $S \subseteq N \backslash \{i\}$; if in addition $v(\{i\}) = 0$, then i is called a **null player** in v; players $i, j \in N$, $i \neq j$, are called **symmetric** in v if $v(S \cup \{i\}) = v(S \cup \{j\})$, and (mutually) **dependent** (Nowak and Radzik, 1995) in v if $v(S \cup \{i\}) = v(S) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

A **coalition structure** $\mathcal{B}$ on N is a partition of the player set N, i.e. a collection of nonempty, pairwise disjoint, and mutually exhaustive subsets of N. Each $B \in \mathcal{B}$ is called a **component** and $\mathcal{B}(i)$ denotes the component that contains a player $i \in N$. A **level structure** (Winter, 1989) on N is a finite sequence $\underline{\mathcal{B}} := \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ of coalition structures $\mathcal{B}^r$, $0 \leq r \leq h+1$, on N such that:

- $\mathcal{B}^0 = \big\{\{i\} : i \in N\big\}$.
- $\mathcal{B}^{h+1} = \{N\}$.
- For each r, $0 \leq r \leq h$, $\mathcal{B}^r$ is a refinement of $\mathcal{B}^{r+1}$, i. e. $\mathcal{B}^r(i) \subseteq \mathcal{B}^{r+1}(i)$ for all $i \in N$.

$\mathcal{B}^r$ is called the r-th **level** of $\underline{\mathcal{B}}$; $\overline{\mathcal{B}}$ is the set of all components $B \in \mathcal{B}^r$ of all levels $\mathcal{B}^r \in \underline{\mathcal{B}}$, $0 \leq r \leq h$; $\mathcal{B}^r(B^k)$ is the component of the r-th level which contains the component $B^k \in \mathcal{B}^k$, $0 \leq k \leq r \leq h+1$.

The collection of all level structures with player set N is denoted by $\mathbb{L}^N$. A TU-game $(N, v) \in \mathbb{V}^N$ together with a level structure $\underline{\mathcal{B}} \in \mathbb{L}^N$ is an **LS-game** $(N, v, \underline{\mathcal{B}})$. The set of all LS-games on N is defined by $\mathbb{VL}^N$. Note that each TU-game (N, v) corresponds to an LS-game $(N, v, \underline{\mathcal{B}_0})$ with a **trivial level structure** $\underline{\mathcal{B}_0} := \{\mathcal{B}^0, \mathcal{B}^1\}$ and we would like to say that each LS-game $(N, v, \underline{\mathcal{B}_1})$, $\underline{\mathcal{B}_1} := \{\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2\}$, corresponds to a game with coalition structure (Aumann and Drèze, 1974), also known as "games with a priori unions" (Owen, 1977).

Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$ and $T \in \Omega^N$. A difference to TU-games in LS-games is that also components can operate as players. So we define for each level r, $0 \leq r \leq h$, the level structure $\underline{\mathcal{B}^r} := \big\{\mathcal{B}^{r^0}, ..., \mathcal{B}^{r^{h+1-r}}\big\} \in \mathbb{L}^{\mathcal{B}^r}$ as the induced ***r*-th level structure** from $\underline{\mathcal{B}}$ by considering the components $B \in \mathcal{B}^r$ as players. There all levels from the original level structure lower then r are dropped. In the k-th level $\mathcal{B}^{r^k}$ of $\underline{\mathcal{B}^r}$, $0 \leq k \leq h+1-r$, we have for each component $B^{r+k} \in \mathcal{B}^{r+k}$ of the $(r+k)$-th level in the original level structure $\underline{\mathcal{B}}$ a related component $B^{r^k} \in \mathcal{B}^{r^k}$. This component $B^{r^k} \in \mathcal{B}^{r^k}$ contains the components $B \in \mathcal{B}^r$ as players which are subsets of the original component $B^{r+k} \in \mathcal{B}^{r+k}$ so that we have $\mathcal{B}^{r^k} := \{\{B \in \mathcal{B}^r : B \subseteq B^{r+k}\}$ for all $B^{r+k} \in \mathcal{B}^{r+k}\}$[2].

If a coalition $T = \bigcup_{B \subseteq T} B$, $B \in \mathcal{B}^r$, is the union of components of the r-th level from $\underline{\mathcal{B}}$ and we want to stress this property, T is denoted by T^r. Each such T^r is related to a coalition of all players $B \in \mathcal{B}^r$, $B \subseteq T^r$, in the induced r-th level structure, denoted by $\mathcal{T}^r := \{B \in \mathcal{B}^r : B \subseteq T^r\}$ and vice versa. The induced **r-th level game** $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}) \in \mathbb{VL}^{\mathcal{B}^r}$, where $\mathcal{B}^r$ is the player set with $B \in \mathcal{B}^r$ as players, is given by

$$v^r(\mathcal{T}^r) := v(T^r) \text{ for all } \mathcal{T}^r \in \Omega^{\mathcal{B}^r}.\text{[3]} \tag{3}$$

A **TU-value** ϕ is an operator that assigns to any $(N, v) \in \mathbb{V}^N$ a payoff vector $\phi(N, v) \in \mathbb{R}^N$, an **LS-value** φ is an operator that assigns payoff vectors $\varphi(N, v, \underline{\mathcal{B}}) \in \mathbb{R}^N$ to all LS-games $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$.

