A Path Integral Approach to Interacting Economic Systems with Multiple Heterogeneous Agents

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Abstract

This paper presents an analytical treatment of economic systems with an arbitrary number of agents that keeps track of the systems' interactions and agent's complexity. The formalism does not seek to aggregate agents: it rather replaces the standard optimization approach by a probabilistic description of the agents' behaviors and of the whole system. This is done in two distinct steps.

A first step considers an interacting system involving an arbitrary number of agents, where each agent's utility function is subject to unpredictable shocks. In such a setting, individual optimization problems need not be resolved. Each agent is described by a time-dependent probability distribution centered around its utility optimum.

The whole system of agents is thus defined by a composite probability depending on time, agents' interactions, relations of strategic dominations, agents' information sets and expectations. This setting allows for heterogeneous agents with different utility functions, strategic domination relations, heterogeneity of information, etc.

This dynamic system is described by a path integral formalism in an abstract space – the space of the agents' actions – and is very similar to a statistical physics or quantum mechanics system. We show that this description, applied to the space of agents' actions, reduces to the usual optimization results in simple cases. Compared to the standard optimization, such a description markedly eases the treatment of a system with a small number of agents. It becomes however useless for a large number of agents.

In a second step therefore, we show that, for a large number of agents, the previous description is equivalent to a more compact description in terms of field theory. This yields an analytical, although approximate, treatment of the system. This field theory does not model an aggregation of microeconomic systems in the usual sense, but rather describes an environment of a large number of interacting agents. From this description, various phases or equilibria may be retrieved, as well as the individual agents' behaviors, along with their interaction with the environment. This environment does not necessarily have a unique or stable equilibrium and allows to reconstruct aggregate quantities without reducing the system to mere relations between aggregates.

For illustrative purposes, this paper studies several economic models with a large number of agents, some presenting various phases. These are models of consumer/producer agents facing binding constraints, business cycle models, and psycho-economic models of interacting and possibly strategic agents.

Key words: path integrals, statistical field theory, phase transition, non trivial vacuum, effective action, Green function, correlation functions, business cycle, budget constraint, aggregation, forward-looking behavior, heterogeneous agents, multi-agent model, strategical advantage, interacting agents, psycho-economic models, integrated structures, emergence.

JEL Classification: C02,C60, E00, E1.

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1 Introduction

In many instances, representative agent models have proven unrealistic, lacking both collective and emerging effects resulting from the agents' interactions. To remedy these pitfalls, various paths have been explored: complex systems, networks, agent based systems or econophysics (for a review of these topics see [1][2] and references therein).

However agent based models and networks rely on microeconomic relations that may be too simplistic or lack microeconomic justifications. In these type of settings, agents are typically defined by, and follow, various set of rules. They allow for equilibrium and dynamics to emerge that would otherwise remain unaccessible to the representative agent setup. However these approaches are highly numerical and model-dependent. Econophysics, for its part, rely heavily on statistical facts as well as empirical, aggregate rules to derive some macroeconomic laws, that ultimately should pose similar problems than ad hoc macroeconomics. Indeed ad hoc macroeconomic models are prone to the Lucas critique, that led to the introduction of micro-foundations in macroeconomic theory.

A gap remains between microeconomic foundations and multi agent systems. This paper develops a setup that models micro, individual interactions along with statistic uncertainty and recovers macroeconomic, aggregate relationships using physics-like methods to replicate interaction systems involving multiple agents.

This paper presents an analytical treatment of a broad class of economic systems with an arbitrary number of agents, while keeping tracks of the system's interactions and complexity at the individual level. In this respect, our approach is similar to the Agent-Based one, in that it does not seek to aggregate all agents, and considers the interaction system in itself. However, we depart from the Agent Based Model in that we do not aggregate the agents in several different types and aim at considering the system as a whole set of large number of interacting agents. This point of view is close to the Econophysics approach, in which agents are often considered as a statistical system. Nevertheless, our objective is to translate, at the level of these statistical systems, the main characteristics of a system of optimizing agents. The goal of this work is to introduce, at the (possibly approximate) statistical level, the agents' forward looking behaviors, the individual constraints, the heterogeneity of agents or information, the strategic dominations relations.

In that, our approach is at the crossroads of statistical and economics models. From the statistical models we keep the idea of dealing with a large number of degrees of freedom of a system without aggregating quantities. From standard economic models, we keep the relevant concepts developed in the past decades to describe the behaviors of rational, or partly rational agents. A natural question arising in that context is the relevance of these concepts at the scale of the statistical system, i.e. the macro level. It is actually known that some microscopic feature may fade away at large scales, whereas some others may become predominant at the macroeconomic or macroscopic scale. The relevance or irrelevance - in the physical sense - of some micro interactions when moving from a micro to a macro scale could indirectly shed some lights on the aggregation problem in economics.

Our work is an attempt and a first step toward an answer to this matter. Although preliminary, it demonstrates that translating standard economic models into statistical ones requires introducing some statistical field models that partly differ from those used usually for physical systems. The models introduced keep track of individual behaviors. Behaviors in turn influence the description in terms of fields, as well as the results, at the macro scale.

The advantage of statistical field theories are threefold. First, they allow, at least approximatively, to deal analytically with systems with large degrees of freedom, without reducing it first to an aggregate. Second, they provide a transcription of micro relations into macro ones. Last but not least, they display features that would otherwise be hidden in an aggregate context. Actually, they allow switching from micro description to macro ones, and vice-versa, and to interpret one scale at the light of the other. Moreover, and relevantly for economic systems, these model may exhibit phase transition. Depending on the parameters of the model, the system may experience structural changes in behaviors, at the individual and collective scale. In that, they allow to approach the question of multiple equilibria.

The statistical approach of economic systems presented here is a two-step process. First, the usual model of optimizing agent is replaced by a probabilistic point of view. We consider an interacting system, involving an arbitrary number of agents, in which each agent is still represented by an intertemporal utility function, or any quantity to optimize depending on an arbitrary number of variables. However we assume that each agent's utility function is subject to unpredictable shocks. In such a setting, individual optimization problems need not be resolved. Each agent is described by a time-dependent probability distribution centered around its utility optimum. Unpredictable shocks deviate each agent from its optimal action, depending on the shocks' variances. When these variances are null, we recover the standard optimization result. It furthermore takes into account the statistic nature of a system of several agents by including uncertainty on the agents' behavior. It nonetheless preserves the analytical treatment by slightly modifying the agents' standard optimization problem.

Note that this form of modelling is close to the usual optimization of an agent when some unpredictable schocks are introduced. In the limit of no uncertainty, standard optimization equations can, in some cases, be recovered. However, the uncertainty introduce is not the one usually considered in economic models, but rather an internal uncertainty about the agent's behavior, goals, or some unobservable shocks. As such it is inherent to the model, and should not be considered as a random and external perturbation.

The system composed by the set of all agents is consequently defined by a composite probability depending on time, agents' interactions, relations of strategic dominations, agents' information sets and expectations. This setting allows for heterogeneous agents with different utility functions, strategic domination relations, heterogeneity of information, etc.

This dynamic system is described by a stochastic process whose characteristics - mean, variance, etc.- determine the sytem's transition probabilities and mean values. For example, the process mean value at time t describes the mean state of each agent at time t. Besides, we can define transition probabilities that describe the evolution of the system from t to t+1.

This setup is actually a path integral formalism in an abstract space – the space of the agents' actions – and is very similar to the statistical physics or quantum mechanics techniques. We show that this description, applied to the space of all agents' actions, reduces to the usual optimization results in simple cases, inasmuch as the unpredictable shocks' variances are null. This description is furthermore a good approximation of standard descriptions and allows to solve otherwise intractable problems. Compared to the standard optimization, such a description markedly eases the treatment of a system with a small number of agents. As a consequence, this approach is in itself consistent and useful, and provides an alternative to the standard modelling in the case of a small number of interacting agents. It allows to recover an average dynamics, which is close, or in some cases even identical to, the standard approach, and study the dynamics of the set of agents, as well as it's fluctuations if we introduce some external shocks. Our main examples will be the models developed in [3][4][5][6] describing systems of interacting agents, or structures in interactions,where some of them have information and strategic advantage. We show through this examples the possibilities of our approach in term of resolution.

However, this formalism becomes useless for a large number of agents. It can nonetheless be modified into another one, based on statistical fields, that will be more efficient in that case. Nevertheless, this first step was necessary since the statistical fieds model is grounded on our preliminary probabilistic description. Actually, this one, by it's form in terms of path integrals for a small number of interacting agents, can be transformed in a straightforward way in a description for large systems. As a consequence, the first step is also a preparatory one, needed for our initial goal, a model of large number of interacting agents.

The second step to reach this goal, therefore, consists in replacing the agents' path integrals description by a model of field theory that replicates the properties of the system when N, the number of agents, is large. Actually, in that case, we can show that the previous description is equivalent to a more compact description in terms of field theory. It allows an analytical, although approximate, treatment of the system. This transformation adapts some methods previously developped in statistical field theory to our context [7].

Hence, a double transformation, with respect to the usual optimization models has been performed. The usual optimization system is first described by a statistical system of n agents. It can then itself be replaced by a specific field theory with a large number of degrees of freedom. This field theory does not represent an aggregation of microeconomic systems in the usual sense, but rather describes an environment of an infinity of agents, from which various phases or equilibria may be retrieved, as well as the behavior of the agent(s), and the way they are influenced by, or interact with, their environment.

This double transformation allows first, for a small number of agents, to solve a system without recurring to aggregation, and second, for a large number of agents, to aggregate them so as to shape an environment whose characteristics will in turn induce, and impact, agents' interactions. This environment, or “medium”, allows to reconstruct some aggregate quantities, without reducing the system to mere relations between aggregates. Indeed, the fundamental environment from which these quantities are drawn can witness fluctuations that may invalidate relations previously established. The environment is not macroscopic in itself, but rather describes a multitude of agents in interaction. It does not necessarily have a unique or stable equilibrium. Relations between macroeconomic quantities ultimately depend on the state or "phase" of this environment ("medium"), and can vary with the state of the environment. This phenomenon is the so-called "phase transition" in field theory: The configuration of the ground state represents an equilibrium for the whole set of agents, and shapes the characteristics of interactions and individual dynamics. Various forms for this ground state, depending on the parameters of the system, may change drastically the description at individual level.

For illustrative purposes, this paper presents several economic models of consumer/producer agents facing binding constraints in competitive markets, generalized to a large number of agents and presenting various phases or equilibria.

The first section presents a probabilistic formalism for a system with N economic agents, heterogenous in goals and information. Agents are described by intertemporal utility functions, or any intertemporal quantities. However instead of optimizing these utilities, agents choose a path for their action that is randomly distributed around the usual optimal path. More precisely, the weight describing the agent behavior is an exponential of the intertemporal utility, which concentrates the probability around the optimal path. This feature models some internal uncertainty, as well as non measurable shocks. Gathering all agents yields a probabilistic description of the system in terms of effective utilities. The latters are utility functions internalizing the forward looking behavior, the interactions and the information pattern of each agent. We also show that if we reduce the internal uncertainty to 0 one recovers for most cases including the case of quadratic utilities, in principle if not in practice, the solution usual optimization problem. We end the section by solving explicitly a basic two agents example to illustrate the main points of the method.

The second section develops a class of models applying the method presented previously. This class of model has already been used previously by the authors to model single individual agents as an aggregate of several sub-structures, some having strategical advantages on others. This class of model is quite general and allows to describe systems with small number of heterogenous agents in interactions. We then provide some applications to check that our method allows a simpler resolution than the usual optimization, but also to recover, in good approximation, the results of the last one as the average path of the system.

Section three further details the effective utility of the whole system of agents, as composed of individual utilities plus possibly some additional contributions. This section stresses the fact that this global effective utility differs from a collection of individual ones. The agents as whole, are not independent from each other.

Section four turns to the probabilistic aspect of our models. We compute the transition functions of the stochastic process associated with a system of N economic agents. These transition functions have the form of euclidean path integrals. We show that, in first approximation, for agents with quadratic utilities, the transition functions are those of a set of interacting harmonic oscillators. Some non quadratic interactions may be added as perturbation expansions. Once diagonalized, the directions corresponding to the harmonic frequencies correspond to mixed, or fundamental structures, that represent independent agents.

Section five introduces constraints relevant for individual agents, such as budget constraints. We show that these individual constraints translate in the path integrals defining the system, into adding some non local contributions. Some of them may be approximated by inertial terms, i.e. "kinetic energy" contributions. Moreover, if constraints depend on other agents behaviors, these additional contributions consist of non local interaction terms.

Section six provides some elements about the Laplace transform of Green functions. It also establishes that general non local interactions must be considered, even when there are no constraints in the model. These considerations will prove useful in the next sections.

Section seven modifies our formalism to systems with a large number of agents. It shows that, in that case, the transition functions is computed as correlation functions of a field theory whose action is directly defined by individual agents' effective utilities. The section provides a back and forth interpretation between micro quantitities - individual behavior - and macro computations, i.e. collective behavior defined by the fields. It shows how some features of field theory, such as non trivial vacuum and/or phase transition, are relevant to our context. We also introduce non local individual interactions such as constraints at the field level. We show how they modify the Green functions of the system, and thus the individual agents' transition functions.

Section eight applies our formalism to several standard economic models with a large number of interacting agents. The optimizing consumers/producers model and a simple business cycle model are studied. In the first case, interactions appear through the budget constraint, an din the second case, through the interest rate determined by capital productivity. For consumers/producers, we compute the correlation functions of the field version of the system and interpret it at the individual level. We recover the usual consumption smoothing, but we can also track the effect of the interaction between agents that increase the fluctuations of an individual behavior. In the business cycle model, we show that a non trivial vacuum may appear: for some values of the parameters, the equilibrium may be shifted in a non continuous manner. The system enters another phase, with different individual behaviors.]

Building on previous results, Section nine details the mechanisms of non trivial vacuua for the field theoretic version of models presented in section two. Stabilization effects between structures may appear in field theoretic formulation through a stabilization potential. This stabilization allows to describe the system as sets of integrated structures. Unstable patterns that would otherwise be short lived may use others to stabilize and form larger and more stable structures. The vacuum configuration corresponding to these integrated structures is different from the initials' and new features may be present in the resulting system. The section also developps the notion of effective actions. When several types of agents are present, the actions of some of them may be integrated out, to be absorbed in the effective action describing the remaining agents. "Hidden" agents are thus included as external conditions shaping the environment and inducing possibly some phase transition.

Section ten sketches a method to compute macro quantities from micro ones in the context of the field formalism. Introducing a macro time scale may allow, in some cases, to recover approximate macroeconomics relations between aggregate quantities.

2 Method

2.1 Principle

In this paper, the usual optimization problem of each agent dynamics within the system is replaced by a probabilistic description of the whole system. Several conditions must be satisfied to keep track of the system of agents' main features. First, at least in some basic cases, the optimization equations in average should be recovered. Moreover, this probabilistic description needs to take into account the individual characteristic of the agents. In a context of economic modelling, it means to include each agent constraint, interactions with other, and last by not least, ability to anticipate other's agents' actions.

This probabilistic description involves a probability density for the state of the system at each period t. In a system composed of N agents, each defined by a vector of action $X_i(t)$, we will define a probability density $P\left((X_i(t))_{i=1..N}\right)$ for the set of actions $(X_i(t))_{i=1..N}$ which describes the state of the system at t. Importantly, for a large number of agents at least, working with a probability distribution is easier than solving some, often untractable, optimization equations. This probability distribution may often be designed to be gaussian and centered around the optimal solution of the utility problem. In that case, if the variance of this distribution is proportional to an exogenous parameter, one may expect, at least for some particular cases, that when this parameter goes to 0, then the probability distribution will be peaked around the optimal, or "classical solution". Then, such a probabilistic description can be seen as a generalization of the usual optimization problem where some internal uncertainty in agents behavior, uncertainty of each of them with respect to the others, as in an imperfect information problem, but also to themselves. We justify this "blured" behavior by the inherent complexity of all agents, their goals and behavior being modified at ech period by some internal, unobservable and individual shocks, the classical case beeing retrieved when this uncertainty is neglected.

To develop this point, consider first the intertemporal utility of an agent i:

$$U_t^{(i)} = \sum_{n\geqslant 0} \beta^n u_{t+n}^{(i)} \left(X_i(t+n), (X_j(t+n-1))_{j\neq i}\right)$$

where $u_{t+n}^{(i)}$ is the instantaneous utility at time $t+n$. In the optimization setup, the agent i optimizes on the control variables $X_i(t+n)$. The variables in parenthesis: $(X_j(t+n-1))_{j\neq i}$ represent the actions of other agents. We will also denote $(X_j(t+n-1))$ the actions for the set of all agents.

Remark at this point that the term utility used here is convenient for any quantity optimized. It can encompass a production function, for example in oligopoly models, and/or production and utility functions, in consumer/producer models. Moreover, this type of model may describe the interaction of several substructures within an individual agent. See for example models of heterogeneous interacting agents L,GL, GLW, or models of motion decision and control in neurosciences.

Now we will explain how to switch toward a probabilistic representations that satisfies our requirements. We start with a simple example and then generalize the procedure. Assume first that agent i has no information about the others, so that their actions are perceived as random shocks by agent i. We then postulate that rather optimizing $U_t^{(i)}$ on $X_i(t)$, agent i will choose an action $X_i(t)$ and a plan (that is recalculated period after period) $X_i(t+n)$, $n>0$, for it's future actions that follow a conditional probabilistic law proportional to:

$$\exp\left(U_t^{(i)}\right)=\exp\left(\sum_{n\geqslant 0}\beta^n u_{t+n}^{(i)}\left(X_i(t+n),(X_j(t+n-1))_{j\neq i}\right)\right)$$

This is a probabilistic law for $X_i(t)$ and the plan $X_i(t+n)$, $n>0$. It is conditional to the action variables $X_j(t+n-1)$ of the other agents, that are perceived as exogenous by agent i.

Remark that, for a usual convex utility with a maximum, the closest the choices of the $X_i(t+n)$ to $U_t^{(i)}$ optimum, the higher the probability associated to $X_i(t+n)$. Thus, this choice of utility is coherent with a probability peaked around the optimization optimum. This choice of utility is therefore coherent with a probability peaked around the optimization optimum.

To better understand the principle of the probabilistic description, we will start with the simplest case, in which one agent has no information about the others. In that case, the variables $X_j(t+n)$ will be considered as random noises. Thus, agent i will integrate out other agents actions as random noises. The probability for $X_i(t)$ and $X_i(t+n)$, $n>0$ will then be

$$\int\exp\left(U_t^{(i)}\right)\exp\left(-\frac{X_j^2(s)}{\sigma_j^2}\right)\prod_{j\neq i}\prod_{s\geqslant t}dX_j(s)$$

$\exp\left(-\frac{X_j^2(s)}{\sigma_j^2}\right)$ being the subjective weight attributed to the $X_j(s)$ by i. In general if there is no information at all, we can assume the $\sigma_j^2\rightarrow\infty$, $\exp\left(-\frac{X_j^2(s)}{\sigma_j^2}\right)\rightarrow\delta(X_j(s))$ where $\delta(X_j(s))$ is the dirac delta function so that other agents may be considered either as inert or, in lack of any further information, as random perturbations. Their future actions are set to 0 by agent i, or, which is equivalent, discarded from the agent planification.

When there are no constraint and no inertia in $u_t^{(i)}$ - or, alternatively - when $u_t^{(i)}$ solely depends on $X_i(t)$ and other agents' previous actions $(X_j(t-1))_{j\neq i}$, the periods are independent. Consequently, $\exp\left(U_t^{(i)}\right)$ is a product of term of the kind $\exp\left(\beta^n u_{t+n}^{(i)}\left(X_i(t+n),(X_j(t+n-1))_{j\neq i}\right)\right)$ that are also independent. As a consequence, the probability associated to the action $X_i(t)$ is:

$$\int\left(\int\exp\left(U_t^{(i)}\right)\exp\left(-\frac{X_j^2(s)}{\sigma_j^2}\right)\prod_{j\neq i}\prod_{s\geqslant t}dX_j(s)\right)\prod_{s>t}dX_i(s)\propto\exp\left(u_t^{(i)}\left(X_i(t),(X_j(t-1))_{j\neq i}\right)\right)$$

Each agent is described by its instantaneous utility: the lack of information induces a short sighted behavior. Each term $\exp\left(u_{t+n}^{(i)}\left(X_i(t+n),(X_j(t+n-1))_{j\neq i}\right)\right)$ is the probability for a random term whose integral on $X_j(s+n)$ is set to 1. In absence of any period overlap, i.e. without any constraint, the behavior of agent i is described by a random distribution peaked around the optimum of $u_t^{(i)}\left(X_i(t),(X_j(t-1))_{j\neq i}\right)$ which models exactly the optimal behavior of an agent influenced by individual random shocks.

Having understood the principle of the probabilistic scheme with this simple example, we can now complexify the information pattern, to account for the agents' heterogeneity. The knowledge that some agents may have about others' utilities affects the statistical weight describing the agent's behavior. Actually, if agent i has some information about agent j utility, it would be able to forecast it's influence on agent j through $X_i(t)$ and in turn the delayed reactions $X_j(t+n)$ of agent j.

Let us more precisely consider, as before, the conditional probability for $X_i(t)$ and the $X_i(t+n)$, $n>0$, depending on the $(X_j(s-1))_{j\neq i}$. For $s \geqslant t$ we conveniently define this probability to be proportional to $\exp\left(U_t^{(i)}\right)$:

$$\begin{aligned}
& P\left(X_i(t), X_i(t+1), ..., X_i(t+n), ... \mid (X_j(s))_{j\neq i, s\geqslant t}\right) \\
\propto\ & \exp\left(U_t^{(i)}\right) \\
=\ & \exp\left(\sum_{n\geqslant 0} \beta^n u_{t+n}^{(i)}\left(X_i(t+n), (X_j(t+n-1))_{j\neq i}\right)\right)
\end{aligned}$$

and this determines the (statistical) behavior of agent i, given other agents' future actions. At first sight integrating this expression over the $(X_j(t+n-1))_{j\neq i, s\geqslant t}$ and $X_i(t+n)$, $n>0$ would yield a statistical weight:

$$P\left(X_i(t) \mid X_i(t-1), (X_j(t-1))_{j\neq i, s\geqslant t}\right) \tag{1}$$

for each agent i. However, we cannot procced this way to find $P\left(X_i(t) \mid X_i(t-1), (X_j(t-1))_{j\neq i, s\geqslant t}\right)$. We will rather show that all the $P\left(X_i(t) \mid X_i(t-1), (X_j(t-1))_{j\neq i, s\geqslant t}\right)$ have to be found jointly, as a system of equations. Actually, in the previous equations, the probabilities

$$P\left(X_i(t), X_i(t+n) \mid (X_j(s))_{j\neq i, s\geqslant t}\right)$$

are conditional to the actions of other agents $(X_j(t+n-1))_{j\neq i}$, as in the simple case of no information. But now, these variables are themselves forecasted by agent i as depending on $X_i(t)$. One then needs to take into account this interconnexion to find $P\left(X_i(t), X_i(t+n) \mid (X_j(s))_{j\neq i, s\geqslant t}\right)$. It leads us to define (agent i's expectation of) the conditional probability of other agents actions given $X_i(t)$:

$$\begin{aligned}
& P_i\left((X_j(t))_{j\neq i}, ..., (X_j(t+n))_{j\neq i}, ... \mid X_i(t)\right) \qquad (2)\\
=\ & E_i^t \prod_{k\geqslant 0} P\left((X_j(t+k))_{j\neq i} \mid (X_j(t+k-1))\right)
\end{aligned}$$

where E_i^t denotes agent i's expectation at time t.

Equation (2) means that agent i, forecasts the probabilities $P\left((X_j(t+k))_{j\neq i} \mid (X_j(t+k-1))\right)$ for other agents, including it's dependence in $X_i(t+k)$ and take into account in it's computations of it's future path. Now, we assume that agent i attributes the weight (2) to the path $(X_j(t))_{j\neq i}, ..., (X_j(t+n))_{j\neq i}, ...$. Then rather than defining a conditional expectation $P\left(X_i(t), X_i(t+n) \mid (X_j(s))_{j\neq i, s\geqslant t}\right)$ we will define a joint probability:

$$\exp\left(U_t^{(i)}\right) E_i^t \prod_{k\geqslant 0} P\left((X_j(t+k))_{j\neq i} \mid (X_j(t+k-1))\right)$$

which describes the probability attributed by agent i to the joint path:

$$X_i(t), ..., X_i(t+n), ... (X_j(t))_{j\neq i}, ..., (X_j(t+n))_{j\neq i}, ...$$

Then, once this weight is attributed, it takes ito account the interrelations between the paths $X_i(t+n)$ and $(X_j(t+n))_{j\neq i}$. One can now integrate on the $(X_j(t+n))_{j\neq i}$ to find the probability for a path

$X_i(t), ..., X_i(t+n), ...$:

$$\begin{aligned}
& P\left(X_i(t), ..., X_i(t+n), ... \mid X_i(t-1), (X_j(t-1))_{j\neq i}\right) \\
= & \int \exp\left(U_t^{(i)}\right) E_i^t \prod_{k\geqslant 0} P\left((X_j(t+k))_{j\neq i} \mid (X_j(t+k-1))\right) d(X_j(t+k))_{j\neq i} \\
= & \int \exp\left(\sum_{n\geqslant 0} \beta^n u_{t+n}^{(i)}\left(X_i(t+n), (X_j(t+n-1))_{j\neq i}\right)\right) \\
& \times E_i^t \prod_{k\geqslant 0} P\left((X_j(t+k))_{j\neq i} \mid (X_j(t+k-1))\right) d(X_j(t+k))_{j\neq i}
\end{aligned}$$

where $d(X_j(t+k))_{j\neq i}$ stands for $\prod_{j\neq i} dX_j(t+k)$. As before, we need to express the behavior of agent i at time t given past actions:

$$P\left(X_i(t) \mid X_i(t-1), (X_j(s-1))_{j\neq i, s\geqslant t}\right)$$

that describes the probability for $X_i(t)$ as a function of $X_i(t-1)$ and $(X_j(t-1))_{j\neq i}$. To do so, we can now integrate

$$P\left(X_i(t), ..., X_i(t+n), ... \mid X_i(t-1), (X_j(t-1))_{j\neq i}\right)$$

over $X_i(t+k+1)$ with $k \geqslant 0$, and this will yield $P\left(X_i(t) \mid X_i(t-1), (X_j(s-1))_{j\neq i, s\geqslant t}\right)$.

$$\begin{aligned}
& P\left(X_i(t) \mid X_i(t-1), (X_j(s-1))_{j\neq i}\right) \qquad (3) \\
= & \int \exp\left(U_t^{(i)}\right) E_i^t \prod_{k} P\left((X_j(t+k))_{j\neq i} \mid (X_j(t+k-1))\right) d(X_j(t+k))_{j\neq i}\, dX_i(t+k+1) \\
= & \int \exp\left(\sum_{n\geqslant 0} \beta^n u_{t+n}^{(i)}\left(X_i(t+n), (X_j(t+n-1))_{j\neq i}\right)\right) \\
& \times E_i^t \prod_{k\geqslant 0} P\left((X_j(t+k))_{j\neq i} \mid (X_j(t+k-1))\right) d(X_j(t+k))_{j\neq i}\, dX_i(t+k+1)
\end{aligned}$$

and the set of these equations with $i = 1, ...k$ where k is the number of agents, defines the set of statistical weights $P\left(X_i(t) \mid X_i(t-1), (X_j(t-1))_{j\neq i}\right)$.

As such, the system of equations (3) depends on agents expectations and this ones have to be defined to solve (3). To do so, we first define the effective utility for agent i at time t, written $U_{eff}(X_i(t))$ as:

$$P\left(X_i(t) \mid X_i(t-1), (X_j(t-1))_{j\neq i}\right) = \frac{\exp(U_{eff}(X_i(t)))}{\mathcal{N}_i} \qquad (4)$$

where the normalization factor $\mathcal{N}_i$ is defined as:

$$\mathcal{N}_i = \int \exp(U_{eff}(X_i(t)))\, dX_i(t)$$

The interpretation of $U_{eff}(X_i(t))$ is straightforward given our procedure. We express the statistical weight describing the behavior of agent i at time t as a the exponential of an utility function that has included all expectations of this agent about the future. In a classical interpretation, the first order condition applied to $U_{eff}(X_i(t))$, that would express $X_i(t)$ as a function of the $X_j(t-1)$, $j \neq i$ and $X_i(t-1)$ corresponds to the solution of the dynamics equation for agent i. Given our approach, this is of course not the case, but we show in Appendix 1, that for quadratic utilities, $U_{eff}(X_i(t))$ encompasses this classical result and allows to recover the optimization solution in the limit of no internal uncertainty.

Remark that our definition (4) does not include directly the normalization term $\mathcal{N}_i$. It implies that $U_{eff}(X_i(t))$ is not uniquely defined by (4) since it allows to include any term independent from $X_i(t)$. However, it allows to work with $U_{eff}(X_i(t))$ without being careful with the normalization of this function, and to add the needed factor only when it is necessary, i.e. when computing some expectations. We will come back to this point later in this section.

The previous definition (4) will allow to rewrite the conditional probabilities in (3) as:

$$\begin{aligned} & P\left(\left(X_j(t)\right),...,\left(X_j(t+n)\right),... \mid X_i(t-1)\right) \\ = \quad & E_i^t\left(\prod_{k\geqslant 0} P\left(\left(X_j(t+k)\right)_{j\neq i} \mid \left(X_j(t+k-1)\right)\right)\right) \\ \equiv \quad & E_i^t \exp\left(\sum_{k\geqslant 0}\sum_{j\neq i} U_{eff}\left(X_j(t+k)\right)\right) \end{aligned}$$

where:

$$E_i^t \exp\left(\sum_{k\geqslant 0}\sum_{j\neq i} U_{eff}\left(X_j(t+k)\right)\right) = \prod_{j\neq i} E_i^t \exp\left(\sum_{k\geqslant 0} U_{eff}\left(X_j(t+k)\right)\right)$$

is the expectation at time t of agent i given its own set of information (the upperscript t in E_i^t is sometimes understood when there is no ambiguity). Then equation (3) becomes:

$$\exp\left(U_{eff}\left(X_i(t)\right)\right) = \int \exp\left(U_t^{(i)}\right)\prod_{j\neq i} E_i^t \exp\left(\sum_{s\geqslant t} U_{eff}\left(X_j(s)\right)\right)\prod_{s\geqslant t} dX_j(s)\prod_{s\geqslant t} dX_i(s+1) \tag{5}$$

Equation (5) is a system making interdependent the statistical behavior of each agent. In order to solve (5) and find the effective utility $U_{eff}(X_i(t))$ one needs to compute the expectations $E_i^t \exp\left(\sum_{s\geqslant t} U_{eff}\left(X_j(s)\right)\right)$ and to do so, we have to introduce some assumptions about the expectations formulation. Basically, we generalize what was said before and will consider two cases, that will be sufficient for most cases (some alternative hypothesis could be developped as well). We will distinguish the agents by their relation with respect to the information they have about the others. An agent i has an information domination (or strategic domination) over j, if it knows the parameters, or some parameters of the agent j utility and if j has no information about $i's$ set of parameters. This allows i to forecast agent j's actions and take into account how it can influence j as explained above in (3). On it's side, agent j perceives agent i's actions as random noises. Moreover, we say that two agents i and j have no information domination on each other, if they have both information (or both no information) on the other one's utility.

It is convenient for the sequel to define the rank of an agent with respect to the others in the following way: When an agent i has an information domination over an agent j one says that $rk(j) < rk(i)$ (or $j < i$ when there is no ambiguity). We also set $rk(i) = rk(j)$ (or $i \nless j$ or $j \nless i$) if there is no information domination relation between i and j.

If i has no information about j, an arbitrary weight $\exp\left(-\frac{X_j^2(s)}{\sigma_j^2}\right)$ is assignated to j. As explained above, it results in simply discarding the variable $X_j(t+k)$ in the problem in consideration. We will use this point below. If i has an information domination over j, $rk(j) < rk(i)$ then we define:

$$E_i^t \exp\left(\sum_{k\geqslant 0} U_{eff}\left(X_j(t+k)\right)\right) = \frac{\exp\left(\sum_{k\geqslant 0}\frac{U_{eff}^{t_i}\left(X_j(t+k)\right)}{\sigma_j^2}\right)}{\mathcal{N}_{ij}} \tag{6}$$

with $\mathcal{N}_{ij} = \int \exp\left(\sum_{k\geqslant 0}\frac{U_{eff}^{t_i}\left(X_j(t+k)\right)}{\sigma_j^2}\right) dX_i(t)$.

The function $U_{eff}^{t_i}\left(X_j(t+k)\right)$ is the i-th truncated effective utility of j, the effective utility $U_{eff}\left(X_j(t+k)\right)$ in which all the variables $X_k(t+k)$ with $rk(k) > rk(i)$ and some (depending on the precise form of the model) of the $X_k(t+k)$ with $rk(k) = rk(i)$ are set to 0. It reflects the fact that in that case, agent i has no information about agents k with $rk(k) > rk(i)$ and for some agents k with $rk(k) = rk(i)$, and as a consequence, no information on the way k impacts j. Note that $\mathcal{N}_{ij}$ depends implicitely on past variables. To simplify the notations we will redefine

$$U_{eff}^{t_i}(X_j(t+k)) - \sigma_j^2 \ln \mathcal{N}_{ij} \rightarrow U_{eff}^{t_i}(X_j(t+k))$$

and (6) becomes:

$$E_i^t \exp\left(\sum_{k \geqslant 0} U_{eff}(X_j(t+k))\right) = \exp\left(\sum_{k \geqslant 0} \frac{U_{eff}^{t_i}(X_j(t+k))}{\sigma_j^2}\right) \tag{7}$$

The parameter σ_j^2 is a measure of agent i's uncertainty about agent j future actions. Remark that σ_j^2 should in fact be written $\sigma_j^2(i)$, but, since σ_j^2 will only appear in intermediate computations of $U_{eff}(X_i(t))$, this dependence in i can be understood. For $\sigma_j^2 \rightarrow 0$, one recovers the full certainty about the agent that behaves as the usual optimizer. For σ_j^2 increasing, this behavior becomes only an average behavior. For $\sigma_j^2 \rightarrow \infty$, agent's action is perceived as random. Thus σ_j^2 introduces the measure of uncertainty about agents behavior, i.e. the measure of external shocks. Concerning agent i'expectations E_i^t, we will also make the assumption that all agents are perceived as independent, that is, given $(X_j(t-1))$, one has:

$$E_i^t P\left((X_j(t))_{j \neq i} \mid (X_j(t-1))\right) = \prod_{j \neq i} P(X_j(t) \mid (X_j(t-1)))$$

and more generally:

$$E_i^t P\left((X_j(t+k))_{j \neq i} \mid (X_j(t+k-1))\right) = \prod_{j \neq i} P(X_j(t+k) \mid (X_j(t+k-1)))$$

Other assumptions could be made, the actions of some agents at time t could be bound together, but those hypothesis would be equivalent to consider some agents as a whole, which would mean to regroup some utility functions from the beginning.

Ultimately, we will also assume that each agent faces an uncertainty about it's own future action. This is modeled by the fact that in (3), we replace $\exp\left(U_t^{(i)}\right)$ by $\exp\left(\frac{U_t^{(1)}}{\sigma_i^2}\right)$ where σ_i^2 measures the degree of uncertainty of i about itself, as σ_j^2 measures the uncertainty about other agents. In fact, as we will se, in most case, the factor σ_i^2 can be rescaled to 1, but it's presence, at least in the beginning, allows to interpret the results more clearly.

The expression of the conditional probabilities appearing in (3), in terms of the $U_{eff}(X_j(t+k))$ allows to write the conditional probablities as intertemporal sums. To find recursively each agent effective utility $U_{eff}(X_i(t))$, we introduce the system of all agents effective utility in the previous formula.

Given our assumptions (5) rewrites:

$$\begin{aligned} & \exp(U_{eff}(X_i(t))) \\ = \; & P\left(X_i(t) \mid X_i(t-1), (X_j(t-1))_{j \neq i}\right) \\ = \; & \int \exp\left(\frac{U_t^{(i)}}{\sigma_i^2}\right) E_i^t \prod_{k \geqslant 0} P\left((X_j(t+k))_{j \neq i} \mid (X_j(t+k-1))\right) \\ & \times d(X_j(t+k))_{j \neq i} \, dX_i(t+k+1) \end{aligned} \tag{8}$$

or, replacing the expectations E_i^t:

$$\begin{aligned} \exp(U_{eff}(X_i(t))) = & \int \exp\left(\frac{U_t^{(i)}}{\sigma_i^2}\right) \\ & \times \exp\left(\sum_{k \geqslant 0} \sum_{j \neq i} \frac{U_{eff}^{t_i}(X_j(t+k))}{\sigma_j^2}\right) \prod_{k \geqslant 0} d(X_j(t+k))_{j \neq i} \, d(X_i(t+k+1)) \end{aligned} \tag{9}$$

with:

$$U_t^{(i)} = \sum_{k\geqslant 0} \beta^n u_{t+n}^{(i)} \left(X_i(t+k), (X_j(t+k-1))_{j\neq i}\right)$$

The system (8) (or (9) defines the $U_{eff}(X_i(t))$ that determine ultimately the probabilities (1) describing the system.

We show in Appendix 1 that, for quadratic utilities, when $\sigma_j^2 \to 0$ and then $\sigma_i^2 \to 0$, one recover the optimization equations of the standard utility maximizing agent. In other words, the agent behavior, is peaked on the usual optimal path. For non quadratic utilities, one would recover the same results but with condition to replace the effective utilities $U_{eff}(X_j(t+k))$ in the right hand side of (8) by their by quadratic approximation around the saddle point solution. More precisely, if we were rather defining the effective utilities as satisfying:

$$\exp(U_{eff}(X_i(t))) = \int \exp\left(\frac{U_t^{(i)}}{\sigma_i^2}\right) \tag{10}$$
$$\times \exp\left(\sum_{k\geqslant 0}\sum_{j\neq i} \frac{\hat{U}_{eff}^{t_i}(X_j(t+k))}{\sigma_j^2}\right) \prod_{k\geqslant 0} d(X_j(t+k))_{j\neq i}\, dX_i(t+k+1)$$

where:

$$U_t^{(i)} = \sum_{k\geqslant 0} \beta^n u_{t+n}^{(i)} \left(X_i(t+k), (X_j(t+k-1))_{j\neq i}\right)$$

is the intertemporal utility of agent j and:

$$\hat{U}_{eff}^{t_i}(X_j(t), (X_k(t-1))) = -\frac{1}{2}(X_j(t) - X_j[(X_k(t-1))])^t A_{jj}(X_j(t) - X_j[(X_k(t-1))])$$

is the quadratic approximation of $U_{eff}^{t_i}(X_j(t), (X_k(t-1)))$ at $X_j[(X_k(t-1))]$. Here, $X_j[(X_k(t-1))]$ is the solution of the optimization problem of $U_{eff}^{t_i}(X_j(t), (X_k(t-1)))$ in the variable X_j. That is, $X_j[(X_k(t-1))]$ satisfies:

$$0 = \left(\frac{\partial}{\partial X_j(t)} U_{eff}^{t_i}(X_j(t), (X_k(t-1)))\right)_{X_j(t)=X_j[(X_k(t-1))]}$$

for any $(X_k(t-1))$. Then, Appendix 1 shows that in that case, the integrals in (10) are peaked around the classical optimization solution when $\sigma_j^2 \to 0$ and then $\sigma_i^2 \to 0$.

We do not choose this representation (10), and rather stay with (8), since we present a different formalism from the standard one, and the form (8) seems both more natural and more convenient. It is sufficient for our purpose to know that we can recover the standard approach as a particular case for the case of quadratic utilities, and as a quadratic approximation for general cases.

Note also that for utilities of homogenous form, and of the same degre in the $X_i(s)$, one can rescale $X_i'(s) = \frac{X_i'(s)}{(\sigma_i^2)^{\frac{1}{\alpha}}}$, $X_j'(s) = \frac{X_j'(s)}{(\sigma_i^2)^{\frac{1}{\alpha}}}$ where α is the degree of the homogenous utility. In this case this is equivalent to set $\sigma_i^2 = 1$ and to redefine σ_j^2 to be equal to $\frac{\sigma_j^2}{\sigma_i^2}$. If we assume that all the σ_j^2 are equal to σ^2, thus we will replace σ^2 by $\frac{\sigma^2}{\sigma_i^2}$. The integrals in (8) include some irrelevant constant factor that are powers of σ_i^2 that will be absorbed in the normalization of the statistical weight $\exp(U_{eff}(X_i(t)))$. As a consequence, after integrations (8) reduces to:

$$\exp(U_{eff}(X_i(t))) = \exp\left(U_t^{(i)}\right) \times \exp\left(\sum_{k\geqslant 0}\sum_{j\neq i} \frac{U_{eff}(X_j(t+k))}{\sigma^2}\right) d(X_j(t+k))_{j\neq i}\, d(X_i(t+k+1))$$

which is a more convenient representation. In that context, retrieving the usual optimization description corresponds still to let $\sigma_j^2 \to 0$ (These optimization equations are in fact for the variables $X_i'(s)$, but due to the homogenous form of the utilities, the factors in powers of $\left(\sigma_i^2\right)^{\frac{1}{\alpha}}$ cancel and one retrieves the equations for the $X_i(s)$).

The system (8) is solved given our assumptions on the agents information sets and the form of the expectations E_i^t. Given our assumptions on the expectations E_i^t, the computation of $\exp\left(U_{eff}\left(X_i(t)\right)\right)$ will involve only the structures on which i has an advantage of information or those that are in a relation of non domination with i. Actually, as said before, the structures about which structure i has no information, are considered as random shocks and not included in agent i computation, that is, if $rk(i) < rk(j)$ $E_i^t \exp\left(\sum_{s\geqslant t} U_{eff}\left(X_j(s)\right)\right) = 1$. In other words, agent i integrates only in its behavior all substructures possible paths. Its choice, for a given set of $X_j(s)$,$X_i(s)$, $j < i$, $s \geqslant t$ is $\exp\left(U_t^{(i)}\right)$ weighted by $\exp\left(\sum_{s\geqslant t} U_{eff}^{t_i}\left(X_j(s)\right)\right)$.

The resolution for the $U_{eff}\left(X_i(t)\right)$ consists then first, by ranking the agents by their strategic advantages. The $U_{eff}\left(X_i(t)\right)$ are found recursively for each set of agents with the same rank. Second, the effective utility just found are reintroduced in the system of equation defining the effective utility of higher rank.

Among a set for a given rank (we use the rescaling $\sigma_i^2 = 1$ described above) (8) rewrites:

$$\begin{aligned} &\exp\left(U_{eff}\left(X_i(t)\right)\right) \\ = &\int \exp\left(U_t^{(i)}\right) \prod_{rk(j)<rk(i)} \exp\left(\sum_{s\geqslant t} \frac{U_{eff}^{t_i}\left(X_j(s)\right)}{\sigma_j^2}\right) \prod_{s\geqslant t} dX_j(s) \\ &\times \left(\prod_{rk(j)=rk(i)} E_i^t \exp\left(\sum_{s\geqslant t} U_{eff}\left(X_j(s)\right)\right) \prod_{s\geqslant t} dX_j(s)\right) \prod_{s\geqslant t} dX_i(s+1) \end{aligned} \tag{11}$$

The $U_{eff}\left(X_j(s)\right)$ with $rk(j) < rk(i)$ are given by hypothesis, and so are the $U_{eff}^{t_i}\left(X_j(s)\right)$ which are obtained from the $U_{eff}\left(X_j(s)\right)$ by truncation. We are thus left with a set of functional equations between the $U_{eff}\left(X_i(t)\right)$ of the same rank.

The resolution depends on the model, and on the formation of expectations for $rk(j) = rk(i)$. Several hypothesis are possible in this case. For example:

$$\begin{aligned} E_i^t \exp\left(\sum_{s\geqslant t} U_{eff}\left(X_j(s)\right)\right) &= 1 \\ E_i^t \exp\left(\sum_{s\geqslant t} U_{eff}\left(X_j(s)\right)\right) &= \exp\left(\sum_{s\geqslant t} \frac{U_{eff}\left(X_j(s)\right)}{\sigma_j^2}\right) \\ E_i^t \exp\left(\sum_{s\geqslant t} U_{eff}\left(X_j(s)\right)\right) &= \exp\left(\sum_{s\geqslant t} \frac{U_{eff}\left(X_i(s)\right)_{X_i(s)\rightarrow X_i(s)}}{\sigma_j^2}\right) \end{aligned}$$

In the first case, structures of the same rank share no information at all. In the second case, they fully share their information. In the third and last case agents are identical: take agent i utility and replace $X_i(s)$ by $X_j(s)$ (assuming thus that j is identical to i).

We keep the first and simplest case $E_i^t \exp\left(\sum_{s\geqslant t} \frac{U_{eff}(X_j(s))}{\sigma_j^2}\right) = 1$ when $rk(j) = rk(i)$. It implies that in the truncation procedure the $X_j(s)$ with $rk(j) = rk(i)$ are set to 0 in the $U_{eff}^{t_i}\left(X_k(s)\right)$. It means that in the sequel of the paper we will work with the following recursive system of equations for the effective utilities:

$$\begin{aligned} &\exp\left(U_{eff}\left(X_i(t)\right)\right) \\ = &\int \exp\left(U_t^{(i)}\right) \prod_{rk(j)<rk(i)} \exp\left(\sum_{s\geqslant t} \frac{U_{eff}^{t_i}\left(X_j(s)\right)}{\sigma_j^2}\right) \prod_{s\geqslant t} dX_j(s) \prod_{s\geqslant t} dX_i(s+1) \end{aligned} \tag{12}$$

Once the $U_{eff}(X_i(t))$ satisfying (11), or more generally (12), are found, we can consider the entire system as being described by an overall weight:

$$P(X(t+k) \mid X(t)) \equiv P((X_i(t+1)) \mid (X_i(t)))$$

and more generally, by the transition probabilities of the system over k periods:

$$P(X(t+k) \mid X(t)) \equiv P((X_i(t+k)) \mid (X_i(t))) \tag{13}$$

where $X(s) = (X_i(s))$ is the concatenation of the $X_i(s)$. Equation (13) models the random path of the whole system. This will be the point of view used in the next sections of this work. There are several ways to define (13), all of them depending on some additional hypothesis. If we assume that the individual transition functions $P(X_i(t+k) \mid (X(t)))$ are independent, one has:

$$\begin{aligned} P(X(t+k) \mid X(t)) &= P(X(t+k) \mid X(t+k-1)) ... P(X(t+1) \mid X(t)) \\ &= \left(\prod_i P(X_i(t+k) \mid (X(t+k-1)))\right) ... \left(\prod_i P(X_i(t+1) \mid (X(t)))\right) \end{aligned} \tag{14}$$

These probabilities can be computed through the effective utilities. Yet, more care must be given to the normalization factors. We first assume that a non-normalized particular effective utility function satifying (11) has been chosen (recall that all such functions differ by a function of all variables except $X_i(t)$). Since the statistical weight defining an agent is proportional to the exponential of the effective utility, one has:

$$P(X_i(t) \mid (X(t-1))) = \frac{\exp(U_{eff}(X_i(t)))}{\mathcal{N}_i} \tag{15}$$

where the normalization factor is defined by:

$$\mathcal{N}_i = \int \exp(U_{eff}(X_i(t)))\, dX_i(t)$$

When gathering all agents, we have to take into account that $\mathcal{N}$ depends on $(X(t-1))$, as seen in (15). As a consequence, we will write:

$$P(X_i(t) \mid (X(t-1))) = \frac{\exp(U_{eff}(X_i(t)))}{\mathcal{N}_i(X(t-1))} \tag{16}$$

and (14) will be given by successive integrals:

$$\begin{aligned} & P((X_i(t+k)) \mid (X_i(t))) \\ = & \int \exp\left(\sum_{l=1}^{k} \sum_i (U_{eff}(X_i(t+l), (X_i(t+l-1))) - \ln \mathcal{N}_i(X(t-1)))\right) \prod_{l=1}^{k} d(X_i(t+l)) \end{aligned} \tag{17}$$

However, other possibilities to define $P((X_i(t+k)) \mid (X_i(t)))$ may be more relevant. Since we are looking at the entire system, one may assume that that the independence hypothesis (14) does not hold any more. Some additional interactions, internal constraints unknown to the individual components, may be relevant to the system and invalidate (17). One should thus modify the effective utility accordingly. Yet there is a simpler way to bind all the components of the system, and it is related again to the normalization problem. We have seen that $U_{eff}(X_i(t+l), (X_i(t+l-1)))$ is defined up to any function independent of $X_i(t+l)$ (see the discussion after (4)). If we chose a particular form for $U_{eff}(X_i(t+l), (X_i(t+l-1)))$ and do not impose (14), we can define the transition function for the system as:

$$P((X_i(t+k)) \mid (X_i(t))) = \frac{1}{\mathcal{N}'} \int \exp\left(\sum_{l=1}^{k} \sum_i U_{eff}(X_i(t+l), (X_i(t+l-1)))\right) \prod_{l=1}^{k} d(X_i(t+l)) \tag{18}$$

where $\mathcal{N}'$ is a global normalization factor for the entire system and the whole path between t and $t+k$. Such a formula distinguishes between collective and individual behaviors, the utility attributed to the system being the sum of individual utilities. Formula (18) will depend on the particular form of $U_{eff}\left(X_i\left(t+l\right),\left(X_i\left(t+l-1\right)\right)\right)$ among the solutions of (11). The choice of $U_{eff}\left(X_i\left(t+l\right),\left(X_i\left(t+l-1\right)\right)\right)$ depends on the particular system studied, but one has to remind that the effective utility $U_{eff}\left(X_i\left(t\right),\left(X_i\left(t-1\right)\right)\right)$ encompasses all forward looking and strategic aspects of agent i's behavior. Thus, all these aspects being integrated out, $U_{eff}\left(X_i\left(t\right),\left(X_i\left(t-1\right)\right)\right)$ should only take into account one period effect, and describe how agent i's reacts to $\left(X_i\left(t-1\right)\right)$. A coherent choice for $U_{eff}\left(X_i\left(t\right),\left(X_i\left(t-1\right)\right)\right)$ is thus to impose that it should not include any contributions independent from $X_i\left(t\right)$. Introducing such terms would actually model a concern for other periods, and that was ruled out from the definition of $U_{eff}\left(X_i\left(t\right),\left(X_i\left(t-1\right)\right)\right)$. As a consequence, in the sequel we will keep this choice when working with (18).

