Hedging under square loss

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Abstract

The framework of minimum-variance hedging rests on a restrictive foundation. This study shows that the objective of variance minimization is only justifiable when variance coincides with expected squared forecast error. Nevertheless, the classical framework is routinely applied when the condition fails, giving rise to inaccurate risk assessments and suboptimal hedging decisions. This study proposes a new, improved framework of hedging which relaxes the condition at no tangible cost. It derives a new objective function, an optimal hedge ratio, and a measure of hedging effectiveness under square loss. Their superior performance is demonstrated from a theoretical standpoint and by applying them to hedging the price risk of oil and natural gas. Simple yet general, the new framework is well suited to replace the classical one and facilitates adequate risk measurement and improved hedging decisions. It also provides fundamental insight into dealing with uncertainty under square loss and beyond.

(**JEL:** D81, G11, G32, Q02)

Keywords: minimum-variance hedging, hedging effectiveness, optimal hedge ratio, risk, uncertainty, square loss, quadratic loss, forecast error.

1. Introduction

Hedging is a classical means of risk reduction in financial markets. It exploits the idea that negative surprises in the future price of an asset can be mitigated by investing in related assets with opposite shocks. The future price of a portfolio formed this way is more certain than the future price of the original asset. A popular measure of risk or uncertainty associated with price is its variance. Variance measures the spread of a random variable around its expected value and hence is a natural and appropriate measure of risk under square loss provided that the expected value is known.[2] This underlies the classical framework of minimum-variance hedging due to Johnson (1960) and Stein (1961) and motivates the use of relative reduction in variance as a measure of hedging effectiveness suggested by Johnson (1960) and Ederington (1979).[3]

However, financial variables such as share or commodity prices do not have known expected values, rendering variance an inappropriate measure of risk under square loss (or any other loss function). Indeed, a variable with zero variance but unknown expected value is in principle less predictable and may produce more uncertainty than a variable with a known expected value and moderate variance. This undermines the use of variance as a risk measure, and variance minimization as a proxy for risk minimization. Employing variance as a risk measure in absence of a known expected value may result in grave miscalculation of uncertainty and inferior hedging decisions, particularly in short hedging horizons and for prices that have a predictable component, as will be illustrated below both theoretically and empirically. Therefore, a replacement risk measure is needed.

In general, our uncertainty over an outcome of a random variable is characterized by the distribution of the difference between our beliefs, or our prediction of the value to be realized, and

[2] An alternative term for *square loss* is *quadratic loss*.

[3] The discussion of risk and uncertainty can be phrased either in terms of price or of return (i.e. price change). The two formulations are mathematically equivalent and yield identical implications in the present context. Hence, without loss of generality, only one of them is entertained here.

the actual random variable. Under square loss, this uncertainty is reflected by the expected squared forecast error, a measure that applies regardless of whether the expected value of the random variable is known or not. As such, the expected squared forecast error is a valid substitute for variance for measuring hedging effectiveness under square loss.

The purpose of this paper is twofold. First, it is to identify, expose, and illustrate the problems with the classical minimum-variance hedging framework in financial markets, and to relate them to several of their symptoms known from the past. Second, it is to introduce a new, appropriate framework of hedging under square loss based on minimizing the expected squared forecast error. The new framework consists of a new objective function, an optimal hedge ratio, and a measure of hedging effectiveness, all seamlessly generalizing their classical counterparts due to the minimum-variance framework.

The new framework will primarily benefit hedgers by enabling them to properly measure risk and adequately assess and compare the performance of alternative hedging strategies, allowing for optimal hedging decisions to be made. It will also facilitate policymakers' better understanding of risk management and may lead to improved regulations and incentive schemes favoring efficient strategies of uncertainty reduction. Therefore, current users of the classical minimum-variance hedging can only gain from adopting the new framework, avoiding the pitfalls inherent in the classical one.

The remainder of the paper is structured as follows. Section 2 reviews measuring uncertainty in general and under square loss in particular. Section 3 presents the minimum-variance hedging framework and traces some of its problems identified in the literature. Section 4 introduces the new framework of hedging under square loss. Two special cases are considered in Section 5; one where the expected values of prices are known and are used as point forecasts, and another where the expected values are additionally known to equal the current prices. Section 6 provides empirical examples from oil and natural gas markets illustrating the failure of the classical hedging framework and the adequacy of the new one. A conclusion and a discussion of the broader implications of the main results are supplied in Section 7.

2. Measuring uncertainty

2.1 Uncertainty and the forecast error

Hedging pertains to reduction of uncertainty, and thus it is important to clearly delineate the latter. Uncertainty reflects an agent's lack of knowledge about the future price of an asset. It involves two basic building blocks, the agent's beliefs about the future price (formulated as a point or a density forecast) and the future price itself (a random variable). The gap between the two, i.e. the mismatch between the beliefs about the random variable and the actual properties of the variable, or the distance between the forecast and the target, characterize uncertainty. Hence, uncertainty cannot be defined without a reference to beliefs held by the agent facing it. For example, information on the future price alone is not sufficient to characterize uncertainty if the information on the agent's beliefs (her forecast) is missing. Therefore, any definitions and/or measures of uncertainty based solely on the price itself, such as its variance, are ill conceived. In contrast, relevant definitions and/or measures of uncertainty reflect the discrepancy between the beliefs and the reality, hence, the forecast error.

An agent cares about what the price of an asset will be in the future, but she does not know it, hence the uncertainty. Currently, at time t, the price of the asset Y is s_t. In the future, at time $t+h$, where $h>0$, the price will be s_{t+h}. At time t, the agent does not know s_{t+h}, but she has some idea of what it could be. She may have a density forecast or at least a point forecast for s_{t+h}. Let us consider the point forecast and let us denote it $\hat{s}_{t+h|t}$, indicating that it is a point forecast of s_{t+h} made at time t.

The difference between the actual realization of the future price and the forecast constitutes a forecast error $err_{t+h|t} \coloneqq s_{t+h} - \hat{s}_{t+h|t}$. The error is realized (becomes known) at time $t+h$; before that, at time t, $err_{t+h|t}$ is a random variable. The probabilistic properties of $err_{t+h|t}$ are fundamental in characterizing the price uncertainty that the agent is facing at time t. These properties allow us to investigate the uncertainty from a quantitative perspective, to measure it, and to link the practical interpretation of uncertainty with its mathematical characterization.

Any and all probabilistic properties of $err_{t+h|t}$, including those characterizing uncertainty, can be extracted from the probability distribution function and the probability density function of $err_{t+h|t}$. For example, if large (negative and/or positive) errors are relatively likely to be realized, i.e. their probability density is high, then the uncertainty is high. If large errors are unlikely, i.e. their probability density is low, the uncertainty is low. Conversely, high uncertainty means that large (negative and/or positive) errors are relatively likely to be realized, i.e. their probability density is high. Meanwhile, low uncertainty means that such errors are unlikely, i.e. their probability density is low. This is how the probability distribution of $err_{t+h|t}$ is interpreted in terms of uncertainty, and also how the practical understanding of uncertainty translates into probabilistic statements about $err_{t+h|t}$.

2.2 Moments that reflect uncertainty

However, it is not always convenient to work with the probability distribution function or the probability density function of a random variable. Instead, some summary characteristics may be simpler to handle yet still serve the purpose of characterizing uncertainty; see Table 1 for a schematic overview. For example, one such characteristic is the first absolute moment $\mathrm{E}_t(|err_{t+h|t}|)$, where $\mathrm{E}_t(\cdot)$ denotes the mathematical expectation conditional on the information available at time t. When $\mathrm{E}_t(|err_{t+h|t}|)$ is large, the probability density of large (negative and/or positive) errors must be high and hence the uncertainty is high; when $\mathrm{E}_t(|err_{t+h|t}|)$ is small, the probability density of large errors must be low and hence the uncertainty is low; Figure 1 illustrates the point. Conversely, high uncertainty translates into high probability density of large errors and thus large values of $\mathrm{E}_t(|err_{t+h|t}|)$; and low uncertainty translates into low probability density of large errors and hence small values of $\mathrm{E}_t(|err_{t+h|t}|)$. Thus $\mathrm{E}_t(|err_{t+h|t}|)$ is informative of the magnitude of uncertainty and in general is a sensible measure of uncertainty.

[Table 1 about here]

[Figure 1 about here]

Another example is the second moment $\mathrm{E}_t(err_{t+h|t}^2)$. When $\mathrm{E}_t(err_{t+h|t}^2)$ is large, the probability density of large errors must be high and hence the uncertainty is high; when $\mathrm{E}_t(err_{t+h|t}^2)$ is small, the density must be low and the uncertainty too. Conversely, high uncertainty and the corresponding high probability density of large errors produce large values of $\mathrm{E}_t(err_{t+h|t}^2)$; and low uncertainty produces small values. Just like the first absolute moment, also the second moment is informative of the magnitude of uncertainty and thus is a sensible measure of uncertainty. Whether to use the first absolute moment or the second moment depends on the loss function that is relevant for a particular application. The first absolute moment is applicable under absolute loss, while the second moment applies under square loss. Other summary characteristics of the probability distribution, such as value at risk or expected shortfall, are relevant under other loss functions.

2.3 Moments that fail to reflect uncertainty

Not all summary characteristics of the error distribution adequately reflect uncertainty. That is, some or all values of these characteristics are not informative its magnitude. The first moment, or the mathematical expectation $\mathrm{E}_t(err_{t+h|t})$, is the simplest example. If $\mathrm{E}_t(err_{t+h|t})$ is large (negative or positive), the uncertainty is high, because a large $\mathrm{E}_t(err_{t+h|t})$ implies that large errors are relatively likely to be realized and that large positive errors do not outweigh large negative errors nor the other way around. But if $\mathrm{E}_t(err_{t+h|t})$ is small (close to zero), the uncertainty may be either low or high. E.g. the error distribution may be light-tailed and symmetric around zero, which corresponds to low probability density of large errors and hence low uncertainty; see panel a) of Figure 1. Alternatively, the error distribution may be heavy-tailed and symmetric around zero, indicating high uncertainty because large errors are likely; see panel b) of Figure 1. Since a small $\mathrm{E}_t(err_{t+h|t})$ is perfectly compatible with both low and high uncertainty, it is not informative of the magnitude thereof. Hence, $\mathrm{E}_t\left(err_{t+h|t}\right)$ is not a sensible measure of uncertainty.

A similar case can be made for the second central moment, or variance $\mathrm{Var}_t\left(err_{t+h|t}\right) \coloneqq \mathrm{E}_t\left(\left(err_{t+h|t} - \mathrm{E}_t\left(err_{t+h|t}\right)\right)^2\right)$. When it is large, large errors (either negative or positive, or both, depending on $\mathrm{E}_t\left(err_{t+h|t}\right)$) are likely and the uncertainty is high; an example is provided in panel b) of Figure 1. But if variance is small, the uncertainty may be either low or high. E.g. the error distribution may be light-tailed and symmetric around zero, which corresponds to low probability density of large errors and hence low uncertainty; see panel a) of Figure 1. On the other hand, the error distribution may be light-tailed and symmetric around a large value (either negative or positive), indicating high uncertainty because large errors are likely; refer to panel c) of Figure 1. A more extreme stylized example is also possible and is depicted in panel d) of Figure 1. There, variance is zero, i.e. the smallest possible, but the error distribution has all of its mass concentrated at a large positive value, meaning high uncertainty because large errors are guaranteed. In summary, small variance is perfectly compatible with both low and high uncertainty, and thus it is not informative of the magnitude thereof. Hence, variance is generally not a valid measure of uncertainty.

Since the second moment is the sum of variance and squared first moment, variance of the forecast error reflects forecast precision (the spread of forecasts around their center) but not forecast accuracy (the closeness of the center of the forecasts to the target). Low uncertainty requires high precision and high accuracy simultaneously, but low variance only ensures the former. Using variance of the forecast error as a measure of uncertainty is akin to drawing the bullseye right at the center of all shots after they have been fired. Clearly, this is not an adequate measure of the overall skill of the shooter.

However, there is one condition under which variance becomes a sensible measure of uncertainty; this condition is that the mathematical expectation be zero. In the context of forecast errors, this translates into a condition of perfect forecast accuracy (though not necessarily perfect precision). If the expectation is zero, variance equals the second moment: $\mathrm{E}_t(err_{t+h|t}) = 0$ $\Rightarrow \mathrm{Var}_t(err_{t+h|t}) = \mathrm{E}_t(err_{t+h|t}^2)$. Since the second moment adequately reflects uncertainty, the expectation being zero ensures that variance does, too. The expectation being zero is an important special case in which variance turns from an otherwise invalid measure of uncertainty into a valid one; it happens, as variance becomes the second moment. Consequently, for all practical purposes of measuring uncertainty, it is always safer and simpler to use the second moment in place of variance. First, if the two coincide, there is no loss in using the second moment. Second, if they do not coincide, it is only the second moment that appropriately measures uncertainty while variance does not.

