Minimax theorem and Nash equilibrium of symmetric three-players zero-sum game with two strategic variables

Masahiko Hattori and Atsuhiro Satoh and Yasuhito Tanaka

26 March 2018

Minimax theorem and Nash equilibrium of symmetric three-players zero-sum game with two strategic variables

Masahiko Hattori*
Faculty of Economics, Doshisha University,
Kamigyo-ku, Kyoto, 602-8580, Japan,
Atsuhiro Satoh†
Faculty of Economics, Hokkai-Gakuen University,
Toyohira-ku, Sapporo, Hokkaido, 062-8605, Japan,
and
Yasuhito Tanaka‡
Faculty of Economics, Doshisha University,
Kamigyo-ku, Kyoto, 602-8580, Japan.

Abstract

We consider a symmetric three-players zero-sum game with two strategic variables. Three players are Players A, B and C. Two strategic variables are t_i and s_i, $i = A, B, C$. They are related by invertible functions. Using the minimax theorem by Sion (1958) and the fixed point theorem by Glicksberg (1952) we will show that Nash equilibria in the following four states are equivalent.

1. All players, Players A, B and C choose t_i, $i = A, B, C$, (as their strategic variables).
2. Two players choose t_i's, and one player chooses s_i.
3. One player chooses t_i, and two players choose s_i's.
4. All players, Players A, B and C choose s_i, $i = A, B, C$.

Keywords: symmetric three-person zero-sum game, Nash equilibrium, two strategic variables

JEL Classification: C72

---

*mhattori@mail.doshisha.ac.jp
†atsatoh@hgu.jp
‡yasuhito@mail.doshisha.ac.jp

1 Introduction

We consider a symmetric three-players zero-sum game with two strategic variables. Three players are Players A, B and C. Two strategic variables are t_i and s_i, $i = A, B, C$. They are related by invertible functions. Using the minimax theorem by Sion (1958) and the fixed point theorem by Glicksberg (1952) we will show that Nash equilibria in the following four states are equivalent.

1. All players, Players A, B and C choose t_i, $i = A, B, C$, (as their strategic variables).

2. Two players choose t_i's, and one player chooses s_i.

3. One player chooses t_i, and two players choose s_i's.

4. All players, Players A, B and C choose s_i, $i = A, B, C$.

In the next section we present a model of this paper and prove some preliminary results which are variations of Sion's minimax theorem. In Section 3 we will show the main results. An example of three-players zero-sum game is a relative profit maximization game in a three firms oligopoly with differentiated goods. See Section 4.

2 The model and the minimax theorem

We consider a symmetric three-players zero-sum game with two strategic variables. Three players are Players A, B and C. Two strategic variables are t_i and s_i, $i = A, B, C$. t_i is chosen from T_i and s_i is chosen from S_i. T_i and S_i are convex and compact sets in linear topological spaces, respectively, for each $i \in \{A, B, C\}$. The relations of the strategic variables are represented by

$$s_i = f_i(t_A, t_B, t_C),\ i = A, B, C,$$

and

$$t_i = g_i(s_A, s_B, s_C),\ i = A, B, C.$$

(f_A, f_B, f_C) and (g_A, g_B, g_C) are continuous invertible functions, and so they are one-to-one and onto functions. When one of the players, for example, Player C chooses s_C, t_C is determined according to

$$t_C = g_C(f_A(t_A, t_B, t_C), f_B(t_A, t_B, t_C), s_C).$$

We denote this t_C by $t_C(t_A, t_B, s_C)$.

When two players, for example, Player B and C choose s_B and s_C, t_B and t_C are determined according to

$$\begin{cases} t_B = g_B(f_A(t_A, t_B, t_C), s_B, s_C) \\ t_C = g_C(f_A(t_A, t_B, t_C), s_B, s_C). \end{cases}$$

We denote these t_B and t_C by $t_B(t_A, s_B, s_C)$ and $t_C(t_A, s_B, s_C)$.

When all players choose s_A, s_B and s_C, t_A, t_B and t_C are determined according to

$$t_A = g_A(s_A, s_B, s_C),\ t_B = g_B(s_A, s_B, s_C),\ t_C = g_C(s_A, s_B, s_C).$$

Denote these t_A, t_B and t_C by $t_A(s_A, s_B, s_C)$, $t_B(s_A, s_B, s_C)$ and $t_C(s_A, s_B, s_C)$.

The payoff function of Player i is $u_i,\ i = A, B, C$. It is written as

$$u_i(t_A, t_B, t_C).$$

We assume

> u_i for each $i \in \{A, B, C\}$ is continuous on $T_1 \times T_2 \times T_3$. Thus, it is continuous on $S_1 \times S_2 \times S_3$ through $f_i,\ i = A, B, C$. It is quasi-concave on T_i and S_i for a strategy of each other player, and quasi-convex on $T_j,\ j \neq i$ and $S_j,\ j \neq i$ for each t_i and s_i.

We do not assume differentiability of the payoff functions.

Symmetry of the game means that the payoff functions of all players are symmetric and in the payoff function of each Player i, Players j and $k,\ j, k \neq i$, are interchangeable. f_A, f_B and f_C are symmetric, and g_A, g_B and g_C are also symmetric. Since the game is a zero-sum game, the sum of the values of the payoff functions of the players is zero. All T_i's are identical, and all S_i's are identical. Denote them by T and S.

Sion's minimax theorem (Sion (1958), Komiya (1988), Kindler (2005)) for a continuous function is stated as follows.

Lemma 1. *Let X and Y be non-void convex and compact subsets of two linear topological spaces, and let $f : X \times Y \to \mathbb{R}$ be a function that is continuous and quasi-concave in the first variable and continuous and quasi-convex in the second variable. Then*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

We follow the description of Sion's theorem in Kindler (2005).

Applying this lemma to the situation of this paper, we have the following relations.

$$\max_{t_A \in T} \min_{t_B \in T} u_A(t_A, t_B, t_C) = \min_{t_B \in T} \max_{t_A \in T} u_A(t_A, t_B, t_C),\ \max_{t_B \in T} \min_{t_A \in T} u_B(t_A, t_B, t_C) = \min_{t_A \in T} \max_{t_B \in T} u_B(t_A, t_B, t_C).$$

$$\max_{t_A \in T} \min_{t_B \in T} u_A(t_A, t_B, t_C(t_A, t_B, s_C)) = \min_{t_B \in T} \max_{t_A \in T} u_A(t_A, t_B, t_C(t_A, t_B, s_C)),$$
$$\max_{t_B \in T} \min_{t_A \in T} u_B(t_A, t_B, t_C(t_A, t_B, s_C)) = \min_{t_A \in T} \max_{t_B \in T} u_B(t_A, t_B, t_C(t_A, t_B, s_C)).$$

$$\max_{s_C\in S}\min_{t_A\in T} u_C(t_A,t_B,t_C(t_A,t_B,s_C)) = \min_{t_A\in T}\max_{s_C\in S} u_C(t_A,t_B,t_C(t_A,t_B,s_C)),$$
$$\max_{t_A\in T}\min_{s_C\in S} u_A(t_A,t_B,t_C(t_A,t_B,s_C)) = \min_{s_C\in S}\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C)).$$