Recall that in the definition of the effective utilities $U_{eff}\left(X_i\left(t+l\right),\left(X_i\left(t+l-1\right)\right)\right)$, a measure of the uncertainty about the agents appears through the variances σ_j^2 (see (12). Let assume that all the σ_j^2 are of same order σ^2. We have seen that in the limit of no uncertainty $\sigma^2 \rightarrow 0$, one recovers, at least for the quadratic approximation, the usual optimization dynamics. In addition to the fact that the classical case can be seen as a particular case of our model, we compare the advanges of the two approaches. Usually, one writes the first order condition for each $X_i\left(t\right)$, then postulates a form for the equilibrium dynamics, and solve the equation. The difficulty comes from the fact that, even if there is no optimization on the $X_i\left(t+n\right)$, $n>0$, those variables enters the dynamic equations, as a consequence of agents anticipations and possible information domination of some agents, and have to be replaced by the dynamic form of the solution. There is thus a circularity that implies difficulties to identify, analyticaly, the coefficients of this equilibrium dynamics.

Working with statistical weights avoids computing the solution for each agents. The probabilistic weights' exponential form ensures that actions are taken so that the action $X_i\left(t\right)$ and the planned action $X_i\left(t+n\right)$, for $n>0$, will be chosen in probability, close to their expected optimum. The process is performed each period again, with no commitment to previous expectations. In the end, this results in modeling the all system by the overall weight (18) and a dynamic centered around the classical optimum. The total effective utility includes the partial resolution of the agents expectations and strategic interaction with others.

Several use of the weight (18) (or 17) can be made. First, it can be seen as the exponential of an effective utility for the system, and as such, it can be used, to find the average path of the system. Actually, the probability $P\left(\left(X_i\left(t\right)\right) \mid \left(X_i\left(t-1\right)\right)\right)$ concentrates on its saddle point value which is given by the set of equations:

$$\nabla_{X_i(t)} U_{eff}\left(X_i\left(t\right),\left(X_i\left(t-1\right)\right)\right)=0 \tag{19}$$

where again, i runs over the set of agents. This is a usual Euler Lagrange type of equation, and as said before for quadratic utilities it leads to the usual linear dynamic solution. The computation of the eigenvalues of the dynamical system being in principle straightforward. Note that the solutions of (19) are different from the usual optimization paths for a non nul value of σ^2.

Some external shocks may also be directly included in this set up. Rather than considering:

$$\sum_i U_{eff}\left(X_i\left(t\right),\left(X_i\left(t-1\right)\right)\right)$$

as a full effective utility of the system, one can includes some perturbation terms:

$$\sum_i U_{eff}\left(X_i\left(t\right),\left(X_i\left(t-1\right)\right)\right)+X_i\left(t\right) L_{ik} \varepsilon_k\left(t\right)$$

where $\varepsilon_k\left(t\right)$ are some random external perturbations, and L_{ik} the response to this shocks for agent i. The dynamic equation thus becomes:

$$\nabla_{X_i(t)} U_{eff}\left(X_i\left(t\right),\left(X_i\left(t-1\right)\right)\right)+L_{ik} \varepsilon_k\left(t\right)=0$$

and in the case of linearized dynamics, the response to $\varepsilon_k\left(t\right)$ is simply:

$$\left(X_i\left(t+m\right)\right)=\left(X_i\left(t\right)\right)+D_{eff}^m\left(\varepsilon_i\left(t\right)\right)$$

where D_{eff} is the matrix describing the linear solution $\left(X_i\left(t+1\right)\right)=D_{eff}\left(X_i\left(t\right)\right)$ and the parenthesis $\left(\varepsilon_i\left(t\right)\right)$ denotes the vector of concatenated shocks.

There is however a second way to use the previous probabilistic description of the system that will be more central in this paper. Rather than focusing on the mean path approximation (18), we can come back to the transition probabilities and write them in the continuous approximation. Actually, whatever the normalization chosen for $U_{eff}\left(X_i(t),(X_i(t-1))\right)$ in (18), we can replace the lag variables $(X_i(t+l-1))$ by $((X_i(t+l-1))-(X_i(t+l)))+(X_i(t+l))$ and identify the difference $((X_i(t+l-1))-(X_i(t+l)))$ with minus the derivative $\frac{d}{dt}(X_i(t+l))=\left(\dot{X}_i(t+l)\right)$. We then obtain $P\left(\left(X_i^0(t+k)\right)\mid\left(X_i^0(t)\right)\right)$ in terms of the variables $(X_i(t+l))$, $\left(\dot{X}_i(t+l)\right)$:

$$P\left(\left(X_i^0(t+k)\right)\mid\left(X_i^0(t)\right)\right)=\int_{X_i(t)=\left(X_i^0\right)}^{X_i(t+k)=\left(X_i^0\right)}\exp\left(\int\sum_i U_{eff}\left((X_i(t)),\left(\dot{X}_i(t)\right)\right)\right)\mathcal{D}(X_i(t)) \tag{20}$$

for two given values of the initial and final state of the system $\left(X_i^0(t+k)\right)$ and $\left(X_i^0(t)\right)$. The integrand $\mathcal{D}(X_i(t))$ denotes the sum over all paths from $\left(X_i^0(t+k)\right)$ to $\left(X_i^0(t)\right)$ and the probability of transition between $\left(X_i^0(t+k)\right)$ and $\left(X_i^0(t)\right)$ is expressed as a path integral between those two points. We will come back to this approach in the third section. This formalism, familiar in theoretical physics appear in a wide range of models, ranging from Quantum Mechanics to statistical physics, and allows to go beyond, the "classical", or in our context, the average dynamics. The system may then be considered as a fully stochastic process, whose transition functions are given by (20). Such integrals are usually difficult to compute, except in the quadratic case. They can however yield many information on the probabilitic nature of the system, notably through several techniques such as perturbation theory, or Feynman graph expansion. Besides, path integrals have already been used in finance, to study the dynamics of stock market prices for example [8].

2.2 Basic example. Comparison with intertemporal optimization

Before developping some more general models, we start with a basic example and consider a system with two agents, with time t utility:

$$\begin{aligned}
u_y(y_t) &= -\left(\frac{1}{2}y_t^2-y_t x_{t-1}\right)\\
u_x(x_t) &= -\left(\frac{1}{2}x_t^2+\frac{1}{2}y_{t-1}^2-\alpha x_t y_{t-1}\right)
\end{aligned}$$

Note that this is the model developped in [4] where we considered a two agents interaction model:

$$\begin{aligned}
U_t^{(1)}(a_1(t)) &= -\frac{1}{2}(a_1(t)-a_0)^2-a_1(t)\,a_2(t-1)\\
U_t^{(2)}(a_2(t)) &= -\frac{\gamma}{2}(a_1(t-1))^2+\alpha a_1(t-1)\,a_2(t)-\frac{1}{2}(a_2(t))^2
\end{aligned}$$

where we set $\gamma=0$, $a_0=0$, to focus on the method of resolution. For comments and interpretations of the model, see [4]. The agents intertemporal utilities are:

$$\begin{aligned}
U_y(y_t) &= \sum_n \beta^n u_y(y_{t+n})\\
U_x(x_t) &= \sum_n \beta^n u_x(x_{t+n})
\end{aligned}$$

x_t has a strategic advantage on y_t which traduces here as a strategic - information - advantage. Agent x knows the utility of agent y and it's impact on y (coefficient -1) , as well as the impact of y on him (coefficient $-\alpha$). Agent y has no knowledge of agent x utility. It only knows the impact of x on itself, and this impact is perceived as the action of a random shock. This kind of model of interaction will be generalized in the next section. Let us remark that this type of model can also represent a dynamic version of the Stackelberg duopoly model. Actually, in a Stackelberg duopoly, the payoff are quadratics:

$$\begin{aligned}
\pi_1 &= Pq_1-c_1q_1\\
\pi_2 &= Pq_2-c_2q_2
\end{aligned}$$

Where the price is P and c_1, c_2 the costs, q_1 and q_2 are the quantities produced. Using the inverse demand function:

$$P = A - q_1 - q_2$$

One is lead to:

$$\begin{aligned}\pi_1 &= (A - q_1 - q_2 - c_1) q_1 \\ \pi_2 &= (A - q_1 - q_2 - c_2)\end{aligned}$$

In a dynamic version, agents would optimize the following functions. Given that in the Stackelberg setup, agent 2 has a strategic advantage and anticipates future actions of the first agent, the time t rewards become:

$$\begin{aligned}\pi_1 (t) &= (A - q_1 (t-1) - q_2 (t) - c_1) q_1 (t-1) \\ \pi_2 (t) &= (A - q_1 (t-1) - q_2 (t) - c_2) q_2 (t)\end{aligned}$$

The lag in $q_1 (t-1)$ transcripts the fact that agent 1 having a strategic advantage, it fixes first its quantity to match the demand at time t. Up to some constant and normalization, the functions $\pi_i (t)$ have the form of the model considered in this paragraph, except for the term $q_2 (t) q_1 (t-1)$ in $\pi_1 (t)$ that would need slight modification of our basic model (inducing some time translation in the computations of the effective utility for the first agent), but this is not our purpose here and this will be discussed in the next section.

Back to the resolution of our example, in the optimization set up, this model is solved with standard methods for optimization with rational expectations (here perfect information). Solving first for y_t

$$y_t = x_{t-1}$$

leads to an effective utility for x_t:

$$\frac{1}{2}x_t^2 + \frac{1}{2}x_{t-2}^2 - \alpha x_t x_{t-2}$$

and an intertemporal utility for x_t:

$$U_x (x_t) = \sum \beta^t \left(\frac{1}{2}x_t^2 \left(1+\beta^2\right) - \alpha x_t x_{t-2} - \alpha\beta^2 x_t x_{t+2}\right)$$

leads to the optimization equation:

$$x_t \left(1+\beta^2\right) - \alpha x_{t-2} - \alpha\beta^2 x_{t+2} = 0 \tag{21}$$

Postulating a solution of the type:

$$x_t = d x_{t-1}$$

leads to the characteristic equation:

$$\left(1+\beta^2\right) d^2 - \alpha - \alpha\beta^2 d^4$$

whose solution is:

$$d = \pm\sqrt{\frac{1}{2\alpha\beta^2}\left(1+\beta^2 - \sqrt{\left(1+\beta^2\right)^2 - 4\alpha^2\beta^2}\right)} \tag{22}$$

On the other side, we apply the formalization scheme developped in the previous paragraph, and then compare the results with the dynamic solution (22). We then need to compute the effective utilities for both agents x and y. We start with y and consider it's intertemporal utility:

$$U_y (y_t) = \sum_n \beta^n u_y (y_{t+n})$$

Given that y_t has no information about x, it will behave according to the statitical weight defined by:

$$\begin{aligned}\exp\left(U_{eff,y} (y_t)\right) &= \int \exp\left(\sum_n \beta^n u_y (y_{t+n})\right) \exp\left(-\frac{x_t^2}{\sigma^2}\right) \prod_{n\geqslant 0} dx_{t+n} dy_{t+n+1} \\ &= \int \exp\left(\sum_{n\geqslant 0} \beta^n \left(\frac{1}{2}y_{t+n}^2 - y_{t+n}x_{t+n-1}\right)\right) \exp\left(-\frac{x_{t+n-1}^2}{\sigma^2}\right) \prod_{n>0} dx_{t+n-1} dy_{t+n}\end{aligned}$$

The integrals
$$\int \exp\left(\beta^n\left(\frac{1}{2}y_{t+n}^2 - y_{t+n}x_{t+n-1}\right)\right)\exp\left(-\frac{x_{t+n-1}^2}{\sigma^2}\right)\prod_{n>0} dx_{t+n-1}dy_{t+n}$$
give a constant result, set to 1 after normalization, so that:
$$\int \exp\left(\sum_{n\geqslant 0}\beta^n\left(\frac{1}{2}y_{t+n}^2 - y_{t+n}x_{t+n-1}\right)\right)\exp\left(-\frac{x_{t+n-1}^2}{\sigma^2}\right)\prod_{n>0} dx_{t+n-1}dy_{t+n} = \exp\left(\left(\frac{1}{2}y_t^2 - y_t x_{t-1}\right)\right)$$
which translates in terms of effective utility:
$$\left(U_{eff,y}\left(y_t\right)\right) = \frac{1}{2}y_t^2 - y_t x_{t-1} = u\left(y_t\right)$$
The result previously stated is retrieved: the effective utility of an agent with no information is the initial time t utility.

Now we can compute the effective utility for agent x. Starting with it's intertemporal utility:
$$\begin{aligned}
U_x\left(x_t\right) &= \sum_{n\geqslant 0}\beta^n u_x\left(x_{t+n}\right)\\
&= \sum_{n\geqslant 0}\beta^n\left(\frac{1}{2}x_{t+n}^2 + \frac{1}{2}y_{t+n-1}^2 - \alpha x_{t+n}y_{t+n-1}\right)\\
&= \sum_{n\geqslant 0}^{n}\left(\frac{1}{2}\hat{x}_{t+n}^2 + \frac{1}{2}\beta\hat{y}_{t+n-1}^2 - \alpha\sqrt{\beta}\hat{x}_t\hat{y}_{t-1}\right)
\end{aligned}$$
where we changed the variables:
$$\begin{aligned}
\hat{x}_{t+n} &= \left(\sqrt{\beta}\right)^n x_{t+n}\\
\hat{y}_{t+n} &= \left(\sqrt{\beta}\right)^n y_{t+n}
\end{aligned}$$
we apply (11) and we are seeking for $\left(U_{eff,x}\left(x_t\right)\right)$ defined by:
$$\exp\left(U_{eff,x}\left(\hat{x}_t\right)\right) = \int \exp\left(U_x\left(\hat{x}_t\right)\right)\exp\left(\left(\sum_{n\geqslant 0}\frac{U_{eff,y}\left(\hat{y}_{t+n}\right)}{\sigma^2}\right)\right)\prod_{n>0} d\hat{y}_{t+n-1}d\hat{x}_{t+n} \tag{23}$$
where $U_{eff,y}\left(\hat{y}_{t+n}\right)$ has to be normalized. We set:
$$\exp\left(U_{eff,y}\left(y_t\right)\right) = \frac{\exp\left(\frac{1}{2}y_t^2 - y_t x_{t-1}\right)}{\mathcal{N}}$$
and impose:
$$\int \exp\left(U_{eff,y}\left(y_t\right)\right)dy_t = 1$$
which leads ultimately to find:
$$\exp\left(U_{eff,y}\left(y_t\right)\right) = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{1}{2}y_t^2 - y_t x_{t-1} + \frac{1}{2}x_{t-1}^2\right)$$

The factor $\frac{1}{\sqrt{2\pi}}$ is a constant factor and can be discarded from the computations and (23) becomes:

$$\begin{aligned}\exp\left(U_{eff,x}\left(\hat{x}_t\right)\right) &= \int \exp\left(U_x\left(\hat{x}_t\right)\right)\exp\left(\left(\sum_{n\geqslant 0}\frac{U_{eff,y}\left(\hat{y}_{t+n}\right)}{\sigma^2}\right)\right)\prod_{n>0}d\hat{y}_{t+n-1}d\hat{x}_{t+n} \qquad (24)\\ &= \int \exp\left(\sum_{n\geqslant 0}\left(\frac{1}{2}\hat{x}_{t+n}^2+\frac{1}{2}\beta\hat{y}_{t+n-1}^2-\alpha\sqrt{\beta}\hat{x}_{t+n}\hat{y}_{t+n-1}\right)\right)\\ &\quad\times\exp\left(\sum_{n\geqslant 0}^{n}\left(\frac{\frac{1}{2}\hat{y}_{t+n}^2-\sqrt{\beta}\hat{y}_{t+n}\hat{x}_{t+n-1}}{\sigma^2}+\frac{\beta\hat{x}_{t+n-1}^2}{2\sigma^2}\right)\right)\prod_{n>0}d\hat{y}_{t+n-1}d\hat{x}_{t+n}\end{aligned}$$

or, when the variables at time t and those at time $t+n$ are separated:

$$\begin{aligned}\exp\left(U_{eff,x}\left(\hat{x}_t\right)\right) &= \exp\left(\frac{1}{2}\left(1+\frac{\beta}{\sigma^2}\right)\hat{x}_t^2+\frac{1}{2}\left(\frac{1}{\sigma^2}+\beta\right)\hat{y}_t^2+\frac{1}{2}\beta\hat{y}_{t-1}^2-\alpha\sqrt{\beta}\hat{x}_t\hat{y}_{t-1}-\frac{\sqrt{\beta}}{\sigma^2}\hat{y}_t\hat{x}_{t-1}\right)\\ &\quad\times\int\exp\left(\sum_{n>0}\left(\frac{1}{2}\left(\left(1+\frac{\beta}{\sigma^2}\right)\hat{x}_{t+n}^2+\left(\frac{1}{\sigma^2}+\beta\right)\hat{y}_{t+n}^2\right)-\alpha\sqrt{\beta}\hat{x}_{t+n}\hat{y}_{t+n-1}-\frac{\sqrt{\beta}}{\sigma^2}\hat{y}_{t+n}\hat{x}_{t+n-1}\right)\right)\\ &\quad\times\prod_{n>0}d\hat{y}_{t+n}d\hat{x}_{t+n}d\hat{y}_t\end{aligned}$$

Now, define:

$$Y_t=\begin{pmatrix}\hat{x}_t\\ \hat{y}_t\end{pmatrix}$$

and the effective utility for $\hat{x}_t$ is written as:

$$\begin{aligned}\exp\left(U_{eff,x}\left(\hat{x}_t\right)\right) &= \exp\left(\frac{1}{2}Y_t^t\begin{pmatrix}\left(1+\frac{\beta}{\sigma^2}\right) & 0\\ 0 & \left(\frac{1}{\sigma^2}+\beta\right)\end{pmatrix}Y_t-\sqrt{\beta}Y_t^t\begin{pmatrix}0 & \alpha\\ \frac{1}{\sigma^2} & 0\end{pmatrix}Y_{t-1}\right) \qquad (25)\\ &\quad\times\int\exp\left(\sum_{n>0}\left(\frac{1}{2}Y_{t+n}^t\begin{pmatrix}\left(1+\frac{\beta}{\sigma^2}\right) & 0\\ 0 & \left(\frac{1}{\sigma^2}+\beta\right)\end{pmatrix}Y_{t+n}-\sqrt{\beta}Y_{t+n}^t\begin{pmatrix}0 & \alpha\\ \frac{1}{\sigma^2} & 0\end{pmatrix}Y_{t+n-1}\right)\right)\\ &\quad\times\prod_{n>0}d\hat{y}_{t+n}d\hat{x}_{t+n}d\hat{y}_t\end{aligned}$$

To compute the integrals we use a result about gaussian integrals for a path of variables $\left\{Y_{t+n}=\begin{pmatrix}\hat{x}_{t+n}\\ \hat{y}_{t+n}\end{pmatrix}\right\}_{n>0}$. This result states that the gaussian integrals $\prod_{n>0}d\hat{y}_{t+n}d\hat{x}_{t+n}$ are known to be equal to the (exponential of the) saddle point value of the integrand in the second exponential of (25), with initial condition $(\hat{x}_t,\hat{y}_t)$ and final value $(0,0)$ at $t=\infty$. More precisely,

$$\begin{aligned}&\int\exp\left(\sum_{n>0}\left(\frac{1}{2}Y_{t+n}^t\begin{pmatrix}\left(1+\frac{\beta}{\sigma^2}\right) & 0\\ 0 & \left(\frac{1}{\sigma^2}+\beta\right)\end{pmatrix}Y_{t+n}-\sqrt{\beta}Y_{t+n}^t\begin{pmatrix}0 & \alpha\\ \frac{1}{\sigma^2} & 0\end{pmatrix}Y_{t+n-1}\right)\right)\\ &\times\prod_{n>0}d\hat{y}_{t+n}d\hat{x}_{t+n}d\hat{y}_t\\ =\ &\exp\left(\text{Saddle point of}\left(S\right)\right)\end{aligned}$$

with:

$$S=\frac{1}{2}Y_{t+n}^t\begin{pmatrix}\left(1+\frac{\beta}{\sigma^2}\right) & 0\\ 0 & \left(\frac{1}{\sigma^2}+\beta\right)\end{pmatrix}Y_{t+n}-\sqrt{\beta}Y_{t+n}^t\begin{pmatrix}0 & \alpha\\ \frac{1}{\sigma^2} & 0\end{pmatrix}Y_{t+n-1}$$

where the saddle point solution satisfies the initial condition given just above.

To compute this saddle point value, define three matrices A, B and C with A symetric, and C antisymetric that allow to rewrite the integrand in the exponential as.

$$A = \begin{pmatrix} 0 & \alpha + \frac{1}{\sigma^2} \\ \alpha + \frac{1}{\sigma^2} & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & \alpha - \frac{1}{\sigma^2} \\ -\alpha + \frac{1}{\sigma^2} & 0 \end{pmatrix}$$

so that:

$$A + C = 2 \begin{pmatrix} 0 & \alpha \\ \frac{1}{\sigma^2} & 0 \end{pmatrix} \text{ and } A - C = 2 \begin{pmatrix} 0 & \frac{1}{\sigma^2} \\ \alpha & 0 \end{pmatrix}$$

The matrix B is defined by:

$$(B - A) = \begin{pmatrix} \left(1 + \frac{\beta}{\sigma^2}\right) & 0 \\ 0 & \left(\frac{1}{\sigma^2} + \beta\right) \end{pmatrix}$$

so that the quantity in the second exponential of the right hand side (25) is written as:

$$S = \frac{1}{2} \sum_{n>0} \left(Y_{t+n}^t (B - A) Y_{t+n} - \sqrt{\beta} Y_{t+n}^t (A + C) Y_{t+n-1} \right)$$

The saddle point equation is then:

$$2 (B - A) Y_{t+n} - \sqrt{\beta} A (Y_{t+n-1} + Y_{t+n+1}) - \sqrt{\beta} C ((Y_{t+n-1} - Y_{t+n+1})) \tag{26}$$

We look for a solution of this equation under the form:

$$Y_{t+n} = D Y_{t+n+1} \tag{27}$$

and the matrix D satisfies

$$-\sqrt{\beta} (A - C) D^2 + 2 (B - A) D - \sqrt{\beta} (A + C) = 0 \tag{28}$$

One can check that the solution D of (28) has the form:

$$D = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

and (28) leads to two equations for a and b:

$$\begin{aligned} a \left(\frac{1}{\sigma^2} \beta + 1 \right) - \alpha \sqrt{\beta} - \frac{1}{\sigma^2} ab \sqrt{\beta} &= 0 \\ b \left(\beta + \frac{1}{\sigma^2} \right) - \frac{1}{\sigma^2} \sqrt{\beta} - ab\alpha \sqrt{\beta} &= 0 \end{aligned}$$

whose solutions are:

$$a = \frac{1}{2\alpha\sqrt{\beta} (\sigma^2 + \beta)} \left(1 + \sigma^2\beta + \sigma^2\alpha^2\beta + \beta^2 - \sqrt{\left(\sigma^2\beta + \beta^2 - 2\alpha\beta - \sigma^2\alpha^2\beta + 1\right)\left(\sigma^2\beta + \beta^2 + 2\alpha\beta - \sigma^2\alpha^2\beta + 1\right)} \right) \tag{29}$$

$$\begin{aligned} b &= \frac{1}{a\sqrt{\beta}} \left(a\beta + \sigma^2 a - \sigma^2\alpha\sqrt{\beta} \right) \\ &= \left(\sigma^2 + \beta\right) \frac{\sigma^2\beta + \beta^2 - \sqrt{\left(\sigma^2\beta + \beta^2 - 2\alpha\beta - \sigma^2\alpha^2\beta + 1\right)\left(\sigma^2\beta + \beta^2 + 2\alpha\beta - \sigma^2\alpha^2\beta + 1\right)} - \sigma^2\alpha^2\beta + 1}{\sqrt{\beta}\left(\sigma^2\beta + \beta^2 - \sqrt{\left(\sigma^2\beta + \beta^2 - 2\alpha\beta - \sigma^2\alpha^2\beta + 1\right)\left(\sigma^2\beta + \beta^2 + 2\alpha\beta - \sigma^2\alpha^2\beta + 1\right)} + \sigma^2\alpha^2\beta + 1 \right)} \end{aligned}$$

Having found D, we replace these expressions in the saddle point solution (27). The effective utility $U_{eff,x}(\hat{x}_t)$ can then be obtained by:

$$\exp\left(U_{eff,x}(\hat{x}_t)\right) = \int \exp\left(\frac{1}{2} \left(\sum_{n>0} \left(Y_{t+n}^t (B - A) Y_{t+n} - \sqrt{\beta} Y_{t+n}^t (A + C) Y_{t+n-1} \right) \right) \right) d\hat{y}_t$$

where Y_{t+n} satisfies (27). The whole integrand

$$S=\frac{1}{2}\sum_{n>0}\left(Y_{t+n}^{t}\left(B-A\right)Y_{t+n}-\sqrt{\beta}Y_{t+n}^{t}\left(A+C\right)Y_{t+n-1}\right)\tag{30}$$

can then be simplified via the dynamic equation (28). This dynamic equation (28) rewrites:

$$\left(A-C\right)D^{2}+2\left(B-A\right)D+\left(A+C\right)=0$$

or, since D is invertible:

$$\left(A+C\right)D^{-1}=-\left(A-C\right)D-2\left(B-A\right)$$

the sum (30) simplifies as:

$$\begin{aligned}
&\frac{1}{2}\sum_{n>0}\left(Y_{t+n}^{t}\left(B-A\right)Y_{t+n}-\sqrt{\beta}Y_{t+n}^{t}\left(A+C\right)Y_{t+n-1}\right)\\
=\;&-\frac{\sqrt{\beta}}{2}Y_{t+1}^{t}\left(A+C\right)Y_{t}\\
&+\sum_{n>1}\frac{1}{2}Y_{t+n}^{t}\left(B-A\right)Y_{t+n}-\sum_{n>1}\frac{\sqrt{\beta}}{2}Y_{t+n}^{t}AY_{t+n-1}-\sum_{n>1}\frac{\sqrt{\beta}}{2}Y_{t+n}^{t}CY_{t+n-1}\\
=\;&-\frac{\sqrt{\beta}}{4}Y_{t+1}^{t}\left(A+C\right)Y_{t}\\
&+\sum_{n>1}\frac{1}{2}Y_{t+n}^{t}\left(\left(B-A\right)Y_{t+n}-\frac{\sqrt{\beta}}{2}A\left(Y_{t+n-1}+Y_{t+n+1}\right)-\frac{\sqrt{\beta}}{2}C\left(Y_{t+n-1}-Y_{t+n+1}\right)\right)\\
=\;&Y_{t+1}^{t}\left(A+C\right)Y_{t}\\
=\;&-\frac{\sqrt{\beta}}{4}Y_{t}^{t}\left(A-C\right)Y_{t+1}=-\frac{\sqrt{\beta}}{4}Y_{t}^{t}\left(A-C\right)DY_{t}
\end{aligned}$$

The second term vanishes, as a consequence of the dynamic equation (26). Then:

$$\begin{aligned}
&\sum_{n>0}\left(\frac{1}{2}Y_{t+n}^{t}\left(B-A\right)Y_{t+n}-\sqrt{\beta}Y_{t+n}^{t}\left(A+C\right)Y_{t+n-1}\right)\\
=\;&-\frac{\sqrt{\beta}}{2}Y_{t}^{t}\left(A-C\right)DY_{t}
\end{aligned}$$

and (24) rewrites:

$$\begin{aligned}
\exp\left(U_{eff,x}\left(\hat{x}_{t}\right)\right)&=\int\exp\left(\left(\frac{1}{2}Y_{t}^{t}\left(B-A\right)Y_{t}-\frac{\sqrt{\beta}}{2}Y_{t}^{t}\left(A+C\right)Y_{t-1}-\frac{\sqrt{\beta}}{4}Y_{t}^{t}\left(A-C\right)DY_{t}\right)\right)d\hat{y}_{t}\\
&=\int\exp\left(-\frac{1}{2}\left(Y_{t}^{t}\left(\left(B-A\right)-\frac{1}{2}\left(A-C\right)D\right)Y_{t}\right)-\frac{\sqrt{\beta}}{2}Y_{t}^{t}\left(A+C\right)Y_{t-1}\right)d\hat{y}_{t}
\end{aligned}$$

We can use again the dynamic equation for D:

$$\left(2\left(B-A\right)-\sqrt{\beta}\left(A-C\right)D\right)=\left(A+C\right)D^{-1}$$

and the previous relation becomes:

$$\begin{aligned}
\exp\left(U_{eff,x}\left(\hat{x}_{t}\right)\right)&=\int\exp\left(\frac{1}{4}\left(Y_{t}^{t}\left(A+C\right)D^{-1}Y_{t}\right)-\frac{1}{2}Y_{t}^{t}\left(A+C\right)Y_{t-1}\right)d\hat{y}_{t}\\
&=\int\exp\left(\frac{1}{4}\left(\left(Y_{t}-DY_{t-1}\right)^{t}\left(A+C\right)D^{-1}\left(Y_{t}-DY_{t-1}\right)\right)\right)d\hat{y}_{t}
\end{aligned}$$

The integration on $\hat{y}_t$ then leads to the following compact expression for $U_{eff}(x_t)$:

$$\begin{aligned} U_{eff}(x_t) &= \left(x_t - (DY_{t-1})_x\right)\left(\left((A+C)D^{-1}\right)_{xx} - \left((A+C)D^{-1}\right)_{xy}\left((A+C)D^{-1}\right)_{yy}\left((A+C)D^{-1}\right)_{yx}\left(x_t - (DY_{t-1})_x\right)\right. \\ &= (x_t - ay_{t-1}) N_{xx} (x_t - ay_{t-1}) \end{aligned}$$

where the subscript x means the coordinate of a vector (or a matrix) in the x direction. The matrix N_{xx} is defined by:

$$\begin{aligned} N_{xx} &= \left((A+C)D^{-1}\right)_{xx} - \left((A+C)D^{-1}\right)_{xy}\left((A+C)D^{-1}\right)_{yy}\left((A+C)D^{-1}\right)_{yx} \\ &= \frac{\alpha}{a} \end{aligned}$$

As a consequence, the full system is finally described by the probability weight:

$$\begin{aligned} &\exp\left(-U_{eff}(x_t) - U_{eff}(y_t)\right) \\ = &\exp\left(-(x_t - ay_{t-1})\frac{\alpha}{a}(x_t - ay_{t-1}) - \left(\frac{1}{2}y_t^2 - y_t x_{t-1}\right)\right) \end{aligned}$$

whose minimum is given by the dynamic equation:

$$\begin{aligned} x_t &= ay_{t-1} \\ y_t &= x_{t-1} \end{aligned}$$

thats is:

$$x_t = ax_{t-2}$$

At this point we have obtained the following result. All computations performed, the mean path followed by agent x is similar to the classical case, but with a different coefficient and this has to be compared with the usual resolution we obtained previously:

$$x_t = dx_{t-1}$$

and the coefficients a and d^2 were given by (22) and (29).

We perform the comparison through a power series expansion in β which allows to compare the effect of forward looking behavior in both models. Actually, as said previously, we know that both approach are identical for $\beta = 0$. This is checked directly here. Actually, at the fourth order:

$$\begin{aligned} d^2 &= \alpha + \beta^2\alpha\left(\alpha^2 - 1\right) + \beta^4\left(2\alpha^2 - 1\right)\alpha\left(\alpha^2 - 1\right) + O\left(\beta^5\right) \\ a &= \alpha + \beta^2\alpha\left(\alpha^2 - 1\right) + \sigma^2\alpha^2\left(\alpha^2 - 1\right)^2\beta^3 + \alpha\left(\alpha^2 - 1\right)\left(2\alpha^2 + \sigma^4\left(\alpha^2 - 1\right)^2 - 1\right)\beta^4 + O\left(\beta^5\right) \end{aligned}$$

For $\sigma^2 = 0$, d^2 and a coincide at all orders, and the usual result is recovered as announced in the previous paragraph. It corresponds to a system with no internal uncertainty and the usual optimization problem is recovered. For $\sigma^2 = 1$, which corresponds include an uncertainty in agent's behavior one finds:

$$a = \alpha + \beta^2\alpha\left(\alpha^2 - 1\right) + o\left(\beta^3\right) = d$$

To the second order, both approaches coincide. The case $\sigma^2 = 1$ is equivalent to the case in which the dominant agent x has full information about y. His knowledge about y's fluctuation are of same amplitude as his own, i.e. he knows the most that can be known about y.

At the third and fourth order, for $\sigma^2 > 0$, the results diverge, and $a > d^2$, this is the consequence of the inherent uncertainty of our model. Whatever the external signals, an internal randomness has been introduced in each agent behavior. This induces in turn fluctuations that destabilizes slightly the system compared to the usual analysis. Only when $\sigma^2 = 0$, For $\beta \to 0$, the two solutions coincide, as explained in the first section. The reason is straigntforward. For $\beta = 0$, in both formalization, agents only care about period t, and whatever their way to produce future forecasts, perfect, ordefined by statistical weight, it will be irrelevant.

For σ^2 large, the previous series expansion for a breaks down and we have to come back to:

$$\begin{aligned}
a &= \frac{1}{\alpha\beta\left(\sigma^2+\beta\right)}\left(\frac{1}{2}\sigma^2\beta+\frac{1}{2}\beta^2-\frac{1}{2}\sqrt{\left(\sigma^2\beta+\beta^2-2\alpha\beta-\sigma^2\alpha^2\beta+1\right)\left(\sigma^2\beta+\beta^2+2\alpha\beta-\sigma^2\alpha^2\beta+1\right)}+\frac{1}{2}\sigma^2\alpha^2\beta+\frac{1}{2}\right) \\
&= \frac{1}{\alpha}\left(\frac{1}{2}-\frac{1}{2}\sqrt{\left(1-\alpha^2\right)^2}+\frac{1}{2}\alpha^2\right) \text{ for } \left(1-\alpha^2\right)>0 \text{ and } \frac{1}{\alpha}\left(\frac{1}{2}+\frac{1}{2}\sqrt{\left(1-\alpha^2\right)^2}+\frac{1}{2}\alpha^2\right) \text{ for } \left(1-\alpha^2\right)<0 \\
&= \alpha
\end{aligned}$$

which is the result expected under no information. This is coherent: agent x information is of low relevance when σ^2 is large. This coincides also with the result for $\beta = 0$, since in that case agents dicard next periods and the consequences of their own actions.

Varying the parameter σ^2 therefore allows to interpolate between the full and no information schemes or, equivalently in this context, between a dynamic Stackelberg and a dynamic Cournot game.

This example suggests two conclusions. First, our scheme allows to switch continuously between a model with no internal uncertainty (the usual optimization problem) to another model including internal uncertainty about agents behavior. In other words, it allows to consider the quality of information at disposal for the agents as a parameter and interpolate between full and no information cases.

Our second conclusion concerns the resolution method. From the exposition above, the standard optimization method seems to yield a mire straightforward answer for the dynamics in the case of no internal uncertainty. From this standpoint, our formalism, eventhough more general, seems tedious in the $\sigma^2 \to 0$ case. However, its advantages become clear when the number of agents increases. Whereas solving the optimization equation (21) becomes harder when the number of agents increases, the dynamic equation (28) will keep the same form. This first order matricial equation will be easier to solve for some particular values of σ^2, such as $\sigma^2 = 1$, thus providing a tool to describe analyticaly the behavior of the agents in a whole range of systems. The dynamics thus obtained would differ from an optimization problem, but will remain centered around the classical solution, and can be seen as an approximation of this one. Let us also note that, however approximate, this "probability-based" solution is no less valid nor realsitic than the standard description of the agent behavior.

3 Application: Several interacting agents defined by a graph

3.1 Static model of several interacting agents.

Having presented the general formalism and described a representative example, we can now apply the above formalism to a general class of models that fit well with our approach. These type of models describe interactions between n heterogenous agents, some agents dominating informationally and strategically others. They are described by a graph ordering the agents by the relations of strategic domination among them (see [5]). They are equivalent to some dynamic games models, and are close to monopoly or oligopoly models. These models can also be used to describe dynamic patterns of decision for agents composed of several sub-structures (see [3][4][5][6]).

We will first present the static version of this class of model to introduce the agents' utility functions along with the domination graph that commands the resolution. We will then develop the dynamic version that will be used. Each agent's effective utilities are computed, to derive the whole system's effective utility. We then consider several examples.

3.2 Strategic relations between agents

The agents' strategic relations define the model setup. An oriented graph Γ whose vertices are labelled by the agents involved describe these relations. When *Agent* i has a strategic advantage over *Agent* j, we draw an oriented edge from i to j and write $i \rightarrow j$. If there exists an oriented path from i to j, we write the relation $i \rightarrowtail j$, and state that *Agent* i dominates directly or indirectly Agent j or, equivalently, that Agent j is subordinated to Agent i. If there is no oriented path from i to j, we write $j \not\leftarrowtail i$, where it is always understood that $i \neq j$. In the following, we merely consider connected graphs without loops.

3.3 Matricial formalism

Agents' utilities are described by the following matricial formalism. Agents' actions are encompassed in a vector of actions, or control variables. The number of possible actions determine the size of the vector. Utilities being quadratic, matrices may be associated with them.

Let $X_i \in R^{n_i}$ be Agent i's vector of control variables, and $\tilde{X}_j^{(i)} \in R^{n_i}$ the vector of goals associated with the variables X_j, as expected by agent i. We normalize $\tilde{X}_j^{(i)}$ to 0, so that Agent i wishes to achieve $X_i = 0$ and $X_j = \tilde{X}_j^{(i)}$. Agent $i's$ utility is given by:

$$\begin{aligned} U_i &= -\frac{1}{2}{}^tX_iA_{ii}^{(i)}X_i - \frac{1}{2}\sum_{j\leftarrowtail i}{}^t\left(X_j - \tilde{X}_j^{(i)}\right)A_{jj}^{(i)}\left(X_j - \tilde{X}_j^{(i)}\right) \\ &\quad -\sum_{j\leftarrowtail i}{}^tX_iA_{ij}^{(i)}\left(X_j - \tilde{X}_j^{(i)}\right) - \sum_{j\not\leftarrowtail i}{}^t\left(X_i - \tilde{X}_i^{(j)}\right)A_{ij}^{(i)}X_j \end{aligned} \tag{31}$$

In the absence of any interaction, Agent $i's$ utility is given by the term

$$-\frac{1}{2}{}^tX_iA_{ii}^{(i)}X_i$$

The variables X_i are normalized so that $A_{ii}^{(i)}$ is a $f_i \times n_i$ diagonal matrix whose coefficients are 1 or 0.

If Agent $i's$ subordinate agents' actions X_j depart from $\tilde{X}_j^{(i)}$, Agent $i's$ will experience a loss of utility of the form :

$$\sum_{j\leftarrowtail i}{}^t\left(X_j - \tilde{X}_j^{(i)}\right)A_{jj}^{(i)}\left(X_j - \tilde{X}_j^{(i)}\right)$$

The $f_j \times n_j$ matrix $A_{jj}^{(i)}$ of parameters is of course symmetric.

The impact of *Agent* j's action on *Agent* i's utility is

$$\sum_{j\leftarrow i} {}^tX_i A_{ij}^{(i)} X_j - \sum_{j\not\leftarrow i} {}^t\left(X_i - \tilde{X}_i^{(j)}\right) A_{ij}^{(i)} X_j$$

where $j \leftarrow i$ can be seen as the impact of Agent $j's$ action on Agent i. In our model, Agent j does not know the agents to whom he is subordinated, and processes their signals as external ones. The second term models the strain imposed on Agent i by Agent j to achieve its own objectives for X_i.

Remark 1 *Since the linear term in X_j disappears during the resolution,*

$$\sum_{j\not\leftarrow i} {}^tX_i A_{ij}^{(i)} X_j$$

is equivalent to

$$\sum_{j\not\leftarrow i} {}^t\left(X_i - \tilde{X}_i^{(j)}\right) A_{ij}^{(i)} X_j$$

Notation 2 *By convention, for the $n_i \times n_j$ parameters matrices $A_{ij}^{(i)}$, we will write ${}^tA_{ij}^{(i)} = A_{ji}^{(i)}$.*

3.4 Dynamic version

This section describes the general model for dynamics interacting structures. We adapt the the procedure of the previous paragraph by transforming the matricial static utilities in a dynamic context, and assuming each agent optimizes a forward-looking intertemporal utility function, given it's own information set.
The intertemporal utility is of the form :

$$V_i(t) = \sum_{m\geq 0} \beta_i^m E_i^t U_i\left(t+m\right)$$

where β_i is *Agent* i's discount factor, and E_i^t his conditional expectation at time t. Agents compute their expectations according to the following information pattern. $U_i\left(t+m\right)$ is period $t+m$ utility and is a dynamic version of the static form (31), where the previous remark allows to set $\tilde{X}_i^{(j)} = 0$.

$$\begin{aligned} U_i\left(t+m\right) &= -\frac{1}{2} X_i^t\left(t+m\right) A_{ii}^{(i)} X_i\left(t+m\right) \\ &\quad -\frac{1}{2}\sum_{j\leftarrow i}\left(X_j^t\left(t+m-1\right)-\tilde{X}_j^{(i)}\right) A_{jj}^{(i)}\left(X_j\left(t+m-1\right)-\tilde{X}_j^{(i)}\right) \\ &\quad -\sum_{j\leftarrow i} X_i^t\left(t+m\right) A_{ij}^{(i)}\left(X_j\left(t+m-1\right)-\tilde{X}_j^{(i)}\right) \\ &\quad -\sum_{j\not\leftarrow i} X_i^t\left(t+m\right) A_{ij}^{(i)} X_j\left(t+m-1\right) \end{aligned} \tag{32}$$

Which is, up to some constant irrelevant term, a straightforward generalization of the static model utility function. Actually, in a dynamic context, we consider that agent i perceives external and other agents' signals with a one period delay.

Concatenating $X_i(t+k)$ and the vectors $X_j(t+k)$ for all $j \leftarrow i$ in one normalized column vector, we rewrite the utilities:

$$Y_i\left(t+k\right) = \left(\beta^{\frac{k}{2}}\left(X_j\left(t+k\right)-\bar{X}_j^{(i)}\right)_{j\leq i}\right)$$

where, by convention $\bar{X}_i^{(i)} = 0$, $\bar{X}_j^{(i)} = \tilde{X}_j^{(i)}$, $j < i$. We work now with the system of variables $Y_i\left(t\right)$. For all $i \geqslant j$, $i = j$, one has the following map

$$\iota : X_j\left(t+k\right) \hookrightarrow Y_i\left(t+k\right)$$

defined by: $\iota\left(\beta^{\frac{k}{2}}X_j\left(t+k\right)\right) = \left(0,...,\beta^{\frac{k}{2}}\left(X_j\left(t\right)_{j\geqslant i}-\bar{X}_j^{(i)}\right),0,...0\right)$. Similarly, we define the injection $\iota':Y_j\left(t+k\right)\hookrightarrow Y_i\left(t+k\right)$, given by $\iota'\left(Y_j\left(t+k\right)\right)=\left(\beta^{\frac{k}{2}}\left(X_j\left(t+k\right)_{k\geqslant j}-\bar{X}_j^{(i)}\right),0,...0\right)$.

When there is no ambiguity, we will still write $X_i(t+k)$ and $X_j(t+k)$ for the images of these vectors by these injections. In other words $X_i(t+k)=\left(Y_i\left(t+k\right)\right)_i$ and $X_j(t+k)=\left(Y_i\left(t+k\right)\right)_j$ are the i-th et j-th components of $Y_i\left(t+k\right)$ respectively.

With these conventions, the utilities rewrite:

$$
\begin{aligned}
U_t^{(i)} &= \sum_{k\geqslant 0}\beta^k\left(\begin{array}{c} X_i\left(t+k\right)A_{ii}^{(i)}X_i\left(t+k\right)+\sum_{j<i}\left(\left(X_j\left(t+k-1\right)-\bar{X}_j^{(i)}\right)A_{jj}^{(i)}\left(X_j\left(t+k-1\right)-\bar{X}_j^{(i)}\right)\right) \\ +2X_i\left(t+k\right)A_{ij}^{(i)}\left(X_j\left(t+k-1\right)-\bar{X}_j^{(i)}\right)+\sum_{j>i}2X_i\left(t+k\right)A_{ij}^{(i)}\left(X_j\left(t+k-1\right)\right)\end{array}\right) \\
&= \sum_{k\geqslant 0}Y_i\left(t+k\right)\left(\begin{array}{cc} A_{ii}^{(i)} & 0 \\ 0 & 0\end{array}\right)Y_i\left(t+k\right)+Y_i\left(t+k-1\right)\left(\begin{array}{cc} 0 & 0 \\ 0 & \beta A_{\{jj\}}^{(i)}\end{array}\right)Y_i\left(t+k-1\right) \\
&\quad +Y_i\left(t+k\right)\left(\begin{array}{cc} 0 & \beta^{\frac{1}{2}}A_{ij}^{(i)} \\ \beta^{\frac{1}{2}}A_{ji}^{(i)} & 0\end{array}\right)Y_i\left(t+k-1\right) \\
&\quad +\sum_{j>i}2X_i\left(t+k\right)A_{ij}^{(i)}\left(X_j\left(t+k-1\right)\right)
\end{aligned}
$$

We will also add possibility for an inertia term:

$$-X_i\left(t\right)\epsilon_{ii}^{(i)}X_i\left(t-1\right)$$

to obtain:

$$
\begin{aligned}
U_t^{(i)} &= \sum_{k\geqslant 0}Y_i\left(t+k\right)\left(\begin{array}{cc} A_{ii}^{(i)} & 0 \\ 0 & 0\end{array}\right)Y_i\left(t+k\right)+Y_i\left(t+k-1\right)\left(\begin{array}{cc} 0 & 0 \\ 0 & \beta A_{\{jj\}}^{(i)}\end{array}\right)Y_i\left(t+k-1\right) \\
&\quad +Y_i\left(t+k\right)\left(\begin{array}{cc} -\beta^{\frac{1}{2}}\epsilon_{ii}^{(i)} & \beta^{\frac{1}{2}}A_{ij}^{(i)} \\ \beta^{\frac{1}{2}}A_{ji}^{(i)} & 0\end{array}\right)Y_i\left(t+k-1\right) \\
&\quad +\sum_{j>i}2X_i\left(t+k\right)A_{ij}^{(i)}\left(X_j\left(t+k-1\right)\right)
\end{aligned}
$$

3.5 Pattern of information

The full resolution of the model relies on agents' expectations, that is agents' information sets or parameters knowledge. The pattern of information over the domination graph we propose describes how agents perform their forecasts. Each agent knows the domination relations of the subtree he strategically dominates, but ignores the reactivity of the subtree's agents to external, non dominated agents. In other words, Agent i knows the values of the $A_{k\ell}^{(k)}$ for $i\rightarrowtail k$ and $i\rightarrowtail \ell$. The remaining coefficients $A_{k\ell}^{(k)}$ are forecasted to 0 for this agent. Remark that, under our assumptions, agents do not attribute a probability to the coefficients they forecast, but rather a fixed value.

We moreover assume that, at each period t, Agent i knows the signals $X_j(t-1)$ for $i\rightarrowtail j$ and $X_j(t-1)$ for $j\not\leftarrowtail i$ by which he is affected. From these hypotheses, we can infer some results about the agents' forecasts. First, Agent i forecasts to 0 all the actions of agents he does not dominate. That is, for $j\not\leftarrowtail i$ and $m\geq 0$ one has:

$$E_i^t X_j\left(t+m\right)=0$$

This condition will allow to simplify some computations when computing the effective action of agent "i". The action variables $X_j\left(t+m\right)$ for $j\not\leftarrowtail i$ will be discarded.

We conclude this paragraph by remarking that In the case of oligopoly interpretation, the pattern of information chosen ulimately determines which kind of game is played, Stackelberg, Cournot...

