Dictatorship on Top-circular Domains

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Abstract

We consider domains with a natural property called top-circularity. We show that if such a domain satisfies either the maximal conflict property or the weak conflict property, then it is dictatorial. We obtain the result in Sato (2010) as a corollary. Further, it follows from our results that the union of a top-connected single-peaked domain and a top-connected single-dipped domain is dictatorial.

Keywords: Dictatorial domains, Top-circularity, Maximal conflict property, Weak conflict property

JEL Classification Codes: D71, D82.

1. Introduction

1.1 Motivation

The coincidence of strategy-proofness and dictatorship has always been an intriguing question since Alan Gibbard and Mark Satterthwaite proposed their impossibility result (Gibbard (1973), Satterthwaite (1975)) - famously known as the Gibbard-Satterthwaite (GS) Theorem - which states that every unanimous and strategy-proof social choice function (SCF) defined over the unrestricted domain of preferences (provided that there are at least three alternatives) is dictatorial. However, the unrestricted domain assumption in the GS theorem is far from being the necessary condition for dictatorship. A domain of preferences is called *dictatorial* if every unanimous and strategy-proof SCF on it is dictatorial.

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Apart from being a generalization of the GS theorem, dictatorial domains have garnered a lot of interest in the literature because dictatorial rules satisfy a desirable property called *tops-onlyness* and a stronger incentive requirement called *group strategy-proofness*. At present, there is a sizeable literature on dictatorial domains as seen in the works of Barberà and Peleg (1990), Aswal et al. (2003), Sato (2010), and Pramanik (2015). However, the existing results on dictatorial domains are mostly of theoretical interest and not of much practical use. Hence, the main motivation of this paper is to find dictatorial domains with some natural structure so that they can be applied to some economic and political environment.

1.2 Our Contribution

A crucial property of a dictatorial domain is that for every alternative a, there must be at least two preferences $ab\ldots$ and $ac\ldots$ in the domain, where $b \neq c$.[1,2] A domain of practical importance of such type is the one whose *top-graph* comprises of a maximal cycle.[3] We call such a domain a *top-circular domain*.

We prove by means of an example that the top-circular domains are not dictatorial. In view of that, we identify two conditions called the *maximal conflict property* and the *weak conflict property* such that if a top-circular domain satisfies either of these two conditions, then it becomes a dictatorial domain.[4] We obtain the dictatorial result in Sato (2010) as a corollary of our result.

We apply this result to the problem of locating a public facility. For certain public facilities such as metro stations, hospitals etc., it is known to the social planner that agents have single-peaked preferences as they want the facility to be located closer to their own locations. On the other hand, for facilities like garbage dumps or nuclear plants, it is known to the social planner that the agents have single-dipped preferences. For both these cases, it is well-known that one can design non-dictatorial rules that satisfy unanimity and strategy-proofness.[5]

[1] We denote by $ab\ldots$ a preference which places a at the top and b at the second-ranked position.

[2] Roy and Storcken (2016) shows that this property is necessary and sufficient for dictatorship on a large class of domains which they call *short-path-connected* domains. However, the domains that we consider are not short-path-connected.

[3] The *top-graph* of a domain is defined as the graph where nodes are alternatives and there is an edge between two alternatives a, b if there are preferences $ab\ldots$ and $ba\ldots$ in the domain.

[4] Several domains of practical importance such as the maximal single-peaked domain, the maximal single-dipped domain, and maximal single crossing domains satisfy the maximal conflict property. Also, maximal single-peaked domain satisfies the weak conflict property.

[5] Moulin (1980), Barberà et al. (1993) and Weymark (2011) characterize the unanimous and strategy-proof SCFs on the single-peaked domains as *min-max rules*. Peremans and Storcken (1999) and Manjunath (2014) characterize the unanimous and strategy-proof SCFs on the single-dipped domains as *voting by extended committees*.

However, for facilities like shopping malls, factories etc., the social planner may not have clear knowledge on whether the agents want it to be closer or farther away. This is because, some individuals may be concerned about the resulting congestion, pollution etc., whereas some others may want to minimize their commuting distance. In such a situation, the relevant admissible domain is the union of a single-peaked and a single-dipped domain.[6] Our result shows that every unanimous and strategy-proof SCF on such a domain is dictatorial.

1.3 Remainder

The rest of the paper is organized as follows. We describe the usual social choice framework in Section 2. Section 3 presents our main results and Section 4 discusses applications of the same. The last section concludes the paper. All the omitted proofs are collected in Appendix A.

2. The Model

Let $N = \{1, ..., n\}$ be a set of agents, who collectively choose an element from a finite set $X = \{x_1, x_2, \ldots, x_m\}$ of at least three alternatives. A *preference* P over X is a complete, transitive, and antisymmetric binary relation (also called a linear order) defined on X. We denote by $\mathbb{L}(X)$ the set of all preferences over X. An alternative $x \in X$ is called the k^{th} *ranked alternative* in a preference $P \in \mathbb{L}(X)$, denoted by $r_k(P)$, if $|\{a \in X \mid aPx\}| = k - 1$. For ease of presentation, by $ab \ldots c \ldots d \ldots$, we denote a preference P where $r_1(P) = a, r_2(P) = b$ and cPd. Also, by $ab \ldots c$, we denote a preference P where $r_1(P) = a$, $r_2(P) = b$, and $r_m(P) = c$. We denote by $\mathcal{D} \subseteq \mathbb{L}(X)$ a set of admissible preferences over X. A preference profile, denoted by P_N, is defined as an element of $\mathcal{D}^n$.

