On strategy-proofness and single-peakedness: median-voting over intervals

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Abstract

We study *solutions* that choose an interval of alternatives when agents have single-peaked preferences. Similar to Klaus and Storcken (2002), we ordinally extend these preferences over intervals. Loosely speaking, we extend the results of Moulin (1980) to our setting and show that the results of Ching (1997) cannot always be similarly extended. Our main results are the following. First, *strategy-proofness* and *peaks-onliness* characterize the class of generalized median solutions. Second, although *peaks-onliness* cannot be replaced by the "weaker" property of *continuity* in our first result -as is the case in Ching (1997)- this equivalence is achieved when *voter-sovereignty* is also required. Finally, if preferences are symmetric and single-peaked, *strategy-proofness* and *voter-sovereignty* characterize the class of *efficient* generalized median solutions.

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1 Introduction

We study the problem where an interval of alternatives is chosen on the interval $[0, 1]$ based on the preferences of a finite number of agents. This interval can be considered as the political spectrum, while the chosen interval can in turn be considered as the legislative constitution or the governmental coalition (in the sense that some "extreme" views are not accounted for by the constitution or are not represented by any member(s) of the governmental coalition). We assume that agents have *single-peaked preferences* defined over all alternatives on $[0, 1]$; that is, an agent's welfare is strictly increasing up to his "peak" (his favorite alternative), and is strictly decreasing thereafter. Moreover, we assume that agents, when comparing two intervals, only consider their best (most favorite) alternative and their worst (least favorite) alternative(s) on each interval. Finally, we look into the situation where the voting mechanism choosing the interval of alternatives guarantees that the agents announce their true preferences; in other words, we are interested in voting mechanisms -which we call (choice) *solutions*- that are *strategy-proof*.

Although the classic result of Gibbard and Satterthwaite establishes that on the full domain of preferences -with more than two alternatives available- *strategy-proofness* and *non-dictatorship* are incompatible (Gibbard, 1973; Satterthwaite, 1975), this is not true for the domain of single-peaked preferences, the domain of interest in this paper.

This compatibility between the two aforementioned properties has been well studied in the context of (choice) *rules* and for infinite sets of alternatives, where following the announcement of the agents' (single-peaked) preferences one alternative is chosen. Specifically, it has been shown that *strategy-proofness* and *peaks-onliness* (the agents only announce their peak) characterize the class of generalized median rules (described in Section 3.1) (Moulin, 1980). Moreover, when also requiring the property of either *efficiency* (in the Pareto sense), or *anonymity* (the names of the agents don't matter), or both to be satisfied, the sub-classes of either efficient generalized median rules (Section 3.1), or median rules (Section 3.2), or efficient median rules (Section 3.2) are characterized (Moulin, 1980). A similar result also holds for the one-dimensional case, when the range of the rule is closed and not connected (Barberà and Jackson, 1994). In addition, on the smaller *domain of quadratic and separable preferences*[1] and on dimensions equal or larger to 1, *peaks-onliness* can be substituted by

[1]In this domain, an agent's welfare depends on the distance of the alternative chosen from his peak, projected

unanimity (when a common best alternative exists, it is chosen) (Border and Jordan, 1983); furthermore, it turns out that in these results two of the required properties can be weakened; specifically, *peak-onliness* and *efficiency* can be substituted by *continuity* (a small change in the announced preferences does not change the outcome a lot) and *voter-sovereignty* (no alternative is a priori excluded from being chosen) respectively (Ching, 1997).[2] Finally, a measure of manipulability was recently proposed, that can be used to compare two generalized median rules (via some necessary and sufficient conditions) (Arribillaga and Massó, 2016).

For the case where a single alternative is chosen among a finite set (of alternatives), *strategy-proofness* and *voter-sovereignty* characterize, on the domain of strict preferences, a class of rules similar to the class of efficient generalized median rules (Barberà et al., 1993). Moreover, the admissible preferences of all agents being *top-connected*[3] characterize the maximal domain in which (i) every *strategy-proof* and *unanimous* rule is a generalized median rule, and (ii) every generalized median rule is *strategy-proof* (Achuthankutty and Roy, 2017).

When departing from the setting where agents have single-peaked preferences and one alternative is chosen, a few more results should be mentioned. First, in the case of *probabilistic rules*,[4] where the agents' single-peaked preferences are ordinarily extended over probability distributions via first-order stochastic dominance, similar results to Moulin's results (1980) were achieved (Ehlers et al., 2002). Moreover, if the agents' preferences are single-peaked but two alternatives can be chosen, the properties of *strategy-proofness*, *continuity*, *anonymity*, and *users-only*[5] characterize the class of *double median rules*[6] (Heo, 2013). Finally, if agents

in every dimension. Specifically, the larger the sum of all such projected distances, the smaller the welfare gained.

[2]Although technically *continuity* is not weaker than *peaks-onliness*, loosely speaking, it imposes fewer restrictions on the result.

[3]For every agent and every pair of "neighboring" alternatives (a, b) there exist admissible preferences such that a is the most favorite alternative and b is the second most favorite alternative.

[4]Given the agents' preferences, a probability distribution over all alternatives is chosen.

[5]For each pair of chosen alternatives (a, b) the choice of a does not depend on agents preferring b over a.

[6]A double median rule can be decomposed into two median rules, where for each preference profile each one selects one alternative.

have *single-dipped preferences*,[7] *strategy-proofness* and *unanimity* characterize the class of *collections of 0-decisive sets with a tie-breaker*[8] (Manjunath, 2014).

In line with the related literature, our main results also make use of either the property of *peaks-onliness* or a version of *continuity* adapted for our context (i.e., where an interval of alternatives is chosen). In addition, we also study the sub-cases where solutions are either *efficient*, or *anonymous*, or both. Concisely, our results are the following. First, in the domain of single-peaked preferences, *strategy-proofness* and *peaks-onliness* characterize the class of generalized median solutions (Theorem 1); and if *anonymity* is also required, then the sub-class of median solutions is characterized (Theorem 2). Second, neither of these results holds in the domain of symmetric and single-peaked preferences, nor can in these results *continuity* substitute *peaks-onliness* (the counter-example on page 19). Third, in the domain of single-peaked preferences, *strategy-proofness*, *voter-sovereignty*, and either *peaks-onliness* or *continuity* characterize the class of efficient generalized median solutions (Theorem 3); and if *anonymity* is also required, then the sub-class of efficient median solutions is characterized (Theorem 4). Finally, in the domain of symmetric and single-peaked preferences, the classes of efficient generalized median solutions and efficient median solutions can be similarly characterized with one difference; due to *single-peakedness* being inherent in the domain, *continuity* plays no role.

The paper proceeds as follows. Section 2 explains the model and states a preliminary result. Section 3 includes the definitions of choice rules and solutions, as well as the definition of the classes of such rules and solution we characterize. Section 4 contains the properties we are interested in and some further preliminary results. Finally, Section 5 contains all non-preliminary results and characterizations, as well as a table summarizing our results.

[7] An agent's welfare is strictly decreasing up to his "dip" (his least favorite alternative), and is strictly increasing thereafter.

[8] Each such rule chooses either the minimum or the maximum alternative. Loosely speaking, if all agents are indifferent between the two alternatives the choice depends on the preference profile (over all other alternatives). Otherwise, the choice depends on the number of agents preferring the minimum over the maximum alternative, their identities, and their preferences.

2 The model

Consider a *coalition* (of agents) $N \equiv \{1, \ldots, n\}$, such that $n \geq 2$, and a set of *alternatives* $A \equiv [0, 1]$.[9] We denote generic agents by i and j, and generic alternatives by x and y. Each i is equipped with *preferences* R_i, defined over A, that are *complete*, *transitive*, and *reflexive*. As usual, $x\,R_i\,y$ is interpreted as "x is at least as desirable as y", $x\,P_i\,y$ as "x is preferred to y", and $x\,I_i\,y$ as "x is indifferent to y". Moreover, for preferences R_i there exists an alternative $p_i \in A$, called the *peak* of i, with the following property: if either $y < x \leq p_i$ or $y > x \geq p_i$, we have $x\,P_i\,y$. We call such preferences *single-peaked* and denote the *domain of single-peaked preferences* by $\mathcal{R}$. Furthermore, if for preferences $R_i \in \mathcal{R}$, $|x - p_i| = |y - p_i|$ implies $x\,I_i\,y$, then we say these preferences are *symmetric* and denote the *domain of symmetric preferences* by $\mathcal{S}$.

In the sequel, all notation and definitions refer to domain $\mathcal{R}$ but also apply to domain $\mathcal{S}$. Moreover, all results presented in this section hold in both domains.

Let $\mathcal{R}^N$ be the set of profiles $R \equiv (R_i)_{i \in N}$ such that for each $i \in N$, $R_i \in \mathcal{R}$. Given $R \in \mathcal{R}^N$ and $j \in N$, we also use R and (R_{-j}, R_j) interchangeably. For each $R \in \mathcal{R}^N$, we denote the *vector of peaks* of R by $p \equiv (p_i)_{i \in N}$. Let the *smallest peak* in R be $\underline{p} \equiv \min(\{p_i\}_{i \in N})$ and the *largest peak* in R be $\bar{p} \equiv \max(\{p_i\}_{i \in N})$. Finally, let the *convex hull* of peaks of R be $\text{Conv}(p) \equiv [\underline{p}, \bar{p}]$.

Let the class of closed intervals in A be denoted by $\mathcal{A}$. We denote generic sets in $\mathcal{A}$ by X and Y. We denote the *minimum* of X by $\underline{X} \equiv \min(X)$ and the *maximum* of X by $\bar{X} \equiv \max(X)$. For each $R_i \in \mathcal{R}$, we denote the *best alternative(s)* of i in X by $b_{R_i}(X) \equiv \{x \in X\colon \text{for each } y \in X,\ x\ R_i\ y\}$ and the *worst alternative(s)* of i in X by $w_{R_i}(X) \equiv \{x \in X\colon \text{for each } y \in X,\ y\ R_i\ x\}$. Note that single-peakedness of R_i and non-emptiness of X imply that the sets $b_{R_i}(X)$ and $w_{R_i}(X)$ contain one or two elements; specifically, if $b_{R_i}(X)$ (respectively, $w_{R_i}(X)$) contains two elements, agent i is indifferent between them. It is with some abuse of notation that we treat sets $b_{R_i}(X)$ and $w_{R_i}(X)$ as if they are points and for each $x \in X$, we write $b_{R_i}(X)\ R_i\ x\ R_i\ w_{R_i}(X)$.

We extend all preferences $R_i \in \mathcal{R}$, defined over A, to preferences defined over $\mathcal{A}$ according

[9] The set of alternatives is chosen without loss of generality. Essentially, our results hold for any closed interval in $\mathbb{R}$.

to the "best-worst" extension of preferences characterized by Barberà et al. (1984).[10] Specifically, when comparing two sets, an agent only considers his best and his worst point(s) in each of them. Therefore, an agent prefers X to Y if he prefers his best point(s) in X to his best point(s) in Y and his worst point(s) in X to his worst point(s) in Y. The following definition also covers three more cases arising when an agent is indifferent between his best or worst point(s) in the two sets.

With some abuse of notation, we use the same symbols to denote preferences over alternatives and preferences over sets of alternatives.

Best-worst extension of preferences. For each $i \in N$ with preferences $R_i \in \mathcal{R}$, and each pair $X, Y \in \mathcal{A}$,

$$X \; R_i \; Y \text{ if and only if } \begin{cases} b_{R_i}(X) \; R_i \; b_{R_i}(Y) \\ \qquad\quad \text{and} \\ w_{R_i}(X) \; R_i \; w_{R_i}(Y). \end{cases}$$

and

$$X \; P_i \; Y \text{ if and only if } X \; R_i \; Y \text{ and } \begin{cases} b_{R_i}(X) \; P_i \; b_{R_i}(Y) \\ \qquad\quad \text{or} \\ w_{R_i}(X) \; P_i \; w_{R_i}(Y). \end{cases}$$

This extension of preferences is *transitive*: for each triple $X, Y, Z \in \mathcal{A}$, if $X \, R_i \, Y$ and $Y \, R_i \, Z$, then $X \; R_i \; Z$. However, it is not *complete*: there exist $X, Y \in \mathcal{A}$ such that neither $X \; R_i \; Y$ nor $Y \; R_i \; X$. To be precise, we now introduce the following definition.

Comparability. Given preferences $R_i \in \mathcal{R}$, sets $X, Y \in \mathcal{A}$ are *comparable* if and only if $[b_{R_i}(X) \, P_i \, b_{R_i}(Y)$ implies $w_{R_i}(X) \, R_i \, w_{R_i}(Y)]$ and $[w_{R_i}(X) \, P_i \, w_{R_i}(Y)$ implies $b_{R_i}(X) \, R_i \, b_{R_i}(Y)]$.

Based on the best-worst extension of preferences, we now define (Pareto) *efficient* sets.

[10]Preferences $R_i^{\mathcal{A}}$ defined over $\mathcal{A}$ satisfy *weak-dominance* ($x \; P_i^{\mathcal{A}} \; y$ implies $\{x\} \; P_i^{\mathcal{A}} \; \{x, y\} \; P_i^{\mathcal{A}} \; \{y\}$) and *weak-independence* (given triple $X, Y, Z \in \mathcal{A}$ such that $[X \cap Z] = [Y \cap Z] = \emptyset$, $X \, P_i^{\mathcal{A}} \, Y$ implies $[X \cup Z] \, R_i^{\mathcal{A}} \, [Y \cup Z]$) if and only if i compares sets in $\mathcal{A}$ according to the "best-worst" extension of preferences. Examples illustrating the reasoning behind requiring these properties (in a slightly different model) are provided in Klaus and Protopapas (2016).

Efficient sets. Given profile $R \in \mathcal{R}^N$, set $X \in \mathcal{A}$ is *efficient* if and only if there is no set $Y \in \mathcal{A}$ such that for each $i \in N$, $Y\, R_i\, X$, and for at least one $j \in N$, $Y\, P_j\, X$; we denote the class containing all *efficient* sets at R by $\mathrm{E}(R)$.

We now present a characterization of efficient sets in this setting that follows from Klaus and Protopapas (2016). Note that the original result is a little more complicated since it holds for all compact sets.

Proposition 1 (Klaus and Protopapas (2016))**.** *At profile $R \in \mathcal{R}^N$, a closed interval is efficient if and only if it is a subset of the convex hull of peaks in R.*

3 Choice rules and solutions

In the sequel, all notation and definitions refer to domain $\mathcal{R}$ but also apply to domain $\mathcal{S}$. Moreover, all results presented in this section hold in both domains.

Each $i \in N$, *announces* preferences $V_i \in \mathcal{R}$ with associated *announced peak* $v_i \in A$. Given (true) profile $R \in \mathcal{R}^N$, if $V_i = R_i$, we say that i is *sincere*; otherwise, if $V_i \neq R_i$, we say that i *deviates*. All terminology, notation, and results of Section 2, defined for preferences $R_i \in \mathcal{R}$, are carried over to announced preferences $V_i \in \mathcal{R}$ by replacing R and p by V and v respectively, and adding the term "announced" as necessary. For example, since in profile $R \in \mathcal{R}^N$ the smallest peak is denoted by $\underline{p} \equiv \min(\{p_i\}_{i \in N})$, in *announced profile* $V_N \in \mathcal{R}^N$ the *smallest announced peak* is denoted by $\underline{v} \equiv \min(\{v_i\}_{i \in N})$.

A *(choice) solution* F assigns to each $V \in \mathcal{R}^N$ a set $F(V) \in \mathcal{A}$, i.e., $F\colon \mathcal{R}^N \to \mathcal{A}$. Given $V \in \mathcal{R}^N$, let the *minimum* of $F(V)$ be $\underline{F}(V) \equiv \min\{F(V)\}$ and the *maximum* of $F(V)$ be $\bar{F}(V) \equiv \max\{F(V)\}$. We denote the family of solutions by $\mathcal{F}$. Moreover, if a solution $F \in \mathcal{F}$ assigns to each $V \in \mathcal{R}^N$ an interval consisting of a single point we will refer to it as a *rule* and use notation $f \in \mathscr{f}$, i.e., $f\colon \mathcal{R}^N \to A$.

Before defining in Sections 3.1 and 3.2 two classes of rules and solutions that our results revolve around, the following definition is necessary: for each odd and positive integer k, and each vector $T \in \mathbb{R}^k$, label the coordinates of T such that $t_1 \leq \cdots \leq t_k$; we define the *median* (coordinate) of T by $\mathrm{med}(T) \equiv t_{\frac{k+1}{2}}$

Finally, we would like the reader to notice that the classes of generalized median rules and solutions, defined in Section 3.1, are as the name suggests, a generalization of the classes of median rules and solutions, defined in Section 3.2. Loosely speaking, this generalization boils down to the agents influencing the chosen interval non-symmetrically. This is formally shown in Lemma 4 (Section 3.2). The reason behind this sequencing is simple: our results for the classes of generalized median rules and solutions can be easily shown to hold for the subclasses of median rules and solutions respectively.

3.1 Generalized median rules and solutions

The first class of rules we consider was introduced under the name *strategy-proof voting schemes* and characterized by *strategy-proofness*[11] and *peaks-onliness*[12] (Moulin, 1980, Proposition 3). It was later shown that *peaks-onliness* can be substituted with the "weaker" property of *continuity*[13] (Ching, 1997, Theorem). In order to provide a useful intuition in understanding this class, we present an example inspired by the one provided in Arribillaga and Massó (2016, p. 564).

Example 1. Consider the two agent case, i.e., $N = \{1, 2\}$ and choose a 4-dimensional vector $\alpha = (\alpha_\emptyset, \alpha_{\{1\}}, \alpha_{\{2\}}, \alpha_N)$ such that $\alpha_N \leq \alpha_{\{1\}} \leq \alpha_{\{2\}} \leq \alpha_\emptyset$. Next, define the rule $f^\alpha \in \not{f}$ as follows. For each $V \in \mathcal{R}^N$, if $v_1 \leq v_2$, choose $\tilde{\alpha} = (\alpha_\emptyset, \alpha_{\{1\}}, \alpha_N)$ and set $f^\alpha(V) = \text{med}(\tilde{\alpha}, v)$, and if $v_1 > v_2$, choose $\tilde{\alpha} = (\alpha_\emptyset, \alpha_{\{2\}}, \alpha_N)$ and set $f^\alpha(V) = \text{med}(\tilde{\alpha}, v)$.

Notice that if $\alpha_{\{1\}} \neq \alpha_{\{2\}}$, then the agents have an asymmetric power in influencing the chosen alternative. Before discussing further this asymmetry, in an effort to shed more light on the behavior of f^α, we first provide a second definition for it.

[11]No agent gains by deviating.

[12]The chosen alternative only depends on the vector of announced peaks.

[13]If the announced preferences change a 'little', the chosen alternative does not change "a lot".

$$\text{For each } V \in \mathcal{R}^N,\ f^\alpha(V) = \begin{cases} \alpha_N & \text{if } v_1, v_2 \leq \alpha_N \\ v_2 & \text{if } v_1 \leq \alpha_N \leq v_2 \leq \alpha_{\{1\}} \\ \alpha_{\{1\}} & \text{if } v_1 \leq \alpha_N \leq \alpha_{\{1\}} \leq v_2 \\ \text{med}(v_1, v_2, \alpha_{\{1\}}) & \text{if } \alpha_N \leq v_1 \leq \alpha_{\{1\}} \\ v_1 & \text{if } \alpha_{\{1\}} \leq v_1 \leq \alpha_{\{2\}} \\ \text{med}(v_1, v_2, \alpha_{\{2\}}) & \text{if } \alpha_{\{2\}} \leq v_1 \leq \alpha_\emptyset \\ \alpha_{\{2\}} & \text{if } v_2 \leq \alpha_{\{2\}} \leq \alpha_\emptyset \leq v_1 \\ v_2 & \text{if } \alpha_{\{2\}} \leq v_2 \leq \alpha_\emptyset \leq v_1, \text{ and} \\ \alpha_\emptyset & \text{if } \alpha_\emptyset \leq v_1, v_2. \end{cases}$$

It is easy to see from this second definition that the range of f^α equals $[\alpha_N, \alpha_\emptyset]$. Hence, this rule can be interpreted as one assigning to agents 1 and 2 the power to choose an alternative from the interval $[\alpha_N, \alpha_\emptyset]$. Furthermore, as already briefly discussed, this power is not symmetric among the agents but depends on the choice of $\alpha_{\{1\}}$ and $\alpha_{\{2\}}$. For instance in this example, since $\alpha_{\{1\}} \leq \alpha_{\{2\}}$, agent 1 has a greater power than agent 2 in influencing the chosen alternative.