$$\max_{s_C\in S}\min_{t_A\in T} u_C(t_A,t_B(t_A,s_B,s_C),t_C(t_A,s_B,s_C)) = \min_{t_A\in T}\max_{s_C\in S} u_C(t_A,t_B(t_A,s_B,s_C),t_C(t_A,s_B,s_C)),$$
$$\max_{t_A\in T}\min_{s_C\in S} u_A(t_A,t_B(t_A,s_B,s_C),t_C(t_A,s_B,s_C)) = \min_{s_C\in S}\max_{t_A\in T} u_A(t_A,t_B(t_A,s_B,s_C),t_C(t_A,s_B,s_C)).$$

$$\max_{s_C\in S}\min_{t_B\in T} u_C(t_A,t_B,t_C(t_A,t_B,s_C)) = \min_{t_B\in T}\max_{s_C\in S} u_C(t_A,t_B,t_C(t_A,t_B,s_C)),$$
$$\max_{t_B\in T}\min_{s_C\in S} u_B(t_A,t_B,t_C(t_A,t_B,s_C)) = \min_{s_C\in S}\max_{t_B\in T} u_B(t_A,t_B,t_C(t_A,t_B,s_C)).$$

$$\max_{s_C\in S}\min_{t_B\in T} u_C(t_A(s_A,t_B,s_C),t_B,t_C(s_A,t_B,s_C)) = \min_{t_B\in T}\max_{s_C\in S} u_C(t_A(s_A,t_B,s_C),t_B,t_C(s_A,t_B,s_C)),$$
$$\max_{t_B\in T}\min_{s_C\in S} u_B(t_A(s_A,t_B,s_C),t_B,t_C(s_A,t_B,s_C)) = \min_{s_C\in S}\max_{t_B\in T} u_B(t_A(s_A,t_B,s_C),t_B,t_C(s_A,ts_B,s_C)).$$

$$\max_{t_A\in T}\min_{s_B\in S} u_A(t_A,t_B(t_A,s_B,s_C),t_C(t_A,s_B,s_C)) = \min_{s_B\in S}\max_{t_A\in T} u_A(t_A,t_B(t_A,s_B,s_C),t_C(t_A,s_B,s_C)),$$
$$\max_{s_B\in S}\min_{t_A\in T} u_B(t_A,t_B(t_A,s_B,s_C),t_C(t_A,s_B,s_C)) = \min_{t_A\in T}\max_{s_B\in S} u_B(t_A,t_B(t_A,s_B,s_C),t_C(t_A,s_B,s_C)).$$

Also, relations which are symmetric to them hold.

Further we show the following results.

Lemma 2.

$$\max_{t_C\in T}\min_{t_A\in T} u_C(t_A,t_B,t_C) = \max_{s_C\in S}\min_{t_A\in T} u_C(t_A,t_B,t_C(t_A,t_B,s_C))$$
$$= \min_{t_A\in T}\max_{s_C\in S} u_C(t_A,t_B,t_C(t_A,t_B,s_C)) = \min_{t_A\in T}\max_{t_C\in T} u_C(t_A,t_B,t_C),$$

and

$$\max_{t_C\in S}\min_{t_B\in T} u_C(t_A,t_B,t_C) = \max_{s_C\in S}\min_{t_B\in T} u_C(t_A,t_B,t_C(t_A,t_B,s_C))$$
$$= \min_{t_B\in T}\max_{s_C\in S} u_C(t_A,t_B,t_C(t_A,t_B,s_C)) = \min_{t_B\in T}\max_{t_C\in S} u_C(t_A,t_B,t_C).$$

Proof. $\min_{t_A\in T} u_C(t_A,t_B,t_C(t_A,t_B,s_C))$ is the minimum of u_C with respect to t_A given t_B and s_C. Let $\tilde{t}_A(s_C) = \arg\min_{t_A\in T} u_C(t_A,t_B,t_C(t_A,t_B,s_C))$, and fix the value of t_C at

$$t_C^0 = g_C(f_A(\tilde{t}_A(s_C),t_B,t_C^0), f_B(\tilde{t}_A(s_C),t_B,t_C^0), s_C). \tag{1}$$

Then, we have

$$\min_{t_A\in T} u_C(t_A,t_B,t_C^0) \le u_C(\tilde{t}_A(s_C),t_B,t_C^0) = \min_{t_A\in T} u_C(t_A,t_B,t_C(t_A,t_B,s_C)),$$

where $\min_{t_A \in T} u_C(t_A, t_B, t_C^0)$ is the minimum of u_C with respect to t_A given the value of t_C at t_C^0. We assume that $\tilde{t}_A(s_C) = \arg\min_{t_A \in T} u_C(t_A, t_B, t_C(t_A, t_B, s_C))$ is single-valued. By the maximum theorem and continuity of u_C, $\tilde{t}_A(s_C)$ is continuous. Then, any value of t_C^0 can be realized by appropriately choosing s_C given t_B according to (1). Therefore,

$$\max_{t_C \in T} \min_{t_A \in T} u_C(t_A, t_B, t_C) \leq \max_{s_C \in S} \min_{t_A \in T} u_C(t_A, t_B, t_C(t_A, t_B, s_C)). \tag{2}$$

On the other hand, $\min_{t_A \in T} u_C(t_A, t_B, t_C)$ is the minimum of u_C with respect to t_A given t_B and t_C. Let $\tilde{t}_A(t_C) = \arg\min_{t_A \in T} u_C(t_A, t_B, t_C)$, and fix the value of s_C at

$$s_C^0 = f_C(\tilde{t}_A(t_C), t_B, t_C). \tag{3}$$

Then, we have

$$\min_{t_A \in T} u_C(t_A, t_B, t_C(t_A, t_B, s_C^0)) \leq u_C(\tilde{t}_A(t_C), t_B, t_C(t_A, t_B, s_C^0)) = \min_{t_A \in T} u_C(t_A, t_B, t_C),$$

where $\min_{t_A \in T} u_C(t_A, t_B, t_C(t_A, t_B, s_C^0))$ is the minimum of u_C with respect to t_A given the value of s_C at s_C^0. We assume that $\tilde{t}_A(t_C) = \arg\min_{t_A \in T} u_C(t_A, t_B, t_C)$ is single-valued. By the maximum theorem and continuity of u_C, $\tilde{t}_A(t_C)$ is continuous. Then, any value of s_C^0 can be realized by appropriately choosing t_C given t_B according to (3). Therefore,

$$\max_{s_C \in S} \min_{t_A \in T} u_C(t_A, t_B, t_C(t_A, t_B, s_C))) \leq \max_{t_C \in S} \min_{t_A \in T} u_C(t_A, t_B, t_C). \tag{4}$$

Combining (2) and (4), we get

$$\max_{s_C \in S} \min_{t_A \in T} u_C(t_A, t_B, t_C(t_A, t_B, s_C)) = \max_{t_C \in T} \min_{t_A \in T} u_C(t_A, t_B, t_C).$$

