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Generating Functions for $\beta_1(n)$ and $\beta_2(n)$

Sabuj Das

Senior Lecturer, Department Of Mathematics
Raozan University College, Bangladesh.
Email: sabujdas.ctg@gmail.com

Haradhan Kumar Mohajan[2]

Assistant Professor, Premier University, Chittagong, Bangladesh
Email: haradhan_km@yahoo.com

ABSTRACT

This paper shows how to prove the two Theorems, which are related to the terms $\beta_1(n)$ and $\beta_2(n)$ respectively Theorem: $N(0,5,5n+1)= \beta_1(n)+N(5,5,5n+1)$ and Theorem: $N(1,5,5n+1)= \beta_2(n)+N(2,5,5n+2)$.

Keywords: Generating functions, Jecobi's triple product.

1. INTRODUCTION

We give the definitions of π, Rank of partition, $N(m,n)$, $N(m,t,n)$, z, $(x)_\infty$ $(zx)_\infty$, $(x^n)_m$, $\beta_1(n)$, $\beta_2(n)$, $(x^k;x^5)_m$ collected from Partitions Yesterday and Today (Garvan 1979), Generalizations of Dyson's rank (Garvan 1986), Ramanujan's Lost Notebook (Andrews 1979). We generate the generating functions for $\beta_1(n)$, $\beta_2(n)$ (Andrews 1979) and prove the Theorems $N(0,5,5n+1)= \beta_1(n)+$ $N(2,5,5n+1)$ and $N(1,5,5n+1)= \beta_2(n)+$ $N(2,5,5n+2)$. Finally we give two examples, which are related to the Theorem 1 and Theorem 2 respectively when n =2.

2. DEFINITIONS

π: A partition.

Rank of partition: The largest part of a partition π minus the number of parts of π.

$N(m,n)$: The number of partitions of *n* with rank *m*.

$N(m,t,n)$: The number of partition of *n* with rank congruent to m modulo *t*.

$\rho_0(n)$: The number of partitions of *n* with unique smallest part and all other parts ≤ the double of the smallest part.

$\rho_1(n)$: The number of partitions of *n* with unique smallest part and all other parts ≤ one plus the double of the smallest part.

z: The set of complex numbers.

$(x)_\infty$: The product of infinite factors is defined as follows:

$$(x)_\infty = (1-x)(1-x^2)(1-x^3) \ldots \infty .$$

$(zx)_\infty$: The product of infinite factors is defined as follows:

$$(zx)_\infty = (1-zx)(1-zx^2)(1-zx^3) \ldots \infty .$$

$(x^n)_m$: The product of *m* factors is defined as follows:

$$(x^n)_m = (1-x^n)(1-x^{n+1})(1-x^{n+2}) \ldots (1-x^{n+m-1}).$$

$(x^k; x^5)_m$: The product of *m* factors is defined as follows:

$$(x^k; x^5)_m = (1-x^k)(1-x^{k+5})(1-x^{k+10}) \ldots (1-x^{k+(m-1)\,5}).$$

$\beta_1(n)$: The number of partitions of *n* into 1's and parts congruent to 0 or –1 modulo 5 with the largest part $\equiv 0 \pmod 5 \le 5$ times the number of 1's ≤ the smallest part $\equiv -1 \pmod 5$.

$\beta_2(n)$: The number of partitions of *n* into 2's and parts congruent to 0 or – 2 modulo 5 with the largest part $\equiv 0 \pmod 5 \le 5$ times the number of 2's ≤ the smallest part $\equiv -2 \pmod 5$.

