The Rogers-Ramanujan Identities

Fazlee Hossain and Sabuj Das

Assistant Professor, Premier University, Chittagong, Bangladesh.

12 March 2015

Turkish Journal of Analysis and Number Theory, 2015, Vol. 3, No. 2, 37-42

The Rogers-Ramanujan Identities

Professor Md. Fazlee Hossain
Department of Mathematics, University of Chittagong, Bangladesh.

Sabuj Das
Senior Lecturer, Department of Mathematics
Raozan University College, Bangladesh
Email: sabujdas.ctg@gmail.com

Haradhan Kumar Mohajan*
Assistant Professor, Premier University, Chittagong, Bangladesh
*Corresponding author: haradhan1971@gmail.com

Abstract *In 1894, Rogers found the two identities for the first time. In 1913, Ramanujan found the two identities later and then the two identities are known as The Rogers-Ramanujan Identities. In 1982, Baxter used the two identities in solving the Hard Hexagon Model in Statistical Mechanics. In 1829 Jacobi proved his triple product identity; it is used in proving The Rogers-Ramanujan Identities. In 1921, Ramanujan used Jacobi's triple product identity in proving his famous partition congruences. This paper shows how to generate the generating function for $C'(n)$, $C_1'(n)$, $C''(n)$ and $C_1''(n)$, and shows how to prove the Corollaries 1 and 2 with the help of Jacobi's triple product identity. This paper shows how to prove the Remark 3 with the help of various auxiliary functions and shows how to prove The Rogers-Ramanujan Identities with help of Ramanujan's device of the introduction of a second parameter a.*

Keywords: *At most, auxiliary function, convenient, expansion, minimal difference, operator, Ramanujan's device.*

1. Introduction

In this article, we give some related definitions of $P(n)$, $C'(n)$, $P_m(n-m^2)$, $C_1'(n)$, $C''(n)$, $P_m(n-m(m+1))$ and $C_1''(n)$. We describe the generating functions for $C'(n)$, $P_m(n-m^2)$, $C_1'(n)$, $C''(n)$, $P_m(n-m(m+1))$ and $C_1''(n)$, and establish the Remarks 1 and 2 with numerical examples and also prove the Corollaries 1 and 2 with the help of Jacobi's triple product identity [3]. We transfer the auxiliary function into another auxiliary function with the help of Ramanujan's device of the introduction of a second parameter a [5],

i.e.,

$$G_k(a,x)=\prod_{n=0}^{\infty}(-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}}\left(1-a^k x^{2kn}\right)C_n$$

to

$$G_1(x,x)=\sum_{m=0}^{\infty}\left(1-x^{5m+1}\right)\left(1-x^{5m+4}\right)\left(1-x^{5m+5}\right),$$

where $k=1$, and $a=x$, it is used in proving The Rogers-Ramanujan Identity 1. We prove The Rogers-Ramanujan Identities with the help of auxiliary functions.

2. Some Related Definitions

$P(n)$ [7]: The number of partitions of n like: 4, 3+1, 2+2, 2+1+1, 1+1+1+1 $\therefore P(4)=5$.

$C'(n)$ [6]: The number of partitions of n into parts each of which is of one of the forms $5m+1$ and $5m+4$.

$P_m(n-m^2)$: The number of partitions of $n-m^2$ into m parts at most.

$C''(n)$: The number of partitions of n into parts of the forms $5m+2$ and $5m+3$.

$C_1'(n)$: The number of partitions of n into parts without repetitions or parts whose minimal difference is 2.

$P_m(n-m(m+1))$: The number of partitions of $n-m(m+1)$ into m parts at most.

$C_1''(n)$: The number of partitions of n into parts not less than 2 and with minimal difference 2.