To see this, fist consider agent 1. He can make sure that the chosen alternative is not larger than $\alpha_{\{1\}}$ and not smaller than v_1 (by announcing $v_1 \leq \alpha_{\{1\}}$), or that it is not larger than v_1 and not smaller than $\alpha_{\{1\}}$ (by announcing $v_1 \geq \alpha_{\{1\}}$). In addition, he is a dictator on the interval $[\alpha_{\{1\}}, \alpha_{\{2\}}]$.

Next, consider agent 2. He only has the power to influence the chosen alternative if agent 1 "allows" him to do so. That is, if $\alpha_N \leq v_1 \leq \alpha_{\{1\}}$, then agent 2 can pinpoint the chosen alternative on the interval $[v_1, \alpha_{\{1\}}]$, and if $v_1 \leq \alpha_N \leq \alpha_{\{1\}}$, then agent 2 can pinpoint the chosen alternative on the interval $[\alpha_N, \alpha_{\{1\}}]$. Similarly, if $\alpha_{\{2\}} \leq v_1 \leq \alpha_\emptyset$, then agent 2 can pinpoint the chosen alternative on the interval $[\alpha_{\{2\}}, v_1]$, and if $\alpha_{\{2\}} \leq \alpha_\emptyset \leq v_1$, then agent 2 can pinpoint the chosen alternative on the interval $[\alpha_{\{2\}}, \alpha_\emptyset]$.

The general n-agent case works as follows. First, take a vector α in A^{2^n}, i.e., the dimension of α equals the number of all sub-coalitions in N (including the empty set). Specifically, let $\alpha \equiv (\alpha_M)_{M \subseteq N}$, such that for each $L \subseteq M$, $\alpha_L \geq \alpha_M$. Next, for an announced profile

V with associated vector of announced peaks v, label the agents such that $v_{\bar{1}} \leq \cdots \leq v_{\bar{n}}$.[14] Finally, construct vector $\tilde{\alpha}$ in A^{n+1} such that $\tilde{\alpha} = (\alpha_{\emptyset}, \alpha_{\{\bar{1}\}}, \alpha_{\{\bar{1},\bar{2}\}}, \ldots, \alpha_N)$ and notice that by construction, $\alpha_N \leq \cdots \leq \alpha_{\{\bar{1},\bar{2}\}} \leq \alpha_{\{\bar{1}\}} \leq \alpha_{\emptyset}$. The *generalized median rule* associated with vector α chooses alternative $\mathrm{med}(v, \tilde{\alpha})$.

Generalized median rules. *Let vector $\alpha \in A^{2^n}$ be such that $\alpha \equiv (\alpha_M)_{M \subseteq N}$, where for each pair $L, M \subseteq N$ with $L \subsetneq M$, $\alpha_L \geq \alpha_M$. Also, for each $V \in \mathcal{R}^N$, let bijection $\pi\colon N \to N$ be such that $v_{\pi(1)} \leq \cdots \leq v_{\pi(n)}$ and construct vector $\tilde{\alpha} = (\alpha_{\emptyset}, \alpha_{\{\pi(1)\}}, \alpha_{\{\pi(1),\pi(2)\}}, \ldots, \alpha_N)$. We denote the generalized median rule associated with vector α by f_G^{α}, where for each $V \in \mathcal{R}^N$, $f_G^{\alpha}(V) \equiv \mathrm{med}(v, \tilde{\alpha})$. Finally, we denote the class of generalized median rules by $\mathcal{F}_G$.*[15]

Clearly, if all agents announce different peaks, a unique ordering of them by their announced peak exists. The next result shows (for generalized median rules) that if such a unique ordering does not exist, then the chosen alternative does not depend on the particular ordering chosen.

Lemma 1. *For each generalized median rule f_G^{α}, if vector $\tilde{\alpha}$ is not unique, then the chosen alternative does not depend on the specific choice of $\tilde{\alpha}$.*

Proof. Let $f_G^{\alpha} \in \mathcal{F}_G$ and $V \in \mathcal{R}^N$ be such that for some agents $i, j \in N$, $v_i = v_j$. Without loss of generality, label the agents such that $v_{\bar{1}} \leq \cdots \leq v_{\bar{n}}$. Hence, by the definition of $\mathcal{F}_G$, we can construct $\tilde{\alpha} = (\alpha_{\emptyset}, \ldots, \alpha_{\{\bar{1},\ldots,i-1\}}, \alpha_{\{\bar{1},\ldots,i\}}, \alpha_{\{\bar{1},\ldots,i,j\}}, \ldots, \alpha_N)$ and $\tilde{\alpha}' = (\alpha_{\emptyset}, \ldots, \alpha_{\{\bar{1},\ldots,i-1\}}, \alpha_{\{\bar{1},\ldots,j\}}, \alpha_{\{\bar{1},\ldots,i,j\}}, \ldots, \alpha_N)$. Clearly, if $\alpha_{\{\bar{1},\ldots,i\}} \neq \alpha_{\{\bar{1},\ldots,j\}}$, then $\tilde{\alpha} \neq \tilde{\alpha}'$.

Let $M = \{\bar{1}, \ldots, i\}$, that is, $\alpha_M = \alpha_{\{\bar{1},\ldots,i\}}$. Let $f_G^{\alpha} = \mathrm{med}(v, \tilde{\alpha})$. We first show that if $f_G^{\alpha} = \alpha_M$, then $\alpha_M = v_i = v_j$. Let $|M| = m$. By definition of $\mathcal{F}_G$, $\alpha_N \leq \alpha_{\{\bar{1},\ldots,\bar{n}-1\}} \leq \cdots \leq \alpha_{\{\bar{1}\}} \leq \alpha_{\emptyset}$, implying there are at least $n+1-m$ coordinates of $\tilde{\alpha}$ not larger than α_M (i.e., coordinates $\alpha_M, \ldots, \alpha_N$) and at least $m+1$ coordinates of $\tilde{\alpha}$ not smaller than α_M (i.e., coordinates $\alpha_{\emptyset}, \ldots, \alpha_M$). Hence, $f_G^{\alpha} = \mathrm{med}(v, \tilde{\alpha})$ implies at least m agents announce peaks not larger than α_M (i.e., agents $\bar{1}, \ldots, i$) and at least $n-m$ agents announce peaks not smaller than α_M (i.e., agents $j, \ldots, \bar{n}$). Therefore, since $v_i = v_j$, $\alpha_M = v_i = v_j$.

[14] Whenever two agents announce the same peak, no unique way to label the agents exists. However, as is shown in Lemma 1, the specific choice of labels does not affect the chosen alternative.

[15] It should be noted that in the literature a generalized median rule $f_G^{\alpha} \in \mathcal{F}_G$ is often described as follows: For each $V \in \mathcal{R}^N$, each $M \subseteq N$, and each $i \in M$, $f_G^{\alpha} = \max\limits_{M \subseteq N} \min\limits_{i \in M} \{p_i, \alpha_M\}$.

By symmetric arguments, it can be shown that if $f_G^\alpha = \mathrm{med}(v, \tilde{\alpha}') = \alpha'_{\{\bar{1},\dots,j\}}$, then $\alpha'_{\{\bar{1},\dots,j\}} = v_i = v_j$. Therefore, $f_G^\alpha = v_i = v_j = \alpha_M = \alpha'_{\{\bar{1},\dots,j\}}$ contradicts that $\alpha_M \neq \alpha'_{\{\bar{1},\dots,j\}}$. $\square$

The first class of solutions we characterize in Section 5 extends the spirit of generalized median rules to solutions. Specifically, take two vectors $\alpha \leq \beta$, each of dimension 2^n, such that $\alpha \equiv (\alpha_M)_{M \subseteq N}$ and $\beta \equiv (\beta_M)_{M \subseteq N}$. Next, for an announced profile V with associated vector of announced peaks v, label the agents such that $v_1 \leq \cdots \leq v_n$ and construct vectors $\tilde{\alpha}$ and $\tilde{\beta}$, each of dimension $n+1$, such that $\tilde{\alpha} = (\alpha_\emptyset, \alpha_{\{1\}}, \alpha_{\{1,2\}}, \dots, \alpha_N)$ and $\tilde{\beta} = (\beta_\emptyset, \beta_{\{1\}}, \beta_{\{1,2\}}, \dots, \beta_N)$. The *generalized median solution* associated with vectors α and β chooses the interval where the minimum alternative is $\mathrm{med}(v, \tilde{\alpha})$ and the maximum alternative is $\mathrm{med}(v, \tilde{\beta})$.

Generalized median solutions. *Let vectors $\alpha, \beta \in A^{2^n}$ be such that $\alpha \equiv (\alpha_M)_{M \subseteq N}$ and $\beta \equiv (\beta_M)_{M \subseteq N}$, with $\alpha \leq \beta$, and for each pair $L, M \subseteq N$, with $L \subsetneq M$, $\alpha_L \geq \alpha_M$ and $\beta_L \geq \beta_M$. Also, for each $V \in \mathcal{R}^N$, let bijection $\pi \colon N \to N$ such that $v_{\pi(1)} \leq \cdots \leq v_{\pi(n)}$ and construct vectors $\tilde{\alpha} = (\alpha_\emptyset, \alpha_{\{\pi(1)\}}, \alpha_{\{\pi(1),\pi(2)\}}, \dots, \alpha_N)$ and $\tilde{\beta} = (\beta_\emptyset, \beta_{\{\pi(1)\}}, \beta_{\{\pi(1),\pi(2)\}}, \dots, \beta_N)$. We denote the generalized median solution associated with vectors α and β by $F_G^{\alpha,\beta}$, where for each $V \in \mathcal{R}^N$, $F_G^{\alpha,\beta}(V) \equiv [\mathrm{med}(v, \tilde{\alpha}), \mathrm{med}(v, \tilde{\beta})]$. Finally, we denote the class of generalized median solutions by $\mathcal{F}_G$.*

Remark 1. By definition of $\mathcal{F}_G$ and $\mathscr{f}_G$, for each profile V, $F_G^{\alpha,\beta}(V) \equiv [\mathrm{med}(v, \tilde{\alpha}), \mathrm{med}(v, \tilde{\beta})] = [f_G^\alpha(V), f_G^\beta(V)]$. Therefore, a generalized median solution $F_G^{\alpha,\beta}$ can be decomposed into two generalized median rules f_G^α and f_G^β.

The next result considers *single-valued* generalized median solutions.

Lemma 2. *A generalized median solution $F_G^{\alpha,\beta}$ is single-valued if and only if $\alpha = \beta$. Moreover, in this case $F_G^{\alpha,\beta}$ is essentially a generalized median rule.*[16]

Proof. Let $F_G^{\alpha,\beta} \in \mathcal{F}_G$ and $f_G^\alpha, f_G^\beta \in \mathscr{f}_G$. Let $V \in \mathcal{R}^N$. By Remark 1, $F_G^{\alpha,\beta}(V) = [f_G^\alpha(V), f_G^\beta(V)]$. If $\alpha = \beta$, then $F_G^{\alpha,\beta} = \{f_G^\alpha(V)\}$. Hence, $F_G^{\alpha,\beta}$ is *single-valued*.

If $F_G^{\alpha,\beta}$ is *single-valued*, then $F_G^{\alpha,\beta} = [f_G^\alpha(V), f_G^\beta(V)]$ implies $f_G^\alpha(V) = f_G^\beta(V)$. Assuming $\alpha \neq \beta$ results in a contradiction as follows. Since $\alpha \neq \beta$, there exists a coalition $M \subseteq N$ such that

[16]To be precise, a single-valued generalized median solution assigns singleton sets of alternatives while the corresponding generalized median rule assigns the alternatives contained in these sets.

$\alpha_M \neq \beta_M$. Let $|M| = m$ and specify V such that for each agent $i \in M$, $v_i = 0$, and for each agent $j \in N \setminus M$, $v_j = 1$. Hence, at profile V, the mth coordinate of vectors $\tilde{\alpha}$ and $\tilde{\beta}$ will be α_M and β_M respectively. Moreover, by definition of $\mathcal{F}_G$, $\alpha_N \leq \alpha_{\{1,\ldots,n-1\}} \leq \cdots \leq \alpha_{\{1\}} \leq \alpha_\emptyset$ and $\beta_N \leq \beta_{\{1,\ldots,n-1\}} \leq \cdots \leq \beta_{\{1\}} \leq \beta_\emptyset$. Thus, there are at least $n+1-m$ coordinates of $\tilde{\alpha}$ not larger than α_M (i.e., coordinates $\alpha_M, \ldots, \alpha_N$) and at least $m+1$ coordinates of $\tilde{\alpha}$ not smaller than α_M (i.e., coordinates $\alpha_\emptyset, \ldots, \alpha_M$). Similarly, there are at least $n+1-m$ coordinates of $\tilde{\beta}$ not larger than β_M and at least $m+1$ coordinates of $\tilde{\beta}$ not smaller than β_M. Hence, $F_G^{\alpha,\beta} = [\text{med}(v, \tilde{\alpha}), \text{med}(v, \tilde{\beta})] = [\alpha_M, \beta_M]$ contradicting that $F_G^{\alpha,\beta}$ is *single-valued*. Therefore, $F_G^{\alpha,\beta}(V)$ being single-valued implies $\alpha = \beta$. □

Our results in Section 5 will also concern *efficient generalized median solutions*. Formally, given $F_G^{\alpha,\beta} \in \mathcal{F}_G$, if for each $V \in \mathcal{R}^N$, $F_G^{\alpha,\beta} \in \text{E}(V)$, we say that $F_G^{\alpha,\beta}$ is an *efficient generalized median solution* and denote the *class of efficient generalized median solutions* by $\mathcal{F}_{EG}$. The next result concerns this class of solutions.

Lemma 3. *A generalized median solution $F_G^{a,b}$ is an efficient generalized median solution if and only if vectors α, β are such that $\alpha_N = \beta_N = 0$ and $\alpha_\emptyset = \beta_\emptyset = 1$.*

Proof. Let $F_G^{\alpha,\beta} \in \mathcal{F}_G$. Assuming that $F_G^{\alpha,\beta} \in \mathcal{F}_{EG}$ such that α, β are not as described above, results in a contradiction as follows.

If $\alpha_N \neq 0$ or $\beta_N \neq 0$, choose $V \in \mathcal{R}^N$ such that $v = (0, \ldots, 0)$. By Proposition 1, $\text{E}(V) = 0$ and by the definition of $\mathcal{F}_G$, $F_G^{\alpha,\beta}(V) = [\alpha_N, \beta_N]$. Hence, $F_G^{\alpha,\beta}(V) \notin \text{E}(V)$. Similarly, if $\alpha_\emptyset \neq 1$ or $\beta_\emptyset \neq 1$, choose $V \in \mathcal{R}^N$ such that $v = (1, \ldots, 1)$. Again, by Proposition 1, $\text{E}(V) = 1$ and by the definition of $\mathcal{F}_G$, $F_G^{\alpha,\beta}(V) = [\alpha_\emptyset, \beta_\emptyset]$. Hence, $F_G^{\alpha,\beta}(V) \notin \text{E}(V)$. Finally, if $\alpha_N = \beta_N = 0$ and $\alpha_\emptyset = \beta_\emptyset = 1$, then for each $V \in \mathcal{R}^N$, $\text{med}(v, \tilde{\alpha}) \in \text{Conv}(v)$ and $\text{med}(v, \tilde{\beta}) \in \text{Conv}(v)$. Hence, by the definition of $\mathcal{F}_G$, $F_G^{\alpha,\beta}(V) \subseteq \text{Conv}(v)$, and thus by Proposition 1, $F_G^{\alpha,\beta}(V) \in \text{E}(V)$. □

3.2 Median rules and solutions

The second class of rules we consider was introduced under the name *strategy-proof and anonymous voting schemes* and characterized by *strategy-proofness*, *peaks-onliness*, and *anonymity*[17] (Moulin, 1980, Proposition 2). This class of rules is a simplification of generalized median rules since now all agents possess the same power in influencing the chosen alternative. For the 2-agent case, it suffices to set $\alpha_{\{1\}} = \alpha_{\{2\}}$ in Example 1 (page 8).

The general n-agent case works as follows. Take a vector a in A^{n+1}. For an announced profile V with associated vector of announced peaks v, the *median rule* associated with a chooses alternative $\text{med}(v, a)$.

Median rules. *Let vector $a \in A^{n+1}$ be such that $a \equiv (a_1, \dots, a_{n+1})$, where $a_1 \leq \dots \leq a_{n+1}$. We denote the median rule associated with vector a by f^a_M, where for each $V \in \mathcal{R}^N$, $f^a_M(V) \equiv \text{med}(v, a)$. Finally, we denote the class of median rules by $\mathscr{f}_M$.*

The second class of solutions we characterize in Section 5 extends the spirit of median rules to solutions. Specifically, take two vectors $a \leq b$, each of dimension $n+1$. For an announced profile V with associated vector of announced peaks v, the *median solution* associated with a and b chooses the interval where the minimum alternative is $\text{med}(v, a)$ and the maximum alternative is $\text{med}(v, b)$.

Median solutions. *Let vectors $a, b \in A^{n+1}$ be such that $a \equiv (a_1, \dots, a_{n+1})$ and $b \equiv (b_1, \dots, b_{n+1})$, with $a \leq b$, $a_1 \leq \dots \leq a_{n+1}$, and $b_1 \leq \dots \leq b_{n+1}$. We denote the median solution associated with vectors a and b by $f^{a,b}_M$, where for each $V \in \mathcal{R}^N$, $F^{a,b}_M(V) \equiv [\text{med}(v, a), \text{med}(v, b)]$. Finally, we denote the class of median solutions by $\mathcal{F}_M$.*

Remark 2. By definition of $\mathcal{F}_M$ and $\mathscr{f}_M$, for each profile V, $F^{a,b}_M(V) \equiv [\text{med}(v, a), \text{med}(v, b)] = [f^a_M(V), f^b_M(V)]$. Therefore, a median solution $F^{a,b}_M$ can be decomposed into two median rules f^a_M and f^b_M.

The next result formally proves that median rules and solutions are special cases of generalized median rules and solutions respectively.

Lemma 4. *The class of median rules (solutions) is a subclass of the class of generalized median rules (solutions).*

Proof. Let $F^{a,b}_M \in \mathcal{F}_M$ and $f^a_M, f^b_M \in \mathscr{f}_M$. Let $V \in \mathcal{R}^N$. By Remark 2, $F^{a,b}_M(V) = [f^a_M(V), f^b_M(V)]$. Hence, by Remark 1, it suffices to show that the class of median rules is a subclass of the class of generalized median rules.

[17] The names of the agents do not affect the chosen alternative.