For simplicity, we do not use braces for singleton sets, for instance, we use the notation i to mean $\{i\}$.

Definition 2.1. A *social choice function* (SCF) f on a domain $\mathcal{D}$ is defined as a mapping $f : \mathcal{D}^n \to X$.

[6] Alternative models that consider similar practical situations exist in the literature. For instance, Thomson (2008) and Feigenbaum and Sethuraman (2014) partition the set of agents into those who can only have single-peaked preferences and those that can only have single-dipped preferences. On the other hand, Unzu and Vorsatz (2015) considers a situation where the social planner is informed about the location of the agents but agents can have single-peaked preferences with the peak at her location or single-dipped preferences with the dip at her location. Though the domain restriction considered in the aforementioned models are close in spirit with ours, they admit non-dictatorial, unanimous, and strategy-proof SCFs.

Definition 2.2. An SCF $f : \mathcal{D}^n \to X$ is *unanimous* if for all $P_N \in \mathcal{D}^n$ such that $r_1(P_i) = x$ for all $i \in N$ and some $x \in X$, we have $f(P_N) = x$.

Definition 2.3. An SCF $f : \mathcal{D}^n \to X$ is *manipulable* if there exists a profile $P_N \in \mathcal{D}^n$, an agent $i \in N$, and a preference $P'_i \in \mathcal{D}$ of agent i such that $f(P'_i, P_{-i})P_i f(P_N)$. An SCF f is *strategy-proof* if it is not manipulable.

Definition 2.4. Given an SCF $f : \mathcal{D}^2 \to X$, we define the *option set* of agent $i \in \{1,2\}$ at preference $P_j \in \mathcal{D}$ of agent $j \in \{1,2\} \setminus i$, denoted by $O_i(P_j)$, as $O_i(P_j) = \bigcup\limits_{P_i \in \mathcal{D}} f(P_i, P_j)$.

REMARK 2.1. Note that if an SCF $f : \mathcal{D}^2 \to X$ is unanimous, then $r_1(P_j) \in O_i(P_j)$ for all $P_j \in \mathcal{D}$. Furthermore, if f is strategy-proof, then for all $i, j \in \{1,2\}; i \neq j$ and all $(P_1, P_2) \in \mathcal{D}^2$, $f(P_1, P_2) = \max\limits_{P_i} O_i(P_j)$, where $\max\limits_{P_i} O_i(P_j) = x$ if and only if $x \in O_i(P_j)$ and xP_iy for all $y \in O_i(P_j) \setminus x$.

Definition 2.5. An SCF $f : \mathcal{D}^n \to X$ is *dictatorial* if there exists an agent $i \in N$ such that for all profiles $P_N \in \mathcal{D}^n$, $f(P_N) = r_1(P_i)$.

REMARK 2.2. Note that an SCF $f : \mathcal{D}^2 \to X$ is dictatorial if and only if there is $i \in \{1,2\}$ such that $O_i(P_j) = \{r_1(P_j)\}$ for all $P_j \in \mathcal{D}$.

Definition 2.6. A domain $\mathcal{D}$ is called *dictatorial* if every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to X$ is dictatorial.

Definition 2.7. A domain $\mathcal{D}$ is *regular* if for all $x \in X$, there exists $P \in \mathcal{D}$ such that $r_1(P) = x$.

REMARK 2.3. All the domains we consider in this paper are regular.

Now, we introduce a few graph theoretic notions. A *graph* G is defined as a pair $\langle V, E \rangle$, where V is the set of *nodes* and $E \subseteq \{\{u, v\} \mid u, v \in V \text{ and } u \neq v\}$ is the set of *edges*. All the graphs we consider in this paper are of the kind $G = \langle X, E \rangle$, i.e., whose node set is the set of alternatives.

Definition 2.8. The *top-graph* of a domain $\mathcal{D}$ is defined as the graph $\langle X, E \rangle$ such that $\{x, y\} \in E$ if and only if there exist two preferences $P, P' \in \mathcal{D}$ with $r_1(P) = r_2(P') = x$ and $r_2(P) = r_1(P') = y$.

Now, we introduce the notion of a top-circular domain.

Definition 2.9. A domain $\mathcal{C}$ with top-graph $\langle X, E \rangle$ is called *top-circular* if $\{x_i, x_j\} \in E$ for all i, j with $|i - j| \in \{1, m-1\}$.

Below, we present a top-circular domain and its top-graph.

Example 2.1. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$. Consider the domain given in Table 1. Figure 1 presents the top-graph of this domain. Note that this graph contains a maximal cycle given by $(x_1, x_2, \ldots, x_5, x_1)$. Further, note that such a graph may contain some additional edges like $\{x_1, x_3\}$ and $\{x_2, x_5\}$.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_5}$	$\mathbf{x_1}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_5}$
$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_4}$	$\mathbf{x_1}$	$\mathbf{x_5}$	$\mathbf{x_3}$	$\mathbf{x_1}$	$\mathbf{x_5}$	$\mathbf{x_2}$
x_5	x_4	x_5	x_4	x_2	x_1	x_3	x_1	x_2	x_3	x_5	x_4	x_3	x_4
x_4	x_5	x_1	x_1	x_1	x_5	x_1	x_3	x_3	x_4	x_2	x_5	x_4	x_3
x_3	x_3	x_4	x_5	x_5	x_2	x_2	x_2	x_4	x_2	x_4	x_2	x_1	x_1