Since any value of s_C can be realized by appropriately choosing t_C given t_A and t_B, we have

$$\max_{s_C \in S} u_C(t_A, t_B, t_C(t_A, t_B, s_C)) = \max_{t_C \in T} u_C(t_A, t_B, t_C).$$

Thus,

$$\min_{t_A \in T} \max_{s_C \in S} u_C(t_A, t_B, t_C(t_A, t_B, s_C)) = \min_{t_A \in T} \max_{t_C \in T} u_C(t_A, t_B, t_C).$$

Therefore,

$$\max_{t_C \in T} \min_{t_A \in T} u_C(t_A, t_B, t_C) = \max_{s_C \in S} \min_{t_A \in T} u_C(t_A, t_B, t_C(t_A, t_B, s_C))$$
$$= \min_{t_A \in T} \max_{s_C \in S} u_C(t_A, t_B, t_C(t_A, t_B, s_C)) = \min_{t_A \in T} \max_{t_C \in T} u_C(t_A, t_B, t_C),$$

given t_B.

By similar procedures, we can show

$$\max_{t_C \in T} \min_{t_B \in T} u_C(t_A, t_B, t_C) = \max_{s_C \in S} \min_{t_B \in T} u_C(t_A, t_B, t_C(t_A, t_B, s_C))$$
$$= \min_{t_B \in T} \max_{s_C \in S} u_C(t_A, t_B, t_C(t_A, t_B, s_C)) = \min_{t_B \in T} \max_{t_C \in T} u_C(t_A, t_B, t_C),$$

given t_A. $\square$

Lemma 3.

$$\min_{t_C\in T}\max_{t_A\in T} u_A(t_A,t_B,t_C) = \min_{s_C\in S}\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C))$$
$$= \max_{t_A\in T}\min_{s_C\in S} u_A(t_A,t_B,t_C(t_A,t_B,s_C)) = \max_{t_A\in T}\min_{t_C\in T} u_A(t_A,t_B,t_C),$$

and

$$\min_{t_C\in T}\max_{t_B\in T} u_B(t_A,t_B,t_C) = \min_{s_C\in S}\max_{t_B\in T} u_B(t_A,t_B,t_C(t_A,t_B,s_C))$$
$$= \max_{t_B\in T}\min_{s_C\in S} u_B(t_A,t_B,t_C(t_A,t_B,s_C)) = \max_{t_B\in T}\min_{t_C\in T} u_B(t_A,t_B,t_C).$$

Proof. $\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C))$ is the maximum of u_A with respect to t_A given t_B and s_C. Let $\tilde{t}_A(s_C) = \arg\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C))$, and fix the value of t_C at

$$t_C^0 = g_C(f_A(\tilde{t}_A(s_C),t_B,t_C^0), f_B(\tilde{t}_A(s_C),t_B,t_C^0), t_C(t_A,t_B,s_C)). \tag{5}$$

Then, we have

$$\max_{t_A\in T} u_A(t_A,t_B,t_C^0) \geq u_A(\tilde{t}_A(s_C),t_B,t_C^0) = \max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C)),$$

where $\max_{t_A\in T} u_A(t_A,t_B,t_C^0)$ is the maximum of u_A with respect to t_A given the value of t_C at t_C^0. We assume that $\tilde{t}_A(s_C) = \arg\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C))$ is single-valued. By the maximum theorem and continuity of u_A, $\tilde{t}_A(s_C)$ is continuous. Then, any value of t_C^0 can be realized by appropriately choosing s_C given t_B according to (5). Therefore,

$$\min_{t_C\in T}\max_{t_A\in T} u_A(t_A,t_B,t_C) \geq \min_{s_C\in S}\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C)). \tag{6}$$

On the other hand, $\max_{t_A\in T} u_A(t_A,t_B,t_C)$ is the maximum of u_A with respect to t_A given t_B and t_C. Let $\tilde{t}_A(t_C) = \arg\max_{t_A\in T} u_A(t_A,t_B,t_C)$, and fix the value of s_C at

$$s_C^0 = f_C(\tilde{t}_A(t_C),t_B,t_C). \tag{7}$$

Then, we have

$$\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C^0)) \geq u_A(\tilde{t}_A(s_C),t_B,t_C(t_A,t_B,s_C^0)) = \max_{t_A\in T} u_A(t_A,t_B,t_C),$$

where $\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C^0))$ is the maximum of u_A with respect to t_A given the value of s_C at s_C^0. We assume that $\tilde{t}_A(t_C) = \arg\max_{t_A\in T} u_A(t_A,t_B,t_C)$ is single-valued. By the maximum theorem and continuity of u_A, $\tilde{t}_A(t_C)$ is continuous. Then, any value of s_C^0 can be realized by appropriately choosing t_C given t_B according to (7). Therefore,

$$\min_{s_C\in S}\max_{t_A\in T} u_A(t_A,t_B,t_C(t_A,t_B,s_C)) \geq \min_{t_C\in S}\max_{t_A\in T} u_A(t_A,t_B,t_C). \tag{8}$$

Combining (6) and (8), we get

$$\min_{s_C \in S} \max_{t_A \in T} u_A(t_A, t_B, t_C(t_A, t_B, s_C)) = \min_{t_C \in S} \max_{t_A \in T} u_A(t_A, t_B, t_C).$$

Since any value of s_C can be realized by appropriately choosing t_C given t_A and t_B, we have

$$\min_{s_C \in S} u_A(t_A, t_B, t_C(t_A, t_B, s_C)) = \min_{t_C \in S} u_A(t_A, t_B, t_C).$$

Thus,

$$\max_{t_A \in T} \min_{s_C \in S} u_A(t_A, t_B, t_C(t_A, t_B, s_C)) = \max_{t_A \in T} \min_{t_C \in S} u_A(t_A, t_B, t_C).$$

Therefore,

$$\begin{aligned}\min_{t_C \in T} \max_{t_A \in T} u_A(t_A, t_B, t_C) &= \min_{s_C \in S} \max_{t_A \in T} u_A(t_A, t_B, t_C(t_A, t_B, s_C)),\\ = \max_{t_A \in T} \min_{s_C \in S} u_A(t_A, t_B, t_C(t_A, t_B, s_C)) &= \max_{t_A \in T} \min_{t_C \in T} u_A(t_A, t_B, t_C),\end{aligned}$$

given t_B.

By similar procedures, we can show

$$\begin{aligned}\min_{t_C \in T} \max_{t_B \in T} u_B(t_A, t_B, t_C) &= \min_{s_C \in S} \max_{t_B \in T} u_B(t_A, t_B, t_C(t_A, t_B, s_C)),\\ = \max_{t_B \in T} \min_{s_C \in S} u_B(t_A, t_B, t_C(t_A, t_B, s_C)) &= \max_{t_B \in T} \min_{t_C \in T} u_B(t_A, t_B, t_C),\end{aligned}$$

given t_A. □

Similarly, we obtain the following results.