3. Minimum-variance hedging and its problems

3.1 Minimum-variance hedging framework

Minimum-variance hedging is one of the oldest and most popular approaches to hedging, introduced by Johnson (1960) and Stein (1961).[4] According to Johnson (1960), "“price risk” can be considered a reflection of the variance <…> of a subjective probability distribution (or a subjective probability density function) for price change from t_1 to t_2 <…> where actual price from t_1 to t_2 is treated as a random variable". This gives rise to declaring variance minimization the hedger's objective:

$$\mathrm{Var}_t(p_{t+h}) = \mathrm{E}_t\left(\left(p_{t+h} - \mathrm{E}_t(p_{t+h})\right)^2\right) \to \min_{\beta}, \tag{1}$$

where $p = p(\beta) = s - \beta f$ is the price of the hedge portfolio $\Pi = Y - \beta X$; s is the price of the original asset Y; f is the price of the hedging instrument X; and $-\beta$ is the portfolio weight of the hedging instrument; β is also known as the hedge ratio. The (negative of the) hedge ratio reflects the hedger's exposure to the price of the hedging instrument as a fraction (or a multiple) of the exposure to the price of the original asset.

[4] Historically, the role and definition of hedging has been investigated in numerous papers, e.g. Hardy and Lyon (1923), Hoffman (1931), Kaldor (1940), and Working (1953, 1962), among other.

Johnson (1960) and later Ederington (1979) suggested assessing hedging effectiveness by calculating the relative reduction in variance due to hedging,

$$\mathrm{RRV}_t(s_{t+h}, p_{t+h}) \coloneqq \frac{\mathrm{Var}_t(s_{t+h}) - \mathrm{Var}_t(p_{t+h})}{\mathrm{Var}_t(s_{t+h})} = 1 - \frac{\mathrm{Var}_t(p_{t+h})}{\mathrm{Var}_t(s_{t+h})}, \tag{2}$$

where the absolute reduction in variance due to hedging, $\mathrm{Var}_t(s_{t+h}) - \mathrm{Var}_t(p_{t+h})$, is measured as a fraction of the variance of the original asset, $\mathrm{Var}_t(s_{t+h})$, to arrive at the relative reduction in variance $\mathrm{RRV}_t(s_{t+h}, p_{t+h})$. The higher the relative reduction in variance, the higher the effectiveness.

The final element to complete the minimum-variance hedging framework is the optimal hedge ratio, $\beta^*_{h,MV}$ (where the subscript h indicates the hedging horizon and MV stands for *minimum variance*), which is defined as the argument minimizing the objective function,

$$\beta^*_{h,MV} \coloneqq \arg\min_{\beta} \mathrm{Var}_t(p_{t+h}). \tag{3}$$

The triplet {objective, optimal hedge ratio, effectiveness measure} constitutes the hedging framework.

3.2 Problems with minimum-variance hedging

Johnson (1960) acknowledges that his concept of risk is different from traditional theory since it is based on subjective rather than objective probability. He introduces a measure of hedging effectiveness that is also defined in terms of subjective probability and does not account for the actual distribution nor the actual realizations of price. However, the minimum-variance hedging framework has been repeatedly employed with (estimated) objective probabilities in settings where the price distribution and particularly the expected value (the first moment of the distribution) may be unknown to the agent. This has led to numerous puzzling results and apparent paradoxes in the literature.

Lien (2008) notes that persistent confusion permeates the literature on minimum-variance hedging and on measuring hedging effectiveness. He observes that the empirical findings are often counterintuitive. Multiple authors (Lindahl, 1989; Lien, 2005a; Alexander & Barbosa, 2007) state that relative reduction in variance is applicable and meaningful only under rather restrictive assumptions on the price changes, or returns[5], of the original asset and of the hedging instrument. Lindahl (1989) identifies a problem with using the classical measure when there is predictable development of the basis, i.e. of the difference between the price of the original asset and that of the hedging instrument. According to her, focusing exclusively on the unexpected changes in the basis would yield a more precise definition of basis risk. However, she acknowledges that distinguishing between expected and unexpected changes is difficult and therefore does not apply this idea in her work. Lien (2005a) identifies a mismatch between the aim of minimizing the conditional variance of portfolio returns when estimating a model in sample and the evaluation of the model performance by measuring the unconditional variance out of sample. Further, he calls the widespread use of the relative reduction in variance "redundant and uninformative" when comparing out-of-sample hedging effectiveness between different hedging strategies. Indeed, Lien (2005b) discourages the readers from using the classical effectiveness measure except under restrictive assumptions on the optimal hedge ratio. He argues that the measure only applies when the hedge ratio is obtained as the slope coefficient in a least-squares regression of the spot returns on the futures returns. Should this fail to be the case, Lien (2005b) raises the idea of using variance of the unpredictable components of portfolio returns instead of that of the raw returns when assessing hedging effectiveness. However, he immediately identifies a weakness that undermines the applicability of this approach: if the model is misspecified, the variance of the model residuals is not economically meaningful. Alexander and Barbosa (2007) refer to the criticism of the classical measure in Lien (2005b) and propose to employ fitted time-varying variance in place of the regular variance. This is intended to address the discrepancy between the objective of minimizing conditional variance and the effectiveness measure that uses unconditional variance. However, allowing for time variation does not make variance a valid

[5] Returns may be nominal or relative. Here and in the remainder of the text, the term *returns* denotes nominal returns that are nothing else than price changes. This should not be confused with *relative returns* that are price changes divided by the price level.

measure of uncertainty as long as it does not coincide with the expected squared forecast error; hence, the quandary persists.

Kahl (1983) does not specifically criticize the minimum-variance framework but contains a useful idea for improving it. She mentions in passing that risk may be measured by "forecast variance, in particular, the mean square error", but does not elaborate on why and how, nor does she acknowledge the fundamental discrepancy between the two measures. Nevertheless, realizing that the forecast error – rather than the portfolio price – is the variable reflecting risk is an important contribution to better understanding the nature of the problem. Similarly, Hauser et al. (1990) consider a setting in which "the hedger compares the variance of an expected to realized hedged price ratio to the variance of an expected to realized unhedged price ratio". Here, the term *expected price* might be interpreted as the price forecast. Focusing on the discrepancy between the forecast and the realized value, Hauser et al. (1990) are approaching a fruitful solution to the problem, though it remains largely implicit. In summary, the problem of measuring uncertainty and hedging effectiveness in the classical framework is widely acknowledged but incompletely understood, with no general remedies available.

Another strand of literature focuses on the relevance of raw versus unexpected returns for estimating the optimal hedge ratio. Hilliard (1984) distinguishes between expected and unexpected returns and only considers the latter in an application on hedging interest rate risk. Bell and Krasker (1986) refer to confusion in the literature regarding the estimation of the optimal hedge ratio and provide a clear and enlightening treatment aimed at clarifying some prevalent misconceptions. Importantly, they note that only the unexpected returns should enter the definition of the optimal hedge ratio as only the conditional rather than marginal distributions of prices matter. Similarly, Myers and Thompson (1989) emphasize the use of unexpected prices (or returns, or relative returns) and derive a generalized estimator for the optimal hedge ratio that should be valid under a variety of price generating processes. Agreeing with the studies above, Viswanath (1993) commends the use of unexpected returns in minimum-variance hedging as he proposes an improvement to the Myers and

Thompson's procedure, namely, to incorporate information on the basis into the model of the spot price. He also notes that minimization of unconditional variance is impossible without minimizing conditional variance, but the difference between the two can be eliminated by investing into bonds to counteract the expected price movements in the original asset. Ederington and Salas (2008) build on these works and examine the adverse effects of considering raw returns in place of the unexpected ones onto the estimated hedge ratio and measures of uncertainty and hedging effectiveness. They find that estimates of optimal hedge ratio are inefficient while those of riskiness and of hedging effectiveness are biased when spot returns are partially predictable.

The shortcomings in the classical measure of uncertainty, the optimal hedge ratio, and the measure of hedging effectiveness are real and important. However, while the suggested remedies are generally helpful, they are not entirely satisfactory for the following two reasons. First, the studies addressing the difference between unexpected and raw returns do not seem to clearly distinguish between the notions of mathematical expectation of a random variable and a point forecast. Consequently, none of the papers (with the possible exception of Kahl, 1983) manage to explicitly operationalize the concept of unexpected returns by forecasts errors. Second, the uncertainty arising from forecast bias in addition to variance of the forecast error is neglected. Therefore, neither the optimal hedge ratios nor the measures of uncertainty or hedging effectiveness proposed above are fully adequate in the general case under square loss. Hence, the classical approach to hedging remains problematic in spite of the suggested improvements. The new hedging framework presented in the next section gets to the heart of the problem and offers a simple yet complete solution to it.

4. Hedging under square loss: the general case

4.1 Objective function

Under square loss, uncertainty is measured by the expected squared forecast error (see Section 2.2), denoted $\mathrm{ESFE}_t(\cdot)$. The goal of a hedger is to minimize uncertainty, i.e. to minimize the expected squared forecast error, by selecting a relevant hedging instrument X among the available set of instruments $\boldsymbol{X}$[6] and an optimal portfolio weight β. Formally, the objective function is

$$\mathrm{ESFE}_t\big(p_{t+h}(\beta,X)\big) \coloneqq \mathrm{E}_t\left(\big(p_{t+h}(\beta,X) - \hat{p}_{t+h|t}(\beta,X)\big)^2\right) \rightarrow \min_{\beta\in\mathbf{R},X\in\mathbf{X}}, \tag{4}$$

where, as before, $p_{t+h} = s_{t+h} + \beta f_{t+h}$ and $\hat{p}_{t+h|t} = \hat{s}_{t+h|t} + \beta \hat{f}_{t+h|t}$, s is the price of the original asset Y, f is the price of the hedging instrument X, and hats denote point forecasts. For a given instrument X, the objective function reduces to

$$\mathrm{ESFE}_t\big(p_{t+h}(\beta)\big) = \mathrm{E}_t\left(\big(p_{t+h}(\beta) - \hat{p}_{t+h|t}(\beta)\big)^2\right) \rightarrow \min_{\beta\in\mathbf{R}}. \tag{5}$$

Equation (5) forms the basis of the new framework of hedging under square loss.

4.2 Optimal hedge ratio

The uncertainty-minimizing portfolio weight, or the optimal hedge ratio, for hedging h periods ahead is denoted $\beta^*_{h,\mathrm{EFSE}}$ and is defined as

$$\beta^*_{h,\mathrm{EFSE}} \coloneqq \arg\min_{\beta\in\mathbf{R}}\Big(\mathrm{ESFE}_t\big(p_{t+h}(\beta)\big)\Big) = \arg\min_{\beta\in\mathbf{R}}\left(\mathrm{E}_t\left(\big(p_{t+h} - \hat{p}_{t+h|t}\big)^2\right)\right). \tag{6}$$

Under regularity conditions on the distribution of p_{t+h} as a function of β, the optimal hedge ratio is obtained by taking the derivative of the objective function in equation (5) with respect to the hedge ratio and setting it to zero:

$$\beta^*_{h,\mathrm{EFSE}} = \left\{\beta\colon \frac{d\mathrm{ESFE}_t\big(p_{t+h}(\beta)\big)}{d\beta} = 0\right\}. \tag{7}$$

This yields

$$\beta^*_{h,\mathrm{ESFE}} \coloneqq -\frac{\mathrm{E}_t(s_{t+h}f_{t+h}) - \hat{s}_{t+h|t}\mathrm{E}_t(f_{t+h}) - \hat{f}_{t+h|t}\mathrm{E}_t(s_{t+h}) + \hat{s}_{t+h|t}\hat{f}_{t+h|t}}{\mathrm{E}_t(f^2_{t+h}) - 2\hat{f}_{t+h|t}\mathrm{E}_t(f_{t+h}) + \hat{f}^2_{t+h|t}}. \tag{8}$$

(see Appendix for the derivation).