3. GENERATING FUNCTIONS (FROM RAMANUJAN'S LOST NOTE BOOK)

From Ramanujan's Lost Note Book (Andrews 1979), Mock Theta Functions (2) (Watson 1937), G. E. Andrews and F. G. Garvan (Andrews and Garvan 1989), we quote the relations as follows:

$$F(x) = \frac{(1-x)(1-x^2)(1-x^3) \ldots \infty}{(1-2x\cos\frac{2n\pi}{5}+x^2)(1-2x^2\cos\frac{2n\pi}{5}+x^4) \ldots \infty}$$

$$f'(x) = 1 + \frac{x}{1 - 2x\cos\frac{2n\pi}{5} + x^2} + \frac{x^4}{(1 - 2x\cos\frac{2n\pi}{5} + x^2)(1 - 2x^2\cos\frac{2n\pi}{5} + x^4)} + \ldots \infty$$

$$, n = 1 \text{ or } 2.$$

$$F(x^{\frac{1}{5}}) = A(x) - 4x^{\frac{1}{5}}\cos\frac{2n\pi}{5}B(x) + 2x^{\frac{2}{5}}\cos\frac{4n\pi}{5}C(x) - 2x^{\frac{3}{5}}\cos\frac{2n\pi}{5}D(x). \quad (1)$$

$$f'(x^{\frac{1}{5}}) = \left\{A(x) - 4\sin^2\frac{2n\pi}{5}\Phi(x)\right\} + x^{\frac{1}{5}}B(x) + 2x^{\frac{2}{5}}\cos\frac{2n\pi}{5}C(x) - 2x^{\frac{3}{5}}\cos\frac{2n\pi}{5}\left\{D(x) + 4\sin^2\frac{2n\pi}{5}.\frac{\psi(x)}{x}\right\}. \quad (2)$$

$$A(x) = \frac{1 - x^2 - x^3 + x^9 + \ldots\infty}{(1-x)^2(1-x^4)^2(1-x^6)^2\ldots\infty},$$

$$B(x) = \frac{(1-x^5)(1-x^{10})(1-x^{15})\ldots\infty}{(1-x)(1-x^4)(1-x^6)\ldots\infty},$$

$$C(x) = \frac{(1-x^5)(1-x^{10})(1-x^{15})\ldots\infty}{(1-x^2)(1-x^3)(1-x^7)\ldots\infty},$$

$$D(x) = \frac{1 - x - x^4 + x^7 + \ldots\infty}{(1-x^2)^2(1-x^3)^2(1-x^7)^2\ldots\infty},$$

$$\phi(x) = -1 + \left\{\frac{1}{1-x} + \frac{x^5}{(1-x)(1-x^4)(1-x^6)} + \frac{x^{20}}{(1-x)(1-x^4)(1-x^6)(1-x^9)(1-x^{11})} + \ldots\infty\right\}.$$

But we get;

$$A(x^5) - 4x\cos\frac{2\pi}{5}B(x^5) + 2x^2\cos\frac{4\pi}{5}C(x^5) - 2x^3\cos\frac{2\pi}{5}D(x^5)$$

$$= 1 - 4x\cos^2\frac{2\pi}{5} + 2x^2\cos\frac{4\pi}{5} - 2x^3\cos\frac{2\pi}{5} + 2x^5 - 4x^6\cos^2\frac{2\pi}{5} + 2x^8\cos\frac{2\pi}{5} - x^{10} + \ldots\infty$$

$$\Psi(x) = -1 + \left\{\frac{1}{1-x^2} + \frac{x^5}{(1-x^2)(1-x^3)(1-x^7)} + \frac{x^{20}}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)(1-x^{12})} + \ldots\infty\right\}.$$

Now,

$$\frac{x}{1-x} + \frac{x^3}{(1-x^2)(1-x^3)} + \frac{x^5}{(1-x^3)(1-x^4)(1-x^5)} + \ldots\infty$$

$$= 3\phi(x) + 1 - A(x).$$

And,

$$\frac{x}{1-x} + \frac{x^2}{(1-x^2)(1-x^3)} + \frac{x^3}{(1-x^3)(1-x^4)(1-x^5)} + \ldots \infty$$

$$= 3\Psi(x) + xD(x).$$

We assume without loss of generality that *n* = 1. Let $\zeta = \exp^{\frac{2\pi i}{5}}$, then we may write the definitions of *F*(*x*) and *f′*(*x*) as;

$$F(x) = \frac{(x)_\infty}{(\zeta x)_\infty (\zeta^{-1} x)_\infty}$$

and,

$$f'(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{n^2}}{(1-\zeta x)(1-\zeta^{-1}x)\ldots(1-\zeta x^n)(1-\zeta^{-1}x^n)}$$

$$= \sum_{n=1}^{\infty} \frac{x^{n^2}}{(\zeta x)_n (\zeta^{-1} x)_n},$$

where we have used the relations;