Table 1: A top-circular domain

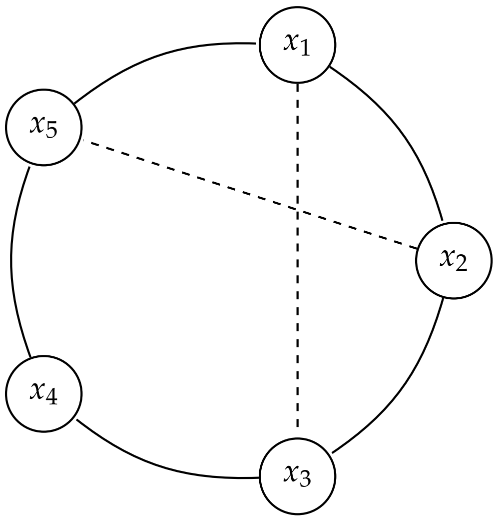

Figure 1: Top-graph of a top-circular domain

3. Main Result

Our following example shows that a top-circular domain admits unanimous, strategy-proof, and non-dictatorial rules.

Example 3.1. Let $X = \{x_1, x_2, x_3, x_4\}$. By $P = x_1x_2x_3x_4$, we mean a preference P such that

$x_1Px_2Px_3Px_4$. Consider the following domain:

$$\mathcal{D} = \{x_1x_2x_4x_3, x_2x_1x_3x_4, x_2x_3x_4x_1, x_3x_2x_4x_1, x_3x_4x_1x_2, x_4x_3x_1x_2, x_4x_1x_3x_2, x_1x_4x_2x_3\}.$$

It can be easily verified that the two-agent SCF on the domain $\mathcal{D}$ given in Table 2 is unanimous, strategy-proof, and non-dictatorial.

P_1 \ P_2	$x_1x_2x_4x_3$	$x_2x_1x_3x_4$	$x_2x_3x_4x_1$	$x_3x_2x_4x_1$	$x_3x_4x_1x_2$	$x_4x_3x_1x_2$	$x_4x_1x_3x_2$	$x_1x_4x_2x_3$
$x_1x_2x_4x_3$	x_1	x_2	x_2	x_3	x_3	x_4	x_4	x_1
$x_2x_1x_3x_4$	x_2	x_2	x_2	x_3	x_3	x_3	x_3	x_2
$x_2x_3x_4x_1$	x_2	x_2	x_2	x_3	x_3	x_3	x_3	x_2
$x_3x_2x_4x_1$	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3
$x_3x_4x_1x_2$	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3
$x_4x_3x_1x_2$	x_4	x_3	x_3	x_3	x_3	x_4	x_4	x_4
$x_4x_1x_3x_2$	x_4	x_3	x_3	x_3	x_3	x_4	x_4	x_4
$x_1x_4x_2x_3$	x_1	x_2	x_2	x_3	x_3	x_4	x_4	x_1

Table 2: A non-dictatorial rule on a top-circular domain

In view of Example 3.1, we present below two conditions, and show that if a top-circular domain satisfies either of the two, then it is dictatorial.

Definition 3.1. A domain $\mathcal{D}$ satisfies the *maximal conflict property* if there exist $P, P' \in \mathcal{D}$ such that $r_k(P) = r_{m-k+1}(P') = x_k$ for all $k = 1, \ldots, m$.

Definition 3.2. A domain $\mathcal{D}$ satisfies the *weak conflict property* if

(i) $\{x_1x_2 \ldots x_m, x_mx_{m-1} \ldots x_1\} \subseteq \mathcal{D}$, and

(ii) for all $k = 2, \ldots, m-1$, there are two preferences $P = x_kx_{k-1} \ldots x_1 \ldots x_{k+1} \ldots$ and $P' = x_kx_{k+1} \ldots x_m \ldots x_{k-1} \ldots$ in the domain $\mathcal{D}$.

In the following, we present an example of a top-circular domain with the maximal conflict property.

Example 3.2. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. Then, the domain $\mathcal{C} = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}\}$ as given in Table 3 is a top-circular domain satisfying the maximal conflict property.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_6}$	$\mathbf{x_7}$	$\mathbf{x_7}$	$\mathbf{x_1}$
$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_4}$	$\mathbf{x_6}$	$\mathbf{x_5}$	$\mathbf{x_7}$	$\mathbf{x_6}$	$\mathbf{x_1}$	$\mathbf{x_7}$
$\mathbf{x_3}$	x_6	x_5	x_6	x_2	x_1	x_7	x_1	x_7	x_7	x_3	$\mathbf{x_5}$	x_5	x_4
$\mathbf{x_4}$	x_5	x_1	x_1	x_6	x_7	x_1	x_3	x_3	x_1	x_5	$\mathbf{x_4}$	x_3	x_2
$\mathbf{x_5}$	x_3	x_4	x_4	x_5	x_2	x_2	x_2	x_4	x_4	x_4	$\mathbf{x_3}$	x_4	x_3
$\mathbf{x_6}$	x_7	x_7	x_7	x_1	x_6	x_6	x_7	x_1	x_2	x_1	$\mathbf{x_2}$	x_2	x_6
$\mathbf{x_7}$	x_4	x_6	x_5	x_7	x_5	x_3	x_6	x_2	x_3	x_2	$\mathbf{x_1}$	x_6	x_5

Table 3: A top-circular domain satisfying the maximal conflict property

Now, we present an example of a top-circular domain with the weak conflict property.