3. Generating Functions for $C'(n)$ and $C''(n)$

In this section we describe the generating functions for $C'(n)$ and $C''(n)$ respectively. The generating function for $C'(n)$ is of the form [5];

$$\sum_{m=0}^{\infty}\frac{1}{(1-x^{5m+1})(1-x^{5m+4})}$$

$$=\frac{1}{(1-x)(1-x^4)(1-x^6)(1-x^9)\ldots\infty}$$

$$=1+x+x^2+x^3+2x^4+2x^5+3x^6+\ldots\infty$$

$$=1+\sum_{n=1}^{\infty}C'(n)x^n \qquad (1)$$

where the coefficient $C'(n)$ of x^n is the number of partitions of n into parts each of which is of one of these forms $5m+1$ and $5m+4$.

Now we consider a special function, which is given below:

$$\frac{x^{m^2}}{(1-x)(1-x^2)\ldots(1-x^m)}$$

$$=x^{m^2}\sum_{n=m^2}^{\infty}P_m(n-m^2)x^{n-m^2}$$

$$=\sum_{n=m^2}^{\infty}P_m(n-m^2)x^n.$$

It is convenient to define $P_m(0)=1$. The coefficient $P_m(n-m^2)$ of x^n in the above expansion is the number of partitions of $n-m^2$ into m parts at most. Another special function, which is defined as;

$$1+\sum_{m=1}^{\infty}\frac{x^{m^2}}{(1-x)(1-x^2)\ldots(1-x^m)}$$

$$=1+\frac{x}{1-x}+\frac{x^4}{(1-x)(1-x^2)}+\frac{x^9}{(1-x)(1-x^2)(1-x^3)}+\ldots\infty$$

$$=1+x+x^2+x^3+2x^4+2x^5+3x^6+3x^7+\ldots\infty$$

$$=1+\sum_{n=1}^{\infty}C_1'(n)x^n \qquad (2)$$

where the coefficient $C_1'(n)$ is the number of partitions of n into parts without repetitions or parts, whose minimal difference is 2.

From (1) and (2) we can establish the following Remark:

Remark 1: $C_1'(11) = C'(11)$ (3)
i.e., the number of partitions of n with minimal difference 2 is equal to the number of partitions of n into parts of the forms $5m + 1$ and $5m + 4$.

Example 1: For $n = 11$, there are 7 partitions of 11 that are enumerated by $C_1'(n)$ of above statement, which are given bellow [6]:

11, 10 + 1, 9 + 2, 8 + 3, 7 + 4, 7 + 3 + 1, 6 + 4 + 1,

$\therefore\ C_1'(11) = 7$.

There are 7 partitions of 11 are enumerated by $C_1'(n)$ of above statement, which are given bellow:

11, 9 + 1 + 1, 6 + 4 + 1, 6 + 1 + 1 + 1 + 1 + 1,
4 + 4 + 1 + 1 + 1, 4 + 1 + 1 + 1 + 1 + 1 + 1 + 1,
1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1,
$\therefore\ C'(11) = 7$.

Hence, $C_1'(11) = C'(11)$.

We can conclude that, $C_1'(11) = C'(11)$.

$$1 + \sum_{n=1}^{\infty} C'(n)\, x^n = 1 + \sum_{n=1}^{\infty} C_1'(n)\, x^n .$$

$$1 + \sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x)(1-x^2)\ldots(1-x^m)}$$

$$= \sum_{m=0}^{\infty} \frac{1}{(1-x^{5m+1})(1-x^{5m+4})},$$

which will be proved later as identity 1, it is known as The Rogers-Ramanujan identity 1.

The generating function for $C''(n)$ is of the form [1];

$$\sum_{m=0}^{\infty} \frac{1}{(1-x^{5m+2})(1-x^{5m+3})}$$

$$= \frac{1}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)\ldots\infty}$$

$$= 1 + 0.x + x^2 + x^3 + x^4 + x^5 + 2x^6 + 2x^7 + \ldots\infty$$

$$= 1 + \sum_{n=1}^{\infty} C''(n)\, x^n \qquad (4)$$

where the coefficient $C''(n)$ is the number of partitions of n into parts of the forms $5m + 2$ and $5m + 3$.