Let $f_G^\alpha \in \mathscr{f}_G$ by choosing vector $\alpha \in A^{2^n}$ such that the weight of each coalition only depends on its cardinality. That is, for each $M \subseteq N$, choose $\alpha_M = a_{n+1-|M|}$ (i.e., choose $\alpha_\emptyset = a_{n+1}$, for each $i \in N$, choose $\alpha_{\{i\}} = a_n$, for each $i, j \in N$ with $i \neq j$, choose $\alpha_{\{i,j\}} = a_{n-1}$, and so on). Thus, $\tilde{\alpha} = a$. Therefore, $\text{med}(v, \tilde{\alpha}) = \text{med}(v, a)$ and by definition of $\mathscr{f}_G$ and $\mathscr{f}_M$, $f_G^\alpha(V) = f_M^a(V)$. $\square$

The next result considers single-valued median solutions.

Lemma 5. *A median solution $F_M^{a,b}$ is single-valued if and only if $a = b$. Moreover, in this case $F_M^{a,b}$ is essentially a median rule.*[18]

The proof of Lemma 5 follows from Lemmas 2 and 4.

Our results in Section 5 also concern *efficient median solutions*. Formally, given $F_M^{a,b} \in \mathcal{F}_M$, if for each $V \in \mathcal{R}^N$, $F_M^{a,b} \in \text{E}(V)$, we say that $F_M^{a,b}$ is an *efficient median solution* and denote the *class of efficient median solutions* by $\mathcal{F}_{EM}$. The next result concerns this class of solutions.

Lemma 6. *A median solution $F_M^{a,b}$ is an efficient median solution if and only if vectors a, b are such that $a_1 = b_1 = 0$ and $a_{n+1} = b_{n+1} = 1$.*

The proof of Lemma 6 follows from Lemmas 3 and 4.

4 Properties of solutions

In the sequel, all properties are defined for solutions in domain $\mathcal{R}$ but also apply to solutions in domain $\mathcal{S}$. Moreover, all results presented in this section hold in both domains.

The two first properties we consider are related; the first is our *efficiency* notion for solutions while the second, being weaker than the first, requires no alternative in A to be a priori excluded from being selected.

Efficiency. For each $V \in \mathcal{R}^N$, $F(V) \in \text{E}(V)$.

Voter-sovereignty. For each $x \in A$, there exists $V \in \mathcal{R}^N$ such that $F(V) = \{x\}$.

[18] To be precise, a single-valued median solution assigns singleton sets of alternatives while the corresponding median rule assigns the alternatives contained in these sets.

The next property, which is central in our results, requires no agent to gain by deviating. Moreover, it implies *comparability* between the chosen sets before and after an agent's deviation.

Strategy-proofness. For each $i \in N$, each $R_i \in \mathcal{R}$, and each $V \in \mathcal{R}^N$, $F(V_{-i}, R_i)\ R_i\ F(V_{-i}, V_i)$.

The next property requires the chosen set to depend only on the vector of announced peaks.

Peaks-onliness. For each pair $V, V' \in \mathcal{R}^N$ such that $v = v'$, $F(V) = F(V')$.

Loosely speaking, the next property requires when the announced preferences of an agent change "a little", the minimum and maximum alternatives chosen to not change "a lot". Before describing it formally, we must first define the three following notions. First, the "indifference relation", which -loosely speaking- given preferences $V_i \in \mathcal{R}$, maps each alternative x to an alternative y, that i finds indifferent to x, according to V_i. Formally, for each $V_i \in \mathcal{R}$, the *indifference relation* $r_{V_i} \colon [0,1] \to [0,1]$ is defined as follows. For each $x \in [0, v_i]$, $r_{V_i}(x) = y$ if $y \in [v_i, 1]$ exists such that $y\ I_i\ x$, or $r_{V_i}(x) = 1$ otherwise; while for each $x \in [v_i, 1]$, $r_{V_i}(x) = y$ if $y \in [0, v_i]$ exists such that $y\ I_i\ x$, or $r_{V_i}(x) = 0$ otherwise. Second, the distance between a pair $V_i, V_i' \in \mathcal{R}$, which is measured using the indifference relation. Formally, it is defined to be $\mathrm{d}(V_i, V_i') \equiv \max_{x \in [0,l]} |r_{V_i}(x) - r_{V_i'}(x)|$. Finally, the notion of convergence. Specifically, for $k \in \mathbb{N}^+$, a sequence $\{V_i^k\}$ in $\mathcal{R}$ converges to V_i, if $k \to \infty$ implies the distance $\mathrm{d}(V_i, V_i^k) \to 0$. We denote this convergence by $V_i^k \to V_i$.

Min/max continuity. For each $V \in \mathcal{R}^N$, each $i \in N$, and each $\{V_i^k\}$ in $\mathcal{R}$,

$$\text{if } V_i^k \to V_i, \text{ then } \begin{cases} \underline{F}(V_{-i}, V_i^k) \to \underline{F}(V), \text{ and} \\ \bar{F}(V_{-i}, V_i^k) \to \bar{F}(V). \end{cases}$$

Notice that *min/max continuity* for rules is equivalent to the regular *continuity* property for rules (with respect to the preference profile).

Lemma 7. *The following two statements for solutions are equivalent.*

(i) Min/max continuity is satisfied.

(ii) Upper-hemi continuity and lower-hemi continuity are satisfied.[19]

[19]Both properties are formally defined in Appendix A.

We prove Lemma 7 in Appendix A.

The next property requires that the agents' identities do not matter.

Anonymity. For each bijection $\sigma\colon N \to N$ and each pair $V, V' \in \mathcal{R}^N$ such that for each $i \in N$, $V_i = V'_{\sigma(i)}$, $F(V) = F(V')$.

The last property we consider depends only on the announced peaks of the agents. Loosely speaking, following a change in an agent's announced preferences, if before and after this change both announced peaks lie on the same side of the minimum (maximum) chosen alternative, then the minimum (maximum) chosen alternative does not change.

Uncompromisigness. For each $i \in N$ and each pair $V, V' \in \mathcal{R}^N$ such that $V'_{-i} = V_{-i}$,

$$\text{if } \begin{cases} v_i < \underline{F}(V) \text{ and } v'_i \leq \underline{F}(V) \text{ or} \\ v_i > \underline{F}(V) \text{ and } v'_i \geq \underline{F}(V), \end{cases} \quad \text{then } \underline{F}(V) = \underline{F}(V'), \text{ and}$$

$$\text{if } \begin{cases} v_i < \bar{F}(V) \text{ and } v'_i \leq \bar{F}(V) \text{ or} \\ v_i > \bar{F}(V) \text{ and } v'_i \geq \bar{F}(V), \end{cases} \quad \text{then } \bar{F}(V) = \bar{F}(V').$$

When a solution does not satisfy *uncompromisingness*, we say that it is *compromised*.

5 Results

We begin by presenting in Section 5.1 results concerning interrelations between the properties presented in Section 4. Said results are then used in our characterization results presented in Sections 5.2 and 5.3. Loosely speaking, in Section 5.2, we extend the characterizations of Moulin (1980, Propositions 2 and 3)[20] to solutions; while in Section 5.3, we show that in some -but not all- of these characterizations, *peaks-onliness* can be substituted by *min/max continuity*. Finally, in section 5.4 we show that the properties in all our characterization results are independent.

[20]Proposition 2: A rule satisfies *strategy-proofness*, *peaks-onliness*, and *anonymity* in $\mathcal{R}$ if and only if it is a median rule. Proposition 3: A rule satisfies *strategy-proofness* and *peaks-onliness* in $\mathcal{R}$ if and only if it is a generalized median rule.

5.1 Interrelations between properties

Our first result in this section holds in domain $\mathcal{S}$. It shows that if *strategy-proofness* is satisfied, then *efficiency* and *voter-sovereignty* are equivalent.

Proposition 2. *For strategy-proof solutions efficiency and voter-sovereignty are equivalent in domain $\mathcal{S}$.*

We prove Proposition 2 in Appendix B.

A similar equivalence result holds in domain $\mathcal{R}$, albeit slightly weaker since *peaks-onliness* or *min/max continuity* is also required. By Proposition 2, this result trivially holds in domain $\mathcal{S}$ as well.

Proposition 3. *The following two statements for strategy-proof solutions hold.*

(i) If peaks-onliness is satisfied, then efficiency and voter-sovereignty are equivalent.

(ii) If min/max continuity is satisfied, then efficiency and voter-sovereignty are equivalent.

We prove Proposition 3 in Appendix B.

The next result holds only in domain $\mathcal{R}$.[21] It shows that *strategy-proofness* and *peaks-onliness* are equivalent with *uncompromisingness*.

Proposition 4. *The following two statements for solutions are equivalent in domain $\mathcal{R}$.*

(i) Strategy-proofness and peaks-onliness are satisfied.

(ii) Uncompromisingness is satisfied.

We prove Proposition 4 in Appendix C.

The next result is in the spirit of Proposition 4 and holds in domain $\mathcal{S}$.

Proposition 5. *The following two statements for solutions are equivalent in domain $\mathcal{S}$.*

(i) Strategy-proofness and voter-sovereignty are satisfied.

(ii) Uncompromisingness and voter-sovereignty are satisfied.

We prove Proposition 5 in Appendix C.

Our final result also concerns *uncompromisingness*. It holds in both domains $\mathcal{R}$ and $\mathcal{S}$.

[21] An example of Proposition 4 not holding in domain $\mathcal{S}$ is illustrated by the counter-example on page 19.

Proposition 6. *Each solution satisfying* strategy-proofness, min/max continuity, *and* voter-sovereignty *also satisfies* uncompromisingness.

We prove Proposition 6 in Appendix C.

5.2 Results in the single-peaked domain $\mathcal{R}$

We now present our characterization results for (generalized) median solutions, as well as a counter-example justifying the absence of such results in some cases. All results hold in domain $\mathcal{R}$. The extension of these results in domain $\mathcal{S}$ is discussed in Section 5.3.

Our first result concerns the class of generalized median solutions.

Theorem 1. *The following three statements for a solution $F \in \mathcal{F}$ are equivalent.*

(i) F satisfies strategy-proofness *and* peaks-onliness.

(ii) F satisfies uncompromisingness.

(iii) F is a generalized median solution.

The equivalence of statements (i) and (ii) follows from Propostion 4. We prove the equivalence of statements (ii) and (iii) in Appendix D; note that this part of the proof also holds in domain $\mathcal{S}$.

Our second result concerns the class of median solutions.

Theorem 2. *The following three statements for a solution $F \in \mathcal{F}$ are equivalent.*

(i) F satisfies strategy-proofness, peaks-onliness, *and* anonymity.

(ii) F satisfies uncompromisingness *and* anonymity.

(iii) F is a median solution.

Proof. The equivalence of statements (i) and (ii) follows from Propostion 4. We proceed by showing the equivalence of statements (ii) and (iii) in two steps; note that this part of the proof also holds in domain $\mathcal{S}$.

Step 1 - (statement (ii) implies statement (iii)): Let $F \in \mathcal{F}$ satisfy *uncompromisingness* and *anonymity*. By Theorem 1, $F(V) = F_G^{\alpha,\beta}(V)$. Moreover, *anonymity* implies for $\alpha, \beta \in A^{2^n}$ that for each pair $L, M \subseteq N$, if $|L| = |M|$, then $\alpha_L = \alpha_M$ and $\beta_L = \beta_M$. Thus, for each $M \subseteq N$, let $a_{n+1-|M|} = \alpha_M$ and $b_{n+1-|M|} = \beta_M$, to effectively construct vectors $a, b \in A^{n+1}$. Next, let $F_M^{a,b} \in \mathcal{F}_M$ and notice that for each $V \in \mathcal{R}^N$, $F_G^{\alpha,\beta}(V) = [\text{med}(v, \tilde{\alpha}), \text{med}(v, \tilde{\beta})] = [\text{med}(v, a), \text{med}(v, b)] = F_M^{a,b}(V)$.

Step 2 - (statement (iii) implies statement (ii)): Let $F_M^{a,b} \in \mathcal{F}_M$. In addition, let $F_G^{\alpha,\beta} \in \mathcal{F}_G$ by choosing vectors $\alpha, \beta \in A^{2^n}$ such that the weight of each coalition only depends on its cardinality; specifically, for each $M \subseteq N$, $\alpha_M = a_{n+1-|M|}$ and $\beta_M = b_{n+1-|M|}$. Hence, for each $V \in \mathcal{R}^N$, $F_G^{\alpha,\beta}(V) = [\text{med}(v, \tilde{\alpha}), \text{med}(v, \tilde{\beta})] = [\text{med}(v, a), \text{med}(v, b)] = F_M^{a,b}(V)$. Therefore, by Theorem 1, $F_M^{a,b}$ satisfies *uncompromisingness* and by the definition of $\mathcal{F}_M$, $F_M^{a,b}$ satisfies *anonymity*. □

Next, we show that in Theorem 1 and Theorem 2 *peaks-onliness* cannot be substituted with *min/max continuity*. We illustrate this in the counter-example that follows by proposing a solution satisfying *strategy-proofness*, *min/max continuity*, and *anonymity* and violating *voter-sovereignty* -and more importantly- *uncompromisingness*; which, as shown in the aforementioned theorems, is satisfied by both classes of generalized median solutions and median solutions. Moreover, as explained in Section 5.3, this example also illustrates that the aforementioned theorems cannot be extended in domain $\mathcal{S}$.

Counter-example. *Let $|N| \geq 1$ and define $r_V^* \equiv \max\{r_{V_i}(0)\}_{i \in N}$, that is, at announced profile V, among the indifferent announced alternatives to 0 of each agent $i \in N$, r_V^* is the largest one. Next, define $F^* \in \mathcal{F}$ as follows. For each $V \in \mathcal{R}^N$, $F^*(V) = [0, r_V^*]$. By definition, it follows that F^* satisfies min/max continuity and anonymity, and that it violates voter-sovereignty. We proceed in 2 steps*

Step 1: *We show F^* satisfies strategy-proofness. Let $V \in \mathcal{R}^N$ ($V \in \mathcal{S}^N$) be such that $i \in N$ is sincere, i.e., $V_i = R_i$. Also, let $V_i' \in \mathcal{R}$ ($V_i' \in \mathcal{S}$) such that $V_i' \neq V_i$. There are two cases.*

Case 1. Let $r_{V_i}(0) = r_V^*$. By single-peakedness, $b_{R_i}(F^*(V)) = \{p_i\}$, implying i's best point does not improve by deviating at V, and $0 \in w_{R_i}(F^*(V))$. By the definition of F^*, $0 \in F^*(V_{-i}, V_i')$, hence i's worst point(s) does not improve by deviating at V. Therefore, $F^*(V) \; R_i \; F^*(V_{-i}, V_i')$.

Case 2. Let $r_{V_i}(0) < r_V^*$. By single-peakedness, $b_{R_i}(F^*(V)) = \{p_i\}$, implying i's best point does not improve by deviating at V, and $w_{V_i}(F^*(V)) = \{r_V^*\}$. By the definition of F^*, $r_V^* \in F^*(V_{-i}, V_i')$, hence i's worst point does not improve by deviating at V. Therefore, $F^*(V) \; R_i \; F^*(V_{-i}, V_i')$.

Step 2: *We show that F^* can be compromised. Let $N = \{1, \dots\}$. Let pair $V, V' \in \mathcal{S}^N$ be defined as follows: $V_{-1} = V'_{-1}$, $v_1 = 0.2$, $v'_1 = 0.3$, and for each $i \in N \setminus \{1\}$, $v_i = 0$. Hence, $r^*_V = r_{V_1}(0) = 0.4$ and $r^*_{V'} = r_{V'_1}(0) = 0.6$. Therefore, $F(V) = [0, 0.4]$ and $F(V') = [0, 0.6]$. Clearly, F is compromised.*

We conclude this section by presenting the "efficient versions" of Theorems 1 and 2. Notice that now *peaks-onliness* and *min/max continuity* become substitutable.

Theorem 3. *The following four statements for a solution $F \in \mathcal{F}$ are equivalent.*

(i) F satisfies strategy-proofness, peaks-onliness, and voter-sovereignty.

(ii) F satisfies uncompromisingness and voter-sovereignty.

(iii) F satisfies strategy-proofness, min/max continuity, and voter-sovereignty.

(iv) F is an efficient generalized median solution.

Proof. The equivalence of statements (i) and (ii) follows from Propostion 4. The equivalence of statement (ii) and (iv) is shown as follows. By Theorem 1, statement (ii) implies $F \in \mathcal{F}_G$. Hence, by the definition of $\mathcal{F}_G$, F satisfies *peaks-onliness*. Thus, Proposition 3 and the definition of $\mathcal{F}_{EG}$ imply $F \in \mathcal{F}_{EG}$, i.e., statement (ii) implies statement (iv). Moreover, by Theorem 1 and $\mathcal{F}_{EG} \subsetneq \mathcal{F}_G$, statement (iv) implies F satisfies *uncompromisingness*; in addition, by the definition of $\mathcal{F}_{EG}$, statement (iv) implies F satisfies *efficiency* and therefore *voter-sovereignty*, i.e., statement (iv) implies statement (ii). Finally, notice that this equivalence of statements (ii) and (iv) also holds in domain $\mathcal{S}$.

Next, by Proposition 6, statement (iii) implies statement (ii). We complete the proof by showing statement (ii) implies statement (iii). By Step 1 of the proof of Theorem 1(statement (ii) implies statement (iii)) on page 42, if $F \in \mathcal{F}$ satisfies *uncompromisingness*, then for $V \in \mathcal{R}^N$ and each $i \in N$ the following holds. If V_i^0 is such that $v_i^0 = 0$ and V_i^1 is such that $v_i^1 = 1$, then $\underline{F}(V) = \text{med}(\underline{F}(V), \underline{F}(V_{-i}, V_i^0), \underline{F}(V_{-i}, V_i^1))$ and $\bar{F}(V) = \text{med}(\bar{F}(V), \bar{F}(V_{-i}, V_i^0), \bar{F}(V_{-i}, V_i^1))$. It follows by the median operator, that statement (ii) implies F satisfies *min/max continuity*. Finally, by the equivalence of statements (i) and (ii), statement (ii) implies F satisfies *strategy-proofness*. □

Theorem 4. *The following four statements for a solution $F \in \mathcal{F}$ are equivalent.*

(i) F satisfies strategy-proofness, peaks-onliness, anonymity, and voter-sovereignty.

(ii) F satisfies uncompromisingness, anonymity, and voter-sovereignty.

(iii) F satisfies strategy-proofness, min/max continuity, anonymity, and voter-sovereignty.

(iv) F is an efficient median solution.

Proof. The equivalence of statements (i), (ii), and (iii) follows from Theorem 3. The equivalence of statements (ii) and (iv) is shown as follows. By Theorem 2, statement (ii) implies $F \in \mathcal{F}_M$. Hence, Proposition 3 and the definition of $\mathcal{F}_{EM}$ imply $F \in \mathcal{F}_{EM}$, i.e., statement (ii) implies statement (iv). Moreover, by Theorem 2 and $\mathcal{F}_{EM} \subsetneq \mathcal{F}_M$, statement (iv) implies F satisfies *uncompromisingness*, and *anonymity*; in addition, by the definition of $\mathcal{F}_{EM}$, statement (iv) implies F satisfies *efficiency* and therefore *voter-sovereignty*, i.e., statement (iv) implies statement (ii). Finally, notice that this equivalence of statements (ii) and (iv) also holds in domain $\mathcal{S}$. □

5.3 Results in the single-peaked and symmetric domain $\mathcal{S}$

We now show the characterizations in domain $\mathcal{S}$ that are -loosely speaking- equivalent to those presented in Section 5.2 for domain $\mathcal{R}$. Specifically, the non-efficient characterizations in domain $\mathcal{R}$ (Theorems 1 and 2) cannot be extended in domain $\mathcal{S}$. This is illustrated by the counter-example presented on page 19, where the suggested solution satisfies *strategy-proofness*, *anonymity*, and *min/max continuity* in domain $\mathcal{S}$ but violates *uncompromisingness*. This violation is of importance because as shown by the proof of Theorem 1 (statement (iii) implies statement (ii)) on page 39 -which also holds in domain $\mathcal{S}$- both classes of generalized median solutions and median solutions satisfy *uncompromisingness*.