$(a)_0 = 1$, $(a)_n = (1-a)(1-ax)\ldots(1-ax^{n-1})$, for $n \geq 1$

and,

$$(a)_\infty = \lim_{n\to\infty} (a)_n = \prod_{n=1}^{\infty} (1-ax^{n-1}).$$

After replacing *x* by x^5 we see that (1) and (2) are identities for *F*(*x*) and *f′*(*x*). We note that the numerators in the definitions of *A*(*x*) and *D*(*x*) are theta series in *x* and hence may be written as infinite products using Jecobi’s triple product identity;

$$\prod_{n=1}^{\infty} (1-zx^n)(1-z^{-1}x^{n-1})(1-x^n)$$

$$= \prod_{n=-\infty}^{\infty} (-1)^n z^n x^{\frac{n(n+1)}{2}} \tag{3}$$

$$= ... + z^{-2}x - z^{-1} + 1 - zx + z^2x^3 - ...\infty .$$

where $z \neq 0$ and $|x| < 1$.

Replacing x by x^5 and z by x^{-3} we get from (3);

$$\prod_{n=1}^{\infty}\left(1-x^{5n-3}\right)\left(1-x^{5n-2}\right)\left(1-x^{5n}\right)$$

$$= ... + x^{11} - x^3 + 1 - x^2 + x^9 - ... \infty$$

$$= 1 - x^2 - x^3 + x^9 + x^{11} - ...\infty .$$

Again replacing x by x^5 and z by x^{-3} equation (3) becomes;

$$\prod_{n=1}^{\infty}\left(1-x^{5n-4}\right)\left(1-x^{5n-1}\right)\left(1-x^{5n}\right)$$

$$= ... + x^{13} - x^4 + 1 - x + x^7 - ...\infty$$

$$= 1 - x - x^4 + x^7 + x^{13} - ...\infty .$$

In fact we have;

$$A(x) = \prod_{n=1}^{\infty} \frac{\left(1-x^{5n-3}\right)\left(1-x^{5n-2}\right)\left(1-x^{5n}\right)}{\left(1-x^{5n-4}\right)^2\left(1-x^{5n-1}\right)^2},$$

$$B(x) = \prod_{n=1}^{\infty} \frac{\left(1-x^{5n}\right)}{\left(1-x^{5n-4}\right)\left(1-x^{5n-1}\right)},$$

$$C(x) = \prod_{n=1}^{\infty} \frac{\left(1-x^{5n}\right)}{\left(1-x^{5n-3}\right)\left(1-x^{5n-2}\right)},$$

$$D(x) = \prod_{n=1}^{\infty} \frac{\left(1-x^{5n-4}\right)\left(1-x^{5n-1}\right)\left(1-x^{5n}\right)}{\left(1-x^{5n-3}\right)\left(1-x^{5n-2}\right)} .$$

3.1 Rank of a Partition

The rank of a partition is defined as the largest part minus the number of parts. Thus the partition 6 + 5 + 2 + 1 + 1 + 1 + 1 of 17 has rank, 6–7 = –1 and the conjugated partition, 7 + 3 + 2 + 2 + 2 + 1 has rank, 7–6 = 1. i.e., the rank of a partition and that of the conjugate partition differ only in sign. The rank of a partition of 5 belongs to any one of the residues (mod 5) and we have exactly 5 residues. There is similar result for all partitions of 7 leading to (mod 7).