The hedge ratio in equation (8) is a theoretical optimal hedge ratio as it involves moments of the underlying (conditional) probability distribution of the price vector (s_{t+h}, f_{t+h}). Since these moments

[6] This setting allows for hedging with several instruments at once since the elements of $\boldsymbol{X}$ may be both individual instruments and instrument portfolios.

are not normally known to the hedger, $\beta^*_{h,\mathrm{ESFE}}$ is not a feasible hedge ratio. However, a feasible hedge ratio $\hat{\beta}^*_{h,\mathrm{ESFE}}$ may be obtained by substituting the true moments with their estimates: $\mathrm{E}_t(s_{t+h})$ with $\hat{s}_{t+h|t}$, $\mathrm{E}_t(f_{t+h})$ with $\hat{f}_{t+h|t}$, $\mathrm{E}_t(s_{t+h}f_{t+h})$ with $\widehat{\mathrm{E}}_t(s_{t+h}f_{t+h})$, and $\mathrm{E}_t\big(f^2_{t+h}\big)$ with $\widehat{\mathrm{E}}_t\big(f^2_{t+h}\big)$, to yield

$$\begin{aligned}
\hat{\beta}^*_{h,\mathrm{ESFE}} &\coloneqq -\frac{\widehat{\mathrm{E}}_t(s_{t+h}f_{t+h}) - \hat{s}_{t+h|t}\hat{f}_{t+h|t} - \hat{f}_{t+h|t}\hat{s}_{t+h|t} + \hat{s}_{t+h|t}\hat{f}_{t+h|t}}{\widehat{\mathrm{E}}_t(f^2_{t+h}) - 2\hat{f}_{t+h|t}\hat{f}_{t+h|t} + \hat{f}^2_{t+h|t}} \\
&= -\frac{\widehat{\mathrm{E}}_t(s_{t+h}f_{t+h}) - \hat{s}_{t+h|t}\hat{f}_{t+h|t}}{\widehat{\mathrm{E}}_t(f^2_{t+h}) - \hat{f}^2_{t+h|t}} \\
&= -\frac{\widehat{\mathrm{Cov}}_t(s_{t+h}, f_{t+h})}{\widehat{\mathrm{Var}}_t(f_{t+h})},
\end{aligned} \tag{9}$$

where $\widehat{\mathrm{Cov}}_t(s_{t+h}, f_{t+h})$ and $\widehat{\mathrm{Var}}_t(f_{t+h})$ are estimates of covariance $\mathrm{Cov}_t(s_{t+h}, f_{t+h})$ and variance $\mathrm{Var}_t(f_{t+h})$, respectively. The hedge ratio in equation (9) may be obtained from sample data without reference to moments of the true probability distribution of the price vector (s_{t+h}, f_{t+h}) and thus constitutes a feasible hedge ratio, i.e. a ratio that can be constructed from the available data.[7] The theoretical and empirical optimal hedge ratios in the general case and under additional assumptions are listed in Table 2.

[Table 2 about here]

4.3 Measures of hedging effectiveness

A natural measure of success (or lack thereof) of hedging is the value of the objective function, $\mathrm{ESFE}_t\big(p_{t+h}(\beta)\big)$, at the chosen hedge ratio β (e.g. at $\hat{\beta}^*_{h,\mathrm{ESFE}}$). It reflects the level of uncertainty over the future portfolio price in terms of square loss. This study proposes to use $\mathrm{ESFE}_t\big(p_{t+h}(\beta)\big)$ as the *theoretical absolute* measure of hedging effectiveness. Since $\mathrm{ESFE}_t\big(p_{t+h}(\beta)\big)$ is a function of the

[7] While $\hat{\beta}^*_{h,\mathrm{ESFE}}$ is derived in a standard way, by substituting the population quantities with their sample counterparts, no optimality is claimed for it as an estimator of $\beta^*_{h,\mathrm{ESFE}}$. Finding optimal estimators for $\beta^*_{h,\mathrm{ESFE}}$ within different classes of estimators is an interesting problem in itself, but is left for future studies.

typically unknown true probability distribution of p_{t+h}, the theoretical measure cannot be applied in practice, and an empirical counterpart is needed.

Consider sets of h-period-ahead point forecasts of the asset price and the price of the portfolio, denoted $\{\hat{s}_{t+h|t}\}_{t=1}^{T}$ and $\{\hat{p}_{t+h|t}\}_{t=1}^{T}$, and the corresponding realized prices, $\{s_{t+h}\}_{t=1}^{T}$ and $\{p_{t+h}\}_{t=1}^{T}$. Hedging effectiveness can be assessed empirically by mean squared forecast error of the portfolio, $\mathrm{MSFE}(p_{+h})$, which is the empirical counterpart of $\mathrm{ESFE}_t(p_{t+h})$ over the time period $t = 1, \dots, T$:

$$\mathrm{MSFE}(p_{+h}) \coloneqq \frac{1}{\mathrm{T}} \sum_{t=1}^{T} \left(p_{t+h} - \hat{p}_{t+h|t}\right)^2. \tag{10}$$

Here, the subscript to the argument of MSFE is $+h$, indicating the forecast horizon. $\mathrm{MSFE}(p_{+h})$ is the *empirical absolute* measure of hedging effectiveness under square loss.

Hedging performance can also be gauged in relative terms by comparing the loss under hedging to the benchmark of no hedging. Let us define the *theoretical relative* measure of hedging effectiveness as the relative reduction in expected squared forecast error (RRESFE) when using the portfolio in comparison with the case of no hedging,

$$\mathrm{RRESFE}_t(p_{t+h}, s_{t+h}) \coloneqq \frac{\mathrm{ESFE}_t(s_{t+h}) - \mathrm{ESFE}_t(p_{t+h})}{\mathrm{ESFE}_t(s_{t+h})} = 1 - \frac{\mathrm{ESFE}_t(p_{t+h})}{\mathrm{ESFE}_t(s_{t+h})}. \tag{11}$$

RRESFE has an upper bound of unity which corresponds to complete absence of uncertainty, or perfect predictability of the portfolio price and hence perfect hedging performance: $\mathrm{RRESFE}_t(p_{t+h}, s_{t+h}) = 1 \Leftrightarrow \mathrm{ESFE}_t(p_{t+h}) = 0$. The measure is unbounded from below. A value between zero and one suggests that hedging is somewhat effective as it helps reduce the uncertainty without completely eliminating it: $\mathrm{RRESFE}_t(p_{t+h}, s_{t+h}) > 0 \Leftrightarrow 0 < \mathrm{ESFE}_t(p_{t+h}) < \mathrm{ESFE}_t(s_{t+h})$. A value of zero indicates that the uncertainty over the portfolio price is as great as that of the unhedged position, and therefore hedging is completely ineffective: $\mathrm{RRESFE}_t(p_{t+h}, s_{t+h}) = 0$ $\Leftrightarrow \mathrm{ESFE}_t(p_{t+h}) = \mathrm{ESFE}_t(s_{t+h})$. A value below zero indicates that hedging increases the expected squared forecast error and is thus detrimental to the goal of uncertainty reduction: $\mathrm{RRESFE}_t(p_{t+h}, s_{t+h}) < 0 \Leftrightarrow \mathrm{ESFE}_t(p_{t+h}) > \mathrm{ESFE}_t(s_{t+h})$. RRESFE is undefined when

$\text{ESFE}_t(s_{t+h}) = 0$, but that also implies there is no uncertainty in the future price of the original asset to begin with, rendering hedging irrelevant.

Similarly, let us define the *empirical relative* measure of hedging effectiveness as the relative reduction in mean squared forecast error (RRMSFE) when using the portfolio compared to no hedging,

$$\text{RRMSFE}(p_{+h}, s_{+h}) := \frac{\text{MSFE}(s_{+h}) - \text{MSFE}(p_{+h})}{\text{MSFE}(s_{+h})} = 1 - \frac{\text{MSFE}(p_{+h})}{\text{MSFE}(s_{+h})}, \tag{12}$$

where $\text{MSFE}(s_{+h}) := \frac{1}{\text{T}}\sum_{t=1}^{T}\left(s_{t+h} - \hat{s}_{t+h|t}\right)^2$. The bounds and the interpretation of values of RRMSFE are analogous to those of RRESFE. The theoretical and empirical, absolute and relative measures of hedging effectiveness in the general case and under additional assumptions are presented in Table 3.

[Table 3 about here]

4.4 Statistical significance of hedging effectiveness

When measuring hedging effectiveness empirically, estimation errors are unavoidable; hence, the measured values are imperfect reflections of the true underlying values. Given a measured value, one may be interested in whether the corresponding true value is different from zero. This is equivalent to asking whether hedging has any genuine effect on price uncertainty. The null hypothesis of equally great price uncertainty under hedging versus no hedging can be tested by the Diebold-Mariano test of equal predictive ability (Diebold, 2015; Diebold & Mariano, 1995; Harvey et al., 1997). A rejection of the null hypothesis would attest that the effect of hedging is genuine, whereas a failure to reject would indicate that the evidence is insufficient to conclude so. Testing whether the hedging effectiveness of two competing strategies is equally great, or assessing the difference between the true effectiveness and an arbitrary value other than zero are trivial extensions.

5. Hedging under square loss: two special cases

5.1 Expected values of prices are known

Let us consider a special case of hedging under square loss where the mathematical expectations of the future prices s_{t+h} and p_{t+h} of the original asset Y and the hedge portfolio Π, respectively, are assumed to be known at time t and are used as point forecasts. (Under square loss, expectations are optimal point forecasts, therefore it would be suboptimal to use any other forecasts when expectations are available.)

Assumption 1. *The conditional expectations of s_{t+h} and p_{t+h}, denoted $\mu_{s,t+h} \coloneqq \mathrm{E}_t(s_{t+h})$ and $\mu_{p,t+h} \coloneqq \mathrm{E}_t(p_{t+h})$, are known as of time t and are used as point forecasts, $\hat{s}_{t+h|t} = \mu_{s,t+h}$ and $\hat{p}_{t+h|t} = \mu_{p,t+h}$.*

Corollary 1. *Under Assumption 1, the conditional expectation of f_{t+h}, denoted $\mu_{f,t+h} \coloneqq \mathrm{E}_t(f_{t+h})$, is known as of time t.*

For example, if Y and X are exchange rates in a liquid currency exchange market or shares of highly-traded companies, their future prices (at least for a relatively short time period h) can reasonably be assumed to have a true population mean at their current values, $\mu_{s,t+h} = s_t$ and $\mu_{f,t+h} = f_t$. In other words, a martingale property can be conjectured for s_t and f_t in short horizons. Or if Y is a share of a highly-traded company and there is a 1:10 share split scheduled for some point in time between t and $t+h$, one may assume $\mu_{s,t+h} = 0.1 \cdot s_t$, except perhaps for a rounding error. Consequently, the mathematical expectation of the future portfolio price for a given hedge ratio β is known today and is $\mu_{p,t+h} = \mu_{s,t+h} + \beta\mu_{f,t+h}$.

Under Assumption 1, $\hat{s}_{t+h|t}$ and $\hat{p}_{t+h|t}$ are equal to $\mu_{s,t+h}$ and $\mu_{p,t+h}$, respectively, and the expected squared forecast error of the portfolio price becomes the variance of the portfolio price, $\mathrm{ESFE}_t(p_{t+h}) = \mathrm{E}_t\left(\left(p_{t+h} - \mu_{p,t+h}\right)^2\right) = \mathrm{Var}_t(p_{t+h})$ according to the definition of variance. Then the objective function in equation (5) turns into

$$\mathrm{Var}_t\big(p_{t+h}(\beta)\big) \to \min_{\beta \in \mathbf{R}}, \tag{13}$$

yielding the well-known objective of variance minimization that stems from the classical framework of Johnson (1960) and Stein (1961). The optimal hedge ratio due to equation (13) is

$$\beta^*_{h,\mathrm{Var}} = -\frac{\mathrm{Cov}_t(s_{t+h}, f_{t+h})}{\mathrm{Var}_t(f_{t+h})}, \tag{14}$$

and its sample counterpart is

$$\hat{\beta}^*_{h,\mathrm{Var}} = -\frac{\widetilde{\mathrm{Cov}}_t(s_{t+h}, f_{t+h})}{\widetilde{\mathrm{Var}}_t(f_{t+h})} \tag{15}$$

(see Appendix for derivations). Here, $\widetilde{\mathrm{Cov}}_t(s_{t+h}, f_{t+h}) \coloneqq \widehat{\mathrm{E}}_t(s_{t+h}f_{t+h}) - \mu_{s,t+h}\mu_{f,t+h}$ and $\widetilde{\mathrm{Var}}_t(f_{t+h}) \coloneqq \widehat{\mathrm{E}}_t\left(f^2_{t+h}\right) - \mu^2_{f,t+h}$ are estimators of covariance and variance, respectively, which employ the true first moments $\mu_{s,t+h}$ and $\mu_{f,t+h}$ rather than their sample counterparts $\widehat{\mathrm{E}}_t(s_{t+h})$ and $\widehat{\mathrm{E}}_t(f_{t+h})$. The presence of $\widetilde{\mathrm{Cov}}_t(s_{t+h}, f_{t+h})$ and $\widetilde{\mathrm{Var}}_t(f_{t+h})$ in the definition of $\hat{\beta}^*_{h,\mathrm{Var}}$ under Assumption 1 is in contrast to $\widehat{\mathrm{Cov}}_t(s_{t+h}, f_{t+h})$ and $\widehat{\mathrm{Var}}_t(f_{t+h})$ in equation (9) for the optimal hedge ratio in the general case.