Now we consider a special function, which is of the form [1];

$$\frac{x^{m(m+1)}}{(1-x)(1-x^2)\ldots(1-x^m)}$$

$$= x^{m(m+1)} \sum_{n=m(m+1)}^{\infty} P_m(n - m(m+1))\, x^{n-m(m+1)}$$

$$= \sum_{n=m(m+1)}^{\infty} P_m(n - m(m+1))\, x^n ,$$

where the coefficient $P_m(n - m(m+1))$ of x^n in the above expansion is the number of partitions of $n - m(m+1)$ into m parts at most.

Another special function, which is defined as;

$$1 + \sum_{m=1}^{\infty} \frac{x^{m(m+1)}}{(1-x)(1-x^2)\ldots(1-x^m)}$$

$$= 1 + \frac{x^2}{1-x} + \frac{x^6}{(1-x)(1-x^2)} + \frac{x^{12}}{(1-x)(1-x^2)(1-x^3)} + \ldots\infty$$

$$=1+x^2+x^3+x^4+x^5+2x^6+2x^7+3x^8+\ldots\infty$$

$$=1+\sum_{n=1}^{\infty}C_1''(n)\,x^n\,, \qquad (5)$$

where the coefficient $C_1''(n)$ is the number of partitions of n into parts not less than 2 and with minimal difference 2.

From (4) and (5) we can establish the following Remark:

Remarks 2: $C_1''(n)=C''(n)$, (6)

i.e., the number of partitions of n into parts not less than 2 and with minimal difference 2 is equal to the number of partitions of n into parts of the forms $5m + 2$ and $5m + 3$.

Example 2: If $n = 11$, the four partitions of 11 into parts not less than 2 and with minimal difference 2 are given below:

11, 9 + 2, 8 + 3, 7 + 4.

Hence, $C_1''(11)=4$.

Again the four partitions of 11 into parts of the form $5m + 2$ and $5m + 3$ are given as;

8 + 3, 7 + 2 + 2, 3 + 3 + 3 + 2, 3 + 2 + 2 + 2 + 2.

Hence, $C''(11)=4$.

$\therefore\ C_1''(11)=C''(11)$.

We can conclude that, $C_1''(n)=C''(n)$.

i.e., $1+\sum_{m=1}^{\infty}C_1''(n)\,x^n \;=1+\sum_{m=1}^{\infty}C''(n)\,x^n$

$$1+\sum_{m=1}^{\infty}\frac{x^{m(m+1)}}{(1-x)(1-x^2)\ldots(1-x^m)}$$
$$=\prod_{m=0}^{\infty}\frac{1}{(1-x^{5m+2})(1-x^{5m+3})},$$

which will be proved later as identity 2, it is known as The Rogers-Ramanujan identity 2.

Now we give two Corollaries, which are related to the Jacobi's triple product identity [3].

Corollary 1: $\prod_{n=0}^{\infty}(1-x^{5n+1})(1-x^{5n+4})(1-x^{5n+5})$

$$=\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(5n+3)}{2}}$$

Proof: From Jacobi's Theorem [2] we have;

$$\prod_{n=0}^{\infty}\left\{(1-x^{2n})(1+x^{2n+1}z)(1+x^{2n-1}z^{-1})\right\}$$
$$=\sum_{n=-\infty}^{\infty}x^{n^2}z^n\,,$$

for all z except $z = 0$, if $|x|<1$.

If we write $x^{5/2}$ for x, $-x^{3/2}$ for z and replace n by $n + 1$ on the left hand side we obtain;

$$\prod_{n=0}^{\infty}(1-x^{5n+1})(1-x^{5n+4})(1-x^{5n+5})$$

$$=1-x-x^4+x^7+x^{13}-\ldots\infty$$

$$=\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(5n+3)}{2}}\,.$$

Hence, the Corollary.