We define $\mathrm{W}^N := \{f : N \to \mathbb{R}_{++}\}$ with $\mathrm{w}_i := \mathrm{w}(i)$ for all $\mathrm{w} \in \mathrm{W}^N$ and $i \in N$ as the set of all positive weight systems on the player set N; for all level structures $\underline{\mathcal{B}}$ we define $\mathcal{W}^{\overline{\mathcal{B}}} := \{f : \overline{\mathcal{B}} \to \mathbb{R}_{++}\}$ with $w_B := w(B)$ for all $w \in \mathcal{W}^{\overline{\mathcal{B}}}$ and $B \in \overline{\mathcal{B}}$ as the set of all positive weight systems on the components of all levels r, $0 \leq r \leq h$, of a level structure $\underline{\mathcal{B}}$. For a level structure $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathbb{L}^N$ and an induced r-th level structure $\underline{\mathcal{B}^r}$ related components have the same weights. So we have for all r, k, $0 \leq r \leq k \leq h$, $B^{r^{k-r}} \in \mathcal{B}^{r^{k-r}}$, $\mathcal{B}^{r^{k-r}} \in \underline{\mathcal{B}^r}$, $B^k \in \mathcal{B}^k$, $\mathcal{B}^k \in \underline{\mathcal{B}}$,

$$w_{B^{r^{k-r}}} = w_{B^k} \text{ with } B^{r^{k-r}} := \{B \in \mathcal{B}^r : B \subseteq B^k\} \text{ and } w_{B^{r^{k-r}}} \in \mathcal{W}^{\overline{\mathcal{B}^r}}, \, w_{B^k} \in \mathcal{W}^{\overline{\mathcal{B}}}. \tag{4}$$

Let $(N, v) \in \mathbb{V}^N$ and $\mathrm{w} \in \mathrm{W}$. The (simply) **weighted Shapley value**[4] Sh^{w} (Shapley, 1953a) is defined by

$$Sh_i^{\mathrm{w}}(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{\mathrm{w}_i}{\sum_{j \in S} \mathrm{w}_j} \Delta_v(S) \quad \text{for all } \ i \in N.$$

A special case of a weighted Shapley value, all weights are equal, is the **Shapley value** Sh (Shapley, 1953b), defined by

$$Sh_i(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{\Delta_v(S)}{|S|} \quad \text{for all } \ i \in N.$$

The best-known LS-value is the Shapley levels value[5] (Winter, 1989). We introduce this value here with a formula presented in Calvo, Lasaga and Winter (1996, eq. (1)):

Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, and for all $T \in \Omega^N$, $T \ni i$,

$$K_T(i) := \prod_{r=0}^{h} K_T^r(i), \quad \text{where}$$

$$K_T^r(i) := \frac{1}{|\{B \in \mathcal{B}^r : B \subseteq \mathcal{B}^{r+1}(i),\, B \cap T \neq \emptyset\}|}.$$

[2] Loosely speaking, from the r-th level upwards the components in both level structures are related in such a way that the same original players from the player set N are somehow the underlying part of two related components.

[3] Vaguely formulated, all coalitions of the r-th level game and the related coalitions of the original game, they contain the same players of the original player set N in some manner, have the same worth.

[4] We desist from possibly null weights as in Shapley (1953a) or Kalai and Samet (1987).

[5] The value is also known as level(s) structure value or Winter's (Shapley type) value. Our designation is used e. g. in Álvarez-Mozos et al. (2017).

The **Shapley Levels value** Sh^L is given by

$$Sh_i^L(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N, T \ni i} K_T(i)\Delta_v(T) \text{ for all } i \in N.$$

It is easy to see that Sh^L coincides with Sh if $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$.

All values above are or coincide with payoff vectors from the **Harsanyi set** (Hammer, 1977; Vasil'ev, 1978), also called **selectope** (Derks, Haller and Peters, 2000), where the payoffs are obtained by distributing the dividends. The payoffs ϕ_i^p in this set, titled **Harsanyi payoffs**, are defined by

$$\phi_i^p(N, v) := \sum_{S \subseteq N, S \ni i} p_i^S \Delta_v(S), i \in N,$$

where the p_i^S are non-negative weights in a sharing system $p = (p_i^S)_{S \in \Omega^N, i \in S}$ and sum up to 1 for each coalition S. The collection P^N on N of all such **dividend share systems** p is given by

$$P^N := \left\{ p = (p_i^S)_{S \in \Omega^N, i \in S} \Big| \sum_{i \in S} p_i^S = 1 \text{ and } p_i^S \geq 0 \text{ for each } S \in \Omega^N \text{ and all } i \in S \right\}.$$

3 Axioms

We refer to the following axioms for LS-values which are adaptions of standard-axioms:

Efficiency, E. For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, we have $\sum_{i \in N} \varphi_i(N, v, \underline{\mathcal{B}}) = v(N)$.

Null player, N. For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$ and $i \in N$ a null player in v, we have $\varphi_i(N, v, \underline{\mathcal{B}}) = 0$.

Additivity, A. For all $(N, v, \underline{\mathcal{B}}), (N, v', \underline{\mathcal{B}}) \in \mathbb{VL}^N$, we have
$\varphi(N, v, \underline{\mathcal{B}}) + \varphi(N, v', \underline{\mathcal{B}}) = \varphi(N, v + v', \underline{\mathcal{B}})$.

Marginality, M. For all $(N, v, \underline{\mathcal{B}}), (N, v', \underline{\mathcal{B}}) \in \mathbb{VL}^N$ and $i \in N$ such that $MC_i^v(S) = MC_i^{v'}(S)$ for all $S \subseteq N \backslash \{i\}$, we have $\varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_i(N, v', \underline{\mathcal{B}})$.

Coalitional strategic equivalence, CSE. For all $(N, v, \underline{\mathcal{B}}), (N, v', \underline{\mathcal{B}}) \in \mathbb{VL}^N$ such that for any $T \in \Omega^N$, $c \in \mathbb{R}$ and all $S \subseteq N$,

$$v(S) = \begin{cases} v'(S) + c, & \text{if } S \supseteq T, \\ v'(S), & \text{else,} \end{cases} \tag{5}$$

we have $\varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_i(N, v', \underline{\mathcal{B}})$ for all $i \in N \backslash T$.

It follow axioms which are typical for level structures. In the first one the sum of the payoffs to all players of a component equals the sum of the payoffs to all players of another component if both components are in the same level r, are subsets of the same component one level higher and both components are symmetric players in the r-th level game.

Symmetry between components, SymBC[6] (Winter, 1989)**.** For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \leq r \leq h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$ and B_k, B_ℓ are

[6]This axiom is called coalitional symmetry in Winter (1989).

symmetric in $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}) \in \mathbb{VL}^{\mathcal{B}^r}$, we have

$$\sum_{i \in B_k} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B_\ell} \varphi_i(N, v, \underline{\mathcal{B}}).$$

A dependent player acts in a game in all coalitions which not contain the other dependent player like a dummy, his marginal contribution to such coalitions is always zero. We introduce a new axiom for level structures (and so also for coalition structures) that uses this property. Here the sum of the payoffs to all players of a component divided by the weight of this component equals the sum of the payoffs to all players of another component divided by the weight of the other component if both components are in the same level r, are subsets of the same component one level higher and both components are dependent players in the r-th level game.