Under Assumption 1, the measures of hedging effectiveness collapse as follows. First, the *theoretical absolute* measure, i.e. the expected squared forecast error, becomes the variance of the portfolio price,

$$\mathrm{ESFE}_t(p_{t+h}) = \mathrm{Var}_t(p_{t+h}), \tag{16}$$

which is analogous to the change in the objective function from equation (5) to equation (13). Second, the *empirical absolute* measure, the mean squared forecast error, becomes the empirical variance of the portfolio price that employs the true first moment rather than its sample counterpart:

$$\mathrm{MSFE}(p_{+h}) = \widetilde{\mathrm{Var}}(p_{+h}), \tag{17}$$

where $\widetilde{\mathrm{Var}}(p_{+h}) \coloneqq \frac{1}{T}\sum_{t=1}^{T}\left(p_{t+h} - \mu_{p,t+h}\right)^2$. An alternative, widespread measure of empirical variance, $\widehat{\mathrm{Var}}(p_{+h}) \coloneqq \frac{1}{T-1}\sum_{t=1}^{T}\left(p_{t+h} - \frac{1}{T}\sum_{t=1}^{T} p_{t+h}\right)^2$, does not make for a meaningful measure of portfolio variance (or absolute hedging effectiveness) as it replaces the true expected values with an estimate of their average, i.e. the mean of realizations $\{p_{t+h}\}_{t=1}^{T}$, thereby introducing an error. $\widehat{\mathrm{Var}}(p_{+h})$ is an upward-biased and inconsistent estimator of the true variance (and hence an ill-suited measure of hedging effectiveness) unless the true first moments are equal for all t between 1 and T: $\mu_{p,1+h} = \cdots = \mu_{p,T+h}$. In the latter case, $\widehat{\text{Var}}(p_{+h})$ is a valid effectiveness measure, though inferior to $\widetilde{\text{Var}}(p_{+h})$, which is more efficient. The use of $\widehat{\text{Var}}(\cdot)$ in place of $\widetilde{\text{Var}}(\cdot)$ in empirical studies may explain some of the counterintuitive findings in the literature.

Third, the *theoretical relative* measure of hedging effectiveness becomes

$$\begin{aligned} \text{RRESFE}_t(p_{t+h}, s_{t+h}) &= \frac{\text{ESFE}_t(s_{t+h}) - \text{ESFE}_t(s_{t+h})}{\text{ESFE}_t(s_{t+h})} \\ &= \frac{\text{Var}_t(s_{t+h}) - \text{Var}_t(p_{t+h})}{\text{Var}_t(s_{t+h})} \\ &=: \text{RRV}_t(p_{t+h}, s_{t+h}), \end{aligned} \tag{18}$$

where RRV stands for "relative reduction in variance", a term and effectiveness measure proposed by Johnson (1960) and Ederington (1979). RRV is justified as the *theoretical relative* measure of hedging effectiveness whenever Assumption 1 holds. Fourth, the *empirical relative* measure of hedging effectiveness turns into

$$\begin{aligned} \text{RRMSFE}(p_{+h}, s_{+h}) &= \frac{\text{MSFE}(s_{+h}) - \text{MSFE}(p_{+h})}{\text{MSFE}(s_{+h})} \\ &= \frac{\widetilde{\text{Var}}(s_{+h}) - \widetilde{\text{Var}}(p_{+h})}{\widetilde{\text{Var}}(s_{+h})} \\ &=: \widetilde{\text{RRV}}(p_{+h}, s_{+h}). \end{aligned} \tag{19}$$

where $\widetilde{\text{RRV}}(p_{+h}, s_{+h})$ is a plug-in estimator of $\text{RRV}_t(p_{t+h}, s_{t+h})$. Just as in the case of the *theoretical relative* measure, also here the use of $\widehat{\text{Var}}(\cdot)$ in place of $\widetilde{\text{Var}}(\cdot)$ is unwarranted following the same argumentation as above. It may be responsible for the prevalent confusion surrounding the minimum-variance framework.

5.2 Expected values of prices equal current prices

When the expected future price of an asset is known in advance, it may typically equal the last observed price as in the examples of the foreign exchange and stock markets in Section 5.1, but unlike the example involving a share split. Thus a relevant assumption can be formulated as follows.

Assumption 2. *The conditional expectations of* s_{t+h} *and* p_{t+h} *equal the last observed values of* s *and* p*, respectively,* $\mu_{s,t+h} = s_t$ *and* $\mu_{p,t+h} = p_t$*.*

Corollary 2. *Under Assumption 2, the conditional expectation of* f_{t+h} *equals the last observed value of* f*,* $\mu_{f,t+h} = f_t$*.*

When the expected price coincides with the last observed price and is used as a point forecast, the h-step-ahead forecast error is nothing else than the realized price change, or return, from time t to $t + h$. This allows replacing the forecast error with the return in the expressions for the objective function, the optimal hedge ratio, and the measures of hedging effectiveness. Under Assumptions 1 and 2, the expected squared forecast error of the portfolio price becomes the expected squared portfolio return, $\mathrm{ESFE}_t(p_{t+h}) = \mathrm{E}_t((p_{t+h} - p_t)^2) =: \mathrm{ESR}_t(p_{t+h})$, leading to the objective function

$$\mathrm{ESR}_t\big(p_{t+h}(\beta)\big) \rightarrow \min_{\beta\in\mathbf{R}}, \tag{20}$$

which is to minimize the expected squared portfolio return with respect to the hedge ratio β. The theoretical optimal hedge ratio is implicitly defined as

$$\beta^*_{h,\mathrm{ESR}} \coloneqq \arg\min_{\beta\in\mathbf{R}}\big(\mathrm{ESR_t}(p_{t+h})\big) = \arg\min_{\beta\in\mathbf{R}}\big(\mathrm{E}_t((p_{t+h} - p_t)^2)\big), \tag{21}$$

which yields the explicit expression

$$\beta^*_{h,\mathrm{ESR}} = -\frac{\mathrm{E}_t(s_{t+h}f_{t+h}) - s_t f_t}{\mathrm{E}_t(f^2_{t+h}) - f^2_t}. \tag{22}$$

The corresponding empirical optimal hedge ratio is

$$\hat{\beta}^*_{h,\mathrm{MSR}} = -\frac{\widehat{\mathrm{E}}_t(s_{t+h}f_{t+h}) - s_t f_t}{\widehat{\mathrm{E}}_t(f^2_{t+h}) - f^2_t} \tag{23}$$

(see Appendix for derivations). The *theoretical absolute* measure of hedging effectiveness is the expected squared return,

$$\mathrm{ESFE}_t(p_{t+h}) = \mathrm{ESR}_t(p_{t+h}), \tag{24}$$

and the *empirical absolute* measure is mean squared return of the portfolio, $\mathrm{MSR}(p_{+h})$,

$$\mathrm{MSR}(p_{+h}) \coloneqq \frac{1}{\mathrm{T}}\sum_{t=1}^{T}(p_{t+h} - p_t)^2. \tag{25}$$

The *theoretical relative* measure of hedging effectiveness becomes the relative reduction in expected squared return,

$$\mathrm{RRESR}_t(p_{t+h}, s_{t+h}) \coloneqq \frac{\mathrm{ESR}_t(s_{t+h}) - \mathrm{ESR}_t(p_{t+h})}{\mathrm{ESR}_t(s_{t+h})}, \tag{26}$$

and the corresponding *empirical relative* measure is the relative reduction in mean squared return when holding the portfolio as compared to holding the original asset alone:

$$\mathrm{RRMSR}(p_{+h}, s_{+h}) \coloneqq \frac{\mathrm{MSR}(s_{+h}) - \mathrm{MSR}(p_{+h})}{\mathrm{MSR}(s_{+h})}. \tag{27}$$

Here, $\mathrm{MSR}(s_{+h}) = \frac{1}{\mathrm{T}}\sum_{t=1}^{T}(s_{t+h} - s_t)^2$ is the mean squared return on the original asset. Of course, it would generally be wrong to use the measures given in equations (24) to (27) when at least one of the Assumptions 1 and 2 is violated. The discussion on the inadequacy of $\widehat{\mathrm{Var}}$ in place of $\widetilde{\mathrm{Var}}$ in Section 5.1 applies here, too, with MSR substituting for $\widetilde{\mathrm{Var}}$. That is, the use of $\widehat{\mathrm{Var}}$ instead of MSR can only be justified when $\mu_{p,1+h} = \cdots = \mu_{p,T+h}$, but even then $\widehat{\mathrm{Var}}$ is less efficient than MSR as the estimator for ESR.

5.3 Why minimum-variance framework does not always fail

Even though the classical minimum-variance hedging framework has been demonstrated to fail in absence of stringent assumptions, it often delivers seemingly sensible results. This section investigates how this might come about. First, the empirical optimal hedge ratio might nearly or fully coincide between the classical and the new framework, as the empirical counterpart of the classical optimal hedge ratio in equation (3) is quite similar to the new empirical optimal hedge ratio in equation (9). Depending on what estimators are used for the former, the two may even be equal. Hence, the optimal hedge ratio derived within the classical framework might often be unproblematic.

Second, the classical *empirical relative* measure of hedging effectiveness, the $\widehat{\mathrm{RRV}} \coloneqq \frac{\widehat{\mathrm{Var}}_t(s_{t+h}) - \widehat{\mathrm{Var}}_t(p_{t+h})}{\widehat{\mathrm{Var}}_t(s_{t+h})}$, might nearly or fully coincide with the new measure, the RRMSFE. This may happen when the price forecasts of the original asset and portfolio are their last observed values in all periods, and the forecast bias happens to average to zero or close to zero for both of them in the given test sample. In such a case, the estimated relative reduction in variance closely matches the relative reduction in mean squared forecast error. In other words, the reduction in uncertainty as measured in the classical framework is about the same as in the new framework.

While these observations might be taken as evidence in favour of continued use of the classical hedging framework, it should be remembered that minimum-variance hedging may or may not fail depending on the application at hand. On the other hand, the new framework is unconditionally guaranteed to deliver correct results.

6. Empirical examples

This section exemplifies the main theoretical considerations of Sections 3 to 5 by demonstrating the performance of the new and the classical hedging frameworks when hedging the spot price risk of two major commodities, oil (WTI) and natural gas (Henry Hub). It reveals how the estimated uncertainty and hedging effectiveness differ across the frameworks and shows whether the same or different hedging strategies are favoured by the relative reduction in mean squared forecast error versus the relative reduction in variance.[8]

Hedging the risk of the monthly spot price with a futures contract is considered for the one-month horizon and for two hedge ratios, the naïve 1:1 and the estimated optimal hedge ratio due to equation (9). A bivariate time series of spot and futures prices is modelled in 120-month-long rolling windows. In the oil market, the spot price is predicted with an error-correction model that allows it to adjust towards the futures price. The futures price of a given contract is treated as a martingale, hence its forecast equals the last observed value. In the natural gas market, the futures price of the relevant contract is taken as a point forecast for both spot and futures prices in the next month. The optimal hedge ratio is obtained from a bivariate GARCH(1,1)-DCC(1,1) model (Bollerslev, 1986; Engle, 2002) applied on in-sample forecast errors of the two price series.

[8] Another application of the new framework is available in Bloznelis (2018), who considers hedging the spot price risk in the market for Norwegian farmed salmon.

The spot price data is obtained from the WIKI Commodities Prices database at Quandl, and the futures prices are taken from Chicago Mercantile Exchange via Quandl. The oil price data covers May 1983 to August 2017 (412 data points), while the natural gas price data covers May 1990 to August 2017 (328 data points). Figures 2 to 5 depict the spot and futures prices of oil and natural gas together with their returns and forecast errors, alongside the returns and the forecast errors of the hedge portfolios.