Concerning the efficient characterizations in domain $\mathcal{R}$, these do extend in domain $\mathcal{S}$. Moreover, since *single-peakedness* is an inherent property of domain $\mathcal{S}$, *min/max continuity* is unnecessary in these characterizations.

Theorem 5. *The following three statements for a solution $F \in \mathcal{F}$ are equivalent.*

(i) F satisfies strategy-proofness and voter-sovereignty.

(ii) F satisfies uncompromisingness and voter-sovereignty.

(iii) F is an efficient generalized median solution.

Proof. The equivalence of statements (i) and (ii) follows from Proposition 5. The equivalence of statements (ii) and (iii) follows from Theorem 3 (recall that as noted on page 20, statements (ii) and (iv) of Theorem 3 are also equivalent in domain $\mathcal{S}$). □

Theorem 6. *The following three statements for a solution $F \in \mathcal{F}$ are equivalent.*

(i) F satisfies strategy-proofness, voter-sovereignty, and anonymity.

(ii) F satisfies uncompromisingness, voter-sovereignty, and anonymity.

(iii) F is an efficient median solution.

Proof. The equivalence of statements (i) and (ii) follows from Proposition 5. The equivalence of statements (ii) and (iii) follows from Theorem 4 (recall that as noted on page 21, statements (ii) and (iv) of Theorem 4 are also equivalent in domain $\mathcal{S}$). □

5.4 Independence of properties

Concerning the independence of the properties used in all our results, consider the following four solutions. First, solution F^* proposed by the counter-example on page 19. Second, let solution $F_1 \in \mathcal{F}$ choose the minimum announced peak when more than two agents prefer it against the maximum announced peak, and choose the maximum announced peak otherwise. Hence in domain $\mathcal{R}$, F_1 satisfies *strategy-proofness*, *anonymity*, and *voter-sovereignty* but violates *peaks-onliness*. Third, for a small and positive value ε let solution $F_2 \in \mathcal{F}$ choose the minimum of: (a) the minimum announced peak plus ε and (b) the maximum peak. Hence in both domains $\mathcal{R}$ and $\mathcal{S}$, F_2 satisfies *peaks-onliness*, *min/max continuity*, *anonymity*, and *voter-sovereignty* but violates *strategy-proofness*. Finally, let $F_3 \in \mathcal{F}$ be the "constant" solution that always chooses 0. Hence in both domains $\mathcal{R}$ and $\mathcal{S}$, F_3 satisfies *anonymity* and *uncompromisingness*.

The summary table that follows has a double purpose. First, columns $\mathcal{F}_G$, $\mathcal{F}_{EG}$, $\mathcal{F}_M$, and $\mathcal{F}_{EM}$, denoting the classes of generalized median solutions, efficient generalized median solutions, median solutions, and efficient median solutions respectively, summarize our characterization results. For example, in the column referring to $\mathcal{F}_{EG}$, the circle containing number 4 shows that in domain $\mathcal{S}$, *strategy-proofness* and *voter-sovereignty* characterize the class of efficient generalized median solutions. The table also shows the independence of our properties in all our characterization results. Specifically, all combinations of properties that need to be checked to check the aforementioned independence, are satisfied by at least one of the four non-median solutions proposed in the previous paragraph, as shown by columns F^*, F_1, F_2, and F_3; this is shown by a check mark in each respective column. Finally, notice that for the reader's convenience, all results in domain $\mathcal{R}$ are shown using circled black numbers in a white background, while all results in domain $\mathcal{S}$ are shown using circled white numbers in a black background.

	$\mathcal{F}_G$	$\mathcal{F}_{EG}$	$\mathcal{F}_M$	$\mathcal{F}_{EM}$	F^*	F_1	F_2	F_3
Domain $\mathcal{R}$ ○	① ②	① ② ③	① ②	① ② ③	✓	✓	✓	
Domain $\mathcal{S}$ ●		❹ ❺		❹ ❺	✓		✓	✓
Strategy-proofness	①	① ② ❹	①	① ② ❹	✓	✓		
Peaks-onliness	①	①	①	①			✓	
Min/max continuity		②		②	✓		✓	
Voter-sovereignty		① ② ③ ❹ ❺		① ② ③ ❹ ❺		✓	✓	
Anonymity			① ②	① ② ③ ❹ ❺	✓	✓	✓	✓
Uncompromisingness	②	③ ❺	②	③ ❺				✓

$\mathcal{F}_G$: class of generalized median solutions, $\mathcal{F}_{EG}$: class of efficient generalized median solutions, $\mathcal{F}_M$: class of median solutions, and $\mathcal{F}_{EM}$: class of efficient median solutions.

Appendices

A Proof of Lemma 7

In the sequel, all properties are defined for solutions in domain $\mathcal{R}$ but also apply to solutions in domain $\mathcal{S}$. Moreover, all results presented in this section, hold in both domains.

Upper-hemi continuity. For each $V \in \mathcal{R}^N$, each $i \in N$, each $\{V_i^k\}$ in $\mathcal{R}$ such that $V_i^k \to V_i$, and each $\{x^k\}$ in A such that $x^k \to x$, the following holds. For each $k \in \mathbb{N}^+$, if $x^k \in F(V_{-i}, V_i^k)$, then $x \in F(V)$.

Lower-hemi continuity. For each $V \in \mathcal{R}^N$, each $i \in N$, and each $\{V_i^k\}$ in $\mathcal{R}$ such that $V_i^k \to V_i$, the following holds. If $x \in F(V)$, then there exists $\{x^k\}$ in A such that $x^k \to x$ and for each $k \in \mathbb{N}^+$, $x^k \in F(V_{-i}, V_i^k)$.

Proof of Lemma 7. Let $F \in \mathcal{F}$. If F satisfies *upper-hemi continuity* and *lower-hemi continuity* then trivially, it also satisfies *min/max continuity*. Next, let F satisfy *min/max continuity*, $V \in \mathcal{R}^N$, and $\{V_i^k\}$ be in $\mathcal{R}$. We show that F satisfies *upper-hemi continuity* and *lower-hemi continuity* in two steps.

Step 1. We show F satisfies *upper-hemi continuity*. Let $\{x^k\}$ in A such that $x^k \to x$ and for each $k \in \mathbb{N}^+$, $x_k \in F(V_{-i}, V_i^k)$. Hence, for each $k \in \mathbb{N}^+$, $\underline{F}(V_{-i}, V_i^k) \leq x^k \leq \bar{F}(V_{-i}, V_i^k)$. Moreover, by *min/max continuity*, $\underline{F}(V_{-i}, V_i^k) \to \underline{F}(V)$ and $\bar{F}(V_{-i}, V_i^k) \to \bar{F}(V)$, which implies $\underline{F}(V) \leq x \leq \bar{F}(V)$ since otherwise, *min/max continuity* would imply that there exists $k^* \in \mathbb{N}^+$ such that $x_{k^*} \notin F(V_{-i}, V_i^{k^*})$. Therefore, $x \in F(V)$.

Step 2. We show F satisfies *lower-hemi continuity*. Let $x \in F(V)$. By *min/max continuity*, $\underline{F}(V_{-i}, V_i^k) \to \underline{F}(V)$ and $\bar{F}(V_{-i}, V_i^k) \to \bar{F}(V)$. Moreover, there exists $\{x^k\}$ in A, such that for each $k \in \mathbb{N}^+$, $\underline{F}(V_{-i}, V_i^k) \leq x^k \leq \bar{F}(V_{-i}, V_i^k)$. Therefore, by *min/max continuity*, $x^k \to x$. □

B Proofs of Propositions 2 and 3

Before proceeding to the proofs of Propositions 2 and 3, we show the following. When strategy-proofness is satisfied, *voter sovereignty* is equivalent with *unanimity*; a property stronger than *voter-sovereignty* but weaker than *efficiency*, that requires when all agents announce the same peak, only this peak to be chosen.[22] This result holds in both domains.

Lemma 8. *For strategy-proof solutions, voter-sovereignty and unanimity are equivalent.*

Proof. Let $F \in \mathcal{F}$. Trivially, *unanimity* implies *voter sovereignty*. Hence, let F satisfy *strategy-proofness* and *voter-sovereignty*. We show that F satisfies *unanimity*.

Let $a \in A$ and $R \in \mathcal{R}^N$ be such that $p = (a, \ldots, a)$. By *voter-sovereignty*, there exists $V \in \mathcal{R}^N$ such that $F(V) = a$. Let $M \subseteq N$ contain all the agents in N whose announced peak at V is not a, i.e., for each $i \in M$, $v_i \neq a$, and for each $j \in N \setminus M$, $v_j = p_j = a$. Without loss of generality, index the agents in N such that $M = \{1, \ldots, m\}$. Next, consider profile $V^1 = (V_{-1}, R_1)$. By *strategy-proofness*, $F(V^1)\ R_1\ F(V)$. Hence, single-peakedness

[22]Formally, given $F \in \mathcal{F}$, for each $V \in \mathcal{R}^N$ such that $v = (x, \ldots, x)$, $F(V) = \{x\}$.

and $F(V) = a = p_1$ imply $F(V^1) = a$. Finally, for each $k = \{2, \dots, m\}$ in increasing indexing order, consider profile $V^k = (V^{k-1}_{-k}, R_k)$. By the arguments presented for V^1, $F(V^k) = F(V) = a$. Therefore, since $V^m = R$, $F(V^m) = F(R) = a$. $\square$

We proceed with the proof of Proposition 2, which makes use of Lemma 8. This proof holds only in domain $\mathcal{S}$ because it makes use of the inherent *peaks-onliness* of this domain.

Proof of Proposition 2. Let $F \in \mathcal{F}$ satisfy *strategy-proofness*. Let $V \in \mathcal{S}^N$ and without loss of generality, assume $v_1 \leq \cdots \leq v_n$. The equivalence of *unanimity* and *voter-sovereignty* follows by Lemma 8. In addition, it is trivial to show that *efficiency* implies *unanimity*. Therefore, it remains to show that *unanimity* implies *efficiency*. We do so by contradiction; specifically, we show that if $F(V) \notin \mathrm{E}(V)$, then *unanimity* is violated. There are two cases.

Case 1. Let $\bar{v} < \bar{F}(V)$. For all agents $i \in N$, define $V'_i \in \mathcal{S}$ to be such that $v'_i = v_n$.

First, consider agent 1, where $\underline{v} = v_1 \leq \bar{v} < \bar{F}(V)$. By single-peakedness, either $[\bar{F}(V) \subseteq w_{V_1}(F(V))$ and thus $w_{V_1}(F(V)) \notin \mathrm{E}(V)]$ or $[\underline{F}(V) = w_{V_1}(F(V))$ and thus $v_1 = \underline{v} < \bar{F}(V)$ and single-peakedness imply $\underline{F}(V) < \underline{v}$, and therefore, $w_{V_1}(F(V)) \notin \mathrm{E}(V)]$. Next, recall the indifference relation r_{V_i} and let $x_1 = r_{V_1}(w_{V_1}(F(V)))$.[23] If $\bar{F}(V) \subseteq w_{V_1}(F(V))$, then $v_1 < \bar{F}(V)$ and single-peakedness imply $x_1 < v_1 = \underline{v}$ and hence, $x_1 \notin \mathrm{E}(V)$. Similarly, if $\underline{F}(V) = w_{V_1}(F(V))$, then $\underline{F}(V) < \underline{v} \leq \bar{v} < \bar{F}(V)$ and single-peakedness imply $x_1 > \bar{v}$ and hence, $x_1 \notin \mathrm{E}(V)$. Finally, assume $R_1 = V_1$ and consider $V^1 = (V_{-1}, V'_1)$. By *strategy-proofness*, $w_{R_1}(F(V))\ R_1\ w_{R_1}(F(V^1))$; hence, $w_{V_1}(F(V)) \notin \mathrm{E}(V)$, $x_1 \notin \mathrm{E}(V)$, and single-peakedness implies $w_{R_1}(F(V^1)) \notin \mathrm{E}(V)$. Therefore, $\mathrm{E}(V^1) \subseteq \mathrm{E}(V)$ implies $w_{R_1}(F(V^1)) \not\subseteq \mathrm{E}(V^1)$; and thus, $F(V^1) \notin \mathrm{E}(V^1)$.

Next, consider agent 2 at profile V^1 and recall that $w_{R_1}(F(V^1)) \notin \mathrm{E}(V^1)$ and $x_1 \notin \mathrm{E}(V)$. Let $x^2 = r_{V_2}(w_{V_2}(F(V^1)))$. If $\underline{F}(V^1) \subseteq w_{R_1}(F(V^1))$, then $\underline{F}(V^1) < v_1$; hence, $v_1 \leq v_2$, single-peakedness, and $V_2 \in \mathcal{S}$ imply $\underline{F}(V^1) \subseteq w_{V_2}(F(V^1))$ and $x^2 \geq \bar{F}(V)$.[24] Thus, $w_{V_2}(F(V^1)), x_2 \notin \mathrm{E}(V^1)$. If $\bar{F}(V^1) = w_{R_1}(F(V^1))$, then $\bar{v}^1 < \bar{F}(V^1)$; hence, $v_2 \leq \bar{v}^1$ and

[23] To be precise, if agent 1 has two worst points on $F(V)$, then with some abuse of notation, assume $w_{V_1}(F(V))$ is the smallest of the two worst points, which implies that x_1 is then the largest of the two worst points.

[24] To be precise, if agent 2 has two worst points on $F(V^1)$, then with some abuse of notation, assume $w_{V_2}(F(V^1))$ is the smallest of the two worst points, which implies that x_2 is then the largest of the two worst points.

single-peakedness imply either $[\bar{F}(V^1) \subseteq w_{V_2}(F(V^1))$ and $x_2 < \underline{v}_1]$ or $[\underline{F}(V^1) = w_{V_2}(F(V^1))$ and $\bar{v}^1 < \bar{F}(V^1) < x_2]$. Thus, $w_{V_2}(F(V^1)), x_2 \notin \mathrm{E}(V^1)$. Therefore, by the arguments presented for V^1, $F(V^2) \notin \mathrm{E}(V^2)$.

Finally, for each $k \in \{3, \ldots, n-1\}$, in increasing order, consider profile $V^k = (V^{k-1}_{-k}, V'_k)$. By the arguments presented for agents 1 and 2 above, $F(V^k) \notin \mathrm{E}(V^k)$. Therefore, at profile V^{n-1} where $v^{n-1} = (v_n, \ldots, v_n)$, $F(V^{n-1}) \notin \mathrm{E}(V^{n-1})$ implying $F(V^{n-1}) \neq \{v_n\}$ which contradicts *unanimity*.

Case 2. Let $\underline{v} > \underline{F}(V)$. The proof is symmetric to Case 1. $\square$

Notice that although for didactic reasons Proposition 3 proceeds Proposition 4 in the main text, the proof of Proposition 3 makes use of Proposition 4 (proof in Appendix C). Recall that this result holds in both domains.

Proof of Proposition 3. Let $F \in \mathcal{F}$ satisfy *strategy-proofness*. By Lemma 8, *unanimity* and *voter-sovereignty* are equivalent. In addition, it is easy to show that *efficiency* implies *unanimity*.

The proof proceeds in three steps. In Step 1 we show that if in addition to *strategy-proofness*, F satisfies *peaks-onliness* and *unanimity*, the following holds. Given an announced profile where an efficient set is chosen, if an agent with the minimum -but not unique- announced peak changes his announcement by moving his announced peak to the right, an efficient set is chosen again. Step 2 shows the same result but for the case where in addition to *strategy-proofness*, F satisfies *min/max continuity* and *unanimity*. Finally in Step 3, by *unanimity* and the intermediate results of Steps 1 and 2, we show that F satisfies *efficiency*.

Step 1. In addition to *strategy-proofness*, let F satisfy *peaks-onliness* and *unanimity*. By Proposition 4, F also satisfies *uncompromisingness*. Let $V \in \mathcal{R}^N$ and $i \in N$ be such that $F(V) \in \mathrm{E}(V)$ and $v_i = \underline{v}$ but where i does not have the unique minimum peak. Hence, by Proposition 1, $F(V) \subseteq \mathrm{Conv}(v)$. In addition, let $V'_i \in \mathcal{R}$ be such that $v'_i \geq \bar{v}$. Assuming $\underline{F}(V_{-i}, V'_i) < \underline{v}$ implies $\underline{F}(V_{-i}, V'_i) < v_i \leq v'_i$. Hence, by uncompromisingness, $\underline{F}(V_{-i}, V'_i) = \underline{F}(V) < \underline{v}$, which contradicts $F(V) \in \mathrm{E}(V)$. Similarly, assuming $\bar{F}(V_{-i}, V'_i) > v'_i$ implies $\bar{F}(V_{-i}, V'_i) > v'_i \geq v_i$. Hence, by uncompromisingness, $\bar{F}(V_{-i}, V'_i) = \bar{F}(V) > \bar{v}$, which contradicts $F(V) \in \mathrm{E}(V)$. Therefore, $F(V_{-i}, V'_i) \in \mathrm{E}(V)$.

Step 2. In addition to *strategy-proofness*, let F satisfy *min/max continuity* and *unanimity*. Define $V \in \mathcal{R}^N$ and $V_i' \in \mathcal{R}$ as in Step 1. By single-peakedness, $w_{V_i'}(F(V)) = \underline{F}(V)$. We show that $F(V_{-i}, V_i') \in \mathrm{E}(V_{-i}, V_i')$ by discrediting all three cases where $F(V_{-i}, V_i') \notin \mathrm{E}(V_{-i}, V_i')$.

Case 1. Let $\underline{F}(V_{-i}, V_i') < \underline{v}$. In addition, let $R_i = V_i'$. By single-peakedness, $\underline{F}(V) P_i \underline{F}(V_{-i}, V_i')$, hence $w_{R_i}(F(V)) = \underline{F}(V)$ implies $w_{R_i}(F(V)) P_i w_{R_i}(F(V_{-i}, V_i'))$. Therefore, if at profile (V_{-i}, V_i') agent i deviates by announcing V_i, his worst point improves. This contradicts *strategy-proofness*.

Case 2. Let $\underline{F}(V_{-i}, V_i') > v_i'$. Since $\underline{F}(V) \leq \bar{F}(V) \leq v_i'$, by *min/max continuity*, there exists some profile $V_i'' \in \mathcal{R}$ such that $V_i'' \neq V_i'$ and $v_i' \in F(V_{-i}, V_i'')$. Let $R_i = V_i'$; hence, $b_{R_i}(F(V_{-i}, V_i'')) = v_i'$ implying $b_{R_i}(F(V_{-i}, V_i'')) P_i b_{R_i}(F(V_{-i}, V_i'))$. Therefore, if at profile (V_{-i}, V_i') agent i deviates by announcing V_i'', his best point improves. This contradicts *strategy-proofness*.

Case 3. Let $\underline{v} \leq \underline{F}(V_{-i}, V_i') \leq v_i'$ and $\bar{F}(V_{-i}, V_i') > v_i'$. In the following, we describe a series of actions that when performed in sequence construct -after a finite number of "moves"- profile V', such that $v' = (v_i', \ldots, v_i')$ and $F(V') \neq v_i'$, i.e., a profile at which *unanimity* is violated.

