Bifurcation theory of a racetrack economy in a spatial economy model

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Abstract Racetrack economy is a conventional spatial platform for economic agglomeration in spatial economy models. Studies of this economy up to now have been conducted mostly on 2^k cities, for which agglomerations proceed via so-called spatial period doubling bifurcation cascade. This paper aims at the elucidation of agglomeration mechanisms of the racetrack economy in a general setting of an arbitrary number of cities. First, an attention was paid to the existence of invariant solutions that retain their spatial distributions when the transport cost parameter is changed. A complete list of possible invariant solutions, which are inherent for replicator dynamics and are dependent on the number of cities, is presented. Next, group-theoretic bifurcation theory is used to describe bifurcation from the uniform state, thereby presenting an insightful information on spatial agglomerations. Among a plethora of theoretically possible invariant solutions, those which actually become stable for spatial economy models are obtained numerically. Asymptotic agglomeration behavior when the number of cities become very large is studied.

Keywords Bifurcation · Economic agglomeration · Racetrack economy · Replicator dynamics · Spatial economy model

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1 Introduction

Although the two-city model is the most customary spatial platform in spatial economics, its insufficiency for the implementation of indirect spatial effects has been corroborated using empirical evidence (Bosker et al. 2010). Elucidating the mechanism of spatial economic agglomeration among a system of cities subject to indirect spatial effects has persisted as an important topic (e.g., Behrens and Thisse 2007). Racetrack economy, which comprises a series of cities with identical microeconomic environments situated on a circle, has come to be used extensively as a spatial platform for a system of cities. An increase in the number of cities to be considered, however, entails a rapid increase of the possible spatial patterns, which is quite problematic. To overcome this problem, it is desirable to develop a theory of spatial agglomeration for a general setting of an arbitrary number of cities. Nevertheless, theoretical studies of the racetrack economy were conducted mainly on 2^k cities because of analytical tractability, as noted in a review in Section 2.

In light of the above, this paper is intended to elucidate the agglomeration mechanism of the racetrack economy of replicator (ad hoc) dynamics in a general setting of *an arbitrary number of cities*. We are particularly interested in the asymptotic agglomeration behavior when the number of cities becomes very large. Development of a systematic methodology to address an arbitrary number of cities is a novel contribution of this paper. This methodology is based on the following two tools: (1) theory on invariant solutions and (2) group-theoretic bifurcation theory.

First, attention is devoted to the existence of invariant solutions that retain their spatial distributions even when the value of the transport cost parameter is changed. Invariant solutions have already been observed implicitly in the two-city model of replicator dynamics for spatial economy models. There are two invariant solutions: (i) the state of two identical cities and (ii) the state of a core city gathering all mobile population and a peripheral city with no population. These states satisfy the governing equation for any value of transport cost and become stable and sustainable equilibria for some microeconomic environments. Knowledge related to these solutions, which is inherent for replicator dynamics, has provided insightful information related to spatial agglomerations. As a novel contribution of this paper, a complete list of possible invariant solutions for a racetrack economy with an arbitrary number of cities is advanced as an extension of the study for 2^k cities (Ikeda et al. 2017b).

Next, we specifically address bifurcating agglomeration patterns from the uniform state. For this purpose, we resort to group-theoretic bifurcation theory, which is an established tool to describe bifurcation of symmetric systems (e.g., Ikeda and Murota 2010). Since the symmetry of racetrack economy is dependent on the number of cities, as are bifurcating agglomeration patterns. Although the bifurcation mechanism for 2^k cities is available in the literature (Ikeda et al. 2012), we newly consider prime-numbered cities and composite-numbered cities. The prime-numbered cities have been proved to encounter spontaneous loss of spatial periodicity upon bifurcation. The composite-numbered cities are shown to display diverse bifurcations leading to spatial period doubling, tripling, and so on, dependent on the way the number is factorized. The mechanism of bifurcation obtained herein is general in that it is independent on micro-economic modeling and on the dynamics employed.

Using the methodology presented above, several theoretically possible patterns are now readily accessible. One can undertake the advanced mission of finding patterns of interest that actually become stable equilibria. Because their stability is dependent on microeconomic modeling, stability must be investigated model by model. We conducted comparative static analysis to obtain stable equilibria of spatial economy models by Forslid and Ottaviano (2003) and Pflüger (2004). They are compared with known agglomeration properties in the Krugman model [e.g., Fujita et al. (1999) for the two-city model and Ikeda et al. (2012) for 4, 6, 8, and 16 cities]. As a result, it can be shown that knowledge related to invariant patterns provides insightful information related to spatial agglomeration of spatial economy models.

This paper is organized as follows. Reports of studies of the racetrack economy are reviewed in Section 2. Spatial economy models are introduced in Section 3. A theory of invariant solutions is developed in Section 4. Bifurcation mechanisms of racetrack economy are advanced in Section 5. Agglomeration behaviors of two, three, and four cities are investigated in Section 6. Numerical analysis is conducted in Section 7.

2 Studies of racetrack economy

For racetrack economy, Krugman (1993) found that fewer larger agglomerations out of uniformity are engendered. For racetrack economy with 2^k cities, a spatially alternation of a core place with a large population and a peripheral place with a small population were observed for spatial economy models (e.g., Picard and Tabuchi 2010; Tabuchi and Thisse 2011). Its mechanism was explained in terms of the spatial period doubling bifurcation cascade, which produces fewer larger agglomerations through repeated doubling of the spatial period of agglomerated cities (Ikeda et al. 2012; Akamatsu et al. 2012; Osawa et al. 2017). Anas (2004) demonstrated the presence of other agglomeration patterns, such as balanced agglomeration, concentrated agglomeration, and de-agglomeration.

The importance of the theoretical study of racetrack economy in the elucidation of agglomeration mechanism in other economies has come to be acknowledged. The racetrack economy was studied comparatively with an economy on a line segment (a long narrow economy) by Mossay and Picard (2011) for Beckman's CBD formation model (1976) in a continuous space to display the difference in agglomeration patterns. An analogy of the agglomerations in the racetrack economy to long narrow economy and square lattice economy was studied in Ikeda et al. (2017a,b). This paper offers more general results by extending this study with 2^k cities to an arbitrary number of cities.

3 Spatial economic models

As an application of the present theory, we employ a pair of multi-regional spatial economic models whose frameworks follow Forslid and Ottaviano (2003) and Pflüger

(2004) (defined as FO and Pf, respectively). The basic assumptions of the multi-regional spatial economic model are the same as those of the FO and Pf models except for the number of regions, but we provide them here for completeness.

3.1 Basic Assumptions

The economy is composed of n regions indexed by $i = 1, ..., n$, two factors of production and two sectors. The two factors of production are skilled and unskilled labor while the workers supply one unit of each type of labor inelastically. The total endowment of skilled and unskilled workers is H and L, respectively. The skilled worker is mobile across regions and λ_i denotes the number of them located in region i. The total endowment of skilled workers is normalized as $H = 1$. The unskilled worker is immobile and equally distributed across all regions (i.e., the number of unskilled workers in each region is $l \equiv L/n$). The two sectors consist of agriculture (abbreviated by A) and manufacturing (abbreviated by M). The A-sector output is homogeneous and each unit is produced using a unit of unskilled labor under perfect competition. The M-sector output is a horizontally differentiated product that is produced using both skilled and unskilled labor under increasing returns to scale and Dixit-Stiglitz monopolistic competition.

The goods of both sectors are transported, but the transportation of A-sector goods is frictionless while the transportation of M-sector goods is inhibited by iceberg transportation costs. That is, for each unit of M-sector goods transported from region i to $j(\neq i)$, only a fraction $1/T_{ij} < 1$ arrives. We assume that $T_{ii} = 1$ for all $i \in \{1, \cdots, n\}$ and that $T_{ij} = T_{ij}(\tau)$ is a function in a transport cost parameter $\tau > 0$ as

$$T_{ij} = \exp(\tau\, m(i, j)), \tag{1}$$

where $m(i, j)$ expresses the shortest distance between regions i and j.

All workers have identical preferences U over both M- and A-sector goods. The utility of an individual in region i is given by

$$\text{[FO model]} \quad U(C_i^{\mathrm{M}}, C_i^{\mathrm{A}}) = \mu \ln C_i^{\mathrm{M}} + (1-\mu) \ln C_i^{\mathrm{A}} \qquad (0 < \mu < 1), \tag{2}$$

$$\text{[Pf model]} \quad U(C_i^{\mathrm{M}}, C_i^{\mathrm{A}}) = \mu \ln C_i^{\mathrm{M}} + C_i^{\mathrm{A}} \qquad (\mu > 0), \tag{3}$$

$$C_i^{\mathrm{M}} \equiv \sum_j \left(\int_0^{n_j} q_{ji}(k)^{(\sigma-1)/\sigma} dk \right)^{\sigma/(\sigma-1)}, \tag{4}$$

where C_i^{A} is the consumption of A-sector goods in region i; C_i^{M} represents the manufacturing aggregate in region i; $q_{ji}(k)$ is the consumption of variety $k \in [0, n_j]$ produced in region j and n_j is the number of varieties produced in region j; μ is a constant parameter and σ is the constant elasticity of substitution between any two varieties. The budget constraint is given by

$$p_i^{\mathrm{A}} C_i^{\mathrm{A}} + \sum_j \int_0^{n_j} p_{ji}(k) q_{ji}(k) dk = y_i, \tag{5}$$

where p_i^{A} is the price in region i of A-sector goods, $p_{ji}(k)$ denotes the price in region i of the M-sector goods produced in region j, and y_i denotes the income of an individual in region i. The incomes (wages) of the skilled and the unskilled workers are represented, respectively, by w_i and w_i^{u}.