[Figures 2, 3, 4, and 5 around here]

In the case of oil, the forecast errors of spot and portfolio prices are visibly smaller in magnitude than the respective returns, suggesting that there is a material discrepancy between uncertainty measured as a function of returns (in the classical framework) versus forecast errors (in the new framework). This can be seen from the first, third, fourth, and last columns of Table 4 that contains the hedging results for oil. The variances of the spot and portfolio prices are considerably (up to three times) higher than the corresponding mean squared forecast errors. Hence, variance and expected squared forecast error are far from interchangeable, attesting that the former is not an adequate proxy for the latter. Indeed, variance is an ill-suited measure of uncertainty under square loss, because not all variability in returns is unpredictable, and the magnitude of returns overestimates the inherent uncertainty. The exception here is the futures price; the model treats the returns on futures contracts as entirely unpredictable, which makes them coincide with the forecast errors.

[Table 4 around here]

The last two columns of Table 4 display the readings of the RRV and the RRMSFE, which can be either similar, as in the case of the 1:1 hedge ratio, or quite different, as in the case of the optimal hedge ratio. Therefore, the RRV cannot be used in place of the RRMSFE as an innocuous alternative or an approximation. Moreover, the two measures may also have different subject-matter implications. E.g. a potential hedger might find a 51% reduction in uncertainty (due to RRV) insufficient to warrant investing her time and effort in hedging, while a 74% reduction (due to

RRMSFE) might seem large enough to spur her into action. Thus relying on RRV in place of RRMSFE might make her miss the opportunity to effectively reduce the price uncertainty.

The case of natural gas underscores the qualitative difference in implications of the two frameworks even more clearly; compare the last two columns in Table 5. While RRV is negative and hence suggests hedging is detrimental, i.e. it increases uncertainty, RRMSFE is positive and thus indicates hedging is moderately helpful. The two measures point to opposite directions, and once again, following the irrelevant measure might lead to suboptimal choices in risk management.

[Table 5 around here]

In summary, real-world hedging applications in major commodity markets highlight pronounced differences between the classical and the new hedging frameworks. They confirm that the variance of the portfolio price can be very different from the corresponding expected squared forecast error, and that RRV and RRMSFE need not be alike. As such, the classical framework of minimum variance hedging may realistically lead to different implications and hedging decisions than the new framework of minimum expected squared forecast error. Since hedging is fundamentally concerned with risk minimization, it is only the latter framework that is adequate for the purpose.

7. Conclusion

Minimum-variance hedging framework has been a highly popular approach to hedging both in the financial world and in the academic literature starting from the early 1960s. Nearly six decades after its origins in Johnson (1960) and Stein (1961), it remains the starting point for introducing non-naïve hedging strategies with futures contracts in finance textbooks (Hull, 2012, p. 57; McDonald, 2013, p. 114). Despite its continued popularity among practitioners and the extensive academic research addressing its theoretical as well as applied facets, a key weakness of the minimum-variance hedging framework has gone unidentified and unexplained so far. It consists of two elements: (1) the fact that uncertainty over a future price may be better characterized by the probability distribution of its forecast error rather than that of the price itself; and (2) the fact that variance is not a generally sensible measure of uncertainty, whereas the second moment is. Even though broadly overlooked, this weakness has created considerable confusion, as the intuitively perceived uncertainty would repeatedly fail to match the formalized uncertainty estimated in the minimum-variance framework. While the variance of the hedge portfolio price would be successfully minimized, the perceived uncertainty (reflected by the magnitude of the expected squared forecast error) would still be looming large. Also, hedging strategies that incorporate additional information in the price models would paradoxically yet routinely be found inferior to ones that ignore it.

This work has pinned down and explicitly formulated the previously elusive problem with the classical minimum-variance hedging framework. It also offers a simple and complete solution to it in the form of a new hedging framework that generalizes and extends the classical one. Instead of aiming to minimize the variance of the portfolio price, the actual goal of a hedger might indeed be to minimize the expected squared forecast error. Once an appropriate objective is adapted, it gives rise to a new optimal hedge ratio and a new measure of hedging effectiveness. Taken together, the objective, the ratio, and the effectiveness measure constitute the new framework of hedging under square loss. This framework applies without restrictions as long as the objective is relevant, and it contains the minimum-variance framework as a special case, namely, under the assumption that the true conditional expectations of the future prices of the original asset and the portfolio are known in advance and are used as point forecasts.

The implications of replacing the classical framework with the new one are substantial. First, confusion is eliminated as the formalized hedging objective now properly reflects the hedger's goal. In turn, measurement of uncertainty is now intuitive and well defined, so that intuitively perceived and measured effectiveness agree. Second, decisions on the choice of best hedging strategies are better informed, since hedging strategies that emerge as optimal under the new framework generally differ from those due to the classical one. Empirical examples from hedging in the oil and natural gas markets serve to illustrate these points, with the implications for hedgers differing considerably depending on which framework is referred to. Overall, the new framework should thus be of immediate interest to hedgers and policymakers in commodity and financial markets.

Appropriate measurement of uncertainty introduced in this study is relevant in a much broader context than hedging alone. Uncertainty underlies significant subfields of finance, economics, and operations research, among other disciplines. For example, modern portfolio theory, i.e. the mean-variance framework (Markowitz, 1952), relies on the variance of the portfolio return as representing the risk. Should variance be replaced by the expected squared forecast error, the theory would need to be revised. Uncertainty is also a critical element in decision-making problems. Therefore, improved understanding of uncertainty and the availability of a proper measure thereof can be instrumental in making better decisions. To conclude, the new measure of uncertainty opens new avenues for improvement in the context of hedging under square loss, hedging in general, and beyond. Embracing the new measure and the ensuing hedging framework and examining the broad range of implications are directions for future research.

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Bibliography

Alexander, C., & Barbosa, A. (2007). Effectiveness of minimum-variance hedging. *The Journal of Portfolio Management, 33*(2), 46-59.

Bell, D. E., & Krasker, W. S. (1986). Estimating hedge ratios. *Financial Management, 15*(2), 34-39.

Bloznelis, D. (2018). Hedging salmon price risk. Aquaculture Economics & Management, 22(2), 168-191.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics, 31*(3), 307-327.

Diebold, F. X. (2015). Comparing predictive accuracy, twenty years later: A personal perspective on the use and abuse of Diebold–Mariano tests. *Journal of Business & Economic Statistics, 33*(1), 1-9.

Diebold, F. X. & Mariano, R. S. (1995). Comparing predictive accuracy. *Journal of Business & Economic Statistics, 13*(3), 253-263.

Ederington, L. H. (1979). The hedging performance of the new futures markets. *The Journal of Finance, 34*(1), 157-170.

Ederington, L. H., & Salas, J. M. (2008). Minimum variance hedging when spot price changes are partially predictable. *Journal of Banking & Finance, 32*(5), 654-663.

Engle, R. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of Business & Economic Statistics, 20*(3), 339-350.

Hardy, C. O., & Lyon, L. S. (1923). The theory of hedging. *Journal of Political Economy, 31*(2), 276-287.

Harvey, D., Leybourne, S., & Newbold, P. (1997). Testing the equality of prediction mean squared errors. *International Journal of Forecasting, 13*(2), 281-291.

Hauser, R. J., Garcia, P., & Tumblin, A. D. (1990). Basis expectations and soybean hedging effectiveness. *North Central Journal of Agricultural Economics, 12*(1), 125-136.

Hilliard, J. E. (1984). Hedging interest rate risk with futures portfolios under term structure effects. *The Journal of Finance, 39*(5), 1547-1569.

Hoffman, G. W. (1931). The Hedging of Grain. *The Annals of the American Academy of Political and Social Science, 155*, 7-22.

Hull, J. C. (2012). *Options, futures, and other derivatives*. London: Pearson Education Limited.

Johnson, L. L. (1960). The theory of hedging and speculation in commodity futures. *The Review of Economic Studies, 27*(3), 139-151.

Kahl, K. H. (1983). Determination of the recommended hedging ratio. *American Journal of Agricultural Economics, 65*(3), 603-605.

Kaldor, N. (1940). A note on the theory of the forward market. *The Review of Economic Studies, 7*(3), 196-201.

Lien, D. (2005a). A note on the superiority of the OLS hedge ratio. *Journal of Futures Markets, 25*(11), 1121-1126.

Lien, D. (2005b). The use and abuse of the hedging effectiveness measure. *International Review of Financial Analysis, 14*(2), 277-282.

Lien, D. (2008). A further note on the optimality of the OLS hedge strategy. *Journal of Futures Markets, 28*(3), 308-311.

Lindahl, M. (1989). Measuring hedging effectiveness with R2: A note. *Journal of Futures Markets, 9*(5), 469-475.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance, 7*(1), 77-91.

McDonald, R. L. (2013). *Derivatives* markets. Boston: Pearson Education.

Myers, R. J., & Thompson, S. R. (1989). Generalized optimal hedge ratio estimation. *American Journal of Agricultural Economics, 71*(4), 858-868.

Stein, J. L. (1961). The simultaneous determination of spot and futures prices. *The American Economic Review, 51*(5), 1012-1025.

Viswanath, P. V. (1993). Efficient use of information, convergence adjustments, and regression estimates of hedge ratios. *Journal of Futures Markets, 13*(1), 43-53.

Working, H. (1953). Futures Trading and Hedging. *The American Economic Review, 43*(3), 314-343.

Working, H. (1962). New concepts concerning futures markets and prices. *The American Economic Review, 52*(3), 431-459.

Appendix A: Tables

Table 1 Some measures of uncertainty and their characteristics

Measure (moment of forecast error distribution)	Value	Magnitude of uncertainty	Is informative of the magnitude of uncertainty	Minimizing the measure is equivalent to minimizing uncertainty
First moment, i.e. mathematical expectation	Close to zero	Unknown	**No**	**No**
	Far from zero	High	Yes	
First absolute moment	Close to zero	Low	Yes	Yes
	Far from zero	High	Yes	
Second moment	Close to zero	Low	Yes	Yes
	Far from zero	High	Yes	
Second central moment, i.e. variance	Close to zero	Unknown	**No***	**No***
	Far from zero	High	Yes	

Note: * except when the first moment is zero.

Table 2 Theoretical and empirical optimal hedge ratios under different assumptions on expected prices

Assumptions	Theoretical and empirical optimal hedge ratios
General case (no assumptions)	$\beta^*_{h,\text{ESFE}} = -\frac{\mathrm{E}_t(s_{t+h}f_{t+h}) - \hat{s}_{t+h\|t}\mathrm{E}_t(f_{t+h}) - \hat{f}_{t+h\|t}\mathrm{E}_t(s_{t+h}) + \hat{s}_{t+h\|t}\hat{f}_{t+h\|t}}{\mathrm{E}_t(f^2_{t+h}) - 2\hat{f}_{t+h\|t}\mathrm{E}_t(f_{t+h}) + \hat{f}^2_{t+h\|t}}$
	$\hat{\beta}^*_{h,\text{ESFE}} = -\frac{\widehat{\mathrm{E}}_t(s_{t+h}f_{t+h}) - \hat{s}_{t+h\|t}\hat{f}_{t+h\|t}}{\widehat{\mathrm{E}}_t(f^2_{t+h}) - \hat{f}^2_{t+h\|t}}$
Assumption 1	$\beta^*_{h,\text{Var}} = -\frac{\mathrm{E}_t(s_{t+h}f_{t+h}) - \mathrm{E}_t(s_{t+h})\mathrm{E}_t(f_{t+h})}{\mathrm{E}_t(f^2_{t+h}) - \left(\mathrm{E}_t(f_{t+h})\right)^2} = -\frac{\mathrm{Cov}_t(s_{t+h}, f_{t+h})}{\mathrm{Var}_t(f_{t+h})}$
	$\tilde{\beta}^*_{h,\text{Var}} = -\frac{\widehat{\mathrm{E}}_t(s_{t+h}f_{t+h}) - \mathrm{E}_t(s_{t+h})\mathrm{E}_t(f_{t+h})}{\widehat{\mathrm{E}}_t(f^2_{t+h}) - \left(\mathrm{E}_t(f_{t+h})\right)^2} = -\frac{\widetilde{\mathrm{Cov}}_t(s_{t+h}, f_{t+h})}{\widetilde{\mathrm{Var}}_t(f_{t+h})}$
Assumptions 1 and 2	$\beta^*_{h,\text{MSR}} = -\frac{\mathrm{E}_t(s_{t+h}f_{t+h}) - s_t f_t}{\mathrm{E}_t(f^2_{t+h}) - f^2_t}$
	$\hat{\beta}^*_{h,\text{MSR}} = -\frac{\widehat{\mathrm{E}}_t(s_{t+h}f_{t+h}) - s_t f_t}{\widehat{\mathrm{E}}_t(f^2_{t+h}) - f^2_t}$

Note: $\widetilde{\mathrm{Cov}}_t(s_{t+h}, f_{t+h}) \coloneqq \widehat{\mathrm{E}}_t(s_{t+h}f_{t+h}) - \mathrm{E}_t(s_{t+h})\mathrm{E}_t(f_{t+h})$ and $\widetilde{\mathrm{Var}}_t(f_{t+h}) \coloneqq \widehat{\mathrm{E}}_t\left(f^2_{t+h}\right) - \left(\mathrm{E}_t(f_{t+h})\right)^2$ are the estimators of covariance and variance, respectively, that employ the true first moments $\mathrm{E}_t(s_{t+h})$ of s_{t+h} and $\mathrm{E}_t(f_{t+h})$ of f_{t+h} rather than their respective sample counterparts $\widehat{\mathrm{E}}_t(s_{t+h})$ and $\widehat{\mathrm{E}}_t(f_{t+h})$.