3.6 Effective utility

As explained in the previous section, each agent j behaves at time t with a so called effective utility $U_{eff}(X_j(t)) \equiv U_{eff}(X_j)$ whose form is found recursively. As shown before, for the less informed agents - those for which $X_i(t) = Y_i(t)$ - the non normalized effective utility reduces to time t utility:

$$U_{eff}(X_i(t)) = \sum_t Y_i(t) A_{ii}^{(i)} Y_i(t) - \sqrt{\beta} Y_i(t-1) \epsilon_{ii}^{(i)} Y_i(t-1) + \sum_{j>i} 2X_i(t) A_{ij}^{(i)} (X_j(t-1))$$

The previous section has shown that $(U_{eff}(X_i(t)))$, the effective utility that determines the probability of behavior of agents who dominate others informationially is given by (11):

$$\exp(U_{eff}(X_i(t))) = \int \exp\left(U_t^{(i)}\right) \prod_{rk(j)<rk(i)} \prod_{s\geqslant t} \exp\left(\sum_{s\geqslant t} \frac{U_{eff}(X_j(s))}{\sigma_j^2}\right) dX_j(s) \tag{33}$$

Appendix 2 proves that, after coming back to the variable $X_i(t)$, the non-normalized effective utilities solving (33) have the form:

$$\begin{aligned} U_{eff}(X_j(s)) &= Y_j^e(s) \begin{pmatrix} N_{ii} & 0 \\ 0 & 0 \end{pmatrix} Y_j^e(s) - 2Y_j^e(s) \begin{pmatrix} M_{ii} & M_{ij} \\ 0 & 0 \end{pmatrix} Y_j^e(s-1) \\ &+ \sum_{i\geqslant k>j} 2\beta^{s-t} X_j(s) A_{jk}^{(j)} (X_k(s-1)) \end{aligned}$$

with:

$$Y_j^{(e)}(t+k) = \left(\beta^{\frac{k}{2}} \left(X_k(t+k) - \bar{X}_k^{(j)e}\right)_{k\leq j}\right) \tag{34}$$

where $\bar{X}_k^{(j)e}$ is the effective goal of j for k. Appendix 2 provides a formula for the effective goal given the parameters of the model, and proves that $U_{eff}(X_i(t))$ is given by:

$$\begin{aligned} U_{eff}(X_i(t)) &= \frac{1}{2}\left(X_i(t) - \bar{X}_i^{(i)e}\right) N_{ii} \left(X_i(t) - \bar{X}_i^{(i)e}\right) - \left(X_i(t) - \bar{X}_i^{(i)e}\right) \frac{M_{ii}}{\sqrt{\beta}} \left(X_i(t-1) - \bar{X}_i^{(i)e}\right) \\ &- \sum_{j<i} \left(X_i(t) - \bar{X}_i^{(i)e}\right) \frac{M_{ij}}{\sqrt{\beta}} \left(X_j(t-1) - \bar{X}_j^{(i)e}\right) + \sum_{j>i} X_i(t) A_{ij}^{(i)} (X_j(t-1)) \end{aligned} \tag{35}$$

The matrices M_{ii}, M_{ij}, N_{ii}, also computed in Appendix 2, are:

$$\begin{aligned} N_{ii} &= ((A-C)(D-2)+2B)_{ii}^S \\ &- ((A-C)(D-2)+2B)_{ij}^S \left(((A-C)(D-2)+2B)_{jj}^S\right)^{-1} \left(((A-C)(D-2)+2B)_{ji}^S\right) \\ M_{ii} &= (N_{ii}) \left(\left(((A-C)(D-2)+2B)^S\right)^{-1} (A+C)\right)_{ii} \\ M_{ij} &= (N_{ii}) \left(\left(((A-C)(D-2)+2B)^S\right)^{-1} (A+C)\right)_{ij} \end{aligned} \tag{36}$$

where S stands for the symetrized matrix, and with:

$$A = \sqrt{\beta}\begin{pmatrix} -\epsilon_{ii}^{(i)} & A_{ij}^{(i)}+A_{ij}^{(j)} \\ A_{ji}^{(i)}+A_{ji}^{(j)} & \left\{\begin{array}{c} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ A_{\{kj\}i>k>j}^{(j)}, A_{\{jk\}i>k>j}^{(j)} \end{array}\right\} \end{pmatrix}$$

$$B = \begin{pmatrix} A_{ii}^{(i)}+B_{11}-\sqrt{\beta}\epsilon_{ii}^{(i)} & \left\{\sqrt{\beta}\left(A_{ij}^{(i)}+A_{ij}^{(j)}\right), B_{12}\right\} \\ \left\{\sqrt{\beta}\left(A_{ji}^{(i)}+A_{ji}^{(j)}\right), B_{12}^{t}\right\} & \left\{\begin{array}{c} \beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22} \\ \sqrt{\beta}\left\{\begin{array}{c} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2} \\ , A_{\{kj\}i>k>j}^{(j)}, A_{\{jk\}i>k>j}^{(j)} \end{array}\right\} \end{array}\right\} \end{pmatrix}$$

$$C = \sqrt{\beta}\begin{pmatrix} 0 & A_{ij}^{(i)}-A_{ij}^{(j)} \\ -\left(A_{ji}^{(i)}-A_{ji}^{(j)}\right) & \left\{\begin{array}{c} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ A_{\{kj\}i>k>j}^{(j)}, -A_{\{jk\}i>k>j}^{(j)} \end{array}\right\} \end{pmatrix}$$

and

$$\begin{aligned} B_{11} &= \beta A_{ij}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{ji}^{(j)} \\ B_{12} &= \left\{\beta A_{ij}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{jk}^{(j)}, \beta\left(A_{ij}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\left(\begin{array}{cc}\left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff}\end{array}\right)\right)\right\} \\ B_{22} &= \left\{\begin{array}{c} \beta A_{lj}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{jk}^{(j)}, \\ \beta\left(\begin{array}{cc}\left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff}\end{array}\right)^{t}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\left(\begin{array}{cc}\left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff}\end{array}\right), \\ \beta\left(A_{kj}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\left(\begin{array}{cc}\left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff}\end{array}\right)\right)^{S} \end{array}\right\} \end{aligned}$$

It is shown in Appendix 3 that the matrix D satisfies the dynamic equation:

$$(A-C)D^{2}+2(B-A)D+(A+C)=0 \tag{37}$$

The notation {} used here is convenient to describe concatenated blocks of matrices such as for example $\left\{\begin{array}{c} \beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22} \\ \sqrt{\beta}\left\{\begin{array}{c} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2} \\ , A_{\{kj\}i>k>j}^{(j)}, A_{\{jk\}i>k>j}^{(j)} \end{array}\right\} \end{array}\right\}$ to refer to matrices $\beta A_{jj}^{(i)}$, B_{22},... that are concatenated in a larger one, say M. The matrix M is built by concatenating the matrices $\beta A_{jj}^{(i)}$, B_{22}, that are pasted given their indices. The dimension of M will thus be implicitely determined by its constituing matrices. For example $\beta A_{jj}^{(i)}$ has elements along the coordinates (j,j). When several matrices have elements at the same place in M, these elements are simply added.

Alternatively one can also represent the effective utility as:

$$\begin{aligned} U_{eff}(X_i(t)) = & \frac{1}{2}\dot{X}_i(t)M_{ii}\dot{X}_i(t)-\left(X_i(t)-\left(\hat{Y}_i^{(1)}\right)_i\right)M_{ij}\left(X_j(t-1)-\left(\hat{Y}_i^{(1)}\right)_j\right) \\ & +\frac{1}{2}\left(X_i(t)-\left(\hat{Y}_i^{(1)}\right)_i\right)\hat{N}_{ii}\left(X_i(t)-\left(\hat{Y}_i^{(1)}\right)_i\right) \end{aligned}$$

with:

$$\hat{N}_{ii}=N_{ii}+\frac{1}{2}M_{ii}$$

$\dot{X}_i(t)$ refers to the discrete derivative, that is $\dot{X}_i(t)=X_i(t)-X_i(t-1)$.

Remark that (35) is not in a normalized form. The normalization can be achieved by imposing that:

$$\int \exp\left(-U_{eff}\left(X_i\left(t\right)\right)\right) dX_i\left(t\right) = 1$$

and this implies:

$$\begin{aligned} U_{eff}\left(X_i\left(t\right)\right) &= \frac{1}{2}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right) N_{ii}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right) \\ &-\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)\frac{M_{ii}}{\sqrt{\beta}}\left(X_i\left(t-1\right)-\bar{X}_i^{(i)e}\right) \\ &-\sum_{j<i}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)\frac{M_{ij}}{\sqrt{\beta}}\left(X_j\left(t-1\right)-\bar{X}_j^{(i)e}\right)+\sum_{j>i}X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right) \\ &-\frac{1}{2}\left(\frac{M_{ii}}{\sqrt{\beta}}\left(X_i\left(t-1\right)-\bar{X}_i^{(i)e}\right)+\sum_{j<i}\frac{M_{ij}}{\sqrt{\beta}}\left(X_j\left(t-1\right)-\bar{X}_j^{(i)e}\right)+\sum_{j>i}A_{ij}^{(i)}\left(X_j\left(t-1\right)\right)\right)^t \\ &\times\left(N_{ii}\right)^{-1}\times\left(\frac{M_{ii}}{\sqrt{\beta}}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)+\sum_{j<i}\frac{M_{ij}}{\sqrt{\beta}}\left(X_j\left(t-1\right)-\bar{X}_j^{(i)e}\right)+\sum_{j>i}A_{ij}^{(i)}\left(X_j\left(t-1\right)\right)\right) \\ &-\ln\det\left(N_{ii}\right) \end{aligned} \tag{38}$$

However the terms depending on contributions for $j > i$ may be discarded due to our pattern of information, in which $X_j\left(t\right)$ with $j > i$ is considered as a random noise by agent i. We are then left with:

$$\begin{aligned} U_{eff}\left(X_i\left(t\right)\right) &= \frac{1}{2}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right) N_{ii}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right) \\ &-\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)\frac{M_{ii}}{\sqrt{\beta}}\left(X_i\left(t-1\right)-\bar{X}_i^{(i)e}\right) \\ &-\sum_{j<i}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)\frac{M_{ij}}{\sqrt{\beta}}\left(X_j\left(t-1\right)-\bar{X}_j^{(i)e}\right)+\sum_{j>i}X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right) \\ &-\frac{1}{2}\left(\frac{M_{ii}}{\sqrt{\beta}}\left(X_i\left(t-1\right)-\bar{X}_i^{(i)e}\right)+\sum_{j<i}\frac{M_{ij}}{\sqrt{\beta}}\left(X_j\left(t-1\right)-\bar{X}_j^{(i)e}\right)\right)^t \\ &\times\left(N_{ii}\right)^{-1}\times\left(\frac{M_{ii}}{\sqrt{\beta}}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)+\sum_{j<i}\frac{M_{ij}}{\sqrt{\beta}}\left(X_j\left(t-1\right)-\bar{X}_j^{(i)e}\right)\right)-\ln\det\left(N_{ii}\right) \end{aligned} \tag{39}$$

and this more precise form is used when needed to compute conditional expectations.

More about this point and the derivation of the normalization is given in Appendix 2. But let us now consider an application of (39). The important point is that the effective utility remains quadratic, after integrating both anticipations and interactions between agents.

The probability associated to that utility is then:

$$\propto \exp\left(U_{eff}\left(Y_i\left(t\right)\right)\right)$$

Remark that the effective utilities for $X_i\left(t\right)$ depend on, and implicitely include the discount factor that was previously absorbed in the definition of, $Y_i\left(t\right)$. Considering again (11) and using (4) means that (recall the notation $X_i(t+k) = (Y_i(t+k))_i$ and $X_j(t+k) = (Y_i(t+k))_j$):

$$\begin{aligned}
& P\left((X_j(t+1))_{j\neq i}, ..., (X_j(t+n))_{j\neq i}, ... \mid X_i(t)\right) \\
= \; & E_i^t \prod_k P\left((X_j(t+k+1))_{j\neq i} \mid (X_j(t+k))_{j\neq i}\right) d(X_j(t+k))_{j\neq i} \\
= \; & \exp\left(\sum_{j<i}\left(\sum_{k\geqslant 0} Y_j(t)\begin{pmatrix} N_{ii} & 0 \\ 0 & 0 \end{pmatrix} Y_j(t) - 2Y_j(t)\begin{pmatrix} M_{ii} & M_{ij} \\ 0 & 0 \end{pmatrix} Y_j(t-1)\right.\right. \\
& \left.\left. + \sum_{i>k>j} 2(Y_j(t))_j A_{jk}^{(j)} (Y_k(t-1))_k\right)\right)
\end{aligned}$$

then rewriting this expression in terms of the initial variables X_i, X_j and including the normalization:

$$\begin{aligned}
& P\left((X_j(t+1))_{j\neq i}, ..., (X_j(t+n))_{j\neq i}, ... \mid X_i(t)\right) \\
= \; & \frac{1}{2}\left(X_j(t) - \bar{X}_j^{(j)e}\right)\left(N_{jj} - M_{jj}(N_{jj})^{-1} M_{jj}\right)\left(X_j(t) - \bar{X}_j^{(j)e}\right) \\
& -\frac{1}{2}\sum_{i\geqslant k\geqslant j}\left(X_j(t) - \bar{X}_j^{(k)e}\right) N_{kk}\left(X_j(t) - \bar{X}_j^{(k)e}\right) \\
& -\frac{1}{2}\sum_{i\geqslant k\geqslant j}\left(X_k(t) - \bar{X}_k^{(k)e}\right)^t M_{kk}N_{kk}M_{kj}\left(X_j(t) - \bar{X}_j^{(k)e}\right) \\
& -\frac{1}{2}\sum_{i\geqslant k\geqslant j}\left(X_j(t) - \bar{X}_j^{(k)e}\right)^t M_{jk}N_{kk}M_{kk}\left(X_k(t) - \bar{X}_k^{(k)e}\right) \\
& -\left(X_j(t) - \bar{X}_j^{(j)e}\right)\frac{M_{jj}}{\sqrt{\beta}}\left(X_j(t-1) - \bar{X}_j^{(j)e}\right) \\
& -\sum_{k<j}\left(X_j(t) - \bar{X}_j^{(j)e}\right)\frac{M_{jk}}{\sqrt{\beta}}\left(X_k(t-1) - \bar{X}_k^{(j)e}\right)
\end{aligned}$$

that is, the probability of future values $X_j(t+k)$, $j \leqslant i$ presents a discount behavior. The uncertainty for future values is increased by the relative absence of concern for future periods.

3.7 Effective action for the system

Having found the non normalized form for agent i effective utility in (35):

$$\begin{aligned}
U_{eff}(X_i(t)) \; = \; & \frac{1}{2}\left(X_i(t) - \bar{X}_i^{(i)e}\right) N_{ii}\left(X_i(t) - \bar{X}_i^{(i)e}\right) - \left(X_i(t) - \bar{X}_i^{(i)e}\right)\frac{M_{ii}}{\sqrt{\beta}}\left(X_i(t-1) - \bar{X}_i^{(i)e}\right) \\
& -\sum_{j<i}\left(X_i(t) - \bar{X}_i^{(i)e}\right)\frac{M_{ij}}{\sqrt{\beta}}\left(X_j(t-1) - \bar{X}_j^{(i)e}\right) + \sum_{j>i} X_i(t) A_{ij}^{(i)}(X_j(t-1))
\end{aligned}$$

we form the effective utility for the set of all agents by summing over i:

$$\sum_i U_{eff}(X_i(t))$$

At this point some precisions have to be added. In the previous expression, one could sum over the normalized utilities defined by:

$$\int \exp\left(-U_{eff}(X_i(t))\right) dX_i(t) = 1 \tag{40}$$

Normalizing the effective utilities was legitimate when computing $U_{eff}(X_i(t))$. Actually to perform its "random" optimization process each agent was attributing a probability to each other agent's action, so that the normalization was needed. But now, all computations done, $U_{eff}(X_i(t))$ describes the utility of a "blind" agent, since all anticipations are included in the form of $U_{eff}(X_i(t))$. These agents participate to a system composed of N interconnected parts, and for this global system the different periods are connected. This is similar, at the individual level, to our procedure attributing a single weight corresponding to the intertemporal utility.

One can check that imposing (40) would correspond, on average, to let all agents optimize $U_{eff}(X_i(t))$ independently. In other words, the normalization condition amounts to consider independent agents. However, once the effective utilities have been computed, the agents' forward-lookingness, computational skills and rationality have been fully taken into account and are included within the form of the effective utility. From this point onward, agents cannot be considered as independent anymore, but must rather be considered as integral and "blind" parts of a global system, whose elements are interconnected through the different periods.

In probability terms, it means that each agent utility at each period can't be normalized independently from the others, but only the probability defined by the all path. As such, only a joint probability has to be defined, and the normalization is performed over all agents and the all set of periods. As a consequence, at the utility level, we will consider the intertemporal effective utility for the system as

$$\sum_t \sum_i U_{eff}(X_i(t))$$

where, in the previous expression, we use the non normalized individual utilities. The global probability weight considered, will be, up to a global normalization:

$$\exp\left(\sum_t \sum_i U_{eff}(X_i(t))\right)$$

it describes he system as a whole, whose weight relates all parts of it and all periods as related. Of course, summing over all agents except i and all periods after t would lead us to retrieve $U_{eff}(X_i(t))$ (plus past contribution that would disappear in a normalization) as needed.

Remark also that this effective utility can be modified by adding also interaction terms between the agents, that were not taken into account in the derivation of effective utility for any of them. It represents a system where each agent has adapted his behavior given it's information, but this one about the all system is incomplete, even for the most informed agents.

By summing over i the expressions in (35) and reordering the sums over agents, one obtains the following expression for the global weight a time t:

$$\begin{aligned}
&U_{eff}((X_i(t))) \\
= \;& \sum_i \left(\frac{1}{2}\left(X_i(t)-\bar{X}_i^{(i)e}\right)N_{ii}\left(X_i(t)-\bar{X}_i^{(i)e}\right)-\left(X_i(t)-\bar{X}_i^{(i)e}\right)\frac{M_{ii}}{\sqrt{\beta}}\left(X_i(t-1)-\bar{X}_i^{(i)e}\right)\right. \\
&\left.-\sum_{j<i}\left(X_i(t)-\bar{X}_i^{(i)e}\right)\frac{M_{ij}}{\sqrt{\beta}}\left(X_j(t-1)-\bar{X}_j^{(i)e}\right)+\sum_{j>i}X_i(t)A_{ij}^{(i)}(X_j(t-1))\right)
\end{aligned}$$

Define the $\bar{X}_i^e$ as the stationnary solution of the saddle point equation. They satisfy the following system

$$\left(\frac{1}{2}N_{ii}-\frac{M_{ii}}{\sqrt{\beta}}\right)\left(\bar{X}_i^e-\bar{X}_i^{(i)e}\right)+\sum_{j>i}\left(A_{ij}^{(i)}\bar{X}_j^e-\frac{M_{ij}}{\sqrt{\beta}}\left(\bar{X}_j^e-\bar{X}_j^{(j)e}\right)\right)+\sum_{j<i}\left(A_{ij}^{(j)}\bar{X}_j^e-\frac{M_{ij}}{\sqrt{\beta}}\left(\bar{X}_j^e-\bar{X}_j^{(i)e}\right)\right)=0$$

that can be rewritten as:

$$\left(\frac{1}{2}N_{ii}-\frac{M_{ii}}{\sqrt{\beta}}\right)\bar{X}_i^e+\sum_{j\neq i}\left(A_{ij}^{(i)}-\frac{M_{ij}}{\sqrt{\beta}}\right)\bar{X}_j^e=-\left(\sum_{j>i}\frac{M_{ij}}{\sqrt{\beta}}\bar{X}_j^{(j)e}+\sum_{j<i}\frac{M_{ij}}{\sqrt{\beta}}\bar{X}_j^{(i)e}\right)$$

It can be solved as:

$$\bar{X}_i^e = -\left(G^{-1}\right)_{ik}\left(\sum_{j>k}\frac{M_{kj}}{\sqrt{\beta}}\bar{X}_j^{(j)e} + \sum_{j<k}\frac{M_{kj}}{\sqrt{\beta}}\bar{X}_j^{(k)e}\right)$$

with G the concatenated matrix defined by:

$$G_{ij} = \left(\frac{1}{2}N_{ii} - \frac{M_{ii}}{\sqrt{\beta}}\right)\delta_{ij} + \left(1-\delta_{ij}\right)\left(A_{ij}^{(i)} - \frac{M_{ij}}{\sqrt{\beta}}\right)$$

Then define $X(t)$, the concatenation of the $X_i(t)$ and $\bar{X}^e$ the concatenation of the $\bar{X}_i^e$. Then, the total effective action rewrites:

$$\begin{aligned}
U_{eff}\left(X\left(t\right)\right) &= \frac{1}{2}\left(X\left(t\right)-\bar{X}^e\right)N\left(X\left(t\right)-\bar{X}^e\right) \\
&\quad -\left(X\left(t\right)-\bar{X}^e\right)\frac{M+O}{\sqrt{\beta}}\left(X\left(t-1\right)-\bar{X}^e\right)+U_{eff}\left(\bar{X}^e\right) \\
&= \frac{1}{2}\left(X\left(t\right)-\bar{X}^e\right)N\left(X\left(t\right)-\bar{X}^e\right)-\left(X\left(t\right)-\bar{X}^e\right)\frac{M+O}{\sqrt{\beta}}\left(X\left(t-1\right)-\bar{X}^e\right) \\
&\quad +\left(\frac{1}{2}\bar{X}^eN\bar{X}^e-\bar{X}^e\frac{M+O}{\sqrt{\beta}}\bar{X}^e-\sum_{j<i}\bar{X}_i^e\frac{M_{ij}}{\sqrt{\beta}}\left(\bar{X}_j^e-\bar{X}_j^{(i)e}\right)\right)
\end{aligned} \tag{41}$$

with:

$$\begin{aligned}
N &= \left(N_{ii}\right) \\
M &= \left(M_{ii}\right) \\
O_{ij} &= A_{ij}^{(j)} - M_{ij}^{(i)} \text{ if } j<i \\
O_{ij} &= A_{ij}^{(i)} - M_{ij}^{(j)} \text{ if } j>i
\end{aligned}$$

The second term coming from the general property of a quadratic form plus linear term:

$$q\left(X\right) = XAX + XBX_0$$

for X_0 a constant vector. If $\bar{X}$ is the saddle point of $q(X)$, one can rewrite:

$$q\left(X\right) = \left(X-\bar{X}\right)A\left(X-\bar{X}\right)+q\left(\bar{X}\right)$$

The quadratic term $U_{eff}\left(\bar{X}^e\right)$ is constant and irrelevant when considering the dynamic over a given time span T. Its contibution to the effective utility is a constant $TU_{eff}\left(\bar{X}^e\right)$ that can be discarded. However, later we will look at a statistical set of processes with a variable time span T. In that case this term will play a role when comparing and averagit over these processes. Note ultimately that $TU_{eff}\left(\bar{X}^e\right)$ can be negative, which will be the most interesting case for us. It corresponds to a lowered effective utility, with respect to 0 as a benchmark case, consequence of internal tension between the different elements composing the system.

Having found the general form for the effective utility, we now describe several examples including different patterns of strategic dominations.

3.8 Example: N non strategic agents

Consider the simplest example/case where N agents have no information nor strategic advantage. In this "N non strategic agents case", which is actually equivalent to a Cournot oligopoly, the utility of each agent is:

$$U_t^{(i)} = -\sum_t \beta^t\left(X_i\left(t\right)A_{ii}^{(i)}X_i\left(t\right)\right)+\sum_{j\neq i}2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right)$$

where the individual goals of any agent has been set to 0 for the sake of simplicity. The agents being non strategic, other agents' actions are perceived as mere external perturbations. In that situation, $X_j(s)$ for

$s>t$ is seen as a variable independent from $X_i(t)$. As such the integrals over these variables does not affect the part of the utility depending on $X_i(t)$ and, as explained in the first section:

$$U_{eff}(X_i(t)) = -X_i(t)A_{ii}^{(i)}X_i(t) + \sum_{j\neq i} 2X_i(t)A_{ij}^{(i)}(X_j(t-1))$$

So that the global weight is:

$$\exp\left(\sum_t\sum_i U_{eff}(X_i(t))\right) = \exp\left(\sum_t\sum_i\left(-X_i(t)A_{ii}^{(i)}X_i(t) + \sum_{j\neq i}2X_i(t)A_{ij}^{(i)}(X_j(t-1))\right)\right)$$

As a consequence the probability for the system path is centered around the minimum of:

$$\sum_t\sum_i\left(-X_i(t)A_{ii}^{(i)}X_i(t) + \sum_{j\neq i}2X_i(t)A_{ij}^{(i)}(X_j(t-1))\right)$$

and this minimum satisfies:

$$A_{ii}^{(i)}X_i(t) = \sum_{j\neq i}A_{ij}^{(i)}(X_j(t-1))$$

for all t. This dynamic equation is the usual optimization of individual utilities. Our method thus reproduces the classical optimization problem, including, through the probability distribution, a modelling of random perturbations on the system. The reason is the following: the absence of any information about the others leads the agents to behave independently from the others. Arguably, under no information, agents tend to behave independently, inducing their actions to be randomly distributed around the individual optimums.

3.9 Example: N+1 agents. Domination of one on the others

This case is a generalization of the basic example of section one. It could be interpreted as a Stackelberg oligopoly with one dominant agent. For the first, least strategic, type of agent, the procedure is the same as in the previous example, and its effective utility will be its time t utility:

$$U_{eff}(X_j(t)) = -X_1(t)A_{jj}^{(j)}X_1(t) + 2X_j(t)A_{jk}^{(j)}(X_k(t-1)) + 2X_j(t)A_{j1}^{(j)}(X_1(t-1))$$

we assume that $A_{jj}^{(j)} = A_{jj}^{(1)} = 1$, $A_{kj}^{(j)} = \alpha$ for all j and k, including $j=1$ or $k=1$.

For the strategic agent, on the other hand, the effective action (35):

$$\begin{aligned}U_{eff}(X_i(t)) &= \frac{1}{2}\left(X_i(t)-\bar{X}_i^{(i)e}\right)N_{ii}\left(X_i(t)-\bar{X}_i^{(i)e}\right) - \left(X_i(t)-\bar{X}_i^{(i)e}\right)\frac{M_{ii}}{\sqrt{\beta}}\left(X_i(t-1)-\bar{X}_i^{(i)e}\right)\\ &\quad -\sum_{j<i}\left(X_i(t)-\bar{X}_i^{(i)e}\right)\frac{M_{ij}}{\sqrt{\beta}}\left(X_j(t-1)-\bar{X}_j^{(i)e}\right) + \sum_{j>i}X_i(t)A_{ij}^{(i)}(X_j(t-1))\end{aligned}$$

is computed using the formula (36) given in the previous paragraph. The matrices M_{ii}, M_{ij}, N_{ii} are computed in Appendix 2 and listed above in (36).

We show in Appendix 5 that we obtain (we record the results for $N>1$ and the case $N=1$ is presented in the same Appendix):

$$
\begin{aligned}
N_{11} &= \left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1} \\
&\quad -\beta^2\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2 \\
&\quad \times\left(\frac{\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2N\frac{NV+W}{N-1}}\right)N^2}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2N\frac{NV+W}{N-1}}\right)\right)}\right. \\
&\quad \left.-\frac{N\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2N\frac{NV+W}{N-1}}\right)\right)}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2N\frac{NV+W}{N-1}}\right)\right)}\right) \\
M_{11} &= -\left(N_{11}\right)\frac{\alpha\sqrt{\beta}\beta N\left(1+2\beta-\beta W\right)\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)}{\left(1+\beta\alpha^2\right)+\beta\alpha^2N\frac{NV+W}{N-1}}}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2N\frac{NV+W}{N-1}}\right)\right)} \\
M_{ij} &= -\left(N_{11}\right)\left(1,..,1\right)\frac{\sqrt{\beta}\left(N-1\right)\beta\left(1+2\beta-\beta W\right)\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)}{\left(1+\beta\alpha^2\right)+\beta\alpha^2N\frac{NV+W}{N-1}}}{\left(\left(1+2\beta\right)-W\right)\left(\left(1+2\beta\right)-W+N\left(\left(V\left(N-1\right)+W+\beta\left(N-2\right)\right)-\frac{\left(\alpha\left(NV+W\right)+\beta\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\alpha^2N\frac{NV+W}{N-1}}\right)\right)}
\end{aligned}
$$

with:

$$
W=\frac{1}{2\beta}\left(1+2\beta-\sqrt{4\beta^2+1}\right)
$$

and V satisfies:

$$
\begin{aligned}
&N\beta\frac{\left(N-1\right)^2+N\alpha^2\left(1+\beta\right)}{N-1}V^2 \\
&+\frac{2\beta\left(\left(N-1\right)^2+N\alpha^2\left(1+\beta\right)\right)+\left(\left(2+\beta\right)N\left(N-1\right)\alpha^2\beta+\left(N-3\right)N^2\beta+\left(4N-2\right)\beta+N-1\right)}{N-1}V \\
&+\frac{\left(\left(N-1\right)\beta+\left(1+\beta\right)\alpha^2\beta\right)W^2+\beta\left(N-1\right)\left(N+2\alpha^2+\alpha^2\beta-2\right)W+\left(N-1\right)\left(\left(N-1\right)\alpha^2\beta+1\right)}{N-1}
\end{aligned}
$$

The full action for the system of agents is thus:

$$
\begin{aligned}
U_{eff}\left(X_j\left(t\right)\right)+U_{eff}\left(X_i\left(t\right)\right) &= \sum_{j<1}\left(-X_j\left(t\right)A_{jj}^{(j)}X_j\left(t\right)+2X_j\left(t\right)A_{jk}^{(j)}\left(X_k\left(t-1\right)\right)+2X_j\left(t\right)A_{j1}^{(j)}\left(X_1\left(t-1\right)\right)\right) \\
&\quad +\frac{1}{2}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)N_{ii}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)-\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)\frac{M_{ii}}{\sqrt{\beta}}\left(X_i\left(t-1\right)-\bar{X}_i^{(i)e}\right) \\
&\quad -\sum_{j<i}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)\frac{M_{ij}}{\sqrt{\beta}}\left(X_j\left(t-1\right)-\bar{X}_j^{(i)e}\right)
\end{aligned}
$$

The average dynamics is the saddle path of the previous global effective utility and is thus given by the dynamic evolution:

$$
\left(\begin{array}{c} X_i\left(t\right)-\bar{X}_i^{(i)e} \\ X_j\left(t\right)\end{array}\right)=M_1\left(\begin{array}{c} X_i\left(t-1\right)-\bar{X}_i^{(i)e} \\ X_j\left(t-1\right)-\bar{X}_j^{(i)e}\end{array}\right)+M_2\left(\begin{array}{c} X_i\left(t-1\right) \\ X_j\left(t-1\right)\end{array}\right)
$$

with:

$$
\begin{aligned}
M_1 &= \left(\begin{array}{cc} \left(N_{11}\right)^{-1}M_{11} & \left(N_{11}\right)^{-1}M_{1j} \\ 0 & 0\end{array}\right) \\
M_2 &= \left(\begin{array}{cc} 0 & 0 \\ \alpha\left(1,..,1\right)^t & \left(1\right)-1\end{array}\right)
\end{aligned}
$$

where we denote by (1) the matrix filled with 1 in every row. We are mainly interested in the dynamical pattern of the system and we will thus set $\bar{X}_i^{(i)e} = \bar{X}_j^{(i)e} = 0$, so that the equilibrium is for $X_i(t) = X_i(t) = 0$.

The dynamical pattern is then determined by M and it's eigenvalues, and Appendix 5 shows that:

$$\begin{aligned} M &= \begin{pmatrix} (N_{11})^{-1} M_{11} & (N_{11})^{-1} M_{1j} \\ \alpha & (1) - 1 \end{pmatrix} \\ &= \begin{pmatrix} -\alpha\sqrt{\beta} N m & -(1,..,1)(N-1)m \\ \alpha(1,..,1)^t & (1) - 1 \end{pmatrix} \end{aligned}$$

with:

$$m = \frac{(2\beta - W + 1)\frac{(\alpha(NV+W)+\beta(N-1))}{\alpha^2 N \frac{NV+W}{N-1} + (1+\beta\alpha^2)}}{((1+2\beta) - W)\left((1+2\beta) - W + N\left((V(N-1) + W + \beta(N-2)) - \frac{(\alpha(NV+W)+\beta(N-1))^2}{(1+\beta\alpha^2)+\alpha^2 N\frac{NV+W}{N-1}}\right)\right)}$$

The eigenvalues of M are:

$$-1, \frac{1}{2}(a+1) \pm \frac{1}{2}\sqrt{a^2 - 2(N-1)a + \frac{4N(N-1)}{\sqrt{\beta}}a}$$

with:

$$a = -\frac{\alpha\sqrt{\beta}N(2\beta - W + 1)\frac{(\alpha(NV+W)+\beta(N-1))}{\alpha^2 N \frac{NV+W}{N-1} + (1+\beta\alpha^2)}}{((1+2\beta) - W)\left((1+2\beta) - W + N\left((V(N-1) + W + \beta(N-2)) - \frac{(\alpha(NV+W)+\beta(N-1))^2}{(1+\beta\alpha^2)+\alpha^2 N\frac{NV+W}{N-1}}\right)\right)}$$

The full study of the dynamical pattern as a function of the parameters being beyond the scope of this paper, we will merely draw the main characteristics of the results. First, the fact that eigenvalues are propotional to α means that interactions between dominated agents create instability in the system. Second, when β is relatively small, $W \simeq 1 - \beta$ and $(2\beta - W + 1) \simeq 3\beta$. This implies that a grows with β, at least for relatively low values of this parameter. Interactions may thus become unstable when agents grow more forwardlooking and attempt to drive the system toward their optimum. Finally, the larger is N, the more unstable the system is, as shown by the term proportional to $N(N-1)$ in the square root, and the fact that a can be proved to be of constant magnitude when N increases. Moreover, for large N, the eigenvalues become imaginary, so that the system presents an oscillatory pattern. The interpretation is that a large number of dominated agents produces fluctuations further amplified by mutual interactions. Under such a setting, no single dominating agent may stabilize the system.

3.10 Example: the three structure model.

This case considers GLW model involving three agents ranked by their relations of strategic advantage.

Each agent optimizes, given it's own information set, a forward-looking intertemporal utility function of the form:

$$V_i(t) = \sum_{m \geq 0} \beta_i^m E_i^t U_i(t+m)$$

The forecasts by Agent i of future quantities is computed given its information set.
The utilities take the following dynamic form:

$$\begin{aligned} U_B(t) &= -\frac{1}{2}(n(t) + 1 - w(t-1))^2 - \alpha n(t) s_n(t-1) \qquad (42) \\ U_U(t) &= -\frac{1}{2}\rho\left(1 - w(t-1) - \widetilde{f}\right)^2 - \frac{1}{2}\gamma(w(t-1) - \widetilde{w})^2 - \frac{1}{2}s_n^2(t) - \frac{1}{2}s_f^2(t) - \frac{1}{2}s_w^2(t) \\ U_C(t) &= -\frac{1}{2}(w(t) - w_0)^2 - \frac{1}{2}\delta n^2(t-1) - \upsilon n(t-1) w(t) - \kappa s_f(t-1)\left(1 - w(t) - \widetilde{f}\right) - \eta s_w(t-1)\left(w(t) - \widetilde{t}\right) \end{aligned}$$

under the constraint: $w+f=1$.

Note that in each of the above utilities the agent own action variables appear with a time index t, as expected for utility at time t, whereas other agents' action variables appear with a time index $t-1$

Utilities are quadratic and normalized so that the terms containing the square control variables have coefficients of $-\frac{1}{2}$ or 0.

The reasons for these choices, as well as the interpretation of the variables is detailled in reference GLW. We give a short acount now.

The utility of the body The body, being an automaton, has no specific goals, and its utility function U_{B} merely describes its reaction to other agents' actions[1]. Without any interaction with the unconscious U, the body would, in first approximation, react linearly to the conscious C action, "feeding" :

$$-\frac{1}{2}\left(n\left(t\right)+1-w\left(t-1\right)\right)^{2}$$

The unconscious influences the body by perturbating its signal

$$-\alpha n s_{n}$$

Whereas in the absence of the unconscious, the body's optimum would be reached for

$$n=-f=0$$

This result being suboptimal for *Agent U*, he will tilt the equilibrium toward its own goal $\tilde{f}$.

Recall that the task performed by the conscious w is not physically demanding, and has no impact on the body's response n. Indeed, we do not model physical efforts per se, but rather seek to understand how the unconscious can manipulate an existing equilibrium between the body and the conscious, i.e. the use of body signals by the unconscious to reach its own goals. By convention α is positive, so that a positive strain will respond to a positive feeding.

The utility of the conscious In the absence of both the unconscious and the body, the conscious' utility would be :

$$-\frac{1}{2}\left(w-w_{0}\right)^{2}$$

so that in the absence of any constraint set on w, *Agent C* would optimally choose $w=w_{0}>0$.

Body needs affect *Agent C* through

$$-\frac{1}{2}\delta n^{2}-\nu n w$$

so that the higher is the need, the more painful is the task.

In the absence of *Agent U*, *Agent C* sets $w=0$ by adjusting the feeding to the anticipated need. The need is in itself painful since:

$$-\frac{1}{2}\delta n^{2}$$

so we set

$$\delta>0$$

[1] In this setting, endowing the body with specific goals would have allowed it to manipulate the conscious, which was not our purpose here.

The above assumption is a direct consequence of dismissing any cost to the feeding f. Here we depart from standard models where costs, or constraints, are imposed to an agent's tasks.Without *Agent U*, *Agent B* and f could be discarded from *Agent C*'s equilibrium. Once *Agent U* is included in the system, it indirectly manipulates *Agent C* through *Agent B* by assigning a strategic role to f. However we impose a binding constraint on the feeding by considering f and w as complementary activities within a given time span, and set $f+w=1$, as previously mentioned. The unconscious imposes its goals $\tilde{f}$ and $\tilde{w}$ on the conscious through perturbation terms:

$$-\kappa s_f(f-\tilde{f})-\eta s_w\left(w-\tilde{w}\right)$$

driving *Agent C*'s actions away from 0 and towards $\tilde{f}$ and $\tilde{w}$.

Some additional technical conditions on U_{C} will prove convenient. We will ensure that U_{C} is negative definite and has an optimum by setting :

$$\delta-\nu^2>0$$

Furthermore, excessive working combined with unsatisfied needs should induce a loss in *Agent C* utility. This is implemented by imposing:

$$\nu>0 \text{ for } n>0 \text{ and } w>0$$

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The utility of the unconscious Agents, conscious or unconscious, build their interpretation of a situation - and thus its utility function - through an own, specific, grid of lecture. See ([4]) for further details.
Agent U and *Agent C* will therefore have two completely different interpretations of a single situation. And while *Agent C* will consider f and w as optimal, *Agent U* will consider other levels of the conscious' activity, $\tilde{f}$, $\tilde{w}$ as optimal.
Agent U's goals with respect to *Agent C*'s activity are:

$$-\frac{1}{2}\rho(f-\tilde{f})^2-\frac{1}{2}\gamma(w-\tilde{w})^2$$

To insure that U_{U} can have an optimum, we further impose ρ and γ to be positive.

Since the three agents are sub-structures of one single individual, a strain inflicted by one agent ends up being painful for all. The costs incurred are :

$$-\frac{1}{2}\left(s_n^2+s_f^2+s_w^2\right)$$

The information setup follows the order of domination among agents. For the sake of clarity we do not present here the information set up. It will be fully described in the resolution of the general model. *Agent B*, the less informed of all agents, is only aware of the strains he's affected by. *Agent C* is aware of it's own influence on *Agent B*, and of the strains *Agent U* puts on him. *Agent U*, the most informed of all agents, knows the utilities function of both *Agent C* and *Agent B*.
The instantaneous utility $U_i(t+m)$ at time $t+m$ reproduces the model described in previous papers. We assume that each action taken at time t by any agent will only be perceived by the other agents at time $t+1$.

3.10.1 Resolution

Following our general procedure in this case presents the same pattern as in the previous example. We compute first the effective utility for the least informed agent, namely B, then for agent C and ultimately for agent U. Then all these effective utilities are gathered to form the effective utility of the all system. All computations are performed in Appendix 6, and they result in the following..The effective utility for the system is:

$$\begin{aligned} U_{eff} &= \left(s(t)-\left(s^{(3)}\right)_{eff}\right)N_{ii}\left(s(t)-\left(s^{(3)}\right)_{eff}\right)-\left(s(t)-\left(s^{(3)}\right)_{eff}\right)M_{ij}\left(\begin{array}{c} w(t-1)-\left(w^{(3)}\right)_{eff} \\ n(t-1)-\left(n^{(3)}\right)_{eff}\end{array}\right) \\ &\quad +(1-c)\,w^2(t)+2\nu w(t)\,n(t-1)+\kappa s_f(t-1)\left(1-w(t)-\tilde{f}\right)+\eta s_w(t-1)\left(w(t)-\tilde{t}\right) \\ &\quad +(n(t))^2-2n(t)\,w(t-1)+2\alpha n(t)\left(\begin{array}{ccc} 1 & 0 & 0\end{array}\right)s(t-1)\end{aligned}$$

Appendix 6 displays the computations leading to coefficients matrices N_{ii}, M_{ij} and c and constants $\left(s^{(3)}\right)_{eff}$, $\left(w^{(3)}\right)_{eff}$. The average dynamics for such system has the standard form

$$X(t)\,MX(t-1) \tag{43}$$

and the matrix M has three nul eigenvalues, and the two others satisfy:

$$\lambda=\pm\sqrt{\sigma^2\nu\left(d+\beta\nu^2-bd\nu^2\right)}\times\sqrt{\frac{Num}{Den}}$$

with:

$$\begin{aligned} Num &= d\alpha\beta\omega\sigma^4+\left(d^2\alpha+f\omega d\alpha^2\beta+d\alpha\beta+fd+\omega\alpha^3\beta^2\right)\sigma^2 \\ &\quad +\left(d\alpha^3\beta-d\alpha^3\beta^2-d^2\alpha^3\beta-fd\alpha^2\beta+\alpha^3\beta^2+f\alpha^2\beta\right)\end{aligned}$$

$$\begin{aligned} Den &= d\left(\left(bd\nu^2-\beta\nu^2-d\right)\sigma^2-r^2\left(\beta-bd\right)\right) \\ &\quad \times\left(d\alpha\beta\omega\sigma^4+\left(d^2\alpha+f\omega d\alpha^2\beta+d\alpha\beta+fd+\omega\alpha^3\beta^2\right)\sigma^2\right. \\ &\quad \left.+\left(fd^2\alpha^2-d\alpha^3\beta^2+d\alpha^3\beta-fd\alpha^2\beta+\alpha^3\beta^2+f\alpha^2\beta\right)\right)\end{aligned}$$

where σ^2 is the degree of uncertainty in agents behavior defined before when designing the effective utilities.

The interpretation is similar to [5][6] : Agent B reacts to Agent C's feeding in a 1 to 1 ratio, and Agent C's will react to Agent B's need with a ratio ν, so that both agents' actions will be multiplied by over a two-period horizon. Agent U's action paying only over a two to three-periods horizon, it is irrelevant when $\beta=0$, and prevents Agent U from taking it. The myopic behavior among agents leads to an oscillatory dynamics. Each agent, reacting sequentially, adjusts its action to undo other agents' previous actions. This describes cyclical and apparently inconsistent or irrational behaviors in the dual agent. These oscillations may diverge or fade away with time, depending on the value of ν. When β is different from 0 but relatively small, the system is still oscillatory. When β increases, the time concern will have an ambiguous effect on its stability. Agent U would tend to stabilize the system through the indirect chanel, but the sensitivity of agent C, may impair this possibility and the stability of the system depends on the relative strength of the parameters.

However, as explained previously, our method providing an interpolation between full certainty and full uncertainty, one can study how the parameter σ^2 influences the results. To do so, we compare the results for the classical dynamics for various degree of uncertainty σ^2 in agents behaviors. We look at three examples, mild uncertainty $\sigma^2=1$, full uncertainty, $\sigma^2\rightarrow\infty$, no uncertainty $\sigma^2\rightarrow 0$, which converges to the classical case. The most interesting case for us will be $\sigma^2=1$, the two others one being bechmarks cases. The parameters and eigenvalues of the model for these cases are listed in Appendix 6, we only keep here the main results.

For $\sigma^2\rightarrow 0$, one finds for the system's eigenvalues, to the second order in β:

$$\lambda=\pm\sqrt{-\frac{\nu}{d}}=\pm\sqrt{-\nu}\left(1-\frac{\beta^2}{2}\left(\delta-\nu^2\right)\right)+O\left(\beta^3\right)$$

and we recover the classical results as needed. This confirms the fact that in the case of no uncertainty, one recover usual optimization results. For the interpretation of this result, see ([6]).

For $\sigma^2 \to \infty$, one obtains:

$$\lambda = \pm\sqrt{-\nu}$$

and the interpretation is straightforward: this results is the same as for $\sigma^2 = 0$, $\beta = 0$. When the agents are facing a full uncertainty concerning the future behaviors, it behaves with a myopic reaction: reacting only to past signals, and not anticipating about the future.

As said before, the case for $\sigma^2 = 1$ is the most interesting for us, since in general it corresponds to what we aim at modeling: agents anticipating other agents, but taking into account for uncertain intrinsic behaviors. The computations to the second order in β, simplify to yield the following values for the parameters:

$$\lambda = \pm\sqrt{-\nu} - \frac{1}{2}\beta^2\sqrt{-\nu}\left(\omega r^2 + \delta - \nu^2\right) + O\left(\beta^3\right)$$

In that case, with respect to the benchmark case $\sigma^2 \to 0$, the amplitude of the oscillations increase. The agents forecasts others, and take into account their behavior in their action. But the increased internal uncertainties increase in turn the internal fluctuations between the agents. The more uncertain the future actions, the more agents react to the information at their diposal.

4 General form for the effective action

Previous sections show that each agent is described by an effective utility $U_{eff}\left(X_i\left(t\right), X_j\left(t-1\right)\right)$ and a probability $\exp\left(U_{eff}\left(X_i\left(t\right), X_j\left(t-1\right)\right)\right)$. We have seen that $U_{eff}\left(X_i\left(t\right), X_j\left(t-1\right)\right)$ can be computed explicitly for a quadratic utility and is then itself quadratic. If agent's utility $U_t^{(i)}$ is not quadratic, the successive integrals defining $U_{eff}\left(X_i\left(t\right), X_j\left(t-1\right)\right)$ do not simplify, but we propose an approximate formula for the effective utility that we will justify from the model point of view.

Relaxing the condition of quadratic utility, we set the following intertemporal utility:

$$\begin{aligned} U_t^{(i)} &= -\sum_k \beta^k \left(\left(V_i^{(i)}\left(X_i\left(t+k\right)\right) + \sum_{j<i}\left(V_j^{(i)}\left(X_j\left(\left(t+k\right)-1\right)\right)\right) + 2X_i\left(\left(t+k\right)\right)A_{ij}^{(i)}\left(X_j\left(\left(t+k\right)-1\right)\right)\right)\right) \\ &\quad + \sum_{j>i} 2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right) \end{aligned}$$

Where $V_i^{(i)}\left(X_i\left(t\right)\right)$ and $V_j^{(i)}\left(X_j\left(t-1\right)\right)$ are agents i and j arbitrary utilities. We have kept quadratic interaction terms (or linear response) between agents. We assume that each agent respond linearly to the external perturbations.

It is useful to rewrite $U_t^{(i)}$ with the variables $Y_i\left(t\right)$ introduced in the previous section, adding the possibility of an inertia term $\epsilon_{ii}^{(i)}$:

$$\begin{aligned} U_t^{(i)} &= \sum_{k\geqslant 0} Y_i\left(t+k\right)\begin{pmatrix} A_{ii}^{(i)} & 0 \\ 0 & 0 \end{pmatrix} Y_i\left(t+k\right) + Y_i\left(t+k-1\right)\begin{pmatrix} -\epsilon_{ii}^{(i)} & 0 \\ 0 & \beta A_{\{jj\}}^{(i)} \end{pmatrix} Y_i\left(t+k-1\right) \qquad (44) \\ &\quad + Y_i\left(t+k\right)\begin{pmatrix} 0 & \beta^{\frac{1}{2}} A_{ij}^{(i)} \\ \beta^{\frac{1}{2}} A_{ji}^{(i)} & 0 \end{pmatrix} Y_i\left(t+k-1\right) \\ &\quad + \sum_{j>i} 2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right) + \sum_{k\geqslant 0}\beta^k V_i^{(i)}\left(\frac{X_i\left(t+k\right)}{\beta^{\frac{k}{2}}}\right) + \sum_{j<i}\left(V_j^{(i)}\left(\frac{Y_j\left(\left(t+k\right)-1\right)}{\beta^{\frac{k-1}{2}}}\right)\right) \end{aligned}$$

Using the procedure given in the first section, we find recursively the effective utility $U_{eff}\left(X_i\left(t\right)\right)$. It is computed trough the integrals in (11):

$$\exp\left(U_{eff}\left(X_i\left(t\right)\right)\right) = \int \exp\left(U_t^{(i)}\right)\prod_{rk(j)<rk(i)}\prod_{s\geqslant t}\exp\left(\sum_{s\geqslant t}\frac{U_{eff}\left(X_j\left(s\right)\right)}{\sigma^2}\right) dX_j\left(s\right)dX_i\left(s+1\right)$$

and depends on the effective utility $U_{eff}(X_j(s))$ where $rk(j) < rk(i)$. We prove in appendix 7 that $U_{eff}(X_j(s))$ has the form:

$$\begin{aligned} U_{eff}(X_i(t)) &= Y_j(t)\begin{pmatrix} N_{ii} & 0 \\ 0 & 0 \end{pmatrix} Y_j(t) - 2Y_j(t)\begin{pmatrix} M_{ii} & M_{ij} \\ 0 & 0 \end{pmatrix} Y_j(t-1) + V_{eff}^{(j)}(Y_j(t)) \\ &\quad + \sum_{k>j} 2X_j(t) A_{jk}^{(j)}(X_k(t-1)) \end{aligned} \tag{45}$$

where $V_{eff}^{(j)}(X_j(t))$ is some function of $X_j(t)$ that depends on the potentials $V_i^{(i)}(X_i(t))$ and $V_j^{(i)}(X_j(t-1))$. This is very similar to the quadratic case, where an additional potential has been added. The proof is similar to the one given in Appendix 2.