The generating function for the rank is of the form (Garvan 1986);

$$\sum_{n=1}^{\infty}(-1)^{n-1}x^{\frac{n}{2}(3n-1)+|m|n}\left(1-x^{n}\right)\prod_{j=1}^{\infty}\left(1-x^{j}\right)^{-1}$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\left\{x^{\frac{n}{2}(3n+2+|m|-1)}-x^{\frac{n}{2}(3n+2|m|+1)}\right\}\sum_{k=0}^{\infty}P(k)x^{k}$$

$$=\left(x^{|m|+1}+0.x^{|m|+2}+x^{|m|+3}+\ldots\infty\right)-\left(x^{2|m|+5}+x^{2|m|+6}+\ldots\infty\right)$$

$$=\sum_{n=0}^{\infty}N(m,n)x^{n}\cdot$$

The generating function for $N(m,t,n)$ is of the form;

$$\sum_{\substack{n=-\infty\\ n\neq 1}}^{\infty}(-1)^{n}x^{\frac{n}{2}(3n+1)}\frac{\left(x^{mn}+x^{n(t-m)}\right)}{1-x^{tn}}\prod_{j=1}^{\infty}\left(1-x^{j}\right)^{-1}$$

$$=\sum_{\substack{n=-\infty\\ n\neq 1}}^{\infty}(-1)^{n}x^{\frac{n}{2}(3n+1)}\left(x^{mn}+x^{n(t-m)}\right)\times\left(1+x^{tn}+x^{2tn}+\ldots\infty\right)\sum_{k=0}^{\infty}P(k)x^{k}$$

$$=\sum_{n=0}^{\infty}N(m,t,n)x^{n};$$

which shows that all the coefficients of x^{-n} (where *n* is any positive integer) are zero.

Now we define the generating function;

$r_a(d)$ for $N(a,t,tn+d)$

where $r_a(d) = r_a(d,t) = \prod_{n=0}^{\infty} N(a,t,tn+d)\,x^n$, and

$$r_{a,b}(d) = r_{a,b}(d,t) = r_a(d) - r_b(d).$$

$$= \prod_{n=0}^{\infty} \{N(a,t,tn+d) - N(b,t,tn+d)\} x^n .$$

The generating function $\phi(x)$ is of the form;

$$\phi(x) = -1 + \left\{ \frac{1}{1-x} + \frac{x^5}{(1-x)(1-x^4)(1-x^6)} + \frac{x^{20}}{(1-x)(1-x^4)(1-x^6)(1-x^9)(1-x^{11})} + \ldots\infty \right\},$$

$$= -1 + (1 + x + x^2 + \ldots\infty) + x^5(1 + x + x^2 + \ldots\infty) \times (1 + x^4 + \ldots\infty)(1 + x^6 + \ldots\infty) + \ldots\infty$$

$$= x + x^2 + x^3 + x^4 + 2x^5 + 2x^6 + 2x^7 + 2x^8 + \ldots\infty$$

$$= \sum_{n=0}^{\infty} \{N(1,5,5n) - N(2,5,5n)\} x^n$$

$$= r_{1,2}(0).$$

The generating function $A(x)$ is defined as;

$$A(x) = \frac{1 - x^2 - x^3 + x^9 + \ldots\infty}{(1-x)^2(1-x^4)^2(1-x^6)^2\ldots\infty}$$

$$= (1 - x^2 - x^3 + x^9 + \ldots\infty)(1 + 2x + 3x^2 + \ldots\infty) \times (1 + 2x^4 + 3x^8 + \ldots\infty)\ldots\infty$$

$$= 1 + 2x + 2x^2 + x^3 + 2x^4 + \ldots\infty$$

$$= 1 + \sum_{n=0}^{\infty} \{N(0,5,5n) - N(2,5,5n) + N(1,5,5n) - 2N(2,5,5n)\} x^2$$

$$= 1 + \sum_{n=0}^{\infty} \{N(0,5,5n) - N(2,5,5n)\} x^n + 2\sum_{n=0}^{\infty} \{N(1,5,5n) - N(2,5,5n)\} x^n$$