Example 3.3. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. Then, the domain $\mathcal{C} = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}\}$ as given in Table 4 is a top-circular domain satisfying the weak conflict property.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_6}$	$\mathbf{x_7}$	$\mathbf{x_7}$	$\mathbf{x_1}$
$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_4}$	$\mathbf{x_6}$	$\mathbf{x_5}$	$\mathbf{x_7}$	$\mathbf{x_6}$	$\mathbf{x_1}$	$\mathbf{x_7}$
x_3	x_6	x_5	x_6	$\mathbf{x_7}$	$\mathbf{x_1}$	$\mathbf{x_7}$	$\mathbf{x_1}$	x_2	$\mathbf{x_1}$	x_3	x_4	x_5	x_4
x_6	$\mathbf{x_3}$	$\mathbf{x_7}$	$\mathbf{x_1}$	x_6	x_7	x_1	$\mathbf{x_6}$	x_3	$\mathbf{x_7}$	x_2	x_2	x_3	x_5
x_4	x_4	x_4	x_5	x_5	$\mathbf{x_5}$	x_2	x_2	$\mathbf{x_7}$	x_4	x_4	x_5	x_4	$\mathbf{x_2}$
x_5	x_7	$\mathbf{x_1}$	$\mathbf{x_4}$	x_1	x_6	$\mathbf{x_3}$	x_7	x_1	x_2	x_1	x_3	x_2	x_6
$\mathbf{x_7}$	x_5	x_6	x_7	$\mathbf{x_2}$	x_2	x_6	x_3	$\mathbf{x_4}$	x_3	$\mathbf{x_5}$	$\mathbf{x_1}$	$\mathbf{x_6}$	x_3

Table 4: A top-circular domain satisfying the weak conflict property

Now, we proceed to present our main results.

Theorem 3.1. *Let $\mathcal{C}$ be a top-circular domain satisfying the maximal conflict property. Then, $\mathcal{C}$ is a dictatorial domain.*

Theorem 3.2. *Let $\mathcal{C}$ be a top-circular domain satisfying the weak conflict property. Then, $\mathcal{C}$ is a dictatorial domain.*

The proofs of Theorem 3.1 and 3.2 are relegated to Appendix A.

4. APPLICATIONS

4.1 LOCATING A PUBLIC FACILITY

In this section, we consider the problem of locating a public facility when the social planner does not have any information whether it generates positive or negative externality for the agents. As argued in Section 1, the relevant domain restriction in such problems is the union of a single-peaked and a single-dipped domain. In what follows, we describe such domains formally, and show that they are dictatorial.

Definition 4.1. A preference $P \in \mathbb{L}(X)$ is called *single-peaked* if $r_1(P) = x_i$ and $[j < k \leq i$ or $i \leq k < j]$ imply $x_k P x_j$.

Definition 4.2. A domain $\mathcal{D}_p$ is called a *top-connected single-peaked domain* if

(i) every preference in $\mathcal{D}_p$ is single-peaked, and

(ii) for every two alternatives $x_i, x_{i+1} \in X$, there are two preferences $P, P' \in \mathcal{D}$ such that $r_1(P) = r_2(P') = x_i$ and $r_2(P) = r_1(P') = x_{i+1}$.

Definition 4.3. A preference $P \in \mathbb{L}(X)$ is called *single-dipped* if $r_m(P) = x_i$ and $[j < k \leq i$ or $i \leq k < j]$ imply $x_j P x_k$.

Definition 4.4. A domain $\mathcal{D}_d$ is called a *top-connected single-dipped domain* if

(i) every preference in $\mathcal{D}_d$ is single-dipped, and

(ii) there are two preferences $P, P' \in \mathcal{D}$ such that $r_1(P) = r_2(P') = x_1$ and $r_2(P) = r_1(P') = x_m$.

A domain $\mathcal{D}$ is called the union of a top-connected single-peaked and a top-connected single-dipped domain if $\mathcal{D} = \mathcal{D}_p \cup \mathcal{D}_d$, where $\mathcal{D}_p$ is a top-connected single-peaked and $\mathcal{D}_d$ is a top-connected single-dipped domain. It is easy to verify that the union of a single-peaked and a single-dipped domain is a top-circular domain satisfying the maximal conflict property. Thus, we have the following corollary of Theorem 3.1.

Corollary 4.1. *Let $\mathcal{D}$ be the union of a top-connected single-peaked and a top-connected single-dipped domain. Then, $\mathcal{D}$ is a dictatorial domain.*

4.2 Circular Domains

The notion of circular domains is introduced in Sato (2010), where he shows that a circular domain is dictatorial. However, we obtain this result as a corollary of our result.

Definition 4.5. A domain $\mathcal{D}$ is called *circular* if it is a top-circular domain satisfying the property that for all $k = 1, \ldots, m$, there are two preferences $x_k x_{k+1} \ldots x_{k-1}$ and $x_k x_{k-1} \ldots x_{k+1}$ in the domain $\mathcal{D}$.

Note that a circular domain is a top-circular domain satisfying the weak conflict property. Thus, we have the following corollary of Theorem 3.2.

Corollary 4.2 (Sato (2010)). *Let $\mathcal{D}$ be a circular domain. Then, $\mathcal{D}$ is a dictatorial domain.*

5. Concluding Remarks

In this paper, we prove that any unanimous and strategy-proof social choice rule on a top-circular domain satisfying either the maximal conflict property or the weak conflict property is dictatorial. Our result is independent from the existing results on dictatorial domains.