3.2 Expression of indirect utility function

With resort to utility maximization, profit maximization, and short-run equilibrium, the indirect utility of each of the multi-region FO and Pf models is expressed as (see Ikeda et al. 2014 for details)

$$\text{[FO model]} \quad v_i = \frac{\mu}{\sigma - 1} \ln[\textstyle\sum_j d_{ji}\lambda_j] + \ln[w_i], \tag{6a}$$

$$\text{[Pf model]} \quad v_i = \frac{1}{\sigma - 1} \ln[\textstyle\sum_j d_{ji}\lambda_j] + \frac{1}{\sigma} w_i. \tag{6b}$$

Here $d_{ji} \equiv T_{ji}^{1-\sigma}$ is the trade friction between the regions i and j and w_i is the equilibrium wage.

3.3 Long-run equilibrium and adjustment dynamics

In the long run, the skilled workers are mobile across regions and will move to the region where their indirect utility is higher. The long-run equilibrium is defined as the spatial distribution of the mobile workers $\boldsymbol{\lambda}$ that satisfies the following conditions:

$$\begin{cases} v^* - v_i(\boldsymbol{\lambda}) = 0 & \text{if} \quad \lambda_i > 0, \\ v^* - v_i(\boldsymbol{\lambda}) \geq 0 & \text{if} \quad \lambda_i = 0, \end{cases} \tag{7a}$$

$$\sum_i \lambda_i = 1, \tag{7b}$$

where v^* denotes the equilibrium utility level. The condition (7a) means that a long-run equilibrium arises when no worker may get a higher utility level by moving to another region.

As guaranteed in Sandholm (2010), it is possible to replace the problem to obtain a set of stable spatial equilibria by another problem to find a set of stable stationary points of the replicator dynamics:

$$\frac{d\boldsymbol{\lambda}}{dt} = \boldsymbol{F}(\boldsymbol{\lambda}, \tau), \tag{8}$$

where $\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = (F_i(\boldsymbol{\lambda}, \tau) \mid i = 1, \dots, n)$, and

$$F_i(\boldsymbol{\lambda}, \tau) = (v_i(\boldsymbol{\lambda}, \tau) - \bar{v}(\boldsymbol{\lambda}, \tau))\lambda_i. \tag{9}$$

Here, $\bar{v} = \sum_{i=1}^n \lambda_i v_i$ is the average utility. Stationary points (rest points) $\boldsymbol{\lambda}^*(\tau)$ of the replicator dynamics (8) are defined as those points which satisfy the static governing equation

$$\boldsymbol{F}(\boldsymbol{\lambda}^*, \tau) = \boldsymbol{0}. \tag{10}$$

Using the eigenvalues of the Jacobian matrix

$$J(\boldsymbol{\lambda}^*, \tau) = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\lambda}}(\boldsymbol{\lambda}^*, \tau),$$

we classify stability as

$$\begin{cases} \text{linearly stable:} & \text{every eigenvalue has a negative real part,} \\ \text{linearly unstable:} & \text{at least one eigenvalue has a positive real part.} \end{cases}$$

A stationary point is asymptotically stable or unstable according to whether it is linearly stable or unstable.

4 Theory of invariant solutions

A bifurcation theory on the replicator dynamics is introduced. By virtue of its product form (9), this dynamics has a number of invariant solutions that retain their spatial patterns when the transport cost τ changes. After introducing classifications of stationary points, we formulate a symmetry condition for the existence of invariant solutions.

4.1 Classifications of stationary points

Stationary points $(\boldsymbol{\lambda}, \tau)$ of the replicator dynamics are classified in preparation for the description of its bifurcation mechanism. First, these points are classified into an *interior solution*, for which all cities have positive population, and a *corner solution*, for which some cities have zero population.

A solution can be expressed, without loss of generality, by appropriately rearranging the order of independent variables $\boldsymbol{\lambda}$ as

$$\hat{\boldsymbol{\lambda}} = \begin{bmatrix} \boldsymbol{\lambda}_+ \\ \boldsymbol{\lambda}_0 \end{bmatrix} \tag{11}$$

with $\boldsymbol{\lambda}_+ = \{\lambda_i > 0 \mid i = 1, \ldots, m\}$ and $\boldsymbol{\lambda}_0 = \boldsymbol{0}$. Note that $\boldsymbol{\lambda}_0$ is absent for an interior solution. Stability and sustainability conditions for corner solutions are given in Appendix A.1.

Next, critical points are classified into a *break bifurcation point* with singular J_+ and a *non-break point* with $v_i - \bar{v} = 0$ for some place i ($i = m + 1, \ldots, n$); a sustain point is a special kind of non-break point. There is another type of critical point with singular J_+, called a limit point of τ (Ikeda et al. 2012); however, this type of point is exceptional in the present study. A bifurcating solution with reduced symmetry branches at a break point (Section 5), whereas population of some cities vanishes at a non-break (sustain) point.

Last, stationary points are classified into an *invariant solution* $(\boldsymbol{\lambda}, \tau)$ for which $\boldsymbol{\lambda}$ remains constant for any $\tau \in (0, \infty)$ and a *non-invariant solution* $(\boldsymbol{\lambda}, \tau)$ for which $\boldsymbol{\lambda}$ changes with τ. Existence of invariant solutions of various kinds is a special feature of the replicator dynamics. Invariant solutions in racetrack economy were studied in Castro et al. (2012) and Ikeda et al. (2012).

Proposition 1 *The flat earth equilibrium (dispersion)* $\boldsymbol{\lambda} = \frac{1}{n}(1, \ldots, 1)^\top$ *is an invariant solution.*

Proof Because we have $v_1 = \cdots = v_n = \bar{v}$ for the flat earth, the condition (7a) for a spatial equilibrium is satisfied for any τ.

4.2 Symmetry of racetrack economy

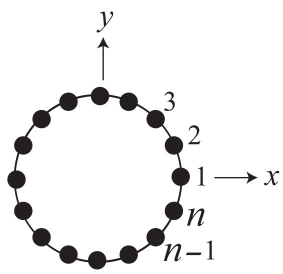


Fig. 1: Racetrack economy with n cities

We consider a racetrack economy with n cities that are equally spread around the circumference of a circle (Fig. 1). The symmetry of these cities can be described by the dihedral group $G = \mathrm{D}_n$ of degree n expressing regular n-gonal symmetry. This group is defined as

$$\mathrm{D}_n = \{e, r, \ldots, r^{n-1}, s, sr, \ldots, sr^{n-1}\}, \tag{12}$$

where $\{\cdot\}$ denotes a group consisting of the geometrical transformations in the parentheses, e is the identity transformation, s is the reflection with respect to the x-axis, and r^j is a counterclockwise rotation about the center of the circle at an angle of $2\pi j/n$ $(j = 0, 1, \ldots, n-1)$.

Bifurcated solutions from the D_n-symmetric racetrack economy have partial symmetries that are labeled by subgroups of D_n. These subgroups are dihedral and cyclic groups that are given respectively as

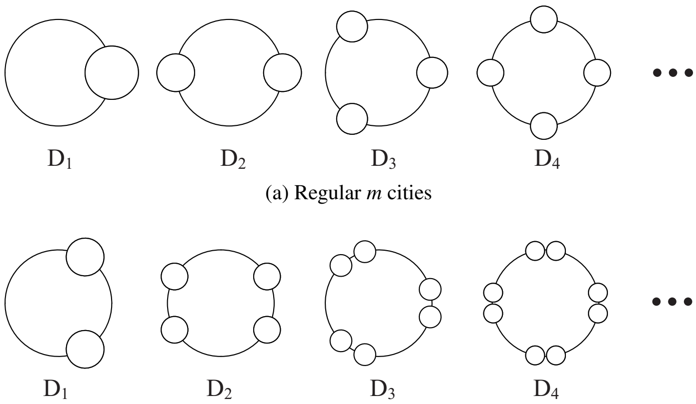
$$\mathrm{D}_m = \{r^{in/m}, sr^{in/m} \mid i = 0, 1, \ldots, m-1\},$$
$$\mathrm{C}_m = \{r^{in/m} \mid i = 0, 1, \ldots, m-1\}.$$

Therein, the subscript m $(= 1, \ldots, n/2)$ is an integer that divides n and C_m denotes cyclic symmetry at an angle of $2\pi/m$. Since m is a divisor of n, the prime factorization of n is influential on the bifurcation.