Gathering the terms in the exponentials, the whole system is modelled by the probability weight:

$$\begin{aligned} &\exp\left(\sum_j \sum_s U_{eff}(X_j(s))\right) \qquad (46) \\ = &\exp\left(\sum_j \sum_s \left(\begin{array}{c} Y_j(t)\begin{pmatrix} N_{ii} & 0 \\ 0 & 0 \end{pmatrix} Y_j(t) - 2Y_j(t)\begin{pmatrix} M_{ii} & M_{ij} \\ 0 & 0 \end{pmatrix} Y_j(t-1) + V_{eff}^{(j)}(Y_j(t)) \\ + \sum_{k>j} 2X_j(t) A_{jk}^{(j)}(X_k(t-1)) \end{array} \right)\right) \end{aligned}$$

as needed to show the recursive form of (45). The fact that the effective action is very similar to the one obtained for the quadratic case, allows to find directly the effective action fo the system as a whole (without normalization). It is obtained by adding to the quadratic action the corrections due to the effective potentials:

$$U_{eff}(X_j(t)) = \frac{1}{2}\left(X(t) - \bar{X}^e\right) N \left(X(t) - \bar{X}^e\right) - \left(X(t) - \bar{X}^e\right)\frac{M+O}{\sqrt{\beta}}\left(X(t-1) - \bar{X}^e\right) + V_{eff}(X(t)) \tag{47}$$

where:

$$V_{eff}(X(t)) = \sum_j V_{eff}^{(j)}(X_j(t))$$

The inclusion of an intertemporal constraint will be modeled in ad hoc way by adding a term

$$\sum_j \int X_j(s) X_i(t)\, ds dt$$

in the effective utility, for a final result:

$$U_{eff}(X_i(t)) = \sum_{j \leqslant i} -\frac{1}{2}\dot{X}_i(t) M_{ij} \dot{X}_j(t) - V_{eff}^{(i)}(X_i(t)) + \sum_j \int X_j(s) X_i(t)\, ds dt$$

4.1 Extensions: measure of uncertainty and optimal control

Our formalism allows to recover, in the limit of no "internal uncertainty" for the agents, the usual optimization dynamics of system. But our formalism may encompass other kinds of models : actually, models including an exogenous dynamics for a state variable which is accessible only through an indicator variable would fit our set up provided that we extend our basic model of interaction between agents. This extension will include a particular type of uncertainty of information for every agent about other structures which is an intermediate possiblity between full/no information.

4.1.1 Exogenous dynamics, indicator variables and Kalman filters

Consider a dynamic system for an arbitrary variable $X_j(t)$ (the "state of the world"):

$$X_j(t) = AX_j(t-1) + BX_i(t-1) + \varepsilon_j(t) \tag{48}$$

with gaussian shocks $\varepsilon_j(t)$ of variance covariance matrix Σ. The vector $X_i(t)$ is the control variable for an agent "i" that influences $X_j(t)$ and is in turn influenced by $X_j(t)$. This type of model appears for example in neuroscience motor control theory. Agent i has an - instantenous - objective function

$$X_i(t)\,A_{ii}^{(i)}X_i(t)+X_j(t-1)\,A_{ii}^{(j)}X_j(t-1)$$

similar to the one studied in Appendix 4. However the difference here is that agent i does not measure directly $X_j(t-1)$ at time t, but only an indicator function $Z_j(t-1)$ related to $X_j(t)$ through:

$$Z_j(t)=HX_j(t)+\omega_j(t)$$

where $\omega_j(t)$ is gaussian of variance covariance matrix Ω.

This model fits in our context providing few modifications. First, the state of the world $X_j(t)$ can be considered as describing a single non strategic agent - or equivalently as an aggregate of such agents - and as such have no forward looking plan with respect to "i". The statistic weight associated with (48) is:

$$\exp\left(-\left(X_j(t)-AX_j(t-1)-BX_i(t-1)\right)^t\Sigma^{-1}\left(X_j(t)-AX_j(t-1)-BX_i(t-1)\right)\right) \tag{49}$$

Actualy, $(X_j(t)-AX_j(t)-BX_i(t))$ is gaussian with variance covariance matrix Σ. The probability associated to $X_j(t)$ is thus proportionnal to (49).

This set up is thus encompassed in the two agents model developed in Appendix 4. Since the weight (49) represents a probability at time t , the method used to derive (35) can be applied here, and the contributions depending only on $t-1$ in (49) can be discarded. As a consequence, (49) is equivalent to:

$$\exp\left(-\left(X_j(t)\right)^t\Sigma^{-1}X_j(t)+2\left(X_j(t)\right)^t\left(\Sigma^{-1}A\right)^SX_j(t-1)+2\left(X_j(t)\right)^t\left(\Sigma^{-1}B\right)^SX_i(t-1)\right) \tag{50}$$

where $\left(\Sigma^{-1}A\right)^S$ and $\left(\Sigma^{-1}B\right)^S$ are the symetrization of $\Sigma^{-1}A$ and $\Sigma^{-1}B$.

Since agent j is not strategic, its effective utility (50) can be rewritten as:

$$\left(X_j(t)\right)^tA_{jj}^{(j)}X_j(t)+2\left(X_j(t)\right)^t\epsilon_{jj}^{(j)}X_j(t-1)+2\left(X_j(t)\right)^tA_{ji}^{(j)}X_i(t-1)$$

with:

$$\begin{aligned}\epsilon_{jj}^{(j)} &= \left(\Sigma^{-1}A\right)^S\\ A_{ji}^{(j)} &= \left(\Sigma^{-1}B\right)^S\end{aligned}$$

The effective action for agent i can thus be directly taken from Appendix 4, except that $X_j(t-1)$ being unknown, it will be replaced by $X_j(t-1\mid t-1)$, agent i forecast of $X_j(t-1)$ given all its information at the beginning of period t, i.e. $Z_j(t-1)$ and $X_i(t-1)$.

$$U^{eff}(X_i(t))=-\left((X_i(t))_i\frac{M_{ii}^S}{\sqrt{\beta}}X_i(t-1)\right)-\left((X_i(t))\frac{M_{ij}^S}{\sqrt{\beta}}X_j(t-1\mid t-1)\right)+\frac{1}{2}\left(X_i(t)\right)^t\left(N_{ii}\right)X_i(t) \tag{51}$$

with:

$$
\begin{aligned}
N_{ii} &= \sqrt{\beta}A_{ij}^{(j)}\left(\left(G\left(V_l\right)^t\right)\left(\left(V_l A_{ji}^{(j)}\right)^t\right)^{-1}\left(\left(\left(V_m^{(1)},0\right)A_{ji}^{(j)}\right)^t\right),G\left(\left(0,V_l^{(1)}\right)\right)^t\right)+1 \\
&\quad -\sqrt{\beta}A_{ij}^{(j)}\left(\Theta+\Delta\left(\left\{\frac{-\epsilon_{jj}^{(j)}}{2}\right\}\right)^{-1}\left(\left(V_l A_{ij}^{(j)}\right)^t\right)\left(G\left(V_l\right)^t\right)^{-1}\right)^{-1} \\
&\quad \times\sqrt{\beta}\left\{\frac{-\epsilon_{jj}^{(j)}}{2}\right\}\left(\left(G\left(V_l\right)^t\right)\left(\left(V_l A_{ji}^{(j)}\right)^t\right)^{-1}\left(\left(\left(V_m^{(1)},0\right)A_{ji}^{(j)}\right)^t\right),G\left(\left(0,V_l^{(1)}\right)\right)^t\right) \\
M_{ii} &= -\left(N_{ii}\right)\Gamma\Theta^{-1}\left(\Delta\left(G\left(V_l\right)^t\right)\left(\left(V_l A_{ij}^{(j)}\right)^t\right)^{-1}\left\{\frac{-\epsilon_{jj}^{(j)}}{2}\right\}\left(\Theta^t\right)^{-1}+1\right)\Gamma^t \\
M_{ij} &= -\left(N_{ii}\right)\Gamma\Theta^{-1}\left(\Delta\left(G\left(V_l\right)^t\right)\left(\left(V_l A_{ij}^{(j)}\right)^t\right)^{-1}\left\{\frac{-\epsilon_{jj}^{(j)}}{2}\right\}+\Theta^t\right)
\end{aligned}
$$

where the matrices E, F, G are defined as a function of H:

$$
E=\left(\sqrt{\beta}A_{ij}^{(j)}\left(\left(\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)\right)^{-1}\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}G+\sqrt{\beta}A_{ji}^{(j)}\right)\right)-B_{12}G\right) \tag{52}
$$

$$
\begin{aligned}
F &= \sqrt{\beta}A_{ij}^{(j)}\left(\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1} \\
&\quad \times\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}H+\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)-B_{12}H
\end{aligned} \tag{53}
$$

$$
G=H\left(\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}A_{ji}^{(j)} \tag{54}
$$

and H satisfies a quadratic equation. Defining:

$$
H'=H\left(\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}+\sqrt{\beta}\left(A_{jj}^{(j)}\right)_{eff}^{-1}
$$

the relation defining H' and then H is:

$$
\begin{aligned}
&\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}+\sqrt{\beta}\left(H'-\sqrt{\beta}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)^{-1}\right) \\
&\times\left(\left\{\frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1} \\
&= \beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}-\sqrt{\beta}\left(\sqrt{\beta}A_{ji}^{(j)}A_{ij}^{(j)}+\left(H'\right)^{-1}\right)^{-1}
\end{aligned} \tag{55}
$$

The matrix V_l is defined such that $\dim(V_l) = \dim\left(A_{ji}^{(i)}\right) = m \times (m+k)$ (m and k are given by the problem), and V_l is the concatenation in column of a nul $m \times k$ matrix and $m \times m$ identity. The matrix $\left(V_m^{(1)}, 0\right)$ is the concatenation in column of $V_m^{(1)}$ which is $m \times k$ matrix with the $m \times m$ nul matrix. The matrix $V_m^{(1)}$ is the concatenation in line of a $k \times k$ identity and a $(m-k) \times k$ nul matrix if $m \geqslant k$. Otherwise it is the concatenation in column of a $m \times m$ identity and a $m \times (k-m)$ nul matrix if $m < k$.

We also define:

$$\begin{aligned}
\left(G\left(V_l\right)^t\right) &= X^{-1}\left(V_l A_{ij}^{(j)}\right)^t, \, G = X^{-1}\left(A_{ij}^{(j)}\right)^t \\
\Gamma &= \sqrt{\beta} A_{ij}^{(j)}, \, \Theta = \sqrt{\beta}\left\{\frac{-\epsilon_{jj}^{(j)}}{2}\right\} \\
\Delta &= \left\{\beta A_{jj}^{(i)} + \left(A_{jj}^{(j)}\right) + B_{22}\right\} \\
B_{22} &= \beta\left(\left(\epsilon_{jj}^{(j)}\right)\right)^t \left(A_{jj}^{(j)}\right)^{-1}\left(\epsilon_{jj}^{(j)}\right)
\end{aligned}$$

where the matrix X solves:

$$\begin{aligned}
&\left(\left(\sqrt{\beta} X^{-1} + \beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right) A_{ji}^{(j)} A_{ij}^{(j)} + 1\right) \\
&\times\left(\left(\left\{\frac{-\epsilon_{jj}^{(j)}}{2}\right\}\right)^{-1}\left(\left(\left\{\beta A_{jj}^{(i)} + \left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}\right) + \sqrt{\beta} X\right)\left(\left\{\frac{-\epsilon_{jj}^{(j)}}{2}\right\}\right)^{-1} - \beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right) \\
= &-\left(\beta\left(A_{jj}^{(j)}\right)_{eff}^{-1} + \sqrt{\beta} X^{-1}\right)
\end{aligned}$$

The solution is unique, since it is imposed to have a series expansion in β that fits with the $\beta = 0$ case.

With matrices N_{ii} M_{ii}^S and M_{ij}^S at hand, we find the usual reaction function for agent i by assuming full certainty about agent i's behavior. Under the assumption of the variance of its effective action being nul, agent's i action is given by it's quadratic action minimum, and its response to agent j is given by the optimization of (51)

$$\begin{aligned}
X_i(t) &= \left(\frac{\left(N_{ii}\right)^{-1} M_{ii}^S}{\sqrt{\beta}} X_i(t-1)\right) + \left(\frac{\left(N_{ii}\right)^{-1} M_{ij}^S}{\sqrt{\beta}} X_j(t-1 \mid t-1)\right) \\
&= \Xi X_i(t-1) + \Upsilon X_j(t-1 \mid t-1)
\end{aligned} \tag{56}$$

supplemented by:

$$X_j(t) = A X_j(t-1) + B X_i(t-1) + \varepsilon_j(t) \tag{57}$$

$$Z_j(t) = H X_j(t) + \omega_j(t) \tag{58}$$

with $\omega_j(t)$ an unknown error of (known) variance matrix Ω.

These three equations, respectively for the state variable $X_j(t)$, the indicator variable $Z_j(t)$ and the reaction function $X_i(t)$ for agent i, describe the system in interaction.

We also assume, as is usually done in this type of model, that expectations for $X_j(t)$ are updated through a linear projection ([9] 13.12.13):

$$X_j(t \mid t) = X_j(t \mid t-1) + K\left(Z_j(t) - Z_j(t \mid t-1)\right) \tag{59}$$

with:

$$\begin{aligned}
K &= E\left(\left(X_j(t) - X_j(t \mid t-1)\right)\left(Z_j(t) - Z_j(t \mid t-1)\right)^t\right) \times \left\{E\left(\left(Z_j(t) - Z_j(t \mid t-1)\right)\left(Z_j(t) - Z_j(t \mid t-1)\right)^t\right)\right\}^{-1} \\
&= P_{t|t-1} H\left(H^t P_{t|t-1} H + \Omega\right)^{-1}
\end{aligned}$$

and $P_{t|t-1}$ is defined as:

$$P_{t|t-1} = E\left(\left(X_j\left(t\right) - X_j\left(t \mid t-1\right)\right) \left(X_j\left(t\right) - X_j\left(t \mid t-1\right)\right)^t \right)$$

and where (58) has been used.

Given (58), equation (59) is also equivalent to:

$$X_j\left(t \mid t\right) = X_j\left(t \mid t-1\right) + KH\left(X_j\left(t\right) - X_j\left(t \mid t-1\right)\right) + K\omega_j\left(t\right) \tag{60}$$

To solve the dynamics of system, we proceed by finding the Kalman matrix K and the form of the expectations:

To find $P_{t|t-1}$ and K, we follow [9] and first define an other squared expectation denoted $P_{t|t}$, given by:

$$P_{t|t} = E\left(\left(X_j\left(t\right) - X_j\left(t \mid t\right)\right) \left(X_j\left(t\right) - X_j\left(t \mid t\right)\right)^t \right)$$

Using eq. 4.5.31 and 13.12.16 in [9]

$$\begin{aligned} & P_{t|t} = E\left(\left(X_j\left(t\right) - X_j\left(t \mid t\right)\right) \left(X_j\left(t\right) - X_j\left(t \mid t\right)\right)^t \right) \\ = \ & E\left(\left(X_j\left(t\right) - X_j\left(t \mid t-1\right)\right) \left(X_j\left(t\right) - X_j\left(t \mid t-1\right)\right)^t \right) \\ & -E\left(\left(X_j\left(t\right) - X_j\left(t \mid t-1\right)\right) \left(Z_j\left(t\right) - Z_j\left(t \mid t-1\right)\right)^t \right) \\ & \times \left(E\left(\left(Z_j\left(t\right) - Z_j\left(t \mid t-1\right)\right) \left(Z_j\left(t\right) - Z_j\left(t \mid t-1\right)\right)^t \right)\right)^{-1} \\ & \times E\left(\left(Z_j\left(t\right) - Z_j\left(t \mid t-1\right)\right) \left(X_j\left(t\right) - X_j\left(t \mid t-1\right)\right)^t \right) \end{aligned}$$

or, using (58):

$$P_{t|t} = P_{t|t-1} - P_{t|t-1} H \left(H^t P_{t|t-1} H + \Omega \right)^{-1} H^t P_{t|t-1} \tag{61}$$

To find the terms $P_{t|t}$ and $P_{t|t-1}$, we first use (57)

$$X_j\left(t+1\right) = AX_j\left(t\right) + BX_i\left(t\right) + \varepsilon_j\left(t+1\right)$$

and introduce the dynamic equation (56), which leads to:

$$X_j\left(t+1\right) = AX_j\left(t\right) + B\left(\Xi X_i\left(t-1\right) + \Upsilon X_j\left(t-1 \mid t-1\right)\right) + \varepsilon_j\left(t+1\right)$$

and then, one obtains an expression for $P_{t+1|t}$ as a function of $P_{t|t}$ and an expression for $P_{t+1|t}$ as a function of $P_{t|t}$:

$$\begin{aligned} P_{t+1|t} &= E\left(\left(X_j\left(t+1\right) - X_j\left(t+1 \mid t\right)\right) \left(X_j\left(t+1\right) - X_j\left(t+1 \mid t\right)\right)^t \right) \\ &= AE\left(\left(X_j\left(t\right) - X_j\left(t \mid t\right)\right) \left(X_j\left(t\right) - X_j\left(t \mid t\right)\right)^t \right) A^t + \Sigma \\ &= AP_{t|t} A^t + \Sigma \end{aligned}$$

which leads, using (61), to the dynamic equation for $P_{t+1|t}$:

$$P_{t+1|t} = A\left(P_{t|t-1} - P_{t|t-1} H \left(H^t P_{t|t-1} H + \Omega \right)^{-1} H^t P_{t|t-1} \right) A^t + \Sigma$$

Given our system we look for a stationary solution that is $P_{t+1|t} = P$ which satisfies:

$$P = A\left(P_{t|t-1} - P_{t|t-1} H \left(H^t P_{t|t-1} H + \Omega \right)^{-1} H^t P_{t|t-1} \right) A^t + \Sigma$$

The Kalman Matrix is then given by:

$$K = PH\left(H^{t}PH+\Omega\right)^{-1}$$

Having found K, the system reduces to:

$$\begin{aligned}
X_i(t) &= \Xi X_i(t-1)+\Upsilon X_j(t-1\mid t-1)\\
X_j(t) &= AX_j(t-1)+BX_i(t-1)+\varepsilon_j(t)\\
Z_j(t) &= HX_j(t)+\omega_j(t)\\
X_j(t\mid t) &= X_j(t\mid t-1)+K\left(Z_j(t)-Z_j(t\mid t-1)\right)\\
&= X_j(t\mid t-1)+KH\left(X_j(t)-X_j(t\mid t-1)\right)+K\omega_j(t)\\
&= (1-KH)X_j(t\mid t-1)+KHX_j(t)+K\omega_j(t)
\end{aligned}$$

The variable $X_j(t\mid t-1)$ is found by taking the expectation by agent i at time $t-1$ of equation (57):

$$X_j(t\mid t-1)=AX_j(t-1\mid t-1)+BX_i(t-1)$$

We are thus left with a system with three dynamic variables:

$$\begin{aligned}
X_i(t) &= \Xi X_i(t-1)+\Upsilon X_j(t-1\mid t-1)\\
X_j(t) &= AX_j(t-1)+BX_i(t-1)+\varepsilon_j(t)\\
X_j(t\mid t) &= (1-KH)AX_j(t-1\mid t-1)+(1-KH)BX_i(t-1)+KHX_j(t)+K\omega_j(t)\\
&= (1-KH)AX_j(t-1\mid t-1)+KHAX_j(t-1)+BX_i(t-1)+K\omega_j(t)+KH\varepsilon_j(t)
\end{aligned}$$

of matricial form:

$$\begin{pmatrix} X_i(t)\\ X_j(t)\\ X_j(t\mid t)\end{pmatrix}=\begin{pmatrix}\Xi & 0 & \Upsilon\\ B & A & 0\\ B & KHA & (1-KH)A\end{pmatrix}\begin{pmatrix} X_i(t-1)\\ X_j(t-1)\\ X_j(t-1\mid t-1)\end{pmatrix}+\begin{pmatrix}0\\ \varepsilon_j(t)\\ K\omega_j(t)+KH\varepsilon_j(t)\end{pmatrix}$$

whose solution for dynamic starting at $t=0$ is:

$$\begin{pmatrix} X_i(t)\\ X_j(t)\\ X_j(t\mid t)\end{pmatrix}=\sum_{s=0}^{t}\begin{pmatrix}\Xi & 0 & \Upsilon\\ B & A & 0\\ B & KHA & (1-KH)A\end{pmatrix}^{t-s}\begin{pmatrix}0\\ \varepsilon_j(s)\\ K\omega_j(s)+KH\varepsilon_j(s)\end{pmatrix}$$

4.1.2 Uncertainty in observations and agents interactions

We have used our formalism to model the interaction between an uncertain exogenous medium and an optimizing agent. The reverse point of view is straightforward to develop, in order to introduce some uncertainty of measurement in our formalism.

For the general form of effective utility (62) in the quadratic case, introducing uncertainty in the information agent i receives from other agents j amounts to replacing past actions $X_j(t-1)$ by $X_j(t-1\mid t-1)$. We thus obtain

$$\begin{aligned}
U_{eff}(X_i(t)) &= Y_j(t)\begin{pmatrix}N_{ii} & 0\\ 0 & 0\end{pmatrix}Y_j(t)-2Y_j(t)\begin{pmatrix}M_{ii} & M_{ij}\\ 0 & 0\end{pmatrix}Y_j\left(t-1\mid(t-1)^{i}\right)\\
&\quad+\sum_{k>i}2X_j(t)A_{jk}^{(j)}X_k\left(t-1\mid(t-1)^{i}\right)
\end{aligned}\tag{62}$$

where $Y_j\left(t-1\mid(t-1)^{i}\right)$ denotes agent i forecast of $Y_j(t-1)$ at $t-1$. The statistical weight $\exp\left(U_{eff}(X_i(t))\right)$ associated to agent $X_i(t)$ implies that the reaction function of agent i is given by:

$$X_i(t)=(N_{ii})^{-1}M_{ii}X_i(t-1)+\sum_{j<i}(N_{ii})^{-1}M_{ij}X_j\left(t-1\mid(t-1)^{i}\right)+\sum_{k>i}A_{jk}^{(j)}X_k\left(t-1\mid(t-1)^{i}\right)+\varepsilon_i(t)\tag{63}$$

with $\varepsilon_i(t)$ of variance $(N_{ii})^{-1}$.

The forecasts $X_j\left(t-1 \mid (t-1)^i\right)$ and $X_k\left(t-1 \mid (t-1)^i\right)$ are obtained as in the previous paragraph through indicator variables and Kalman matrices. We also assume indicator variables for $X_j(t-1)$ and $X_k(t-1)$:

$$\begin{aligned} Z_j(t) &= H_j X_j(t) + \omega_j(t) \\ Z_k(t) &= H_k X_k(t) + \omega_k(t) \end{aligned} \tag{64}$$

where $\omega_j(t)$ and $\omega_k(t)$ have variances Ω_j and Ω_k respectively. For the sake of simplicity we will assume all agents have common indicator variables. However some specialized indicators to some of agents could be introduced. To be consistent with our previous assumptions, we assume that agent i has no information about $X_k(t)$ apart from $Z_k(t)$, and that:

$$X_k\left(t-1 \mid (t-1)^i\right) = Z_k(t) = H_k X_k(t) + \omega_k(t) \tag{65}$$

a random variable of variance:

$$H\hat{\Omega}_k H^t + (N_{kk})^{-1}$$

Up to some details, the forecasting procedure is thus the same. Agent i faces an exogenous dynamic given agents j, $j<i$ and k is perceived as a random shock. For i the dynamic of the "state of world" is then:

$$X_j^{(i)}(t) = (N_{jj})^{-1} M_{jj} X_j(t-1) + \sum_{l<j} (N_{jj})^{-1} M_{jl} X_l\left(t-1 \mid (t-1)^j\right) + \sum_{i\geqslant l>j} A_{jl}^{(j)} X_l\left(t-1 \mid (t-1)^j\right) + \varepsilon_j(t) \tag{66}$$

Given our initial (first section) assumptions, the actions of agents $l>i$, being unknown to i, are discarded. The vector $X_j^{(i)}(t)$ is the dynamic for j anticipted by i which is different from $X_j(t)$, given the terms for $l>i$ that have been discarded)). Then:

$$\begin{aligned} X_j^{(i)}\left(t \mid (t-1)^i\right) &= (N_{jj})^{-1} M_{jj} X_j\left(t-1 \mid (t-1)^i\right) + \sum_{l<i} (N_{jj})^{-1} M_{jl} X_l^{(i)}\left(t-1 \mid (t-1)^j\right) \\ &\quad + \sum_{i\geqslant l>j} A_{jl}^{(j)} X(i)_l\left(t-1 \mid (t-1)^j\right) \end{aligned}$$

Note that agent i having more information than agent j we have used that $X_l\left(t-1 \mid (t-1)^j \mid (t-1)^i\right) = X_l\left(t-1 \mid (t-1)^j\right)$ in the previous expression.

As before, the actualization of forecast is given by (we remove temporarily the upperscript (i) in the forecast):

$$X_j(t \mid t) = X_j(t \mid t-1) + K\left(Z_j(t) - Z_j(t \mid t-1)\right) \tag{67}$$

with:

$$\begin{aligned} K &= E\left(\left(X_j(t) - X_j(t \mid t-1)\right)\left(Z_j(t) - Z_j(t \mid t-1)\right)^t\right) \\ &\quad \times \left\{E\left(\left(Z_j(t) - Z_j(t \mid t-1)\right)\left(Z_j(t) - Z_j(t \mid t-1)\right)^t\right)\right\}^{-1} \\ &= P_{t|t-1} H_j \left(H_j^t P_{t|t-1} H_j + \Omega\right)^{-1} \end{aligned}$$

and $P_{t|t-1}$ is defined as:

$$P_{t|t-1} = E\left(\left(X_j(t) - X_j(t \mid t-1)\right)\left(X_j(t) - X_j(t \mid t-1)\right)^t\right)$$

and where (64) has been used. Given (64), equation (67) is also equivalent to:

$$X_j(t \mid t) = X_j(t \mid t-1) + K H_j\left(X_j(t) - X_j(t \mid t-1)\right) + K\omega_j(t) \tag{68}$$

Following the same procedure as in the previous paragraph, one finds the Kalman matrix K, by defining $P_{t|t}$ which is given by:

$$P_{t|t} = E\left(\left(X_j\left(t\right) - X_j\left(t \mid t\right)\right)\left(X_j\left(t\right) - X_j\left(t \mid t\right)\right)^t\right)$$

that satisfies

$$P_{t|t} = P_{t|t-1} - P_{t|t-1}H_j\left(H_j^t P_{t|t-1}H_j + \Omega\right)^{-1} H_j^t P_{t|t-1} \tag{69}$$

Now, we use the dynamics equations (66) and (63) to find $P_{t|t}$ and $P_{t|t-1}$. Starting with (66)

$$\begin{aligned} X_j^{(i)}\left(t\right) &= \left(N_{jj}\right)^{-1} M_{jj}X_j\left(t-1\right) + \sum_{l<j}\left(N_{jj}\right)^{-1} M_{jl}X_l^{(i)}\left(t-1 \mid (t-1)^j\right) \\ &\quad + \sum_{i\geqslant l>j} A_{jl}^{(j)} X_l^{(i)}\left(t-1 \mid (t-1)^j\right) + \varepsilon_j\left(t\right) \end{aligned} \tag{70}$$

and then, since

$$\sum_{l<j}\left(N_{jj}\right)^{-1} M_{jl}X_l^{(i)}\left(t-1 \mid (t-1)^j\right) + \sum_{i\geqslant l>j} A_{jl}^{(j)} X_l^{(i)}\left(t-1 \mid (t-1)^j\right)$$

is known to agent i at time $t-1$ (agent i has more information than agent j), then

$$\begin{aligned} &\left(\left(\sum_{l<j}\left(N_{jj}\right)^{-1} M_{jl}X_l^{(i)}\left(t-1 \mid (t-1)^j\right) + \sum_{i\geqslant l>j} A_{jl}^{(j)} X_l^{(i)}\left(t-1 \mid (t-1)^j\right)\right) \mid (t-1)^i\right) \\ = &\sum_{l<j}\left(N_{jj}\right)^{-1} M_{jl}X_l^{(i)}\left(t-1 \mid (t-1)^j\right) + \sum_{i\geqslant l>j} A_{jl}^{(j)} X_l^{(i)}\left(t-1 \mid (t-1)^j\right) \end{aligned}$$

one thus obtain an expression for $P_{t+1|t}$ as a function of $P_{t|t}$:

$$\begin{aligned} P_{t+1|t} &= E\left(\left(X_j^{(i)}\left(t+1\right) - X_j^{(i)}\left(t+1 \mid t\right)\right)\left(X_j^{(i)}\left(t+1\right) - X_j^{(i)}\left(t+1 \mid t\right)\right)^t\right) \\ &= \left(\left(N_{jj}\right)^{-1} M_{jj}\right) E\left(\left(X_j^{(i)}\left(t\right) - X_j^{(i)}\left(t \mid t\right)\right)\left(X_j^{(i)}\left(t\right) - X_j^{(i)}\left(t \mid t\right)\right)^t\right)\left(\left(N_{jj}\right)^{-1} M_{jj}\right)^t + \left(N_{jj}\right)^{-1} \\ &= \left(\left(N_{jj}\right)^{-1} M_{jj}\right) P_{t|t}^t \left(\left(N_{jj}\right)^{-1} M_{jj}\right) + \left(N_{jj}\right)^{-1} \end{aligned}$$

Leads, using (69), to the dynamic equation for $P_{t+1|t}$. We reintroduce now an index j to recall that the probabitity $P_{t+1|t}$ is computed for X_j and an index i to stand for the fact that the expectations are computed by agent i:

$$P_{t+1|t}^{i,j} = \left(\left(N_{jj}\right)^{-1} M_{jj}\right)\left(P_{t|t-1}^{i,j} - P_{t|t-1}^{i,j}H_j\left(H_j^t P_{t|t-1}^{i,j}H_j + \Omega\right)^{-1} H_j^t P_{t|t-1}^{i,j}\right)\left(\left(N_{jj}\right)^{-1} M_{jj}\right)^t + \left(N_{jj}\right)^{-1}$$

Given our system, we look for a stationary solution, $P_{t+1|t} = P$, which satisfies:

$$P^{i,j} = \left(\left(N_{jj}\right)^{-1} M_{jj}\right)\left(P^{i,j} - P^{i,j}H_j\left(H_j^t P^{i,j}H_j + \Omega\right)^{-1} H_j^t P^{i,j}\right)\left(\left(N_{jj}\right)^{-1} M_{jj}\right)^t + \left(N_{jj}\right)^{-1}$$

The Kalman Matrix is given by:

$$K^{i,j} = P^{i,j}H_j\left(H_j^t P^{i,j}H_j + \Omega\right)^{-1}$$

which produces the forecast

$$\begin{aligned} X_j^{(i)}\left(t \mid (t)^i\right) &= X_j^{(i)}\left(t \mid (t-1)^i\right) + K^{i,j}\left(Z_j\left(t\right) - Z_j\left(t \mid (t-1)^i\right)\right) \\ &= X_j^{(i)}\left(t \mid (t-1)^i\right) + K^{i,j}H_j\left(X_j\left(t\right) - X_j^{(i)}\left(t \mid (t-1)^i\right)\right) + K^{i,j}H_j\omega_j\left(t\right) \end{aligned}$$

which, using (70) and (65), is equal to:

$$
\begin{aligned}
X_j^{(i)}\left(t \mid (t)^i\right) &= X_j^{(i)}\left(t \mid (t-1)^i\right) + K^{i,j}\left(Z_j(t) - Z_j\left(t \mid (t-1)^i\right)\right) \\
&= X_j^{(i)}\left(t \mid (t-1)^i\right) + K^{i,j} H_j (N_{jj})^{-1} M_{jj}\left(X_j^{(i)}(t-1) - X_j^{(i)}\left(t-1 \mid (t-1)^i\right)\right) \\
&\quad + K^{i,j} H_j (\omega_j(t) + \varepsilon_i(t)) \\
&= (N_{jj})^{-1} M_{jj} X_j^{(i)}\left(t-1 \mid (t-1)^i\right) + \sum_{l<j} (N_{jj})^{-1} M_{jl} X_l^{(i)}\left(t-1 \mid (t-1)^j\right) \\
&\quad + \sum_{i \geqslant l > j} A_{jl}^{(j)} X_l^{(i)}\left(t-1 \mid (t-1)^j\right) \\
&\quad + K^{i,j} H_j (N_{jj})^{-1} M_{jj}\left(X_j^{(i)}(t-1) - X_j^{(i)}\left(t-1 \mid (t-1)^i\right)\right) \\
&\quad + K^{i,j} H_j (\omega_j(t) + \varepsilon_i(t)) \\
&= K^{i,j} H_j (N_{jj})^{-1} M_{jj} X_j^{(i)}(t-1) + \left(1 - K^{i,j} H_j\right)(N_{jj})^{-1} M_{jj} X_j^{(i)}\left(t-1 \mid (t-1)^i\right) \\
&\quad + \sum_{l<j} (N_{jj})^{-1} M_{jl} X_l^{(i)}\left(t-1 \mid (t-1)^j\right) + \sum_{i \geqslant l > j} A_{jl}^{(j)} (H_l X_l(t-1) + \omega_l(t-1)) \\
&\quad + K^{i,j} H_j (\omega_j(t) + \varepsilon_i(t))
\end{aligned}
$$

and, supplemented by the three equations:

$$
\begin{aligned}
X_i(t) &= (N_{ii})^{-1} M_{ii} X_i(t-1) + \sum_{j<i} (N_{ii})^{-1} M_{ij} X_j^{(i)}\left(t-1 \mid (t-1)^i\right) \\
&\quad + \sum_{k>i} A_{jk}^{(j)} X_k\left(t-1 \mid (t-1)^i\right) + \varepsilon_i(t) \\
&= (N_{ii})^{-1} M_{ii} X_i(t-1) + \sum_{j<i} (N_{ii})^{-1} M_{ij} X_j^{(i)}\left(t-1 \mid (t-1)^i\right) \\
&\quad + \sum_{k>i} A_{jk}^{(j)} (H_k X_k(t-1) + \omega_k(t-1)) + \varepsilon_i(t)
\end{aligned}
\tag{71}
$$

$$
\begin{aligned}
X_j^{(i)}(t) &= (N_{jj})^{-1} M_{jj} X_j^{(i)}(t-1) + \sum_{l<j} (N_{jj})^{-1} M_{jl} X_l^{(i)}\left(t-1 \mid (t-1)^j\right) \\
&\quad + \sum_{i \geqslant l > j} A_{jl}^{(j)} X_l^{(i)}\left(t-1 \mid (t-1)^j\right) + \varepsilon_j(t) \\
&= (N_{jj})^{-1} M_{jj} X_j^{(i)}(t-1) + \sum_{l<j} (N_{jj})^{-1} M_{jl} X_l^{(i)}\left(t-1 \mid (t-1)^j\right) \\
&\quad + \sum_{i \geqslant l > j} A_{jl}^{(j)} (H_l X_l(t-1) + \omega_l(t-1)) + \varepsilon_j(t)
\end{aligned}
$$

$$
\begin{aligned}
X_{l \leqslant j}^{(i)}\left(t \mid (t)^j\right) &= X_{l \leqslant j}^{(i)}\left(t \mid (t-1)^j\right) + K^{j,l}\left(Z_l(t) - Z_l\left(t \mid (t-1)^l\right)\right) \\
&= K^{j,l} H_l (N_{ll})^{-1} M_{ll} X_l^{(i)}(t-1) + \left(1 - K^{j,l} H_l\right)(N_{ll})^{-1} M_{ll} X_l^{(i)}\left(t-1 \mid (t-1)^j\right) \\
&\quad + \sum_{p<l} (N_{ll})^{-1} M_{lp} X_l^{(i)}\left(t-1 \mid (t-1)^j\right) \\
&\quad + \sum_{j \geqslant p > l} A_{lp}^{(j)} (H_p X_p(t-1) + \omega_p(t-1)) + K^{j,l} H_l (\omega_l(t) + \varepsilon_j(t))
\end{aligned}
$$

leads to the dynamic system:

$$\begin{pmatrix} X_i(t) \\ X^{(i)}_{\{j\}}(t) \\ X^{(i)}_{\{j\}}\left(t \mid (t)^i\right) \\ X^{(i)}_{\{l\leqslant j\}}\left(t \mid (t)^j\right) \end{pmatrix}$$

$$= \begin{pmatrix} (N_{ii})^{-1} M_{ii} & 0 & (N_{ii})^{-1} M_{ij} & 0 \\ A^{(j)}_{ji} H_i & \left\{ \begin{array}{c} (N_{jj})^{-1} M_{jj}, \\ \left\{ A^{(j)}_{\{jl\}} H_l \right\}_{l>j} \end{array} \right\} & 0 & \left\{ \left(N_{\{jj\}}\right)^{-1} M_{\{jl\}_{l<j}} \right\} \\ A^{(j)}_{ji} H_i & \left\{ \begin{array}{c} K^{i,j} H_j (N_{jj})^{-1} M_{jj}, \\ \left\{ A^{(j)}_{\{jl\}} H_l \right\}_{l>j} \end{array} \right\} & \left\{ \begin{array}{c} \left(1 - K^{i,j} H_j\right) (N_{jj})^{-1} M_{jj}, \\ \left(N_{\{jj\}}\right)^{-1} M_{\{jl\}_{l<i}} \end{array} \right\} & \left\{ \left(N_{\{jj\}}\right)^{-1} M_{\{jl\}_{l<j}} \right\} \\ 0 & \left\{ \begin{array}{c} K^{j,l} H_l (N_{ll})^{-1} M_{ll}, \\ \left\{ A^{(j)}_{lp} H_p \right\}_{j\geqslant p>l} \end{array} \right\} & 0 & \left\{ \begin{array}{c} \left\{ (N_{ll})^{-1} M_{lp} \right\}_{p<l}, \\ \left(1 - K^{j,l} H_l\right) (N_{ll})^{-1} M_{ll} \end{array} \right\} \end{pmatrix}$$

$$\times \begin{pmatrix} X_i(t-1) \\ X^{(i)}_{\{j\}}(t-1) \\ X^{(i)}_{\{j\}}\left(t-1 \mid (t-1)^i\right) \\ X^{(i)}_{\{l\leqslant j\}}\left(t-1 \mid (t-1)^j\right) \end{pmatrix} + \begin{pmatrix} \sum_{k>i} A^{(j)}_{jk} \left(H_k X_k(t-1) + \omega_k(t-1)\right) \\ \sum_{i\geqslant l>j} A^{(j)}_{jl} \omega_l(t-1) + \varepsilon_j(t) \\ \sum_{i\geqslant l>j} A^{(j)}_{jl} \omega_l(t-1) + K^{i,j} H_j \left(\omega_j(t) + \varepsilon_i(t)\right) \\ \sum_{j\geqslant p>l} A^{(j)}_{lp} \omega_p(t-1) + K^{j,l} H_l \left(\omega_l(t) + \varepsilon_j(t)\right) \end{pmatrix}$$

5 Transition functions (Green functions)

5.1 General form for the transition function

As explained previously, the mean path dynamics, i.e. the mean time evolution of the interacting agents, is obtained as the saddle path solution of the effective action of the interacting system. This saddle path is relatively easy to compute since all anticipations and forwardlookingness have been absorbed in the effective action. However we have also seen that (20) the path integral of the effective action allows to model the stochastic nature of the interacting system. It provides more precise results about the agents actions' fluctuations and their transition probability between two states, thus allowing to represent the stochastic paths associated to the system. Moreover, Because this approach will also prove important when we shift to the field representation for a large number of agents, this section will detail the form of the transition functions, and their interpretation.

To do so, let us start with the system as a whole. As in (20) we define:

$$X(t) = (X_i(t))$$

the concatenated vector of all the $X_i(t)$ with i running on the set of all agents. Moreover, $U_{eff}\left((X_i(t)), \left(\dot{X}_i(t)\right)\right)$ is the total effective action found in the first section (see (41)):

$$\exp U_{eff}(X(t)) = \exp\left\{ -\left(\frac{1}{2}\left(X(t) - \bar{X}^e\right) N \left(X(t) - \bar{X}^e\right) - \left(X(t) - \bar{X}^e\right) M \left(X(t-1) - \bar{X}^e\right) \right) \right\}$$

where we redefined $\frac{O+M}{\sqrt{\beta}}$ as M. The quantity:

$$P\left(\left(X^0(t+k)\right) \mid \left(X^0(t)\right)\right) = \int_{X_i(t)=\left(X_i^0\right)}^{X_i(t+k)=\left(X_i^0(t+k)\right)} \exp\left(\int U_{eff}\left(X(t), \dot{X}(t)\right)\right) \mathcal{D}(X(t)) \qquad (72)$$

is the transition probability from a state $\left(X^0\right)$ of the global system at time t, to a state $\left(X^0\right)$ at time $t+k$.

To understand better this quantity, it is useful to use a continuous time representation. To do so, we first rewrite the quadratic effective utility in a convenient manner. In the formula (41):

$$U_{eff}\left(X\left(t\right)\right)=-\left(\frac{1}{2}\left(X\left(t\right)-\bar{X}^{e}\right)N\left(X\left(t\right)-\bar{X}^{e}\right)-\left(X\left(t\right)-\bar{X}^{e}\right)\frac{M}{\sqrt{\beta}}\left(X\left(t-1\right)-\bar{X}^{e}\right)\right)$$

Decompose $M=M^{S}+M^{A}$ where M^{S} and M^{A} are symetric and antisymetric respectively. Then, since $U_{eff}\left(X\left(t\right)\right)$ will be summed over t, rewrite the first contribution to $\sum_{t}U_{eff}\left(X\left(t\right)\right)$:

$$\sum_{t}\frac{1}{2}\left(X\left(t\right)-\bar{X}^{e}\right)N\left(X\left(t\right)-\bar{X}^{e}\right)$$

as:

$$\begin{aligned}
&\sum_{t}\frac{1}{2}\left(X\left(t\right)-\bar{X}^{e}\right)N\left(X\left(t\right)-\bar{X}^{e}\right)\\
=&\sum_{t}\left(\frac{1}{4}\left(X\left(t\right)-\bar{X}^{e}\right)N\left(X\left(t\right)-\bar{X}^{e}\right)+\frac{1}{4}\left(X\left(t+1\right)-\bar{X}^{e}\right)N\left(X\left(t+1\right)-\bar{X}^{e}\right)\right)\\
=&\sum_{t}\left(\frac{X\left(t\right)+X\left(t+1\right)}{2}-\bar{X}^{e}\right)N\left(\frac{X\left(t\right)+X\left(t+1\right)}{2}-\bar{X}^{e}\right)+\frac{1}{4}\left(X\left(t+1\right)-X\left(t\right)\right)N\left(X\left(t+1\right)-X\left(t\right)\right)
\end{aligned}$$

On the other hand, the second contribution in $U_{eff}\left(X\left(t\right)\right)$ can be transformed by expressing the symetric part of $\left(X\left(t\right)-\bar{X}^{e}\right)\frac{M}{\sqrt{\beta}}\left(X\left(t-1\right)-\bar{X}^{e}\right)$ as:

$$\begin{aligned}
&\left(X\left(t\right)-\bar{X}^{e}\right)\frac{M^{S}}{\sqrt{\beta}}\left(X\left(t-1\right)-\bar{X}^{e}\right)\\
=&\left(\frac{X\left(t\right)+X\left(t+1\right)}{2}-\bar{X}^{e}\right)\frac{M^{S}}{\sqrt{\beta}}\left(\frac{X\left(t\right)+X\left(t+1\right)}{2}-\bar{X}^{e}\right)-\frac{1}{4}\left(X\left(t+1\right)-X\left(t\right)\right)\frac{M^{S}}{\sqrt{\beta}}\left(X\left(t+1\right)-X\left(t\right)\right)
\end{aligned}$$

Ultimately, the remaining term in $U_{eff}\left(X\left(t\right),\dot{X}\left(t\right)\right)$

$$\left(X\left(t\right)-\bar{X}^{e}\right)\frac{M^{A}}{\sqrt{\beta}}\left(X\left(t-1\right)-\bar{X}^{e}\right)$$

can be rewritten:

$$\begin{aligned}
&\left(X\left(t\right)-\bar{X}^{e}\right)\frac{M^{A}}{\sqrt{\beta}}\left(X\left(t-1\right)-\bar{X}^{e}\right)\\
=&\frac{1}{2}\left(X\left(t\right)-\bar{X}^{e}\right)\frac{M^{A}}{\sqrt{\beta}}\left(X\left(t-1\right)-\bar{X}^{e}\right)+\frac{1}{2}\left(X\left(t\right)-\bar{X}^{e}\right)\frac{M^{A}}{\sqrt{\beta}}\left(X\left(t-1\right)-\bar{X}^{e}\right)\\
=&\frac{1}{2}\left(X\left(t\right)-\bar{X}^{e}+\left(X\left(t-1\right)-\bar{X}^{e}\right)\right)\frac{M^{A}}{\sqrt{\beta}}\left(X\left(t-1\right)-\bar{X}^{e}\right)+\frac{1}{2}\left(X\left(t\right)-\bar{X}^{e}\right)\frac{M^{A}}{\sqrt{\beta}}\left(X\left(t-1\right)-\bar{X}^{e}+\left(X\left(t\right)-\bar{X}^{e}\right)\right)\\
=&\left(\frac{X\left(t\right)+X\left(t-1\right)}{2}-\bar{X}^{e}\right)\frac{M^{A}}{\sqrt{\beta}}\left(X\left(t-1\right)-\bar{X}^{e}\right)+\left(X\left(t\right)-\bar{X}^{e}\right)\frac{M^{A}}{\sqrt{\beta}}\left(\frac{X\left(t\right)+X\left(t-1\right)}{2}-\bar{X}^{e}\right)
\end{aligned}$$

since M^{A} is antisymetric. And thus,

$$\begin{aligned}
&\left(X\left(t\right)-\bar{X}^{e}\right)\frac{M^{A}}{\sqrt{\beta}}\left(X\left(t-1\right)-\bar{X}^{e}\right)\\
=&\left(X\left(t\right)-X\left(t-1\right)\right)\frac{M^{A}}{\sqrt{\beta}}\left(\frac{X\left(t\right)+X\left(t-1\right)}{2}-\bar{X}^{e}\right)
\end{aligned}$$

Gathering these terms allow to write ultimately:

$$\sum_t U_{eff}(X(t)) = \left(\frac{X(t)+X(t+1)}{2} - \bar{X}^e\right)\left(N - \frac{M^S}{\sqrt{\beta}}\right)\left(\frac{X(t)+X(t+1)}{2} - \bar{X}^e\right)$$
$$+\frac{1}{4}(X(t+1)-X(t))\left(N+\frac{M^S}{\sqrt{\beta}}\right)(X(t+1)-X(t))$$
$$-(X(t)-X(t-1))\frac{M^A}{\sqrt{\beta}}\left(\frac{X(t)+X(t-1)}{2} - \bar{X}^e\right)$$

We can then switch to a continuous time formulation of the effective action by using the mid point approximation between $X(t)$ and $X(t+1)$, that is replacing $\frac{X(t)+X(t+1)}{2}$ with $X(t)$ (and t is a continuous variable) and introducing

$$\dot{X}(t) = X(t) - X(t-1)$$

so that $\sum_t U_{eff}(X(t))$ becomes:

$$\int \left[\left(X(t)-\bar{X}^e\right)\left(N-\frac{M^S}{\sqrt{\beta}}\right)\left(X(t)-\bar{X}^e\right) + \frac{1}{4}\dot{X}(t)\left(N+\frac{M^S}{\sqrt{\beta}}\right)\dot{X}(t) + \left(X(t)-\bar{X}^e\right)\frac{M^A}{\sqrt{\beta}}\dot{X}(t)\right] dt$$

If we add a potential $V_{eff}(X(t))$ with:

$$V_{eff}(X(t)) = \sum_j V_{eff}^{(j)}(X_j(t))$$

then (we include the factor $\left(\sqrt{\beta}\right)^{-1}$ in the definition of M^S and M^A):

$$\begin{aligned} U_{eff}(X(t)) &= \int \left(\frac{1}{4}\dot{X}(t)\left(N+M^S\right)\dot{X}(t) + \left(X(t)-\bar{X}^e\right)\left(N-M^S\right)\left(X(t)-\bar{X}^e\right)\right. \\ &\quad \left. + \left(X(t)-\bar{X}^e\right)M^A\dot{X}(t) + V_{eff}(X(t))\right) dt \end{aligned} \tag{73}$$

and the path integral defining the transition probability between two states is:

$$\begin{aligned} &P\left(X^1, t+s \mid X^0, t\right) \\ = &\int \exp\left(\int_{X(t)=X^0}^{X(t+s)=X^1} \sum_i \left(\frac{1}{4}\dot{X}(t)\left(N+M^S\right)\dot{X}(t) + \left(X(t)-\bar{X}^e\right)\left(N-M^S\right)\left(X(t)-\bar{X}^e\right)\right.\right. \\ &\left.\left. + \left(X(t)-\bar{X}^e\right)M^A\dot{X}(t) + V_{eff}(X(t))\right)\right)\mathcal{D}(X(t)) \end{aligned} \tag{74}$$

External perturbations - shocks - may be added by the mean of a linear term $X(t)J(t)$ often refered to as "the source terme". It describes the linear response of the system to a general and arbitrary external perturbation. The form of the transition function, or Green function, in (74) allows to compute, analyticaly for a quadratic effective action, or as a series expansion (see below) when $V_{eff}(X(t))$ is introduced, the stochastic pattern of a system deviating from it's static equilibrium $\bar{X}^e$.