Corollary 2:

$$\prod_{n=0}^{\infty}\left(1-x^{5n+2}\right)\left(1-x^{5n+3}\right)\left(1-x^{5n+5}\right)$$

$$=\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(5n+1)}{2}}$$

Proof: From Jacobi's Theorem we have;

$$\prod_{n=0}^{\infty}\left(1-x^{2n}\right)\left(1+x^{2n+1}z\right)\left(1+x^{2n-1}z^{-1}\right)$$

$$=\sum_{n=-\infty}^{\infty}x^{n^2}z^n,$$

for all z except $z = 0$, when $|x|<1$.

If we write $x^{5/2}$ for x, $-x^{1/2}$ for z and replace n by $n + 1$ on the left hand side we obtain;

$$\prod_{n=0}^{\infty}\left(1-x^{5n+2}\right)\left(1-x^{5n+3}\right)\left(1-x^{5n+5}\right)$$

$$=1-x^2-x^3+x^9+x^{11}-\ldots\infty$$

$$=\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(5n+1)}{2}}.$$

Hence the Corollary.

4.The Rogers-Ramanujan Identities

First we transfer the following auxiliary function into another auxiliary function. Let us consider the auxiliary function [1, 2] with $|x|<1$ and $|a|<1$.

$$G_k(a,x)=\prod_{n=0}^{\infty}(-1)^n a^{2n}x^{\frac{n(5n+1)-2kn}{2}}\left(1-a^k x^{2kn}\right)C_n \qquad (7)$$

it is known as Ramanujan's device of the introduction of a second parameter a, where k is 0, 1 or 2 and $C_0=1$,

$$C_n=\frac{(1-a)(1-ax)\ldots\left(1-ax^{n-1}\right)}{(1-x)\left(1-x^2\right)\ldots\left(1-x^n\right)}.$$

Hence,

$$G_k(a,x)=\sum_{n=0}^{\infty}\left[(-1)^n a^{2n}x^{\frac{n(5n+1)-2kn}{2}}\left(1-a^k x^{2kn}\right)\times\right.$$

$$\left.\frac{(1-a)(1-ax)\ldots\left(1-ax^{n-1}\right)}{(1-x)\left(1-x^2\right)\ldots\left(1-x^n\right)}\right].$$

$$\frac{G_k(a,x)}{(1-a)(1-ax)\ldots\infty}$$

$$=\sum_{n=0}^{\infty}\left[(-1)^n a^{2n}x^{\frac{n(5n+1)-2kn}{2}}\times\right.$$

$$\left.\frac{1-a^k x^{2kn}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^n\right)\left(1-ax^n\right)\left(1-ax^{n+1}\right)\ldots\infty}\right]$$

$$=\sum_{n=0}^{\infty}(-1)^n a^{2n}x^{\frac{n(5n+1)-2kn}{2}}\left(1-a^k x^{2kn}\right)P_nQ_n(a),$$

where $P_n=\prod_{r=1}^{n}\frac{1}{1-x^r}$,

$$Q_n(a)=\prod_{r=n}^{\infty}\frac{1}{1-ax^r}=H_k(a,x) \qquad (8)$$

which is another auxiliary function, and it is used in proving The Rogers-Ramanujan Identities [1].

But from (7) we can easily verify that with $k = 1, 2$ and $a = x$.

$$G_1(x,x)=1-x-x^4+x^7+x^{13}-\ldots\infty$$

$$G_1(x,x)=\prod_{n=0}^{\infty}\left(1-x^{5n+1}\right)\left(1-x^{5n+4}\right)\left(1-x^{5n+5}\right)$$

(by Corollary 1). (9)

$$G_2(x,x)=1-x^2-x^3+x^9+x^{11}-\ldots\infty$$

$$G_2(x,x)=\prod_{m=0}^{\infty}\left(1-x^{5m+2}\right)\left(1-x^{5m+3}\right)\left(1-x^{5m+5}\right)$$

(by Corollary 2). (10)