$$= 1 + r_{0,2}(0) + 2r_{1,2}(0).$$

The generating function is of the form;

$$\prod_{n=1}^{\infty}\frac{1-x^{5n}}{\left(1-x^{5n-4}\right)\left(1-x^{5n-1}\right)}$$

$$=\prod_{n=1}^{\infty}\left(1-x^{5n}\right)\left(1+x^{5n-4}+\ldots\infty\right)\left(1+x^{5n-1}+\ldots\infty\right)$$

$$=(1-0)+(3-2)x+(12-11)x^2+x^3+2x^4+\ldots\infty$$

$$=\sum_{n=0}^{\infty}\{N(0,5,5n+1)-N(2,5,5n+1)\}x^n$$

$$=r_{0,2}(1).$$

The generating function is of the form;

$$\prod_{n=1}^{\infty}\frac{1-x^{5n}}{\left(1-x^{5n-3}\right)\left(1-x^{5n-2}\right)}$$

$$=\prod_{n=1}^{\infty}\left(1-x^{5n}\right)\left(1+x^{5n-3}+x^{10n-6}+\ldots\infty\right)$$

$$=(1-0)+(3-3)x+(16-15)x^2+\ldots\infty$$

$$=\sum_{n=0}^{\infty}\{N(0,5,5n+2)-N(2,5,5n+2)\}x^n\times\left(1+x^{5n-2}+x^{10n-4}+\ldots\infty\right)$$

$$=r_{1,2}(2).$$

The generating function $\Psi(x)$ is of the form;

$$\Psi(x)=-1+\left\{\frac{1}{1-x^2}+\frac{x^5}{\left(1-x^2\right)\left(1-x^3\right)\left(1-x^7\right)}+\frac{x^{20}}{\left(1-x^2\right)\left(1-x^3\right)\left(1-x^7\right)\left(1-x^8\right)\left(1-x^{12}\right)}+\ldots\infty\right\}$$

$$=-1+\left(1+x^2+x^4+\ldots\infty\right)+x^5\left(1+x^2+\ldots\infty\right)\left(1+x^3+x^6+\ldots\infty\right)\left(1+x^7+\ldots\infty\right)+\ldots\infty$$

$$=x^2+x^4+x^6+x^7+2x^8+x^9+2x^{10}+\ldots\infty.$$

Hence,

$$\frac{\Psi(x)}{x} = x + x^3 + x^4 + x^5 + x^6 + 2x^7 + x^8 + 2x^9 + \ldots\infty$$

$$= \sum_{n=0}^{\infty} \{N(2,5,5n+3) - N(0,5,5n+3)\} x^n$$

$$= r_{2,0}(3)$$

and,

$$r_{0,2}(3) = -\frac{\Psi(x)}{x}.$$

The generating function $D(x)$ is of the form;

$$D(x) = \frac{1 - x - x^4 + x^7 + \ldots\infty}{(1-x^2)^2(1-x^3)^2(1-x^7)^2\ldots\infty}$$

$$= (1 - x - x^4 + x^7 + \ldots\infty)(1 + 2x^2 + 3x^4 + \ldots\infty)\times(1 + 2x^3 + \ldots\infty)(1 + 2x^7 + \ldots\infty)\ldots\infty$$

$$= 1 - x + 2x^2 + 0.x^3 + \ldots\infty$$

$$= \sum_{n=0}^{\infty} \{N(0,5,5n+3) - N(1,5,5n+3) + N(0,5,5n+3) - N(2,5,5n+3)\} x^n$$

$$= \sum_{n=0}^{\infty} \{N(0,5,5n+3) - N(1,5,5n+3)\} x^n + \sum_{n=0}^{\infty} \{N(0,5,5n+3) - N(2,5,5n+3)\} x^n$$