Weighted proportionality between components, WPBC[7]**.** For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $w \in \mathcal{W}^{\overline{\mathcal{B}}}$, $B_k, B_\ell \in \mathcal{B}^r$, $0 \le r \le h$, such that $B_\ell \subseteq \mathcal{B}^{r+1}(B_k)$, and B_k, B_ℓ are dependent in $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}) \in \mathbb{VL}^{\mathcal{B}^r}$, we have

$$\sum_{i \in B_k} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_k}} = \sum_{i \in B_\ell} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_\ell}}.$$

Our last axiom asserts that the sum of the payoff to all players in a component coincides with the payoff to this component if this component is regarded as a player in an induced level game.

Level game property, LG (Winter, 1989)**.** For all $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $B \in \mathcal{B}^r$, $0 \le r \le h + 1$, we have

$$\sum_{i \in B} \varphi_i(N, v, \underline{\mathcal{B}}) = \varphi_B(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}). \tag{6}$$

4 The Shapley levels value

Winter (1989) used the Owen value (Owen, 1977) as a starting point for his LS-value. Therefore Winter upgraded the efficiency, null player, symmetry and additivity axioms to axioms for level structures where symmetry is splitted up in coalitional symmetry and individual symmetry. If a level structure is defined as above, meaning that the singletons are the elements of the lowest level, in Winter (1989, remark 1.6) is pointed out that the individual symmetry can be omitted. In this sense we present Winter's first axiomatization of the Shapley levels value[8].

Theorem 4.1. *(Winter, 1989) Sh^L is the unique LS-value that satisfies **E**, **N**, **SymBC** and **A**.*

It should be noted that there exist some further axiomatizations of the Shapley levels value (see Calvo, Lasaga and Winter 1996, Khmelnitskaya and Yanovskaya 2007 and Casajus 2010).

[7]In Nowak and Radzik (1995) the basic version of this axiom for TU-values is called ω-mutual dependence. We call it **weighted proportionality.**

[8]Winter (1989) introduced his value axiomatically and used this axiomatization as a definition.

5 Weighted Shapley support levels values

If we allow that not only elements of the player set N but also coalitions can act as players, e. g., as it is feasible by the Shapley set value, the question arises that symmetric or dependent players (acting components) should no longer be treated equally if there exist some convincing weights which are not included in the coalition function. The following value gives one possibility to deal with such a situation, especially if two components of the same level r which are subsets of the same component one level higher are dependent in the r-th level game. Another characterizing part of this value is that each player is "supported" for her share of the related dividends by the weights of all components including her that leads to the naming of this LS-value.

Definition 5.1. *Let* $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $w \in \mathcal{W}^{\overline{\mathcal{B}}}$ *and for all* $T \in \Omega^N$, $T \ni i$,

$$K_{w,T}(i) := \prod_{r=0}^{h} K^r_{w,T}(i), \text{ where} \tag{7}$$

$$K^r_{w,T}(i) := \frac{w_{\mathcal{B}^r(i)}}{\sum_{\substack{B \in \mathcal{B}^r:\, B \subseteq \mathcal{B}^{r+1}(i),\\ B \cap T \neq \emptyset}} w_B}.$$

The ***weighted Shapley support levels value*** Sh^{wSL} *is given by*

$$Sh^{wSL}_i(N, v, \underline{\mathcal{B}}) := \sum_{T \subseteq N,\, T \ni i} K_{w,T}(i) \Delta_v(T) \text{ for all } i \in N. \tag{8}$$

Remark 5.2. *We see that the Shapley levels value is a weighted Shapley support levels value where all components have the same weight.* Sh^{wSL} *coincides with* Sh^{w} *if* $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$ *and, if* $\underline{\mathcal{B}} = \underline{\mathcal{B}_1}$*, the* $K_{w,T}(i)$ *coincide with the "*λ_i^S*" given in Dragan (1992, sec. 2(e)). Therefore, in this case, the* Sh^{wSL} *coincide with the McLean weighted coalition structure values (Dragan, 1992; Levy and McLean, 1989; McLean, 1991).*

The weighted Shapley support levels values match a lot of convincing axioms, especially those which are used in our axiomatizations.

Theorem 5.3. *The weighted Shapley support levels values* Sh^{wSL} *satisfy* ***E****,* ***N****,* ***A****,* ***M/CSE****,* ***LG*** *and* ***WPBC****.*

For the proof, see appendix 7.2.1.

5.1 A characterization similar to Winter

The first axiomatization of the weighted Shapley values in Nowak and Radzik (1995) is based on efficiency, null player, weighted proportionality and linearity. The following theorem 5.4 shows that additivity can substitute linearity also in TU-games (the theorem holds naturally also for a level structure with $\underline{\mathcal{B}} = \underline{\mathcal{B}_0}$). So weighted proportionality replaces symmetry in the classical axiomatization of the Shapley value (Shapley, 1953b) by efficiency, null player , symmetry and additivity. Our axiomatization is based on the same axioms extended to LS-games, replacing symmetry between components in theorem 4.1 by weighted proportionality between components. So we have a "weighted" analogue to theorem 4.1.

Theorem 5.4. *Sh^{wSL} is the unique LS-value that satisfies **E**, **N**, **WPBC** and **A**.*

For the proof[9], see appendix 7.2.2.

5.2 A characterization similar to Khmelnitskaya and Yanovskaya

Khmelnitskaya and Yanovskaya (2007) characterized the Shapley levels value in the sense of Young (1985) by efficiency, symmetry between components[10] and marginality[11]. In Casajus and Huettner (2008) is shown that coalitional strategic equivalence and marginality are equivalent in TU-games. Their proof obviously holds for LS-games too. We obtain a characterisation which is also an extension of theorem 2.3 in Nowak and Radzik (1995).

Theorem 5.5. *Sh^{wSL} is the unique LS-value that satisfies **E**, **WPBC** and **M/CSE**.*

For the proof, see appendix 7.2.3.