From (8) we can also find that, if $k=1$ and $a=x$, then;

$$H_1(x,x)=\frac{G_1(x,x)}{(1-x)\left(1-x^2\right)\left(1-x^3\right)\ldots\infty}$$

$$=\frac{\prod_{m=0}^{\infty}\left(1-x^{5m+1}\right)\left(1-x^{5m+4}\right)\left(1-x^{5m+5}\right)}{(1-x)\left(1-x^2\right)\left(1-x^3\right)\ldots\infty}$$

$$=\prod_{m=0}^{\infty}\frac{1}{\left(1-x^{5m+2}\right)\left(1-x^{5m+3}\right)}. \qquad (11)$$

Again for $k=2$ and $a=x$, we get;

$$H_2(x,x)=\frac{G_2(x,x)}{(1-x)\left(1-x^2\right)\left(1-x^3\right)\ldots\infty}$$

$$=\frac{\prod_{m=0}^{\infty}\left(1-x^{5m+2}\right)\left(1-x^{5m+3}\right)\left(1-x^{5m+5}\right)}{(1-x)\left(1-x^2\right)\left(1-x^3\right)\ldots\infty}$$

$$=\prod_{m=0}^{\infty}\frac{1}{\left(1-x^{5m+1}\right)\left(1-x^{5m+4}\right)}. \qquad (12)$$

Now we can consider the following Remark [2].

Remark 3: $H_k-H_{k-1}=a^{k-1}\eta H_{3-k}$, where the operator η is defined by $\eta\, f(a)=f(ax)$, and $k=1$ or 2.

Proof: From (8) we have;

$$H_k=H_k(a,x)$$
$$=\sum_{n=0}^{\infty}(-1)^n a^{2n}x^{\frac{n(5n+1)-2kn}{2}}\left(1-a^k x^{2kn}\right)P_n Q_n(a),$$

where $P_n=\prod_{r=1}^{n}\frac{1}{1-x^r}$, and $Q_n(a)=\prod_{r=n}^{\infty}\frac{1}{1-ax^r}$.

It is convenient to define $P_0=1$, $H_0=1$. We have;

$$H_k-H_{k-1}=\sum_{n=0}^{\infty}\left\{(-1)^n a^{2n}x^{\frac{n(5n+1)}{2}}\times\right.$$
$$\left.\left[x^{-kn}-a^k x^{kn}-x^{(1-k)n}+a^{k-1}x^{n(k-1)}\right]P_nQ_n\right\}$$

$$=\sum_{n=0}^{\infty}\left\{(-1)^n a^{2n}x^{\frac{n(5n+1)}{2}}\times\right.$$
$$\left.\left[x^{-kn}-a^k x^{kn}-x^{(1-k)n}+a^{k-1}x^{n(k-1)}\right]P_nQ_n\right\}$$

$$=\sum_{n=0}^{\infty}(-1)^n a^{2n}x^{\frac{n(5n+1)}{2}}\times$$
$$\left[a^{k-1}x^{n(k-1)}\left(1-ax^n\right)+x^{-kn}\left(1-x^n\right)\right]P_nQ_n.$$

Now we have, $\left(1-ax^n\right)Q_n=Q_{n+1}$ and $\left(1-x^n\right)P_n=P_{n-1}$, hence,

$$H_k-H_{k-1}$$

$$=\sum_{n=0}^{\infty}(-1)^n a^{2n+k-1}x^{\frac{n(5n+1)+2n(k-1)}{2}}P_nQ_{n+1}$$

$$+\sum_{n=0}^{\infty}(-1)^n a^{2n}x^{\frac{n(5n+1)-2kn}{2}}P_{n-1}Q_n.$$

In the second sum on the right hand side of the Identity we change n into $n+1$. Thus,

$$H_k - H_{k-1}$$

$$= \sum_{n=0}^{\infty} (-1)^n a^{2n+k-1} x^{\frac{n(5n+1)+2n(k-1)}{2}} P_n Q_{n+1}$$