Since dictatorial rules are tops-only, Theorem 3.1 and 3.2 imply that top-circular domains satisfying either the maximal conflict property or the weak conflict property are tops-only. Chatterji and Sen (2011) provides sufficient conditions for a domain to be tops-only, however, our domain restrictions do not satisfy their condition. Moreover, since dictatorial rules are also group-strategy-proof, it follows that the notions of strategy-proofness and group-strategy-proofness are equivalent for the domains we consider.

Appendix A. Proofs

In this section, we prove Theorem 3.1 and Theorem 3.2. The following proposition in Aswal et al. (2003) allows us to restrict our attention to the case of two agents.

Proposition A.1 (Aswal et al. (2003)). *Let $\mathcal{D}$ be a regular domain such that every unanimous and strategy-proof SCF $f : \mathcal{D}^2 \to X$ is dictatorial. Then, every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to X$ is dictatorial.*

The following proposition in Sanver (2007) allows us to restrict our attention to *minimal* top-circular domains satisfying either the maximal conflict or the weak conflict property.[7]

Proposition A.2 (Sanver (2007))**.** *A superset of a regular dictatorial domain is also dictatorial.*

For all the subsequent results, let $\mathcal{C}$ be a minimal top-circular domain. Suppose $f : \mathcal{C}^2 \to X$ is a unanimous and strategy-proof SCF and $O_i(P_j)$ is the corresponding option set of agent i at a preference P_j of agent $j \in \{1, 2\} \setminus i$. We prove a sequence of lemmas that we use in the proofs of Theorem 3.1 and Theorem 3.2.

The following lemma establishes a property of a minimal top-circular domain. We assume for this lemma that $0 \equiv m$ and $m + 1 \equiv 1$.

Lemma A.1. *Let $\mathcal{C}$ be a minimal top-circular domain and let $P_2, P_2' \in \mathcal{C}$ be such that $r_1(P_2) = r_1(P_2') = x_k$. Then, for all $j \in \{k-1, k+1\}$, $x_j \in O_1(P_2)$ if and only if $x_j \in O_1(P_2')$.*

Proof. Assume for contradiction that there exist $P_2, P_2' \in \mathcal{C}$ with $r_1(P_2) = r_1(P_2') = x_k$ such that $x_j \in O_1(P_2)$ and $x_j \notin O_1(P_2')$ for some $j \in \{k-1, k+1\}$. Consider $P_1 \in \mathcal{C}$ such that $r_1(P_1) = x_j$ and $r_2(P_1) = x_k$. Such a preference exists in $\mathcal{C}$ as $|j - k| = 1$. Then, by the strategy-proofness of f, $f(P_1, P_2) = x_j$ and $f(P_1, P_2') = x_k$. This means agent 2 manipulates at (P_1, P_2) via P_2', a contradiction. This completes the proof of the lemma. ■

The subsequent lemmas establish few crucial properties of a minimal top-circular domain $\mathcal{C}$ such that $\{x_1x_2 \ldots x_m, x_mx_{m-1} \ldots x_1\} \subseteq \mathcal{C}$. Note that if a minimal top-circular domain $\mathcal{C}$ satisfies either the maximal conflict property or the weak conflict property, then such two preferences are there in $\mathcal{C}$.

Lemma A.2. *Let $\mathcal{C}$ be a minimal top-circular domain such that $\{x_1x_2 \ldots x_m, x_mx_{m-1} \ldots x_1\} \subseteq \mathcal{C}$. Then, for all $P_2 \in \{x_1x_2 \ldots x_m, x_mx_{m-1} \ldots x_1\}$, $r_m(P_2) \notin O_1(P_2)$ implies $O_1(P_2) = \{r_1(P_2)\}$.*

Proof. We prove the lemma for the case where $P_2 = x_1x_2 \ldots x_m \in \mathcal{C}$, the proof of the same for the other case is analogous. Let $P_2 = x_1x_2 \ldots x_m \in \mathcal{C}$ and let $r_m(P_2) = x_m \notin O_1(P_2)$. We show $O_1(P_2) = \{r_1(P_2)\}$. Assume for contradiction that $x_j \in O_1(P_2)$ for some $j \neq 1, m$. Let $P_2' \in \mathcal{C}$ be such that $r_1(P_2') = x_1$ and $r_2(P_2') = x_m$. Since $x_m \notin O_1(P_2)$, by Lemma A.1, $x_m \notin O_1(P_2')$. Let $P_1 = x_mx_{m-1} \ldots x_1$. By unanimity and strategy-proofness, we must have $f(P_1, P_2') \in \{x_1, x_m\}$ as otherwise, agent 2 manipulates at (P_1, P_2') via a preference which places x_m at the top. Also, since

[7] A top-circular domain is *minimal* if none of its subsets is top-circular.

$x_m \notin O_1(P_2')$, we have $f(P_1, P_2') = x_1$. However, since $x_j \in O_1(P_2)$ and $x_j P_1 x_1$, it must be that $f(P_1, P_2) \neq x_1$. Because $r_1(P_2) = x_1 = r_1(P_2')$, this means agent 2 manipulates at (P_1, P_2) via P_2', a contradiction. This completes the proof of the lemma. ■

Lemma A.3. *Let $\mathcal{C}$ be a minimal top-circular domain such that $\{x_1x_2 \ldots x_m, x_mx_{m-1} \ldots x_1\} \subseteq \mathcal{C}$ and let $O_1(P_2) \in \{\{r_1(P_2)\}, X\}$ for all $P_2 \in \{x_1x_2 \ldots x_m, x_mx_{m-1} \ldots x_1\}$. Suppose $\hat{P}_2, \bar{P}_2 \in \mathcal{C}$ is such that $r_1(\hat{P}_2) = x_1$ and $r_1(\bar{P}_2) = x_m$. Then, $O_1(\hat{P}_2) = \{x_1\}$ if and only if $O_1(\bar{P}_2) = \{x_m\}$.*

Proof. Let $\hat{P}_2, \bar{P}_2 \in \mathcal{C}$ be such that $r_1(\hat{P}_2) = x_1$ and $r_1(\bar{P}_2) = x_m$. It is sufficient to show that $O_1(\hat{P}_2) = \{x_1\}$ implies $O_1(\bar{P}_2) = \{x_m\}$. By strategy-proofness, it is enough to show that $O_1(\bar{P}_2) = \{x_m\}$ where $\bar{P}_2 = x_mx_{m-1} \ldots x_1$.