5.2 Transition function for the quadratic case

Putting aside the perturbations $V(X_i(t)) + X(t)J(t)$, but keeping the quadratic potential term which is relevant for usual dynamic systems, the Green function associated to

$$\int_0^t dt U_{eff}^{quad}(X(t)) = \frac{1}{4}\dot{X}(t)\left(N+M^S\right)\dot{X}(t) + \left(X(t)-\bar{X}^e\right)\left(N-M^S\right)\left(X(t)-\bar{X}^e\right) + \dot{X}(t)M^A\left(X(t)-\bar{X}^e\right) \tag{75}$$

is obtained in a way similar to the discrete case arising in the individual agent problem (basic example of the first section or Appendix 2). Since the effective utility (75) is quadratic, the computation of (74) reduces to a saddle point computation. We thus need to compute (75) for a classical solution X^c of the Euler Lagrange equation :

$$\frac{1}{2}\left(N+M^S\right)\ddot{X}(t)+\left(\left(M^{(A)}\right)^t-M^{(A)}\right)\dot{X}(t)-2\left(N-M^S\right)\left(X(t)-\left(\tilde{X}\right)\right)=0 \tag{76}$$

That will be inserted in the action:

$$-\left(\frac{1}{4}\dot{X}(t)\left(N+M^S\right)\dot{X}(t)+\left(X(t)-\bar{X}^e\right)\left(N-M^S\right)\left(X(t)-\bar{X}^e\right)+\dot{X}(t)M^A\left(X(t)-\bar{X}^e\right)\right)$$

with initial conditions:

$$X(t)=X^0 \text{ and } X(t+s)=X^1$$

and the exponential of the result, after a suitable normalization, will be $P\left(X^1,t+s\mid X^0,t\right)$.

$$\begin{aligned}
&\int_0^t dt U_{eff}^{quad}\left(X^c(t)\right)\\
=&-\int_0^t\left(\frac{1}{4}\dot{X}(t)\left(N+M^S\right)\dot{X}(t)+\left(X(t)-\bar{X}^e\right)\left(N-M^S\right)\left(X(t)-\bar{X}^e\right)+\dot{X}(t)M^A\left(X(t)-\bar{X}^e\right)\right)dt\\
=&-\left[\frac{1}{4}\dot{X}^c(t)\left(N+M^S\right)X^c(t)\right]_0^t\\
&+\int_0^t\left(\frac{1}{4}X^c(t)\left(N+M^S\right)\ddot{X}^c(t)-\left(X^c(t)-\bar{X}^e\right)M^A\dot{X}^c(t)-\left(X^c(t)-\bar{X}^e\right)\left(N-M^S\right)\left(X^c(t)-\bar{X}^e\right)\right)dt
\end{aligned}$$

Given, the equation of motion for $X^c(t)$, the second term becomes

$$\begin{aligned}
&\int_0^t\left(\frac{1}{4}X^c(t)\left(N+M^S\right)\ddot{X}^c(t)-\left(X^c(t)-\bar{X}^e\right)M^A\dot{X}^c(t)-\left(X^c(t)-\bar{X}^e\right)\left(N-M^S\right)\left(X^c(t)-\bar{X}^e\right)\right)dt\\
=&\int_0^t\left(\frac{1}{2}\left(X^c(t)-\bar{X}^e\right)\left(-\left(\left(M^A\right)^t-M^A\right)\dot{X}^c(t)+2\left(N-M^S\right)\left(X^c(t)-\left(\tilde{X}\right)\right)\right)\right.\\
&\left.-\left(X^c(t)-\bar{X}^e\right)M^A\dot{X}^c(t)-\left(X^c(t)-\bar{X}^e\right)\left(N-M^S\right)\left(X^c(t)-\bar{X}^e\right)\right)dt\\
=&-\frac{1}{2}\int_0^t\left(X^c(t)-\bar{X}^e\right)\left(\left(\left(M^A\right)^t+M^A\right)\dot{X}^c(t)\right)dt\\
=&\ 0
\end{aligned}$$

since $M^{(A)}$ is antisymetric, and we are led to:

$$\int_0^t dt U_{eff}^{quad}\left(X^c(t)\right)=-\frac{1}{4}\left[\left(X^c(t)-\bar{X}^e\right)\left(N+M^S\right)\dot{X}^c(t)\right]_0^t$$

To find this last expression one needs to compute $X^c(t)$. We rewrite (76) as:

$$\ddot{X}(t)+A\dot{X}(t)+B\left(X^c(t)-\bar{X}^e\right)=0$$

with:

$$\begin{aligned}
A&=\left(\frac{1}{2}\left(N+M^S\right)\right)^{-1}\left(\left(M^A\right)^t-M^A\right)=-4\left(\left(N+M^S\right)\right)^{-1}M^A\\
B&=-\left(\frac{1}{2}\left(N+M^S\right)\right)^{-1}\left(N-M^S\right)
\end{aligned} \tag{77}$$

and set $\left(X^c(t)-\bar{X}^e\right)=\exp\left(-\frac{At}{2}\right)X'(t)$ so that $X'(t)$ satisfies:

$$\begin{aligned}
\left(\frac{A}{2}\right)^2X'(t)+\ddot{X}'(t)-\frac{A^2}{2}X'(t)+BX'(t)&=0\\
\ddot{X}'(t)+\left(B-\frac{A^2}{4}\right)X'(t)&=0
\end{aligned}$$

Diagonalizing $\frac{A^2}{4}-B$ allows to find $\sqrt{\frac{A^2}{4}-B}$ and

$$\left(X^c(s)-\bar{X}^e\right)=\exp\left(-\frac{As}{2}\right)\left(\exp\left(\sqrt{\frac{A^2}{4}-B}s\right)\alpha+\exp\left(-\sqrt{\frac{A^2}{4}-B}s\right)\beta\right)$$

Now we can use the initial conditions:

$$\begin{aligned} X^c(0) &= x \\ X^c(t) &= y \end{aligned}$$

to find the coefficients α and β:

$$\begin{aligned} x-\left(\tilde{X}\right) &= \alpha+\beta \\ y-\left(\tilde{X}\right) &= \exp\left(-\frac{At}{2}\right)\left(\exp\left(\sqrt{\frac{A^2}{4}-B}t\right)\alpha+\exp\left(-\sqrt{\frac{A^2}{4}-B}t\right)\beta\right) \end{aligned}$$

and ultimately, the classical solution is:

$$\left(X^c(s)-\bar{X}^e\right)=\exp\left(-\frac{As}{2}\right)\left(\frac{\sinh\left(\sqrt{\frac{A^2}{4}-B}\,(t-s)\right)}{\sinh\left(\sqrt{\frac{A^2}{4}-B}t\right)}\left(x-\left(\tilde{X}\right)\right)+\exp\left(\frac{At}{2}\right)\frac{\sinh\left(\sqrt{\frac{A^2}{4}-B}s\right)}{\sinh\left(\sqrt{\frac{A^2}{4}-B}t\right)}\left(y-\left(\tilde{X}\right)\right)\right)$$

Therefore the statistical weight we are looking for is:

$$\begin{aligned} & \int_0^t dt U_{eff}^{quad}\left(X^c(t)\right) \\ = & -\frac{1}{4}\left[\left(X^c(t)-\left(\tilde{X}\right)\right)\left(N+M^S\right)\dot{X}^c(t)\right]_0^t \\ = & -\frac{1}{4}\left(y-\left(\tilde{X}\right)\right)\left(N+M^S\right)\left(\begin{array}{c} -\frac{A}{2}\left(y-\left(\tilde{X}\right)\right)-\exp\left(-\frac{At}{2}\right)\frac{\sqrt{\frac{A^2}{4}-B}}{\sinh\left(\sqrt{\frac{A^2}{4}-Bt}\right)}\left(x-\left(\tilde{X}\right)\right) \\ +\cosh\left(\sqrt{\frac{A^2}{4}-B}t\right)\frac{\sqrt{\frac{A^2}{4}-B}}{\sinh\left(\sqrt{\frac{A^2}{4}-Bt}\right)}\left(y-\left(\tilde{X}\right)\right) \end{array}\right) \\ & +\frac{1}{4}\left(x-\left(\tilde{X}\right)\right)\left(N+M^S\right)\left(\begin{array}{c} -\frac{A}{2}\left(x-\left(\tilde{X}\right)\right)-\cosh\left(\sqrt{\frac{A^2}{4}-B}t\right)\frac{\sqrt{\frac{A^2}{4}-B}}{\sinh\left(\sqrt{\frac{A^2}{4}-Bt}\right)}\left(x-\left(\tilde{X}\right)\right) \\ +\exp\left(\frac{At}{2}\right)\frac{\sqrt{\frac{A^2}{4}-B}}{\sinh\left(\sqrt{\frac{A^2}{4}-Bt}\right)}\left(y-\left(\tilde{X}\right)\right) \end{array}\right) \end{aligned}$$

which can be written:

$$\begin{aligned}
&\int_0^t dt U_{eff}^{quad}\left(X^c(t)\right) \qquad (78)\\
&= -\frac{\left(y-\left(\tilde{X}\right)\right)}{2}\left[\left(N+M^S\right)\left(\cosh\left(\sqrt{\frac{A^2}{4}-B}t\right)\frac{\sqrt{\frac{A^2}{4}-B}}{\sinh\left(\sqrt{\frac{A^2}{4}-B}t\right)}\right)\right]\frac{\left(y-\left(\tilde{X}\right)\right)}{2}\\
&-\frac{\left(x-\left(\tilde{X}\right)\right)}{2}\left[\left(N+M^S\right)\left(\cosh\left(\sqrt{\frac{A^2}{4}-B}t\right)\frac{\sqrt{\frac{A^2}{4}-B}}{\sinh\left(\sqrt{\frac{A^2}{4}-B}t\right)}\right)\right]\frac{\left(x-\left(\tilde{X}\right)\right)}{2}\\
&+\frac{\left(y-\left(\tilde{X}\right)\right)}{2}\left(\left(\left(N+M^S\right)\exp\left(-\frac{At}{2}\right)\frac{\sqrt{\frac{A^2}{4}-B}}{\sinh\left(\sqrt{\frac{A^2}{4}-B}t\right)}\right)+\left(\left(N+M^S\right)\exp\left(\frac{At}{2}\right)\frac{\sqrt{\frac{A^2}{4}-B}}{\sinh\left(\sqrt{\frac{A^2}{4}-B}t\right)}\right)^t\right)\\
&\times\frac{\left(x-\left(\tilde{X}\right)\right)}{2}
\end{aligned}$$

The normalization can be now introduced, as usually done for propagation of quadratic potential:

$$P\left(y,t+s\mid x,t\right)=\frac{1}{\sqrt{\det\left(\frac{M}{\pi}\right)}}\exp\left(\int_0^t dt U_{eff}^{quad}\left(X^c(t)\right)\right)$$

Where M is the matrix defined by:

$$\int_0^t dt U_{eff}^{quad}\left(X^c(t)\right)=\begin{pmatrix} x-\left(\tilde{X}\right)\\ y-\left(\tilde{X}\right)\end{pmatrix}^t M\begin{pmatrix} x-\left(\tilde{X}\right)\\ y-\left(\tilde{X}\right)\end{pmatrix}$$

This is a direct expression of the propagation kernel for a time span of t. We will give below an example of computation for the transition function $P\left(y,t+s\mid x,t\right)$ in the two agents model previously studied. However before doing so, and to ease the interpretation, it will be useful to separate this expression in two types of contribution.

5.3 Interpretation: harmonic oscillations around the equilibrium

In the previous expressions for $\int_0^t dt U_{eff}^{quad}\left(X^c(t)\right)$, a change of variable:

$$\left(X^c(s)-\tilde{X}\right)'=U\sqrt{\frac{\left(N+M^S\right)}{2}}\left(X^c(s)-\tilde{X}\right) \qquad (79)$$

where U diagonalizes $\sqrt{\frac{A^2}{4}-B}$, i.e. $\sqrt{\frac{A^2}{4}-B} = U\Lambda U^{-1}$ leads to replace the relevant quantities in $\int_0^t dtU_{eff}^{quad}(X^c(t))$ by:

$$\begin{aligned}
\frac{\left(N+M^S\right)}{2} &\rightarrow 1 \\
A &\rightarrow -2\left(\sqrt{\frac{(N+M^S)}{2}}\right)^{-1} M^A \left(\sqrt{\frac{(N+M^S)}{2}}\right)^{-1} \\
B &\rightarrow -\left(\sqrt{\frac{(N+M^S)}{2}}\right)^{-1} \left(N-M^S\right) \left(\sqrt{\frac{(N+M^S)}{2}}\right)^{-1} \\
\sqrt{\frac{A^2}{4}-B} &\rightarrow \Lambda \\
\exp\left(-\frac{At}{2}\right) &\rightarrow U^{-1}\exp\left(-\frac{At}{2}\right)U
\end{aligned}$$

so that the effective quadratic action becomes:

$$\begin{aligned}
\int_0^t dtU_{eff}^{quad}(X^c(t)) &= -\frac{1}{2}\left(y'-\tilde{X}'\right)\left[\frac{\Lambda}{\tanh(\Lambda t)}\right]\left(y'-\tilde{X}'\right) \\
&\quad -\frac{1}{2}\left(x'-\tilde{X}'\right)\left[\frac{\Lambda}{\tanh(\Lambda t)}\right]\left(x'-\tilde{X}'\right)+\left[\left(y'-\tilde{X}'\right)\left(\frac{\Lambda}{\sinh(\Lambda t)}\right)\left(x'-\tilde{X}'\right)\right] \\
&\quad +\frac{1}{2}\left[\left(y'-\tilde{X}'\right)\left(\left(\exp\left(-\frac{At}{2}\right)-1\right)\frac{\Lambda}{\sinh(\Lambda t)}+\frac{\Lambda}{\sinh(\Lambda t)}\left(\exp\left(-\frac{At}{2}\right)-1\right)\right)\left(x'-\tilde{X}'\right)\right]
\end{aligned} \tag{80}$$

The last term in the right hand side represents the interaction between structures induced by the interaction term A. It can be neglected if M^A, which measures the asymetry between the various agents, is relatively small with respect to the other parameters of the system. If we do so, the three first terms on the right hand side describe a sum of harmonic oscillators whose frequencies are given by the eigenvalues of Λ. These oscillator are not the initial structures, but rather some mixed structures involving all the initial agents. They represent some independent and stable patterns arising from the interactions of the system.

This formulation of the effective utility allows in turn to model the system in terms of deep - i.e. fundamental - independent structures whose internal frquencies are given by the $\Lambda_i(t)$. The combination of their fluctuations, plus some interaction leads to the apparent behavior, as an interaction between cycles of different time scales.

5.4 Example of transition function

We will illustrate the computation of the transition functions using the basic example from the first section. The exponential of the effective utility for the two agents' system is then:

$$\begin{aligned}
&\exp\left(-\sum_t\left(U_{eff}(x_t)+U_{eff}(y_t)\right)\right) \\
&= \exp\left(-\sum_t\left(x_t-ay_{t-1}\right)\frac{\alpha}{2a}\left(x_t-ay_{t-1}\right)-\sum_t\left(\frac{1}{2}y_t^2-y_tx_{t-1}\right)\right) \\
&= \exp\left(-\sum_t\left(\frac{\alpha}{2a}x_t^2-\alpha x_ty_{t-1}\right)-\sum_t\left(\frac{1}{2}y_t^2-y_tx_{t-1}\right)\right) \\
&= \exp\left(-\sum_t\left(\frac{\alpha}{2a}x_t^2+\frac{1}{2}y_t^2-\alpha x_ty_{t-1}-y_tx_{t-1}\right)\right)
\end{aligned}$$

where a was defined in (29):

$$a = \frac{1}{2\alpha\beta\left(N+\beta\right)}\left(1+N\beta+N\alpha^2\beta+\beta^2-\sqrt{\left(N\beta+\beta^2-2\alpha\beta-N\alpha^2\beta+1\right)\left(N\beta+\beta^2+2\alpha\beta-N\alpha^2\beta+1\right)}\right)$$

The effective utility has the form of (41) with:

$$\begin{aligned} X\left(t\right) &= \begin{pmatrix} x_t \\ y_t \end{pmatrix}, N = \begin{pmatrix} 1 & 0 \\ 0 & \alpha a \end{pmatrix}, M = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} \\ M^S &= \begin{pmatrix} 0 & \frac{1+\alpha}{2} \\ \frac{1+\alpha}{2} & 0 \end{pmatrix}, M^A = \begin{pmatrix} 0 & \frac{\alpha-1}{2} \\ -\frac{\alpha-1}{2} & 0 \end{pmatrix} \end{aligned} \tag{81}$$

The diagonalized transition function (80)

$$\begin{aligned} \int_0^t dt U_{eff}^{quad}\left(X^c\left(t\right)\right) &= -\frac{1}{2}\left(y'-\tilde{X}'\right)\left[\frac{\Lambda}{\tanh\left(\Lambda t\right)}\right]\left(y'-\tilde{X}'\right) \\ &\quad -\frac{1}{2}\left(x'-\tilde{X}'\right)\left[\frac{\Lambda}{\tanh\left(\Lambda t\right)}\right]\left(x'-\tilde{X}'\right)+\left[\left(y'-\tilde{X}'\right)\left(\frac{\Lambda}{\sinh\left(\Lambda t\right)}\right)\left(x'-\tilde{X}'\right)\right] \\ &\quad +\frac{1}{2}\left[\left(y'-\tilde{X}'\right)\left(\left(\exp\left(-\frac{At}{2}\right)-1\right)\frac{\Lambda}{\sinh\left(\Lambda t\right)}+\frac{\Lambda}{\sinh\left(\Lambda t\right)}\left(\exp\left(-\frac{At}{2}\right)-1\right)\right)\left(x'-\tilde{X}'\right)\right] \end{aligned}$$

can now be computed. The matrices A and B are given by (77) and (81) :

$$\begin{aligned} A &= -4\left(\left(N+M^S\right)\right)^{-1}M^A = -4\begin{pmatrix} 1 & \frac{1+\alpha}{2} \\ \frac{1+\alpha}{2} & \alpha a \end{pmatrix}^{-1}\begin{pmatrix} 0 & \frac{\alpha-1}{2} \\ -\frac{\alpha-1}{2} & 0 \end{pmatrix} \\ &= \begin{pmatrix} -4\frac{(\alpha+1)(\alpha-1)}{-2\alpha+4a\alpha-\alpha^2-1} & -\frac{8a\alpha(\alpha-1)}{-2\alpha+4a\alpha-\alpha^2-1} \\ \frac{8(\alpha-1)}{-2\alpha+4a\alpha-\alpha^2-1} & \frac{4(\alpha+1)(\alpha-1)}{-2\alpha+4a\alpha-\alpha^2-1} \end{pmatrix} \\ B &= -\left(\frac{1}{2}\left(N+M^S\right)\right)^{-1}\left(N-M^S\right) = -2\begin{pmatrix} 1 & \frac{1+\alpha}{2} \\ \frac{1+\alpha}{2} & \alpha a \end{pmatrix}^{-1}\begin{pmatrix} 1 & -\frac{1+\alpha}{2} \\ -\frac{1+\alpha}{2} & \alpha a \end{pmatrix} \\ &= \begin{pmatrix} \frac{2\left(2\alpha+4a\alpha+\alpha^2+1\right)}{2\alpha-4a\alpha+\alpha^2+1} & -\frac{8a\alpha(\alpha+1)}{2\alpha-4a\alpha+\alpha^2+1} \\ -\frac{8(\alpha+1)}{2\alpha-4a\alpha+\alpha^2+1} & \frac{2\left(2\alpha+4a\alpha+\alpha^2+1\right)}{2\alpha-4a\alpha+\alpha^2+1} \end{pmatrix} \end{aligned}$$

$$\frac{A^2}{4}-B = \begin{pmatrix} 2\frac{6\alpha+4a\alpha-\alpha^2-1}{-2\alpha+4a\alpha-\alpha^2-1} & -8a\alpha\frac{\alpha+1}{-2\alpha+4a\alpha-\alpha^2-1} \\ -8\frac{\alpha+1}{-2\alpha+4a\alpha-\alpha^2-1} & 2\frac{6\alpha+4a\alpha-\alpha^2-1}{-2\alpha+4a\alpha-\alpha^2-1} \end{pmatrix}$$

The change of variables (79) is:

$$\left(X^c\left(s\right)-\tilde{X}\right)' = U\sqrt{\frac{\left(N+M^S\right)}{2}}\left(X^c\left(s\right)-\tilde{X}\right)$$

with:

$$\begin{aligned} U &= \begin{pmatrix} 1 & 1 \\ -\frac{1}{a\alpha}\sqrt{a\alpha} & \frac{1}{a\alpha}\sqrt{a\alpha} \end{pmatrix} \\ \sqrt{\frac{\left(N+M^S\right)}{2}} &= \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{2}\sqrt{-\frac{1}{2}X+\frac{1}{2}a\alpha+\frac{1}{2}}\,\frac{X+a\alpha-1}{X} & -\frac{1}{2}\left(\alpha+1\right)\frac{\sqrt{-\frac{1}{2}X+\frac{1}{2}a\alpha+\frac{1}{2}}}{X} \\ \frac{1}{2}\sqrt{\frac{1}{2}X+\frac{1}{2}a\alpha+\frac{1}{2}}\,\frac{X-a\alpha+1}{X} & \frac{1}{2}\left(\alpha+1\right)\frac{\sqrt{\frac{1}{2}X+\frac{1}{2}a\alpha+\frac{1}{2}}}{X} \end{pmatrix} \\ \text{for } X &= \sqrt{a^2\alpha^2-2a\alpha+\alpha^2+2\alpha+2} \end{aligned}$$

and the diagonal matrix Λ is defined by:

$$\sqrt{\frac{A^2}{4}-B} = U\Lambda U^{-1} \text{ with } \Lambda = \begin{pmatrix} \sqrt{2\frac{1+\alpha^2-\left(6\alpha+4a\alpha+4\alpha\sqrt{a\alpha}+4\sqrt{a\alpha}\right)}{1+2\alpha-4a\alpha+\alpha^2}} & 0 \\ 0 & \sqrt{2\frac{1+\alpha^2+4\alpha\sqrt{a\alpha}+4\sqrt{a\alpha}-\left(6\alpha+4a\alpha\right)}{1+2\alpha-4a\alpha+\alpha^2}} \end{pmatrix}$$

For $\alpha > 1$ close to 1 the interaction term

$$\frac{1}{2}\left[\left(y'-\tilde{X}'\right)\left(\left(\exp\left(-\frac{At}{2}\right)-1\right)\frac{\Lambda}{\sinh\left(\Lambda t\right)}+\frac{\Lambda}{\sinh\left(\Lambda t\right)}\left(\exp\left(-\frac{At}{2}\right)-1\right)\right)\left(x'-\tilde{X}'\right)\right]$$

between the two oscillators is negligible, since A is close to 0 so that $\left(\exp\left(-\frac{At}{2}\right)-1\right) << 1$ for any finite span of time. considering α close to 1 is reasonable since it describes mutual interactions between the two agents that are of the same order of magnitude.

We can check that for relatively large degree of uncertainty N and for α close to 1, a is of order α, and the two eigenvalues $2\frac{1+\alpha^2+4\alpha\sqrt{a\alpha}+4\sqrt{a\alpha}-(6\alpha+4a\alpha)}{1+2\alpha-4a\alpha+\alpha^2}$ and $2\frac{1+\alpha^2-\left(6\alpha+4a\alpha+4\alpha\sqrt{a\alpha}+4\sqrt{a\alpha}\right)}{1+2\alpha-4a\alpha+\alpha^2}$ are positive with:

$$\sqrt{2\frac{1+\alpha^2-(6\alpha+4a\alpha+4\alpha\sqrt{a\alpha}+4\sqrt{a\alpha})}{1+2\alpha-4a\alpha+\alpha^2}} > \sqrt{2\frac{1+\alpha^2+4\alpha\sqrt{a\alpha}+4\sqrt{a\alpha}-(6\alpha+4a\alpha)}{1+2\alpha-4a\alpha+\alpha^2}}$$

In our range of parameters the smallest one is close to 0, and the other one is of order 1.

As explained previously, computing the transition function between two states reduces to evaluating the exponential along a "classical" path:

$$P\left(y,t+s\mid x,t\right)=\frac{1}{\sqrt{\det\left(\frac{M}{\pi}\right)}}\exp\left(\int_0^t dtU_{eff}^{quad}\left(X^c\left(t\right)\right)\right)$$

and, given our assumptions, $\int_0^t dtU_{eff}^{quad}\left(X^c\left(t\right)\right)$ reduces approximatively to

$$\begin{aligned}\int_0^t dtU_{eff}^{quad}\left(X^c\left(t\right)\right) &= -\frac{1}{2}\left(y'-\tilde{X}'\right)_1\left[\frac{\Lambda_1}{\tanh\left(\Lambda_1 t\right)}\right]\left(y'-\tilde{X}'\right)_1 \\ &\quad -\frac{1}{2}\left(x'-\tilde{X}'\right)_1\left[\frac{\Lambda_1}{\tanh\left(\Lambda_1 t\right)}\right]\left(x'-\tilde{X}'\right)_1+\left[\left(y'-\tilde{X}'\right)_1\left(\frac{\Lambda_1}{\tanh\left(\Lambda_1 t\right)}\right)\left(x'-\tilde{X}'\right)_1\right]\end{aligned}$$

where Λ_1 is the eigenvalue:

$$\Lambda_1=\sqrt{2\frac{1+\alpha^2-(6\alpha+4a\alpha+4\alpha\sqrt{a\alpha}+4\sqrt{a\alpha})}{1+2\alpha-4a\alpha+\alpha^2}}$$

The subscript 1 assigned to the vectors represents their coordinate along the eigenvector corresponding to Λ_1. This eigenvector describes a mixed structure of x_t and y_t.The transition function for $\left(y'-\tilde{X}'\right)_1=0$ and $\left(x'-\tilde{X}'\right)_1=x$ is proportionnal to

$$\exp\left(-\frac{1}{2}x\left(\frac{\Lambda_1}{\tanh\left(\Lambda_1 t\right)}\right)x\right)$$

A short time approximation looks like a Brownian path with transition function $\exp\left(-\frac{x^2}{2t}\right)$ which describes a diffusion process without interaction. However this approximation is not correct for longer time scales, and the diffusion allows for transitions between far states.

5.5 Non quadratic contributions, perturbation expansion

Up to now we have described the classical - or mean value - dynamics of the whole system of interacting structures, as well as its associated random diffusion process in the case of quadratic utilities through the transition function $P\left(x,y,t\right)$. For non quadratic corrections, the interaction potential $V\left(X_i\left(t\right)\right)+X\left(t\right)J\left(t\right)$ can be introduced as a perturbation. It allows to describe $G_\lambda^{full}\left(x,y\right)$, the Green function for the whole system, as a perturbative series in $V\left(X_i\left(t\right)\right)+X\left(t\right)J\left(t\right)$.

External shocks can aso be introduced through $X(t)J(t)$. Both term are now included in $V_{eff}(X(t))$. The computation of the Green function $P\left(X^1, t+s \mid X^0, t\right)$ is computed by decomposing

$$\begin{aligned}
& P\left(X^1, t+s \mid X^0, t\right) \\
= & \int \exp \int_{X(t)=X^0}^{X(t+s)=X^1}\left(\frac{1}{4} \dot{X}(t) M^S \dot{X}(t)+\left(X(t)-\bar{X}^e\right)\left(N-M^S\right)\left(X(t)-\bar{X}^e\right)\right. \\
& \left.+\dot{X}(t) M^A\left(X(t)-\bar{X}^e\right)+V_{eff}(X(t))\right) \mathcal{D}(X(t)) \\
\equiv & \int \exp \left(\int_{X(t)=X^0}^{X(t+s)=X^1}\left(U_{eff}^{quad}(X(t))+V_{eff}(X(t))\right)\right) \mathcal{D}(X(t))
\end{aligned} \tag{82}$$

and expanding $\exp \left(\int_{X(t)=X^0}^{X(t+s)=X^1}\left(V_{eff}(X(t))\right)\right)$ in series. One then finds $P\left(X^1, t+s \mid X^0, t\right)$ as a sum:

$$\begin{aligned}
& P\left(X^1, t+s \mid X^0, t\right) \\
= & \int\left(\sum_n \frac{1}{n!} \exp \left(\int_{X(t)=X^0}^{X(t+s)=X^1} U_{eff}^{quad}(X(u)) du\right)\left(\int_t^{t+s} V_{eff}(X(u)) du\right)^n\right) \mathcal{D}(X(t)) \\
= & \int\left(\sum_n \frac{1}{n!} \int_{t<u_i<t+s} \prod_{u_i, i=1 \ldots n} du_i \int \exp \left(\int_{X(t)=X^0}^{X(t+s)=X^1} U_{eff}^{quad}(X(u)) du\right) \prod_{u_i, i=1 \ldots n} V_{eff}\left(X\left(u_i\right)\right)\right) \mathcal{D}(X(t)) \\
= & \sum_n \frac{1}{n!} \int_{t<u_i<t+s} \prod_{u_i, i=1 \ldots n} du_i\left(\int \exp \left(\int_{X(t)=X^0}^{X(t+s)=X^1} U_{eff}^{quad}(X(u)) du\right) \prod_{u_i, i=1 \ldots n} V_{eff}\left(X\left(u_i\right)\right) \mathcal{D}(X(t))\right)
\end{aligned} \tag{83}$$

This expression can be simplified by using the convolution properties of:

$$\exp \left(\int_{X(t)=X^0}^{X(t+s)=X^1} U_{eff}^{quad}(X(u)) du\right) \mathcal{D}(X(t)) \equiv P_0\left(X^1, t+s \mid X^0, t\right)$$

which are, in terms of integrals over $X(t)$:

$$P_0\left(X^1, t+s \mid X^0, t\right)=\int P_0\left(X^1, t+s \mid X^{\prime}, t+u\right) P_0\left(X^{\prime}, t+u \mid X^0, t\right) dX^{\prime}$$

and more generally, for arbitrary $u_i, i=1 \ldots n$, with $u_i<u_j$ for $i<j$ and $t<u_j<t+s$:

$$\begin{aligned}
& P_0\left(X^1, t+s \mid X^0, t\right) \\
= & \int\left\{P_0\left(X^1, t+s \mid X_n, u_n\right)\left(\prod_{i=1 \ldots n-1} P_0\left(X_{i+1}, u_{i+1} \mid X_i, u_i\right)\right) P_0\left(X_1, u_1 \mid X^0, t\right)\right\} \prod_{i=1 \ldots n} dX_i
\end{aligned}$$

As a consequence (83) becomes:

$$\begin{aligned}
& \left(\int \exp \left(\int_{X(t)=X^0}^{X(t+s)=X^1} U_{eff}^{quad}(X(u)) du\right) \prod_{u_i, i=1 \ldots n} V_{eff}\left(X\left(u_i\right)\right) \mathcal{D}(X(t))\right) \\
= & \int \prod_{i=1 \ldots n} dX_i\left\{P_0\left(X^1, t+s \mid X_n, u_n\right) V_{eff}\left(X\left(u_n\right)\right)\right. \\
& \left.\times\left(\prod_{i=1 \ldots n-1} P_0\left(X_{i+1}, u_{i+1} \mid X_i, u_i\right) V_{eff}\left(X\left(u_i\right)\right)\right) P_0\left(X_1, u_1 \mid X^0, t\right)\right\}
\end{aligned}$$

and the propagator we are looking for becomes a series of convolutions:

$$\begin{aligned} & P\left(X^1, t+s \mid X^0, t\right) \\ = & \int\left(\sum_n \frac{1}{n!} \exp\left(\int_{X(t)=X^0}^{X(t+s)=X^1} U_{eff}^{quad}\left(X\left(u\right)\right) du\right)\left(\int_t^{t+s} V_{eff}\left(X\left(u\right)\right) du\right)^n\right) \mathcal{D}\left(X\left(t\right)\right) \\ = & \int\left(\sum_n \frac{1}{n!} \int_{t<u_i<t+s} \prod_{u_i, i=1\ldots n} du_i \int \exp\left(\int_{X(t)=X^0}^{X(t+s)=X^1} U_{eff}^{quad}\left(X\left(u\right)\right) du\right) \prod_{u_i, i=1\ldots n} V_{eff}\left(X\left(u_i\right)\right)\right) \mathcal{D}\left(X\left(t\right)\right) \\ = & \sum_n \frac{1}{n!} \int_{t<u_i<t+s} \prod_{u_i, i=1\ldots n} du_i \int \prod_{i=1\ldots n} dX_i \left\{P_0\left(X^1, t+s \mid X_n, u_n\right) V_{eff}\left(X_n\right)\right. \\ & \times\left.\left(\prod_{i=1\ldots n-1} P_0\left(X_{i+1}, u_{i+1} \mid X_i, u_i\right) V_{eff}\left(X_i\right)\right) P_0\left(X_1, t+u_1 \mid X^0, t\right)\right\} \end{aligned} \quad (84)$$

This series can be understood a series of Feynman graph without loops.

For each n, draw $n+1$ lines connecting t, u_1, u_2..., $t+s$. Label each point u_i with $V_{eff}\left(X\left(u_i\right)\right)$. This graph represents the propagation of the system between t and $t+s$. During the intervall of time u_i, u_{i+1}, it propagates "freely" from X_i to X_{i+1}, i.e. with probability $P_0\left(X_{i+1}, t+u_{i+1} \mid X_i, t+u_i\right)$. Then, at u_{i+1}, a perturbation occurs, of magnitude $V_{eff}\left(X\left(u_{i+1}\right)\right)$, and the system propagates again freely between u_i and u_{i+1}. The total contribution to $P\left(X^1, t+s \mid X^0, t\right)$ coming from this graph is then:

$$P_0\left(X^1, t+s \mid X_n, u_n\right) V_{eff}\left(X_n\right)\left(\prod_{i=1\ldots n-1} P_0\left(X_{i+1}, u_{i+1} \mid X_i, u_i\right) V_{eff}\left(X_i\right)\right) P_0\left(X_1, t+u_1 \mid X^0, t\right)$$

The overall transition function is an infinite sum over all possibilities of perturbations at u_i, where the u_i are the times at which the perturbation occurs, and X_i, the points where they occurs.

Let us remark that the previous series can also be obtained through a Laplace transform by defining:

$$G_\lambda\left(x, y\right) = \int dt \exp\left(-\lambda t\right) P\left(x, y, t\right) \quad (85)$$

In that case, the convolutions in time - the integrals over the u_i - are replaced, after Laplace transform, by products of terms. Defining the free propagator:

$$G_\lambda^0\left(x, y\right) = \int dt \exp\left(-\lambda t\right) P_0\left(x, y, t\right)$$

the laplace transform of

$$\begin{aligned} & \sum_n \frac{1}{n!} \int_{t<u_i<t+s} \prod_{u_i, i=1\ldots n} du_i \int \prod_{i=1\ldots n} dX_i \left\{P_0\left(X^1, t+s \mid X_n, u_n\right) V_{eff}\left(X_n\right)\right. \\ & \times\left.\left(\prod_{i=1\ldots n-1} P_0\left(X_{i+1}, u_{i+1} \mid X_i, u_i\right) V_{eff}\left(X_i\right)\right) P_0\left(X_1, t+u_1 \mid X^0, t\right)\right\} \end{aligned}$$

in (84) becomes:

$$= \sum_n \frac{1}{n!} \int \prod_{i=1\ldots n} dX_i G_\lambda^0\left(X^1, X_n\right) V_{eff}\left(X_n\right) \times \left(\prod_{i=1\ldots n-1} G_\lambda^0\left(X_{i+1}, X_i\right) V_{eff}\left(X_i\right)\right) G_\lambda^0\left(X_{i+1}, X^0\right) \quad (86)$$

which is easier to compute. The graphical interpretation is similar to the one developped for (84), Except that the time variable has disappeared. We rather sum over perturbations regardless their time of occurence. The n^{th} term occuring in (86) correspond as before to $n+1$ segments of "free" propagation, purturbed n times by external influences or shocks. ultimately, all these perturbation terms can be formally added, before retrieving the time representation $P\left(x, y, t\right)$ by inverse Laplace transform.

The green function $G_\lambda(x,y)$ not only eases computations : besides its meaning it will prove usefull, for a large number N of agents, to compute the transition function for finitely lived agents whose probability of transition between x and y is a process of random duration t, with Poisson distribution of mean $\frac{1}{\lambda}$. It then describes the mean transition probability for a process with average lifespan of $\frac{1}{\lambda}$ and λ is a characteristic scale for the system with a large number of agents. We will come back to this point later.

6 Introduction of constraints

Up to this point, no constraint has been included in the behavior of the agents. For usual models in Game theory, such as simple oligopolistic models, or independent interacting structure models, this is not a problem. It may however represent a limitation for producers/consumers models, or systems including global constraints in the interactions between independent agents. We will now consider the introduction of constraints, in an exact way for simple cases, or as first approximation in the general case.

To start with an example, we will consider the introduction af a budget constraint for an economic agent optimising a quadratic utility. We will then extend the result to N agents with quadratic utilities and bound by linear arbitrary constraints. We will finally suggest an approach to the general case of arbitrary utility.

6.1 Example: Single agent budget constraint

Consider the example of an agent, endowed with a quadratic utility, whose action vector $X_i(t)$ reduces to its consumption. Successive periods are linked through a current account intertemporal constraint of the following form:

$$C_s = B_s + Y_s - B_{s+1} \tag{87}$$

where Y_s is an exogenous random variable, such as the revenue in the standards optimal control models. For the sake of simplicity, we will discard any discount rate here. The inclusion of a discount rate will be considered later in the context of a large number of agents described by a field theoretic formalism.

Since the successive periods are interconnected through the constraint, when replacing C_s by the state variable B_s, the probability weight studied previously becomes:

$$\exp\left(U(C_s) + \sum_{i>0} U(C_{s+i})\right) = \exp\left(U(B_s + Y_s - B_{s+1}) + \sum_{i>0} U(B_{s+i} + Y_{s+i} - B_{s+i+1})\right) \tag{88}$$

This measures the probability for a choice C_s and C_{s+i}, $i = 1...T$ with T the time horizon, or alternatively the probability for the state variable B, to follow a path $\{B_{s+i}\}_{i\geqslant 0}$ starting from B_s. The time horizon T represents the expectation at time s of the interaction process remaining duration. It should depend decreasingly on s, but will later be supposed a random following a poisson process. As a consequence, the mean expected duration will be a constant written T, whatever s. Integrating over the $\{B_{s+i}\}_{i\geqslant 2}$ yields a probability of transition between B_s and B_{s+1} written $\langle B_{s+1}||B_s\rangle$. The latter is the probability to reach B_{s+1} given B_s and is equal to

$$\langle B_{s+1}||B_s\rangle = \int \prod_{i=2}^{T} dB_{s+i} \exp\left(U(B_s + Y_s - B_{s+1}) + \sum_{i>0} U(B_{s+i} + Y_{s+i} - B_{s+i+1})\right)$$

Computing $\langle B_{s+1}||B_s\rangle$ rather than the transition function for C_s does not change the approach developped previously. It merely has to be applied to the state variable B_s rather than to the control variable C_s. However, due to the overlapping nature of state variables, the probability transition $\langle B_{s+1}||B_s\rangle$ now measures a probability involving two successive periods. The whole point will be to rebuild the probability for the path $\{C_s\}_{i\geqslant 0}$ from the data $\langle B_{s+1}||B_s\rangle$.

To do so, consider a usual quadratic utility function, or at least its second order approximation, of the form, $U(C_s) = -\alpha\left(C_s - \bar{C}\right)^2$, with objective $\bar{C}$. Then rescale $-\alpha\left(C_s - \bar{C}\right)^2 \to -C_s^2$ for the sake of simplicity. The constant $\bar{C}$ can be reintroduced at the end of the computation. The transition probability between two consecutive state variables thus becomes:

$$\begin{aligned}
\langle B_{s+1} | |B_s\rangle &= \int \prod_{i=2}^{T} dB_{s+i} \exp\left(U(C_s) + \sum_{i>0} U(C_{s+i})\right) C_s \\
&= \int \prod_{i=2}^{T} dB_{s+i} \exp\left(-\left(C_s - \bar{C}\right)^2 - \sum_{i>0}\left(C_{s+i} - \bar{C}\right)^2\right) \\
&= \int \prod_{i=2}^{T} dB_{s+i} \exp\left(-\left(B_s + Y_s - B_{s+1} - \bar{C}\right)^2 - \sum_{i>0}\left(B_{s+i} + Y_{s+i} - B_{s+i+1} - \bar{C}\right)^2\right) \\
&= \exp\left(-\left(B_s + Y_s - B_{s+1} - \bar{C}\right)^2 - \frac{1}{T}\left(B_{s+1} + \sum_{i>0}\left(Y_{s+i} - \bar{C}\right)\right)^2\right) \qquad (89)
\end{aligned}$$

with $B_s \to 0$, $s \to T$ to impose the transversality condition. The number of periods, T, is itself unknown, but as said before T is the expected mean process duration.

If Y_{s+i} is centered on $\bar{Y}$ with variance σ^2, $\sum_{i>0} Y_{s+i}$ centered on $\bar{Y}$ with variance $T\sigma^2$, integration over Y_{s+i} yields

$$\begin{aligned}
& \int \prod dY_{s+i} \exp\left(-\frac{1}{T}\left(B_{s+1} + \sum_{i>0}\left(Y_{s+i} - \bar{C}\right)\right)^2 - \frac{1}{\sigma^2}\sum_{i=1}^{T}\left(Y_{s+i} - \bar{Y}\right)^2\right) \\
= & \int \prod dY'_{s+i} \exp\left(-\frac{1}{T}\left(B_{s+1} + \sum_{i=1}^{T}\left(Y'_{s+i} - \left(\bar{C} - \bar{Y}\right)\right)\right)^2 - \frac{1}{\sigma^2}\sum_{i=1}^{T}\left(Y'_{s+i}\right)^2\right)
\end{aligned}$$

with $Y'_{s+i} = Y_{s+i} - \bar{Y}$. The exponential rewrites:

$$\begin{aligned}
& \exp\left(-\frac{1}{T}\left(B_{s+1} + \sum_{i=1}^{T}\left(Y'_{s+i} - \left(\bar{C} - \bar{Y}\right)\right)\right)^2 - \frac{1}{\sigma^2}\sum_{i=1}^{T}\left(Y'_{s+i}\right)^2\right) \\
= & \exp\left(-\frac{1}{T}\left(B_{s+1} - T\left(\bar{C} - \bar{Y}\right)\right)^2 - \frac{2}{T}\left(B_{s+1} - T\left(\bar{C} - \bar{Y}\right)\right)\sum_{i=1}^{T} Y'_{s+i} - \left(\frac{1}{\sigma^2} + \frac{1}{T}\right)\sum_{i=1}^{T}\left(Y'_{s+i}\right)^2\right)
\end{aligned}$$

and the integration over the Y'_{s+i} leads to a weight:

$$\begin{aligned}
& \exp\left(-\frac{1}{T}\left(B_{s+1} - T\left(\bar{C} - \bar{Y}\right)\right)^2 - \frac{2}{T}\left(B_{s+1} - T\left(\bar{C} - \bar{Y}\right)\right)\sum_{i=1}^{T} Y'_{s+i} - \left(\frac{1}{\sigma^2} + \frac{1}{T}\right)\sum_{i=1}^{T}\left(Y'_{s+i}\right)^2\right) \\
= & \exp\left(-\frac{1}{T}\left(B_{s+1} - T\left(\bar{C} - \bar{Y}\right)\right)^2 + \frac{\sigma^2}{T\left(\sigma^2 + T\right)}\left(B_{s+1} - T\left(\bar{C} - \bar{Y}\right)\right)^2\right) \\
= & \exp\left(-\frac{1}{T + \sigma^2}\left(B_{s+1} - T\left(\bar{C} - \bar{Y}\right)\right)^2\right)
\end{aligned}$$

We can now write B_{s+1} as a function of the past variables:

$$B_{s+1} = \sum_{i \leqslant 0} Y_{s+i} - \sum_{i \leqslant 0} C_{s+i} \qquad (90)$$

Along with the expression $B_s + Y_s - B_{s+1} - \bar{C} = C_s - \bar{C}$ to write the global weight (89) as:

$$\exp\left(-\left(C_s-\bar{C}\right)^2-\frac{1}{T+\sigma^2}\left(\sum_{i\leqslant 0}Y_{s+i}-\sum_{i\leqslant 0}C_{s+i}-T\left(\bar{C}-\bar{Y}\right)\right)^2\right)$$
$$\simeq \exp\left(-\left(C_s-\bar{C}\right)^2-\frac{1}{T}\left(\sum_{i\leqslant 0}Y_{s+i}-\sum_{i\leqslant 0}C_{s+i}-T\left(\bar{C}-\bar{Y}\right)\right)^2\right)$$

for a time scale large enough, so that $T>>\sigma^2$. The statistical weight thus becomes:

$$\exp\left(-\left(\frac{T+1}{T}\right)\left(C_s-\frac{T}{T+1}\bar{C}-\frac{1}{T+1}\left(\sum_{i\leqslant 0}Y_{s+i}-\sum_{i<0}C_{s+i}-T\left(\bar{C}-\bar{Y}\right)\right)\right)^2\right)$$
$$= \exp\left(-\left(\frac{T+1}{T}\right)\left(C_s-\frac{1}{T+1}\left(\sum_{i\leqslant 0}Y_{s+i}-\sum_{i<0}C_{s+i}+T\bar{Y}\right)\right)^2\right)$$
$$= \exp\left(-\left(\frac{T+1}{T}\right)\left(C_s-\bar{Y}-\frac{1}{T+1}\left(\sum_{i\leqslant 0}\left(Y_{s+i}-\bar{Y}\right)-\sum_{i<0}\left(C_{s+i}-\bar{Y}\right)\right)\right)^2\right)$$

For $T>>1$, this reduces to:

$$\simeq \exp\left(-\left(C_s-\bar{Y}-\frac{1}{T}\left(\sum_{i\leqslant 0}\left(Y_{s+i}-\bar{Y}\right)-\sum_{i<0}\left(C_{s+i}-\bar{Y}\right)\right)\right)^2\right)$$

and defining $\hat{C}_s=C_s-\bar{Y}$, we are left with:

$$\exp\left(-\left(\hat{C}_s-\frac{1}{T}\left(\sum_{i\leqslant 0}\hat{Y}_{s+i}-\sum_{i<0}\hat{C}_{s+i}\right)\right)^2\right)\propto \exp\left(-\left(\hat{C}_s\right)^2+\frac{2\hat{C}_s}{T}\left(\sum_{i\leqslant 0}\hat{Y}_{s+i}-\sum_{i<0}\hat{C}_{s+i}\right)\right)$$

The global weight, over all periods is then:

$$\exp\left(-\sum_s\left(\hat{C}_s\right)^2-\frac{1}{T}\sum_{s_1,s_2}\hat{C}_{s_1}\hat{C}_{s_2}+\frac{2}{T}\sum_{s_1\geqslant s_2}\hat{C}_{s_1}\hat{Y}_{s_2}\right) \tag{91}$$

As a consequence, the introduction of a constraint is equivalent to the introduction of non local interaction terms. The non local terms may, in some cases, be approximated by some terms in the derivatives of C_s. Actually, remark that the quadratic terms

$$\frac{1}{T}\sum_{s_1,s_2}C_{s_1}C_{s_2}$$

can be approximated by a sum of local terms through a taylor expansion of C_{s_2}. Indeed, writing the n - arbitrary - lag contribution:

$$\frac{1}{T}\sum_{s_1}C_{s_1}C_{s_1-n}=\frac{1}{T}\sum_{s_1}C_{s_1}\left(C_{s_1-n}-C_{s_1-(n-1)}+C_{s_1-(n-1)}-...+C_{s_1}\right) \tag{92}$$

introduces derivatives of C_{s_1}. For example, the term for $n = 0$, i.e. $\frac{1}{T}\sum_{s_1} C_{s_1}C_{s_1}$ shifts the quadratic potential, and the term for $n = 1$

$$\frac{1}{T}\sum_{s_1} C_{s_1}C_{s_1-1} = \frac{1}{4T}\sum_{s_1} (C_{s_1} + C_{s_1-1})^2 - (C_{s_1} - C_{s_1-1})^2$$

becomes in the continuous approximation

$$\frac{1}{T}\int (C_s)^2 \, ds - \frac{1}{4T}\int \left(\frac{d}{ds}C_s\right)^2 ds$$

Similarly, (92) can be written in the continuous approximation as a linear combination of terms;

$$\frac{1}{T}\int C_s \left(\sum_{p=1}^{n} a_p \frac{d^p}{ds^p}C_s\right)$$

with integer coefficients a_p. Integrating by parts and neglecting the border terms we are led to a sum:

$$-\frac{1}{T}\int \left(\sum_{\substack{p=1 \\ p \text{ even}}}^{n} (-1)^p a_p \left(\frac{d^{\frac{p}{2}}}{ds^{\frac{p}{2}}}C_s\right)^2\right) \tag{93}$$

These terms do not, in general, have to be expanded very far. Actually, when several agents interact through short term interactions, some inertia naturally appears. When an inertia term $-\alpha\int\left(\frac{d}{ds}C_s\right)^2 ds$ is added in the utility $-(C_s)^2$, the characteristic time of interaction is of order $\frac{1}{\sqrt{\alpha}}$, and the agent is behaving in first approximation as an oscillator described by an effective utility:

$$-(C_s)^2 - \alpha\int \left(\frac{d}{ds}C_s\right)^2 ds$$

and in that case, in first approximation:

$$C_s \approx C_t \cos\left(\frac{s-t}{\sqrt{\alpha}}\right)$$

so that

$$\int dtdsC_sC_t \approx C_t^2 \cos\left(\frac{s-t}{\sqrt{\alpha}}\right) = \int dtC_t^2$$

as a consequence, the interaction term $\frac{1}{T}\sum_{s_1} C_{s_1}C_{s_1-1}$ reduces to a correction to the quadratic term. We will see later how to deal with the whole contribution $\frac{1}{T+\sigma^2}\sum_{s_1,s_2} C_{s_1}C_{s_2}$ when considering a large number of interacting agents in the context of a field formulation. However, if one is interested in only one agent behavior, one can, in first approximation, keep only the one lag correction term to account for an action including the constraint:

$$\exp\left(-\sum_s \left(C_s - \frac{\sigma^2}{T+\sigma^2}\bar{C}\right)^2 - \frac{1}{T+\sigma^2}\sum_s C_sC_{s-1} + \frac{2}{T+\sigma^2}\sum_{s_1,s_2} C_{s_1}\left(E_{s_1}Y_{s_2}\right)\right)$$

or, which is equivalent, using (93):

$$\exp\left(-\sum_s \left(C_s - \frac{\sigma^2}{T+\sigma^2}\bar{C}\right)^2 - \frac{1}{T+\sigma^2}\sum_s \left(\frac{d}{ds}C_s\right)^2 + \frac{2}{T+\sigma^2}\sum_{s_1,s_2} C_{s_1}\left(E_{s_1}Y_{s_2}\right)\right) \tag{94}$$

6.2 Case of N>>1 agents

Until now, computations in this section were performed under the assumption that the constraint included some exogenous variable Y_s. For a system of N agents however, constraints are more likely imposed on agents by the entire set of interacting agents. For example, in the previous paragraph, the variable Y_s in the constraint (87) represented the agent's revenue. In the context of N interacting agents, this variable depends on others activity, or in our simple model, on their consumption. Actually, in a system of consumer/producer, the others' consumption generates the flow of revenue Y_s. In other word, agent i revenue Y_s^i depends on other agents' consumptions C_s^j - or possibly C_{s-1}^j if we assume a lag between agents actions and their effect. More generally, for a system with a large number of agents, the revenue Y_s^i, may depend on endogenous variables that can still be considered as exogenous in agent i's perspective. Thus our benchmark hypothesis in this section will be that agents are too numerous to be manipulated by a single agent. Therefore the procedure developped in the previous section to introduce a constraint for a single agent remains valid and can be generalized directly. Again, we will impose a constraint for each agent and encode it in Y_s or $\bar{Y}$. First Y_s will be considered as exogenous by the individual agent and thus (94) will apply. Then (94) will be modified to take into account the fact that Y_s depends endogenously on other agents. Assume for example that $Y_s^i = \sum \alpha_j^i C_{s-1}^{(j)}$. The term $\frac{2}{T}\sum_{s_1,s_2} C_{s_1}\left(E_{s_1}^i Y_{s_2}^i\right)$ can then be replaced in (91): $E_{s_1}Y_{s_2} \rightarrow \sum \alpha_j^i C_{s_2-1}^{(j)}$ if $s_2 < s_1$. We will need to find $E_{s_1}^i\ C_{s_2-1}^{(j)}$ for $s_2 > s_1$ If we assume that agents' forecasts $C_{s_2-1}^{(j)}$ have a gaussian random error and the number of agents N is large, the sum of errors in $\sum \alpha_j^i C_{s_2-1}^{(j)}$ cancels out. Note that here we rule out a collective mistake that could otherwise be reintroduced. As a consequence one can replace $\frac{2}{T}\sum_{s_1,s_2} C_{s_1}\left(E_{s_1}^i Y_{s_2}^i\right) \rightarrow \sum_{s_1,s_2} \alpha_j^i C_{s_1} C_{s_2-1}^{(j)}$.