Action 1. Let profile $V^0 = (V_{-i}, V_i')$. Let $N_1 \subsetneq N$ be such that $j \in N_1$ if and only if $v_j^0 \neq v_i'$ and $v_j^0 < \underline{F}(V^0)$. If $N_1 = \emptyset$, then proceed to Action 2. Otherewise, let $j \in N_1$. By $v_j^0 < \underline{F}(V^0) \leq \bar{F}(V^0)$ and single-peakedness, $b_{V_j^0}(F(V^0)) = \underline{F}(V^0)$ and $w_{V_j^0}(F(V^0)) = \bar{F}(V^0)$. Let $V_j^1 = V_i'$ and profile $V^1 = (V_{-j}^0, V_j^1)$. Assume $R_j = V_j^0$. By *strategy-proofness*, $b_{R_j}(F(V^0))\, R_j\, b_{R_j}(F(V^1))$ and $w_{R_j}(F(V^0))\, R_j\, w_{R_j}(F(V^1))$; hence, by single-peakedness, either $[\underline{F}(V^0) \leq \underline{F}(V^1)$ and $\bar{F}(V^0) \leq \bar{F}(V^1)]$ or $[\underline{F}(V^1) \leq \bar{F}(V^1) < v_j^0]$. However, if $\underline{F}(V^1) \leq \bar{F}(V^1) < v_j^0$, then by *min/max continuity* there exist preference $V_j^* \in \mathcal{R}$ such that $v_j^0 \in F(V_{-j}^0, V_j^*)$. This violates *strategy-proofness* since if at profile V^0 agent j deviates by announcing V^*, his best point improves. Hence, $\underline{F}(V^0) \leq \underline{F}(V^1)$ and $\bar{F}(V^0) \leq \bar{F}(V^1)$. Therefore, $v_i' < \bar{F}(V^1)$.

Next, let $N_2 \subsetneq N$ be such that $k \in N_2$ if and only if $v_k^1 \neq v_i'$ and $v_k^1 < \underline{F}(V^1)$. If $N_2 = \emptyset$, then proceed to Action 2. Otherwise, let $k \in N_2$. In addition, let $V_k^2 = V_i'$ and profile $V^2 = (V_{-k}^1, V_k^2)$. By the process described in the previous paragraph for agent j, $v_i' < \bar{F}(V^2)$.

Finally, repeat this process μ times (where μ is smaller than the number of agents, $\mu \leq n-1$) until the following holds. Set $N_\mu \subsetneq N$, constructed similarly to N_1 and N_2, is empty. When this occurs, proceed to Action 2.

Action 2. Let profile $\bar{V}^0 = V^{\mu-1}$. Let $\bar{N}_1 \subsetneq N$ be such that $j \in \bar{N}_1$ if and only if $\bar{v}_j^0 \neq v_i'$ and $\bar{F}(\bar{V}^0) \subseteq w_{\bar{V}_j^0}(F(\bar{V}^0))$. Recall that $\bar{F}(\bar{V}^0) > v_i'$. If $\bar{N}_1 = \emptyset$, then proceed to Action 3. Otherwise, let $j \in \bar{N}_1$ and notice that by $N_\mu = \emptyset$ (as defined in Action 1), the choice of $\bar{N}_1$ implies $\underline{F}(\bar{V}^0) \leq \bar{v}_j^0 < \bar{F}(\bar{V}^0)$. Define $\bar{V}_j^1 \in \mathcal{R}$ such that $\bar{v}_j^1 = v_i'$ and $w_{\bar{V}_j^1}(F(\bar{V}^0)) = \underline{F}(\bar{V}^0)$, and let profile $\bar{V}^1 = (\bar{V}_{-j}^0, \bar{V}_j^1)$. Assume that $R_j = \bar{V}_j^0$. By *strategy-proofness*, $w_{R_j}(F(\bar{V}^0))\ R_j\ w_{R_j}(F(\bar{V}^1))$; hence, by single-peakedness, $\bar{F}(\bar{V}^0) \leq \bar{F}(\bar{V}^1)$ and perhaps, $\underline{F}(\bar{V}^0) > \underline{F}(\bar{V}^1)$. Assume that $R_j = \bar{V}_j^1$. If $\underline{F}(\bar{V}^1) < \underline{F}(\bar{V}^0) < \bar{v}_j^1$, then single-peakedness implies $w_{R_j}(F(\bar{V}^0))\ P_j\ w_{R_j}(F(\bar{V}^1))$. This violates *strategy-proofness* since if at profile $\bar{V}^1$ agent j deviates by announcing $\bar{V}_j^0$, his worst point improves. Therefore, $\bar{F}(\bar{V}^0) \leq \bar{F}(\bar{V}^1)$ and $\underline{F}(\bar{V}^0) \leq \underline{F}(\bar{V}^1)$. Hence, $v_i' < \bar{F}(\bar{V}^1)$.

Next, if $\underline{F}(\bar{V}^0) < \underline{F}(\bar{V}^1)$, perhaps there exist some agents $\bar{j} \in N$ such that $v_{\bar{j}} < \underline{F}(\bar{V}^1) \leq \bar{F}(\bar{V}^1)$. If this is the case, then repeat the process described in Action 1 and denote the resulting profile (again) by $\bar{V}^1$. If no such agents exist, then $\bar{V}^1$ is the profile constructed in the end of the previous paragraph.

Following this, let $\bar{N}_2 \subsetneq N$ be such that $k \in \bar{N}_2$ if and only if $\bar{v}_k^1 \neq v_i'$ and $\bar{F}(\bar{V}^1) \subseteq w_{\bar{V}_k^1}(F(\bar{V}^1))$, where $\bar{F}(\bar{V}^1) > v_i'$. If $\bar{N}_2 = \emptyset$, then proceed to Action 3. Otherwise, let $k \in \bar{N}_2$ and notice that either by $N_\mu = \emptyset$ (as defined in Action 1), or by Action 1 being repeated in the previous paragraph, the choice of $\bar{N}_2$ implies $\underline{F}(\bar{V}^1) \leq \bar{v}_k^1 < \bar{F}(\bar{V}^1)$. Define $\bar{V}_k^2 \in \mathcal{R}$ such that $\bar{v}_k^2 = v_i'$ and $w_{\bar{V}_k^2}(F(\bar{V}^1)) = \underline{F}(\bar{V}^1)$, and let profile $\bar{V}^2 = (\bar{V}_{-k}^1, \bar{V}_k^2)$. By the process described above for agent j, $\bar{F}(\bar{V}^1) \leq \bar{F}(\bar{V}^2)$ and $\underline{F}(\bar{V}^1) \leq \underline{F}(\bar{V}^2)$. Moreover, if $\underline{F}(\bar{V}^1) < \underline{F}(\bar{V}^2)$, perhaps Action 1 needs to be repeated as explained in the previous paragraph. In this case, $\bar{V}^2$ is the resulting profile after repeating Action 1, otherwise, $\bar{V}^2$ remains unchanged. In both cases, $v_i' < \bar{F}(\bar{V}^2)$.

Finally, repeat this process for a finite integer μ (where μ is smaller than the number of agents, $\mu \leq n-1$) until the following holds. Set $\bar{N}_\mu \subsetneq N$, constructed similarly to $\bar{N}_1$ and $\bar{N}_2$, is empty. Notice that $v_i' < \bar{F}(\bar{V}^{\mu-1})$ and proceed to Action 3.

Action 3. Let profile $\hat{V}^0 = \bar{V}^{\mu-1}$ and recall that $v_i' < \bar{F}(\hat{V}^0)$. Let $\hat{N} \subsetneq N$ be such that $j \in \hat{N}$ if and only if $\hat{v}_j^0 \neq v_i'$ and $w_{\hat{V}_j^0}(F(\bar{V}^0)) = \underline{F}(\hat{V}^0)$. Let $j \in \hat{N}$ and notice that by $\hat{v}_j^0 < v_i' < \bar{F}(\hat{V}^0)$ and single-peakedness, $\underline{F}(\hat{V}^0) < \hat{v}_j^0 < \bar{F}(\hat{V}^0)$. Define $\hat{V}_j^1 \in \mathcal{R}$ such that $\hat{v}_j^1 = v_i'$ and $w_{\hat{V}_j^1}(F(\hat{V}^0)) = \bar{F}(\hat{V}^0)$, and let profile $\hat{V}^1 = (\hat{V}_{-j}^0, \hat{V}_j^1)$. Assume that

$R_j = \hat{V}_j^0$. By *strategy-proofness*, $w_{R_j}(F(\hat{V}^0))\ R_j\ w_{R_j}(F(\hat{V}^1))$; hence, by single-peakedness, either $[\underline{F}(\hat{V}^0) \geq \underline{F}(\hat{V}^1)]$ or $[\underline{F}(\hat{V}^0) < \underline{F}(\hat{V}^1)$ and $\bar{F}(\hat{V}^0) < \bar{F}(\hat{V}^1)]$.

Next, assume that $R_j = \hat{V}_j^1$. If $\bar{F}(\hat{V}^1) > \bar{F}(\hat{V}^0) > \hat{v}_j^1$, then single-peakedness implies $w_{R_j}(F(\hat{V}^0))\ P_j\ w_{R_j}(F(\hat{V}^1))$. This violates *strategy-proofness* since if at profile $\hat{V}^1$ agent j deviates by announcing $\hat{V}_j^0$, his worst point improves. In addition, if $\bar{F}(\hat{V}^1) < \hat{v}_j^1 = b_{R_j}(F(\hat{V}^0))$, then single-peakedness implies $b_{R_j}(F(\hat{V}^0))\,P_j\,b_{R_j}(F(\hat{V}^1))$. This violates *strategy-proofness* since if at profile $\hat{V}^1$ agent j deviates by announcing $\hat{V}_j^0$, his best point improves. Therefore, $\underline{F}(\hat{V}^0) \geq \underline{F}(\hat{V}^1)$ and in addition, $v_i' \leq \bar{F}(\hat{V}^1) \leq \bar{F}(\hat{V}^0)$. Hence, $v_i' > \underline{F}(\hat{V}^1)$.

Finally, notice that by single-peakedness, $\underline{F}(\hat{V}^0) \geq \underline{F}(\hat{V}^1)$ and $v_i' \leq \bar{F}(\hat{V}^1) \leq \bar{F}(\hat{V}^0)$ the following holds; for each agent $k \in \hat{N}$, $w_{\hat{V}_k^0}(F(\bar{V}^0)) = \underline{F}(\hat{V}^0)$ implies $w_{\hat{V}_k^1}(F(\bar{V}^1)) = \underline{F}(\hat{V}^1)$. Hence, by the process described above for agent j, the announced peaks of all agents $k \in \hat{N}$, such that $k \neq j$, can be sequentially changed to v_i' and profile $\hat{V}^{|\hat{N}|}$ can be constructed. Therefore, since $\hat{v}^{|\hat{N}|} = (v_i', \ldots, v_i')$, $\hat{V}^{|\hat{N}|} = V'$, and hence, $v_i' > \underline{F}(V')$ implies *unanimity* is violated.

Step 3. Let $V \in \mathcal{R}^N$. Without loss of generality, index the agents in N such that $v_1 \leq \cdots \leq v_n$. Let $V' \in \mathcal{R}^N$ be such that $V' = (V_1, \ldots, V_1)$. By *unanimity*, $F(V') = v_1$, hence by Proposition 1, $F(V') \in \mathrm{E}(V')$. Next, consider profile $V^2 = (V'_{-2}, V_2)$ where $v_2 \geq v_2' = v_1$, $\underline{v}^2 = \underline{v}'$, and $\bar{v}^2 \geq \bar{v}'$. Step 1 or Step 2, and $F(V') \in \mathrm{E}(V')$ imply $F(V^2) \in \mathrm{E}(V^2)$. Finally, for each $k = \{3, \ldots, n\}$, in increasing order, consider profile $V^k = (V_{-k}^{k-1}, V_k)$. By the arguments presented for V^2, $F(V^k) \in \mathrm{E}(V^k)$. Therefore, since $V^n = V$, $F(V) \in \mathrm{E}(V)$. $\square$

C Proofs of Propositions 4, 5, and 6

Before proceeding to the proof of Proposition 4 we present a lemma that holds only in domain $\mathcal{R}$[25] and concerns *strategy-proof* solutions satisfying *peaks-onliness*. Loosely speaking, following an agent's announcement change, there are restrictions on the chosen set.

Lemma 9. *For each $F \in \mathcal{F}$ satisfying strategy-proofness and peaks-onliness, each $i \in N$, and each pair $V, V' \in \mathcal{R}^N$ such that $V_{-i} = V'_{-i}$, the following hold.*

[25]It does not hold in domain $\mathcal{S}$ because the proof makes use of non-symmetrical single-peaked preferences.

(i) If $v_i < \bar{F}(V)$, then $\bar{F}(V) \leq \bar{F}(V')$, and if in addition $v_i < \underline{F}(V)$, then $\underline{F}(V) \leq \underline{F}(V')$.

(ii) If $v_i > \underline{F}(V)$, then $\underline{F}(V) \geq \underline{F}(V')$, and if in addition $v_i > \bar{F}(V)$, then $\bar{F}(V) \geq \bar{F}(V')$.

Proof. We prove statement (i), the proof of statement (ii) is symmetric. Let $F \in \mathcal{F}$ satisfy *strategy-proofness* and *peaks-onliness*. Let pair $V, V' \in \mathcal{R}^N$ and $i \in N$ be such that $V_{-i} = V'_{-i}$ and $v_i < \bar{F}(V)$.

Suppose R_i is such that $p_i = v_i$ and $0 \; P_i \; \bar{F}(V)$. By *peaks-onliness*, $F(V_{-i}, R_i) = F(V)$. Hence, by single-peakedness and the choice of R_i, $w_{R_i}(F(V_{-i}, R_i)) = \bar{F}(V)$. Thus, since $V' = (V_{-i}, V'_i)$, *strategy-proofness* implies $\bar{F}(V) \, R_i \, w_{R_i}(F(V'))$. Therefore, single-peakedness, the choice of R_i, and $0 \leq v_i < \bar{F}(V)$ imply $w_{R_i}(F(V')) \geq \bar{F}(V)$, and hence, $\bar{F}(V') \geq \bar{F}(V)$.

If in addition $v_i < \underline{F}(V)$, then by single-peakedness, $b_{R_i}(F(V_{-i}, R_i)) = \underline{F}(V)$. Thus, since $V' = (V_{-i}, V'_i)$, *strategy-proofness* implies $\underline{F}(V) \, R_i \, b_{R_i}(F(V'))$. Therefore, single-peakedness, $\bar{F}(V') \geq \bar{F}(V)$, and $v_i < \underline{F}(V)$ imply $\underline{F}(V') \geq \underline{F}(V)$. □

The proof of Proposition 4 follows and holds only in domain $\mathcal{R}$ because it makes use of Lemma 9.

Proof of Proposition 4. The proof is split in two parts.

Part 1: We show that *strategy-proofness* and *peaks-onliness* imply *uncompromisingness*.

Let $F \in \mathcal{F}$ satisfy *strategy-proofness* and *peaks-onliness*. Let pair $V, V' \in \mathcal{R}^N$ and $i \in N$ be such that $V_{-i} = V'_{-i}$. If $v_i = v'_i$, then *peaks-onliness* implies *uncompromisingness*. Hence, let $v_i \neq v'_i$, and by symmetry of arguments, let $v_i < v'_i$. There are four cases. Notice that Case 1.1 overlaps with Case 2.1, while Case 1.2 overlaps with Cases 2.1 and 2.2.

Case 1.1. Let $v_i < v'_i \leq \underline{F}(V)$. By Lemma 9(i), $\underline{F}(V) \leq \underline{F}(V')$. Moreover, assuming $\underline{F}(V) < \underline{F}(V')$ results in a contradiction as follows. Since $v'_i < \underline{F}(V')$, Lemma 9(i) implies $\underline{F}(V') \leq \underline{F}(V)$. Therefore, $\underline{F}(V) = \underline{F}(V')$.

Case 1.2. Let $\underline{F}(V) < v_i < v'_i$. By Lemma 9(ii), $\underline{F}(V') \leq \underline{F}(V)$. Hence, $\underline{F}(V') < v'_i$, and by Lemma 9(ii), $\underline{F}(V) \leq \underline{F}(V')$. Therefore, $\underline{F}(V) = \underline{F}(V')$.

Case 2.1. Let $v_i < v'_i \leq \bar{F}(V)$. By Lemma 9(i), $\bar{F}(V) \leq \bar{F}(V')$. Moreover, assuming $\bar{F}(V) < \bar{F}(V')$ results in a contradiction as follows. Since $v'_i < \bar{F}(V')$, Lemma 9(i) implies $\bar{F}(V') \leq \bar{F}(V) < \bar{F}(V')$. Therefore, $\bar{F}(V) = \bar{F}(V')$.

Case 2.2. Let $\bar{F}(V) < v_i < v'_i$. By Lemma 9(ii), $\bar{F}(V') \leq \bar{F}(V)$. Hence, $\bar{F}(V') < v'_i$, and by Lemma 9(ii), $\bar{F}(V) \leq \bar{F}(V')$. Therefore, $\bar{F}(V) = \bar{F}(V')$.

Part 2: We show that *uncompromisingness* implies *strategy-proofness* and *peaks-onliness*. Notice that this part of the proof also hold in domain $\mathcal{S}$.

Let $F \in \mathcal{F}$ satisfy *uncompromisingness*. Let $i \in N$ and pair $V, V' \in \mathcal{R}^N$ be such that $V_{-i} = V'_{-i}$ and $V_i \neq V'_i$. We proceed in two steps.

Step 1. We show that F satisfies *peaks-onliness*.

Let $v_i = v'_i$. If $v_i = \underline{F}(V)$, then assuming $\underline{F}(V') \neq \underline{F}(V)$ results in a contradiction, since $v'_i = v_i \neq \underline{F}(V')$ and *uncompromisingness* imply $\underline{F}(V) = \underline{F}(V')$. Similarly, if $v_i = \bar{F}(V)$, then assuming $\bar{F}(V') \neq \bar{F}(V')$ results in a contradiction, since $v'_i = v_i \neq \bar{F}(V')$ and *uncompromisingness* imply $\bar{F}(V) = \bar{F}(V')$. Finally, if $v_i \neq \underline{F}(V)$ and $v_i \neq \bar{F}(V)$, then by *uncompromisingness*, $F(V) = F(V')$. Therefore, F satisfies *peaks-onliness*.

Step 2. We show that F satisfies *strategy-proofness*. Recall that $V_i \neq V'_i$ and by symmetry of arguments, let $v_i \leq v'_i$. By Step 1, F satisfies *peaks-onliness*, hence, if $v_i = v'_i$, then *strategy-proofness* is satisfied. By symmetry of arguments, let $v_i < v'_i$. We proceed in two stages.

Stage 1. We show that $\underline{F}(V) \leq \underline{F}(V')$ and $\bar{F}(V) \leq \bar{F}(V')$. There are 3 cases.

Case 1. Let $v_i < \underline{F}(V)$. If $v_i < v'_i \leq \underline{F}(V)$, then by *uncompromisingness*, $\underline{F}(V) = \underline{F}(V')$. Otherwise, if $\underline{F}(V) < v'_i$, then consider the following. Assuming $\underline{F}(V') < \underline{F}(V)$ results in a contradiction as follows. Let V_i^1 be such that $v_i^1 = \underline{F}(V')$. Since $v_i < \underline{F}(V)$ and $v_i^1 < \underline{F}(V)$, by *uncompromisingness*, $\underline{F}(V_{-i}, V_i^1) = \underline{F}(V)$. However, since $\underline{F}(V') < v'_i$ and $v_i^1 = \underline{F}(V')$, by *uncompromisingness*, $\underline{F}(V'_{-i}, V_i^1) = \underline{F}(V')$. Hence, $\underline{F}(V) \leq \underline{F}(V')$.

Case 2. Let $v_i = \underline{F}(V) < v'_i$. Assuming $\underline{F}(V') < \underline{F}(V)$ results in a contradiction as follows. Since $\underline{F}(V') < v_i < v'_i$, by *uncompromisingness*, $\underline{F}(V') = \underline{F}(V)$. Hence, $\underline{F}(V) \leq \underline{F}(V')$.