$$= r_{0,1}(3) + r_{0,2}(3)$$

4. THE GENERATING FUNCTIONS FOR $\beta_1(n)$ AND $\beta_2(n)$

First we shall establish the following identity, which is used in proving the Theorems. If a and t are both real numbers with $|a| < 1$ and $|t| < 1$, we have;

$$\frac{(at)_\infty}{(t)_\infty} = \frac{(1-at)(1-atx)(1-atx^2)+\ldots\infty}{(1-t)(1-tx)(1-tx^2)+\ldots\infty}$$

$$=\{(1-at)(1-atx)\ldots\infty\}(1+t+t^2+\ldots\infty)\times(1+tx+t^2x^2+\ldots\infty)(1+t^4x^4+\ldots\infty)\ldots\infty$$

$$=1+t\{(1+x+x^2+\ldots\infty)-a(1+x+x^2+\ldots\infty)\}+t^2\{(1+x+2x^2+2x^3+\ldots\infty)-a(1+2x+3x^2+\ldots\infty)+$$

$$a^2(x+x^2+2x^3+2x^4+\ldots\infty)+\ldots\infty$$

$$=1+(1-a)t(1+x+x^2+\ldots\infty)+(1-a)(1-ax)t^2(1+x+2x^2+2x^3+\ldots\infty)+\ldots\infty$$

$$=1+\frac{(1-a)t}{1-x}+\frac{(1-a)(1-ax)t^2}{(1-x)(1-x^2)}+\frac{(1-a)(1-ax)(1-ax^2)t^3}{(1-x)(1-x^2)(1-x^3)}+\ldots\infty$$

$$=\sum_{n=0}^{\infty}\frac{(a)_n t^n}{(x)_n}$$

i.e., $\frac{(at)_\infty}{(t)_\infty}=\sum_{n=0}^{\infty}\frac{(a)_n t^n}{(x)_n}$. (4)

The generating function for $\beta_1(n)$ is defined as;

$$\sum_{n=0}^{\infty}\frac{x^n}{(x^5;x^5)_n(x^{5n+4};x^5)_\infty}$$

$$=\frac{1}{(1-x^4)(1-x^9)(1-x^{14})\ldots\infty}+\frac{1}{(1-x^5)(1-x^9)(1-x^{14})\ldots\infty}+\frac{x^2}{(1-x^5)(1-x^{10})(1-x^{14})\ldots\infty}+\ldots\infty$$

$$=1+x+x^2+x^3+2x^4+x^5+2x^6+3x^8+\ldots\infty$$

$$=\sum_{n=0}^{\infty}\beta_1(n)\,x^n\,, \qquad (5)$$

were we have assumed $\beta_1(0)=1$.

The generating function for $\beta_2(n)$ is defined as;

$$\sum_{n=0}^{\infty}\frac{x^{2n}}{(x^5;x^5)_n(x^{5n+3};x^5)_\infty}$$

$$= \frac{1}{(1-x^3)(1-x^8)(1-x^{13})\ \ldots\ \infty} + \frac{x^2}{(1-x^5)(1-x^8)(1-x^{13})\ \ldots\ \infty} + \frac{x^4}{(1-x^5)(1-x^{10})(1-x^{13})\ \ldots\ \infty} + \ldots\infty$$

$$= 1 + x^2 + x^3 + x^4 + 2x^6 + x^7 + 2x^8 + \ldots\infty$$

$$= \sum_{n=0}^{\infty} \beta_1(n)\, x^n\ , \qquad (6)$$

were we have assumed $\beta_1(0) = 1$.

Here we give two Theorems, which are related to the terms $\beta_1(n)$ and $\beta_2(n)$ respectively.

Theorem 1: $N(0,5,5n+1) = \beta_1(n) + N(2,5,5n+1)$,
where $\beta_1(n)$ is the number of partitions of *n* into 1's and parts congruent to 0 or –1 modulo 5 with the largest part $\equiv 0 \pmod 5 \leq 5$ times the number of 1's ≤ the smallest part $\equiv -1 \pmod 5$.