Thus the interaction terms for an agent i in (91) becomes:

$$\frac{2}{T}\sum_{s_1,s_2} C_{s_1}^{(i)}\left(E_{s_1}Y_{s_2}\right) \rightarrow \sum_j \int\int \alpha_j^i C_s^{(i)} C_t^{(j)} dsdt \tag{95}$$

To sum up, the introduction of several agents translates the constraints as some non local interactions between agents, and each agent constraint is shaped by the environment others created. Similarly, the quadratic term becomes:

$$-\frac{1}{T}\left(\sum_{s_1,s_2}\left(E_{s_1}Y_{s_2}\right)\right)^2 \rightarrow -\frac{1}{T}\sum_{j_1,j_2}\int\int \alpha_{j_1}^i \alpha_{j_2}^i C_s^{(j_1)} C_t^{(j_2)} dsdt$$

This cannot be integrated out, but yields a contribution to the system's statistical weight:

$$\begin{aligned} &\frac{2}{T}\sum_{i,j}\int\int \alpha_j^i C_s^{(i)} C_t^{(j)} dsdt - \frac{1}{T}\sum_{i,j_1,j_2}\int\int \alpha_{j_1}^i \alpha_{j_2}^i C_s^{(j_1)} C_t^{(j_2)} dsdt \\ = &\ \frac{1}{T}\sum_{i,j}\int\int\left(2\alpha_j^i - \sum_k \alpha_i^k \alpha_i^k\right) C_s^{(i)} C_t^{(j)} dsdt \end{aligned}$$

and consequently, for the system as a whole, including the constraint leads considering the term in the effective utility:

$$\frac{1}{T}\sum_{i,j}\int\int\left(2\alpha_j^i - \sum_k \alpha_i^k \alpha_i^k\right) C_s^{(i)} C_t^{(j)} dsdt \equiv \sum_{i,j}\int V\left(C_s^{(i)}, C_t^{(j)}\right) dsdt \tag{96}$$

Writing the constraints in term of a potential terms $V\left(C_s^{(i)}, C_t^{(j)}\right)$ allow taking into account, when necessary, some non linear constraints modelled by the form of the potential $V\left(C_s^{(i)}, C_t^{(j)}\right)$. Gathering these results leads the global statistical weight for the set of agents as a continuous time version of (91):

$$\exp\left(U^{eff}\right) = \exp\left(-\sum_i \int \left(C_s^i\right)^2 ds - \frac{1}{T}\sum_i \int\int C_s^i C_t^i dsdt - \frac{1}{T}\sum_i \int\int C_s^i C_t^j dsdt - \frac{1}{T}\sum_{i,j}\int\int V\left(C_s^{(i)}, C_t^{(j)}\right) dsdt\right) \tag{97}$$

Keeping only the first contributions of inertia terms $\frac{1}{T}\int\int C_s^i C_t^i dsdt$ as in the previous paragraph would lead to:

$$\begin{aligned}\exp\left(U^{eff}\right) &= \exp\left(-\sum_i\int\left(1+\frac{\alpha}{T}\right)\left(C_s^i\right)^2 ds-\frac{\beta}{T}\int\int\left(\frac{d}{ds}C_s^i\right)^2 ds\right. \quad (98)\\ &\left.-\frac{1}{T}\sum_i\int\int C_s^i C_t^j dsdt-\frac{1}{T}\sum_{i,j}\int\int V\left(C_s^{(i)},C_t^{(j)}\right)dsdt\right)\end{aligned}$$

where α and β are constants depending on the expansion of $\sum_i\int\int C_s^i C_t^j$ and the parameters of the system.

6.3 Quadratic effective utility with constraints, general case for large N

We can now apply these methods to the more general model of interacting agents with quadratic utilities presented above. Recall the form for the effective action without constraint (35)

$$\begin{aligned}U_{eff}\left(X_i(t)\right) &= \frac{1}{2}\left(X_i(t)-\bar{X}_i^{(i)e}\right)N_{ii}\left(X_i(t)-\bar{X}_i^{(i)e}\right)-\left(X_i(t)-\bar{X}_i^{(i)e}\right)\frac{M_{ii}}{\sqrt{\beta}}\left(X_i(t)-\bar{X}_i^{(i)e}\right)\\ &-\sum_{j<i}\left(X_i(t)-\bar{X}_i^{(i)e}\right)\frac{M_{ij}}{\sqrt{\beta}}\left(X_j(t-1)-\bar{X}_j^{(i)e}\right)+\sum_{j>i}2X_i(t)A_{ij}^{(i)}\left(X_j(t-1)\right)\end{aligned}$$

It is found recursively by starting from the less informed agents. Including a linear constraint can be done in the following way. Assume as before a constraint of the form:

$$X_i(t)=B_i(t)+Z_i(t)-B_i(t+1) \quad (99)$$

where $Z_i(t)$ is other agents' exogenous influence. Due to the large number of agents involved in the interaction process, we suppose each agent may at best influence those surrounding agents on which it has a strategic advantage. We can therefore assume that their weight in the whole set of agents is negligible. As a consequence, the term $Z_i(t)$ being the other agents' influence, and beyond the control of any agent, it must be considered exogenous.. Once this is specified, we can then introduce in the effective utility a term $X_i(t)M_i\left(E_t^{(i)}\sum_s Z_i(s)\right)$ with $E_t^{(i)}Z_i(s)=Z_i(s)$ for $s\leqslant t$ and $E_t^{(i)}Z_i(s)=\text{constant}$ for $s>t$, where M_i is found recursively, which yields:

$$\begin{aligned}U_{eff}\left(X_i(t)\right) &= \frac{1}{2}\left(X_i(t)-\bar{X}_i^{(i)e}\right)N_{ii}\left(X_i(t)-\bar{X}_i^{(i)e}\right)-\left(X_i(t)-\bar{X}_i^{(i)e}\right)\frac{M_{ii}}{\sqrt{\beta}}\left(X_i(t)-\bar{X}_i^{(i)e}\right) \quad (100)\\ &-\sum_{j<i}\left(X_i(t)-\bar{X}_i^{(i)e}\right)\frac{M_{ij}}{\sqrt{\beta}}\left(X_j(t-1)-\bar{X}_j^{(i)e}\right)+\sum_{j>i}2X_i(t)A_{ij}^{(i)}\left(X_j(t-1)\right)\\ &+\sum_{j\leqslant i}X_i(t)K_{ij}^{(i)}\left(E_t^{(i)}\sum_s Z_j(s)\right)\end{aligned}$$

and apply the methods presented in Appendix 2 to find $U_{eff}\left(X_i(t)\right)$ given the $U_{eff}\left(X_j(t)\right)$, $j<i$. Note however that we have included agent i constraint by replacing $X_i(t)=B_i(t)+Z_i(t)-B_i(t+1)$, and imposed the transversality condition $B_i(t)\to 0,\ t\to T$). For detailed computations and results, see Appendix 4.

The matrices are given by:

$$
\begin{aligned}
N_{ii} &= \left(P^t\left((A-C)D\right)P+2(B-A)\right)_{ii} \\
&\quad -\left(P^t\left((A-C)D\right)P+2(B-A)\right)_{ij}^{-1}\left(\left(P^t\left((A-C)D\right)P+2(B-A)\right)_{jj}\right)\left(\left(P^t\left((A-C)D\right)P+2(B-A)\right)_{ji}\right) \\
&\quad +\left(P_i^t\left((A-C)D\right)\tilde{P}\left(N_{ii}\right)\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1}+P_i^t\left((A-C)D\right)P_i\right) \\
M_{ii} &= \left(N_{ii}\right)\left(\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1}(A+C)\right)_{ii} \\
&\quad +\left(P_i^t\left((A-C)D\right)\tilde{P}\left(N_{ii}\right)\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1}+P_i^t\left((A-C)D\right)P_i\right) \\
M_{ij} &= \left(N_{ii}\right)\left(\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1}(A+C)\right)_{ij} \\
M_i &= \left(P_i^t\left((A-C)D\right)\tilde{P}\left(N_{ii}\right)\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1}+P_i^t\left((A-C)D\right)P_i\right)
\end{aligned} \tag{101}
$$

$$
\begin{aligned}
K_{ij}^{(i)} &= \begin{pmatrix} N_{ii}+M_{ii} & M_{ij}\end{pmatrix}\begin{pmatrix} A_{ii}^{(i)}+B_{11}-\sqrt{\beta}\epsilon_{ii}^{(i)} & \left\{B_{12},2\sqrt{\beta}\left(A_{ij}^{(i)}\right)^S\right\} \\ \left\{B_{12}^t,2\sqrt{\beta}\left(A_{ji}^{(j)}\right)^S\right\} & \left\{\begin{array}{c}\left(A_{jj}^{(j)}\right)_{eff}+\beta A_{jj}^{(i)},B_{22}, \\ \left(2\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff},2A_{\{kj\}i>k>j}^{(j)}\right)^S\end{array}\right\}\end{pmatrix}^{-1} \\
&\quad\times\begin{pmatrix} \frac{B_{12}^{(3)}}{2}\left(N_{jj}\right)^{-1}K_{ij}^{(i)}\left(\sum_{j\leqslant i}E_t^{(i)}\sum_s Z_j(s)\right) \\ \left\{\left(A_{jj}^{(j)}\right)_{eff},B"_{22},\frac{B_{22}^{(3)}}{2},\sqrt{\beta}\left(\frac{\epsilon_{\{kj\}k\leqslant j}^{(j)}}{2}\right)_{eff}\right\}\left(N_{jj}\right)^{-1}K_{ij}^{(i)}\left(\sum_{j\leqslant i}E_t^{(i)}\sum_s Z_j(s)\right)\end{pmatrix}
\end{aligned}
$$

where D is the solution of (37) and:

$$
\begin{aligned}
P &= \begin{pmatrix} P_i & 0 \\ 0 & 1_j\end{pmatrix},\tilde{P}=\begin{pmatrix} 0 & 0 \\ 0 & 1_j\end{pmatrix} \\
P_i &= \left(\frac{(1-D)}{D\left(1-D^T\right)}\right) \\
1_j &= \text{identity matrix for the block } j<i
\end{aligned}
$$

The effective utility thus obtained includes the constraint $\sum_{j\leqslant i}X_i(t)K_{ij}^{(i)}\left(E_t^{(i)}\sum_s Z_j(s)\right)$ that mixes the agent action with some external dynamic variable, that may include the contribution of the whole set of agents perceived as an externality, as in (95). Note that, compared to (91), a quadratic but non local in time term $X_i(t)X_i(s)$ arises in the effective utility. The reason is that we have considered the same approximation as in the example of the consumer with a budget constraint (the first example of the previous paragraph) and kept only in these quadratic interactions the most relevant terms, $X_i(t)X_i(t-1)$. Appendix 8 shows however that the full analog of (91) as well as an exact effective utility with constraint could be retrieved, for a total result of:

$$
\begin{aligned}
U_{eff}\left(X_i(t)\right) &= \frac{1}{2}\left(X_i(t)-\bar{X}_i^{(i)e}\right)N_{ii}^{(0)}\left(X_i(t)-\bar{X}_i^{(i)e}\right)-\left(X_i(t)-\bar{X}_i^{(i)e}\right)\frac{M_{ii}^{(0)}}{\sqrt{\beta}}\left(X_i(t)-\bar{X}_i^{(i)e}\right) \\
&\quad -\sum_{j<i}\left(X_i(t)-\bar{X}_i^{(i)e}\right)\frac{M_{ij}^{(0)}}{\sqrt{\beta}}\left(X_j(t-1)-\bar{X}_j^{(i)e}\right)+\sum_{j>i}2X_i(t)A_{ij}^{(i)}\left(X_j(t-1)\right) \\
&\quad +\sum_{j\leqslant i}X_i(t)K_{ij}^{(i)}\left(E_t^{(i)}\sum_s Z_j(s)\right)+\sum_{j\leqslant i}\sum_{s<t}X_j(s)\epsilon_{ij}^{(i,n)}X_i(t)
\end{aligned} \tag{102}
$$

that includes the quadratic terms $\sum_{j\leqslant i}\sum_{s<t}X_j(s)\epsilon_{ij}^{(i,n)}X_i(t)$ that were, in (100), reduced to the "one lag" approximation.

6.4 Non quadratic utilities with constraints

We conclude this section by considering some constraints within the context of non quadratic utilities. To do so, we start with a simple example and consider the budgent constraint (87) for a single agent:

$$C_s = B_s + Y_s - B_{s+1} \tag{103}$$

At time t, the agent's statistical weight has the general form (88):

$$\int \prod_{i>1} \exp\left(U\left(B_s + Y_s - B_{s+1}\right) + \sum_{i>0} U\left(B_{s+i} + Y_{s+i} - B_{s+i+1}\right)\right) dB_{s+i} \tag{104}$$

Performing the following change of variables for $i > 1$:

$$\begin{aligned} B_{s+i} &\rightarrow B_{s+i} - \sum_{j\geqslant i} Y_{s+j} \\ B_{s+i} + Y_{s+i} - B_{s+i+1} &\rightarrow B_{s+i} - B_{s+i+1} \end{aligned}$$

the statistical weight (104) become:

$$\int \prod_{i>1} \exp\left(U\left(B_s + Y_s - B_{s+1}\right) + U\left(B_{s+1} - B_{s+2} + \sum_{i\geqslant 1} Y_{s+i}\right) + \sum_{i>1} U\left(B_{s+i} - B_{s+i+1}\right)\right) dB_{s+i} \tag{105}$$

Except for the case of a quadratic utility function, the successive integrals

$$\int \prod_{i>1} \exp\left(U\left(B_{s+1} - B_{s+2} + \sum_{i\geqslant 1} Y_{s+i}\right) + \sum_{i>1} U\left(B_{s+i} - B_{s+i+1}\right)\right) dB_{s+i} \tag{106}$$

arising in (105) cannot be computed exactly, . However, we can still define a function $\check{U}\left(B_{s+1}\right)$ resulting from the convolution integrals (106):

$$\exp\left(\check{U}\left(B_{s+1} + \sum_{i\geqslant 1} Y_{s+i}\right)\right) = \int \exp\left(U\left(B_{s+1} - B_{s+2} + \sum_{i\geqslant 1} Y_{s+i}\right) + \sum_{i>1} U\left(B_{s+i} - B_{s+i+1}\right)\right) \prod_{i>1} dB_{s+i} \tag{107}$$

The function $\check{U}$ can be approximatively computed - we will comment on that later in the paragraph - however its precise form is not needed here. Instead, we use the general formula (107) to write (105) as:

$$\exp\left(U\left(B_s + Y_s - B_{s+1}\right) + \check{U}\left(B_{s+1} + \sum_{i\geqslant 1} Y_{s+i}\right)\right) \tag{108}$$

Here again (see the first paragraph of this section), we can get rid of the variables Y_{s+i} by considering them to be gaussian random variables centered on $\bar{Y}$ for $i \geqslant 1$. The transition probability for B_s is obtained by integrating (108) over the variables Y_{s+i} :

$$\int \prod dY_{s+i} \exp\left(U\left(B_s + Y_s - B_{s+1}\right) + \check{U}\left(B_{s+1} + \sum_{i\geqslant 1} Y_{s+i}\right) - \frac{1}{\sigma^2} \sum_{i>0} \left(Y_{s+i} - \bar{Y}\right)^2\right) \tag{109}$$

This expression can be simplified. Actually, in the gaussian integrals:

$$\int \prod dY_{s+i} \exp\left(\check{U}\left(B_{s+1} + \sum_{i\geqslant 1} Y_{s+i}\right) - \frac{1}{\sigma^2} \sum_{i>0} \left(Y_{s+i} - \bar{Y}\right)^2\right) \tag{110}$$

the variable $\sum_{i\geqslant 1} Y_{s+i}$ has mean $T\bar{Y}$ and variance $T\sigma^2$. As a consequence, if we assume T large enough so that $\sqrt{T} >> \sigma$, then

$$\sum_{i\geqslant 1} Y_{s+i} \simeq T\bar{Y} \pm \sqrt{T}\sigma \simeq T\bar{Y}$$

in first approximation. This allows to simplify (110):

$$\begin{aligned}
&\int \prod dY_{s+i} \exp\left(\check{U}\left(B_{s+1} + \sum_{i\geqslant 1} Y_{s+i}\right) - \frac{1}{\sigma^2}\sum_{i>0}\left(Y_{s+i} - \bar{Y}\right)^2\right) \\
\simeq\ & \exp\left(\check{U}\left(B_{s+1} + T\bar{Y}\right)\right)
\end{aligned}$$

so that, using the constraint (103) to write B_{s+1} as a function of the past variables:

$$\begin{aligned}
B_{s+1} + \sum_{i\geqslant 1} Y_{s+i} &= \sum_{i\leqslant 0} Y_{s+i} - \sum_{i\leqslant 0} C_{s+i} + \sum_{i\geqslant 1} Y_{s+i} \\
&\simeq \sum_{i\leqslant 0} Y_{s+i} - \sum_{i\leqslant 0} C_{s+i} + T\bar{Y}
\end{aligned}$$

the weight (109) results in:

$$\begin{aligned}
&\int \prod dY_{s+i} \exp\left(U\left(B_s + Y_s - B_{s+1}\right) + \check{U}\left(B_{s+1} + \sum_{i\geqslant 1} Y_{s+i}\right) - \frac{1}{\sigma^2}\sum_{i>0}\left(Y_{s+i} - \bar{Y}\right)^2\right) \quad (111) \\
\simeq\ & \exp\left(U\left(B_s + Y_s - B_{s+1}\right) + \check{U}\left(\sum_{i\leqslant 0} Y_{s+i} - \sum_{i\leqslant 0} C_{s+i} + T\bar{Y}\right)\right)
\end{aligned}$$

We can consider that the term:

$$\sum_{i\leqslant 0} Y_{s+i} - \sum_{i\leqslant 0} C_{s+i} + T\bar{Y}$$

has relatively small fluctuations with respect to its average $T\bar{Y}$, we can approximate $\check{U}$ by its second order expansion:

$$\check{U}\left(\sum_{i\leqslant 0} Y_{s+i} - \sum_{i\leqslant 0} C_{s+i} + T\bar{Y}\right) \simeq C - \gamma\left(\sum_{i\leqslant 0} Y_{s+i} - \sum_{i\leqslant 0} C_{s+i} + \bar{Y}\right)^2$$

the values of C and γ depending on (107). Then, up to the irrelevant constant C, (111) simplifies to the second order approximation:

$$\begin{aligned}
&\exp\left(U\left(B_s + Y_s - B_{s+1}\right) - \gamma\left(\sum_{i\leqslant 0} Y_{s+i} - \sum_{i\leqslant 0} C_{s+i} + \bar{Y}\right)^2\right) \quad (112) \\
=\ & \exp\left(U\left(C_s\right) - \gamma\left(\sum_{i\leqslant 0} Y_{s+i} - \sum_{i\leqslant 0} C_{s+i} + \bar{Y}\right)^2\right)
\end{aligned}$$

a result similar to the first example of this section. The constraint can be introduced as a quadratic and non local contribution to the utility $U(C_s)$. This result is not surprising. The constraint being imposed on the whole path of the system, the inclusion of its intertemporal quadratic expansion enforces the constraint on average, as needed.

Ultimately, the system statistical weight (112) can be summed over all periods, to give:

$$\exp\left(\sum_s U\left(C_s\right)-\gamma\sum_s\left(\sum_{i\leqslant 0}Y_{s+i}-\sum_{i\leqslant 0}C_{s+i}+\bar{Y}\right)^2\right)$$

We can modify the previous expression accordingly, as we did in the case of a quadratic utility, when the variables Y_{s+i} and $\bar{Y}$ depend on the interactions with other agents. The procedure leading to (97) can therefore be followed and yields the statistical weight for a set of agents with non quadratic utilities and constraints:

$$\exp\left(U^{eff}\right)=\exp\left(-\sum_i\int U\left(C_s^i\right)ds-\frac{\gamma}{T}\sum_i\int\int C_s^iC_t^i dsdt-\frac{\gamma}{T}\sum_i\int\int C_s^iC_t^j dsdt-\frac{1}{T}\sum_{i,j}\int\int V\left(C_s^{(i)},C_t^{(j)}\right)dsdt\right) \tag{113}$$

In (97), the first terms represent the individual utilities, the second and the third model the constraint binding the agents, and V is an arbitrary potential of interaction between agents.

Let us close this section by quickly discussing the form of the function $\check{U}$ defined by (107):

$$\exp\left(\check{U}\left(B_{s+1}+\sum_{i\geqslant 1}Y_{s+i}\right)\right)=\int\exp\left(U\left(B_{s+1}-B_{s+2}+\sum_{i\geqslant 1}Y_{s+i}\right)+\sum_{i>1}U\left(B_{s+i}-B_{s+i+1}\right)\right)\prod_{i>1}dB_{s+i} \tag{114}$$

These integrals can be approximatively computed with the saddle path approximation technique developped in the first and second sections. The saddle path result is not exact for a non quadratic utility, but constitutes a sufficient approximation for us. The saddle path (114) for the function inside the exponential can be written as a difference equation B_{s+i} with $i>1$:

$$U'\left(B_{s+i}-B_{s+i+1}\right)-U'\left(B_{s+i-1}-B_{s+i}\right)=0 \text{ for } i>2$$

and:

$$U'\left(B_{s+1}-B_{s+2}\right)-U\left(B_{s+1}-B_{s+2}+\sum_{i\geqslant 1}Y_{s+i}\right)\ \text{ for } i=2$$

Once the saddle path $\bar{B}_{s+i}$ is found, it can be introduced in (114) to yield:

$$\exp\left(\check{U}\left(B_{s+1}+\sum_{i\geqslant 1}Y_{s+i}\right)\right)=\exp\left(U\left(B_{s+1}-\bar{B}_{s+2}+\sum_{i\geqslant 1}Y_{s+i}\right)+\sum_{i>1}U\left(\bar{B}_{s+i}-\bar{B}_{s+i+1}\right)\right)$$

and a first approximation for $\check{U}$ is thus:

$$\check{U}\left(B_{s+1}+\sum_{i\geqslant 1}Y_{s+i}\right)=U\left(B_{s+1}-\bar{B}_{s+2}+\sum_{i\geqslant 1}Y_{s+i}\right)+\sum_{i>1}U\left(\bar{B}_{s+i}-\bar{B}_{s+i+1}\right) \tag{115}$$

Some corrections to the saddle path can be included if we expand the RHS to the second order around the saddle point by letting

$$B_{s+i}=\bar{B}_{s+i+1}+\delta B_{s+i}$$

and then integrate over δB_{s+i}:

$$\begin{aligned}
&\exp\left(\check{U}\left(B_{s+1}+\sum_{i\geqslant 1}Y_{s+i}\right)\right) \qquad (116)\\
=\;&\exp\left(U\left(B_{s+1}-\bar{B}_{s+2}+\sum_{i\geqslant 1}Y_{s+i}\right)+\sum_{i>1}U\left(\bar{B}_{s+i}-\bar{B}_{s+i+1}\right)\right)\\
&\times\int\exp\left(U''\left(B_{s+1}-\bar{B}_{s+2}+\sum_{i\geqslant 1}Y_{s+i}\right)\left(\delta B_{s+2}\right)^2\right.\\
&\left.\qquad+\sum_{i>1}U''\left(\bar{B}_{s+i}-\bar{B}_{s+i+1}\right)\left(\delta B_{s+i}-\delta B_{s+i+1}\right)^2\right)\prod_{i>1}d\delta B_{s+i}
\end{aligned}$$

The log of the integrals in (116) will yield some corrections to (115), but we will not inspect further the precise form of these corrections.

7 Fundamental structures and non local interactions: toward large N systems

The system studied until now had a relatively small number of interacting agents. To later adapt the formalism to a system with a large number of agents, two points have to be developped. First we will justify the need for non local (in time) interactions between an arbitrary number of agents, even without constraints. Second, it is usefull to come back to the Laplace transform of the Green function, and give a more accurate account of its necessity.

7.1 Fundamental structures and non local interactions

We have found the transition functions for quadratic effective utilities. The potential term acting as an interaction term was developped perturbatively and provided an expansion for the transition functions for any interaction potential. In the following, we will show how some simplification may arise and reduce the system to sums of independent subsystems, called the fundamental structures.

To do so, rewrite the action (73):

$$\begin{aligned}
U_{eff}\left(X\left(t\right)\right)\;=\;&\int\left(\frac{1}{4}\dot{X}\left(t\right)\left(N+M^{S}\right)\dot{X}\left(t\right)+\left(X\left(t\right)-\bar{X}^{e}\right)\left(N-M^{S}\right)\left(X\left(t\right)-\bar{X}^{e}\right)+\dot{X}\left(t\right)M^{A}\left(X\left(t\right)-\bar{X}^{e}\right)\right.\\
&\left.+V_{eff}\left(X\left(t\right)\right)\right)dt
\end{aligned}$$

and rescale the variables:

$$\begin{aligned}
\sqrt{N+M^{S}}X\left(t\right)&\rightarrow X\left(t\right) \qquad (117)\\
\left(\sqrt{N+M^{S}}\right)^{-1}M^{(A)}\left(\sqrt{N+M^{S}}\right)^{-1}&\rightarrow M^{(A)}\\
\left(\sqrt{N+M^{S}}\right)^{-1}\left(N-M^{S}\right)\left(\sqrt{N+M^{S}}\right)^{-1}&\rightarrow\left(N-M^{S}\right)
\end{aligned}$$

where $\sqrt{N+M^{S}}$ is a square root obtained through the Jordan form of $N+M^{S}$ obtained through the Jordan form. The matrix $\sqrt{N+M^{S}}$ is symmetric. Consequently, the effective utility rewrites:

$$\begin{aligned}
&-\frac{1}{2}\dot{X}\left(t\right)\dot{X}\left(t\right)-\dot{X}\left(t\right)M^{(A)}\left(X\left(t\right)-\left(\tilde{X}\right)\right)-\left(X\left(t\right)-\left(\tilde{X}\right)\right)\left(N-M^{S}\right)\left(X\left(t\right)-\left(\tilde{X}\right)\right)\\
=\;&-\frac{1}{2}\left(\dot{X}\left(t\right)+M^{(A)}\left(X\left(t\right)-\left(\tilde{X}\right)\right)\right)\left(\dot{X}\left(t\right)+M^{(A)}\left(X\left(t\right)-\left(\tilde{X}\right)\right)\right)\\
&-\left(X\left(t\right)-\left(\tilde{X}\right)\right)\left(N-M^{S}-\left(M^{(A)}\right)^{t}\left(M^{(A)}\right)\right)\left(X\left(t\right)-\left(\tilde{X}\right)\right)
\end{aligned}$$

The fact that $M^{(A)}$ and $\left(N-M^{S}\right)$ can be simultaneously diagonalized by blocks - for example if $\left(N-M^{S}\right)$ is proportional to the identity as will be assumed here - leads to a sum of independent subsystems.

$$\begin{aligned}
&\rightarrow && -\frac{1}{2}\left(\dot{\hat{X}}(t)+\left(M_k^{(A)}\right)\left(\hat{X}(t)-\left(\tilde{X}\right)\right)\right)\left(\dot{\hat{X}}(t)+\left(M_k^{(A)}\right)\left(\hat{X}(t)-\left(\tilde{X}\right)\right)\right)\\
&&& -\left(\hat{X}(t)-\left(\tilde{X}\right)\right)\left(\left(N-M^{S}\right)_k-\left(M_k^{(A)}\right)^t\left(M_k^{(A)}\right)\right)\left(\hat{X}(t)-\left(\tilde{X}\right)\right)\\
&= && \sum_k -\frac{1}{2}\left(\dot{\hat{X}}_k(t)+M_k^{(A)}\left(\hat{X}_k(t)-\left(\tilde{X}\right)\right)\right)\left(\dot{\hat{X}}_k(t)+M_k^{(A)}\left(\hat{X}_k(t)-\left(\tilde{X}\right)\right)\right)\\
&&& -\left(\hat{X}_k(t)-\left(\tilde{X}\right)\right)\left(\left(N-M^{S}\right)_k-\left(M_k^{(A)}\right)^t\left(M_k^{(A)}\right)\right)\left(\hat{X}_k(t)-\left(\tilde{X}\right)\right)
\end{aligned}$$

where $\left(M_k^{(A)}\right)$ and $\left(N-M^{S}\right)_k$ are block diagonal matrices, whose blocks are written respectively $M_k^{(A)}$ and $\left(N-M^{S}\right)_k$. Change the coordinates according to the eigenblocks of Λ. Each $\hat{X}_k(t)$ defines an independent structure, or equivalently the whole set of $\left\{\hat{X}_k(t)\right\}$ are of different type or species. These species correspond to mixed structures, combinations of several agents or substructures. In a psycho-economic perspective, they account for both conscious-unconscious structures. Note that it is a vague reminder of the Lacan/Mobius strip. We will call these mixed structures, the fundamental structures. Remark that if each block is itself diagonalized so that $\left(N-M^{S}\right)_k-\left(M_k^{(A)}\right)^t\left(M_k^{(A)}\right)\rightarrow\Lambda_{ef}$ then, by a change of basis

$$-\frac{1}{2}\left(\dot{X}_{ef}(t)-\tilde{M}_{ef}X_{ef}(t)\right)\left(\dot{X}_{ef}(t)-\tilde{M}_{ef}X_{ef}(t)\right)+\frac{1}{2}\left(X_{ef}(t)-\left(\hat{Y}^{(1)}\right)_{ef}\right)\Lambda_{ef}\left(X_{ef}(t)-\left(\hat{Y}^{(1)}\right)_{ef}\right) \tag{118}$$

represents a sum of n independent structures, each having its own fundamental frequencies given by Λ_{ef}. This translates the independence of these structures in terms independent oscillations.

Remark also that, in a more comprehensive setting, the appearance of $\tilde{M}_{ef}$ reminds of the evolution of a system on a curved manifold. The connexion of this space is tracked in $\tilde{M}_{ef}$ and takes into account internal tension inside an independent structure. This tension induces a non trivial, i.e. curved, trajectory. The apparent coherence of motion reflects the independence and internal coherence of each of these structure. Inversely, a break down in coherence, i.e. continuity of the motion may come from a singularity in the metric.

Once the fundamental blocks or structure are isolated, they evolve independently. This is the mark of the stationnarity or stability of the system. The only interactions are local and internal to each block, tracked by the curved classical trajectory.

For psychological agents/structures however, the local in time interaction may not be relevant. Actually, for this type of models we are rather interested in "structures to structures" interactions, independent from any causality. By this, we mean a type of interaction involving the global form of each structures. Mathematically, it translates into a non local interaction involving the whole dynamic path of each interacting structure: i.e. the interaction cannot be reduced to time to time action/reaction schemes.

Besides, we saw that for models including a binding constraint between agents, these constraints where not local, but involved all periods as a whole. In large scale models, each agent participate to others' environment. As such, in this case also, interactions are seen, not as time to time action-reaction scheme, but rather as global.

Large scale or global interaction between structures must therefore be introduced in our formalism. They may take several forms, and describe inter and intra species inter-relations. These interactions will be added through constraints representing long term, and not local in time, interaction.

Thus, whatever the kind of system consider, be it a large N economic system, or a large population of structures with long-term interactions, non-local in time interactions have to be added to the system. We have shown above that these interactions have the form:

$$\sum_n \int V_n\left(X_1(s_1)...X_n(s_n)\right)ds_1...ds_n$$

where the variables $X_i(s_i)$ define the control variables of a fundamental structure "i" and V_n stands for any potential of interaction (including the case of a linear constraint). We will see later how the formalism can be modified to account for these terms when the number of agents is large.

7.1.1 Green function as a kernel of operator

Alternatively, the Green function can be described through an operator formalism that will prove useful for a larger N. Using the generic effective action (73):

$$\begin{aligned} U_{eff}(X(t)) &= \int\left(\frac{1}{4}\dot{X}(t)\left(N+M^S\right)\dot{X}(t)+\left(X(t)-\bar{X}^e\right)\left(N-M^S\right)\left(X(t)-\bar{X}^e\right)+\dot{X}(t)M^A\left(X(t)-\bar{X}^e\right)\right. \\ &\left.+V_{eff}(X(t))\right)dt \end{aligned}$$

and rewriting it to include M^A in the kinetic term.

$$\begin{aligned} U_{eff}(X(t)) &= \int\left(\frac{1}{4}\left(\dot{X}(t)+2\left(N+M^S\right)^{-1}\left(M^A\right)\left(X(t)-\bar{X}^e\right)\right)\left(N+M^S\right)\right. \\ &\times\left(\dot{X}(t)+2\left(N+M^S\right)^{-1}M^A\left(X(t)-\bar{X}^e\right)\right) \\ &\left.-\left(X(t)-\bar{X}^e\right)\left(N-M^S-\left(M^A\right)^t\left(N+M^S\right)^{-1}M^A\right)\left(X(t)-\bar{X}^e\right)+V_{eff}(X(t))\right)dt \end{aligned}$$

The transition function associated to this functional is known satisfy (see [10]) :

$$\begin{aligned} \frac{\partial}{\partial t}P(x,y,s) &= \nabla\left(M^{(S)}+N\right)^{-1}\left(\nabla+\left(M^A\right)\left(x-\bar{X}^e\right)\right) \\ &-\left(x-\bar{X}^e\right)\left(N-M^S-\left(M^A\right)^t\left(N+M^S\right)^{-1}M^A\right)\left(x-\bar{X}^e\right) \end{aligned}$$

A Laplace transform of the above equation replaces the derivative in times by a multiplication by α, and $G(x,y,\alpha)$, the Laplace transform of $P(x,y,s)$ satifies:

$$\begin{aligned} \delta(x-y) &= \left(-\nabla\left(M^{(S)}+N\right)^{-1}\left(\nabla+\left(M^A\right)\left(x-\bar{X}^e\right)\right)\right. \\ &\left.+\left(x-\bar{X}^e\right)\left(N-M^S-\left(M^A\right)^t\left(N+M^S\right)^{-1}M^A\right)\left(x-\bar{X}^e\right)+\alpha\right) \\ &\times G(x,y,\alpha) \end{aligned} \tag{119}$$

Namely, the propagator $G_\lambda(x,y)$ (85) satisfies (119). It is thus the kernel of a differential operator, and as such satisfies:

$$\begin{aligned} G^{-1}(x,y,\alpha) &= \left(-\nabla\left(M^{(S)}+N\right)^{-1}\left(\nabla+\left(M^A\right)\left(x-\bar{X}^e\right)\right)\right. \\ &\left.+\left(x-\bar{X}^e\right)\left(N-M^S-\left(M^A\right)^t\left(N+M^S\right)^{-1}M^A\right)\left(x-\bar{X}^e\right)+\alpha\right)\delta(x-y) \end{aligned} \tag{120}$$

As an example, if we were to specialize to the fundamental structure k that appeared in the previous paragraph and whose effective action (118) after change of variable was

$$-\frac{1}{2}\left(\dot{X}_{ef}(t)-\tilde{M}_{ef}X_{ef}(t)\right)\left(\dot{X}_{ef}(t)-\tilde{M}_{ef}X_{ef}(t)\right)+\frac{1}{2}\left(X_{ef}(t)-\left(\hat{Y}^{(1)}\right)_{ef}\right)\Lambda_{ef}\left(X_{ef}(t)-\left(\hat{Y}^{(1)}\right)_{ef}\right)$$

one would obtain the green function as the inverse of a differential operator for each fundamental structure. More precisely:

$$G^{-1}=-\frac{1}{2}\left(\nabla_k\right)\left(\nabla_k-\tilde{M}_{ef}\left(\hat{X}_k-\left(\tilde{X}\right)_k\right)\right)+\frac{1}{2}\left(\hat{X}_k-\left(\tilde{X}\right)_k\right)\Lambda_{ef}\left(\hat{X}_k-\left(\tilde{X}\right)_k\right)$$

Ultimately, and more generally, the analogy between non quadratic utility and the dynamic on a curved variety mentionned above leads us to consider the possibility of Green function in a more general form:

$$G^{-1} = \frac{1}{2}(\nabla_i) m_{ia}(x)\left(\nabla_j m_{ja}(x) - \tilde{M}'x\right) + \frac{1}{2}\left(x - \left(\hat{Y}^{(1)}\right)\right)\hat{N}\left(x - \left(\hat{Y}^{(1)}\right)\right) \tag{121}$$

where $m_{ia}(x)$ is the vielbein associated to the metric $M_{ij}^{-1} = m_{ia}m_{ja}$. This possibility would stem from non quadratic utility contributions included in the coeficients $m_{ia}(x)$. The idea remains the same however: the internal tension inside each structure induces a kind of "curved" trajectory.

The utility of representing the green function as the inverse of a diferential operator will appear in the next section, but the idea is the following: for a large number of agents, a different point of view is necessary. Rather than describing an assembly of N agents, it is more usefull to consider a medium constituted by the assembly of agents, in which we can study the actions and interactions of an agent with others. The Green function previously described participates to this description. The second order differential operator associated to G^{-1} will model the basic displacement operator, i.e. the diffusion process, associated to an agent in the surrounding.

8 Half Phenomenological model for interactions between large number agents

We now use the results of the previous sections to transform the formalism in a collective representation, in terms of fields, that will allow modelling systems with large number of agents.

8.1 Transition toward field theoretic formulation. Laplace transform

The results of the previous section can be summed up as follows. We described a set of several individual economic agents by a stochastic process defined in a space whose dimension depends of the number of degrees of freedom, that is number of state variables, of the system. For the sake of the exposition, we will choose a simplified version of the model developped previously, in its continuous time version. Each agent's behavior can be represented during a time span of s by a probability weight for each possible path of actions. For a path $x(t)$ of actions - such as consumption, production, signals - for $t \in [0, s]$, the weight is:

$$\exp\left(-\int_{x(0)=x}^{x(s)=y}\left(\frac{\dot{x}^2}{2}(t) + K(x(t))\,dt\right)\right)$$

where $K(x(t))$ is a "potential term" whose form depends explicitely on the agent's utility function, or any other intertemporal function the agent optimizes.

The term $\frac{\dot{x}^2}{2}(t)$ represents an inertia term that may be induced by the externalities, the agents's environment, or some constraint function in first approximation. We may associate to this probability weight the probability of transition between states x and y, that is the sum of these probabilities for all possible paths:

$$P(x, y, s) = \int \mathcal{D}x(t)\exp\left(-\int_{x(0)=x}^{x(s)=y}\left(\frac{\dot{x}^2}{2}(t) + K(x(t))\,dt\right)\right) \tag{122}$$

It represents the probability for an agent to reach y starting from x during the time span s. It is the probability of social mobility - moving from point x to y - for an agent in the social space. Written under this form, the probability transition (122) is given by a path integral: The weight in the exponential includes a random, brownian motion, plus a potential K describing the individual goals as well as social/economical influences. It can be seen as an intertemporal utility whose optimization would yield the usual brownian noise plus some external determinants.

Now we can consider interactions between N agents in two ways. The first one is local. Interactions between agents are direct: an agent's action implies a reaction at the next period, and the weight associated to the system has the form:

$$\exp\left(-\int_{x(0)=x}^{x(s)=y}\left(\sum_i \frac{\dot{x}_i^2}{2}(t)+\sum_i K(x_i(t))\,dt+\sum_{i,j}\alpha_{i,j}x_i(t)\,\dot{x}_j(t)\right)\right)$$

The quadratic interaction term $x_i(t)\,\dot{x}_j(t)$ between agents could be generalized by a potential

$$V(x_1(t),\dot{x}_1(t),...,x_i(t),\dot{x}_i(t),...,x_N(t),\dot{x}_N(t))$$

This type of interaction describes strong interactions as well as possibly strategic domination relations between agents.

This inclusion of local interactions can be set in a more compact form. By concatenating the agents' actions in one vector $X(t)$ whose dimension is the sum of the dimension of the $x_i(t)$. The total weight for $X(t)$ has the form:

$$\exp\left(-\int_{x(0)=x}^{x(s)=y}\left(\frac{1}{2}\dot{X}(t)\,M\dot{X}(t)+K(X(t))\right)dt\right) \tag{123}$$

where the matrix M encompasses the terms with derivatives (inertial or interaction terms). In other words, the whole system can be described by a single path integral in a space of configuration which is the sum of the individual configuration space, reflecting the strong interaction between agents.

The second kind of interactions is non local in time and may arise in two cases. The first one arises from constraints agents impose on others. In standard economic models, the consumption function is subject to the budget constraint, itself determined by a flow of income. This flow of income depends in turn on the overall agents' behavior. This implies interactions between the system's various agents. Besides, when forward looking behavior and usual intertemporal optimization are accounted for, the resulting interaction becomes non local. The action's effective utility then becomes:

$$\exp\left(-\int_{x(0)=x}^{x(s)=y}\left(\sum_i \frac{\dot{x}_i^2}{2}(t)+\sum_i K(x_i(t))\,dt+\int\sum_{i,j}V(x_i(t),x_j(s))\,dsdt\right)\right)$$

and the potential term $V(x_i(t),x_j(s))$ reflects the interaction through the constraint and the potential term $V(x_i(t),x_j(s))$ reflects the interaction through the constraint.

The second case where non local interaction may arise in our context comes back to (123). The effective utility may, in some cases, be diagonalized in some fundamental structures, and written as a sum of independent terms:

$$\frac{1}{2}\dot{X}(t)\,M\dot{X}(t)+K(X(t))=\sum_k\left(\frac{1}{2}\dot{X}_k(t)\,M_k\dot{X}_k(t)+K_k(X_k(t))\right)$$

Since the probability weight of the system is a product of each structure weight, these structures have independent dynamics. However, one may want to include some previously neglected interactions. Since each structure has a long term persistence, one may assume that the whole set of agents shapes the environment of each agent, considered individually. This type of interaction may be modelled by constraints, or more generally non local interactions.

Including these types of interactions yield the following effective utility:

$$-\sum_k\left(\frac{1}{2}\dot{X}_k(t)\,M\dot{X}_k(t)+K(X_k(t))\right)+\sum_{l=1}^{N}\sum_{k_1,...,k_l}V(X_{k_1}(s_1),...,X_{k_l}(s_l))$$

and the path integral:

$$\exp\left(-\sum_k\int_0^s\left(\frac{1}{2}\dot{X}_k(t)\,M_k\dot{X}_k(t)+K_k(X_k(t))\right)ds_k+\sum_l\int_{0<s_i<s}\sum_{k_1,...,k_l}V(X_{k_1}(s_1),...,X_{k_l}(s_l))\right)$$
$$\times\mathcal{D}X_1(s_1)...\mathcal{D}X_n(s_n) \tag{124}$$

where the potential terms include all possible non local interactions between the several fundamental structures. This type of model includes the several cases mentionned just above. The local interactions are included in a system from which some fundamental structures emerge. Then the non local interactions and constraints arise as non local interactions between these fundamental structures.

Our aim would now be to deal with such models, but for a large number of agents. However, since the number of variables $X_k(t)$ increases with N, (124) becomes untractable when N becomes large. As a consequence our formalism needs to be simplified or modified to deal with a large number of agents.

We can do so by first supposing that the agents involved in (124) are not so entirely heterogenous that they would have different effective utilities. We rather expect agents to belong to broad classes or types. Inside each class, differences arise from the internal uncertainty present from the beginning, from interaction terms among a class, or with the other classes. It is these internal uncertainty and interactions that will provide statistical differences results among the various types of agents.