Case 3. Let $v_i > \underline{F}(V)$. Since $v_i < v'_i$, by *uncompromisingness*, $\underline{F}(V) = \underline{F}(V')$.

Stage 2. By Stage 1, $\underline{F}(V) \leq \underline{F}(V')$ and $\bar{F}(V) \leq \bar{F}(V')$. We show that F satisfies *strategy-proofness*. Let $R_i = V_i$. There are five cases.

Case 1. Let $p_i < \underline{F}(V)$. By peaks-onliness, $b_{R_i}(F(V)) = \underline{F}(V)$, $w_{R_i}(F(V)) = \bar{F}(V)$, $b_{R_i}(F(V')) = \underline{F}(V')$, and $w_{R_i}(F(V')) = \bar{F}(V')$. Hence, by single-peakedness, i's best and worst points do not improve by deviating at V. Therefore, $F(V)\ R_i\ F(V')$.

Case 2. Let $p_i = \underline{F}(V)$. By peaks-onliness, $b_{R_i}(F(V)) = p_i$ and $w_{R_i}(F(V)) = \bar{F}(V)$, implying i can't improve on his best point. Regarding his worst point, since $\bar{F}(V) \leq \bar{F}(V')$, by peaks-onliness, $w_{R_i}(F(V))\ R_i\ \bar{F}(V')$. Therefore, $F(V)\ R_i\ F(V')$.

Case 3. Let $\underline{F}(V) < p_i < \bar{F}(V)$. By peaks-onliness, $b_{R_i}(F(V)) = p_i$ and $w_{R_i}(F(V)) \subseteq \{\underline{F}(V), \bar{F}(V)\}$, implying agent i can't improve on his best point. Regarding his worst point(s), since $\underline{F}(V) < p_i \leq v'_i$, by *uncompromisingness*, $\underline{F}(V) = \underline{F}(V')$. Since also $\bar{F}(V) \leq \bar{F}(V')$, by peaks-onliness, $w_{R_i}(F(V))\ R_i\ w_{R_i}(F(V'))$. Therefore, $F(V)\ R_i\ F(V')$.

Case 4. Let $p_i = \bar{F}(V)$. By peaks-onliness, $b_{R_i}(F(V)) = p_i$ and $w_{R_i}(F(V)) = \underline{F}(V)$, implying i can't improve on his best point. Regarding his worst point, if $\underline{F}(V) < \bar{F}(V)$, then since $\underline{F}(V) < p_i \leq v'_i$, and by *uncompromisingness*, $\underline{F}(V) = \underline{F}(V')$. Hence, $\bar{F}(V) \leq \bar{F}(V')$ implies $w_{R_i}(F(V))\ R_i\ w_{R_i}(F(V'))$. Otherwise, if $\underline{F}(V) = \bar{F}(V)$, then $w_{R_i}(F(V)) = p_i$, implying i can't improve on his worst point. Therefore, in both cases $F(V)\ R_i\ F(V')$.

Case 5. Let $p_i > \bar{F}(V)$. Since $\bar{F}(V) < p_i \leq v'_i$, by *uncompromisingness*, $F(V) = F(V')$. Therefore, $F(V)\ I_i\ F(V')$. □

Before proceeding to the proof of Proposition 5 we present two lemmata that hold only in domain $\mathcal{S}$ (because they make use of Proposition 2) and concern *strategy-proof* solutions satisfying *voter-sovereignty*. Loosely speaking, both show cases where following a change in the preferences of some agents, there are restrictions in the minimum and maximum chosen alternatives.

Lemma 10. *For each $F \in \mathcal{F}$ satisfying strategy-proofness and voter-sovereignty, each $V \in \mathcal{S}^N$, and each $x \in A$ the following hold.*
(i) Let $V' \in \mathcal{S}^N$ be as follows. For each each $i \in N$, if $v_i \leq x$, then $v'_i = x$, otherwise $v'_i = v_i$. Then, $x \leq \bar{F}(V)$ implies $\bar{F}(V) \leq \bar{F}(V')$; in addition, $x \leq \underline{F}(V)$ implies $\underline{F}(V) \leq \underline{F}(V')$.
(ii) Let $V' \in \mathcal{S}^N$ be as follows. For each $i \in N$, if $v_i \geq x$, then $v'_i = x$, otherwise $v'_i = v_i$. Then, $x \geq \underline{F}(V)$ implies $\underline{F}(V) \geq \underline{F}(V')$; in addition, $x \geq \bar{F}(V)$ implies $\bar{F}(V) \geq \bar{F}(V')$.

Proof. We prove statement (i), the proof of statement (ii) is symmetric. Let $F \in \mathcal{F}$ satisfy *strategy-proofness* and *voter-sovereignty*. By Proposition 2, F also satisfies *efficiency*. Let $V \in \mathcal{S}^N$ and $x \in A$ be such that $x \leq \bar{F}(V)$; and without loss of generality, let $v_1 \leq \cdots \leq v_n$.

Moreover, let $V' \in \mathcal{S}^N$ be defined as follows. For each $i \in N$, if $v_i \leq x$, then $v'_i = x$, otherwise $v'_i = v_i$. Let $M = \{1, \ldots, m\} \subseteq N$ be such that $i \in M$ implies $v'_i = x$. Hence, $V' = (V_x, \ldots, V_x, V_{m+1}, \ldots, V_n)$, where $V_x \in \mathcal{S}$ and $v_x = x$.

Begin from profile V. By *efficiency* and Proposition 1, $F(V) \subseteq \text{Conv}(V)$. Let $R_1 = V_1$, hence $v_1 \leq x \leq \bar{F}(V)$ implies $w_{R_1}(F(V)) = \bar{F}(V)$. Next, let $V_1^1 = V_x$ and consider profile $V^1 = (V_{-1}, V_1^1)$. By *efficiency* and Proposition 1, $F(V^1) \subseteq \text{Conv}(V^1)$, and by *strategy-proofness*, $w_{R_1}(F(V))\, R_1\, w_{R_1}(F(V^1))$. Therefore, by single-peakedness, $v_1 \leq \bar{F}(V) \leq \bar{F}(V^1)$. If $V^1 = V'$, then we are done. Otherwise, for each $k \in \{2, \ldots, m\}$, in increasing order, consider profile $V^k = (V^{k-1}_{-k}, V^k_k)$. By the arguments presented for V^1, $\bar{F}(V^{k-1}) \leq \bar{F}(V^k)$. Therefore, $V^m = V'$ implies $\bar{F}(V) \leq \bar{F}(V')$.

If in addition $x \leq \underline{F}(V)$, then $b_{R_1}(F(V)) = \underline{F}(V)$. In this case, begin from profile V and construct profile V' as shown above. By the same arguments to the ones presented above, but expressed for the best alternative instead of the worst, it follows that $\underline{F}(V) \leq \underline{F}(V')$. $\square$

Lemma 11. *For each $F \in \mathcal{F}$ satisfying strategy-proofness and voter-sovereignty, and each $V \in \mathcal{S}^N$ the following hold.*
(i) Let $M \subseteq N$ be such that $i \in M$ implies $v_i = \underline{v}$. Let $V' \in \mathcal{S}^N$ be as follows. For each $i \in N$, if $i \in M$, then $v'_i \leq v_i$, otherwise $v'_i = v_i$. Then, $\underline{v} < \bar{F}(V)$ implies $\bar{F}(V) \leq \bar{F}(V')$; in addition, $\underline{v} < \underline{F}(V)$ implies $\underline{F}(V) \leq \underline{F}(V')$.
(ii) Let $M \subseteq N$ be such that $i \in M$ implies $v_i = \bar{v}$. Let $V' \in \mathcal{S}^N$ be as follows. For each $i \in N$, if $i \in M$, then $v'_i \geq v_i$, otherwise $v'_i = v_i$. Then, $\bar{v} > \underline{F}(V)$ implies $\underline{F}(V) \geq \underline{F}(V')$; in addition, $\bar{v} > \bar{F}(V)$ implies $\bar{F}(V) \geq \bar{F}(V')$.

Proof. We prove statement (i), the proof of statement (ii) is symmetric. Let $F \in \mathcal{F}$ satisfy *strategy-proofness* and *voter-sovereignty*, and $V \in \mathcal{S}^N$ be such that $\underline{v} < \bar{F}(V)$. By Proposition 2, F also satisfies *efficiency*; hence, Proposition 1 implies $\underline{v} < \bar{F}(V) \leq \bar{v}$. In addition, let $M \subseteq N$ be such that $i \in M$ implies $v_i = \underline{v}$, and without loss of generality, let $M = (1, \ldots, m)$; hence, $\underline{v} < \bar{v}$ implies $M \subsetneq N$. Moreover, let $V' \in \mathcal{S}^N$ be as follows. For each $i \in N$, if $i \in M$, then $v'_i \leq v_i$, otherwise $v'_i = v_i$. Finally, without loss of generality, let $v'_1 \leq \cdots \leq v'_m < v'_{m+1} \leq \cdots \leq v'_n$.

Begin from profile V and let $\delta = |\underline{v} - \bar{F}(V)| > 0$. Assume $R_1 = V_1$. By single-peakedness, $w_{R_1}(F(V)) = \bar{F}(V)$. Change the announced preferences of agent 1 to $V_1^1 \in \mathcal{S}$ as follows. If $|v'_m - \underline{v}| < \delta$, then set $v_1^1 = v'_m$, otherwise, set $v_1^1 = \underline{v} - \frac{\delta}{2}$. By *efficiency* and Proposition 1,

$F(V_{-1}, V_1^1) \subseteq \text{Conv}(V_{-1}, V_1^1)$. By *strategy-proofness*, $w_{R_1}(F(V)) \, R_1 \, w_{R_1}(F(V_{-1}, V_1^1))$. Therefore, $|v_1^1 - \underline{v}| < \delta$ implies (in domain $\mathcal{S}$) that $\bar{F}(V) \leq \bar{F}(V_{-1}, V_1^1)$. Following this, sequentially repeat this process for all agents $i \in \{2, \ldots, m\}$ (if such agents exist) and construct profile $V^1 = (V_1^1, \ldots, V_m^1, V_{m+1}, \ldots, V_n)$, where $\bar{F}(V) \leq \bar{F}(V^1)$. If $v_1^1 = v'_m$, proceed to the next paragraph. Otherwise, let $\delta^1 = |\underline{v}^1 - \bar{F}(V^1)| > 0$, assume $R_1 = V_1^1$, and repeat the process to construct profile V^2, where $\bar{F}(V) \leq \bar{F}(V^2)$. If $v_1^2 = v'_m$, proceed to the next paragraph. Otherwise, keep repeating this process until the profile $\bar{V}^m = (V'_m, \ldots, V'_m, V_{m+1}, \ldots, V_n)$ has been constructed, where $\bar{F}(V) \leq \bar{F}(\bar{V}^m)$.

Next, repeat the process described above for all agents $i \in \{1, \ldots, m-1\}$ (if such agents exist) and construct profile $\bar{V}^{m-1} = (V'_{m-1}, \ldots, V'_{m-1}, V'_m, V_{m+1}, \ldots, V_n)$, where $\bar{F}(V) \leq \bar{F}(\bar{V}^{m-1})$.

Finally, continue repeating this whole process until the profile $\bar{V}^1 = V' = (V'_1, \ldots, V'_m, V_{m+1}, \ldots, V_n)$ has been constructed, where $\bar{F}(V) \leq \bar{F}(V')$.

If in addition, $\underline{v} < \underline{F}(V)$, begin from profile V, let $\delta = |\underline{v} - \underline{F}(V)| > 0$, and construct profile V' as shown above. By the same arguments to the ones presented above, but expressed for the best alternative instead of the worst, it follows that $\underline{F}(V) \leq \underline{F}(V')$. □

We now proceed with the proof of Proposition 5 that holds only in domain $\mathcal{S}$ because it makes indirect use of Proposition 2 through Lemmas 10 and 11.

Proof of Proposition 5. Let $F \in \mathcal{F}$. Part 2 of Proposition 4 on page 31 (which also holds in $\mathcal{S}$) shows that if F satisfies *uncompromisingness* then it also satisfies *strategy-proofness*. Hence, it follows that statement (ii) implies statement (i). Next, we show that statement (i) implies statement (ii).

Let F satisfy *strategy-proofness* and *voter-sovereignty*. Let $i \in N$ and pair $V, V' \in \mathcal{S}^N$ be such that $V_{-i} = V'_{-i}$. Without loss of generality, assume $v_1 \leq \cdots \leq v_n$. Since $v_i = v'_i$ trivially satisfies *uncompromisingness* in domain $\mathcal{S}$, let $v_i \neq v'_i$. There are six cases.

Case 1.1. Let $v'_i < v_i \leq \underline{F}(V)$. Since $v_i = \underline{F}(V) = \bar{F}(V)$ trivially satisfies *uncompromisingness*, let $v_i < \bar{F}(V)$. In addition, let $M \subsetneq N$ be such that $j \in M$ if and only if $v_j \leq v_i$. Begin from profile V and consider profile V^1 to be such that $V_{-M} = V^1_{-M}$ and where each agent $j \in M$ announces preferences $V_j^1 = V_i$. By construction of V^1 and Lemma 10(i), $\underline{F}(V^1) \geq \underline{F}(V)$ and $\bar{F}(V^1) \geq \bar{F}(V)$. Moreover, begin from profile V^1 and consider profile

V. Since for each $j \in M$, $v_j \leq v_j^1 = \underline{v}^1$, and for each $k \in N \setminus M$, $v_k = v_k^1 > \underline{v}^1$, by Lemma 11(i), $\bar{F}(V) \geq \bar{F}(V^1)$ and in addition, if $v_i < \underline{F}(V)$, then $\underline{F}(V) \geq \underline{F}(V^1)$. Therefore, $\underline{F}(V^1) = \underline{F}(V)$ and $\bar{F}(V^1) = \bar{F}(V)$.

Next, begin from profile V^1 and consider profile V'. Since for each $j \in M$, $v'_j \leq v_j^1 = \underline{v}^1$, and for each $k \in N \setminus M$, $v'_k = v_k^1 > \underline{v}^1$, by Lemma 11(i), $\bar{F}(V') \geq \bar{F}(V^1)$, and in addition, if $v_i < \underline{F}(V^1) = \underline{F}(V)$, then $\underline{F}(V') \geq \underline{F}(V^1)$. Finally, begin from profile V' and consider profile V^1. Since for each $j \in M$, $v'_j \leq v_j^1 = v'_i$, and for each $k \in N \setminus M$, $v'_k = v_k^1 > v'_i$, by Lemma 10(i), $\bar{F}(V^1) \geq \bar{F}(V')$ and $\underline{F}(V^1) \geq \underline{F}(V')$. Therefore, $\bar{F}(V^1) = \bar{F}(V) = \bar{F}(V')$ and $\underline{F}(V^1) = \underline{F}(V) = \underline{F}(V')$.

Case 1.2. Let $v'_i > v_i \geq \bar{F}(V)$. The proof is symmetric to Case 1.1.

Case 2.1. Let $v_i \leq \underline{F}(V)$ and $v_i < v'_i$. If $v'_i > \bar{F}(V)$, then *uncompromisingness* is trivially satisfied; hence, let $v'_i \leq \bar{F}(V)$. In addition, let $M \subsetneq N$ be such that $j \in M$ if and only if $v_j \leq v'_i$. Begin from profile V and consider profile V^1 to be such that $V_{-M} = V^1_{-M}$ and where each agent $j \in M$ announces preferences $V^1_j = V'_i$. By construction of V^1 and Lemma 10(i), $\bar{F}(V^1) \geq \bar{F}(V)$ and in addition, if $v'_i \leq \underline{F}(V)$, then $\underline{F}(V^1) \geq \underline{F}(V)$. Moreover, begin from profile V^1 and consider profile V. Since for each $j \in M$, $v_j \leq v_j^1 = \underline{v}^1$, and for each $k \in N \setminus M$, $v_k = v_k^1 > \underline{v}^1$, by Lemma 11(i), if $v_i < \bar{F}(V^1)$, then $\bar{F}(V) \geq \bar{F}(V^1)$ and in addition, if $v_i < \underline{F}(V^1)$, then $\underline{F}(V) \geq \underline{F}(V^1)$. Therefore, $\bar{F}(V^1) = \bar{F}(V)$ and in addition, if $v'_i \leq \underline{F}(V)$, then $\underline{F}(V^1) = \underline{F}(V)$. There are three sub-cases.

(i) Let $v'_i = \bar{F}(V)$. Assume $R_i = V'_i$. Hence, $b_{R_i}(F(V)) = \bar{F}(V)$. Since $V_{-i} = V'_{-i}$, by *strategy-proofness*, $\bar{F}(V) = v'_i \in F(V')$. Thus, $\bar{F}(V') \geq \bar{F}(V)$. Moreover, begin from profile V' and consider profile V^1. Since for each $j \in M$, $v'_j \leq v_j^1 = v'_i$, and for each $k \in N \setminus M$, $v'_k = v_k^1 > v'_i$, by Lemma 10(i), $\bar{F}(V^1) \geq \bar{F}(V')$. Therefore, $\bar{F}(V^1) = \bar{F}(V) = \bar{F}(V')$.

(ii) Let $v'_i < \bar{F}(V)$. Begin from profile V^1 and consider profile V'. Since for each $j \in M$, $v'_j \leq v_j^1 = \underline{v}^1$, and for each $k \in N \setminus M$, $v'_k = v_k^1 > \underline{v}^1$, by $v'_i < \bar{F}(V)$ and Lemma 11(i), $\bar{F}(V') \geq \bar{F}(V^1)$, and in addition, if $v'_i < \underline{F}(V)$, then $\underline{F}(V') \geq \underline{F}(V^1)$. Moreover, begin from profile V' and consider profile V^1. Since for each $j \in M$, $v'_j \leq v_j^1 = v'_i$, and for each $k \in N \setminus M$, $v'_k = v_k^1 > v'_i$, by $v'_i < \bar{F}(V)$ and Lemma 10(i), $\bar{F}(V^1) \geq \bar{F}(V')$, and in addition, if $v'_i < \underline{F}(V')$, then $\underline{F}(V^1) \geq \underline{F}(V')$. Therefore, $\bar{F}(V^1) = \bar{F}(V') = \bar{F}(V)$ and in addition, if $v'_i < \underline{F}(V)$, then $\underline{F}(V^1) = \underline{F}(V') = \underline{F}(V)$.

(iii) Let $v'_i = \underline{F}(V)$. If $v'_i = \underline{F}(V) = \bar{F}(V)$, then *uncompromisingness* is trivially satisfied; hence, let $v'_i = \underline{F}(V) < \bar{F}(V)$. As shown in the previous sub-case, $\bar{F}(V') = \bar{F}(V)$. Assume

$R_i = V_i$. Since $v_i < v_i'$, by single-peakedness, $b_{R_i}(F(V)) = \underline{F}(V)$ and $w_{R_i}(F(V)) = \bar{F}(V)$. Hence, $V_{-i}' = V_{-i}$ and *strategy-proofness* imply $b_{R_i}(F(V))\, R_i\, b_{R_i}(F(V'))$. Thus, by $\bar{F}(V') = \bar{F}(V)$ and single-peakedness, $b_{R_i}(F(V)) = \underline{F}(V) \leq \underline{F}(V')$. Finally, assume $R_i = V_i'$. Since $v_i' \leq \underline{F}(V') \leq \bar{F}(V')$, by single-peakedness, $b_{R_i}(F(V')) = \underline{F}(V')$ and $w_{R_i}(F(V')) = \bar{F}(V')$. Hence, $V_{-i}' = V_{-i}$ and *strategy-proofness* imply $b_{R_i}(F(V'))\, R_i\, b_{R_i}(F(V))$. Thus, by $\bar{F}(V') = \bar{F}(V)$ and single-peakedness, $b_{R_i}(F(V')) = \underline{F}(V') \leq \underline{F}(V)$. Therefore, $\underline{F}(V) = \underline{F}(V')$.