Assume for contradiction that $O_1(\hat{P}_2) = \{x_1\}$ and $O_1(\bar{P}_2) \neq \{x_m\}$. By the assumption of the lemma, $O_1(\bar{P}_2) \neq \{x_m\}$ implies $O_1(\bar{P}_2) = X$. Consider $\bar{P}_2' \in \mathcal{C}$ such that $r_1(\bar{P}_2') = x_m$ and $r_2(\bar{P}_2') = x_1$. Since $O_1(\bar{P}_2) \neq \{x_m\}$, it follows from strategy-proofness that $O_1(\bar{P}_2') \neq \{x_m\}$. We show $x_j \notin O_1(\bar{P}_2')$ for all $j \neq 1, m$. Suppose not. Then, $f(P_1, \bar{P}_2') = x_j$ for some $P_1 \in \mathcal{C}$ with x_j at the top. However, because $O_1(\hat{P}_2) = \{x_1\}$, agent 2 manipulates at $(P_1, \bar{P}_2')$ via $\hat{P}_2$. Since $O_1(\bar{P}_2') \neq \{x_m\}$ and $x_j \notin O_1(\bar{P}_2')$ for all $j \neq 1, m$, it must be that $O_1(\bar{P}_2') = \{x_1, x_m\}$. However, since $O_1(\bar{P}_2) = X$, which in turn means $x_{m-1} \in O_1(\bar{P}_2)$, by Lemma A.1, we must have $x_{m-1} \in O_1(\bar{P}_2')$, a contradiction. This completes the proof of the lemma. ■

A.1 Proof of Theorem 3.1

In this section, we provide a proof of Theorem 3.1. First, we establish a few properties of a top-circular domain satisfying the maximal conflict property.

Lemma A.4. *Let $\mathcal{C}$ be a minimal top-circular domain satisfying the maximal conflict property. Let $P, P' \in \mathcal{C}$ be such that $r_k(P) = r_{m-k+1}(P') = x_k$ for all $k = 1, \ldots, m$. Then, for all $P_2 \in \{P, P'\}$, $r_m(P_2) \in O_1(P_2)$ implies $O_1(P_2) = X$.*

Proof. We prove this lemma for the case where $P_2 = P$, the proof of the same for the other case is analogous. Let $P_2 = P$. Suppose $x_m \in O_1(P_2)$. We show $O_1(P_2) = X$. We prove this by induction. Since $x_m \in O_1(P_2)$, it is sufficient to show that for all $1 < k \leq m$, $x_k \in O_1(P_2)$ implies $x_{k-1} \in O_1(P_2)$. Assume for contradiction that $x_k \in O_1(P_2)$ but $x_{k-1} \notin O_1(P_2)$ for some $1 < k \leq m$. Consider $P_1 = x_{k-1}x_k \ldots \in \mathcal{C}$. Since $x_k \in O_1(P_2)$ and $x_{k-1} \notin O_1(P_2)$, $f(P_1, P_2) = x_k$. However, this means agent 2 manipulates at (P_1, P_2) via a preference which places x_{k-1} at the top, a contradiction. This completes the proof of the lemma. ■

REMARK A.1. Let $\mathcal{C}$ be a minimal top-circular domain satisfying the maximal conflict property, and let $P, P' \in \mathcal{C}$ be such that $r_k(P) = r_{m-k+1}(P') = x_k$ for all $k = 1, \ldots, m$. Then, it follows from Lemma A.2 that for all $P_2 \in \{P, P'\}$, $r_m(P_2) \notin O_1(P_2)$ implies $O_1(P_2) = \{r_1(P_2)\}$. Again, it follows from Lemma A.4 that for all $P_2 \in \{P, P'\}$, $r_m(P_2) \in O_1(P_2)$ implies $O_1(P_2) = X$. Thus, for all $P_2 \in \{P, P'\}$, we have $O_1(P_2) \in \{\{r_1(P_2)\}, X\}$.

Lemma A.5. *Let $\mathcal{C}$ be a minimal top-circular domain satisfying the maximal conflict property. Further, let $P, P' \in \mathcal{C}$ be such that $r_k(P) = r_{m-k+1}(P') = x_k$ for all $k = 1, \ldots, m$. Then, for all $P_2 \in \{P, P'\}$, $O_1(P_2) = \{r_1(P_2)\}$ implies $O_1(\bar{P}_2) = \{r_1(\bar{P}_2)\}$ for all $\bar{P}_2 \in \mathcal{C}$.*

Proof. It is enough to prove the lemma for the case where $P_2 = P$, the proof for the other case is analogous. Let $P_2 = P$. Suppose $O_1(P_2) = \{r_1(P_2)\}$. We show $O_1(\bar{P}_2) = \{r_1(\bar{P}_2)\}$ for all $\bar{P}_2 \in \mathcal{C}$. By strategy-proofness, this means $O_1(\bar{P}_2) = \{r_1(\bar{P}_2)\}$ for all $\bar{P}_2 \in \mathcal{C}$ with $r_1(\bar{P}_2) = x_1$. Moreover, by Lemma A.3 and Remark A.1, we have $O_1(\bar{P}_2) = \{r_1(\bar{P}_2)\}$ for all $\bar{P}_2 \in \mathcal{C}$ with $r_1(\bar{P}_2) = x_m$. Take $j \neq 1, m$ and $\hat{P}_2 \in \mathcal{C}$ with $r_1(\hat{P}_2) = x_j$. We show $O_1(\hat{P}_2) = \{r_1(\hat{P}_2)\}$.