Proof: From (4) by replacing $(z^{-1}x)$ for *a* and *z* for *t* we have;

$$\frac{(x)_\infty}{(z)_\infty (z^{-1}x)_\infty}$$

$$= \frac{1}{(z^{-1}x)_\infty} \sum_{n=0}^{\infty} \frac{(z^{-1}x)_n z^n}{(x)_n}\ ,$$

where $|z| < 1$ but $z \neq 0$

$$= \frac{1}{(1-z^{-1}x)(1-z^{-1}x^2)\ldots\infty}\left[1 + \frac{(1-z^{-1}x)z}{(1-x)} + \frac{(1-z^{-1}x)(1-z^{-1}x^2)z^2}{(1-x)(1-x^2)} + \ldots\infty\right]$$

$$= \frac{1}{(1-z^{-1}x)(1-z^{-1}x^2)\ \ldots\infty} + \frac{z}{(1-x)(1-z^{-1}x^2)\ \ldots\infty} + \frac{z^2}{(1-x)(1-x^2)(1-z^{-1}x^3)\ \ldots\infty} + \ldots\infty$$

$$= \frac{(1-x)(1-x^2)\ \ldots\ \infty}{(1-z)(1-zx)\ldots(1-z^{-1}x)(1-z^{-1}x^2)\ \ldots\infty}$$

Replacing *x* by x^5 and *z* by *x*, we obtain;

$$\frac{1}{(1-x^4)(1-x^9)\ \ldots\infty} + \frac{x}{(1-x^5)(1-x^9)(1-x^{14})\ \ldots\infty} + \frac{x^2}{(1-x^5)(1-x^{10})(1-x^{14})\ \ldots\infty} + \ldots\infty$$

$$=\frac{\left(1-x^{5}\right)\left(1-x^{10}\right)\ldots\infty}{\left\{(1-x)\left(1-x^{6}\right)\ldots\infty\right\}\left(1-x^{4}\right)\left(1-x^{9}\right)\ldots\infty}.$$

Hence,

$$\sum_{n=0}^{\infty}\frac{x^{n}}{\left(x^{5};x^{5}\right)_{n}\left(x^{5n+4};x^{5}\right)_{\infty}}=\frac{\left(x^{5};x^{5}\right)_{\infty}}{\left(x;x^{5}\right)_{\infty}\left(x^{4};x^{5}\right)_{\infty}}$$

$\therefore \sum_{n=0}^{\infty}\beta_{1}(n)x^{n}=r_{0,2}(1)$, by above;

$$=\sum_{n=0}^{\infty}\left\{N(0,5,5n+1)-N(2,5,5n+1)\right\}x^{n}.$$

Equating the coefficient of x^{n} on both sides, we get;

$\beta_{1}(n)=N(0,5,5n+1)-N(2,5,5n+1).$

Hence the Theorem.

Theorem 2: $N(1,5,5n+1)=\beta_{2}(n)+N(2,5,5n+2)$, where $\beta_{2}(n)$ is the number of partitions of *n* into 2's and parts congruent to 0 or – 2 modulo 5 with the largest part $\equiv 0(\mathrm{mod}\,5)\leq 5$ times the number of 2's ≤ the smallest part $\equiv -2(\mathrm{mod}\,5)$.

Proof: From (4) by replacing $\left(z^{-1}x\right)$ for *a*, and *z* for *t* we have;

$$\frac{(x)_{\infty}}{(z)_{\infty}\left(z^{-1}x\right)_{\infty}}$$

$$=\frac{1}{\left(z^{-1}x\right)_{\infty}}\sum_{n=0}^{\infty}\frac{\left(z^{-1}x\right)_{n}z^{n}}{(x)_{n}},$$

where $|z|<1$ but $z\neq 0$

$$=\frac{1}{\left(1-z^{-1}x\right)\left(1-z^{-1}x^{2}\right)\ldots\infty}\left[1+\frac{\left(1-z^{-1}x\right)z}{(1-x)}+\frac{\left(1-z^{-1}x\right)\left(1-z^{-1}x^{2}\right)z^{2}}{(1-x)\left(1-x^{2}\right)}+\ldots\infty\right]$$