Case 2.2. Let $v_i \geq \bar{F}(V)$ and $v_i > v_i'$. The proof is symmetric to Case 2.1.

Case 3.1. Let $\underline{F}(V) < v_i < \bar{F}(V)$ and $v_i' > v_i$. In addition, let $M \subsetneq N$ be such that $j \in M$ if and only if $v_j \geq v_i'$. Begin from profile V and consider profile V^1 to be such that $V_{-M} = V_{-M}^1$ and where each agent $j \in M$ announces preferences $V_j^1 = V_i'$. By construction of V^1 and Lemma 10(ii), $\underline{F}(V) \geq \underline{F}(V^1)$. Moreover, begin from profile V^1 and consider profile V. Since for each $j \in M$, $v_j \geq v_j^1 = \bar{v}^1$, and for each $k \in N \setminus M$, $v_k = v_k^1 < \bar{v}^1$, by $v_i' > \underline{F}(V^1)$ and Lemma 11(ii), $\underline{F}(V^1) \geq \underline{F}(V')$. Therefore, $\underline{F}(V) = \underline{F}(V^1)$.

Next, begin from profile V^1 and consider profile V'. Since for each $j \in M$, $v_j' \geq v_j^1 = \bar{v}^1$, and for each $k \in N \setminus M$, $v_k' = v_k^1 < \bar{v}^1$, by $\bar{v}^1 > \underline{F}(V^1)$ and Lemma 11(ii), $\underline{F}(V') \leq \underline{F}(V^1)$. Finally, begin from profile V' and consider profile V^1. Since for each $j \in M$, $v_j' \geq v_j^1 = v_i'$, and for each $k \in N \setminus M$, $v_k' = v_k^1 < v_i'$, by Lemma 10(ii), $\underline{F}(V^1) \leq \underline{F}(V')$. Therefore, $\underline{F}(V^1) = \underline{F}(V) = \underline{F}(V')$.

If $v_i' > \bar{F}(V)$, then we are done. If $v_i' \leq \bar{F}(V)$, then let $L \subsetneq N$ be such that $j \in L$ if and only if $v_j \leq v_i'$. Begin from profile V and consider profile V^2 to be such that $V_{-L} = V_{-L}^2$ and where each agent $j \in L$ announces preferences $V_j^2 = V_i'$. By construction of V^2 and Lemma 10(i), $\bar{F}(V) \leq \bar{F}(V^2)$. There are two sub-cases.

(i) Let $v_i' < \bar{F}(V)$. Begin from profile V^2 and consider profile V. Since for each $j \in L$, $v_j' \leq v_j^2 = \underline{v}^2$, and for each $k \in N \setminus L$, $v_k' = v_k^2 > \underline{v}^2$, by $\underline{v}^2 < \bar{F}(V^2)$ and Lemma 11(i), $\bar{F}(V^2) \leq \bar{F}(V)$. Therefore, $\bar{F}(V) = \bar{F}(V^2)$.

Next, begin from profile V' and consider profile V^2. Since for each $j \in L$, $v_j' \leq v_j^2 = v_i'$, and for each $k \in N \setminus L$, $v_k' = v_k^2 < v_i'$, by Lemma 10(i), $\bar{F}(V^2) \geq \bar{F}(V')$. Finally, begin from profile V^2 and consider profile V'. Since for each $j \in L$, $v_j' \leq v_j^2 = \underline{v}^2$, and for each $k \in N \setminus L$, $v_k' = v_k^2 > \underline{v}^2$, by $\underline{v}^2 < \bar{F}(V^2)$ and Lemma 11(i), $\bar{F}(V') \geq \bar{F}(V^2)$. Therefore, $\bar{F}(V^2) = \bar{F}(V) = \bar{F}(V')$.

(ii) Let $v_i' = \bar{F}(V)$. Assume $R_i = V_i'$. By single-peakedness, $b_{R_i}(F(V)) = v_i'$. Hence,

$V_{-i} = V'_{-i}$ and *strategy-proofness* imply $v'_i \in F(V')$. Thus, $\bar{F}(V') \geq \bar{F}(V)$.
Assuming $\bar{F}(V') > \bar{F}(V)$ results in a contradiction as follows. Begin from profile V and consider profile V^2. Since for each $j \in L$, $v_j \leq v^2_j = v'_i$, and for each $k \in N \setminus L$, $v_k = v^2_k > v'_i$, by Lemma 10(i), $\bar{F}(V) \leq \bar{F}(V^2)$. Moreover, begin from profile V^2 and consider profile V. Since for each $j \in L$, $v_j \leq v^2_j = \underline{v}^2$, and for each $k \in N \setminus L$, $v_k = v^2_k > \underline{v}^2$, by $\underline{v}^2 < \bar{F}(V^2)$ and Lemma 11(i), $\bar{F}(V) \geq \bar{F}(V^2)$. Therefore, $\bar{F}(V) = \bar{F}(V^2)$.
Next, begin from profile V' and consider profile V^2 as described in the previous sub-case. Since for each $j \in L$, $v'_j \leq v^2_j = v'_i$, and for each $k \in N \setminus L$, $v'_k = v^2_k > v'_i$, by Lemma 10(i), $\bar{F}(V') \leq \bar{F}(V^2)$. Finally, begin from profile V^2 and consider profile V'. Since for each $j \in L$, $v'_j \leq v^2_j = \underline{v}^2$, and for each $k \in N \setminus L$, $v'_k = v^2_k > \underline{v}^2$, by $\underline{v}^2 < \bar{F}(V^2)$ and Lemma 11(i), $\bar{F}(V') \geq \bar{F}(V^2)$. Therefore, $\bar{F}(V^2) = \bar{F}(V')$. Therefore, $\bar{F}(V^2) = \bar{F}(V) = \bar{F}(V')$.

Case 3.2. Let $\underline{F}(V) < v_i < \bar{F}(V)$ and $v'_i < v_i$. The proof is symmetric to Case 3.1. □

Finally, we present the proof of Proposition 6.

Proof of Proposition 6. Let $F \in \mathcal{F}$ satisfy *strategy-proofness*, *min/max continuity*, and *voter-sovereignty*. By Proposition 3, F also satisfies *efficiency*. Let pair $V, V' \in \mathcal{R}^N$ and $i \in N$ be such that $V_{-i} = V'_{-i}$. There are five cases.

Case 1.1. Let $v_i < \underline{F}(V)$ and $v'_i \leq \bar{F}(V)$. Notice that if $R_i = V_i$, by single-peakedness, $b_{R_i}(F(V)) = \underline{F}(V)$ and $w_{R_i}(F(V)) = \bar{F}(V)$.

Assuming $v_i \geq \underline{F}(V')$ leads to a contradiction as follows. By *min/max continuity*, there exists $V^*_i \in \mathcal{R}$ such that $\underline{F}(V_{-i}, V^*_i) = v_i$. Assume $R_i = V_i$. By single-peakedness, $b_{R_i}(F(V_{-i}, V^*_i)) = v_i \notin F(V)$. Hence, if at profile V agent i deviates by announcing V^*_i, his best point improves. This contradicts *strategy-proofness*. Therefore, $v_i < \underline{F}(V') \leq \bar{F}(V')$.

Next, assuming $\underline{F}(V') < \underline{F}(V)$ or $\bar{F}(V') < \bar{F}(V)$ leads to a contradiction as follows. Assume $R_i = V_i$. By $v_i < \underline{F}(V') \leq \bar{F}(V')$ and single-peakedness, $b_{R_i}(F(V'))P_i b_{R_i}(F(V))$ or $w_{R_i}(F(V'))P_i b_{R_i}(F(V))$. Hence, if at profile V agent i deviates by announcing V'_i, his best point or his worst point improves. This contradicts *strategy-proofness*. Therefore, $\underline{F}(V') \geq \underline{F}(V)$ and $\bar{F}(V') \geq \bar{F}(V)$.

Finally, assuming $\underline{F}(V) < \underline{F}(V')$ or $\bar{F}(V) < \bar{F}(V')$ leads to a contradiction as follows. Assume $R_i = V'_i$. By $v'_i \leq \underline{F}(V) \leq \bar{F}(V)$ and single-peakedness, $b_{R_i}(F(V))P_i b_{R_i}(F(V'))$ or

$w_{R_i}(F(V))P_i b_{R_i}(F(V'))$. Hence, if at profile V' agent i deviates by announcing V_i, his best point or his worst point improves. This contradicts *strategy-proofness*.

Therefore, $\bar{F}(V') = \bar{F}(V)$ and in addition, if $v_i' \leq \underline{F}(V)$, then $\underline{F}(V') = \underline{F}(V)$.

Case 1.2. Let $v_i = \underline{F}(V)$ and $v_i' \leq \bar{F}(V)$. By the same arguments to the ones presented in Case 1.1 for the maximum point chosen, it follows that $\bar{F}(V') = \bar{F}(V)$.

Case 2.1. Let $v_i > \bar{F}(V)$ and $v_i' \geq \underline{F}(V)$. By symmetric arguments to those presented in Case 1.1, it follows that $\underline{F}(V') = \underline{F}(V)$ and in addition, if $v_i' \geq \bar{F}(V)$, then $\bar{F}(V') = \bar{F}(V)$.

Case 2.2. Let $v_i = \bar{F}(V)$ and $v_i' \geq \underline{F}(V)$. By symmetric arguments to those presented in Case 1.2, it follows that $\underline{F}(V') = \underline{F}(V)$.

Case 3. Let $\underline{F}(V) < v_i < \bar{F}(V)$. By symmetry of arguments, let $v_i \geq v_i'$. Without loss of generality, let $v_1 \leq \cdots \leq v_n$ and notice that by *efficiency* and Proposition 1, $F(V) \subseteq \text{Conv}(V)$; hence, agent $i \notin \{1, n\}$. In addition, for each agent $j \in N \setminus \{i\}$, define preferences $\bar{V}_j \in \mathcal{R}$ be such that $\bar{V}_j = V_i'$.

Begin from profile V and consider profile $V^1 = (V_{-1}, \bar{V}_1)$. By *efficiency* and Proposition 1, $F(V) \subseteq \text{Conv}(V)$. Hence, since $v_1 = \underline{v}$, by either Case 1.1 (if $v_1 < \underline{F}(V)$) or Case 1.2 (if $v_1 = \underline{F}(V)$), $\bar{F}(V^1) = \bar{F}(V)$. Moreover, by *efficiency* and Proposition 1, $F(V^1) \subseteq \text{Conv}(V^1)$; hence, $v_2 = \underline{v}^1$. Next, for agents $k \in \{2, \ldots, i\}$, in increasing order, consider profile $V^k = (V^{k-1}_{-k}, \bar{V}_k)$. By the arguments presented for V^1, $\bar{F}(V^k) = \bar{F}(V)$. Therefore, at profile $V^i = (\bar{V}_1, \ldots, \bar{V}_i, V_{i+1}, \ldots, V_n)$, $\bar{F}(V^i) = \bar{F}(V)$. Finally, begin from profile V'. By the same technique as the one described for profile V, change the preferences of agents $k \in \{1, \ldots, i - i\}$, in increasing order, to again construct profile $V^i = (\bar{V}_1, \ldots, \bar{V}_i, V_{i+1}, \ldots, V_n)$. Therefore, $\bar{F}(V^i) = \bar{F}(V') = \bar{F}(V)$.

Similarly, if $v_i' \leq \underline{F}(V)$, then once can show that $\underline{F}(V') = \underline{F}(V)$, by using symmetrical arguments to the ones presented above. Specifically, begin from profile V and change the preferences of agents $k \in \{i, \ldots, n\}$, in decreasing order, and show that $\underline{F}(V_1, \ldots, V_{i-1}, \bar{V}_i, \ldots, \bar{V}_n) = \underline{F}(V)$. Finally, begin from profile V' and change the preferences of agents $k \in \{i+1, \ldots, n\}$, in decreasing order, and show that $\underline{F}(V_1, \ldots, V_{i-1}, \bar{V}_i, \ldots, \bar{V}_n) = \underline{F}(V') = \underline{F}(V)$. $\square$

D Proof of Theorem 1 (equivalence of statements (ii) and (iii))

We first show for Theorem 1 that statement (iii) implies statement (ii) in domain $\mathcal{R}$. Moreover, as discussed in Section 5.3, this result also holds in domain $\mathcal{S}$.

Proof of Theorem 1 (statement (iii) implies statement (ii)). Let $F_G^{\alpha,\beta} \in \mathcal{F}_G$. By the definition of $\mathcal{F}_G$, to show that $F_G^{\alpha,\beta}$ satisfies *uncompromisingness*, it suffices to show that the minimum and maximum chosen alternatives by $F_G^{\alpha,\beta}$ are not compromised. Moreover, by symmetry of arguments, we only need to show that $\underline{F}_G^{\alpha,\beta}(V)$ is not compromised.

Let $V \in \mathcal{R}^N$ and without loss of generality, let $v_1 \leq \cdots \leq v_n$. Let $i \in N$ and $V' \in \mathcal{R}^N$ be such that $V_{-i} = V'_{-i}$. Moreover, let $v_i \neq \underline{F}_G^{\alpha,\beta}(V)$. Hence, $\underline{F}_G^{\alpha,\beta}(V) = \mathrm{med}(v, \tilde{\alpha})$ and $\underline{F}_G^{\alpha,\beta}(V') = \mathrm{med}(v', \tilde{\alpha}')$. There are two cases.

Case 1. Let $j \in N$ and $\underline{F}_G^{\alpha,\beta}(V) = \mathrm{med}(v, \tilde{\alpha}) = v_j$. Hence, $v_i \neq \underline{F}_G^{\alpha,\beta}(V)$ implies $i \neq j$. Since at V, $v_1 \leq \cdots \leq v_n$, at least j agents announce peaks smaller than or equal to v_j and at least $n-j+1$ agents announce peaks larger than or equal to v_j. Thus, since there are n agents in total and $\tilde{\alpha} \in A^{n+1}$, by the median operator, vector $\tilde{\alpha}$ contains at least $n-j+1$ coordinates smaller than or equal to v_j and at least j coordinates larger than or equal to v_j. Therefore, since $\alpha_N \leq \cdots \leq \alpha_\emptyset$, if $j = 1$, $\alpha_{\{1\}} \leq v_j \leq \alpha_\emptyset$, and otherwise, $\alpha_{\{1,\ldots,j\}} \leq v_j \leq \alpha_{\{1,\ldots,j-1\}}$. There are two sub-cases.

(i) Let $v_i < \underline{F}_G^{\alpha,\beta}(V) = v_j$, that is, $i \in \{1, \ldots, j-1\}$. This implies $j \in \{2, \ldots, n\}$ and $\alpha_{\{1,\ldots,j\}} \leq v_j \leq \alpha_{\{1,\ldots,j-1\}}$. In addition, let $v'_i \leq v_j$. Thus, at profile V', at least j agents announce peaks smaller than or equal to v_j (i.e., agents $1, \ldots, j$) and at least $n-j+1$ agents announce peaks larger than or equal to v_j (i.e., agents $j, \ldots, n$). Moreover, $V_{-i} = V'_{-i}$ and $v'_i \leq v_j$ imply that $v'_i \leq v_j \leq v_{j+1} \leq \cdots \leq v_n$, that is, the agents announcing the $j-1$ smallest peaks at V (i.e., agents $1, \ldots, j-1$) also announce the $j-1$ smallest peaks at V'. Similarly, the agents announcing the j smallest peaks at V (i.e., agents $1, \ldots, j$) also announce the j smallest peaks at V'. Hence, coordinates $\alpha_{\{1,\ldots,j\}}$ and $\alpha_{\{1,\ldots,j-1\}}$ are included in vector $\tilde{\alpha}'$. Thus, $\alpha_{\{1,\ldots,j\}} \leq v_j \leq \alpha_{\{1,\ldots,j-1\}}$ and the definition of $\mathcal{F}_G$ implies that vector $\tilde{\alpha}'$ contains at least $n-j+1$ coordinates smaller than or equal to v_j and at least j coordinates larger than or equal to v_j. Therefore, $\underline{F}_G^{\alpha,\beta}(V') = \mathrm{med}(v', \tilde{\alpha}') = v_j = \underline{F}_G^{\alpha,\beta}(V)$.

(ii) Let $v_i > \underline{F}_G^{\alpha,\beta}(V) = v_j$, that is, $i \in \{j+1, \ldots, n\}$. The proof is symmetric to (i).

Case 2. Let $M \subseteq N$ such that $|M| = m$. Let $\underline{F}_G^{\alpha,\beta}(V) = \text{med}(v, \tilde{\alpha}) = \alpha_M$, such that for each $i \in N$, $v_i \neq \alpha_M$. Hence, if $|M| = 0$, $\alpha_M = \alpha_\emptyset$, and otherwise, $\alpha_M = \alpha_{\{1,\ldots,m\}}$. Since $\alpha_N \leq \cdots \leq \alpha_\emptyset$, vector $\tilde{\alpha}$ contains at least $n - m + 1$ coordinates smaller than or equal to α_M (i.e., coordinates $\alpha_{\{1,\ldots,m\}}, \ldots, \alpha_N$) and at least $m+1$ coordinates larger than or equal to α_M (i.e., coordinates $\alpha_\emptyset, \ldots, \alpha_{\{1,\ldots,m\}}$). Thus, since there are n agents in total and none of their announced peaks equals α_M, by the median operator, at V, m agents announce peaks smaller than α_M (i.e., agents $1, \ldots, m$) and $n - m$ agents announce peaks larger than α_M (i.e., agents $m+1, \ldots, n$). Therefore, since $v_1 \leq \cdots \leq v_n$, if $m = 0$, $\alpha_M = \alpha_\emptyset < v_1$, if $m = n$, $\alpha_M = \alpha_N > v_n$, and otherwise, $v_m < \alpha_M = \alpha_{\{1,\ldots,m\}} < v_{m+1}$. There are four sub-cases.

(i) Let $m = 0$. Hence, $\alpha_M = \alpha_\emptyset < v_1 \leq v_i$. In addition, let $\alpha_M = \underline{F}_G^{\alpha,\beta} \leq v'_i$.
Thus, at V', all n agents announce peaks larger than α_M. In addition, since $\alpha_N \leq \cdots \leq \alpha_\emptyset = \alpha_M$, vector $\tilde{\alpha}'$ contains at least $n + 1$ coordinates smaller than or equal to α_M (i.e., coordinates $\alpha_\emptyset, \ldots, \alpha_N$) and at least 1 coordinate larger than or equal to α_M (i.e., coordinate $\alpha_\emptyset$). Therefore, $\underline{F}_G^{\alpha,\beta}(V') = \text{med}(v', \tilde{\alpha}') = \alpha_M = \underline{F}_G^{\alpha,\beta}(V)$.

(ii) Let $m = n$. The proof is symmetric to (i).

(iii) Let $m \in \{1, \ldots, n-1\}$ and $v_i < \alpha_{\{1,\ldots,m\}} = \alpha_M$. Hence, $v_i \leq v_m < \alpha_M < v_{m+1}$. In addition, let $v'_i \leq \alpha_M$. Thus, at V', at least m agents announce peaks smaller than or equal to α_M (i.e., agents $1, \ldots, m$) and $n - m$ agents announce peaks larger than α_M (i.e., agents $m+1, \ldots, n$). Moreover, $V_{-i} = V'_{-i}$ and $v'_i \leq \alpha_M$ imply that $v'_i \leq \alpha_M < v_{m+1} \leq \cdots \leq v_n$, that is, the agents announcing the m smallest peaks at V (i.e., agents $1, \ldots, m$) also announce the m smallest peaks at V'. Hence, coordinate $\alpha_{\{1,\ldots,m\}}$ is included in vector $\tilde{\alpha}'$. Thus, the definition of $\mathcal{F}_G$ implies that vector $\tilde{\alpha}'$ contains at least $n - m + 1$ coordinates smaller than or equal to α_M and at least $m + 1$ coordinates larger than or equal to α_M. Therefore, $\underline{F}_G^{\alpha,\beta}(V') = \text{med}(v', \tilde{\alpha}') = \alpha_M = \underline{F}_G^{\alpha,\beta}(V)$.