Remark 1 There is a series of m-axis symmetric patterns labeled by

$$\mathrm{D}_m^{\ell,n} = \{r^{in/m}, sr^{\ell-1+in/m} \mid i = 0, 1, \ldots, m-1\}.$$

Here the superscript ℓ $(= 1, \ldots, n/m)$ expresses the directions of the reflection axes. To simplify the discussion, these patterns are presented by D_m by redefining the reflection s as $sr^{\ell-1}$ in the remainder of this paper. □



(a) Regular m cities

(b) Skewed $2m$ cities

Fig. 2: Invariant solutions with D_m-invariance

4.3 Invariant solutions

A corner solution with m identical agglomerated cities, i.e.,

$$\hat{\boldsymbol{\lambda}} = \begin{bmatrix} \boldsymbol{\lambda}_+ \\ \boldsymbol{\lambda}_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{m}\mathbf{1} \\ \mathbf{0} \end{bmatrix} \tag{13}$$

is paid special attention in this paper. This is a core–periphery pattern with a two-level hierarchy: Population is agglomerated to m core cities with identical population, while other peripheral cities have no population.

Lemma 1 *A corner solution* $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \tau) = (\frac{1}{m}\mathbf{1}, \mathbf{0}, \tau)$ *that satisfies the following assumption is an invariant solution.*

The corner solution with m identical agglomerated cities in (13) *is invariant to a subgroup G of* D_n *and there is a set of representation matrices $T_+(g)$* $(g \in G)$ *that permutes any two entries of $\boldsymbol{\lambda}_+$ (see Appendix A.2 for the concrete form of $T_+(g)$).*

Proof See Appendix A.3.1.

Among an ensemble of agglomerations to m cities, the following two types with D_m-invariance play an important role in the present discussion: (i) complete agglomeration to m cities with a shape of regular m-gon for some m divides n (Fig. 2(a)) and (ii) that to a pair of cities cyclically repeated m times (Fig. 2(b)). We hereafter call the former the *regular m cities* and the latter the *skewed 2m cities*. For example, two cities of these types are called *regular twin* cities and *skewed twin cities*.

The regular m cities have the inter-agglomerated-city roads of the same length, whereas the skewed $2m$ cities have those of two different lengths. Such difference is influential on stability (Section 7).

Proposition 2 *Invariant solutions are either the regular m cities or the skewed* $2m$ *cities.*

Table 1: Existence and absence of invariant solutions for several values of n (◯ indicates the existence and × denotes the absence)

Number n of cities	Regular m cities						
	$m = 12$	$m = 6$	$m = 5$	$m = 4$	$m = 3$	$m = 2$	$m = 1$
2	×	×	×	×	×	◯	◯
3	×	×	×	×	◯	×	◯
4	×	×	×	◯	×	◯	◯
5	×	×	◯	×	×	×	◯
6	×	◯	×	×	◯	◯	◯
12	◯	◯	×	◯	◯	◯	◯

Number n of cities	Skewed $2m$ cities						
	$m = 12$	$m = 6$	$m = 5$	$m = 4$	$m = 3$	$m = 2$	$m = 1$
2	×	×	×	×	×	×	×
3	×	×	×	×	×	×	◯
4	×	×	×	×	×	×	◯
5	×	×	×	×	×	×	◯
6	×	×	×	×	×	◯	◯
12	×	×	×	◯	◯	◯	◯

Proof See Appendix A.3.2.

Proposition 3 *(i) Regular m cities are an invariant solution when n is divisible by m* ($1 \leq m \leq n$). *(ii) Skewed* $2m$ *cities are an invariant solution when n is divisible by m* ($1 \leq m \leq n/3$).

Proof See Appendix A.3.3.

By virtue of Proposition 3, the invariant solutions of any number of cities can be obtained exclusively. Table 1 lists a variety of invariant solutions for specific values of the number n of cities. For example, invariant solutions for $n = 12$ can be exhausted as shown in Fig. 3. We have the following proposition.

Proposition 4 *(i) An atomic mono-center, twin cities, and the flat earth are invariant solutions that exist for any number of cities. (ii) For prime numbered cities with* $n \geq 3$*, an atomic mono-center, skewed twin cities, and the flat earth are the only invariant solutions.*

Proof See Appendix A.3.4.

5 Bifurcation mechanism: symmetry breaking

Bifurcation mechanism of racetrack economy is described for an arbitrary number of cities, whereas the previous studies on bifurcation mechanism of this economy focused on $n = 2^k$ cities as reviewed in Section 2. This mechanism is endowed with independence on micro-economic modeling and on the dynamics employed.

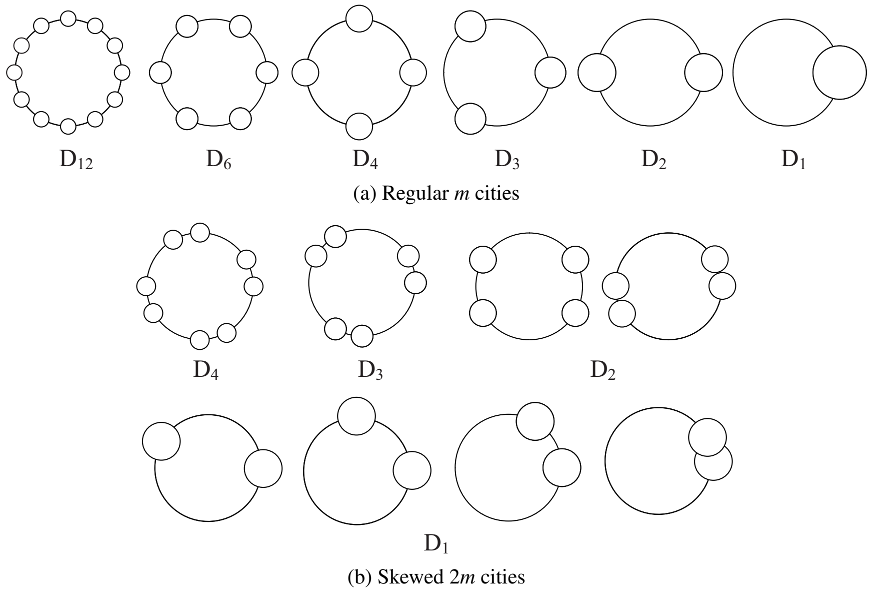


Fig. 3: Invariant solutions for $n = 12$

5.1 Symmetry breaking bifurcation from the flat earth equilibrium

The Jacobian matrix $\boldsymbol{J}= \partial \boldsymbol{F}/\partial \boldsymbol{\lambda}$ on the flat earth was found to take a special form due to its D_n-invariance (Ikeda et al. 2012). To be concrete, the eigenvectors of this matrix are given by discrete Fourier series as

$$\begin{cases} \boldsymbol{\eta}^{(+)} = \frac{1}{\sqrt{n}}(1,\dots,1)^\top, \quad \boldsymbol{\eta}^{(-)} = \frac{1}{\sqrt{n}}(\cos\pi\cdot 0,\dots,\cos\{\pi(n-1)\})^\top, \\ \boldsymbol{\eta}^{(j),1} = \sqrt{\frac{2}{n}}(\cos(2\pi j\cdot 0/n),\dots,\cos\{2\pi j(n-1)/n\})^\top, \\ \boldsymbol{\eta}^{(j),2} = \sqrt{\frac{2}{n}}(\sin(2\pi j\cdot 0/n),\dots,\sin\{2\pi j(n-1)/n\})^\top, \quad j=1,\dots,j_{\max}. \end{cases} \tag{14}$$

Here $j_{\max} = \lfloor (n-1)/2 \rfloor$ ($n \geq 3$) and $\lfloor x \rfloor$ is the maximum integer not exceeding x.

The associated eigenvalues are given as

$$\begin{cases} g^{(+)},\ g^{(-)} & \text{for } n=2, \\ g^{(+)},\ g^{(-)},\ g^{(1)},\ g^{(1)},\ \dots, g^{(j_{\max})},\ g^{(j_{\max})} & \text{for } n=2q+2, \\ g^{(+)},\ g^{(1)},\ g^{(1)},\ \dots,\ g^{(j_{\max})},\ g^{(j_{\max})} & \text{for } n=2q+1, \end{cases} \tag{15}$$

where q is a natural number. $g^{(+)}$ and $g^{(-)}$ are simple eigenvalues that are repeated once, whereas $g^{(1)},\dots,g^{(j_{\max})}$ are double eigenvalues repeated twice.