$$=\frac{1}{\left(1-z^{-1}x\right)\left(1-z^{-1}x^{2}\right)\ldots\infty}+\frac{z}{\left(1-x\right)\left(1-z^{-1}x^{2}\right)\ldots\infty}+\frac{z^{2}}{\left(1-x\right)\left(1-x^{2}\right)\left(1-z^{-1}x^{3}\right)\ldots\infty}+\ldots\infty$$

$$=\frac{\left(1-x\right)\left(1-x^{2}\right)\ldots\infty}{\left(1-z\right)\left(1-zx\right)\ldots\left(1-z^{-1}x\right)\left(1-z^{-1}x^{2}\right)\ldots\infty}.$$

After replacing x by x^5, and z by x^2, we get;

$$\frac{1}{\left(1-x^{4}\right)\left(1-x^{9}\right)\ldots\infty}+\frac{x}{\left(1-x^{5}\right)\left(1-x^{9}\right)\left(1-x^{14}\right)\ldots\infty}+\frac{x^{2}}{\left(1-x^{5}\right)\left(1-x^{10}\right)\left(1-x^{14}\right)\ldots\infty}+\ldots\infty.$$

We get by replacing x by x^5, and z by x^2;

$$\frac{1}{\left(1-x^{3}\right)\left(1-x^{8}\right)\ldots\infty}+\frac{x^{2}}{\left(1-x^{5}\right)\left(1-x^{8}\right)\left(1-x^{13}\right)\ldots\infty}+\frac{x^{4}}{\left(1-x^{5}\right)\left(1-x^{10}\right)\left(1-x^{13}\right)\ldots\infty}+\ldots\infty$$

$$=\frac{\left(1-x^{5}\right)\left(1-x^{10}\right)\left(1-x^{15}\right)\ldots\infty}{\left\{\left(1-x^{2}\right)\left(1-x^{7}\right)\ldots\infty\right\}\left\{\left(1-x^{3}\right)\left(1-x^{8}\right)\ldots\infty\right\}}.$$

Hence,

$$\sum_{n=0}^{\infty}\frac{x^{2n}}{\left(x^{5};x^{5}\right)_{n}\left(x^{5n+3};x^{5}\right)_{\infty}}=\frac{\left(x^{5};x^{5}\right)_{\infty}}{\left(x^{2};x^{5}\right)_{\infty}\left(x^{3};x^{5}\right)_{\infty}}$$

$$\therefore\sum_{n=0}^{\infty}\beta_{2}(n)x^{n}=r_{1,2}(2),\text{ by above;}$$

$$=\sum_{n=0}^{\infty}\left\{N(1,5,5n+2)-N(2,5,5n+2)\right\}x^{n}.$$

Equating the coefficient of x^n on both sides, we get;

$$\beta_{2}(n)=N(1,5,5n+2)-N(2,5,5n+2)$$

$$N(1,5,5n+2)=\beta_{2}(n)+N(2,5,5n+2).$$

Hence the Theorem.

Now we give two examples, which are related to the Theorems respectively.

Example 1: $N(0, 5, 11) = 12$, $N(2, 5, 11) = 11$, $\beta_1(2) = 1$, with the relevant partition is 1 + 1.

$\therefore N(0, 5, 11) = \beta_1(2) + N(2, 5, 11)$.

Example 2: $N(1, 5, 12) = 16$, $N(2, 5, 12) = 15$, $\beta_2(2) = 1$, with the relevant partition is 2.

$\therefore N(1, 5, 12) = \beta_2(2) + N(2, 5, 12)$.

5. CONCLUSION

We have verified for any positive integer of *n*, the two Theorems related to the terms $\beta_1(n)$ and $\beta_2(n)$ respectively. We have also verified the Theorems for $n = 2$.

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