Lemma 4.

$$\begin{aligned}&\max_{t_C \in T} \min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, t_C), t_C) = \max_{s_C \in S} \min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C))\\ &= \min_{t_A \in T} \max_{s_C \in S} u_C(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C)) = \min_{t_A \in T} \max_{t_C \in T} u_C(t_A, t_B(t_A, s_B, t_C), t_C),\end{aligned}$$

and

$$\begin{aligned}&\max_{t_C \in T} \min_{t_B \in T} u_C(t_A(s_A, t_B, t_C), t_B, t_C) = \max_{s_C \in S} \min_{t_B \in T} u_C(t_A(s_A, t_B, s_C), t_B, t_C(s_A, t_B, s_C))\\ &= \min_{t_B \in T} \max_{s_C \in S} u_C(t_A(s_A, t_B, s_C), t_B, t_C(s_A, t_B, s_C)) = \min_{t_B \in T} \max_{t_C \in T} u_C(t_A(s_A, t_B, t_C), t_B, t_C).\end{aligned}$$

Proof. See Appendix A. □

Lemma 5.

$$\begin{aligned}&\min_{t_C \in T} \max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, t_C), t_C) = \min_{s_C \in S} \max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C))\\ &= \max_{t_A \in T} \min_{s_C \in S} u_A(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C)) = \max_{t_A \in T} \min_{t_C \in T} u_A(t_A, t_B(t_A, s_B, t_C), t_C),\end{aligned}$$

and

$$\min_{t_C\in T}\max_{t_B\in T} u_B(t_A(s_A,t_B,t_C),t_B,t_C) = \min_{s_C\in S}\max_{t_B\in T} u_B(t_A(s_A,t_B,s_C),t_B,t_C(s_A,t_B,s_C))$$
$$= \max_{t_B\in T}\min_{s_C\in S} u_B(t_A(s_A,t_B,s_C),t_B,t_C(s_A,t_B,s_C)) = \max_{t_B\in T}\min_{t_C\in T} u_B(t_A(s_A,t_B,t_C),t_B,t_C).$$

Proof. See Appendix B. □

Also, relations which are symmetric to these lemmas hold.

3 The main results

In this section we present the main results of this paper. First we show

Theorem 1. *The equilibrium where all players choose t_i's is equivalent to the equilibrium where one player (Player C) chooses s_C and two players (Players A and B) choose t_i's as their strategic variables.*

Proof. 1. Consider a situation $(t_A, t_B, t_C) = (t, t, t)$. Let

$$s^0(t) = f_C(t, t, t).$$

By symmetry of the game

$$\max_{t_A\in T} u_A(t_A, t, t) = \max_{t_B\in T} u_B(t, t_B, t) = \max_{t_C\in T} u_C(t, t, t_C),$$

and

$$\arg\max_{t_A\in T} u_A(t_A, t, t) = \arg\max_{t_B\in T} u_B(t, t_B, t) = \arg\max_{t_C\in T} u_C(t, t, t_C) \in T.$$

Consider the following function.

$$t \to \arg\max_{t_A\in T} u_A(t_A, t, t).$$

Since this function is continuous and T is compact, there exists a fixed point. Denote it by t^*. Then,

$$t^* \to \arg\max_{t_A\in T} u_A(t_A, t^*, t^*).$$

We have

$$\max_{t_A\in T} u_A(t_A, t^*, t^*) = u_A(t^*, t^*, t^*) = \max_{t_B\in T} u_B(t^*, t_B, t^*) = u_B(t^*, t^*, t^*) = \max_{t_C\in T} u_C(t^*, t^*, t_C) = u_C(t^*, t^*, t^*$$

2. Because the game is zero-sum,

$$u_A(t_A, t^*, t^*) + u_B(t_A, t^*, t^*) + u_C(t_A, t^*, t^*) = 0.$$

By symmetry $u_B(t_A, t^*, t^*) = u_C(t_A, t^*, t^*)$. Thus,

$$u_A(t_A, t^*, t^*) + 2u_C(t_A, t^*, t^*) = 0.$$

This means

$$u_A(t_A, t^*, t^*) = -2u_C(t_A, t^*, t^*),$$

and

$$\max_{t_A \in T} u_A(t_A, t^*, t^*) = -2 \min_{t_A \in T} u_C(t_A, t^*, t^*).$$

From this we get

$$\arg\max_{t_A \in T} u_A(t_A, t^*, t^*) = \arg\min_{t_A \in T} u_C(t_A, t^*, t^*) = t^*.$$

By symmetry of the game

$$\arg\max_{t_A \in T} u_A(t_A, t^*, t^*) = \arg\min_{t_C \in T} u_A(t^*, t^*, t_C) = t^*.$$

We have

$$\max_{t_A \in T} u_A(t_A, t^*, t^*) = \min_{t_C \in T} u_A(t^*, t^*, t_C) = u_A(t^*, t^*, t^*) = 0.$$

Then,

$$\min_{t_C \in T} \max_{t_A \in T} u_A(t_A, t^*, t_C) \leq \max_{t_A \in T} u_A(t_A, t^*, t^*) = \min_{t_C \in T} u_A(t^*, t^*, t_C) \leq \max_{t_A \in T} \min_{t_C \in T} u_A(t_A, t^*, t_C).$$

From Lemma 3 we obtain

$$\min_{t_C \in T} \max_{t_A \in T} u_A(t_A, t^*, t_C) = \max_{t_A \in T} u_A(t_A, t^*, t^*) = \min_{t_C \in T} u_A(t^*, t^*, t_C) \quad (9)$$
$$= \max_{t_A \in T} \min_{t_C \in T} u_A(t_A, t^*, t_C) = \min_{s_C \in S} \max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s_C)) = \max_{t_A \in T} \min_{s_C \in T} u_A(t_A, t^*, t_C(t_A, t^*, s_C)) = 0$$

3. Since any value of s_C can be realized by appropriately choosing t_C,

$$\min_{s_C \in S} u_A(t^*, t^*, t_C(t^*, t^*, s_C)) = \min_{t_C \in T} u_A(t^*, t^*, t_C) = u_A(t^*, t^*, t^*) = 0. \quad (10)$$

Then,

$$\arg\min_{s_C \in S} u_A(t^*, t^*, t_C(t^*, t^*, s_C)) = s^0(t^*).$$

(9) and (10) mean

$$\min_{s_C \in S} \max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s_C)) = \min_{s_C \in S} u_A(t^*, t^*, t_C(t^*, t^*, s_C)) = 0. \quad (11)$$

And we have

$$\max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s_C)) \geq u_A(t^*, t^*, t_C(t^*, t^*, s_C)).$$

Then,

$$\arg\min_{s_C \in S} \max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s_C)) = \arg\min_{s_C \in S} u_A(t^*, t^*, t_C(t^*, t^*, s_C)) = s^0(t^*)$$

Thus, by (11)

$$\min_{s_C \in S} \max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s_C)) = \max_{t_A \in T} u_A(t_A, t^*, t_C(t^*, t^*, s^0(t^*))) = u_A(t^*, t^*, t_C(t^*, t^*, s^0(t^*))) = 0.$$