A bifurcation point emerges when some of these eigenvalues vanish and is classified as

$$\begin{cases} \text{simple bifurcation point:} & \text{for } g^{(-)} = 0 \ \ (n \text{ is even}), \\ \text{double bifurcation point:} & \text{for } g^{(j)} = 0. \end{cases}$$

Here a limit point of τ, which is associated with $g^{(+)} = 0$, does not take place on the flat earth equilibrium. In comparison with the previous study on racetrack economy (Ikeda et al. 2012), it is a novel contribution of this paper to deal with the double bifurcation point.

At a simple bifurcation point with $g^{(-)} = 0$, which exists for n even, we encounter a reduction of symmetry expressed by

$$\mathrm{D}_n \to \mathrm{D}_{n/2}, \tag{16}$$

which leads to a doubling of spatial period: $T_n = 2\pi/n \to T_{n/2} = 4\pi/n$, where $\to$ denotes an occurrence of bifurcation. This bifurcation is symmetric in the sense that $\boldsymbol{\eta}^{(-)}$ and $-\boldsymbol{\eta}^{(-)}$ are identical up to geometrical transformations ($T(g)\boldsymbol{\eta}^{(-)} = -\boldsymbol{\eta}^{(-)}$ holds for some g).

At a double bifurcation point with $g^{(j)} = 0$, which exists for $n \geq 3$, we encounter a reduction of symmetry expressed by

$$\mathrm{D}_n \to \mathrm{D}_{n/\widehat{n}}.$$

Here

$$\widehat{n} = n/\gcd(n, j) \quad (3 \leq \widehat{n} \leq n) \tag{17}$$

is an important index for the double point and $\gcd(n, j)$ is the greatest common divisor of j and n. At the onset of this bifurcation, the spatial period is extended $\widehat{n}$ times as $T_n = 2\pi/n \to 2\widehat{n}\pi/n$.

Lemma 2 *A double bifurcation is symmetric when $\widehat{n}$ is even and is asymmetric when $\widehat{n}$ is odd.*

Proof See Chapter 8 of Ikeda and Murota (2010).

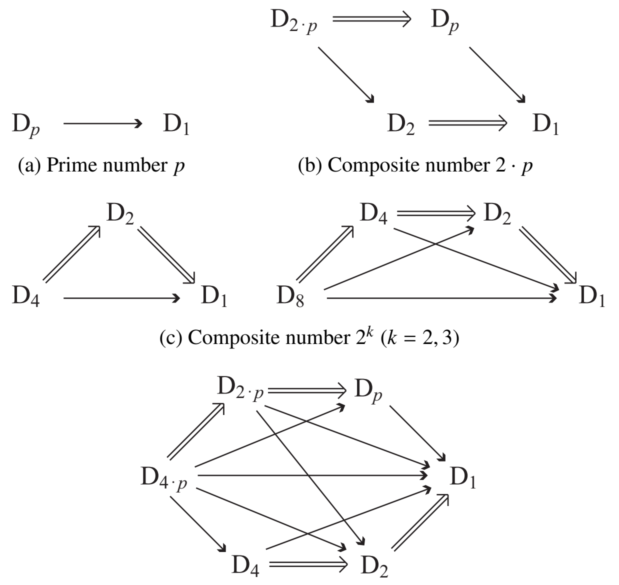
The bifurcation is dependent on the number n of cities. To elucidate this dependence, we consider the numbers of two kinds: (1) prime number and (2) composite number. The composite number is further classified into the type of $n = 2^k$ and others.

Proposition 5 *When n is a prime number, there are only double bifurcations that are asymmetric and lead to the reduction of symmetry* $\mathrm{D}_n \to \mathrm{D}_1$.

Proof See Appendix A.3.5.

Proposition 6 *When n is a composite number and is divisible by m ($1 \leq m \leq n/2$), there are bifurcations that lead to the reduction of symmetry $\mathrm{D}_n \to \mathrm{D}_m$ and extend the spatial period n/m times.*

Proof See Appendix A.3.6.



(a) Prime number p (b) Composite number $2 \cdot p$

(c) Composite number 2^k $(k = 2, 3)$

(d) Composite number $4 \cdot p$

Fig. 4: Hierarchical bifurcation structure of a system invariant to D_n [bold arrow denotes simple bifurcation and thin arrow denotes double bifurcation; bifurcation $D_m \to C_m$ (m divides n) is suppressed in this figure since it is exceptional (see, e.g., Ikeda et al. 2012 for the Krugman model)]

5.2 Recursive bifurcation

Racetrack economy with composite numbered cities can potentially undergo recursive bifurcation (Ikeda and Murota 2010). Its mechanism is dependent the factorization of the city number n, which, in general, takes the form:

$$n = 2^{n_2} 3^{n_3} 5^{n_5} \cdots, \quad n_q \geq 0 \ (q = 2, 3, 5, \ldots). \tag{18}$$

Then in the recursive bifurcation of this economy, we have the following proposition.

Proposition 7 *For the n cities expressed by* (18)*, the existence of the factors* q^{n_q} ($n_q \geq 1$) *indicates that spatial period q times bifurcations can occur* n_q *times at most.*

Proof This is apparent from Proposition 6.

Figure 4 depicts recursive bifurcation, where thick arrows express simple bifurcations and thin arrows denote double bifurcations. Prime numbers p have the simplest hierarchical bifurcation structure $D_p \Longrightarrow D_1$ (Fig. 4(a)). Composite numbers have more complicated ones, such as those for $2 \cdot p$, 2^k, and $4 \cdot p$ depicted in Figs. 4(b)–(d), respectively.

For $n = 2^k$ ($k = 1, 2, \ldots$) cities, simple bifurcations can take place repeatedly (Ikeda et al. 2012; Akamatsu et al. 2012):

$$\mathrm{D}_n \to \mathrm{D}_{n/2} \to \cdots \to \mathrm{D}_2 \to \mathrm{D}_1, \tag{19}$$

and lead to spatial period doubling cascade:

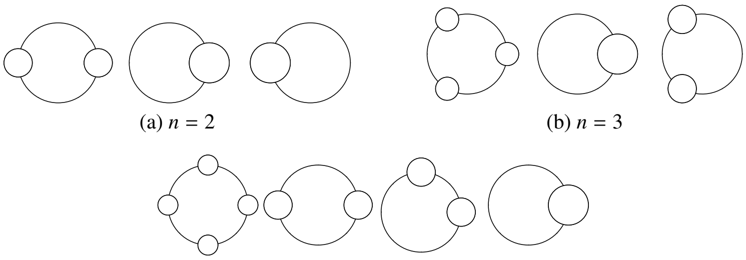
$$T_n = 2\pi/n \to 4\pi/n \to \cdots \to \pi \to 2\pi. \tag{20}$$

6 Agglomeration behavior for a specific number of cities

Based on the theory of racetrack economy presented in Sections 4 and 5, agglomeration behaviors for a specific numbers of cities ($n = 2, 3, 4$) are analyzed in order to demonstrate the usefulness of the theory.

6.1 Two cities

By Proposition 3(i), invariant solutions of two cities are the regular m cities with $m = 1$ and $m = 2$, i.e., the flat earth $\boldsymbol{\lambda} = (1/2, 1/2)^\top$ for $m = 2$ and atomic mono-centers $(1, 0)^\top$ and $(0, 1)^\top$ for $m = 1$ (Fig. 5(a)). The mechanism of progress of agglomeration among these invariant solutions is depicted in Fig. 6(a).



(a) $n = 2$ (b) $n = 3$

(c) $n = 4$

Fig. 5: Invariant solutions for $n = 2$, 3, and 4 cities

On the flat earth (at the left of this figure), a simple break bifurcation with an eigenvector $\boldsymbol{\eta}^{(-)} = \frac{1}{\sqrt{2}}(1, -1)^\top$ entails symmetry reduction of $\mathrm{D}_2 \to \mathrm{D}_1$. This bifurcation is the only possible bifurcation, and is either pitchfork (stable) or tomahawk (unstable). As summarized in Pflüger and Südekum (2008), this bifurcation is pitchfork (stable) for the Pflüger (Pf) model (2004), and is tomahawk (unstable) for the Kurgman model (Fujita et al. 1999) and the Forslid and Ottaviano model (2003).