6 Conclusion

The rapidly increasing volume of collected data and global networking make it possible and necessary to share benefits between cooperating participants, often structured hierarchical. To distribute generated surpluses the presented new class of LS-values is an alternative to the Shapley levels value, founded on convincing axioms, if there exist exogenous given weights for some coalitions. These values extend the McLean weighted coalition structure values, satisfy the level game property, can be axiomatized by adapted classical axiomatizations of the Shapley levels value (Winter, 1989; Khmelnitskaya and Yanovskaya, 2007) and contain the Shapley levels value as well.

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7 Appendix

The following lemma states that each non-empty coalition S is for each level the subset of only one coalition that is a union of components from this level which have a non-empty intersection with S.

[9] Replacing dependent by symmetric and **WPBC** by **SymBC** and using that players $i, j \in N$ are symmetric in v if $\Delta_v(S \cup \{i\}) = \Delta_v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\})$, we get a new proof of theorem 4.1.

[10] Also here individual symmetry can be dropped if the singletons are the elements of the lowest level.

[11] Youngs original axiom is called strong monotonicity. In Chun (1989) the essential part of this axiom for the proof of the uniqueness is named marginality.

7.1 Additional lemmas and a remark, used in the proofs

Lemma 7.1. *Let* $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $\mathcal{B}^r \in \underline{\mathcal{B}}$, $0 \leq r \leq h$. *Each* $S \in \Omega^N$ *is a subset of exactly one coalition* $T^r \in \Omega^N$, $T^r = \bigcup_{\substack{B^r \subseteq T^r,\, B^r \in \mathcal{B}^r,\\ B^r \cap S \neq \emptyset}} B^r$. *Thus each* $S \in \Omega^N$ *is also uniquely referred to as* S_{T^r}.

Proof. Each coalition $T^r \in \Omega^N$ is a union of components $B \in \mathcal{B}^r$. $\mathcal{B}^r$ is a partition and so each player $i \in S$, $S \in \Omega^N$, is contained in only one component $B \in \mathcal{B}^r$. Thus exists for each coalition $S \in \Omega^N$ exactly one coalition $T^r \in \Omega^N$ which is a union of all components $B \in \mathcal{B}^r$ containing at least one player $i \in S$. □

The next lemma shows that for each coalition $\mathcal{T}$ in an induced level structure the dividend in the induced level structure equals the sum of the dividends from all coalitions S in the original level structure which are subsets of a coalition T related to $\mathcal{T}$ and have the property of the previous lemma in relation to the coalition T.

Lemma 7.2. *Let* $N \in \mathcal{N}$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\} \in \mathbb{L}^N$, $v \in \mathbb{VL}^N$, $\mathcal{B}^r \in \underline{\mathcal{B}}$, $0 \leq r \leq h$, *and* S_{T^r} *the coalitions from lemma 7.1 with related coalitions* T^r. *Then we have in the* r*-th level game* $(\mathcal{B}^r, v^r, \underline{\mathcal{B}}^r)$ *for each* $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$, *related to* $T^r \in \Omega^N$,

$$\Delta_{v^r}(\mathcal{T}^r) = \sum_{S_{T^r} \subseteq T^r} \Delta_v(S_{T^r}). \tag{9}$$

Proof. Let $t = |\{B \in \mathcal{B}^r\colon\ B \subseteq T^r\}|$ the number of components $B \in \mathcal{B}^r$ which are subsets from a coalition $T^r \in \Omega^N$ with $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$. We use induction on the size t, $1 \leq t \leq |\mathcal{B}^r|$.

Initialisation: Let $t = 1$. T^r is a component $B \in \mathcal{B}^r$ and $\mathcal{T}^r$ is a player in v^r. We have

$$\Delta_{v^r}(\mathcal{T}^r) \underset{(1)}{=} v^r(\mathcal{T}^r) \underset{(3)}{=} v(T^r) \underset{(1)}{=} \sum_{S \subseteq T^r} \Delta_v(S) \underset{\substack{Lem.\\ 7.1}}{=} \sum_{S_{T^r} \subseteq T^r} \Delta_v(S_{T^r}).$$

Induction step: Assume that eq. (9) holds for an arbitrary $\hat{t} \geq 1$ (IH). Let now $\hat{\mathcal{T}}^r \in \Omega^{\mathcal{B}^r}$ with correlated $\hat{T}^r \in \Omega^N$, $\hat{t} = |\{B \in \mathcal{B}^r : B \subseteq \hat{T}^r\}|$ and $T^r = \hat{T}^r \cup \hat{B}$, $\hat{B} \in \mathcal{B}^r$, $\hat{B} \nsubseteq \hat{T}^r$. We have $t = \hat{t} + 1$ and it follows

$$\begin{aligned}
\Delta_{v^r}(\mathcal{T}^r) \quad &\underset{(1)}{=} \quad \sum_{\mathcal{Q}^r \subseteq \mathcal{T}^r} \Delta_{v^r}(\mathcal{Q}^r) \underset{\substack{(1)\\(3)}}{=} v(T^r) - \sum_{\mathcal{Q}^r \subsetneq \mathcal{T}^r} \Delta_{v^r}(\mathcal{Q}^r) \\
&\underset{\substack{(1)\\(IH)}}{=} \quad \Delta_v(T^r) + \sum_{S \subsetneq T^r} \Delta_v(S) - \sum_{\substack{Q^r \subsetneq T^r,\\ \mathcal{Q}^r \subseteq \mathcal{B}^r}} \sum_{S_{Q^r} \subseteq Q^r} \Delta_v(S_{Q^r}) \\
&\underset{\substack{Lem.\\ 7.1}}{=} \quad \Delta_v(T^r) + \sum_{S \subsetneq T^r} \Delta_v(S) - \sum_{\substack{S \subsetneq T^r,\\ S \neq S_{T^r}}} \Delta_v(S) \\
&= \quad \Delta_v(T^r) + \sum_{S_{T^r} \subsetneq T^r} \Delta_v(S_{T^r}) = \sum_{S_{T^r} \subseteq T^r} \Delta_v(S_{T^r}).
\end{aligned}$$

□

Lemma 7.3. *Players* $i, j \in N$, $i \neq j$, *are dependent in* $v \in \mathbb{V}^N$, *iff* $\Delta_v(S \cup \{k\}) = 0$, $k \in \{i, j\}$, *for all* $S \subseteq N \backslash \{i, j\}$.