Second, since (124) describes an interaction process with a duration - or agents' lifespan s -we might assume that this duration, for a large number of agents, may vary among interacting agents, or group of agents.

To model this, we use the single agent transition function $P(x,y,s)$ and compute its Laplace transform:

$$G_K(x,y,\alpha)=\int\exp(-\alpha s)\int\mathcal{D}x(t)\exp\left(-\int_{x(0)=x}^{x(s)=y}\left(\frac{\dot{x}^2}{2}(t)+K(x(t))\,dt\right)\right)ds$$

This expression models the transition function between x and y for an agent whose lifespan is a Poisson process of average $\frac{1}{\alpha}$. It fits well for a large number of agents whose interaction duration varies among the population. The Poisson law has the advantage, among others, to describe a memory-free process. So that, at each period, the same law will model the probability for the remaining time of interaction. Describing the system in terms of $G_K(x,y,\alpha)$ is a step toward the modelling of large N systems. It models a mean transition function for a set of agents with random lifespan duration (or more generally, the duration of the interaction process), where agents are themselves unaware of the length of this duration.

The green function $G_K(x,y,\alpha)$ is the one worked out in the previous section for an arbitrary effective utility, along with a kinetic term $\frac{\dot{x}^2}{2}(t)$ induced by interactions, inertia, and or constraints. We quoted previously that $G_K(x,y,\alpha)$ can be seen as the inverse of an operator. Actually, it is the laplace transform of $P(x,y,s)$, with $P(x,y,s)$ solving the usual laplacian equation:

$$\frac{\partial}{\partial s}P(x,y,s)=\left(\frac{1}{2}\nabla^2-K(x)\right)P(x,y,s)$$

As a consequence its Laplace transform $G_K(x,y,\alpha)$ satisfies:

$$\left(-\frac{1}{2}\nabla^2+\alpha+K(x)\right)G_K(x,y,\alpha)=\delta(x-y)\tag{125}$$

Considering the description in term of Laplace transforms, the path integral to consider for the whole set of agents becomes:

$$\begin{aligned}&\int\exp(-\alpha s)\,ds\int\exp\left(-\sum_k\int_0^s\left(\frac{1}{2}\dot{X}_k(t)\,M_k\dot{X}_k(t)+K_k(X_k(t))\right)ds_k\right.\\&\left.\qquad+\sum_l\int_{0<s_i<s}\sum_{k_1,\dots,k_l}V\left(X_{k_1}(s_1),\dots,X_{k_l}(s_l)\right)\right)\times\mathcal{D}X_1(s_1)\dots\mathcal{D}X_n(s_n)\end{aligned}\tag{126}$$

Or, if we consider different average lifespan for the various agents:

$$\begin{aligned}&\int\prod_i\exp\left(-\alpha_i s^{(i)}\right)ds_i\int\exp\left(-\sum_k\int_0^s\left(\frac{1}{2}\dot{X}_k(t)\,M_k\dot{X}_k(t)+K_k(X_k(t))\right)ds_k\right.\\&\left.\qquad+\sum_l\int_{0<s_i<s}\sum_{k_1,\dots,k_l}V\left(X_{k_1}(s_1),\dots,X_{k_l}(s_l)\right)\right)\times\mathcal{D}X_1(s_1)\dots\mathcal{D}X_n(s_n)\end{aligned}\tag{127}$$

Up to the Laplace transform, (127) is the description we adopted in the previous sections. The third adaptation we have to perform on the model starts with formula (127). Indeed, the sum of potentials $\sum_{l}\int_{0<s_i<s^{(i)}}\sum_{k_1,...,k_l}V\left(X_{k_1}\left(s_1\right),...,X_{k_l}\left(s_l\right)\right)$ accounts for interactions between several types of agents, some of whom may involve numerous structures. Our description being statistical, it should average over interactions involving a variable number of agents of various types, which would allow to describe both the interactions of a large number of agents in average, and the evolution of a small number of structures in the whole set of agents' environment. This can be performed by resorting to the following device: rather than considering (127) directly with a large number of agents (that is a sum for $k=1,...,N$ with N large), where among the sum, the agents are divided into few classes of identical agents, one will sum over systems with variable number of agents from 1 to $N\rightarrow\infty$. Consider a single type of identical agents. We will generalize the procedure to different types later. The so called Grand Partition Function for a set of N interacting individual paths associated to the partition function (127):

$$\sum_{N}\frac{1}{N!}\prod_{i=1}^{N}\int\exp\left(-\alpha s_i\right)\int\mathcal{D}x_i\left(t\right)\exp\left(-\sum_{i}\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}\left(t\right)+K\left(x_i\left(t\right)\right)dt\right)\right.$$
$$\left.-\sum_{k=2}^{A}\sum_{i_1,...i_k}\int_{x_i(0)=x}^{x_i(s)=y}V_k\left(x_{i_1}\left(t_1\right)...x_{i_k}\left(t_k\right)\right)dt_1...dt_k\right) \tag{128}$$

Up to the sum over N, this is the - Laplace transformed - transition function for a system of N identical agents interacting through the potentials $V_k\left(x_1\left(t\right)...x_k\left(t\right)\right)$. We assume arbitrary interaction processes through the potentials $V_k\left(x_1\left(t\right)...x_k\left(t\right)\right)$, with A standing for the maximal number of agents in interaction. Recall that the Nth term in (128) computes the transition probability between $\left\{x_i\right\}_{i=1...N}$ and $\left\{y_i\right\}_{i=1...N}$ for a system with N agents during a time interval s.

As said before, the sum over N implies the possibility of interaction processes involving a variable number of agents. The $N!$ reflects the fact that agents are identical in that context.

Some difficulties arising from the computation of (128) can be avoided by considering the potential $K\left(x\left(t\right)\right)$ as a source term. To do so, we follow the presentation of [7]. and adapt this one to our context. Starting with the simplest case of no interaction, i.e. $V_k\left(x_1\left(t\right)...x_k\left(t\right)\right)=0$, the function of interest to us is:

$$\sum_{N}\frac{1}{N!}\prod_{i=1}^{N}\int\exp\left(-\alpha s_i\right)\int\mathcal{D}x_i\left(t\right)\exp\left(-\sum_{i}\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}\left(t\right)+K\left(x_i\left(t\right)\right)dt\right)\right) \tag{129}$$

Each of these integrals being independent from each others, the results for (129) is:

$$\sum_{N}\frac{1}{N!}\prod_{i=1}^{N}\int\exp\left(-\alpha s_i\right)\int\mathcal{D}x_i\left(t\right)\exp\left(-\sum_{i}\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}\left(t\right)+K\left(x_i\left(t\right)\right)dt\right)\right)=\sum_{N}\frac{1}{N!}\prod_{i=1}^{N}G_K\left(x_i,y_i,\alpha\right) \tag{130}$$

which is a mixed sum over N of transition functions for N agents. Each product $\frac{1}{N!}\prod_{i=1}^{N}G_K\left(x_i,y_i,\alpha\right)$ computes, as needed, the transition probability from $\left\{x_i\right\}_{i=1...N}$ to $\left\{y_i\right\}_{i=1...N}$ for N ordered agents during a process of mean duration $\frac{1}{\alpha}$. Thus the sum can be seen as a generating series for these probabilities with N agents. However, between identical agents, order is irrelevant, so that the probability of transition of the system from $\left\{x_i\right\}_{i=1...N}$ to $\left\{y_i\right\}_{i=1...N}$ is the sum over the permutations with N elements of the terms on (130) rhs. Since these terms are equal, the "true" probability of transition is $\prod_{i=1}^{N}G_K\left(x_i,y_i,\alpha\right)$. The whole problem at stake is to recover the case with interaction (128) from the "free" case (129). The benchmark case interaction contribution (129) can be recovered using the following method. Using the functional derivative with respect to K we write:

$$\begin{aligned}&\frac{\delta}{\delta K\left(x_{i_1}\right)}\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp\left(-\alpha s_i\right)\int\mathcal{D}x_i\left(t\right)\exp\left(-\sum_i\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}\left(t\right)+K\left(x_i\left(t\right)\right)dt\right)\right)\\=\;&\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp\left(-\alpha s_i\right)\int\mathcal{D}x_i\left(t\right)\exp\left(-\sum_i\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}\left(t\right)+K\left(x_i\left(t\right)\right)dt\right)\right)\\&\times\left\{-\sum_i\int_{x_{i_1}(0)=x_{i_1}}^{x_{i_1}(s)=y_{i_1}}dt\delta\left(x_{i_1}\left(t\right)-x_{i_1}\right)\right\}\end{aligned}$$

where $\delta\left(x_{i_1}\left(t\right)-x_{i_1}\right)$ is the delta of Dirac function. By extension, this generalizes for any function $V\left(x_{i_1}\right)$, to:

$$\begin{aligned}&\int dx_{i_1}V\left(x_{i_1}\right)\frac{\delta}{\delta K\left(x_{i_1}\right)}\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp\left(-\alpha s_i\right)\int\mathcal{D}x_i\left(t\right)\exp\left(-\sum_i\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}\left(t\right)+K\left(x_i\left(t\right)\right)dt\right)\right)\\=\;&\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp\left(-\alpha s_i\right)\int\mathcal{D}x_i\left(t\right)\exp\left(-\sum_i\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}\left(t\right)+K\left(x_i\left(t\right)\right)dt\right)\right)\\&\times\left\{-\sum_i\int_{x_{i_1}(0)=x_{i_1}}^{x_{i_1}(s)=y_{i_1}}dtV\left(x_{i_1}\left(t\right)\right)\right\}\end{aligned}$$

and for any function of several variables, to:

$$\begin{aligned}&\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp\left(-\alpha s_i\right)\int\mathcal{D}x_i\left(t\right)\exp\left(-\sum_i\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}\left(t\right)+K\left(x_i\left(t\right)\right)dt\right)\right)\\&\times\sum_{k\geqslant2}\sum_{i_1,\ldots i_k}\int_{x_i(0)=x}^{x_i(s)=y}V_k\left(x_{i_1}\left(t_k\right)\ldots x_{i_k}\left(t_k\right)\right)dt_1\ldots dt_k\\=\;&\sum_{k\geqslant2}\sum_{i_1,\ldots i_k}\left\{(-1)^k\int dx_{i_1}\ldots dx_{i_k}V_k\left(x_{i_1}\ldots x_{i_k}\right)\frac{\delta}{\delta K\left(x_{i_1}\right)}\ldots\frac{\delta}{\delta K\left(x_{i_k}\right)}\right\}\\&\times\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp\left(-\alpha s_i\right)\int\mathcal{D}x_i\left(t\right)\exp\left(-\sum_i\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}\left(t\right)+K\left(x_i\left(t\right)\right)dt\right)\right)\end{aligned}\tag{131}$$

To find (128) from (129), the next step is to exponentiate (131) as:

$$\begin{aligned}&\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp\left(-\alpha s_i\right)\int\mathcal{D}x_i\left(t\right)\\&\times\exp\left(-\sum_i\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}\left(t\right)+K\left(x_i\left(t\right)\right)dt\right)-\sum_{k=2}^A\sum_{i_1,\ldots i_k}\int_{x_i(0)=x}^{x_i(s)=y}V_k\left(x_{i_1}\left(t_1\right)\ldots x_{i_k}\left(t_k\right)\right)dt_1\ldots dt_k\right)\\=\;&\exp\left(-\int dx_{i_1}\ldots dx_{i_k}V_k\left(x_{i_1}\ldots x_{i_k}\right)\frac{\delta}{\delta K\left(x_{i_1}\right)}\ldots\frac{\delta}{\delta K\left(x_{i_k}\right)}\right)\\&\times\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp\left(-\alpha s_i\right)\int\mathcal{D}x_i\left(t\right)\exp\left(-\sum_i\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}\left(t\right)+K\left(x_i\left(t\right)\right)dt\right)\right)\end{aligned}$$

In other words, using (130) one finds the partition function for the system of agents in interaction:

$$\begin{aligned}
&\sum_N \frac{1}{N!}\prod_{i=1}^N \int \exp\left(-\alpha s_i\right)\int \mathcal{D}x_i\left(t\right)\\
&\times\exp\left(-\sum_i \int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}\left(t\right)+K\left(x_i\left(t\right)\right)dt\right)-\sum_{k=2}^A\sum_{i_1,\dots i_k}\int_{x_i(0)=x}^{x_i(s)=y} V_k\left(x_{i_1}\left(t_1\right)...x_{i_k}\left(t_k\right)\right)dt_1...dt_k\right)\\
=\ &\exp\left(-\int dx_{i_1}...dx_{i_k}V_k\left(x_{i_1}...x_{i_k}\right)\frac{\delta}{\delta K\left(x_{i_1}\right)}...\frac{\delta}{\delta K\left(x_{i_k}\right)}\right)\times\sum_N\frac{1}{N!}\prod_{i=1}^N G_K\left(x_i,y_i,\alpha\right)
\end{aligned}$$

This would allow to compute the transition functions, or average quantities for interactions processes involving all agents. However, there exists a more compact and general way to compute the same results and, eventually, to obtain more results about the nature of the interacting system. This implies a switch in representation from the N agents' system to the collective surrounding description of these N agents. We can actually infer from (125) that the determinant of operator G_K whose kernel is $G_K\left(x,y,\alpha\right)$ can be expressed as an infinite dimensional integral different from the ones studied up to now:

$$\left(\det\left(G_K\right)\right)^{-1}=\int\exp\left(-\Psi\left(x\right)\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)\Psi^\dagger\left(x\right)\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger \tag{132}$$

where the integrals over $\Psi\left(x\right)$ and $\Psi^\dagger\left(x\right)$ are performed over the space of complex-valued functions of one variable x. The function $\Psi^\dagger\left(x\right)$ is the complex conjugate of $\Psi^\dagger\left(x\right)$.

The formula (125) is simply the generalization in infinite dimension of the gaussian integral formula

$$\left(\det\left(M\right)\right)^{-1}=\int\exp\left(-X\left(M\right)X^\dagger\right)\mathcal{D}X\mathcal{D}X^\dagger$$

where (125) is used.

Introducing a source term $J\left(x\right)\Psi^\dagger\left(x\right)+J^\dagger\left(x\right)\Psi\left(x\right)$, we claim that:

$$\begin{aligned}
&\frac{\int\exp\left(-\Psi\left(x\right)\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)\Psi^\dagger\left(x\right)+J\left(x\right)\Psi^\dagger\left(x\right)+J^\dagger\left(x\right)\Psi\left(x\right)\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger}{\int\exp\left(-\Psi\left(x\right)\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)\Psi^\dagger\left(x\right)\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger} \qquad (133)\\
=\ &\exp\left(J\left(x\right)\left(-\frac{1}{2}\nabla^2+\alpha+K\left(x\right)\right)^{-1}J^\dagger\left(x\right)\right)\\
=\ &\exp\left(J\left(x\right)G_K\left(x,y,\alpha\right)J^\dagger\left(x\right)\right)
\end{aligned}$$

This comes directly when changing the variable $\Psi\left(x\right)\rightarrow\Psi\left(x\right)+J\left(x\right)$ in the numerator of (133).

The terms in (129) can thus be recovered from (133). Actually, the transition function for N agents (133):

$$\prod_{i=1}^N G_K\left(x_i,y_i,\alpha\right) \tag{134}$$

providing that $\frac{1}{N!}$ in (130) accounted for a chosen order among agents, and that we multiplied $N!$ to restore the indentity between the agents, can directly be written as:

$$\prod_{i=1}^N G_K\left(x_i,y_i,\alpha\right)=\left[\left(\frac{\delta}{\delta J\left(x_{i_1}\right)}\frac{\delta}{\delta J^\dagger\left(y_{i_1}\right)}\right)...\left(\frac{\delta}{\delta J\left(x_{i_N}\right)}\frac{\delta}{\delta J^\dagger\left(y_{i_N}\right)}\right)\exp\left(J\left(x\right)G_K\left(x,y,\alpha\right)J^\dagger\left(x\right)\right)\right]_{J=J^\dagger=0}$$

Consequently, we now have an infinite dimensionnal integral representation for the transition functions for

N agents:

$$\begin{aligned}
&\prod_{i=1}^{N} G_K(x_i, y_i, \alpha) \\
=& \left[\left(\frac{\delta}{\delta J(x_{i_1})}\frac{\delta}{\delta J^{\dagger}(x_{i_1})}\right)...\left(\frac{\delta}{\delta J(x_{i_N})}\frac{\delta}{\delta J^{\dagger}(x_{i_N})}\right)\right]_{J=J^{\dagger}=0} \\
&.\frac{\int \exp\left(-\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+K(x)\right)\Psi^{\dagger}(x)+J(x)\Psi^{\dagger}(x)+J^{\dagger}(x)\Psi(x)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}}{\int \exp\left(-\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+K(x)\right)\Psi^{\dagger}(x)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}} \\
=& \frac{1}{\int \exp\left(-\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+K(x)\right)\Psi^{\dagger}(x)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}} \\
&\times\left[\left(\frac{\delta}{\delta J(x_{i_1})}\frac{\delta}{\delta J^{\dagger}(x_{i_1})}\right)...\left(\frac{\delta}{\delta J(x_{i_N})}\frac{\delta}{\delta J^{\dagger}(x_{i_N})}\right)\right]_{J=J^{\dagger}=0} \\
&.\int \exp\left(-\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+K(x)\right)\Psi^{\dagger}(x)+J(x)\Psi^{\dagger}(x)+J^{\dagger}(x)\Psi(x)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}
\end{aligned}$$

The normalization factor

$$\frac{1}{\int \exp\left(-\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+K(x)\right)\Psi^{\dagger}(x)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}}$$

is usually implied and will thus be - whenever possible - omitted in the formula. The transition functions are computed by taking the derivatives with respect to $J(x)$ and $J^{\dagger}(x)$ of

$$\int \exp\left(-\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+K(x)\right)\Psi^{\dagger}(x)+J(x)\Psi^{\dagger}(x)+J^{\dagger}(x)\Psi(x)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}$$

However, the source term is also usually implied and only reintroduced ultimately, at the end of the computations. As a consequence,

$$\int \exp\left(-\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+K(x)\right)\Psi^{\dagger}(x)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger} \tag{135}$$

will describe the same system of identical non interacting structures. We will use this representation occasionally.

On the other hand we have seen how to introduce interactions between agents. It amounts to make an operator act, namely

$$\exp\left(-\int dx_{i_1}...dx_{i_k}V_k(x_{i_1}...x_{i_k})\frac{\delta}{\delta K(x_{i_1})}...\frac{\delta}{\delta K(x_{i_k})}\right)$$

on the transition functions. In other words, the quantity

$$\begin{aligned}
&\exp\left(-\int dx_{i_1}...dx_{i_k}V_k(x_{i_1}...x_{i_k})\frac{\delta}{\delta K(x_{i_1})}...\frac{\delta}{\delta K(x_{i_k})}\right) \\
&\times\int \exp\left(-\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+K(x)\right)\Psi^{\dagger}(x)+J(x)\Psi^{\dagger}(x)+J^{\dagger}(x)\Psi(x)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}
\end{aligned}$$

allows to compute, by differentiation with respect to the source terms $J(x)$ and $J^{\dagger}(x)$, the transition functions for a system of N interacting particles. The action of the functional differential operator can be written:

$$\begin{aligned}
&\exp\left(-\int dx_{i_1}...dx_{i_k}V_k(x_{i_1}...x_{i_k})\frac{\delta}{\delta K(x_{i_1})}...\frac{\delta}{\delta K(x_{i_k})}\right) \\
&\times\int \exp\left(-\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+K(x)\right)\Psi^{\dagger}(x)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger} \\
=&\int \exp\left(-\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+K(x)\right)\Psi^{\dagger}(x)-\Psi(x_{i_1})...\Psi(x_{i_k})V_k(x_{i_1}...x_{i_k})\Psi^{\dagger}(x_{i_1})...\Psi^{\dagger}(x_{i_k})\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}
\end{aligned}$$

The above formula can be directly extended by considering all types of interaction process involving k identical agents where $k \geqslant 2$. We can sum up the previous development by asserting that the quantity

$$\int \exp\left(-\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+K(x)\right)\Psi^{\dagger}(x) \right. \tag{136}$$
$$\left. -\sum_{k\geqslant 2}\sum_{i_1,\ldots i_k}\Psi(x_{i_1})\ldots\Psi(x_{i_k})V_k(x_{i_1}\ldots x_{i_k})\Psi^{\dagger}(x_{i_1})\ldots\Psi^{\dagger}(x_{i_k})+J(x)\Psi^{\dagger}(x)+J^{\dagger}(x)\Psi(x)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}$$

computes, by successive derivatives with respect to $J(x)$ and $J^{\dagger}(x)$, the transition functions of a system of infinite number of identical agents, with effective utility $X_i(t)\left(-\frac{1}{2}\nabla^2+K(x)\right)X_i(t)$, and arbitrary, non local in time, interactions $V_k\left(X_{i_1}(t_1)\ldots X_{i_k}(t_k)\right)$ involving k agents, with k arbitrary. The constant α is the characteristic scale of the interaction process, and $\frac{1}{\alpha}$ the mean duration of the interaction process, or alternately the mean lifespan of the agents. The transition functions are given by:

$$G_K(\{x_i\},\{y_i\},\alpha) \tag{137}$$
$$= \left[\left(\frac{\delta}{\delta J(x_{i_1})}\frac{\delta}{\delta J^{\dagger}(y_{i_1})}\right)\ldots\left(\frac{\delta}{\delta J(x_{i_N})}\frac{\delta}{\delta J^{\dagger}(y_{i_N})}\right)\int \exp\left(-\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+K(x)\right)\Psi^{\dagger}(x)\right.\right.$$
$$\left.\left.-\sum_{k\geqslant 2}\sum_{i_1,\ldots i_k}\Psi(x_{i_1})\ldots\Psi(x_{i_k})V_k(x_{i_1}\ldots x_{i_k})\Psi^{\dagger}(x_{i_1})\ldots\Psi^{\dagger}(x_{i_k})+J(x)\Psi^{\dagger}(x)+J^{\dagger}(x)\Psi(x)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}\right]_{J=J^{\dagger}=0}$$

and $G_K(\{x_i\},\{y_i\},\alpha)$ is the probability of transition for N agents from a state $\{x_i\}$ to a state $\{y_i\}$. Remark that this formulation realizes what was announced before. The switch in formulation induces that the transition of the agents, i.e. their dynamical and stochastic properties, takes place in a surrounding. Instead of computing directly the dynamic of the system, we derive this behavior from the global properties of a substratum, the global action for the field $\Psi(x)$. By global action we denote the functional, or action:

$$S(\Psi)=\int dx\left(\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+K(x)\right)\Psi^{\dagger}(x)+\sum_{k\geqslant 2}^{A}\sum_{i_1,\ldots i_k}\Psi(x_{i_1})\ldots\Psi(x_{i_k})V_k(x_{i_1}\ldots x_{i_k})\Psi^{\dagger}(x_{i_1})\ldots\Psi^{\dagger}(x_{i_k})\right)$$

The infinite dimensional integral (137), the so-called "path integral", can be written as a shortcut when the source terms are omitted:

$$\int \exp(-S(\Psi))\,\mathcal{D}\Psi$$

This point of view is usual both in quantum and in statistical field theory. The latter, that is the closest to our approach, deals with system with large degrees of freedom. To reach an analog degree of formalism, we built the notion of effective utility, starting from interacting and strategic agents. This notion has then been used to find the action functional for a field describing a large number of structures. The individual features of the effective utilities render the action functionals more specific than their analog physics. Moreover, the physics and the symetry laws generally at stakes in statistical physics ultimately constrains the form of the global action. These constraint are not present here, and we will see that the form of the problems involved by the systems of socio/ eco interacting agents lead to very different forms of global actions than the one studied in physics. These symmetries are absent here. Besides, systems of socio/economic interacting agents lead to very different forms of global actions than those studied in physics. However, some basic ideas and principles remain valid and will conduct the use of this formalim.

The first application of this formalism asserts that in the expression

$$S(\Psi)=\int\frac{1}{2}\left(\Psi^{\dagger}(x)\left(\nabla^2+\alpha+K(x)\right)\Psi(x)\right)dx+\int\sum_{k=2}^{A}V(x_1,\ldots,x_k)\Psi(x_1)\Psi^{\dagger}(x_1)\ldots\Psi(x_k)\Psi^{\dagger}(x_k)\,dx_1\ldots dx_k \tag{138}$$

the contibutions of the potential $V(x_1,\ldots,x_k)$ to the computation of the two points Green functions can be obtained as a series of Feynman Graphs. This one represents also the "sum over all histories" and will yield the statistic "fate" of a single path through the various interaction processes These graphs actually compute how the path of single agent is perturbed by interaction processes with one, two... and more agents. Each of these interaction processes will contribute in probability to the transition of the agent from one state to another. That is, the series of graphs models the environment impact on the trajectory of a structure.

More generally, n points correlation functions give the probability of transition between 2 sets of states for n agents: given a certain process with n agents and initial values, it yields the probability value for the outcome.

The second application of the formalism of statistical physics is the possibility of non trivial vaccua and phase transition. The system (138) can be studied independently from the system of agents it represents. The functional $S(\Psi)$ may present some non trivial minima, and these minima modify the properties of the correlation functions of the system. The field Ψ for which $S(\Psi)$ reaches it's minimal value describes the phase of the system. Given the parameters of the model, the phase may change and confer different properties to the system. The properties of individual behavior will then depend on the phase of the system as a whole.

Both possibilities will be studied in the next sections, but before doing so, we will conclude this section by generalizing our results to the models developped in the previous sections.

Remark that the first term in (138):

$$\frac{1}{2}\int \left(\Psi^{\dagger}(x)\left(\nabla^2+\alpha+K(x)\right)\Psi(x)\right)$$

can be identified with:

$$\frac{1}{2}\int \Psi^{\dagger}(x)\left(G_K^{-1}(x,y)\right)\Psi(x)$$

Besides, $G_K(x,y)$ is the Green function for the effective utility of a single agent, or a single subset of several interacting structures, or some fundamental structure.

As a consequence of the previous discussion, the formalism may be generalized for curved space of configurations that appeared in the previous section, and which represents the most general form of quadratic effective utility. Actually, consider a single interacting system with effective action (73), in which we now include the term derived in (41) and previously discarded:

$$\begin{aligned} U_{eff}(X(t)) &= \int\left(\frac{1}{4}\dot{X}(t)\left(N+M^S\right)\dot{X}(t)+\left(X(t)-\bar{X}^e\right)\left(N-M^S\right)\left(X(t)-\bar{X}^e\right)\right. \\ &\left. +\left(X(t)-\bar{X}^e\right)M^A\dot{X}(t)+V_{eff}(X(t))+U_{eff}\left(\bar{X}^e\right)\right)dt \end{aligned}$$

The associated Green function $G(x,y,\alpha)$ is the inverse of a differential operator given by (120):

$$\begin{aligned} G^{-1}(x,y,\alpha) &= \left(-\nabla\left(M^{(S)}+N\right)^{-1}\left(\nabla+\left(M^A\right)\left(x-\bar{X}^e\right)\right)\right. \\ &\left. +\left(x-\bar{X}^e\right)\left(N-M^S-\left(M^A\right)^t\left(N+M^S\right)^{-1}M^A\right)\left(x-\bar{X}^e\right)+U_{eff}\left(\bar{X}^e\right)+\alpha\right)\delta(x-y) \end{aligned}$$

and then $G(x,y,\alpha)$ satisfies:

$$\begin{aligned} \delta(x-y) &= \left(-\nabla\left(M^{(S)}+N\right)^{-1}\left(\nabla+\left(M^A\right)\left(x-\bar{X}^e\right)\right)\right. \\ &\left. +\left(x-\bar{X}^e\right)\left(N-M^S-\left(M^A\right)^t\left(N+M^S\right)^{-1}M^A\right)\left(x-\bar{X}^e\right)+U_{eff}\left(\bar{X}^e\right)+\alpha\right) \\ &\times G(x,y,\alpha) \end{aligned}$$

Gathering the potential terms

$$\left(x-\bar{X}^e\right)\left(N-M^S-\left(M^A\right)^t\left(N+M^S\right)^{-1}M^A\right)\left(x-\bar{X}^e\right)+U_{eff}\left(\bar{X}^e\right)+\alpha+V_{eff}(x)\rightarrow m^2+V(x)$$

allows to write the effective utility and associated inverse Green function as:

$$G^{-1}(x,y)=-\nabla\left(M^{(S)}+N\right)^{-1}\left(\nabla+\left(M^A\right)\left(x-\bar{X}^e\right)\right)+V(x) \tag{139}$$

with:

$$m^2 = \alpha + U_{eff}\left(\bar{X}^e\right) \tag{140}$$

note that m^2 can be positive or negative, depending on $U_{eff}\left(\bar{X}^e\right)$ - α is always positive, but we keep this notation by analogy with the mass term in field theory.

Formula (139) leads to the field formulation of large number of interactions:

$$\begin{aligned} S(\Psi) &= \int \frac{1}{2}\left(\Psi^{\dagger}(x)\left(-\nabla\left(M^{(S)}+N\right)^{-1}\left(\nabla+\left(M^{A}\right)\left(x-\bar{X}^e\right)\right)+m^2+V(x)\right)\Psi(x)\right)dx \\ &+\int\sum_{k\geqslant 2}\sum_{i_1,\ldots i_k}\Psi\left(x_{i_1}\right)\ldots\Psi\left(x_{i_k}\right)V_k\left(x_{i_1}\ldots x_{i_k}\right)\Psi^{\dagger}\left(x_{i_1}\right)\ldots\Psi^{\dagger}\left(x_{i_k}\right)dx_1\ldots dx_k \end{aligned} \tag{141}$$

If we change the coordinates $\left(x-\bar{X}^e\right)\rightarrow\sqrt{M^{(S)}+N}\left(x-\bar{X}^e\right)$ to normalize $M^{(S)}+N$ to 1, we have:

$$\begin{aligned} S(\Psi) &= \int \frac{1}{2}\left(\Psi^{\dagger}(x)\left(-\nabla^2+\nabla M^{(A)}\left(x-\bar{X}^e\right)+m^2+V(x)\right)\Psi(x)\right)dx \\ &+\int\sum_{k\geqslant 2}\sum_{i_1,\ldots i_k}\Psi\left(x_{i_1}\right)\ldots\Psi\left(x_{i_k}\right)V_k\left(x_{i_1}\ldots x_{i_k}\right)\Psi^{\dagger}\left(x_{i_1}\right)\ldots\Psi^{\dagger}\left(x_{i_k}\right)dx_1\ldots dx_k \end{aligned} \tag{142}$$

That describes a set of fundamental structures over the whole relevant time span, as well as their potential temporal realizations.

The first contribution describes the dynamic of a set of identical structure whose fundamental state - or classical solution - is bended by its own internal interactions and constraints as explained in the previous sections. The second term represents the possibly non local interactions between agents. Each interaction type creates a surrounding constraining the individual structures.

8.2 Introduction of several type of agents

The previous paragraph has introduced a field theoretic description of a large number of interacting identical agents, or structures. To differentiate between fundamental structures, one introduce the different species, either by diagonalizing the initial system and replicating it, and then adding non local interactions, either directly by introducing some original bricks and their interactions. Each of these types corresponds to a field living in a space whose dimension is given by the dimension of each block $X_k(t)$. We denote them $\Psi^{(k)}\left(\hat{X}_k\right)$. The coordinate $\hat{X}_k$ describes the space, or variety, of characteristic variables of the fundamental structure k.

The treatment of these several species is straightforward given the previous paragraph. Without interaction, each fundamental structure is described by a quadratic action similar to the ones described in the previous paragraph (see (139)). Recall that:

$$\int \exp\left(-S(\Psi)\right)\mathcal{D}\Psi$$

computes the probability weight for a system. Gathering the various systems of identical fundamental structures, the path integral to consider reduces to the product of the weights of each system of fundamental structures. The non interacting blocks Path. Integrals is then:

$$\begin{aligned} &\int\prod_k \mathcal{D}\Psi\left(\hat{X}_k\right)\times \\ &\times\exp\left(\sum_k\int d\hat{X}_k\left(\left(-\frac{1}{2}\Psi^{(k)\dagger}\left(\hat{X}_k\right)\left[\left(\nabla_k\right)\left(M_k^{(S)}+N_k\right)^{-1}\left(\nabla_k-M_k^{(A)}\left(\hat{X}_k-\left(\tilde{X}\right)_k\right)\right)+V\left(\hat{X}_k\right)\right]\Psi^{(k)}\left(\hat{X}_k\right)\right)\right)\right) \end{aligned}$$

where $V\left(\hat{X}_k\right)$ includes $\frac{1}{2}\left(\hat{X}_k-\left(\tilde{X}\right)_k\right)\left(N_k-M_k^{(S)}\right)\left(\hat{X}_k-\left(\tilde{X}\right)_k\right)$.

The inclusion of the interaction potential between fundamental structures follows the same previous steps and leads to the decomposition for the full action of the system with an infinite number of agents divided in M type, or species, of structures:

$$\begin{aligned}
&S\left(\left\{\Psi^{(k)}\right\}_{k=1\ldots M}\right)\\
=&\sum_{k}\int d\hat{X}_{k}\left(\left(-\frac{1}{2}\Psi^{(k)\dagger}\left(\hat{X}_{k}\right)\left[\left(\nabla_{k}\right)\left(N_{k}-M_{k}^{(S)}\right)\left(\nabla_{k}-M_{k}^{(A)}\left(\hat{X}_{k}-\left(\tilde{X}\right)_{k}\right)\right)+m_{k}^{2}+V\left(\hat{X}_{k}\right)\right]\Psi^{(k)}\left(\hat{X}_{k}\right)\right)\right)\\
&+\underbrace{\sum_{k}\sum_{n}V_{n}\left(\left\{\hat{X}_{k}^{(i)}\right\}_{1\leqslant i\leqslant n}\right)\prod_{1\leqslant i\leqslant n}\Psi^{(k)\dagger}\left(\hat{X}_{k}^{(i)}\right)\Psi^{(k)}\left(\hat{X}_{k}^{(i)}\right)}_{\text{intra species interaction}}\\
&+\underbrace{\sum_{m}\sum_{k_{1}\ldots k_{m}}\sum_{n_{1}\ldots n_{m}}V_{(k_{1},n_{1})\ldots(k_{m},n_{m})}\left(\left\{\hat{X}_{k_{j}}^{(i_{n_{j}})}\right\}_{1\leqslant i_{n_{j}}\leqslant n_{j}}\right)\prod_{j=1}^{m}\prod_{1\leqslant i_{n_{j}}\leqslant n_{j}}\Psi^{(k_{j})\dagger}\left(\hat{X}_{k_{j}}^{(i_{n_{j}})}\right)\Psi^{(k_{j})}\left(\hat{X}_{k_{j}}^{(i_{n_{j}})}\right)}_{\text{inter species interaction}}
\end{aligned}\tag{143}$$

The variables $\hat{X}_{k}^{(i)}$ are copies of the coordinates on the fundamental structure k. The intra type/species interaction terms describes then the interactions between several structures of the same kind. The inter-species interaction term rather involves coordinates $\hat{X}_{k_{j}}^{(i_{n_{j}})}$ on different manifolds, and describes then interactions between different types of agents. The potential $V_{(k_{1},n_{1})\ldots(k_{m},n_{m})}$ involves n_1 copies of structures k_1,..., n_m copies of structures k_m.

8.3 Computation of Green functions. Graphs

We start with the system composed an infinite number of agents of one type, whose action is described by (141). Without interaction, we have seen that the Green function for $2n$ independent variables through (137)

$$\begin{aligned}
&G_{K}^{0}\left(\{x_{i}\}_{1\ldots n},\{y_{i}\}_{1\ldots n},\alpha\right)\\
=&\left[\left(\frac{\delta}{\delta J\left(x_{i_{1}}\right)}\frac{\delta}{\delta J^{\dagger}\left(y_{i_{1}}\right)}\right)\ldots\left(\frac{\delta}{\delta J\left(x_{i_{n}}\right)}\frac{\delta}{\delta J^{\dagger}\left(y_{i_{n}}\right)}\right)\right.\\
&\left.\int\exp\left(-\Psi\left(x\right)\left(-\nabla^{2}+\nabla M^{(A)}\left(x-\bar{X}^{e}\right)+m^{2}+V\left(x\right)\right)\Psi^{\dagger}\left(x\right)+J\left(x\right)\Psi^{\dagger}\left(x\right)+J^{\dagger}\left(x\right)\Psi\left(x\right)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}\right]_{J=J^{\dagger}=0}
\end{aligned}\tag{144}$$

The upperscript 0 has been added on $G_{K}^{0}\left(\{x_{i}\}_{1\ldots n},\{y_{i}\}_{1\ldots n},\alpha\right)$ to denote the Green function without interaction potential between the different agents. Equation (144) can also be rewritten as :

$$\begin{aligned}
G_{K}^{0}\left(\{x_{i}\}_{1\ldots n},\{y_{i}\}_{1\ldots n},\alpha\right) &= \int\Psi\left(x_{i_{1}}\right)\Psi^{\dagger}\left(y_{i_{1}}\right)\ldots\Psi\left(x_{i_{n}}\right)\Psi^{\dagger}\left(y_{i_{n}}\right)\\
&\quad\exp\left(-\Psi\left(x\right)\left(-\nabla^{2}+\nabla M^{(A)}\left(x-\bar{X}^{e}\right)+m^{2}+V\left(x\right)\right)\Psi^{\dagger}\left(x\right)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}
\end{aligned}\tag{145}$$

And the left hand side of (145) can be rewritten as a product when there is no interaction potential (see (134)), so that:

$$\begin{aligned}
\sum_{\sigma\in\sigma_{n}}\prod_{j=1}^{n}G_{K}^{0}\left(x_{i_{j}},y_{\sigma(i_{j})},\alpha\right) &= \int\Psi\left(x_{i_{1}}\right)\Psi^{\dagger}\left(y_{i_{1}}\right)\ldots\Psi\left(x_{i_{n}}\right)\Psi^{\dagger}\left(y_{i_{n}}\right)\\
&\quad\exp\left(-\Psi\left(x\right)\left(-\nabla^{2}+\nabla M^{(A)}\left(x-\bar{X}^{e}\right)+m^{2}+V\left(x\right)\right)\Psi^{\dagger}\left(x\right)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}
\end{aligned}\tag{146}$$

This is known as the Wick theorem (see [11]) and is the basis to compute perturbatively the $2n$ points Green function when a potential is added to the action.

Now, consider the full $2n$ points Green function including an interaction potential as in (137), but with the general action (141):

$$
\begin{aligned}
& G_K\left(\left\{x_i\right\},\left\{y_i\right\}, \alpha\right) \\
= & \left[\left(\frac{\delta}{\delta J\left(x_{i_1}\right)} \frac{\delta}{\delta J^{\dagger}\left(y_{i_1}\right)}\right) \ldots\left(\frac{\delta}{\delta J\left(x_{i_N}\right)} \frac{\delta}{\delta J^{\dagger}\left(y_{i_N}\right)}\right)\right. \\
& \times \int \exp \left(-\Psi(x)\left(-\nabla\left(M^{(S)}+N\right)^{-1}\left(\nabla+\left(M^A\right)\left(x-\bar{X}^e\right)\right)+m^2+V(x)\right) \Psi^{\dagger}(x)\right. \\
& \left.\left.-\sum_{k \geqslant 2} \sum_{i_1, \ldots i_k} \Psi\left(x_{i_1}\right) \ldots \Psi\left(x_{i_k}\right) V_k\left(x_{i_1} \ldots x_{i_k}\right) \Psi^{\dagger}\left(x_{i_1}\right) \ldots \Psi^{\dagger}\left(x_{i_k}\right)+J(x) \Psi^{\dagger}(x)+J^{\dagger}(x) \Psi(x)\right) \mathcal{D} \Psi \mathcal{D} \Psi^{\dagger}\right]_{J=J^{\dagger}=0}
\end{aligned}
$$

Exactly in the same way as for the case without interaction, the $2n$ points Green function can also be written:

$$
\begin{aligned}
G_K\left(\left\{x_i\right\},\left\{y_i\right\}, \alpha\right) = & \int \Psi\left(x_{i_1}\right) \Psi^{\dagger}\left(y_{i_1}\right) \ldots \Psi\left(x_{i_n}\right) \Psi^{\dagger}\left(y_{i_n}\right) \\
& \times \exp \left(-\Psi(x)\left(-\nabla\left(M^{(S)}+N\right)^{-1}\left(\nabla+\left(M^A\right)\left(x-\bar{X}^e\right)\right)+m^2+V(x)\right) \Psi^{\dagger}(x)\right. \\
& \left.-\sum_{k \geqslant 2} \sum_{i_1, \ldots i_k} \Psi\left(x_{i_1}\right) \ldots \Psi\left(x_{i_k}\right) V_k\left(x_{i_1} \ldots x_{i_k}\right) \Psi^{\dagger}\left(x_{i_1}\right) \ldots \Psi^{\dagger}\left(x_{i_k}\right)\right) \mathcal{D} \Psi \mathcal{D} \Psi^{\dagger}
\end{aligned} \tag{147}
$$

and given the Wick theorem, this can be computed in the following way as a function of the Green function without interaction $G_K^0\left(x_i, y_{\sigma(i)}, \alpha\right)$.

Actually, expanding the exponential term containing the potential $V_k\left(x_{i_1} \ldots x_{i_k}\right)$:

$$
\begin{aligned}
& \exp \left(-\sum_{k \geqslant 2} \sum_{i_1, \ldots i_k} \Psi\left(x_{i_1}\right) \ldots \Psi\left(x_{i_k}\right) V_k\left(x_{i_1} \ldots x_{i_k}\right) \Psi^{\dagger}\left(x_{i_1}\right) \ldots \Psi^{\dagger}\left(x_{i_k}\right)\right) \\
= & \int \sum_{l=0}^{\infty} \prod_{\substack{k_j \geqslant 2 \\ 1 \leqslant j \leqslant l}}\left\{V_{k_j}\left(x_{i_1}^{(k_j)} \ldots x_{i_{k_j}}^{(k_j)}\right)\left[\Psi\left(x_{i_1}^{(k_j)}\right) \ldots \Psi\left(x_{i_{k_j}}^{(k_j)}\right) \Psi^{\dagger}\left(x_{i_1}^{(k_j)}\right) \ldots \Psi^{\dagger}\left(x_{i_{k_j}}^{(k_j)}\right)\right] d x_{i_1}^{(k_j)} \ldots d x_{i_{k_j}}^{(k_j)}\right\}
\end{aligned}
$$

And using the Wick theorem, contributions like:

$$
\begin{aligned}
& \int \Psi\left(x_{i_1}\right) \Psi^{\dagger}\left(y_{i_1}\right) \ldots \Psi\left(x_{i_n}\right) \Psi^{\dagger}\left(y_{i_n}\right) \exp \left(-\Psi(x)\left(-\nabla\left(M^{(S)}+N\right)^{-1}\left(\nabla+\left(M^A\right)\left(x-\bar{X}^e\right)\right)+m^2+V(x)\right) \Psi^{\dagger}(x)\right) \\
& \times \int \prod_{1 \leqslant j \leqslant l}\left\{V_{k_j}\left(x_{i_1}^{(k_j)} \ldots x_{i_{k_j}}^{(k_j)}\right)\left[\Psi\left(x_{i_1}^{(k_j)}\right) \ldots \Psi\left(x_{i_{k_j}}^{(k_j)}\right) \Psi^{\dagger}\left(x_{i_1}^{(k_j)}\right) \ldots \Psi^{\dagger}\left(x_{i_{k_j}}^{(k_j)}\right)\right] d x_{i_1}^{(k_j)} \ldots d x_{i_{k_j}}^{(k_j)}\right\} \mathcal{D} \Psi \mathcal{D} \Psi^{\dagger}
\end{aligned} \tag{148}
$$

for a given sequence $\{k_j \geqslant 2\}$, $j=1,\ldots,l$, are equal to:

$$
\begin{aligned}
& \sum_{n_1=0}^{n} \sum_{\sigma \in \boldsymbol{\sigma}_n, \sigma' \in \boldsymbol{\sigma}_n} \sum_{\hat{\sigma} \in \boldsymbol{\sigma}_{2N}} \sum_{\{x_1, \ldots, x_{2n}\}=\underset{1 \leqslant j \leqslant l}{\cup} \hat{\sigma}\left\{x_{i_1}^{(k_j)} \ldots x_{i_{k_j}}^{(k_j)}\right\}} G_K^0\left(x_{i_{\sigma(n_1+1)}}, y_{i_{\sigma'(n_1+1)}}, \alpha\right) \ldots G_K^0\left(x_{i_{\sigma(n)}}, y_{i_{\sigma'(n)}}, \alpha\right) \\
& \times G_K^0\left(x_{i_{\sigma(1)}}, x_1, \alpha\right) G_K^0\left(y_{i_{\sigma'(1)}}, x_2, \alpha\right) \ldots G_K^0\left(x_{i_{\sigma(n_1)}}, x_{2n_1-1}, \alpha\right) G_K^0\left(y_{i_{\sigma'(n_1)}}, x_{2n_1}, \alpha\right) \\
& \prod_{p=n_1+1}^{N} G_K^0\left(x_{2p-1}, x_{2p}, \alpha\right) \prod_{\substack{k_j \geqslant 2 \\ 1 \leqslant j \leqslant l}}\left\{V_{k_j}\left(x_{i_1}^{(k_j)} \ldots x_{i_{k_j}}^{(k_j)}\right) d x_{i_1}^{(k_j)} \ldots d x_{i_{k_j}}^{(k_j)}\right\}
\end{aligned} \tag{149}
$$

where $N=\sum_{j=1}^{l} k_j$ and with the convention that the contributions are nul for $2n_1>N$. The Green function is obtained by summing over l from 0 to ∞ and over the sequence $\{k_j \geqslant 2\}$, $j=1,\ldots,l$. Remark that the sums have to be performed only on sequences corresponding to connected graph, as explained just now above.

Actually, these integrals have convenient graph representations. Draw the $2n$ external points labelled by x_{i_1}, then for $j=1$ to l draw l vertices with k_1,...k_l legs and labelled by $V_{k_j}\left(x_{i_1}^{(k_j)}...x_{i_{k_j}}^{(k_j)}\right)$. Then draw $2n$ lines joining the external vertices to the legs of any internal one. Then link all remaining internal legs together in all possible ways labelling them by the points they are joining, such a way that the resulting graph is connected Finally link all remaining internal legs together in all possible ways, and label them by the points they are joining, in such a way that the resulting graph is connected. This gives a series of graphs, each providing a contribution to G_K. The contribution of any graph is computed in the following way:

For each internal or external line, associate a factor $G_K^0\left(x_{i_n}, x_{p_{2n-1}}^{(k_{j_{2n-1}})}, \alpha\right)$ or $G_K^0\left(x_{r_{i_{2j-1}}}^{\left(k_{m_{i_{2j-1}}}\right)}, x_{r_{i_{2j}}}^{\left(k_{m_{i_{2j}}}\right)}, \alpha\right)$ where the variables in the function G_K^0 represents the points the line is connecting. Then multiply by the factors $V_{k_j}\left(x_{i_1}^{(k_j)}...x_{i_{k_j}}^{(k_j)}\right)$ associated to the internal points. Then integrate the results over all internal points. The fact that only contributions corresponding to connected graph is explained for example in [11] but can quickly be understood as follows. Recall that the path integrals for n-points correlation functions like (148) have to be normalized by dividing by the "zero point" correlation functions:

$$\int \exp\left(-\Psi(x)\left(-\nabla\left(M^{(S)}+N\right)^{-1}\left(\nabla+\left(M^{A}\right)\left(x-\bar{X}^{e}\right)\right)+m^{2}+V(x)\right)\Psi^{\dagger}(x)\right) \tag{150}$$
$$\times\int\prod_{1\leqslant j\leqslant l}\left\{V_{k_j}\left(x_{i_1}^{(k_j)}...x_{i_{k_j}}^{(k_j)}\right)\left[\Psi\left(x_{i_1}^{(k_j)}\right)...\Psi\left(x_{i_{k_j}}^{(k_j)}\right)\Psi^{\dagger}\left(x_{i_1}^{(k_j)}\right)...\Psi^{\dagger}\left(x_{i_{k_j}}^{(k_j)}\right)\right]dx_{i_1}^{(k_j)}...dx_{i_{k_j}}^{(k_j)}\right\}\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}$$

and the contributions to (150) given by (149) are precisely given by (any) product of graph made of cycles (due to the fact that there are no external points). These contributions cancel precisely the non connected graphs in (149), that is those containing themselves cycles.

The method of graphs computations can be useful to find corrections to the individual propagators G_K^0. However, given the particular form of our model, it will often be more useful to use some other aspects of the collective field representation, as will be explained later.