A bifurcating branch exists in the direction of $\boldsymbol{\eta}^{(-)} = \frac{1}{\sqrt{2}}(1, -1)^\top$ and another bifurcating branch exists in the opposite direction of $-\boldsymbol{\eta}^{(-)} = \frac{1}{\sqrt{2}}(-1, 1)^\top$. The agglomeration pattern on these bifurcating branches has the spatial period doubling of

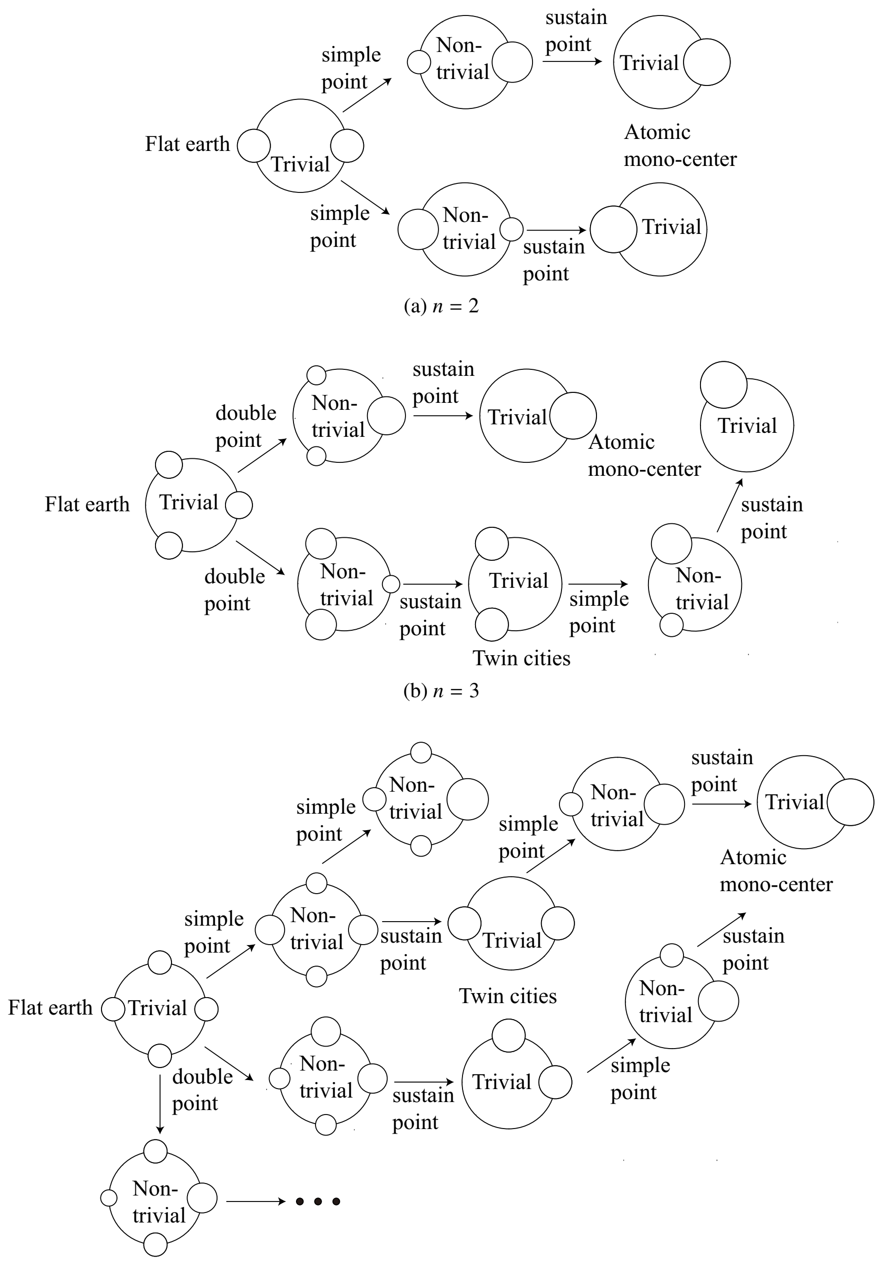


Fig. 6: Agglomeration mechanism for n = 2, 3, and 4 cities

the form

$$\boldsymbol{\lambda} = (1/2 + a, 1/2 - a)^\top \quad (-1/2 \leq a \leq 1/2), \tag{21}$$

which is the one and the only non-invariant solution in the two cities. These branches are connected to an invariant solution $\boldsymbol{\lambda} = (1, 0)^\top$ at a sustain point ($a = 1/2$) and to $\boldsymbol{\lambda} = (0, 1)^\top$ at another sustain point ($a = -1/2$) (at the right of Fig. 6(a)).

A final remark is on the similarity of the geometrical configurations of the two atomic mono-centers. Atomic mono-centers $\boldsymbol{\lambda} = (1, 0)^\top$ and $(0, 1)^\top$ are identical up to the transformation of π rotation: $T(r)(1, 0)^\top = (0, 1)^\top$. We identify agglomeration patterns that are identical up to transformation(s) here and in the remainder of this paper to simplify the discussion.

6.2 Three cities

By Proposition 4(ii), three cities have invariant solutions of three kinds (Fig. 5(b)): the flat earth $\boldsymbol{\lambda} = (1/3, 1/3, 1/3)^\top$, skewed twin cities $(0, 1/2, 1/2)^\top$, and an atomic mono-center $(1, 0, 0)^\top$. The mechanism of progress of agglomeration among these invariant solutions is depicted in Fig. 6(b).

On the flat earth of the three cities, a double bifurcation, which entails symmetry reduction of $\mathrm{D}_3 \to \mathrm{D}_1$, is the only possible bifurcation. This bifurcation is asymmetric and all bifurcating solutions are unstable in the neighborhood of the bifurcation point (Ikeda and Murota 2010).

A bifurcating branch exists in the direction of $\boldsymbol{\eta}^{(1),1} = \frac{1}{\sqrt{6}}(2, -1, -1)^\top$, and another branch exists in the direction of $-\boldsymbol{\eta}^{(1),1} = \frac{1}{\sqrt{6}}(-2, 1, 1)^\top$. These bifurcating branches have a population of

$$\boldsymbol{\lambda} = (1/3 + 2a, 1/3 - a, 1/3 - a)^\top \quad (-1/6 \leq a \leq 1/3)$$

and are connected to an invariant solution $\boldsymbol{\lambda} = (1, 0, 0)^\top$ at a sustain point ($a = 1/3$) and $\boldsymbol{\lambda} = (0, 1, 1)^\top$ at another sustain point ($a = -1/6$).

6.3 Four cities

By Proposition 3, four cities have invariant solutions of four kinds (Fig. 5(c)): the flat earth $\boldsymbol{\lambda} = (1/4, 1/4, 1/4, 1/4)^\top$, regular twin cities $(1/2, 0, 1/2, 0)^\top$, skewed twin cities, $(1/2, 1/2, 0, 0)^\top$, and an atomic mono-center $(1, 0, 0, 0)^\top$. On the flat earth, there is a simple bifurcation and a double bifurcation with $\widehat{n} = 4$ (Fig. 6(c)).

The simple bifurcation engenders a spatial period doubling non-invariant solution of the form

$$\boldsymbol{\lambda} = (1/4 + a, 1/4 - a, 1/4 + a, 1/4 - a)^\top \quad (-1/4 \leq a \leq 1/4), \tag{22}$$

which reaches an invariant solution $(1/2, 0, 1/2, 0)^\top$ at a sustain point ($a = 1/4$). Another simple bifurcation on the non-invariant solution in (22) or on the invariant solution $(1/2, 0, 1/2, 0)^\top$ leads to further spatial period doubling.

7 Stability of agglomeration of spatial economy models

With the aid of the theories in Sections 4 and 5, we obtained spatial agglomeration patterns for various kinds of city numbers. For a spatial economy model by Forslid and Ottaviano (FO) (2003), the stability of these patterns was investigated numerically for various kinds of city numbers (Section 7.1). The agglomeration behavior is shown to be convergent as the number n of cities increases in Section 7.2. The commonality in the stability of invariant solutions of the FO model with that of the Pflüger (Pf) model (2004) is demonstrated in Section 7.3.

We set the elasticity of substitution as $\sigma = 10.0$, the ratio of the manufacturing labor force as $\mu = 0.4$. These parameter values satisfy the so-called no-black-hole condition $\sigma - 1 = 9.0 > \mu = 0.4$ for the FO model (Forslid and Ottaviano 2003). The total population of the skilled worker as $H = 1$.

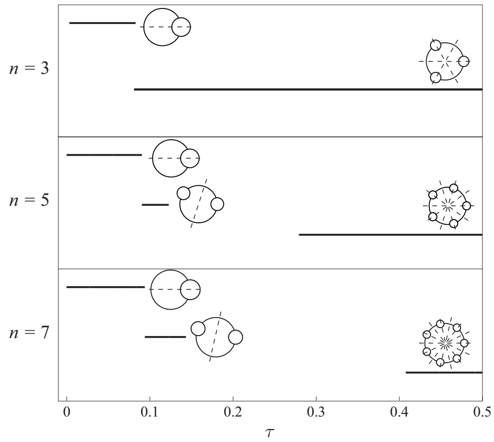
7.1 Agglomeration for various kinds of city numbers

For primed numbered cities of $n = 3$, 5, and 7, we first obtained invariant solutions based on the theory in Section 4 and numerically investigated their stability (Fig. 7(a)); a solution is called *stable* herein if it is stable for some range of τ, and is called *unstable* if it is unstable for any τ. For three cities, stable invariant solutions existed for any τ and there were distinct ranges of τ for the stable flat earth and the atomic mono-center. For the five and seven cities, there were three distinct ranges of τ for which the flat earth, skewed twin cities, and the atomic mono-center were stable. It is to be noted that there were gaps of τ in which stable invariant solutions were absent.