Proof. Let $i, j \in N$, $i \neq j$, and $v \in \mathbb{V}^N$. We show by induction on the size $s := |S|$ of all coalitions $S \subseteq N \backslash \{i, j\}$

$$\Delta_v(S \cup \{k\}) = 0 \quad \Leftrightarrow \quad v(S \cup \{k\}) = v(S) + v(\{k\}). \tag{10}$$

Initialisation: If $S = \emptyset$ and so $s = 0$, statement (10) is satisfied.

Induction step: Assume that equality in (10) and such equivalence hold for all coalitions $\tilde{S} \subseteq N \backslash \{i, j\}$, $|\tilde{S}| \leq s'$, $s' \geq 0$, (IH) and let $s = s' + 1$ and $k \in \{i, j\}$. We get

$$\begin{aligned}
v(S \cup \{k\}) = {} & v(S) + v(\{k\}) \\
\underset{(1)}{\Leftrightarrow} \quad \Delta_v(S \cup \{k\}) + \sum_{R \subsetneq (S \cup \{k\})} \Delta_v(R) = {} & \sum_{R \subseteq S} \Delta_v(R) + \Delta_v(\{k\}) \\
\underset{(IH)}{\Leftrightarrow} \quad \Delta_v(S \cup \{k\}) + \Delta_v(\{k\}) + \sum_{R \subseteq S} \Delta_v(R) = {} & \sum_{R \subseteq S} \Delta_v(R) + \Delta_v(\{k\}) \\
\Leftrightarrow \quad \Delta_v(S \cup \{k\}) = {} & 0.
\end{aligned}$$

□

Remark 7.4. *It is well-known or easy to prove that statement (5) in **CSE** can be replaced equivalently by*

$$\Delta_v(S) = \begin{cases} \Delta_{v'}(T) + c, & \textit{if } S = T, \\ \Delta_{v'}(S), & \textit{otherwise.} \end{cases}$$

7.2 Proofs

Convention 7.5. To avoid cumbersome case distinctions in the proves using **WPBC** if there is only one single player assessed in isolation, she is defined as dependent by herself. Then **WPBC** is trivially satisfied.

7.2.1 Proof of theorem 5.3

Let $(N, v, \underline{\mathcal{B}}), (N, v', \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $w \in \mathcal{W}^{\overline{\mathcal{B}}}$ and $K^r_{w,T}$ the expressions according to def. 5.1.

- **E, N, A, M/CSE**: Let $T \in \Omega^N, j \in T$. It is easy to show, by induction on r, that

$$\sum_{i \in \mathcal{B}^{r+1}(j),\, i \in T} \prod_{\ell=0}^{r} K^r_{w,T}(i) = 1.$$

So $\sum_{i \in T} K_{w,T}(i) = 1$ and, with $K_{w,T}(i) > 0$, $i \in T$, the $K_{w,T}(i)$ form a dividend share system $p \in P^N$ and Sh^{wSL} coincides with a Harsanyi payoff. Therefore Sh^{wSL} satisfies all for level structures simply adapted axioms which are, as related TU-axioms, satisfied by a Harsanyi payoff, in particular **E, N, A** and **M/CSE** are well-known matched axioms.

- **LG**: Let $B^r \in \mathcal{B}^r$, $0 \leq r \leq h+1$. If $r = 0$, eq. (6) trivially is satisfied because the 0-th level game corresponds to the original LS-game, if $r = h+1$, eq. (6) is satisfied by **E**.

Let now $1 \leq r \leq h$. We have for all $S \subseteq N$, $S \cap B^r \neq \emptyset$,

$$\sum_{i \in B^r,\, i \in S} \prod_{\ell=0}^{r-1} K^\ell_{w,S}(i) = 1. \tag{11}$$

In the game $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r})$ we have for all $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$, $B^r \in \mathcal{T}^r$,

$$K_{w,\mathcal{T}^r}(B^r) = \prod_{\ell=r}^{h} K^{\ell-r}_{w,\mathcal{T}^r}(B^r). \tag{12}$$

Let $i \in B^r$, $r \leq \ell \leq h$, and S_{T^r} the coalitions from lemma 7.1 with related coalitions T^r. We have $\mathcal{B}^\ell(i) = \mathcal{B}^\ell(B^r)$. Notice that for each $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$, related to $T^r \in \Omega^N$, if $i \in S_{T^r}$, we have also $B^r \in \mathcal{T}^r$. It follows for all $S_{T^r} \in \Omega^N$, $i \in S_{T^r}$,

$$\begin{aligned} K^\ell_{w,S_{T^r}}(i) &\underset{\substack{Def.\\5.1}}{=} \frac{w_{\mathcal{B}^\ell(i)}}{\sum_{\substack{B\in\mathcal{B}^\ell:B\subseteq\mathcal{B}^{\ell+1}(i),\\ B\cap S_{T^r}\neq\emptyset}} w_B} \underset{\substack{Lem.\\7.1}}{=} \frac{w_{\mathcal{B}^\ell(B^r)}}{\sum_{\substack{B\in\mathcal{B}^\ell:B\subseteq\mathcal{B}^{\ell+1}(B^r),\\ B\cap T^r\neq\emptyset}} w_B} \\ &\underset{(4)}{=} \frac{w_{\mathcal{B}^{r\,\ell-r}(B^r)}}{\sum_{\substack{B\in\mathcal{B}^{r\,\ell-r}:B\subseteq\mathcal{B}^{r\,\ell+1-r}(B^r),\\ B\cap\mathcal{T}^r\neq\emptyset}} w_B} \underset{\substack{Def.\\5.1}}{=} K^{\ell-r}_{w,\mathcal{T}^r}(B^r). \end{aligned} \tag{13}$$