First, we show $O_2(P_1) = O_2(P_1') = X$, where $r_{m-k+1}(P_1) = r_k(P_1') = x_k$ for all $k = 1, \ldots, m$. We show this for P_1, the proof of the same for P_1' is analogous. Since $O_1(P_2) = \{x_1\}$, we have $f(P_1, P_2) = x_1$. Because $r_m(P_1) = x_1$, this means $r_m(P_1) \in O_2(P_1)$. By Lemma A.4, this means $O_2(P_1) = X$.

Now, we complete the proof of the lemma. Assume for contradiction that $x_l \in O_1(\hat{P}_2)$ for some $x_l \neq r_1(\hat{P}_2) = x_j$. Since $r_{m-k+1}(P_1) = r_k(P_1') = x_k$ for all $k = 1, \ldots, m$, we must have either $x_l P_1 x_j$ or $x_l P_1' x_j$. Assume without loss of generality that $x_l P_1 x_j$. Since $O_2(P_1) = X$ and $r_1(\hat{P}_2) = x_j$, $f(P_1, \hat{P}_2) = x_j$. Let $\hat{P}_1 \in \mathcal{C}$ such that $r_1(\hat{P}_1) = x_l$. Since $x_l \in O_1(\hat{P}_2)$ and $r_1(\hat{P}_1) = x_l$, we have $f(\hat{P}_1, \hat{P}_2) = x_l$. This means agent 1 manipulates at $(P_1, \hat{P}_2)$ via $\hat{P}_1$, a contradiction. Therefore, $O_1(\hat{P}_2) = \{r_1(\hat{P}_2)\}$, which completes the proof of the lemma. ■

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. In view of Propositions A.1 and A.2, it sufficient to show that a minimal top-circular domain with the maximal conflict property is dictatorial for two agents. Consider $P_2 \in \mathcal{C}$ such that $r_k(P_2) = x_k$ for all $1 \leq k \leq m$. By Remark A.1, we have $O_1(P_2) \in \{\{r_1(P_2)\}, X\}$. Suppose $O_1(P_2) = \{r_1(P_2)\}$. Then, by Lemma A.5, it follows that $O_1(P_2') = \{r_1(P_2')\}$ for all $P_2' \in \mathcal{C}$, which implies agent 2 is the dictator.

Now, suppose $O_1(P_2) = X$. Consider $P_1 \in \mathcal{C}$ such that $r_1(P_1) = x_m$. Since $O_1(P_2) = X$, we have $f(P_1, P_2) = x_m$. We claim $O_2(P_1) = \{r_1(P_1)\}$. Assume for contradiction that $x_j \in O_2(P_1)$ for some $j \neq m$. Since $r_m(P_2) = x_m$, we have $x_j P_2 x_m$. However, since $x_j \in O_2(P_1)$, agent 2 manipulates at (P_1, P_2) via some preference $\bar{P}_2$ with $r_1(\bar{P}_2) = x_j$. Therefore, $O_2(P_1) = \{r_1(P_1)\}$. By Lemma A.5, this means $O_2(P_1) = \{r_1(P_1)\}$ for all $P_1 \in \mathcal{C}$, which implies agent 1 is the dictator. This completes the proof of the theorem. ■

A.2 Proof of Theorem 3.2

In this section, we provide a proof of Theorem 3.2. First, we establish a few properties of a top-circular domain satisfying the weak conflict property.

Lemma A.6. *Let $\mathcal{C}$ be a minimal top-circular domain satisfying the weak conflict property. Suppose $P_2 \in \{x_1 x_2 \dots x_m, x_m x_{m-1} \dots x_1\} \subseteq \mathcal{C}$. Then, $r_m(P_2) \in O_1(P_2)$ implies $O_1(P_2) = X$.*