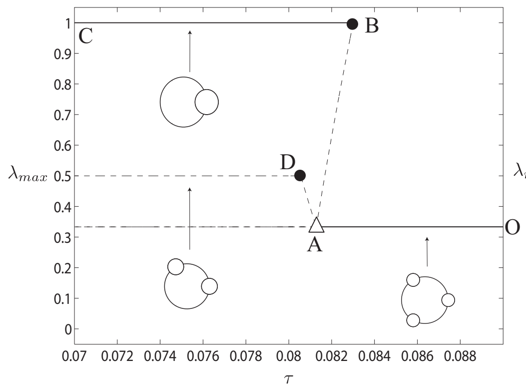
Next, we obtained non-invariant solutions, in addition to invariant ones, in Figs. 7(b)–(d); invariant solutions correspond to the horizontal lines and non-invariant ones to the non-horizontal ones; stable and unstable ones are expressed by the solid and dashed lines, respectively. For the three cities (Fig. 7(b)), there were no stable non-invariant solutions, thereby demonstrating the adequacy of the investigation of stable invariant ones conducted above. For the five cities (Fig. 7(c)), the stable non-invariant solution DE resided on a wide range of τ and played an important role to fill the gap of τ of the stable invariant solutions, unlike the three cities. Such was also the case for the seven cities (Fig. 7(d)).

For composite numbered cities with $n = 2^k$ ($k = 2, 3$), we first numerically obtained stable invariant solutions (Fig. 8(a)): there was a series of stable invariant solutions for the regular one, two, and four cities, which form a chain of *spatial period doubling bifurcation cascade*. On the other hand, skewed $2m$ cities ($m = 1$ and 1) were mostly unstable, except for the ones for $n = 8$ that is stable in a narrow stable range of τ.

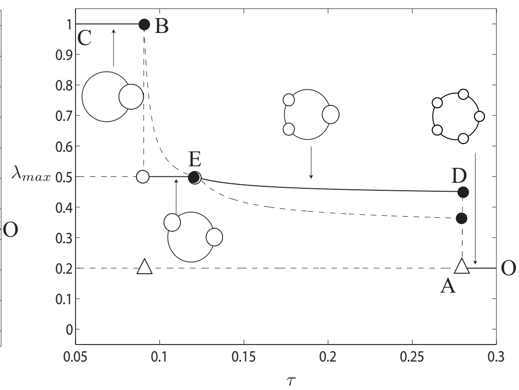
Next, we obtained other solutions (Figs. 8(b) and (c)). There were stable non-invariant solutions with very thin ranges of τ associated with nearly vertical lines, thereby demonstrating again the sufficiency of the investigation of stable invariant ones.



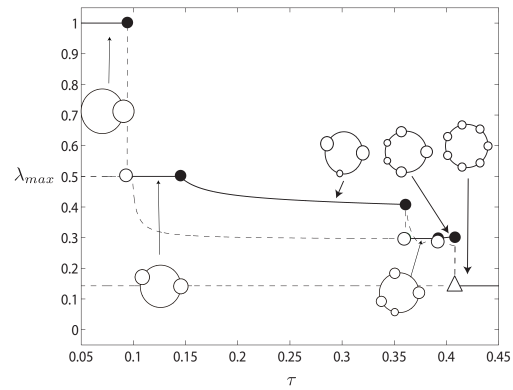
(a) Durations of τ for stable invariant solutions



(b) Three cities

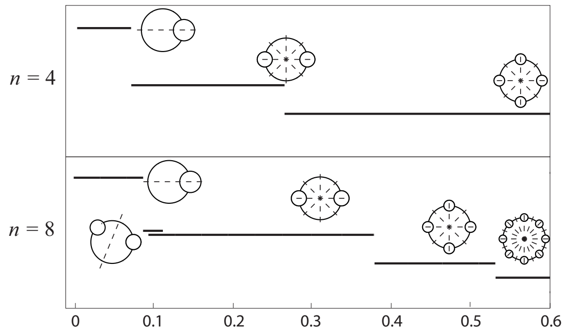


(c) Five cities

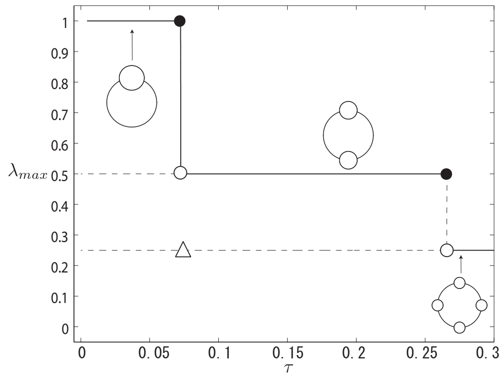


(d) Seven cities

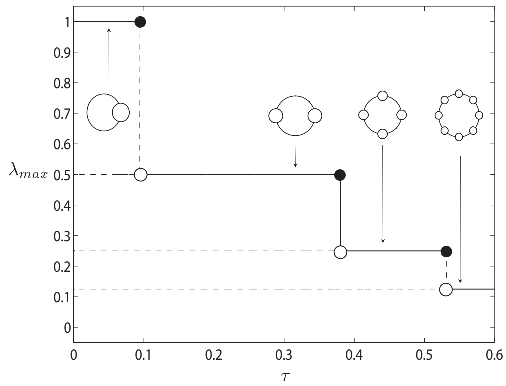
Fig. 7: Durations of τ for stable invariant solutions and progress of agglomeration for three, five, and seven cities (solid curves: stable solutions; dashed curves: unstable solutions; $\lambda_{\max}$: maximum population among the cities; ◯: simple bifurcation; △: double bifurcation; •: sustain point)



(a) Durations of τ for stable invariant solutions

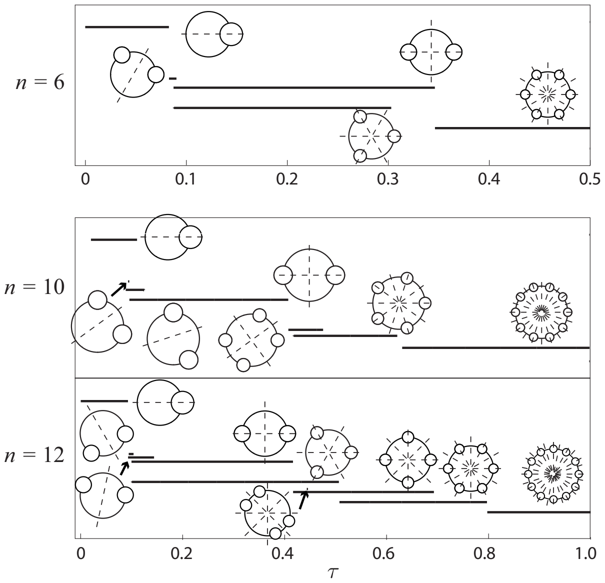


(b) Four cities

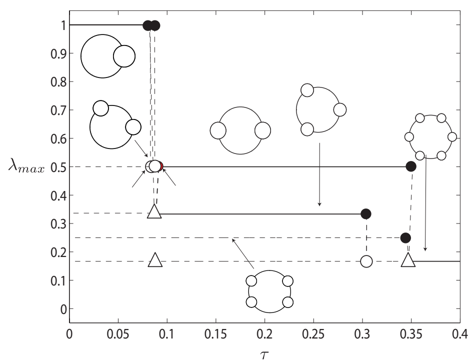


(c) Eight cities

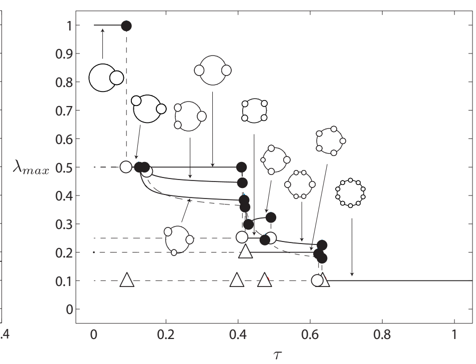
Fig. 8: Durations of τ for stable invariant solutions and progress of agglomeration for four and eight cities (solid curves: stable solutions; dashed curves: unstable solutions; $\lambda_{\max}$: maximum population among the cities; ○: simple bifurcation; △: double bifurcation; •: sustain point)



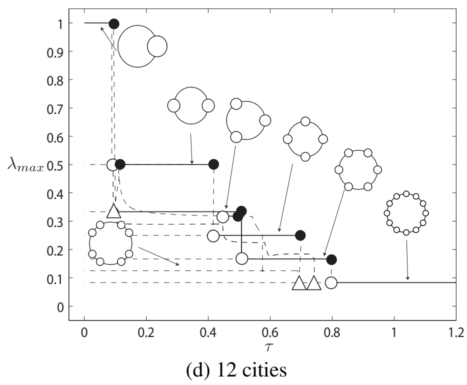
(a) Durations of τ for stable invariant solutions