Thus we have for all $S_{T^r} \in \Omega^N$, $B^r \in \mathcal{T}^r$, $\mathcal{T}^r \in \Omega^{\mathcal{B}^r}$,

$$\begin{aligned} \sum_{\substack{i\in B^r,\\ i\in S_{T^r}}} K_{w,S_{T^r}}(i) &\underset{(7)}{=} \sum_{\substack{i\in B^r,\\ i\in S_{T^r}}} \prod_{\ell=0}^{h} K^\ell_{w,S_{T^r}}(i) \underset{(13)}{=} \sum_{\substack{i\in B^r,\\ i\in S_{T^r}}} \prod_{\ell=0}^{r-1} K^\ell_{w,S_{T^r}}(i) \prod_{\ell=r}^{h} K^{\ell-r}_{w,\mathcal{T}^r}(B^r) \\ &\underset{(11)}{=} \prod_{\ell=r}^{h} K^{\ell-r}_{w,\mathcal{T}^r}(B^r) \underset{(12)}{=} K_{w,\mathcal{T}^r}(B^r). \end{aligned} \tag{14}$$

Finally we get

$$\begin{aligned} \sum_{i\in B^r} Sh^{wSL}_i(N,v,\underline{\mathcal{B}}) &\underset{(8)}{=} \sum_{i\in B^r} \sum_{\substack{S\subseteq N,\\ S\ni i}} K_{w,S}(i)\Delta_v(S) \underset{\substack{Lem.\\7.1}}{=} \sum_{i\in B^r} \sum_{\substack{S_{T^r}\subseteq N,\\ S_{T^r}\ni i}} K_{w,S_{T^r}}(i)\Delta_v(S_{T^r}) \\ &= \sum_{S_{T^r}\subseteq N} \sum_{\substack{i\in B^r,\\ i\in S_{T^r}}} K_{w,S_{T^r}}(i)\Delta_v(S_{T^r}) \underset{(14)}{=} \sum_{\substack{S_{T^r}\subseteq N,\\ \mathcal{T}^r\ni B^r}} K_{w,\mathcal{T}^r}(B^r)\Delta_v(S_{T^r}) \\ &\underset{\substack{Lem.\\7.1}}{=} \sum_{\mathcal{T}^r\subseteq\mathcal{B}^r,\,\mathcal{T}^r\ni B^r} K_{w,\mathcal{T}^r}(B^r) \sum_{S_{T^r}\subseteq T^r} \Delta_v(S_{T^r}) \\ &\underset{\substack{Lem.\\7.2}}{=} \sum_{\mathcal{T}^r\subseteq\mathcal{B}^r,\,\mathcal{T}^r\ni B^r} K_{w,\mathcal{T}^r}(B^r)\Delta_{v^r}(\mathcal{T}^r) \underset{\substack{Def.\\5.1}}{=} Sh^{wSL}_{B^r}(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}). \end{aligned}$$

• **WPBC**: Let $k, \ell \in N$, $0 \leq r \leq h$, $\mathcal{B}^r(\ell) \subseteq \mathcal{B}^{r+1}(k)$ and $\mathcal{B}^r(k), \mathcal{B}^r(\ell)$ be dependent in v^r for the LS-game $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}) \in \mathbb{VL}^{\mathcal{B}^r}$. If $r = 0$, then k, ℓ are dependent in v and we get

$$\begin{aligned} \frac{Sh^{wSL}_k(N,v,\underline{\mathcal{B}})}{w_{\{k\}}} &\underset{(8)}{=} \sum_{T\subseteq N,\,T\ni k} \frac{K_{w,T}(k)}{w_{\{k\}}}\Delta_v(T) \underset{\substack{Lem.\\7.3}}{=} \sum_{T\subseteq N,\,\{k,\ell\}\subseteq T} \frac{K_{w,T}(k)}{w_{\{k\}}}\Delta_v(T) \\ &\underset{\substack{Def.\\5.1}}{=} \sum_{T\subseteq N,\,\{k,\ell\}\subseteq T} \frac{K_{w,T}(\ell)}{w_{\{\ell\}}}\Delta_v(T) = \frac{Sh^{wSL}_\ell(N,v,\underline{\mathcal{B}})}{w_{\{\ell\}}}. \end{aligned}$$

Thus we have also in the r-th level game, $0 \leq r \leq h$,

$$\frac{Sh^{wSL}_{\mathcal{B}^r(k)}(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r})}{w_{\mathcal{B}^r(k)}} = \frac{Sh^{wSL}_{\mathcal{B}^r(\ell)}(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r})}{w_{\mathcal{B}^r(\ell)}}$$

and the claim follows by **LG**. □

7.2.2 Proof of theorem 5.4

Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $w \in \mathcal{W}^{\overline{\mathcal{B}}}$, $S \in \Omega^N$ arbitrary and φ an LS-value which satisfies all axioms of theorem 5.4. Due to theorem 5.3, property (2) and **A**, it is sufficient to show that φ is uniquely defined on the game $v_S := \Delta_v(S) \cdot u_S$.

By lemma 7.1 exists for each level r, $0 \leq r \leq h$, exactly one coalition T_S^r, $\mathcal{T}_S^r \subseteq \mathcal{B}^r$, which is the smallest coalition of all R^r, $R^r \supseteq S$, with correlated $\mathcal{R}^r \subseteq \mathcal{B}^r$ and so in each game $(\mathcal{B}^r, v_S^r, \underline{\mathcal{B}^r}) \in \mathbb{VL}^{\mathcal{B}^r}$ we have $\Delta_{v_S^r}(\mathcal{T}_S^r) = \Delta_v(S)$ and $\Delta_{v_S^r}(\mathcal{R}^r) = 0$ for $\mathcal{R}^r \subseteq \mathcal{B}^r$, $\mathcal{R}^r \neq \mathcal{T}_S^r$. Therefore, by lemma 7.3, possibly using conv. 7.5, all components $B \in \mathcal{B}^r$, $B \cap S \neq \emptyset$, are dependent in v_S^r. If $B \in \mathcal{B}^r$, $B \cap S = \emptyset$, we have $\sum_{i \in B} \varphi_i(N, v_S, \underline{\mathcal{B}}) = 0$ by **N**.

We use induction on the size m, $0 \leq m \leq h$, for all levels r, $0 \leq r \leq h$, with $m := h - r$.