Proof. It is enough to prove the lemma for $P_2 = x_1 x_2 \dots x_m \in \mathcal{C}$, the proof for the other case is analogous. Suppose $x_m \in O_1(P_2)$. We show $O_1(P_2) = X$. We prove this by induction. By unanimity, $x_1 \in O_1(P_2)$. Therefore, it is sufficient to show that for all $1 \leq k < m$, $x_k \in O_1(P_2)$ implies $x_{k+1} \in O_1(P_2)$. Assume for contradiction that $x_k \in O_1(P_2)$ and $x_{k+1} \notin O_1(P_2)$ for some $1 \leq k < m$. Let $\hat{P}_2 = x_k x_{k+1} \dots x_m \dots x_{k-1} \dots \in \mathcal{C}$. Note that since $x_m \in O_1(P_2)$ and $r_m(P_2) = x_m$, by strategy-proofness, it must be that $x_m \in O_1(\hat{P}_2)$. Let $P_1 = x_{k+1} x_{k+2} \dots x_m \dots x_k \dots \in \mathcal{C}$. By unanimity and strategy-proofness, $f(P_1, \hat{P}_2) \in \{x_k, x_{k+1}\}$, as otherwise agent 2 manipulates at $(P_1, \hat{P}_2)$ via some preference with x_{k+1} at the top. Suppose $f(P_1, \hat{P}_2) = x_k$. Since $x_m P_1 x_k$ and $x_m \in O_1(\hat{P}_2)$, this means agent 1 manipulates at $(P_1, \hat{P}_2)$ via some preference with x_m at the top. Therefore, we have $f(P_1, \hat{P}_2) = x_{k+1}$. Now, let $P_1' = x_{k+1} x_k \dots \in \mathcal{C}$. Then, since $f(P_1, \hat{P}_2) = x_{k+1}$ and $r_1(P_1) = r_1(P_1') = x_{k+1}$, by strategy-proofness, $f(P_1', \hat{P}_2) = x_{k+1}$. Also, because $x_k \in O_1(P_2)$ and $x_{k+1} \notin O_1(P_2)$, we have $f(P_1', P_2) = x_k$. Therefore, agent 2 manipulates at $(P_1', \hat{P}_2)$ via P_2, a contradiction. This completes the proof of the lemma. ■

Remark A.2. Let $\mathcal{C}$ be a minimal top-circular domain satisfying the weak conflict property. Then, by using arguments similar to the ones employed in Remark A.1, it follows from Lemma A.2 and Lemma A.6 that for all $P_2 \in \{x_1 x_2 \dots x_m, x_m x_{m-1} \dots x_1\}$, $O_1(P_2) \in \{\{r_1(P_2)\}, X\}$.

Lemma A.7. *Let $\mathcal{C}$ be a minimal top-circular domain satisfying the weak conflict property. Further, let $P_2 \in \{x_1 x_2 \dots x_m, x_m x_{m-1} \dots x_1\}$. Then, $O_1(P_2) = \{r_1(P_2)\}$ implies $O_1(\bar{P}_2) = \{r_1(\bar{P}_2)\}$ for all $\bar{P}_2 \in \mathcal{C}$.*

Proof. We prove this lemma for the case where $P_2 = x_1 x_2 \ldots x_m$, the proof for the case where $P_2 = x_m x_{m-1} \ldots x_1$ is analogous. Let $P_2 = x_1 x_2 \ldots x_m$. Suppose $O_1(P_2) = \{x_1\}$. We show $O_1(\bar{P}_2) = \{r_1(\bar{P}_2)\}$ for all $\bar{P}_2 \in \mathcal{C}$. By strategy-proofness, we have $O_1(\bar{P}_2) = \{r_1(\bar{P}_2)\}$ for all $\bar{P}_2 \in \mathcal{C}$ with $r_1(\bar{P}_2) = x_1$. By Lemma A.3 and Remark A.2, $O_1(P_2) = \{x_1\}$ implies $O_1(\bar{P}_2) = \{x_m\}$ for all $\bar{P}_2 \in \mathcal{C}$ with $r_1(\bar{P}_2) = x_m$. We prove the lemma using induction. Take $1 \leq j < m$. Suppose $O_1(\bar{P}_2) = \{x_j\}$ for all $\bar{P}_2 \in \mathcal{C}$ with $r_1(\bar{P}_2) = x_j$. We show $O_1(\hat{P}_2) = \{x_{j+1}\}$ for all $\hat{P}_2 \in \mathcal{C}$ with $r_1(\hat{P}_2) = x_{j+1}$. Take $\hat{P}_2 \in \mathcal{C}$ with $r_1(\hat{P}_2) = x_{j+1}$. We show $O_1(\hat{P}_2) = \{x_{j+1}\}$. By strategy-proofness, it is enough to show this for $\hat{P}_2 = x_{j+1} x_j \ldots$.

First, we claim $x_k \notin O_1(\hat{P}_2)$ for all $k \neq j, j+1$. Assume for contradiction that $x_k \in O_1(\hat{P}_2)$ for some $k \neq j, j+1$. Then, $f(P_1, \hat{P}_2) = x_k$ for some $P_1 \in \mathcal{C}$ with $r_1(P_1) = x_k$. However, since $O_1(\bar{P}_2) = \{x_j\}$ for all $\bar{P}_2 \in \mathcal{C}$ with $r_1(\bar{P}_2) = x_j$, agent 2 manipulates at $(P_1, \hat{P}_2)$ via some preference $\bar{P}_2$ with $r_1(\bar{P}_2) = x_j$.

Now, we show $x_j \notin O_1(\hat{P}_2)$. Assume for contradiction that $x_j \in O_1(\hat{P}_2)$. Let $\hat{P}'_2 = x_{j+1} x_{j+2} \ldots x_m \ldots x_j \ldots$. Then, by Lemma A.1, $x_j \in O_1(\hat{P}'_2)$. Take $P_1 \in \mathcal{C}$ such that $r_1(P_1) = x_j$. Then, because $x_j \in O_1(\hat{P}'_2)$, $f(P_1, \hat{P}'_2) = x_j$. Now, take $P_2 \in \mathcal{C}$ with $r_1(P_2) = x_m$. Since $O_1(P_2) = \{x_m\}$, we have $f(P_1, P_2) = x_m$. This means agent 2 manipulates at $(P_1, \hat{P}'_2)$ via P_2. This completes the proof of the lemma. ∎

Proof of Theorem 3.2. The proof of Theorem 3.2 follows by using analogous arguments as for the proof of Theorem 3.1. ∎

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