(b) Six cities



(c) Ten cities



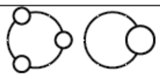
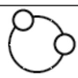
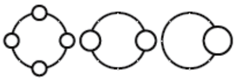
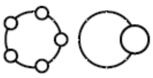
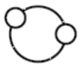
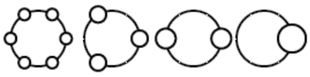
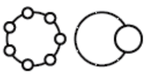
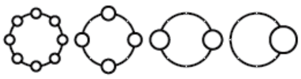
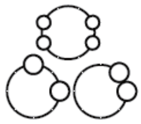
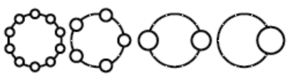
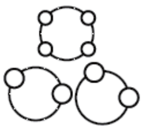
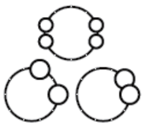
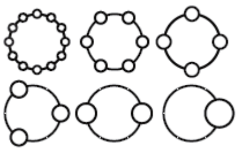
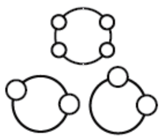
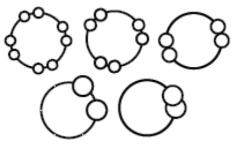
(d) 12 cities

Fig. 9: Durations of τ for stable invariant solutions and progress of agglomeration for six, ten, and 12 cities (solid curves: stable solutions; dashed curves: unstable solutions; $\lambda_{\max}$: maximum population among the cities; ○: simple bifurcation; △: double bifurcation; •: sustain point)

For composite numbered cities with $n = 6$ and 10 ($n = 2 \cdot p$) and $n = 12$ ($n = 4 \cdot p$), stable ranges of τ for invariant solutions are depicted in Fig. 9(a). Unlike other cases, there were multiple stable solutions for the same value of τ. In Figs. 9(b) and (d) for $n = 6$ and 12, non-invariant solutions were not predominant. This may be attributable to the coexistence of spatial period doubling and tripling due to the factors of 2 and 3 of $n = 6$ and 12. On the other hand, in Fig. 9(c) for $n = 10$, there were several stable non-invariant solutions. This is similar to the agglomeration for the five cities (Fig. 7(c)) possibly due to the existence of spatial period five times bifurcation as $n = 10$ has a factor of five.

The stability of these invariant solutions is classified in Table 2. All invariant solutions of regular m cities were stable. On the other hand, not all of invariant solutions of skewed $2m$ cities were stable and were inferior in stability; stable ones had shapes closer to regular twin or four cities. This possibly is attributable to the equi-distantness of the regular m-cities.

Table 2: Stability of invariant solutions

Type	Regular m cities	Skewed $2m$ cities	
Stability	Stable	Stable	Unstable
$n = 3$			
$n = 4$			
$n = 5$			
$n = 6$			
$n = 7$			
$n = 8$			
$n = 10$			
$n = 12$			

7.2 Asymptotic agglomeration behaviors for a large number of cities

Most of the previous studies on the racetrack economy with a large number of cities dealt with $n = 2^k$ cities (Section 2). Yet it is yet to be ensured that such number of cities represents the agglomeration behaviors when the number n becomes large. To ensure this issue, stable curves of transport cost parameter τ and population ratio for $n = 4, 5, 6$ in Fig. 10(a) were compared for those of $n = 8, 9, 10$ in Fig. 10(b). The difference of stable curves of $n = 8, 9, 10$ was apparently smaller that of $n = 4, 5, 6$. It, accordingly, is a logical sequel to depict $n = 2^k$ to represent a large number of cities, since its agglomeration behavior is much simpler than other cases (Section 7.1). Figure 11 for $n = 128 = 2^7$ displays an echelon-like structure of the transition to fewer and larger agglomerations with an orderliness via period doubling that took place seven times in agreement with Proposition 7.

7.3 Comparison with the Pf model

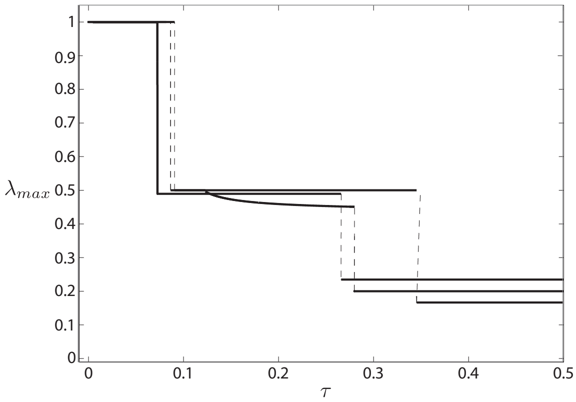
There may be a widespread pessimism on the numerical investigation of stability as it is well known to be model dependent. Nonetheless, we found commonality in the stability of invariant solutions of the FO and Pf models. The stable ranges of τ for invariant solutions of the Pf model shown in Fig. 12 are in good agreement with those of the FO model presented above (Figs. 7(a)–9(a)). Thus it achieves a drastic simplification to observe solely on stable invariant solutions, while a complete numerical analysis of all stable equilibria would demand huge tasks as we have seen in Section 7.2. This shows the importance of the study of invariant patterns presented in this paper.

8 Conclusion

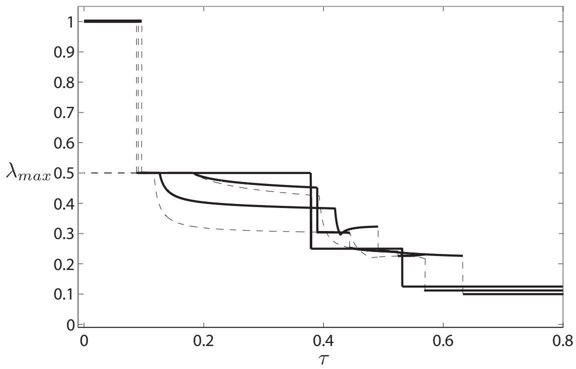
This paper elucidated bifurcation and agglomeration properties of the racetrack economy with an arbitrary number of cities by extending the previous results for $n = 2^k$ (Ikeda et al. 2012). It is ensured that primed numbered cities undergo bifurcation with a very simple hierarchical structure, whereas composite numbered cities undergo very complicated one. An attention was paid to the existence of invariant solutions that retain their spatial distributions when the transport cost parameter is changed. A knowledge on these solutions, which are inherent for replicator dynamics and are dependent on the number of cities, gave an insightful information on spatial agglomerations of spatial economy models.

We have observed two kinds of agglomerations: those of invariant solutions and those of all solutions including non-invariant ones. Although the latters give a complete view of agglomerations, the formers can present a nearly complete view in a much simpler setting. A future study of spatial agglomeration of racetrack economy can be conducted focusing of stable invariant solution as it achieves a drastic simplification to observe solely on stable invariant solutions (Section 7.3).

The difference of stable curves for city numbers of several kinds was demonstrated numerically to diminish as the number of cities increases. It is a logical sequel



(a) $n = 4, 5, 6$



(b) $n = 8, 9, 10$

Fig. 10: Change of stable agglomeration as n increases (solid curves: stable solutions; dashed curves: unstable solutions; $\lambda_{\max}$: maximum population among the cities)

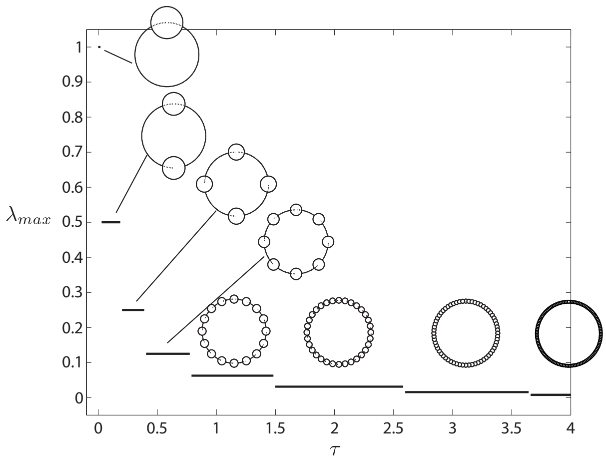


Fig. 11: Spatial period doubling cascade for stable solutions of $n = 128$ (solid lines: stable solutions; $\lambda_{\max}$: maximum population among the cities)