Initialisation: Let $m = 0$ and so $r = h$. We get for an arbitrary $i \in S$

$$\sum_{\substack{B \in \mathcal{B}^h,\\ B \cap S \neq \emptyset}} \sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) \underset{\textbf{(WPBC)}}{=} \sum_{\substack{B \in \mathcal{B}^h,\\ B \cap S \neq \emptyset}} \frac{w_B}{w_{\mathcal{B}^h(i)}} \sum_{j \in \mathcal{B}^h(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) \underset{\textbf{(E)}}{=} \Delta_v(S)$$

$$\Leftrightarrow \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) = \left[\prod_{k=h-m}^{h} \frac{w_{\mathcal{B}^k(i)}}{\sum_{\substack{B \in \mathcal{B}^k:\, B \subseteq \mathcal{B}^{k+1}(i),\\ B \cap S \neq \emptyset}} w_B} \right] \Delta_v(S). \tag{15}$$

Induction step: Assume that eq. (15) holds to φ with an arbitrary $m-1$, $0 \leq m-1 \leq h-1$ (IH). It follows for an arbitrary $i \in S$

$$\begin{aligned}
\sum_{\substack{B \in \mathcal{B}^r,\, B \cap S \neq \emptyset,\\ B \subseteq \mathcal{B}^{r+1}(i)}} \sum_{j \in B} \varphi_j(N, v_S, \underline{\mathcal{B}}) &\underset{\textbf{(WPBC)}}{=} \sum_{\substack{B \in \mathcal{B}^r,\, B \cap S \neq \emptyset,\\ B \subseteq \mathcal{B}^{r+1}(i)}} \frac{w_B}{w_{\mathcal{B}^r(i)}} \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) \\
&\underset{(IH)}{=} \left[\prod_{k=h-m+1}^{h} \frac{w_{\mathcal{B}^k(i)}}{\sum_{\substack{B \in \mathcal{B}^k:\, B \subseteq \mathcal{B}^{k+1}(i),\\ B \cap S \neq \emptyset}} w_B} \right] \Delta_v(S) \\
\Leftrightarrow \sum_{j \in \mathcal{B}^r(i)} \varphi_j(N, v_S, \underline{\mathcal{B}}) &= \left[\prod_{k=h-m}^{h} \frac{w_{\mathcal{B}^k(i)}}{\sum_{\substack{B \in \mathcal{B}^k:\, B \subseteq \mathcal{B}^{k+1}(i),\\ B \cap S \neq \emptyset}} w_B} \right] \Delta_v(S).
\end{aligned}$$

So φ is uniquely defined on v_S (take $m = h$ and so $r = 0$). □

7.2.3 Proof of theorem 5.5

Let $(N, v, \underline{\mathcal{B}}) \in \mathbb{VL}^N$, $\underline{\mathcal{B}} = \{\mathcal{B}^0, ..., \mathcal{B}^{h+1}\}$, $w \in \mathcal{W}^{\overline{\mathcal{B}}}$ and φ an LS-value which satisfies all axioms of theorem 5.5. By theorem 5.3 we have only to show that φ satisfies eq. (8).

We use a first induction I_1 on $t := |\{T \subseteq N : T \text{ is active in } v\}|$.

Initialisation I_1: Let $t = 0$, then for all games $(\mathcal{B}^r, v^r, \underline{\mathcal{B}^r}) \in \mathbb{VL}^{\mathcal{B}^r}$, $0 \leq r \leq h$, v^r is identically zero on all coalitions. So all players, possibly using conv. 7.5, are dependent in each game v^r and for all $B_k^r, B_\ell^r \in \mathcal{B}^r$, $B_\ell^r \subseteq \mathcal{B}^{r+1}(B_k^r)$ we have

$$\sum_{i \in B_k^r} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_k^r}} \underset{\textbf{(WPBC)}}{=} \sum_{i \in B_\ell^r} \frac{\varphi_i(N, v, \underline{\mathcal{B}})}{w_{B_\ell^r}}.$$

We use a second induction I_2 on the size $m := h - r$ to show

$$\sum_{i \in B^r} \varphi_i(N, v, \underline{\mathcal{B}}) = 0 \text{ for all } 0 \leq r \leq h \text{ and } B^r \in \mathcal{B}^r. \tag{16}$$

Initialisation I_2: Let $m = 0$ and so $r = h$. We get for an arbitrary $B_k^h \in \mathcal{B}^h$

$$\sum_{B^h \in \mathcal{B}^h} \sum_{i \in B^h} \varphi_i(N, v, \underline{\mathcal{B}}) \underset{(\mathbf{WPBC})}{=} \sum_{B^h \in \mathcal{B}^h} \frac{w_{B^h}}{w_{B_k^h}} \sum_{i \in B_k^h} \varphi_i(N, v, \underline{\mathcal{B}}) \underset{(\mathbf{E})}{=} 0.$$

Thus follows $\sum_{i \in B^h} \varphi_i(N, v, \underline{\mathcal{B}}) = 0$ for all $B^h \in \mathcal{B}^h$ because $w_{B^h} > 0$ and B_k^h was arbitrary.

Induction step I_2: Assume that eq. (16) holds to φ if $m \geq 0$ (IH_2). We get for an arbitrary $B_k^r \in \mathcal{B}^r$

$$\sum_{\substack{B^r \in \mathcal{B}^r, \\ B^r \subseteq \mathcal{B}^{r+1}(B_k^r)}} \sum_{i \in B^r} \varphi_i(N, v, \underline{\mathcal{B}}) \underset{(\mathbf{WPBC})}{=} \sum_{\substack{B^r \in \mathcal{B}^r, \\ B^r \subseteq \mathcal{B}^{r+1}(B_k^r)}} \frac{w_{B^r}}{w_{B_k^r}} \sum_{i \in B_k^r} \varphi_i(N, v, \underline{\mathcal{B}}) \underset{(IH_2)}{=} 0.$$

It follows $\sum_{i \in B^r} \varphi_i(N, v, \underline{\mathcal{B}}) = 0$ for all $0 \leq r \leq h$ and $B^r \in \mathcal{B}^r$. Therefore we have also $\varphi_i(N, v, \underline{\mathcal{B}}) = 0$ for all $i \in N$ and eq. (8) is satisfied for φ if $t = 0$.