Therefore,

$$\arg\max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) = t^*. \tag{12}$$

By symmetry of the game,

$$\arg\max_{t_B \in T} u_B(t^*, t_B, t_C(t^*, t_B, s^0(t^*))) = t^*. \tag{13}$$

On the other hand, because any value of s_C is realized by appropriately choosing t_C,

$$\max_{s_C \in S} u_C(t^*, t^*, t_C(t^*, t^*, s_C)) = \max_{t_C \in T} u_C(t^*, t^*, t_C) = u_C(t^*, t^*, t^*) = 0.$$

Therefore,

$$\arg\max_{s_C \in S} u_C(t^*, t^*, t_C(t^*, t^*, s_C)) = s^0(t^*) = f_C(t^*, t^*, t^*). \tag{14}$$

From (12), (13) and (14), $(t^*, t^*, t_C(t^*, t^*, s^0(t^*)))$ is a Nash equilibrium which is equivalent to (t^*, t^*, t^*).

□

Now we assume

Assumption 1. *At the equilibrium such that $t_A = t_B = t^*$ and $s_C = s^0(t^*)$, where $t_C = t^*$, the responses of u_B and u_C to a small change in t_A have the same sign.*

u_A is maximized at $t_A = t^*$ given $t_B = t^*$ and $s_C = s^0(t^*)$.

Using this assumption we show the following result.

Theorem 2. *The equilibrium where all players choose t_i's is equivalent to the equilibrium where one player (Player A) chooses t_A and two players (Players B and C) choose s_B and s_C.*

Proof. By Theorem 1

$$\arg\max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) = \arg\max_{t_B \in T} u_B(t^*, t_B, t_C(t^*, t_B, s^0(t^*))) = t^*,$$

$$\arg\max_{s_C \in S} u_C(t^*, t^*, t_C(t^*, t^*, s_C)) = s^0(t^*).$$

Since any value of t_B is realized by appropriately choosing s_B, we get

$$\max_{s_B \in S} u_B(t^*, t_B(t^*, s_B, s^0(t^*)), t_C(t^*, s_B, s^0(t^*))) = \max_{t_B \in T} u_B(t^*, t_B, t_C(t^*, t_B, s^0(t^*))) = u_B(t^*, t^*, t_C(t^*, t_B, s^0(t^*)$$

and

$$\arg\max_{s_B \in S} u_B(t^*, t_B(t^*, s_B, s^0(t^*)), t_C(t^*, s_B, s^0(t^*))) = s^0(t^*). \tag{15}$$

By symmetry

$$\max_{s_C \in S} u_C(t^*, t_B(t^*, s^0(t^*), s_C), t_C(t^*, s^0(t^*), s_C)) = \max_{s_C \in S} u_C(t^*, t^*, t_C(t^*, t^*, s_C)),$$

and

$$\arg\max_{s_C \in S} u_C(t^*, t_B(t^*, s^0(t^*), s_C), t_C(t^*, s^0(t^*), s_C)) = s^0(t^*). \tag{16}$$

Since the game is zero-sum,

$$u_A(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) + u_B(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) + u_C(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) = 0,$$

and so

$$u_A(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) = -(u_B(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) + u_C(t_A, t^*, t_C(t_A, t^*, s^0(t^*)))).$$

Thus,

$$\max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) = -\min_{t_A \in T}[u_B(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) + u_C(t_A, t^*, t_C(t_A, t^*, s^0(t^*)))]$$
$$= u_A(t^*, t^*, t_C(t_A, t^*, s^0(t^*))) = 0.$$

By Assumption 1 since $u_A(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) \leq 0$,

$$u_B(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) \geq 0, \; u_C(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) \geq 0,$$

in any neighborhood of $(t^*, t^*, t_C(t^*, t^*, s^0(t^*)))$. Thus, we have

$$\min_{t_A \in T} u_B(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) = 0, \tag{17a}$$

$$\arg\min_{t_A \in T} u_B(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) = t^*, \tag{17b}$$

$$\min_{t_A \in T} u_C(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) = 0,$$

and

$$\arg\min_{t_A \in T} u_C(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) = t^*.$$

By symmetry (17a) and (17b) mean

$$\min_{t_B \in T} u_A(t^*, t_B, t_C(t^*, t_B, s^0(t^*))) = 0,$$

$$\arg\min_{t_B \in T} u_A(t^*, t_B, t_C(t^*, t_B, s^0(t^*))) = t^*.$$

Thus,

$$\max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) = \min_{t_B \in T} u_A(t^*, t_B, t_C(t^*, t_B, s^0(t^*))) = u_A(t^*, t^*, t_C(t^*, t^*, s^0(t^*))) = 0.$$

Then,

$$\begin{aligned} &\min_{t_B \in T} \max_{t_A \in T} u_A(t_A, t_B, t_C(t_A, t_B, s^0(t^*))) \le \max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) \\ &= \min_{t_B \in T} u_A(t^*, t_B, t_C(t^*, t_B, s^0(t^*))) \le \max_{t_A \in T} \min_{t_B \in T} u_A(t_A, t_B, t_C(t_A, t_B, s^0(t^*))). \end{aligned}$$

From Lemma 5, interchanging B and C, we obtain

$$\begin{aligned} &\min_{t_B \in T} \max_{t_A \in T} u_A(t_A, t_B, t_C(t_A, t_B, s^0(t^*))) = \max_{t_A \in T} u_A(t_A, t^*, t_C(t_A, t^*, s^0(t^*))) \\ &= \max_{t_A \in T} \min_{t_B \in T} u_A(t_A, t_B, t_C(t_A, t_B, s^0(t^*))) = \min_{s_B \in S} \max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, s^0(t^*)), t_C(t_A, s_B, s^0(t^*))) \\ &= \max_{t_A \in T} \min_{s_B \in S} u_A(t_A, t_B(t_A, s_B, s^0(t^*)), t_C(t_A, s_B, s^0(t^*))) = \min_{t_B \in T} u_A(t^*, t_B, t_C(t^*, t_B, s^0(t^*))) = 0. \end{aligned} \tag{18}$$

Since any value of t_B is realized by appropriately choosing s_B,

$$\min_{t_B \in T} u_A(t^*, t_B, t_C(t^*, t_B, s^0(t^*))) = \min_{s_B \in S} u_A(t^*, t_B(t^*, s_B, s^0(t^*))), t_C(t^*, s_B, s^0(t^*))) = 0. \tag{19}$$

Thus,

$$\arg\min_{s_B \in S} u_A(t^*, t_B(t^*, s_B, s^0(t^*))), t_C(t^*, s_B, s^0(t^*))) = s^0(t^*).$$

From (18) and (19)

$$\begin{aligned} &\min_{s_B \in S} \max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, s^0(t^*)), t_C(t_A, s_B, s^0(t^*))) \\ &= \min_{s_B \in S} u_A(t^*, t_B(t^*, s_B, s^0(t^*))), t_C(t^*, s_B, s^0(t^*))) = 0. \end{aligned} \tag{20}$$