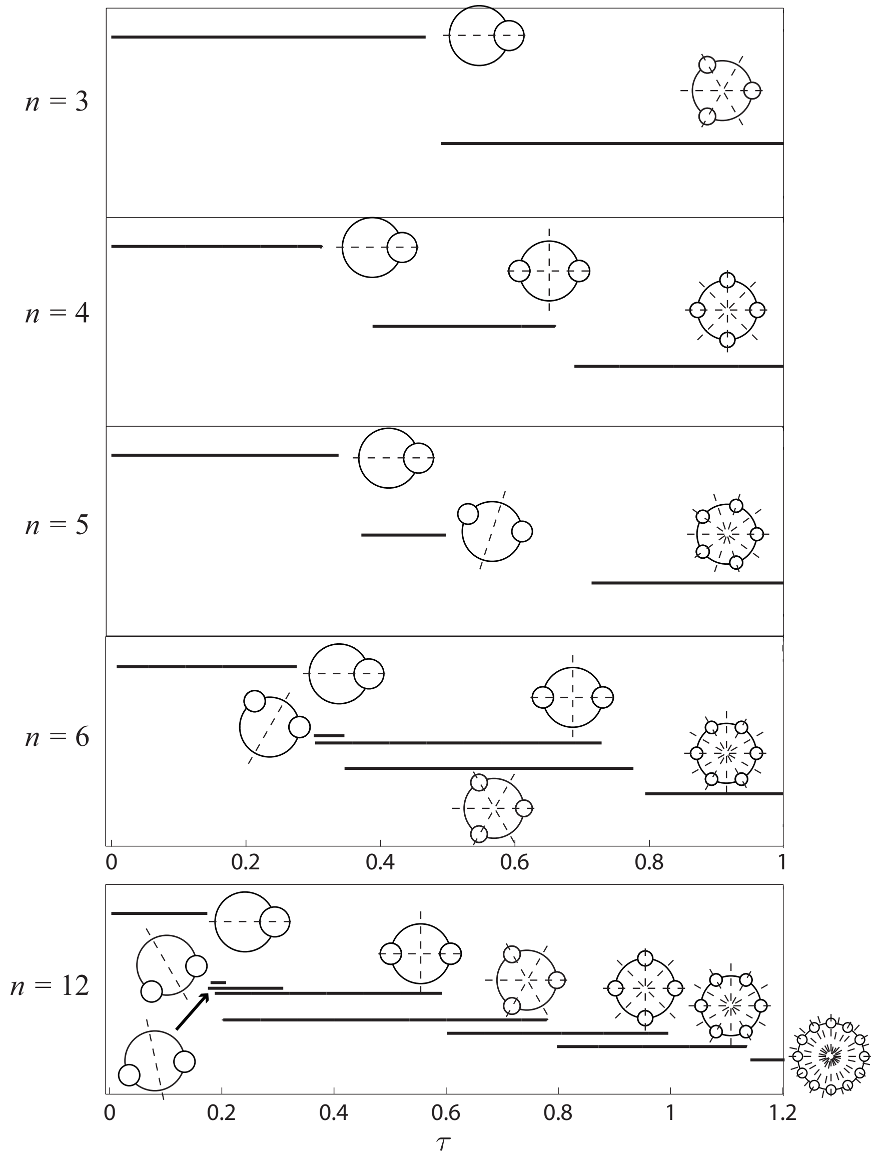


Fig. 12: Durations of τ for stable invariant solutions for the Pf model (solid lines: stable solutions; $\lambda_{\max}$: maximum population among the cities)

to depict $n = 2^k$ as a representative of a large number, since the agglomeration behavior of $n = 2^k$ is much simpler than other cases. This also ensures the validity of the studies of $n = 2^k$ cities conducted up to now (Section 2).

A Theoretical details

A.1 Stability and sustainability conditions for corner solutions

The static governing equation (10) can be rearranged accordingly as

$$\hat{\boldsymbol{F}} = \begin{bmatrix} \boldsymbol{F}_+(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \tau) \\ \boldsymbol{F}_0(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \tau) \end{bmatrix} \tag{23}$$

with the rearranged Jacobian matrix

$$\hat{J} = \begin{bmatrix} J_+ & J_{+0} \\ O & J_0 \end{bmatrix}, \tag{24}$$

where

$$J_+ = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)\,\{\partial(v_i - \bar{v})/\partial \lambda_j \mid i, j = 1, \ldots, m\},$$
$$J_{+0} = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)\,\{\partial(v_i - \bar{v})/\partial \lambda_j \mid i = 1, \ldots, m;\ j = m+1, \ldots, n\},$$
$$J_0 = \mathrm{diag}(v_{m+1} - \bar{v}, \ldots, v_n - \bar{v}).$$

A stable spatial equilibrium is given by a stable stationary solution, for which all eigenvalues of $\hat{J}$ are negative. Such stability condition is decomposed into two conditions:

$$\begin{cases} \text{Stability condition for } \boldsymbol{\lambda}_+\text{:} & \text{all eigenvalues of } J_+ \text{ are negative.} \\ \text{Sustainability condition for } \boldsymbol{\lambda}_0\text{:} & \text{all diagonal entries of } J_0 \text{ are negative.} \end{cases} \tag{25}$$

A.2 Representation matricies for D_n

In our study of a system of n cities on the racetrack economy, each element g of D_n acts as a permutation among city numbers $(1, \ldots, n)$. Consequently, each representation matrix $T(g)$, which expresses the geometrical transformation by g, is a permutation matrix. With the use of the representation matrices for r and s:

$$T(r) = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}, \quad T(s) = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & \iddots & \\ & 1 & & \end{pmatrix},$$

the representation matrices $T(g)$ $(g \in \mathrm{D}_n)$ can be generated as

$$T(r^j) = \{T(r)\}^j, \quad T(sr^j) = T(s)\{T(r)\}^j, \quad (j = 0, 1, \ldots, n-1).$$

A.3 Proofs of Lemma A.3.1 and Propositions A.3.2–A.3.6

A.3.1 Proof of Lemma 1

Since the m places belonging to $\boldsymbol{\lambda}_+$ are permuted each other by $T_+(g)$ $(g \in G)$, we have $v_1 = \cdots = v_m$, as well as $\lambda_1 = \cdots = \lambda_m = 1/m$. Then we have $\bar{v} = \sum_{i=1}^m \lambda_i v_i = v_i$ and, in turn, $v_i - \bar{v} = 0$ $(i = 1, \ldots, m)$, thereby satisfying $\boldsymbol{F}_+(\frac{1}{m}\mathbf{1}, \mathbf{0}, \tau) = \mathbf{0}$. For $n-m$ places with no population, we have $\lambda_j = 0$, thereby satisfying $\boldsymbol{F}_0(\frac{1}{m}\mathbf{1}, \mathbf{0}, \tau) = \mathbf{0}$. This shows that $(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \tau) = (\frac{1}{m}\mathbf{1}, \mathbf{0}, \tau)$ serves as a solution for any τ, i.e., an invariant solution.

A.3.2 *Proof of Proposition 2*

The group G in Lemma 1 is chosen as D_m both for the regular m cities and for the skewed $2m$ cities, thereby ensuring that they are invariant solutions. Skewed $3m$ cites with D_m-invariance do not exist since the rotation permuting three neighboring agglomerated cities is finer than C_m and there is only a reflection permuting only two cities among them. Likewise, regular $4m$ cities and so on do not exist. These suffice for the proof.

A.3.3 *Proof of Proposition 3*

(i) Since the regular m cities are D_m-invariant, n is divisible by m. Conversely, when n is divisible by m ($1 \leq m \leq n$), it is possible to set forth the regular m cities by choosing every n/m cities as agglomerated places.
(ii) Since the skewed $2m$ cities are D_m-invariant, n is divisible by m. A subset containing n/m cities are cyclically repeated m times. Since $n/m = 1$ is related to the regular n cities and $n/m = 2$ is related to the regular $n/2$ cities, we have $n/m \geq 3$ for the skewed $2m$ cities. This gives $1 \leq m \leq n/3$.

A.3.4 *Proof of Proposition 4*

(i) The flat earth is an invariant solution for any n by Proposition 1. An atomic mono-center (concentration) was shown to be an invariant solution in a racetrack economy for any n in Castro et al. (2012) and Ikeda et al. (2012). For the twin cities, Lemma 1 is satisfied by a group G that exchanges the twin cities, thereby showing that the twin cities is an invariant solution for any n.
(ii) A primed number n has two factors: 1 and n. The invariant solutions of regular m cities type are either an atomic mono-center with $m = 1$ and a flat earth with $m = n$ by Proposition 3(i). The invariant solution of skewed $2m$ cities type is skewed twin cities with $m = 1$ by Proposition 3(ii).

A.3.5 *Proof of Proposition 5*

The primed number n is always odd and is expressed as $n = 2j_{\max} + 1$ ($j_{\max} = 2, 3, 5, 6, \ldots$). There is no simple bifurcation. The index of a double bifurcation is always $\widehat{n} = n/\gcd(n, j) = n$ ($j = 1, \ldots, j_{\max}$) and is odd. By Lemma 2, this bifurcation is asymmetric.

A.3.6 *Proof of Proposition 6*

If $n/m = 2$, this is a simple point. Otherwise, we have $3 \leq n/m \leq n$ and set $j = m$ ($1 \leq j < n/2$) that leads to the index for a double point in (17) as $\widehat{n} = n/\gcd(n, j) = n/m$ ($3 \leq \widehat{n} \leq n$). Thus there is a double bifurcation associated with $j = m$. The spatial period elongates from $2\pi/n$ to $2\pi/m$, thereby extending n/m times.

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