Induction step I_1: Assume that eq. (8) holds to φ if $t \geq 0$, (IH_1). Let exactly $t + 1$ coalitions $Q_k \subseteq N$, $1 \leq k \leq t + 1$, active in v and denote

$$Q := \bigcap_{1 \leq k \leq t+1} Q_k.$$

We distinguish two cases: (a) $i \in N \backslash Q$ and (b) $i \in Q$.

(a) Each player $i \in N \backslash Q$ is a member of at most t active coalitions Q_k and v gets at least one active coalition T_i, $i \notin T_i$. Hence exists a coalition function $v_i \in \mathbb{VL}^N$, where all coalitions have the same dividend in v_i as in v, except the coalition T_i, that gets the dividend $\Delta_{v_i}(T_i) = 0$, and there is existing a scalar $c \in \mathbb{R}$, $c \neq 0$, with

$$\Delta_v(S) = \begin{cases} \Delta_{v_i}(T_i) + c, & \text{if } S = T_i, \\ \Delta_{v_i}(S), & \text{else.} \end{cases}$$

By remark 7.4 and **CSE** we get $\varphi_i(v) = \varphi_i(v_i)$ with $i \in N \backslash T_i$ and, because there exists for all $i \in N \backslash Q$ a such T_i, it follows $\varphi_i(v) = \varphi_i(v_i)$ for all $i \in N \backslash Q$. All coalition functions v_i get at most t active coalitions and by (IH_1) we have

$$\varphi_i(v) = Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) \text{ for all } i \in N \backslash Q. \tag{17}$$

(b) Each player $j \in Q$ is a member of all $t + 1$ active coalitions $Q_k \subseteq N$, $1 \leq k \leq t + 1$, and therefore, by lemma 7.3 and conv. 7.5, all players $j \in Q$ are dependent in v. Now we define for each r, $0 \leq r \leq h$, a set

$$\mathcal{B}_Q^r := \{B^r \in \mathcal{B}^r : B^r \cap Q \neq \emptyset\}.$$

Note that all components $B_k^r, B_\ell^r \in \mathcal{B}_Q^r$, $B_\ell^r \subseteq \mathcal{B}^{r+1}(B_k^r)$, are dependent in v^r. We use a third induction I_3 on the size $s := h - r$ to show for all $B_k^r \in \mathcal{B}_Q^r$

$$\sum_{i \in B_k^r} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B_k^r} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}). \tag{18}$$

Initialisation I_3: Let $s = 0$ and so $r = h$. We get for an arbitrary $B_k^h \in \mathcal{B}_Q^h$

$$\begin{aligned}
& \sum_{B^h \in \mathcal{B}_Q^h} \sum_{i \in B^h} \varphi_i(N, v, \underline{\mathcal{B}}) \underset{\textbf{(WPBC)}}{=} \sum_{B^h \in \mathcal{B}_Q^h} \frac{w_{B^h}}{w_{B_k^h}} \sum_{i \in B_k^h} \varphi_i(N, v, \underline{\mathcal{B}}) \\
\underset{\substack{\textbf{(E)} \\ (17)}}{=} & \sum_{B^h \in \mathcal{B}_Q^h} \sum_{i \in B^h} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) \underset{\textbf{(WPBC)}}{=} \sum_{B^h \in \mathcal{B}_Q^h} \frac{w_{B^h}}{w_{B_k^h}} \sum_{i \in B_k^h} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) \\
\Leftrightarrow & \sum_{i \in B_k^h} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B_k^h} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}).
\end{aligned}$$

Induction step I_3: Assume that eq. (18) holds to φ if $s \geq 0$ (IH_3). We get for an arbitrary $B_k^r \in \mathcal{B}_Q^r$ and because $\mathcal{B}^{r+1}(B_k^r) \in \mathcal{B}_Q^{r+1}$

$$\begin{aligned}
& \sum_{\substack{B^r \in \mathcal{B}_Q^r, \\ B^r \subseteq \mathcal{B}^{r+1}(B_k^r)}} \sum_{i \in B^r} \varphi_i(N, v, \underline{\mathcal{B}}) \underset{\textbf{(WPBC)}}{=} \sum_{\substack{B^r \in \mathcal{B}_Q^r, \\ B^r \subseteq \mathcal{B}^{r+1}(B_k^r)}} \frac{w_{B^r}}{w_{B_k^r}} \sum_{i \in B_k^r} \varphi_i(N, v, \underline{\mathcal{B}}) \\
\underset{\substack{(IH_3) \\ (17)}}{=} & \sum_{\substack{B^r \in \mathcal{B}_Q^r, \\ B^r \subseteq \mathcal{B}^{r+1}(B_k^r)}} \sum_{i \in B^r} Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) \underset{\textbf{(WPBC)}}{=} \sum_{\substack{B^r \in \mathcal{B}_Q^r, \\ B^r \subseteq \mathcal{B}^{r+1}(B_k^r)}} \frac{w_{B^r}}{w_{B_k^r}} \sum_{i \in B_k^r} Sh^{wSL}(N, v, \underline{\mathcal{B}}) \\
\Leftrightarrow & \sum_{i \in B_k^r} \varphi_i(N, v, \underline{\mathcal{B}}) = \sum_{i \in B_k^r} Sh_i^{wSL}(N, v, \underline{\mathcal{B}})
\end{aligned}$$

and so finally

$$\varphi_i(N, v, \underline{\mathcal{B}}) = Sh_i^{wSL}(N, v, \underline{\mathcal{B}}) \text{ for all } i \in Q.$$

By Casajus and Huettner (2008) **M** and **CSE** are equivalent in TU-games. Their proof holds obviously also for LS-games and the proof of theorem 5.5 is complete. □

7.3 Logical independence

All axiomatizations must hold if $\underline{\mathcal{B}} = \underline{\mathcal{B}}_0$, too. In this case all axioms, used for axiomatization in this paper, coincide with usual axioms for TU-values. So the given axiomatizations coincide in this case with axiomatizations of the weighted Shapley values. It is well-known or easy to proof that in this case the used axioms are logical independent. Therefore all axioms for LS-values must be also logical independent in the given axiomatizations.

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