$$- \sum_{n=0}^{\infty} (-1)^n a^{2(n+1)} x^{\frac{(n+1)(5n+6)-2k(n+1)}{2}} P_n Q_{n+1}.$$

$$= \sum_{n=0}^{\infty} (-1)^n \left\{ a^{2n+k-1} x^{\frac{n(5n+1)+2n(k-1)}{2}} - a^{2(n+1)} x^{\frac{(n+1)(5n+6)-2k(n+1)}{2}} \right\} P_n Q_{n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \left\{ a^{2n+k-1} x^{\frac{n(5n+1)+2n(k-1)}{2}} \left(1 - a^{3-k} x^{(2n+1)(3-k)}\right) \right\} P_n Q_{n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[a^{k-1} \eta \left\{ a^{2n} x^{\frac{n(5n+1)-2n(3-k)}{2}} \left(1 - a^{3-k} x^{2n(3-k)}\right) \right\} \right] P_n Q_{n+1}$$

.

We have $Q_{n+1} = \eta Q_n$ and so,

$$H_k - H_{k-1}$$

$$= a^{k-1} \eta \sum_{n=0}^{\infty} (-1)^n a^{2n} x^{\frac{n(5n+1)-2n(3-k)}{2}} \left(1 - a^{3-k} x^{2n(3-k)}\right) P_n Q_n$$

$$= a^{k-1} \eta H_{3-k}.$$

Hence, the Remark.

The Rogers-Ramanujan Identities

Identity 1 [4]:

$$1 + \sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x)(1-x^2)\ldots(1-x^m)}$$

$$= \prod_{m=0}^{\infty} \frac{1}{(1-x^{5m+2})(1-x^{5m+3})}$$

Identity 2 [4]:

$$1 + \sum_{m=1}^{\infty} \frac{x^{m(m+1)}}{(1-x)(1-x^2)\ldots(1-x^m)}$$

$$= \prod_{m=0}^{\infty} \frac{1}{(1-x^{5m+2})(1-x^{5m+3})}$$

Proof: From (8) we have;

$$H_k(a,x) = \frac{G_k(a,x)}{(1-a)(1-ax)\ldots\infty} \qquad (13)$$

where $H_0 = 0$.

From above Remark we have;

$$H_k - H_{k-1} = a^{k-1} \eta H_{3-k}$$

where the operator η is defined by $\eta f(a) = f(ax)$, and $k = 1$ or 2. In particular

$$H_1 = \eta H_2,$$

$$H_2 - H_1 = a \eta H_1. \qquad (14)$$

So we have,

$$H_2 = \eta H_2 + a \eta^2 H_2. \qquad (15)$$

Suppose now that;

$$H_2 = 1 + c_1 a + c_2 a^2 + \ldots\infty \qquad (16)$$

where the coefficients depend on x only. Substituting this into (15), we obtain;

$$1 + c_1 a + c_2 a^2 + \ldots\infty$$

$$=1+c_1ax+c_2a^2x^2+\ldots\infty+$$

$$a\left(1+c_1ax^2+c_2a^2x^4+\ldots\infty\right).$$

Hence, equating the coefficients of various powers of a from both sides we get;

$$c_1=\frac{1}{1-x},\ c_2=\frac{x^2}{1-x^2}c_1,\ c_3=\frac{x^4}{1-x^3}c_2,\ \ldots,$$

$$c_n=\frac{x^{n(n-1)}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^n\right)}.$$