Assumption 1. *The conditional expectations of s_{t+h} and p_{t+h}, denoted $\mu_{s,t+h} \coloneqq \mathrm{E}_t(s_{t+h})$ and $\mu_{p,t+h} \coloneqq \mathrm{E}_t(p_{t+h})$, are known as of time t and are used as point forecasts, $\hat{s}_{t+h|t} = \mu_{s,t+h}$ and $\hat{p}_{t+h|t} = \mu_{p,t+h}$.*

Assumption 2. *The conditional expectations of s_{t+h} and p_{t+h} equal the last observed values of s and p, respectively, $\mu_{s,t+h} = s_t$ and $\mu_{p,t+h} = p_t$.*

Table 3 Measures of hedging effectiveness under different assumptions on expected prices

Assumptions	Theoretical absolute measure	Empirical absolute measure	Theoretical relative measure	Empirical relative measure
General case (no assumptions)	Expected squared forecast error, $\mathrm{ESFE}_t(p_{t+h}) = \mathrm{E}_t\left(\left(p_{t+h} - \hat{p}_{t+h\|t}\right)^2\right)$	Mean squared forecast error, $\mathrm{MSFE}(p_{+h}) = \frac{1}{\mathrm{T}}\sum_{t=1}^{T}\left(p_{t+h} - \hat{p}_{t+h\|t}\right)^2$	Relative reduction in expected squared forecast error, $\mathrm{RRESFE}_t(p_{t+h}, s_{t+h}) = \frac{\mathrm{ESFE}_t(s_{t+h}) - \mathrm{ESFE}_t(p_{t+h})}{\mathrm{ESFE}_t(s_{t+h})}$	Relative reduction in mean squared forecast error, $\mathrm{RRMSFE}(p_{+h}, s_{+h}) = \frac{\mathrm{MSFE}(s_{+h}) - \mathrm{MSFE}(p_{+h})}{\mathrm{MSFE}(S_{+1})}$
Assumption 1	Variance*, $\mathrm{Var}_t(p_{t+h}) = \mathrm{E}_t\left(\left(p_{t+h} - \mu_{p,t+h}\right)^2\right)$	Empirical variance*, $\widetilde{\mathrm{Var}}(p_{+h}) = \frac{1}{T}\sum_{t=1}^{T}\left(p_{t+h} - \mu_{p,t+h}\right)^2$	Relative reduction in variance*, $\mathrm{RRV}_t(p_{t+h}, s_{t+h}) = \frac{\mathrm{Var}_t(s_{t+h}) - \mathrm{Var}_t(p_{t+h})}{\mathrm{Var}_t(s_{t+h})}$	Relative reduction in empirical variance*, $\widetilde{RRV}(p_{+h}, s_{+h}) = \frac{\widetilde{\mathrm{Var}}(s_{+h}) - \widetilde{\mathrm{Var}}(p_{+h})}{\widetilde{\mathrm{Var}}(s_{+h})}$
Assumptions 1 and 2	Expected squared return, $\mathrm{ESR}_t(p_{t+h}) = \mathrm{E}_t\left(\left(p_{t+h} - p_{t+h\|t}\right)^2\right)$	Mean squared return, $\mathrm{MSR}(p_{+h}) = \frac{1}{\mathrm{T}}\sum_{t=1}^{T}(p_{t+h} - p_t)^2$	Relative reduction in expected squared return, $\mathrm{RRESR}_t(p_{t+h}, s_{t+h}) = \frac{\mathrm{ESR}_t(s_{t+h}) - \mathrm{ESR}_t(p_{t+h})}{\mathrm{ESR}_t(s_{t+h})}$	Relative reduction in mean squared return, $\mathrm{RRMSR}(p_{+h}, s_{+h}) = \frac{\mathrm{MSR}(s_{+h}) - \mathrm{MSR}(p_{+h})}{\mathrm{MSR}(s_{+h})}$

Note: * denotes variance based on the true first moments $\mu_{p,t+h}$ for the time periods $t = 1, \ldots, T$. For Assumptions 1 and 2, see note under Table 2.

Table 4 Results of hedging the monthly spot price of oil (WTI) with oil futures contracts

Effectiveness measure	Spot	Futures	Portfolio 1:1	Portfolio OHR	Relative reduction 1:1	Relative reduction OHR
MSFE	13.41	27.11	7.52	3.52	0.44	0.74
Variance	22.44	27.10	13.10	10.89	0.42	0.51

Note: The hedging period is from June 1993 to August 2017. Rolling windows of 120 months are used for estimating the optimal hedge ratio. Hedging horizon is 1 month ahead. 1:1 denotes the naive 1:1 hedge ratio; OHR denotes the estimated optimal hedge ratio due to equation (9).

Table 5 Results of hedging the monthly spot price of natural gas (Henry Hub) with natural gas futures contracts

Effectiveness measure	Spot	Futures	Portfolio 1:1	Portfolio OHR	Relative reduction 1:1	Relative reduction OHR
MSFE	0.51	0.91	0.42	0.45	0.18	0.12
Variance	0.70	0.91	0.71	0.75	-0.02	-0.08

Note: The hedging period is from June 2000 to August 2017. Rolling windows of 120 months are used for estimating the optimal hedge ratio. Hedging horizon is 1 month ahead. 1:1 denotes the naive 1:1 hedge ratio; OHR denotes the estimated optimal hedge ratio due to equation (9).

Appendix B: Figures

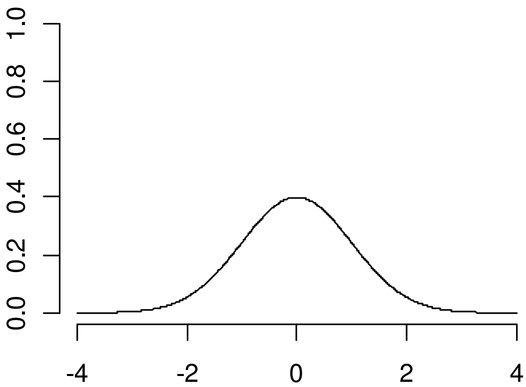
Figure 1 Illustrations of measures of uncertainty for different distributions of forecast error

a) **Low uncertainty**

Light-tailed distribution, symmetric around zero.

$\mathrm{E}_t(err_{t+h|t}) = 0$, $\mathrm{E}_t(err^2_{t+h|t})$ small,

$\mathrm{E}_t(|err_{t+h|t}|)$ small, $\mathrm{Var}_t(err_{t+h|t})$ small

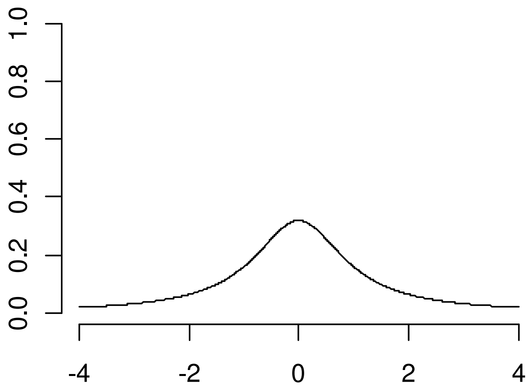


b) **High uncertainty**

Heavy-tailed distribution, symmetric around zero.

$\mathrm{E}_t(err_{t+h|t}) = 0$ (!), $\mathrm{E}_t(err^2_{t+h|t})$ large,

$\mathrm{E}_t(|err_{t+h|t}|)$ large, $\mathrm{Var}_t(err_{t+h|t})$ large

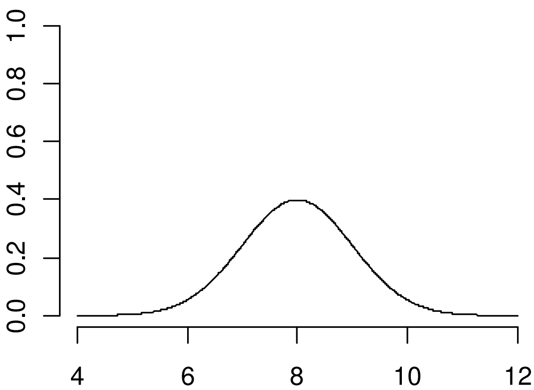


c) **High uncertainty**

Light-tailed distr., symmetric around a large value.

$\mathrm{E}_t(err_{t+h|t})$ large, $\mathrm{E}_t(err^2_{t+h|t})$ large

$\mathrm{E}_t(|err_{t+h|t}|)$ large, $\mathrm{Var}_t(err_{t+h|t})$ small (!)

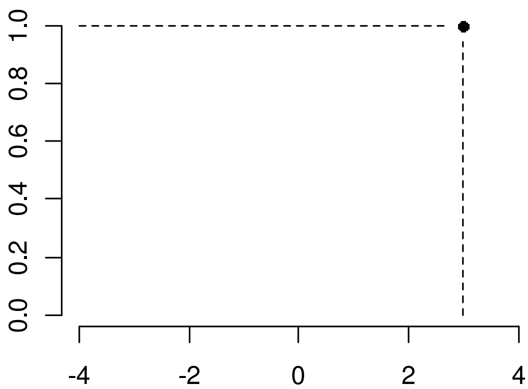


d) **High uncertainty**

Distribution with a unit mass at a large value.

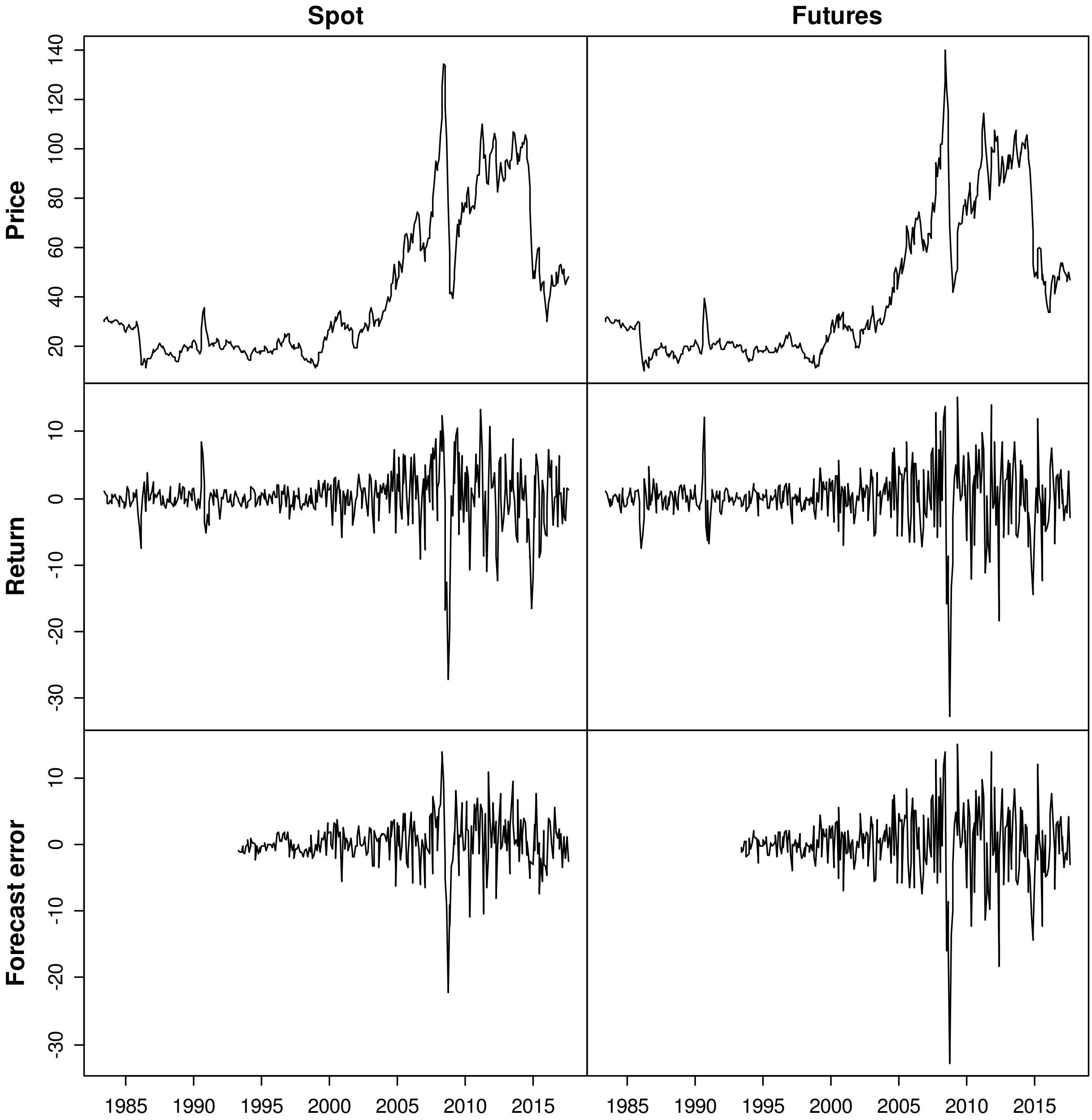
$\mathrm{E}_t(err_{t+h|t})$ large, $\mathrm{E}_t(err^2_{t+h|t})$ large,

$\mathrm{E}_t(|err_{t+h|t}|)$ large, $\mathrm{Var}_t(err_{t+h|t}) = 0$ (!)