And we have

$$\max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, s^0(t^*)), t_C(t_A, s_B, s^0(t^*))) \ge u_A(t_A, t_B(t_A, s^0(t^*), s^0(t^*)), t_C(t_A, s^0(t^*), s^0(t^*))).$$

Then,

$$\begin{aligned} &\arg\min_{s_B \in S} \max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, s^0(t^*)), t_C(t_A, s_B, s^0(t^*))) \\ &= \arg\min_{s_B \in S} u_A(t^*, t_B(t^*, s_B, s^0(t^*))), t_C(t^*, s_B, s^0(t^*))) = s^0(t^*). \end{aligned}$$

Thus, by (20)

$$\min_{s_B\in S}\max_{t_A\in T} u_A(t_A, t_B(t_A, s_B, s^0(t^*)), t_C(t_A, s_B, s^0(t^*))) = \max_{t_A\in T} u_A(t_A, t_B(t_A, s^0(t^*), s^0(t^*)), t_C(t_A, s^0(t^*), s^0(t^*)$$
$$= u_A(t^*, t_B(t^*, s^0(t^*), s^0(t^*)), t_C(t^*, s^0(t^*), s^0(t^*))) = 0.$$

Therefore,

$$\arg\max_{t_A\in T} u_A(t_A, t_B(t_A, s^0(t^*), s^0(t^*)), t_C(t_A, s^0(t^*), s^0(t^*))) = t^*. \qquad (21)$$

From (15), (16) and (21) $(t^*, t_B(t^*, s^0(t^*), s^0(t^*)), t_C(t^*, s^0(t^*), s^0(t^*)))$ is a Nash equilibrium which is equivalent to $(t^*, t^*, t_C(t^*, t^*, s^0(t^*)))$, and hence it is equivalent to (t^*, t^*, t^*). □

Since any value of t_A is realized by appropriately choosing s_A, (21) means

$$\max_{t_A\in T} u_A(t_A, t_B(t_A, s^0(t^*), s^0(t^*)), t_C(t_A, s^0(t^*), s^0(t^*)))$$
$$= \max_{s_A\in S} u_A(t_A(s_A, s^0(t^*), s^0(t^*)), t_B(s_A, s^0(t^*), s^0(t^*)), t_C(s_A, s^0(t^*), s^0(t^*)))$$
$$= u_A(t^*, t_B(t^*, s^0(t^*), s^0(t^*)), t_C(t^*, s^0(t^*), s^0(t^*))),$$

and

$$\max_{s_A\in S} u_A(t_A(s_A, s^0(t^*), s^0(t^*)), t_B(s_A, s^0(t^*), s^0(t^*)), t_C(s_A, s^0(t^*), s^0(t^*))) = s^0(t^*).$$

Therefore, $(t_A(s^0(t^*), s^0(t^*), s^0(t^*)), t_B(s^0(t^*), s^0(t^*), s^0(t^*)), t_C(s^0(t^*), s^0(t^*), s^0(t^*)))$ is a Nash equilibrium which is equivalent to $(t^*, t^*, t_C(t^*, t^*, s^0(t^*)))$ and (t^*, t^*, t^*).

Summarizing the results we have shown

Theorem 3. *Nash equilibria in the following four states are equivalent.*

1. *All players, Players A, B and C choose* $t_i,\ i = A, B, C$.
2. *Two players choose* t_i*'s, and one player chooses* s_i.
3. *One player chooses* t_i*, and two players choose* s_i*'s.*
4. *All players, Players A, B and C choose* $s_i,\ i = A, B, C$.

4 Example of an asymmetric three-players zero-sum game

Consider a relative profit maximization game in an oligopoly with three firms producing differentiated goods[1]. It is an example of three-players zero-sum game with two strategic variables. The firms are A, B and C. The strategic variables are the outputs and the prices of the goods of the firms.

We consider the following four cases.

[1] About relative profit maximization under imperfect competition please see Matsumura, Matsushima and Cato (2013), Satoh and Tanaka (2013), Satoh and Tanaka (2014a), Satoh and Tanaka (2014b), Tanaka (2013a), Tanaka (2013b) and Vega-Redondo (1997)

1. Case 1: All firms determine their outputs.

   The inverse demand functions are

   $$p_A = a - x_A - bx_B - bx_C,$$

   $$p_B = a - x_B - bx_A - bx_C,$$

   and

   $$p_C = a - x_C - bx_A - bx_B,$$

   where $0 < b < 1$. p_A, p_B and p_C are the prices of the goods of Firm A, B and C, and x_A, x_B and x_C are the outputs of them.

2. Case 2: Firms A and B determine their outputs, and Firm C determines the price of its good.

   From the inverse demand functions,

   $$p_A = (1-b)a + b^2 x_B - bx_B + b^2 x_A - x_A + bp_C,$$

   $$p_B = (1-b)a + b^2 x_B - x_B + b^2 x_A - bx_A + bp_C,$$

   and

   $$x_C = a - bx_B - bx_A - p_C$$

   are derived.

3. Case 3: Firms B and C determine the prices of their goods, and Firm A determines its output.

   Also, from the above inverse demand functions, we obtain

   $$p_A = \frac{(1-b)a + 2b^2 x_A - bx_A - x_A + bp_C + bp_B}{1+b},$$

   $$x_B = \frac{(1-b)a + b^2 x_A - bx_A + bp_C - p_B}{(1-b)(1+b)},$$

   and

   $$x_C = \frac{(1-b)a + b^2 x_A - bx_A - p_C + bp_B}{(1-b)(1+b}.$$

4. Case 4: All firms determine the prices of their goods.

   From the inverse demand functions the direct demand functions are derived as follows;

   $$x_A = \frac{(1-b)a - (1+b)p_A + b(p_A + p_C)}{(1-b)(1+2b)},$$

   $$x_B = \frac{(1-b)a - (1+b)p_B + b(p_B + p_C)}{(1-b)(1+2b)},$$

   and

   $$x_C = \frac{(1-b)a - (1+b)p_C + b(p_A + p_B)}{(1-b)(1+2b)}.$$

The (absolute) profits of the firms are

$$\pi_A = p_A x_A - c_A x_A,$$

$$\pi_B = p_B x_B - c_B x_B,$$

and

$$\pi_C = p_C x_C - c_C x_C.$$

c_A, c_B and c_C are the constant marginal costs of Firm A, B and C. The relative profits of the firms are

$$\varphi_A = \pi_A - \frac{\pi_B + \pi_C}{2},$$

$$\varphi_B = \pi_B - \frac{\pi_A + \pi_C}{2},$$

and

$$\varphi_C = \pi_C - \frac{\pi_A + \pi_B}{2}.$$

The firms determine the values of their strategic variables to maximize the relative profits. We see

$$\varphi_A + \varphi_B + \varphi_C = 0,$$

so the game is zero-sum.