(iv) Let $m \in \{1, \ldots, n-1\}$ and $v_i > \alpha_{\{1,\ldots,m\}} = \alpha_M$. The proof is symmetric to (iii). $\square$

Before showing for Theorem 1 that statement (ii) implies statement (iii), we first prove the following intermediate result that holds in both domains $\mathcal{R}$ and $\mathcal{S}$.

Lemma 12. *Let $F \in \mathcal{F}$ satisfy strategy-proofness, peaks-onliness, and uncompromisingness. Then, for each $i \in N$ and each pair $V, V' \in \mathcal{R}^N$ such that $V'_{-i} = V_{-i}$, if $v_i \leq v'_i$, then*

$\underline{F}(V) \leq \underline{F}(V')$ and $\bar{F}(V) \leq \bar{F}(V')$.[26]

Proof. Let $F \in \mathcal{F}$ satisfy *strategy-proofness*, *peaks-onliness*, and *uncompromisingness*. Let $i \in N$ and pair $V, V' \in \mathcal{R}^N$ be such that $V'_{-i} = V_{-i}$. Since by *peaks-onliness*, $v_i = v'_i$ implies $F(V) = F(V')$, let $v_i < v'_i$. There are three cases.

Case 1. Let $v_i < \underline{F}(V)$ and $v_i < v'_i$. Concerning the maximum alternative chosen, if $v'_i \leq \bar{F}(V)$, then by *uncompromisingness*, $\bar{F}(V) = \bar{F}(V')$. Let $V_i^1 \in \mathcal{R}$ be such that $v_i^1 = \bar{F}(V)$. Hence, by *uncompromisingness*, $\bar{F}(V_{-i}, V_i^1) = \bar{F}(V)$. If $v'_i > \bar{F}(V)$, then assuming $\bar{F}(V') < \bar{F}(V)$ leads to a contradiction as follows. Begin from V' and let agent i change his announcement to V_i^1. Since $\bar{F}(V') < v_i^1 < v'_i$, by *uncompromisingness*, $\bar{F}(V'_{-i}, V_i^1) = \bar{F}(V')$. Thus, $\bar{F}(V'_{-i}, V_i^1) = \bar{F}(V_{-i}, V_i^1)$ contradicts $\bar{F}(V') < \bar{F}(V)$. Therefore, in both cases, $\bar{F}(V') \geq \bar{F}(V)$.

Concerning the minimum alternative chosen, assume $R_i = V_i$. Since $v_i < \underline{F}(V)$ and $\bar{F}(V) \leq \bar{F}(V')$, if $\underline{F}(V')) < \underline{F}(V)$, then single-peakedness implies $b_{R_i}(F(V')\; R_i\; b_{R_i}(F(V))$. Hence, if at profile V agent i deviates by announcing V'_i, his best point improves. This contradicts *strategy-proofness*. Therefore, $\underline{F}(V') \geq \underline{F}(V)$

Case 2. Let $\underline{F}(V) \leq v_i < \bar{F}(V)$ and $v_i < v'_i$. Concerning the maximum alternative chosen, by the arguments presented in Case 1, $\bar{F}(V') \geq \bar{F}(V)$. Concerning the minimum alternative chosen, if $\underline{F}(V') < \underline{F}(V)$, then $\underline{F}(V') < v_i < v'_i$ and *uncompromisingness* imply $\underline{F}(V') = \underline{F}(V'_{-i}, V_i) = \underline{F}(V)$. Therefore, $\underline{F}(V') \geq \underline{F}(V)$.

Case 3. Let $\bar{F}(V) \leq v_i$ and $v_i < v'_i$. Concerning the maximum alternative chosen, if $\bar{F}(V') < \bar{F}(V)$, then $\bar{F}(V) \leq v_i < v'_i$ and *uncompromisingness* imply $\bar{F}(V') = \bar{F}(V'_{-i}, V_i) = \bar{F}(V)$. Similarly, concerning the minimum alternative chosen, if $\underline{F}(V') < \underline{F}(V)$, then $\underline{F}(V) \leq v_i < v'_i$ and *uncompromisingness* imply $\underline{F}(V') = \underline{F}(V'_{-i}, V_i) = \underline{F}(V)$. $\square$

The last part of the proof of Theorem 1 follows. Notice that this part holds in both domains $\mathcal{R}$ and $\mathcal{S}$.

[26]Notice that this result simply shows that *strategy-proofness* and *uncompromisingness* imply *peak-monotonicity*, a property that we refrain from introducing formally since it is only used in the "only if" part of Theorem 1. Loosely speaking, this property requires the following: if an agent's announced peak moves to the right (left), then the chosen set also moves to the right (left).

Proof of Theorem 1 (statement (ii) implies statement (iii)). Let $F \in \mathcal{F}$ satisfy *uncompromisingness*. By Proposition 4, F satisfies *strategy-proofness* and *peaks-onliness*. For each $i \in N$, let pair $V_i^{\min}, V_i^{\max} \in \mathcal{R}$ be such that $v_i^{\min} = 0$ and $v_i^{\max} = 1$. We proceed in three steps.

Step 1. We show that at each announced profile $V \in \mathcal{R}^N$ and for each $i \in N$, the minimum chosen alternative is the median of: (i) the announced peak of i at profile V (i.e., v_i), (ii) the minimum chosen alternative if i changes his announcement to $V_i^{\min}$ (i.e., $\underline{F}(V_{-i}, V_i^{\min})$), and (iii) the minimum chosen alternative if i changes his announcement to $V_i^{\max}$ (i.e., $\underline{F}(V_{-i}, V_i^{\max})$). By symmetry of arguments, we do not show the equivalent result for the maximum chosen alternative.

Let $i \in N$ and $V \in \mathcal{R}^N$. Consider profiles $V^{\min} = (V_{-i}, V_i^{\min})$ and $V^{\max} = (V_{-i}, V_i^{\max})$. Since $V_{-i} = V_{-i}^{\min} = V_{-i}^{\max}$ and $v_i^{\min} \leq v_i \leq v_i^{\max}$, by Lemma 12, $\underline{F}(V^{\min}) \leq \underline{F}(V) \leq \underline{F}(V^{\max})$. There are three cases.

Case 1. Let $v_i < \underline{F}(V^{\min}) \leq \underline{F}(V)$. Since $0 = v_i^{\min} \leq v_i < \underline{F}(V)$, *uncompromisingness* implies $\underline{F}(V^{\min}) = \underline{F}(V)$. Therefore, $\underline{F}(V^{\min}) = \underline{F}(V) = \text{med}(\underline{F}(V^{\min}), v_i, \underline{F}(V^{\max}))$.

Case 2. Let $v_i > \underline{F}(V^{\max}) \geq \underline{F}(V)$. Symmetric proof to Case 1.

Case 3. Let $\underline{F}(V^{\min}) \leq v_i \leq \underline{F}(V^{\max})$. Assuming $v_i < \underline{F}(V)$ and thus $\underline{F}(V^{\min}) < \underline{F}(V)$ results in a contradiction as follows. Since $0 = v_i^{\min} \leq v_i < \underline{F}(V)$, *uncompromisingness* implies $\underline{F}(V^{\min}) = \underline{F}(V)$. Similarly, assuming $\underline{F}(V) < v_i$ and thus $\underline{F}(V) < \underline{F}(V^{\max})$ results in a contradiction as follows. Since $\underline{F}(V) < v_i \leq v_i^{\max}$, *uncompromisingness* implies $\underline{F}(V) = \underline{F}(V^{\max})$. Therefore, $\underline{F}(V) = v_i = \text{med}(\underline{F}(V^{\min}), v_i, \underline{F}(V^{\max}))$.

Step 2. We construct two vectors α and β. In this step of the proof and in contrast to the rest of the paper, we will use a different letter to label announced profiles (U instead of V). This is done in an attempt to facilitate the notation used in Step 3 of the proof that follows.

For each $M \subseteq N$, let $U^M \in \mathcal{R}^N$ be such that all agents in M announce 0 as their peak and all other agents announce 1 as their peak, i.e., $u^M = (\underbrace{0, \ldots, 0}_{i \in M}, \underbrace{1, \ldots, 1}_{i \in N \setminus M})$. Next, let vectors $\alpha = (\alpha_M)_{M \subseteq N}$ and $\beta = (\beta_M)_{M \subseteq N}$ be such that $\alpha_M = \underline{F}(U^M)$ and $\beta_M = \bar{F}(U^M)$, hence, $\alpha_M \leq \beta_M$. Moreover, for each $L, M \subseteq N$ such that $L \subsetneq M$ notice the following. For each $i \in M \setminus L$, $u_i^L = 1 > 0 = u_i^M$, and for each $j \notin M \setminus L$, $U_j^L = U_j^M$. Begin from profile U^L and consider that all agents i (sequentially) change their announcements to U_i^M. Since $u_i^L > u_i^M$, by (sequentially) applying Lemma 12,both $\alpha_L \geq \alpha_M$ and $\beta_L \geq \beta_M$.

Step 3. We show that F is a generalized median solution associated with vectors α and β constructed in Step 2.

Let $V \in \mathcal{R}^N$. Without loss of generality, index the agents in N such that $v_1 \leq \cdots \leq v_n$. Recall vectors α, β and profiles U^M, for $M \subseteq N$, defined in Step 2. Let vectors $\tilde{\alpha}, \tilde{\beta} \in A^{n+1}$ be such that $\tilde{\alpha} = (\alpha_\emptyset, \alpha_{\{1\}}, \alpha_{\{1,2\}}, \ldots, \alpha_N)$ and $\tilde{\beta} = (\beta_\emptyset, \beta_{\{1\}}, \beta_{\{1,2\}}, \ldots, \beta_N)$.

Since the coordinates of $\tilde{\alpha}$ are such that $0 \leq \alpha_N \leq \cdots \leq \alpha_\emptyset \leq 1$ and $u^\emptyset = (1, \ldots, 1)$, $\underline{F}(U^\emptyset) = \text{med}(u^\emptyset, \tilde{\alpha}) = \alpha_\emptyset$. Moreover, for each $i \in \{1, \ldots, n\}$, $u^{\{1,\ldots,i\}} = (\underbrace{0, \ldots, 0}_{j \in \{1,\ldots,i\}}, \underbrace{1, \ldots, 1}_{j \in \{i+1,\ldots,n\}})$ implies $\underline{F}(U^{\{1,\ldots,i\}}) = \text{med}(u^{\{1,\ldots,i\}}, \tilde{\alpha}) = \alpha_{\{1,\ldots,i\}}$. Similarly for $\tilde{\beta}$, $\bar{F}(U^\emptyset) = \beta_\emptyset$ and for each $i \in \{1, \ldots, n\}$, $\bar{F}(U^{\{1,\ldots,i\}}) = \beta_{\{1,\ldots,i\}}$.

Next, for each $i \in \{1, \ldots, n\}$, let $V^i \in \mathcal{R}^N$ be such that $V^i = (V_1, \ldots, V_i, V_{i+1}^{\max}, \ldots, V_n^{\max})$ and notice that $V^n = V$. We show that $F(V) = F_G^{\alpha,\beta}(V)$ by induction, in two stages.

Stage 1. We show that $F(V^1) = F_G^{\alpha,\beta}(V^1)$.

Consider profile $V^1 = (V_1, V_2^{\max}, \ldots, V_n^{\max})$. Recall profiles $U^{\{1\}} = (V_1^{\min}, V_2^{\max}, \ldots, V_n^{\max})$ and $U^\emptyset = (V_1^{\max}, \ldots, V_n^{\max})$. Hence, $U^{\{1\}} = (V_{-1}^1, V_1^{\min})$ and $U^\emptyset = (V_{-1}^1, V_1^{\max})$. By Step 1, $\underline{F}(V^1) = \text{med}(\underline{F}(U^{\{1\}}), v_1, \underline{F}(U^\emptyset))$ and $\bar{F}(V^1) = \text{med}(\bar{F}(U^{\{1\}}), v_1, \bar{F}(U^\emptyset))$. Hence, $\underline{F}(V^1) = \text{med}(\alpha_{\{1\}}, v_1, \alpha_{\{\emptyset\}})$ and $\bar{F}(V^1) = \text{med}(\beta_{\{1\}}, v_1, \beta_{\{\emptyset\}})$. Moreover, since $\alpha_N \leq \cdots \leq \alpha_\emptyset \leq v_2 = \cdots = v_n$ and $\beta_N \leq \cdots \leq \beta_\emptyset \leq v_2 = \cdots = v_n$, $\underline{F}(V^1) = \text{med}(v, \tilde{\alpha})$ and $\bar{F}(V^1) = \text{med}(v, \tilde{\beta})$. Therefore, $F(V^1) = F_G^{\alpha,\beta}(V^1)$.

Stage 2. Let $i \in \{2, \ldots, n\}$ be such that $F(V^{i-1}) = F_G^{\alpha,\beta}(V^{i-1})$. We show that $F(V^i) = F_G^{\alpha,\beta}(V^i)$. Notice that we only show $\underline{F}(V^i) = \underline{F}_G^{\alpha,\beta}(V^i)$. The proof showing $\bar{F}(V^i) = \bar{F}_G^{\alpha,\beta}(V^i)$ is symmetric, that is, it can be obtained using the same arguments but after replacing all references to the minimum chosen alternative and $\tilde{\alpha}$ with the equivalent references to the maximum chosen alternative and $\tilde{\beta}$ respectively.

Recall that $V^{i-1} = (V_1, \ldots, V_{i-1}, V_i^{\max}, \ldots, V_n^{\max})$ and $V^i = (V_{-i}^{i-1}, V_i)$. There are three cases.

Case 1. Let $v_i > \underline{F}(V^i)$. Since $V_{-i}^{i-1} = V_{-i}^i$ and $\underline{F}(V^i) < v_i \leq v_i^{\max}$, by *uncompromisingness*, $\underline{F}(V^i) = \underline{F}(V^{i-1}) = \text{med}(v^{i-1}, \tilde{\alpha})$. Thus, $V_{-i}^{i-1} = V_{-i}^i$ and $\text{med}(v^{i-1}, \tilde{\alpha}) < v_i \leq v_i^{\max}$ implies $\underline{F}(V^i) = \text{med}(v^i, \tilde{\alpha}) = \underline{F}_G^{\alpha,\beta}(V^i)$.

Case 2. Let $v_i < \underline{F}(V^i)$ and recall that $U^{\{1,\ldots,i\}} = (V_1^{\min}, \ldots, V_i^{\min}, V_{i+1}^{\max}, \ldots, V_n^{\max})$. Since $v_1 \leq \cdots \leq v_n$ and $v_i < \underline{F}(V^i)$, for each $j \in \{1, \ldots, i\}$, $v_j^{\min} \leq v_j < \underline{F}(V^i)$; hence, by *uncompromisingness*, $\underline{F}(V_{-j}^i, V_j^{\min}) = \underline{F}(V^i)$. Therefore, beginning from profile V^i and considering that all agents $j \in \{1, \ldots, i\}$ (sequentially) change their announcements to $V_j^{\min}$, implies by (sequentially applying) *uncompromisingness*, that $\underline{F}(V^i) = \underline{F}(U^{\{1,\ldots,i\}})$ where as shown above $\underline{F}(U^{\{1,\ldots,i\}}) = \alpha_{\{1,\ldots,i\}}$. Therefore, since at profile V^i, for each $j \in \{1, \ldots, i\}$, $v_j^i < \alpha_{\{1,\ldots,i\}}$, and for each $k \in \{i+1, \ldots, n\}$, $v_k^i = v_k^{\max} = 1 \geq \alpha_{\{1,\ldots,i\}}$, by the median operator, $\underline{F}(V^i) = \mathrm{med}(v^i, \tilde{\alpha}) = \underline{F}_G^{\alpha,\beta}(V^i)$.

Case 3. Let $v_i = \underline{F}(V^i)$. Since $V_{-i}^{i-1} = V_{-i}^i$ and $v_i \leq v_i^{\max}$, by Lemma 12, $\underline{F}(V^i) \leq \underline{F}(V^{i-1})$. Thus, $v_1 \leq \cdots \leq v_n$ and $v_i = \underline{F}(V^i)$, imply $v_{i-1}^i = v_{i-1}^{i-1} \leq \underline{F}(V^{i-1})$. There are two sub-cases.

(i) Let $v_{i-1}^i = \underline{F}(V^{i-1})$. Thus, $v_{i-1}^i = v_i = \underline{F}(V^i) = \underline{F}(V^{i-1})$. Hence, $\underline{F}(V^{i-1}) = \underline{F}_G^{\alpha,\beta}(V^{i-1})$ implies $\mathrm{med}(v^{i-1}, \tilde{\alpha}) = v_i \leq v_i^{\max}$. Therefore, by the median operator, $\underline{F}(V^i) = v_i = \mathrm{med}(v^i, \tilde{\alpha}) = \underline{F}_G^{\alpha,\beta}(V^i)$.

(ii) Let $v_{i-1}^i < \underline{F}(V^{i-1})$. Recall that at profiles $U^{\{1,\ldots,i-1\}} = (V_1^{\min}, \ldots, V_{i-1}^{\min}, V_i^{\max}, \ldots, V_n^{\max})$ and $U^{\{1\ldots,i\}} = (V_{-i}^{\{1,\ldots,i-1\}}, V_i^{\min})$, $\underline{F}(U^{\{1,\ldots,i-1\}}) = \alpha_{\{1,\ldots,i-1\}}$ and $\underline{F}(U^{\{1,\ldots,i\}}) = \alpha_{\{1,\ldots,i\}}$. Since $v_i = \underline{F}(V^i) \leq \underline{F}(V^{i-1})$, it follows that $v_i \leq \alpha_{\{1,\ldots,i-1\}}$.

Next, begin from profile $U^{\{1,\ldots,i\}}$ and consider that all agents $j \in \{1, \ldots, i\}$ (sequentially) change their announcements to V_j, i.e., the final new profile is $V^i = (V_1, \ldots, V_i, V_{i+1}^{\max}, \ldots, V_n^{\max})$. Since $v_j \geq v_j^{\min}$, by (sequentially) applying Lemma 12, $\underline{F}(V^i) \geq \underline{F}(U^{\{1,\ldots,i\}}) = \alpha_{\{1,\ldots,i\}}$. Hence, $v_i \geq \alpha_{\{\bar{1},\ldots,i\}}$ and it follows, that $\alpha_{\{1,\ldots,i\}} \leq v_i \leq \alpha_{\{1,\ldots,i-1\}}$. Thus, since $\alpha_N \leq \cdots \leq \alpha_\emptyset$, vector $\tilde{\alpha}$ contains at least $n+1-i$ coordinates not larger than v_i (i.e., coordinates $\alpha_{\{1,\ldots,i\}}, \ldots, \alpha_N$) and at least i coordinates not smaller than v_i (i.e., coordinates $\alpha_\emptyset, \ldots, \alpha_{\{1,\ldots,i-1\}}$). In addition, since $v_1 \leq \cdots \leq v_n$, at least i agents announce peaks not larger than v_i (i.e., agents $1, \ldots, i$) and $n-i+1$ agents announce peaks not smaller than v_i (i.e., agents $i, \ldots, n$). Therefore, by the median operator, $\underline{F}(V^i) = \mathrm{med}(v^i, \tilde{\alpha}) = v_i = \underline{F}_G^{\alpha,\beta}(V^i)$. $\square$

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