From (13) and (16), we have for $k = 2$;

$$\frac{G_2(a,x)}{(1-a)(1-ax)\ldots\infty}$$

$$=H_2(a,x)$$

$$=1+\frac{a}{1-x}+\frac{a^2x^2}{(1-x)\left(1-x^2\right)}+$$

$$\frac{a^3x^6}{(1-x)\left(1-x^2\right)\left(1-x^3\right)}+\ldots\infty.$$

If $a = x$, then;

$$1+\frac{x}{1-x}+\frac{x^4}{(1-x)\left(1-x^2\right)}+$$

$$\frac{x^9}{(1-x)\left(1-x^2\right)\left(1-x^3\right)}+\ldots\infty$$

$$=\frac{G_2(x,x)}{(1-x)\left(1-x^2\right)\ldots\infty}.$$

Therefore, $1+\sum_{m=1}^{\infty}\frac{x^{m^2}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^m\right)}$

$$=\prod_{m=0}^{\infty}\frac{1}{\left(1-x^{5m+1}\right)\left(1-x^{5m+4}\right)}.$$

Hence the Identity 1.

Again from (13), (14) and (16) we have with $k = 1$,

$$\frac{G_1(a,x)}{(1-a)(1-ax)\ldots\infty}$$

$$=H_1(a,x)=\eta H_2(a,x)$$

$$=1+\frac{ax}{1-x}+\frac{a^2x^4}{(1-x)\left(1-x^2\right)}+$$

$$\frac{a^3x^9}{(1-x)\left(1-x^2\right)\left(1-x^3\right)}+\ldots\infty.$$

If $a = x$, then we have;

$$1+\frac{x^2}{1-x}+\frac{x^6}{(1-x)\left(1-x^2\right)}$$

$$+\frac{x^{12}}{(1-x)\left(1-x^2\right)\left(1-x^3\right)}+\ldots\infty$$

$$=\frac{G_1(x,x)}{(1-x)\left(1-x^3\right)\ldots\infty}.$$

Therefore, $1+\sum_{m=1}^{\infty}\frac{x^{m(m+1)}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^m\right)}$

$$=\prod_{m=0}^{\infty}\frac{1}{\left(1-x^{5m+2}\right)\left(1-x^{5m+3}\right)}.$$

Hence the Identity 2.

5. Conclusion

In this study, we have shown $C_1'(n) = C'(n)$ with the help of a numerical example when *n*=11, and also have shown $C_1''(n) = C''(n)$ with the help of a numerical example when *n* =11. We have transferred the auxiliary function into another auxiliary function with the help of Ramanujan's device of the introduction of a second parameter *a*,

i.e.,

$$G_k(a,x) = \prod_{n=0}^{\infty} (-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}} \left(1 - a^k x^{2kn}\right) C_n$$

to

$$G_2(x,x) = \sum_{m=0}^{\infty} \left(1 - x^{5m+2}\right)\left(1 - x^{5m+3}\right)\left(1 - x^{5m+5}\right),$$

where k =2, and $a = x$, it is used in proving The Rogers-Ramanujan Identity 2. Finally we have proved The Roger-Ramanujan Identities with the help of auxiliary function,

$$H_k(a,x) = \frac{G_k(a,x)}{(1-a)(1-ax)\ldots\infty} \text{ ,where}$$

$H_0 = 0$.

References

[1] Andrews, G.E, "An Introduction to Ramanujan's Lost Notebook", *American Mathmatical Monthly*, 86: 89–108. 1979.

[2] Hardy, G.H. and Wright, E.M. "*Introduction to the Theory of Numbers*", 4th Edition, Oxford, Clarendon Press, 1965.

[3] Jacobi, C.G.J. (1829), "*Fundamenta Nova Theoriae Functionum Ellipticarum (in Latin), Konigsberg Borntraeger*", Cambridge University Press, 2012.

[4] Baxter, R.J., "*Exactly Solved Model in Statistical Models*", London, Academic Press, 1982.

[5] Ramanujan, S., "Congruence Properties of Partitions", Math, Z. 9: 147–153. 1921.

[6] Ramanujan, S., "Some Properties of *P*(*n*), Number of Partitions of *n*", Proc. of the Cam. Philo. Society XIX, 207–210. 1919.

[7] Das, S. and Mohajan, H.K., "Generating Function for P(n,p,*) and P(n, *,p)", Amer. Rev. of Math. and Sta. 2(1): 33–35. 2014.