We compare the the equilibrium outputs of Firm B in four cases. Denote the value of x_B in each case by x_B^1, x_B^2, x_B^3 and x_B^4. Then, we get

$$x_B^1 = \frac{(4-b)a + bc_C - bc_B - 4c_B + bc_A}{(4-b)(2+b)},$$

$$x_B^2 = \frac{8(2-b)a - 3b^3c_C - b^2c_C + 4bc_C + 7b^2c_B - 16c_B + 5b^2c_A + 4bc_A + 3ab^3 - 11ab^2}{(4-b)(1-b)(2+b)(4+3b)},$$

$$x_B^3 = \frac{8(1+2b)a - b^3c_C + 3b^2c_C + 4bc_C + 4b^3c_B + 7b^2c_B - 16bc_B - 16c_B + 2b^3c_A + 9b^2c_A + 4bc_A - 5ab^3 - 19a}{(1-b)(b+2)(b+4)(5b+4)}$$

and

$$x_B^4 = \frac{(4+b)a + 2b^2c_C + bc_C + b^2c_B - 3bc_B - 4c_B + 2b^2c_A + bc_A - 5ab^2}{(1-b)(2+b)(4+5b)}.$$

When $c_C = c_A$, they are

$$x_B^1 = \frac{(4-b) - abc_B - 4c_B + 2bc_A}{(4-b)(2+b)},$$

$$x_B^2 = \frac{8(2-b)a + 7b^2c_B - 16c_B - 3b^3c_A + 4b^2c_A + 8bc_A + 3ab^3 - 11ab^2}{(4-b)(1-b)(2+b)(4+3b)},$$

$$x_B^3 = \frac{8(2+b)a + 4b^3c_B + 7b^2c_B - 16bc_B - 16c_B + b^3c_A + 12b^2c_A + 8bc_A - 5ab^3 - 19ab^2}{(1-b)(2+b)(4+b)(4+5b)},$$

and

$$x_B^4 = \frac{(4+b)a + b^2 c_B - 3bc_B - 4c_B + 4b^2 c_A + 2bc_A - 5ab^2}{(1-b)(2+b)(4+5b)}.$$

Further when $c_C = c_B = c_A$, we get

$$x_B^1 = x_B^2 = x_B^3 = x_B^4 = \frac{a - c_A}{2+b}.$$

We can show the same result for the equilibrium outputs of the other firms. Thus, in a fully symmetric game the four cases are equivalent.

5 Concluding Remarks

In this paper we have shown that a symmetric three-players zero-sum game with two strategic variables, choice of strategic variables is irrelevant to the Nash equilibrium. We want to extend this result to a general multi-person zero-sum game. In an asymmetric situation the Nash equilibrium depends on the choice of strategic variables by players other than two-players case[2].

Acknowledgment

This work was supported by Japan Society for the Promotion of Science KAKENHI Grant Number 15K03481.

Appendices

A Proof of Lemma 4

Proof. $\min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C))$ is the minimum of u_C with respect to t_A given s_B and s_C. Let $\tilde{t}_A(s_C) = \arg\min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C))$, and fix the value of t_C at the value which is derived from the following equations.

$$\begin{cases} t_B^0 = g_B(f_A(t_A(s_C), t_B^0, t_C^0), s_B, s_C) \\ t_C^0 = g_C(f_A(t_A(s_C), t_B^0, t_C^0), s_B, s_C). \end{cases} \tag{22}$$

Then, we have

$$\min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, t_C^0), t_C^0) \leq u_C(\tilde{t}_A(s_C), t_B(t_A, s_B, t_C^0), t_C^0) = \min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C)),$$

where $\min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, t_C^0), t_C^0)$ is the minimum of u_C with respect to t_A given the value of t_C at t_C^0. We assume that $\tilde{t}_A(s_C) = \arg\min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C))$

[2] About two-players case please see Satoh and Tanaka (2017).

is single-valued. By the maximum theorem and continuity of u_C, $\tilde{t}_A(s_C)$ is continuous. Then, any value of t_C^0 can be realized by appropriately choosing s_C given s_B according to (22). Therefore,

$$\max_{t_C \in T} \min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, t_C), t_C) \leq \max_{s_C \in S} \min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C)). \tag{23}$$

On the other hand, $\min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, t_C), t_C)$ is the minimum of u_C with respect to t_A given s_B and t_C. Let $\tilde{t}_A(t_C) = \arg\min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, t_C), t_C)$, and fix the value of s_C at the value which is derived from the following equations.

$$\begin{cases} s_A^0 = f_A(t_A(t_C), g_B(s_A^0, s_B, s_C^0), t_C) \\ s_C^0 = f_C(t_A(t_C), g_B(s_A^0, s_B, s_C^0), t_C). \end{cases} \tag{24}$$

Then, we have

$$\min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, s_C^0), t_C((t_A, s_B, s_C^0))) \leq u_C(\tilde{t}_A(s_C), t_B(t_A, s_B, s_C^0), t_C(t_A, s_B, s_C^0)) = \min_{t_A \in T} u_C(t_A, t_B(t_A, s$$

where $\min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, s_C^0), t_C(t_A, s_B, s_C^0))$ is the minimum of u_C with respect to t_A given the value of s_C at s_C^0. We assume that $\tilde{t}_A(t_C) = \arg\min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, t_C), t_C)$ is single-valued. By the maximum theorem and continuity of u_C, $\tilde{t}_A(t_C)$ is continuous. Then, any value of s_C can be realized by appropriately choosing t_C given s_B according to (24). Therefore,

$$\max_{s_C \in S} \min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C)) \leq \max_{t_C \in S} \min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, t_C), t_C). \tag{25}$$

Combining (23) and (25), we get

$$\max_{s_C \in S} \min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C)) = \max_{t_C \in S} \min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, t_C), t_C).$$

Since any value of s_C can be realized by appropriately choosing t_C given t_A and s_B, we have

$$\max_{s_C \in S} u_C(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C)) = \max_{t_C \in T} u_C(t_A, t_B(t_A, s_B, t_C), t_C).$$

Thus,

$$\min_{t_A \in T} \max_{s_C \in S} u_C(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C)) = \min_{t_A \in T} \max_{t_C \in T} u_C(t_A, t_B(t_A, s_B, t_C), t_C).$$

Therefore,

$$\begin{aligned} &\max_{t_C \in T} \min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, t_C), t_C) = \max_{s_C \in S} \min_{t_A \in T} u_C(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C)) \\ =& \min_{t_A \in T} \max_{s_C \in S} u_C(t_A, t_B, t_C(t_A, t_B, s_C)) = \min_{t_A \in T} \max_{t_C \in T} u_C(t_A, t_B(t_A, s_B, t_C), t_C). \end{aligned}$$

By similar procedures, we can show

$$\begin{aligned} &\max_{t_C \in S} \min_{t_B \in T} u_C(t_A(s_A, t_B, t_C), t_B, t_C) = \max_{s_C \in S} \min_{t_B \in T} u_C(t_A(s_A, t_B, s_C), t_B, t_C(s_A, t_B, s_C)) \\ =& \min_{t_B \in T} \max_{s_C \in S} u_C(t_A(s_A, t_B, s_C), t_B, t_C(s_A, t_B, s_C)) = \min_{t_B \in T} \max_{t_C \in S} u_C(t_A(s_A, t_B, t_C), t_B, t_C). \end{aligned}$$