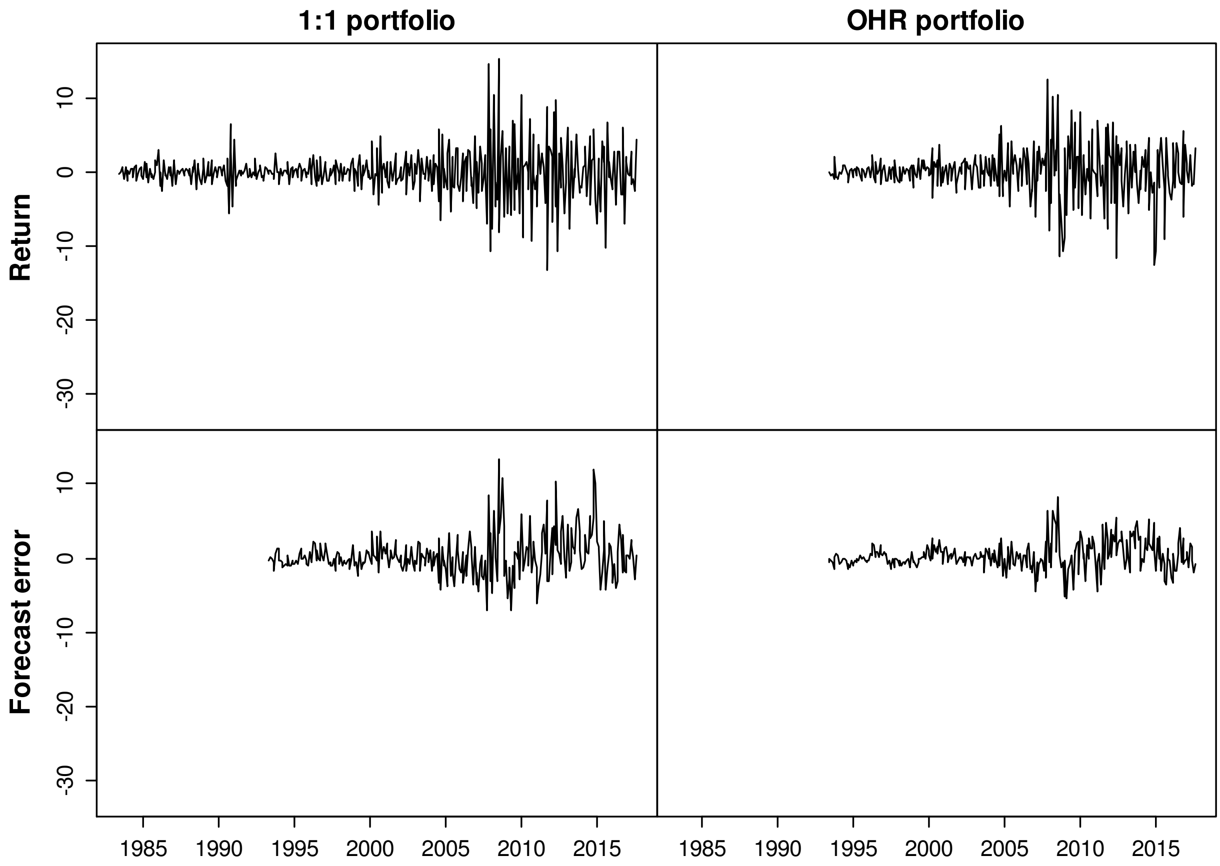
Note: $err_{t+h|t}$ denotes the forecast error resulting from a forecast made at time t for a target variable at time $t + h$. $\mathrm{E}_t(\cdot)$ and $\mathrm{Var}_t(\cdot)$ are the mathematical expectation and variance, respectively, conditional on the information available at time t. $|\cdot|$ is the absolute value operator.

Figure 2 Spot and futures prices, returns, and forecast errors of oil (WTI)



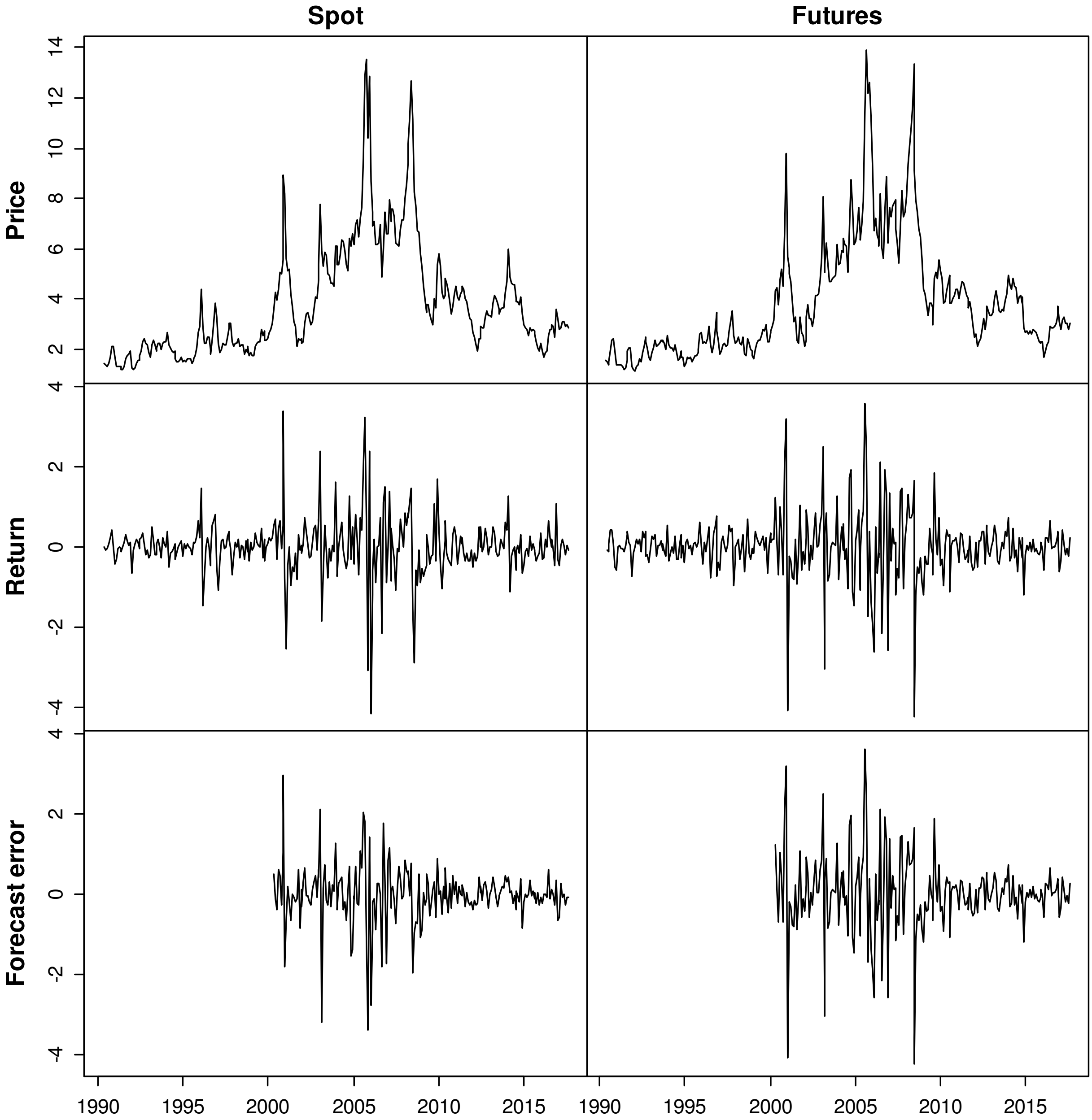
Note: Monthly data from May 1983 to August 2017 (412 data points). Return denotes price change.

Figure 3 Hedge portfolio returns and forecast errors of oil (WTI)



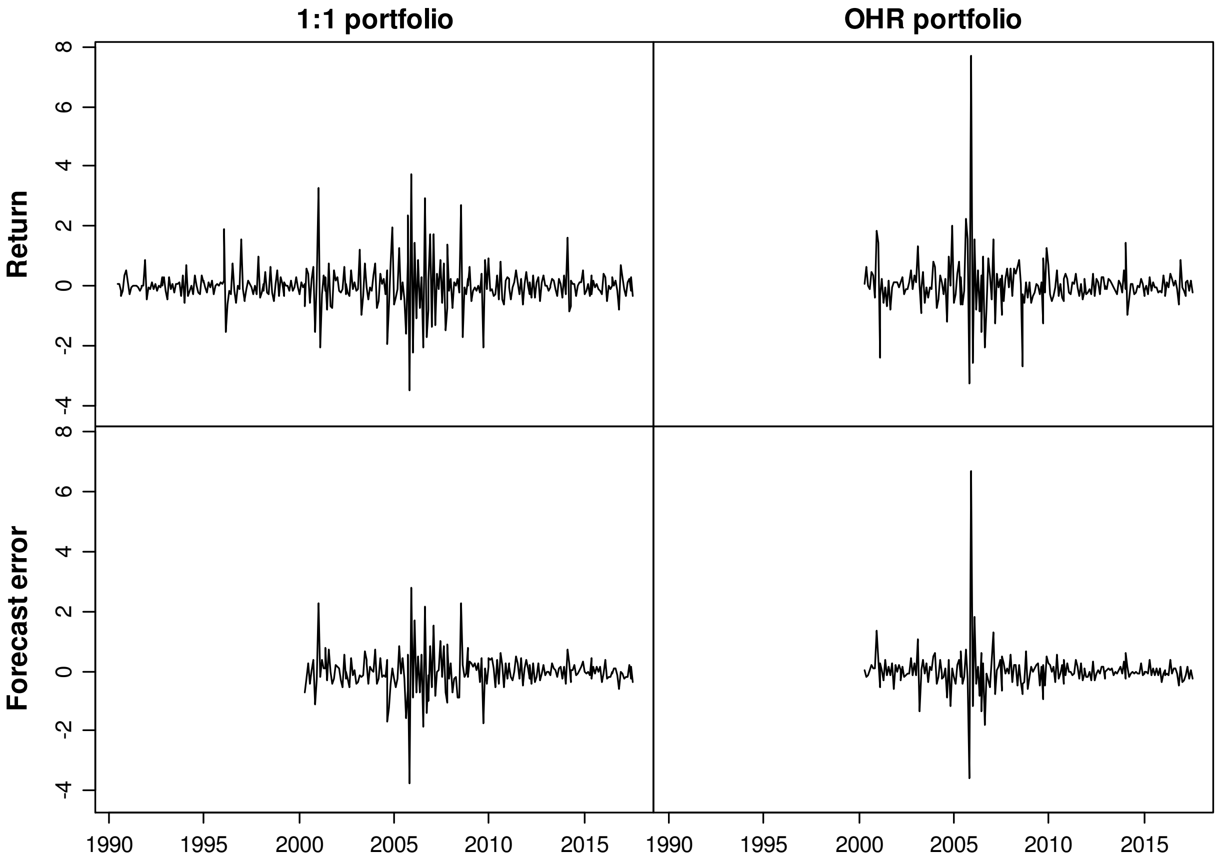
Note: Monthly data from May 1983 to August 2017 (412 data points). Return denotes price change; (1:1) denotes naïve 1:1 hedge ratio; (OHR) denotes estimated optimal hedge ratio due to equation (9).

Figure 4 Spot and futures prices, returns, and forecast errors of natural gas (Henry Hub)



Note: Monthly data from May 1990 to August 2017 (328 data points). Return denotes price change.

Figure 5 Hedge portfolio returns and forecast errors of natural gas (Henry Hub)



Note: Monthly data from May 1990 to August 2017 (328 data points). Return denotes price change; (1:1) denotes naïve 1:1 hedge ratio; (OHR) denotes estimated optimal hedge ratio due to equation (9).

Appendix C: Proofs

Derivation of the optimal hedge ratio in the general case: equation (7) → (8)

$$\frac{d\mathrm{ESFE}_t\big(p_{t+h}(\beta)\big)}{d\beta} = \frac{d}{d\beta}\mathrm{E}_t\left(\left(p_{t+h}(\beta,X) - \hat{p}_{t+h|t}(\beta,X)\right)^2\right) \tag{7}$$

$$= \frac{d}{d\beta}\mathrm{E}_t\left(\left((s_{t+h} + \beta f_{t+h}) - \left(\hat{s}_{t+h|t} + \beta\hat{f}_{t+h|t}\right)\right)^2\right)$$

$$= \frac{d}{d\beta}\mathrm{E}_t(s_{t+h}^2 + 2\beta s_{t+h}f_{t+h} - 2s_{t+h}\hat{s}_{t+h|t} - 2\beta s_{t+h}\hat{f}_{t+h|t} + \beta^2 f_{t+h}^2$$

$$-2\beta f_{t+h}\hat{s}_{t+h|t} - 2\beta^2 f_{t+h}\hat{f}_{t+h|t} + \hat{s}_{t+h}^2 + 2\beta\hat{s}_{t+h|t}\hat{f}_{t+h|t} + \beta^2\hat{f}_{t+h}^2)$$

$$= \frac{d}{d\beta}\mathrm{E}_t(2\beta s_{t+h}f_{t+h} - 2\beta s_{t+h}\hat{f}_{t+h|t} + \beta^2 f_{t+h}^2$$

$$-2\beta f_{t+h}\hat{s}_{t+h|t} - 2\beta^2 f_{t+h}\hat{f}_{t+h|t} + 2\beta\hat{s}_{t+h|t}\hat{f}_{t+h|t} + \beta^2\hat{f}_{t+h}^2)$$

$$= \mathrm{E}_t(2s_{t+h}f_{t+h}) - \mathrm{E}_t\left(2s_{t+h}\hat{f}_{t+h|t}\right) + \mathrm{E}_t\left(2\beta f_{t+h}^2\right)$$

$$- \mathrm{E}_t\left(2f_{t+h}\hat{s}_{t+h|t}\right) - \mathrm{E}_t\left(4\beta f_{t+h}\hat{f}_{t+h|t}\right) + \mathrm{E}_t\left(2\hat{s}_{t+h|t}\hat{f}_{t+h|t}\right) + \mathrm{E}_t\left(2\beta\hat{f}_{t+h}^2\right)$$

$$= 2\mathrm{E}_t(s_{t+h}f_{t+h}) - 2\hat{f}_{t+h|t}\mathrm{E}_t(s_{t+h}) + 2\beta\mathrm{E}_t\left(f_{t+h}^2\right) - 2\hat{s}_{t+h|t}\mathrm{E}_t(f_{t+h})$$

$$- 4\beta\hat{f}_{t+h|t}\mathrm{E}_t(f_{t+h}) + 2\hat{s}_{t+h|t}\hat{f}_{t+h|t} + 2\beta\hat{f}_{t+h}^2$$

$$= 0,$$

$$\mathrm{E}_t(s_{t+h}f_{t+h}) - \hat{f}_{t+h|t}\mathrm{E}_t(s_{t+h}) + \beta\mathrm{E}_t\left(f_{t+h}^2\right) - \hat{s}_{t+h|t}\mathrm{E}_t(f_{t+h}) - 2\beta\hat{f}_{t+h|t}\mathrm{E}_t(f_{t+h})$$

$$+ \hat{s}_{t+h|t}\hat{f}_{t+h|t} + \beta\hat{f}_{t+h}^2 = 0,$$

$$\beta\left(\mathrm{E}_t\left(f_{t+h}^2\right) - 2\hat{f}_{t+h|t}\mathrm{E}_t(f_{t+h}) + \hat{f}_{t+h|t}^2\right) + (\mathrm{E}_t(s_{t+h}f_{t+h}) - \hat{s}_{t+h|t}\mathrm{E}_t(f_{t+h}) - \hat{f}_{t+h|t}\mathrm{E}_t(s_{t+h})$$

$$+ \hat{s}_{t+h|t}\hat{f}_{t+h|t}) = 0,$$

$$\beta_{h,\mathrm{ESFE}}^* \coloneqq -\frac{\mathrm{E}_t(s_{t+h}f_{t+h}) - \hat{s}_{t+h|t}\mathrm{E}_t(f_{t+h}) - \hat{f}_{t+h|t}\mathrm{E}_t(s_{t+h}) + \hat{s}_{t+h|t}\hat{f}_{t+h|t}}{\mathrm{E}_t(f_{t+h}^2) - 2\hat{f}_{t+h|t}\mathrm{E}_t(f_{t+h}) + \hat{f}_{t+h|t}^2}. \tag{8}$$

* * *

Derivation of the theoretical optimal hedge ratio under Assumption 1: equation (8) → (14)

$$\beta_{h,\mathrm{ESFE}}^{*} = -\frac{\mathrm{E}_t(s_{t+h}f_{t+h}) - \hat{s}_{t+h|t}\mathrm{E}_t(f_{t+h}) - \hat{f}_{t+h|t}\mathrm{E}_t(s_{t+h}) + \hat{s}_{t+h|t}\hat{f}_{t+h|t}}{\mathrm{E}_t(f_{t+h}^2) - 2\hat{f}_{t+h|t}\mathrm{E}_t(f_{t+h}) + \hat{f}_{t+h|t}^2} \tag{8}$$

$$= -\frac{\mathrm{E}_t(s_{t+h}f_{t+h}) - \mathrm{E}_t(s_{t+h})\mathrm{E}_t(f_{t+h}) - \mathrm{E}_t(f_{t+h})\mathrm{E}_t(s_{t+h}) + \mathrm{E}_t(s_{t+h})\mathrm{E}_t(f_{t+h})}{\mathrm{E}_t(f_{t+h}^2) - 2\mathrm{E}_t(f_{t+h})\mathrm{E}_t(f_{t+h}) + \left(\mathrm{E}_t(f_{t+h})\right)^2}$$

$$= -\frac{\mathrm{E}_t(s_{t+h}f_{t+h}) - \mathrm{E}_t(s_{t+h})\mathrm{E}_t(f_{t+h})}{\mathrm{E}_t(f_{t+h}^2) - \left(\mathrm{E}_t(f_{t+h})\right)^2}$$

$$= -\frac{\mathrm{Cov}_t(s_{t+h}, f_{t+h})}{\mathrm{Var}_t(f_{t+h})}$$

$$=: \beta_{h,\mathrm{Var}}^{*}\,. \tag{14}$$

* * *

Derivation of the empirical optimal hedge ratio under Assumption 1: equation (9) → (15)

$$\hat{\beta}_{h,\mathrm{ESFE}}^{*} = -\frac{\widehat{\mathrm{E}}_t(s_{t+h}f_{t+h}) - \hat{s}_{t+h|t}\hat{f}_{t+h|t} - \hat{f}_{t+h|t}\hat{s}_{t+h|t} + \hat{s}_{t+h|t}\hat{f}_{t+h|t}}{\widehat{\mathrm{E}}_t(f_{t+h}^2) - 2\hat{f}_{t+h|t}\hat{f}_{t+h|t} + \hat{f}_{t+h|t}^2} \tag{9}$$

$$= -\frac{\widehat{\mathrm{E}}_t(s_{t+h}f_{t+h}) - \hat{s}_{t+h|t}\hat{f}_{t+h|t}}{\widehat{\mathrm{E}}_t(f_{t+h}^2) - \hat{f}_{t+h|t}^2}$$

$$= -\frac{\widehat{\mathrm{E}}_t(s_{t+h}f_{t+h}) - \mathrm{E}_t(s_{t+h})\mathrm{E}_t(f_{t+h})}{\widehat{\mathrm{E}}_t(f_{t+h}^2) - \left(\mathrm{E}_t(f_{t+h})\right)^2}$$

$$= -\frac{\widetilde{\mathrm{Cov}}_t(s_{t+h}, f_{t+h})}{\widetilde{\mathrm{Var}}_t(f_{t+h})}$$

$$=: \tilde{\beta}_{h,\mathrm{Var}}^{*} \tag{15}$$

where $\widetilde{\mathrm{Cov}}_t(s_{t+h}, f_{t+h}) \coloneqq \widehat{\mathrm{E}}_t(s_{t+h}f_{t+h}) - \mathrm{E}_t(s_{t+h})\mathrm{E}_t(f_{t+h})$ and $\widetilde{\mathrm{Var}}_t(f_{t+h}) \coloneqq \widehat{\mathrm{E}}_t(f_{t+h}^2) - \left(\mathrm{E}_t(f_{t+h})\right)^2$ are the estimators of covariance and variance, respectively, that employ the true first moments $\mathrm{E}_t(s_{t+h})$ of s_{t+h} and $\mathrm{E}_t(f_{t+h})$ of f_{t+h} rather than their respective sample counterparts $\widehat{\mathrm{E}}_t(s_{t+h})$ and $\widehat{\mathrm{E}}_t(f_{t+h})$.

* * *

Derivation of the theoretical optimal hedge ratio under Assumptions 1 and 2: equation (8) → (22)

$$\beta_{h,\mathrm{ESFE}}^{*} = -\frac{\mathrm{E}_t(s_{t+h}f_{t+h}) - \hat{s}_{t+h|t}\mathrm{E}_t(f_{t+h}) - \hat{f}_{t+h|t}\mathrm{E}_t(s_{t+h}) + \hat{s}_{t+h|t}\hat{f}_{t+h|t}}{\mathrm{E}_t(f_{t+h}^2) - 2\hat{f}_{t+h|t}\mathrm{E}_t(f_{t+h}) + \hat{f}_{t+h|t}^2} \tag{8}$$

$$= -\frac{\mathrm{E}_t(s_{t+h}f_{t+h}) - s_tf_t - s_tf_t + s_tf_t}{\mathrm{E}_t(f_{t+h}^2) - 2f_tf_t + f_t^2}$$

$$= -\frac{\mathrm{E}_t(s_{t+h}f_{t+h}) - s_tf_t}{\mathrm{E}_t(f_{t+h}^2) - f_t^2}$$

$$=: \beta_{h,\mathrm{MSR}}^* \,. \tag{22}$$

* * *

Derivation of the empirical optimal hedge ratio under Assumptions 1 and 2: equation (9) → (23)

$$\hat{\beta}_{h,\mathrm{ESFE}}^* = -\frac{\widehat{\mathrm{E}}_t(s_{t+h}f_{t+h}) - \hat{s}_{t+h|t}\hat{f}_{t+h|t} - \hat{f}_{t+h|t}\hat{s}_{t+h|t} + \hat{s}_{t+h|t}\hat{f}_{t+h|t}}{\widehat{\mathrm{E}}_t(f_{t+h}^2) - 2\hat{f}_{t+h|t}\hat{f}_{t+h|t} + \hat{f}_{t+h|t}^2} \tag{9}$$

$$= -\frac{\widehat{\mathrm{E}}_t(s_{t+h}f_{t+h}) - \hat{s}_{t+h|t}\hat{f}_{t+h|t}}{\widehat{\mathrm{E}}_t(f_{t+h}^2) - \hat{f}_{t+h|t}^2}$$

$$= -\frac{\widehat{\mathrm{E}}_t(s_{t+h}f_{t+h}) - s_tf_t}{\widehat{\mathrm{E}}_t(f_{t+h}^2) - f_t^2}$$

$$=: \hat{\beta}_{h,\mathrm{MSR}}^* . \tag{23}$$

* * *