$\square$

B Proof of Lemma 5

Proof. $\max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C))$ is the maximum of u_A with respect to t_A given s_B and s_C. Let $\tilde{t}_A(s_C) = \arg\max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C))$, and fix the value of t_C at the value which is derived from the following equations.

$$\begin{cases} t_B^0 = g_B(f_A(t_A(s_C), t_B^0, t_C^0), s_B, s_C) \\ t_C^0 = g_C(f_A(t_A(s_C), t_B^0, t_C^0), s_B, s_C). \end{cases} \tag{26}$$

Then, we have

$$\max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, t_C^0), t_C^0) \geq u_A(\tilde{t}_A(s_C), t_B(t_A, s_B, t_C^0), t_C^0) = \max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C)),$$

where $\max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, t_C^0), t_C^0)$ is the maximum of u_A with respect to t_A given the value of t_C at t_C^0. We assume that $\tilde{t}_A(s_C) = \arg\max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C))$ is single-valued. By the maximum theorem and continuity of u_A, $\tilde{t}_A(s_C)$ is continuous. Then, any value of t_C^0 can be realized by appropriately choosing s_C given s_B according to (26). Therefore,

$$\min_{t_C \in T} \max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, t_C), t_C) \geq \min_{s_C \in S} \max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C)). \tag{27}$$

On the other hand, $\max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, t_C), t_C)$ is the maximum of u_A with respect to t_A given s_B and t_C. Let $\tilde{t}_A(t_C) = \arg\max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, t_C), t_C)$, and fix the value of s_C at the value which is derived from the following equations.

$$\begin{cases} s_A^0 = f_A(t_A(t_C), g_B(s_A^0, s_B, s_C^0), t_C) \\ s_C^0 = f_C(t_A(t_C), g_B(s_A^0, s_B, s_C^0), t_C). \end{cases} \tag{28}$$

Then, we have

$$\max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, s_C^0), t_C(t_A, s_B, s_C^0)) \geq u_A(\tilde{t}_A(s_C), t_B(t_A, s_B, s_C^0), t_C(t_A, s_B, s_C^0)) = \max_{t_A \in T} u_A(t_A, t_B(t_A, s_B$$

where $\max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, s_C^0), t_C(t_A, s_B, s_C^0))$ is the maximum of u_A with respect to t_A given the value of s_C at s_C^0. We assume that $\tilde{t}_A(t_C) = \arg\max_{t_A \in T} u_A(t_A, s_B, t_C)$ is single-valued. By the maximum theorem and continuity of u_A, $\tilde{t}_A(t_C)$ is continuous. Then, any value of s_C^0 can be realized by appropriately choosing t_C given s_B according to (28). Therefore,

$$\min_{s_C \in S} \max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C)) \geq \min_{t_C \in S} \max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, t_C), t_C). \tag{29}$$

Combining (27) and (29), we get

$$\min_{s_C \in S} \max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C)) = \min_{t_C \in S} \max_{t_A \in T} u_A(t_A, t_B(t_A, s_B, t_C), t_C).$$

Since any value of s_C can be realized by appropriately choosing t_C given t_A and s_B, we have

$$\min_{s_C \in S} u_A(t_A, t_B(t_A, s_B, s_C), s_C) = \min_{t_C \in S} u_A(t_A, t_B(t_A, s_B, t_C), t_C).$$

Thus,

$$\max_{t_A\in T}\min_{s_C\in S} u_A(t_A, t_B(t_A, s_B, s_C), t_C(t_A, s_B, s_C)) = \max_{t_A\in T}\min_{t_C\in S} u_A(t_A, t_B(t_A, s_B, t_C), t_C).$$

Therefore,

$$\min_{t_C\in T}\max_{t_A\in T} u_A(t_A(s_A, t_B, t_C), t_B, t_C) = \min_{s_C\in S}\max_{t_A\in T} u_A(t_A(s_A, t_B, s_C), t_B, t_C(s_A, t_B, s_C)),$$
$$= \max_{t_A\in T}\min_{s_C\in S} u_A(t_A(s_A, t_B, s_C), t_B, s_C) = \max_{t_A\in T}\min_{t_C\in T} u_A(t_A(s_A, t_B, t_C), t_B, t_C).$$

By similar procedures, we can show

$$\min_{t_C\in T}\max_{t_B\in T} u_B(t_A(s_A, t_B, t_C), t_B, t_C) = \min_{s_C\in S}\max_{t_B\in T} u_B(t_A(s_A, t_B, s_C), t_B, t_C(s_A, t_B, s_C)),$$
$$= \max_{t_B\in T}\min_{s_C\in S} u_B(t_A(s_A, t_B, s_C), t_B, t_C(s_A, t_B, s_C)) = \max_{t_B\in T}\min_{t_C\in T} u_B(t_A(s_A, t_B, t_C), t_B, t_C).$$

□

References

Glicksberg, I.L. (1952) "A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points." *Proceedings of the American Mathematical Society*, **3**, pp.170-174.

Kindler, J. (2005), "A simple proof of Sion's minimax theorem," *American Mathematical Monthly*, **112**, pp. 356-358.

Komiya, H. (1988), "Elementary proof for Sion's minimax theorem," *Kodai Mathematical Journal*, **11**, pp. 5-7.

Matsumura, T., N. Matsushima and S. Cato (2013) "Competitiveness and R&D competition revisited," *Economic Modelling*, **31**, pp. 541-547.

Satoh, A. and Y. Tanaka (2013) "Relative profit maximization and Bertrand equilibrium with quadratic cost functions," *Economics and Business Letters*, **2**, pp. 134-139, 2013.

Satoh, A. and Y. Tanaka (2014a) "Relative profit maximization and equivalence of Cournot and Bertrand equilibria in asymmetric duopoly," *Economics Bulletin*, **34**, pp. 819-827, 2014.

Satoh, A. and Y. Tanaka (2014b), "Relative profit maximization in asymmetric oligopoly," *Economics Bulletin*, **34**, pp. 1653-1664.

Satoh, A. and Y. Tanaka (2017), "Two person zero-sum game with two sets of strategic variables," MPRA Paper 73272, University Library of Munich, Germany.

Sion, M. (1958), "On general minimax theorems," *Pacific Journal of Mathematics*, **8**, pp. 171-176.

Tanaka, Y. (2013a) "Equivalence of Cournot and Bertrand equilibria in differentiated duopoly under relative profit maximization with linear demand," *Economics Bulletin*, **33**, pp. 1479-1486.

Tanaka, Y. (2013b) "Irrelevance of the choice of strategic variables in duopoly under relative profit maximization," *Economics and Business Letters*, **2**, pp. 75-83, 2013.

Vega-Redondo, F. (1997) "The evolution of Walrasian behavior,", *Econometrica*, **65**, pp. 375-384.