This formula can be generalized for interactions beween various types of structures. Starting from (143), a computation similar to the previous ones yields the following contributions to the transition functions $G_K\left((\{x_i\},\{y_i\})_{n_1},...,(\{x_i\},\{y_i\})_{n_A},\alpha\right)$ for $2n_1$ points of type 1,...., $2n_A$ points of type A:

$$\prod_{B=1}^{A}\left[\sum_{(n_1)_B=0}^{n_B}\sum_{\sigma\in\boldsymbol{\sigma}_{n_B},\sigma'\in\boldsymbol{\sigma}_{n_B}}\sum_{\hat{\sigma}\in\boldsymbol{\sigma}_{2N_B}}\sum_{\{x_1,...,x_{2n}\}=\hat{\sigma}\left(\bigcup_{1\leqslant j\leqslant l}\left\{x_{i_1}^{(k_j)}...x_{i_{k_j}}^{(k_j)}\right\}\right)}G_K^{0,B}\left(x_{i_{\sigma(n_1+1)}},y_{i_{\sigma'(n_1+1)}},\alpha\right)\times\right. \tag{151}$$
$$\times...\times G_K^{0,B}\left(x_{i_{\sigma(n_B)}},y_{i_{\sigma'(n_B)}},\alpha\right)$$
$$\left.\times\, G_K^0\left(x_{i_{\sigma(1)}},x_1,\alpha\right)G_K^0\left(y_{i_{\sigma'(1)}},x_2,\alpha\right)...G_K^0\left(x_{i_{\sigma(n_1)}},x_{2n_1-1},\alpha\right)G_K^0\left(y_{i_{\sigma'(n_1)}},x_{2n_1},\alpha\right)\prod_{p=n_1+1}^{N_B}G_K^0\left(x_{2p-1},x_{2p},\alpha\right)\right]$$
$$\times\prod_{p=1}^{l}\left\{V_{\{(k_1,n_1)...(k_m,n_m)\}_p}\left(\left\{x_{k_j}^{(i_{n_j})}\right\}_{1\leqslant i_{n_j}\leqslant n_j}\right)d\left\{x_{k_j}^{(i_{n_j})}\right\}_{1\leqslant i_{n_j}\leqslant n_j}\right\}$$

with $N_B=\sum_{j=1}^{l}(n_B)_p$ where $(n_B)_p$ is the number of copies of the species k_B appearing in $\{(k_1,n_1)...(k_m,n_m)\}_p$. The Green function is obtained by summing over l from 0 to ∞ and over the sequences $\{(k_1,n_1)...(k_m,n_m)\}_p$, $p=1,...,l$.

8.4 Non trivial vacuum, phase transition (one type of agent)

8.4.1 Principle: Vacuum value and Green function

The previous perturbative computation relies on a development around the Green functions of a system of non interacting agents. However, this expansion may not be valid in (137). The effective action arising in the exponential of (147) may have a non trivial minimum $\Psi_0(x)$ in some cases. Changing the coordinates $\left(x-\bar{X}^e\right) \rightarrow \sqrt{M^{(S)}+N}\left(x-\bar{X}^e\right)$ for the sake of simplicity, the Green functions are then better approximated by expanding:

$$\begin{aligned}
S(\Psi) &= \int \frac{1}{2}\left(\Psi^{\dagger}(x)\left(-\nabla^2+\nabla M^{(A)}\left(x-\bar{X}^e\right)+m^2+V(x)\right)\Psi(x)\right)dx \\
&+\int \sum_{k\geqslant 2}\sum_{i_1,\ldots i_k}\Psi\left(x_{i_1}\right)\ldots\Psi\left(x_{i_k}\right)V_k\left(x_{i_1}\ldots x_{i_k}\right)\Psi^{\dagger}\left(x_{i_1}\right)\ldots\Psi^{\dagger}\left(x_{i_k}\right)dx_1\ldots dx_k
\end{aligned}$$

to the second order around $\Psi_0(x)$. The Green function is then recovered by computing the integral of the second group of terms over $\Psi(x)$, plus higher order contributions. The possibilities of non trivial minima $\Psi_0(x)$, depending on the parameters of the model, is related to the phenomenon of phase transition (for a short account see Pesh). Given the particular form of action functional $S(\Psi)$ involved in this context, its minima are quite different from the one obtained in usual models in field theory. Hower, the principle remains the same. Assume a non zero minimum $\Psi_0(x)$ for

$$\begin{aligned}
&-\Psi(x)\left(-\nabla^2+\nabla M^{(A)}\left(x-\bar{X}^e\right)+m^2+V(x)\right)\Psi^{\dagger}(x) \\
&-\sum_{k\geqslant 2}\sum_{i_1,\ldots i_k}\Psi\left(x_{i_1}\right)\ldots\Psi\left(x_{i_k}\right)V_k\left(x_{i_1}\ldots x_{i_k}\right)\Psi^{\dagger}\left(x_{i_1}\right)\ldots\Psi^{\dagger}\left(x_{i_k}\right)
\end{aligned}$$

i.e.

$$\begin{aligned}
0 &= \left[\frac{\delta}{\delta\Psi(x)}\left(-\Psi(x)\left(-\nabla^2+\nabla M^{(A)}\left(x-\bar{X}^e\right)+m^2+V(x)\right)\Psi^{\dagger}(x)\right.\right. \\
&\left.\left.-\sum_{k\geqslant 2}\sum_{i_1,\ldots i_k}\Psi\left(x_{i_1}\right)\ldots\Psi\left(x_{i_k}\right)V_k\left(x_{i_1}\ldots x_{i_k}\right)\Psi^{\dagger}\left(x_{i_1}\right)\ldots\Psi^{\dagger}\left(x_{i_k}\right)\right)\right]_{\Psi(x)=\Psi_0(x)}
\end{aligned}$$

Then, expanding

$$\begin{aligned}
S(\Psi(x)) &= -\Psi(x)\left(-\nabla^2+\nabla M^{(A)}\left(x-\bar{X}^e\right)+m^2+V(x)\right)\Psi^{\dagger}(x) \\
&-\sum_{k\geqslant 2}\sum_{i_1,\ldots i_k}\Psi\left(x_{i_1}\right)\ldots\Psi\left(x_{i_k}\right)V_k\left(x_{i_1}\ldots x_{i_k}\right)\Psi^{\dagger}\left(x_{i_1}\right)\ldots\Psi^{\dagger}\left(x_{i_k}\right)
\end{aligned}$$

with:

$$\Psi(x)=\Psi_0(x)+\delta\Psi(x)$$

yields:

$$\begin{aligned}
S\left(\Psi_0(x)+\delta\Psi(x)\right) &= S\left(\Psi_0(x)\right)-\delta\Psi(x)\left(-\nabla^2+\nabla M^{(A)}\left(x-\bar{X}^e\right)+m^2+V(x)\right)\delta\Psi^{\dagger}(x) \qquad (152)\\
&-\sum_{k=2}^{A}\sum_{i_1,\ldots i_k}\sum_{i_j}\delta\Psi\left(x_{i_j}\right)\Psi_0\left(x_{i_1}\right)\ldots\hat{\Psi}_0\left(x_{i_j}\right)\ldots\Psi_0\left(x_{i_k}\right)V_k\left(x_{i_1}\ldots x_{i_k}\right) \\
&\times\Psi_0^{\dagger}\left(x_{i_1}\right)\ldots\hat{\Psi}_0^{\dagger}\left(x_{i_j}\right)\ldots\Psi_0^{\dagger}\left(x_{i_k}\right)\delta\Psi^{\dagger}\left(x_{i_j}\right) \\
&+\text{higher order terms in } \delta\Psi\left(x_{i_j}\right)
\end{aligned}$$

where the hat over $\Psi\left(x_{i_j}\right)$ and its conjugate $\Psi^{\dagger}\left(x_{i_j}\right)$ means that these terms are omitted. In other words, the potential term in the individual action has been shifted from $K(x)$ to

$$K(x)-\sum_{k=2}^{A}\sum_{i_1,\ldots i_k}\sum_{i_j}\Psi_0\left(x_{i_1}\right)\ldots\hat{\Psi}_0\left(x_{i_j}\right)\ldots\Psi_0\left(x_{i_k}\right)V_k\left(x_{i_1}\ldots x_{i_k}\right)\Psi_0^{\dagger}\left(x_{i_1}\right)\ldots\hat{\Psi}_0^{\dagger}\left(x_{i_j}\right)\ldots\Psi_0^{\dagger}\left(x_{i_k}\right)$$

yielding a change in the individual Green function and in turn, in the individual effective utility. The influence of a large number of interactions induces a non trivial collective minimum: it shifts the individual behavior. Actually, the new individual action term:

$$-\delta\Psi(x)\left(-\frac{1}{2}\nabla^2+\alpha+K(x)-\sum_{k=2}^{A}\sum_{i_1,\dots i_k}\sum_{i_j}\Psi_0(x_{i_1})\dots\hat{\Psi}_0(x_{i_j})\dots\Psi_0(x_{i_k})V_k(x_{i_1}\dots x_{i_k})\right. \quad (153)$$
$$\left.\times\,\Psi_0^\dagger(x_{i_1})\dots\hat{\Psi}_0^\dagger(x_{i_j})\dots\Psi_0^\dagger(x_{i_k})\right)\delta\Psi^\dagger(x)\,dx_{i_1}\dots\hat{d}x_{i_j}\dots dx_{i_k}$$

modifyies the inverse of the Green function by some "constant", independent from $\delta\Psi(x)$, inducing a damped or extended dynamic. In other words the individual fluctuations can be frozen or magnified, justifying the use of the term phase transition. We will see below that the presence of a non trivial minimum may also shifts the equilibrium values for individual agents.

Remark that the higher order terms in (152) model the effective several agents interactions in the new phase at stake after expansion around $\Psi_0(x)$. These results fit with the change of representation implied by the use of field theory. The study of the set of agents as a continuum substratum leads to modifications of individual transitions as a result of the fluctuations from this medium.

8.4.2 Shift in equilibrium values

The second consequence of a phase transition is the shift in equilibrium value. The expansion around a non trivial vacuum leads to a quadratic term (153) that impacts the agent's effective utility. Actually, considering the reciprocal link between individual dynamics and collective fluctuations, we can assert that the form of the effective action impacts the effective utility. Facing a phase transition, the correction term in the effective action (153) would lead to an individual effective utility of the form:

$$\frac{\dot{x}_i^2}{2}(t)+K(x(t))+\hat{V}(x(t))$$

with:

$$\hat{V}(x(t)) = -\sum_{k=2}^{A}\sum_{i_1,\dots i_k}\sum_{i_j}\int\Psi_0(x_{i_1})\dots\hat{\Psi}_0(x_{i_j})\dots\Psi_0(x_{i_k})V_k(x_{i_1}\dots x(t)\dots x_{i_k})\Psi_0^\dagger(x_{i_1})\dots\hat{\Psi}_0^\dagger(x_{i_j})\dots\Psi_0^\dagger(x_{i_k})$$
$$\times dx_{i_1}\dots\hat{d}x_{i_j}\dots dx_{i_k}$$

This effective utility has a new saddle point x with respect to the individual case, which satisfies:

$$\frac{\delta}{\delta x}\left(K(x)+\hat{V}(x(t))\right)=0$$

As a consequence, the possibility of phase transition, i.e. the existence of non trivial minimum Ψ_0 for $S(\Psi(x))$ depending on the parameters, induces a shift in each agent's individual equilibrium. The collective system impacts directly the individual ones and prescribes A DIFFERENT effective potential THAN the one describing initially the system at the micro level.

8.5 Several possibilities of Interactions

Having described the formalism of collective fields and its possible use, we now detail two examples of interactions between fundamental structures.

8.5.1 Reciprocal interactions between identical agents

By reciprocal interaction we mean the introduction of a symetric potential of any form:

$$V(x_{i_1},x_{i_2},\dots,x_{i_n})$$

between agents of the same species. It models the mutual influence of these agents when none of them have a strategic advantage over the others. The graph expansion for the Green functions with this potential is given by (149) with a single type of agent, i.e.$G_K\left(\{x_i\},\{y_i\},\alpha\right)$:

$$\begin{aligned}
&G_K\left(\{x_i\},\{y_i\},\alpha\right) \qquad (154)\\
=&\int\sum_{\substack{p_1,\ldots,p_{2n}\in\{i_1,\ldots i_n\}\\ l_1,\ldots l_{2n}\in\{i_1,\ldots i_n\}}}\int G_K^0\left(x_{i_1},x_{p_1}^{(l_1)},\alpha\right)G_K^0\left(y_{i_1},x_{p_2}^{(l_2)},\alpha\right)\ldots G_K^0\left(x_{i_n},x_{p_{2n-1}}^{(l_{2m-1})},\alpha\right)G_K^0\left(y_{i_n},x_{p_{2n}}^{(l_{2m})},\alpha\right)\\
&\times V\left(x_{i_1}^{(1)},x_{i_2}^{(1)},\ldots,x_{i_n}^{(1)}\right)\ldots V\left(x_{i_1}^{(m)},x_{i_2}^{(m)},\ldots,x_{i_n}^{(m)}\right)\prod_{\substack{(i,j),(r,s)\\ \cup\left(x_r^{(i)},x_s^{(j)}\right)\cup\left\{x_{p_k}^{(l_k)}\right\}_{k=1,\ldots 2m}\\ =\left\{x_{i_1}^{(1)},x_{i_2}^{(1)},\ldots,x_{i_n}^{(1)},\ldots,x_{i_1}^{(m)},x_{i_2}^{(m)},\ldots,x_{i_n}^{(m)}\right\}}}G_K^0\left(x_r^{(i)},x_s^{(j)},\alpha\right)\\
&\times dx_{i_1}^{(1)}dx_{i_2}^{(1)}\ldots dx_{i_n}^{(1)},\ldots dx_{i_1}^{(m)}dx_{i_2}^{(m)}\ldots dx_{i_n}^{(m)}
\end{aligned}$$

The various individual propagators $G_K^0\left(x_{i_1},x_{p_1}^{(l_1)},\alpha\right)$ ... can be obtained through Laplace transform of the general formula (78). However for later purpose it will be more useful to use a different way.

We consider G_K^0 for an individual effective utility of quadratic form. As shown in the previous section, if we neglect the curvature effects for individual fundamental structures, and if we consider a system of coordinates where the potential is diagonalized (see (118)), the inverse propagator for block k (i.e. i_1 or i_2) is:

$$\left(G_K^0\right)^{-1}=-\nabla_k^2+m_k^2+\left((x_i)_k-\check{Y}_{eff}\right)(\Lambda_i)_k\left((x_i)_k-\check{Y}_{eff}\right)$$

m_k^2 can be positive or negative depending on the parameters of each fundamental system (see (140)). The kernel of this operator can be computed through its egenvalues and eigenfunctions. Actually we can cast the previous differential operator in the form:

$$\begin{aligned}
&-\nabla_k^2+m_i^2+\left((x_i)_k-\left(\check{Y}_{eff}\right)_k\right)(\Lambda_i)_k\left((x_i)_k-\left(\check{Y}_{eff}\right)_k\right)\\
=&\sum_n\psi_n(x)\left(m_i^2+\left(n+\frac{1}{2}\right)(\Lambda_i)_k\right)\psi_n^*(y)
\end{aligned}$$

Such an operator has a kernel (i.e. the Green function) such that:

$$-\nabla_k^2+m_i^2+\left((x_i)_k-\left(\check{Y}_{eff}\right)_k\right)(\Lambda_i)_k\left((x_i)_k-\left(\check{Y}_{eff}\right)_k\right)f\left((x_i)_k\right)=\int G\left((x_i)_k,(y_i)_k\right)f\left((y_i)_k\right)d(y_i)_k$$

for any function $f\left((x_i)_k\right)$. For such operator, the Kernel can be written in terms of its eigenfunctions:

$$\sum_n\psi_n(x)\left(m_i^2+\left(n+\frac{1}{2}\right)(\Lambda_i)_k\right)\psi_n^*(y) \qquad (155)$$

where ψ_n is the nth Hermite polynomials, times a gaussian term with shifted variable $(x_i)_k-\left(\check{Y}_{eff}\right)_k$.

$$\psi_n(x)=\left(\frac{\sqrt{a}}{\pi}\right)^{\frac{1}{4}}\sqrt{\frac{1}{2^n n!}}H_n\left(a^{\frac{1}{4}}x\right)\exp\left(-\frac{\sqrt{a}}{2}x^2\right)$$

where the $H_n\left(a^{\frac{1}{4}}x\right)$ are the Hermite polynomials. Some details are given in Appendix 9. The Green function can thus be found directly and is given by:

$$G(x,y)=\sum_n\frac{\varphi_n(x)\varphi_m^{\dagger}(x)}{(n+1)\sqrt{a}+\alpha}$$

Applying this results to our problem yields $G(x,y)$:

$$\begin{aligned}
\left(G_K^0\right)(x,y)&=\langle x|\frac{1}{-\nabla_k^2+m_i^2+\left((x_i)_k-\left(\check{Y}_{eff}\right)_k\right)(\Lambda_i)_k\left((x_i)_k-\left(\check{Y}_{eff}\right)_k\right)}|y\rangle \qquad (156)\\
&=\sum_n\psi_n(x)\frac{1}{m_i^2+\left(n+\frac{1}{2}\right)(\Lambda_i)_k}\psi_n^*(y)
\end{aligned}$$

This form of Green function is usefull to deal with (149). Actually, the infinite sum here can be truncated if we assume in first approximation that only a finite number of "harmonic" n participate to the dynamic of the system. This kind of truncature, or cut off, will also be used below. We can insert formula (156) in (149). Defining:

$$\begin{aligned}
&\int \prod_{\substack{k_j \geqslant 2\\ 1\leqslant j\leqslant l}} V_{k_j}\left(x_{1+\sum_{m=1}^{j-1}k_j},...,x_{k_j+\sum_{m=1}^{j-1}k_j}\right)\\
&\times\psi_{q_1}\left(x_{i_{\sigma(1)}}\right)\psi^*_{q_1}\left(x_1\right)\psi_{q_2}\left(y_{i_{\sigma'(1)}}\right)\psi^*_{q_2}\left(x_2\right)\\
&\times...\psi_{q_{2n_1-1}}\left(x_{i_{\sigma(n_1)}}\right)\psi^*_{q_{2n_1-1}}\left(x_{2n_1}-1\right)\psi_{q_{2n_1}}\left(y_{i_{\sigma'(n_1)}}\right)\psi^*_{q_{2n_1}}\left(x_{2n_1}\right)dx_1...dx_{2n_1}\\
&\times\prod_{p=n_1+1}^{N}\psi_{q_p}\left(x_{2p-1}\right)\psi^*_{q_p}\left(x_{2p}\right)dx_{2p-1}dx_{2p}\\
= \;& \hat{V}_{\{k_j\geqslant 2,1\leqslant j\leqslant l\}}\left(x_{i_{\sigma(1)}},y_{i_{\sigma'(1)}}...,x_{i_{\sigma(n_1)}},y_{i_{\sigma'(n_1)}},q_1,...,q_N\right)
\end{aligned}$$

one uses the permutation symetry of the V_{k_j} to write (149) as:

$$\begin{aligned}
&\sum_{n_1=0}^{n}\sum_{\sigma\in\boldsymbol{\sigma}_n,\sigma'\in\boldsymbol{\sigma}_n}\sum_{\hat{\sigma}\in\boldsymbol{\sigma}_{2N}} G_K^0\left(x_{i_{\sigma(n_1+1)}},y_{i_{\sigma'(n_1+1)}},\alpha\right)...G_K^0\left(x_{i_{\sigma(n)}},y_{i_{\sigma'(n)}},\alpha\right)\\
&\times\hat{V}_{\{k_j\geqslant 2,1\leqslant j\leqslant l\}}\left(x_{i_{\sigma(1)}},y_{i_{\sigma'(1)}}...,x_{i_{\sigma(n_1)}},y_{i_{\sigma'(n_1)}},q_{2n_1+1},...,q_N\right)\times\prod_{p=1}^{N}\frac{1}{m_i^2+\left(q_p+\frac{1}{2}\right)\left(\Lambda_i\right)_k}
\end{aligned}$$

with $k_j=\sum_{j=1}^{l}k_j$. Then summing over l and the k_j yields:

$$\begin{aligned}
G_K\left(\{x_i\},\{y_i\},\alpha\right) \;=\;& \sum_{l=0}^{\infty}\sum_{\substack{k_j\geqslant 2\\ 1\leqslant j\leqslant l}}\sum_{n_1=0}^{n}\sum_{\sigma\in\boldsymbol{\sigma}_n,\sigma'\in\boldsymbol{\sigma}_n}\sum_{\hat{\sigma}\in\boldsymbol{\sigma}_{2N}} G_K^0\left(x_{i_{\sigma(n_1+1)}},y_{i_{\sigma'(n_1+1)}},\alpha\right)...G_K^0\left(x_{i_{\sigma(n)}},y_{i_{\sigma'(n)}},\alpha\right)\\
&\times\sum_{q_{p_1}=1,...,q_{p_N}=1}^{\infty}\hat{V}_{\{k_j\geqslant 2,1\leqslant j\leqslant l\}}\left(x_{i_{\sigma(1)}},y_{i_{\sigma'(1)}}...,x_{i_{\sigma(n_1)}},y_{i_{\sigma'(n_1)}},q_1,...,q_N\right)\\
&\times\prod_{p=1}^{N=\sum_{j=1}^{l}k_j}\frac{1}{m_i^2+\left(q_p+\frac{1}{2}\right)\left(\Lambda_i\right)_k}
\end{aligned}$$

8.5.2 Non reciprocal interactions

We want to model an interaction potential where one type of agent imposes a stress on another one to drive it towards, or push it away, from a certain equilibrium position.

It is useful for agents with strategic advantage models, such as those presented in the second section. We assume two types of agents, the first one imposing a strain on the second one. We choose:

$$V\left(x_{i_1},x_{i_2}\right)=V\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)$$

where $\hat{x}_{i_2}^{(i_1)}$ is the objective function set for i_2 by i_1. We will later consider an example with

$$V\left(x_{i_1},x_{i_2}\right)=\delta\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2$$

The formula for the Green function (151) simplifies, since the first agent is not involved in the potential, and the Green functions reduces to a product of Green functions for both agents:

$$G_K\left(\left(\{x_i\},\{y_i\}\right)_{n_1},...,\left(\{x_i\},\{y_i\}\right)_{n_2}\alpha\right)=G_K^{(0)}\left(\left(\{x_i\},\{y_i\}\right)_{n_1},\alpha\right)G_K\left(\left(\{x_i\},\{y_i\}\right)_{n_2},\alpha\right)$$

The function $G_K^{(0)}\left(\left(\{x_i\},\{y_i\}\right)_{n_1},\alpha\right)$ is the free Green function with $2n_1$ points for the first type of agents, since there is no potential for this type of agent. while $G_K\left(\left(\{x_i\},\{y_i\}\right)_{n_2},\alpha\right)$ is the Green function with $2n_2$ points for the second agent. The function $G_K\left(\left(\{x_i\},\{y_i\}\right)_{n_2},\alpha\right)$ includes a sum of contributions given by (149) for a potential depending on one variable only.

$$\sum_{n_1=0}^{n}\sum_{\sigma\in\boldsymbol{\sigma}_n,\sigma'\in\boldsymbol{\sigma}_n}\sum_{\hat{\sigma}\in\boldsymbol{\sigma}_{2N}}\sum_{\{x_1,...,x_{2n}\}=\hat{\sigma}\left(\underset{1\leqslant j\leqslant l}{\cup}\left\{x_{i_1}^{(k_j)}\right\}\right)} G_K^0\left(x_{i_{\sigma(n_1+1)}},y_{i_{\sigma'(n_1+1)}},\alpha\right)...G_K^0\left(x_{i_{\sigma(n)}},y_{i_{\sigma'(n)}},\alpha\right)$$
$$\times G_K^0\left(x_{i_{\sigma(1)}},x_1,\alpha\right)G_K^0\left(y_{i_{\sigma'(1)}},x_2,\alpha\right)...G_K^0\left(x_{i_{\sigma(n_1)}},x_{2n_1-1},\alpha\right)G_K^0\left(y_{i_{\sigma'(n_1)}},x_{2n_1},\alpha\right)$$
$$\prod_{p=n_1+1}^{N}G_K^0\left(x_{2p-1},x_{2p},\alpha\right)\prod_{\substack{k_j\geqslant 2\\1\leqslant j\leqslant l}}\left\{V_{k_j}\left(x_{i_1}^{(k_j)}-\hat{x}_{i_2}^{(i_1)}\right)dx_{i_1}^{(k_j)}\right\}$$

$N=\sum_{j=1}^{l}k_j$. However, since the potential depends only on one variable, these contribution can be resummed to produce a free Green function shifted by the potential $V\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)$.

$$\left(G_K\right)^{-1}=-\nabla_k^2+m_k^2+\left((x_i)_k-\check{Y}_{eff}\right)(\Lambda_i)_k\left((x_i)_k-\check{Y}_{eff}\right)+V\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)$$

Thus the system describes a free effective utility for the first agent, and a potential, effective utility for the second agent, that is shifted by a term driving it towards or away $\hat{x}_{i_2}^{(i_1)}$, given the sign of $V\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)$.

8.6 Introduction of constraints

When agents face constraints, like the budget constraint for example, some additive terms have to be added to (143). Recall that, for a set of interacting individual agents, a linear constraint binding the agents implies to include, in the effective utility, a term of the form (97):

$$-\frac{1}{T+\sigma}\sum_i\int_0^s C_s^i ds\int_0^s C_t^i dt-\frac{1}{T+\sigma}\sum_i\int_0^s\int_0^T C_s^iC_t^j dsdt \tag{157}$$

where Agent i is defined by an action C_s^i, and T and σ are some parameters of the model. As explained when (97) was introduced, σ measures the uncertainty about the future, and T is proportional to the characteristic time scale of the interaction process. As explained before, we assume that each agent estimates at each moment the remaining duration of the interaction process by a Poisson process of mean T. We also assume that among the set of interacting agents, the statistical mean of the estimated duration reaches the true value s. That is, we suppose unbiased estimations. We will inspect less restrictive assumptions at the end of this paragraph, and show that this does not modify the result.

If we moreover neglect σ, the fluctuation term with respect to the duration of the interaction process, we are left with the following expression for (97):

$$-\frac{1}{s}\sum_i\int_0^s C_s^i ds\int_0^s C_t^i dt-\frac{1}{s}\sum_i\int_0^s\int_0^s C_s^iC_t^j dsdt$$

and (97) can be generalized for any type of vector of action $X^i(s)$ or constraint:

$$-\frac{1}{s}\sum_i\left(\int_0^s\left(X_1^i(s)\right)ds\right)^2-\frac{1}{s}\sum_i\int_0^s\int_0^s a_{1,2}X_1^i(s)X_2^j(t)dsdt \tag{158}$$

and $a_{i,j}$ describes the interdependence of two different species through the constraint.

The second term in (158) has already been described in the field theoretic formulation. It amounts to include a potential:

$$-a_{1,2}\int\int\left(\Psi^{(1)\dagger}\left(\hat{X}_1\right)\hat{X}_1\Psi^{(1)}\left(\hat{X}_1\right)\right)\left(\Psi^{(2)\dagger}\left(\hat{X}_2\right)\hat{X}_2\Psi^{(2)}\left(\hat{X}_2\right)\right)d\hat{X}_1d\hat{X}_2$$

in the global action. The first term requires some additional computations. We compute the Green function of the individual agents with effective utility including a term:

$$-\sum_{i}\left(\int X_{1}^{i}(s)\,ds\right)^{2}$$

We start with:

$$\int \exp(-\alpha s_i)\int \mathcal{D}x_i(t)\exp\left(-\sum_i\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}(t)+K(x_i(t))\,dt\right)-\frac{1}{s}\left(\int_0^s ds_1 C_{s_1}^{(i)}\right)\int_0^s C_s^i ds\right)$$

and neglecting the potential $K(x_i(t))$ that can be reintroduced as a perturbation term, one thus has to compute:

$$\bar{G}(x,y)=\int \exp(-\alpha s_i)\int \mathcal{D}x_i(t)\exp\left(-\sum_i\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}(t)\,dt\right)-\frac{1}{s}\left(\int_0^s ds_1 C_{s_1}^{(i)}\right)\int_0^s C_s^i ds\right) \tag{159}$$

which is the Green function for an agent under constraint. It can also be written:

$$\begin{aligned}\bar{G}(x,y) &= P(0,s,x_i,y_i)\left\langle \exp\left(-\frac{1}{T}\left(\int_0^s X(u)\,du\right)\left(\int_0^T X(u)\,du\right)\right)\right\rangle \qquad (160a)\\ &= \frac{\exp\left(-\frac{(x-y)^2}{\sigma^2 s}\right)}{\sqrt{s}}\left\langle \exp\left(-\frac{1}{T}\left(\int_0^s X(u)\,du\right)\left(\int_0^T X(u)\,du\right)\right)\right\rangle\end{aligned}$$

Where $X(u)$ a brownian motion starting at x_i at time 0 and reaching y_i at time s and

$$\left\langle \exp\left(-\frac{1}{s}\left(\int_0^s X(u)\,du\right)\left(\int_0^s X(u)\,du\right)\right)\right\rangle$$

is the expectation value of $\exp\left(\frac{1}{s}\left(\int_0^s X(u)\,du\right)^2\right)$ given the process $X(u)$.

The appearance of the factor $P(0,s,x_i,y_i)$ in (160a) comes from the fact that in (159) the measure is not normalized, and (159) is computed for the measure of a free Brownian motion. Thus the global weight for the path starting at x_i at time 0 and reaching y_i at time s is not equal to 1 but to $P(0,s,x_i,y_i)$. We compute $\bar{G}(x,y)$ in Appendix 10, and show that, when $\sigma<\alpha$, s being of order $\frac{1}{\alpha}$, and individual fluctuations measured by $\frac{\sigma}{\sqrt{\alpha}}$ are negligeable with respect to the mean path $\frac{x+y}{2}$ over the all duration of interaction, one has in first approximation:

$$\begin{aligned}\bar{G}(\alpha,x,y) &= \mathcal{L}\left[\left\langle \exp\left(-\frac{1}{s}\left(\int_0^s X(u)\,du\right)\left(\int_0^s X(u)\,du\right)\right)\right\rangle\frac{\exp\left(-\frac{(x-y)^2}{\sigma^2 s}\right)}{\sqrt{s}}\right] \qquad (161)\\ &= \frac{\exp\left(-\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\left|\frac{x-y}{\sigma}\right|\right)}{\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}}\end{aligned}$$

These assumptions are quite always satisfied since $\alpha=\frac{1}{T}$, with T the mean duration of all interaction processes. We furthermore expect the sum of fluctuations on this period, i.e. the sum of the fluctuations on the global time span, to be lower than one, or equivalently, the fluctuation per unit of time σ to be lower than $\frac{1}{T}$. By a similar reasonning, we assume that the fluctuations over the all time span, measured by $\frac{\sigma}{\sqrt{\alpha}}$ or equivalently by $\sigma\sqrt{s}$, are lower than the mean value of the path, i.e. $\frac{x+y}{2}$. The formula (161) has an interpretation in term of individual agents fluctuations. Actually, in (161) $\bar{G}(\alpha,x,y)$ satifies:

$$\bar{G}(\alpha,x,y)=\int \exp\left(-\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)s\right)\frac{\exp\left(-\frac{(x-y)^2}{\sigma^2 s}\right)}{\sqrt{s}}ds \tag{162}$$

This is easy to see: $\bar{G}(\alpha,x,y)$ is the Laplace transform of the usual brownian transition function $\frac{\exp\left(-\frac{(x-y)^2}{\sigma^2 s}\right)}{\sqrt{s}}$ with α shifted to $\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)$. By a change of variable:

$$s'=\frac{\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}{\alpha}s$$

we have:

$$\bar{G}(\alpha,x,y)=\int\exp(-\alpha s)\frac{\exp\left(-\frac{(x-y)^2}{\left(\frac{\alpha}{\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\sigma^2\right)s'}\right)}{\sqrt{\frac{\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}{\alpha}}\sqrt{s}}ds \tag{163}$$

Up to the factor $\sqrt{\frac{\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}{\alpha}}$, which is constant with respect to s, this is the Laplace transform of a gaussian path with variance:

$$(\sigma')^2=\frac{\alpha}{\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\sigma^2$$

Recall that there is usually no inertia in the standard models of utility optimization under constraint. This amounts to setting $\sigma^2\rightarrow\infty$ in our formalism, to model no other interconnexion between periods than the constraint. Recall however that (161) was derived under the assumption that $\sigma^2<<\left(\frac{x+y}{2}\right)^2$. As a consequence, the introduction of the constraint leads us to describe the individual agent following a brownian path with $(\sigma')^2<<1$. Considered at the scale of the overall processes - i.e. compared to the unit of time which is much lower than s - this variance $(\sigma')^2$ is of order 1, and the agent is described by a brownian path with variance of order 1. The introduction of the contraint has thus transformed the individual dynamics into an apparent brownian noise. This replicates the usual result in classical consumption smoothing theory (see [12] for example).

The field theoretic counterpart of the Green function $\bar{G}(\alpha,x,y)$ is obtained by finding a differential operator whose inverse is $\bar{G}(\alpha,x,y)$ or equivalently, a differential equation satisfied by $\bar{G}(\alpha,x,y)$. Appendix 10 shows that $\bar{G}(\alpha,x,y)$ satisfies:

$$\begin{aligned}\delta(x-y) &= \frac{\sigma^2}{2}\left[\left(\nabla^2-2\frac{\alpha+\left(\frac{x+y}{2}\right)^2}{\sigma^2}\right)-\frac{(x+y)^2}{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\left(\left(\frac{3}{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}+\frac{3\left|\frac{x-y}{\sigma}\right|}{\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}}+\left|\frac{x-y}{\sigma}\right|^2\right)\right)\right.\\ &\left.-\frac{1+\left|\frac{x-y}{\sigma}\right|\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}}{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\left(\left(2(x+y)\frac{H(x-y)-H(y-x)}{\sigma}\right)\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}-1\right)\right]\bar{G}(\alpha,x,y)\end{aligned}$$

and the term in brackets is the operator whose kernel is $\bar{G}(\alpha,x,y)$. The Appendix 10 also shows that, given our assumptions, this reduces in the limit of small fluctuations, to:

$$\delta(x-y)=\left(\frac{\sigma^2}{2}\nabla^2-2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)\right)\bar{G}(\alpha,x,y)$$

Reintroducing the potential $K(x)$, the field theoretic formulation of the problem for a single type of agent with effective action (157) reduces to describing the set of individual agents by the effective action:

$$\int\exp\left(-\Psi(x)\left[\left(\bar{G}(\alpha,x,y)\right)^{-1}+K(x)\right]\Psi^{\dagger}(x)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger} \tag{164}$$

which discards temporarilly interactions among agents. Of course, when we remove the constraint, $\bar{G}(\alpha,x,y)$ reduces to $G(\alpha,x,y)$, and $(G(\alpha,x,y))^{-1}=-\frac{1}{2}\nabla^2+\alpha+K(x)$, as in the previous cases. In developped terms, the exponential in (164) becomes:

$$\begin{aligned}
\Psi^{\dagger}(x) \quad & \left[\left(-\sigma^2\frac{\nabla^2}{2}+\frac{\alpha+\left(\frac{x+y}{2}\right)^2}{\sigma^2}\right)+\frac{\sigma^2(x+y)^2}{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\left(\left(\frac{3}{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}+\frac{3\left|\frac{x-y}{\sigma}\right|}{\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}}+\left|\frac{x-y}{\sigma}\right|^2\right)\right)\right. \\
& +\sigma^2\frac{1+\left|\frac{x-y}{\sigma}\right|\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}}{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\times \\
& \left.\times\left(\left(2(x+y)\frac{H(x-y)-H(y-x)}{\sigma}\right)\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}-1\right)+K(x)\right]\Psi(y)
\end{aligned}$$

In the case of $\sigma<<1$ considered here, in which individual fluctuations are relatively small, it remains:

$$\Psi^{\dagger}(x)\left[-\frac{\sigma^2}{2}\nabla^2+2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)\right]\Psi(y) \tag{165}$$

This form of propagator has a direct interpretation in terms of constraint. The first term ensures that the mean of $x+y$ is centered on its expectation value, which is nul here by normalization. The second term ensures that x and y are equal in means. Both contributions thus describe a smoothing behavior, which is characteristic of long-run binding constraints. The path for $X(s)$, apart from a white noise contribution $\epsilon(s)$, is constant in time:

$$X(s)=X(s-1)+\epsilon(s)$$

We also recover the results of (93) and its subsequent formulae. For $x-y<<1$, we recover the series expansion in gradient:

$$\begin{aligned}
& \Psi^{\dagger}(x)\left[\left(\nabla^2-\frac{\alpha}{\sigma^2}\right)\delta(x-y)-\frac{\left(\frac{x+y}{2}\right)^2}{\sigma^2}-2\left|\frac{x-y}{\sigma}\right|^2\right]\Psi(y) \\
\simeq\quad & \Psi^{\dagger}(x)\left[\left(\nabla^2-\frac{\alpha}{\sigma^2}\right)-\frac{x^2}{\sigma^2}+\nabla^2\right]\delta(x-y)\Psi(x)
\end{aligned}$$

Then, introducing the constrained propagator (165) in (143) yields:

$$\begin{aligned}
& S\left(\left\{\Psi^{(k)}\right\}_{k=1\ldots M}\right) \\
= & \frac{1}{2}\sum_k\int d\hat{X}_k^{(1)}d\hat{X}_k^{(2)}\Psi^{(k)\dagger}\left(\hat{X}_k^{(2)}\right)\left[\left[\left(\nabla_{\hat{X}_k^{(1)}}\right)\left(\nabla_{\hat{X}_k^{(1)}}-M_k^{(1)}\left(\hat{X}_k^{(1)}-\left(\tilde{X}\right)_k\right)\right)+m_k^2+V\left(\hat{X}_k^{(1)}\right)\right]\delta\left(\hat{X}_k^{(1)}-\hat{X}_k^{(2)}\right)\right. \\
& \left.+\underbrace{\frac{\alpha+\left(\frac{\hat{X}_k^{(1)}+\hat{X}_k^{(2)}}{2}\right)^2}{\sigma^2}+2\left|\frac{\hat{X}_k^{(1)}-\hat{X}_k^{(2)}}{\sigma}\right|^2}_{\text{constraint, individual level}}\right]\Psi^{(k)\dagger}\left(\hat{X}_k^{(2)}\right)+\underbrace{\sum_k\sum_n V_n\left(\left\{\hat{X}_k^{(i)}\right\}_{1\leqslant i\leqslant n}\right)\prod_{1\leqslant i\leqslant n}\Psi^{(k)\dagger}\left(\hat{X}_k^{(i)}\right)\Psi^{(k)}\left(\hat{X}_k^{(i)}\right)}_{\text{intra species interaction}} \\
& +\underbrace{\sum_m\sum_{k_1\ldots k_m}\sum_{n_1\ldots n_m}V_{n_1\ldots n_m}\left(\left\{\hat{X}_{k_j}^{\left(i_{n_j}\right)}\right\}_{1\leqslant i_{n_j}\leqslant n_j}\right)\prod_{j=1}^m\prod_{1\leqslant i_{n_j}\leqslant n_j}\Psi^{(k_j)\dagger}\left(\hat{X}_{k_j}^{\left(i_{n_j}\right)}\right)\Psi^{(k_j)}\left(\hat{X}_{k_j}^{\left(i_{n_j}\right)}\right)}_{\text{inter species interaction}} \\
& +\underbrace{\sum_{k_1,k_2}a_{k_1,k_2}\int\int\left(\Psi^{(k_1)\dagger}\left(\hat{X}_{k_1}\right)\hat{X}_{k_1}\Psi^{(k_1)}\left(\hat{X}_{k_1}\right)\right)\left(\Psi^{(k_2)\dagger}\left(\hat{X}_{k_2}\right)\hat{X}_{k_2}\Psi^{(k_2)}\left(\hat{X}_{k_2}\right)\right)d\hat{X}_{k_1}d\hat{X}_{k_2}}_{\text{constraint, collective level}}
\end{aligned}$$

Appendix 11 shows how to generalize this result in presence of a discount rate and we show that in that case, in the approximation $\frac{r}{\alpha} >> 1$, which means that the time span of interaction is long enough for the discount rate to be effective:

$$\begin{aligned}
& S\left(\left\{\Psi^{(k)}\right\}_{k=1\ldots M}\right) \qquad (166)\\
= & \frac{1}{2}\sum_{k}\int d\hat{X}_k^{(1)} d\hat{X}_k^{(2)} \Psi^{(k)\dagger}\left(\hat{X}_k^{(2)}\right)\left[\left[\left(\nabla_{\hat{X}_k^{(1)}}\right)\left(\nabla_{\hat{X}_k^{(1)}} - M_k^{(1)}\left(\hat{X}_k^{(1)} - \left(\tilde{X}\right)_k\right)\right) + m_k^2 + V\left(\hat{X}_k^{(1)}\right)\right]\delta\left(\hat{X}_k^{(1)} - \hat{X}_k^{(2)}\right)\right.\\
& \left. + \underbrace{2\frac{\left(\hat{X}_k^{(1)}\right)^2 + \left(\hat{X}_k^{(2)}\right)^2}{r^2} - 4\frac{\sqrt{2\alpha}\hat{X}_k^{(1)}\hat{X}_k^{(2)}}{r\sigma}\left(H\left(\hat{X}_k^{(1)} - \hat{X}_k^{(2)}\right) - H\left(\hat{X}_k^{(2)} - \hat{X}_k^{(1)}\right)\right)}_{\text{constraint, individual level}}\right]\Psi^{(k)\dagger}\left(\hat{X}_k^{(2)}\right)\\
& + \underbrace{\sum_k\sum_n V_n\left(\left\{\hat{X}_k^{(i)}\right\}_{1\leqslant i\leqslant n}\right)\prod_{1\leqslant i\leqslant n}\Psi^{(k)\dagger}\left(\hat{X}_k^{(i)}\right)\Psi^{(k)}\left(\hat{X}_k^{(i)}\right)}_{\text{intra species interaction}}\\
& + \underbrace{\sum_m\sum_{k_1\ldots k_m}\sum_{n_1\ldots n_m} V_{n_1\ldots n_m}\left(\left\{\hat{X}_{k_j}^{(i_{n_j})}\right\}_{1\leqslant i_{n_j}\leqslant n_j}\right)\prod_{j=1}^{m}\prod_{1\leqslant i_{n_j}\leqslant n_j}\Psi^{(k_j)\dagger}\left(\hat{X}_{k_j}^{(i_{n_j})}\right)\Psi^{(k_j)}\left(\hat{X}_{k_j}^{(i_{n_j})}\right)}_{\text{inter species interaction}}\\
& + \underbrace{\begin{array}{c}\sum_{k_1,k_2} a_{k_1,k_2}\int\int\left(\Psi^{(k_1)\dagger}\left(\hat{X}_{k_1}^{(1)}\right)\left(\exp\left(-\left(r+\frac{1}{2}\bar{N}r\right)G\left(\hat{X}_{k_1}^{(1)},\hat{X}_{k_1}^{(2)}\right)\right)\frac{\hat{X}_{k_1}^{(1)}+\hat{X}_{k_1}^{(2)}}{2}\right)\Psi^{(k_1)}\left(\hat{X}_{k_1}^{(2)}\right)\right)d\hat{X}_{k_1}^{(1)}d\hat{X}_{k_1}^{(2)}\\ \times\int\int\left(\Psi^{(k_2)\dagger}\left(\hat{X}_{k_2}^{(1)}\right)\left(\exp\left(\left(r+\frac{1}{2}\bar{N}r\right)G\left(\hat{X}_{k_2}^{(1)},\hat{X}_{k_2}^{(2)}\right)\right)\frac{\hat{X}_{k_2}^{(1)}+\hat{X}_{k_2}^{(2)}}{2}\right)\Psi^{(k_2)}\left(\hat{X}_{k_2}^{(2)}\right)\right)d\hat{X}_{k_2}^{(1)}d\hat{X}_{k_2}^{(2)}\end{array}}_{\text{constraint, collective level}}
\end{aligned}$$

We conclude this paragraph by inspecting other assumptions about the expected time horizon T. Assume that agents have some hint about the true duration s, and, as a consequence the Poisson distribution is no more accurate. These informations translate into the fact that T depends on the time at which it is evaluated. For example $T(v) = s - f(v)$ where f is a slow varying and increasing function. Under this hypothesis, the quadratic term due to the constraints becomes:

$$\exp\left(\int_0^s\left(\frac{1}{s-f(v)}X(v)\int_0^v X(u)\,du\right)dv\right)$$

and since $f(v)$ varies slowly, one can approximate $f(v)$ by its mean over $[0,s]$ in the integral:

$$\begin{aligned}
& \int_0^s\left(\frac{1}{s-f(v)}X(v)\int_0^v X(u)\,du\right)dv\\
\simeq & \int_0^s\left(\frac{1}{s-\bar{f}(s)}X(v)\int_0^v X(u)\,du\right)dv = \frac{1}{1-\frac{\bar{f}(s)}{s}}\frac{1}{s}\int_0^s\left(X(v)\int_0^v X(u)\,du\right)dv
\end{aligned}$$

The term $\frac{1}{s}\int_0^s\left(X(v)\int_0^v X(u)\,du\right)dv$ is the one we dealt with before and $\frac{\bar{f}(s)}{s}$ can be considered as a perturbation. Since $\frac{\bar{f}(s)}{s}$ varies slowly, it can be approximated by $\alpha\ \bar{f}\left(\frac{1}{\alpha}\right)$ where $\frac{1}{\alpha}$ is the mean duration process.

We are thus left with:

$$\begin{aligned}
\bar{G}(\alpha,x,y) &= \mathcal{L}\left[\left\langle \exp\left(\frac{1}{1-\alpha\bar{f}\left(\frac{1}{\alpha}\right)}\frac{1}{s}\left(\int_0^s X(u)\,du\right)\left(\int_0^s X(u)\,du\right)\right)\right\rangle \frac{\exp\left(-\frac{(x-y)^2}{\sigma^2 s}\right)}{\sqrt{s}}\right] \\
&\simeq \mathcal{L}\left[\exp\left(\frac{1}{1-\alpha\bar{f}\left(\frac{1}{\alpha}\right)}s\left(\frac{x+y}{2}\right)^2\right)\frac{\exp\left(-\frac{(x-y)^2}{\sigma^2 s}\right)}{\sqrt{s}}\right] \\
&= :\exp\left(\frac{\left(\left(\frac{x+y}{2}\right)^2\right)}{1-\alpha\bar{f}\left(\frac{1}{\alpha}\right)}\frac{\partial}{\partial\alpha}\right):\frac{\exp\left(-\sqrt{2\alpha}\left|\frac{x-y}{\sigma}\right|\right)}{\sqrt{2\alpha}} \\
&= \frac{\exp\left(-\sqrt{2\left(\alpha+\frac{\left(\frac{x+y}{2}\right)^2}{\left(1-\alpha\bar{f}\left(\frac{1}{\alpha}\right)\right)}\right)}\left|\frac{x-y}{\sigma}\right|\right)}{\sqrt{2\left(\alpha+\frac{\left(\frac{x+y}{2}\right)^2}{\left(1-\alpha\bar{f}\left(\frac{1}{\alpha}\right)\right)}\right)}}
\end{aligned}$$

where the notation $:\exp\left(\frac{\left(\left(\frac{x+y}{2}\right)^2\right)}{1-\alpha\bar{f}\left(\frac{1}{\alpha}\right)}\frac{\partial}{\partial\alpha}\right):$ denotes the ordered product, i.e. all the derivative are set on the right after expansion. As a consequence, the introduction of a varying time horizon shifts the mean path $\left(\frac{x+y}{2}\right)$ to $\left(\frac{x+y}{2\sqrt{1-\alpha\bar{f}\left(\frac{1}{\alpha}\right)}}\right)$ but all the previous results are kept, when this shift is included.

9 Examples

9.1 Consumers/Producers with current account constraint

9.1.1 Case 1: One type of agents

We consider N identical agents that are consumers/producers. Each of them is producing one good that is consumed by other agents in constant proportion.The production/revenue $Y_s^{(i)}$ is proportionnal to other agents consumption (plus some exogenous constant flow):

$$Y_s = \sum_j f\left(C_s^{(j)}\right) + \bar{Y} \tag{167}$$

with $Nf = 1$. Each agent is facing the C.A. balance constraint:

$$C_s^{(i)} = B_s^{(i)} + Y_s^{(i)} - B_{s+1}^{(i)}$$

which rewrites, given (167):

$$C_s = B_s + \sum_j \left(f\left(C_s^{(j)}\right) + \bar{Y}\right) - B_{s+1}$$

We have seen that, with such a constraint, we obtained the following weight (98):

$$\exp\left(U^{eff}\right) = \exp\left(-\sum_i\sum_s\left(C_s^{(i)}\right)^2 - \frac{1}{T}\sum_i\sum_{s_1,s_2}C_{s_1}^{(i)}C_{s_2}^{(i)} + \frac{2}{T}f\sum_{i,j}\sum_{s_1\geqslant s_2}C_{s_1}^{(i)}C_{s_2}^{(j)}\right)$$

Here, the consumption variable was shifted by substracting it's optimum, i.e. as before $-C_s^2$ stands for $-\alpha\left(C_s-\bar{Y}\right)^2$ and Y_s for $Y_s+\bar{Y}$. When we consider a large number of identical agents, we can follow the procedure given above (see (95), (96), (97),(98)). The main point is that Y_s is endogeneized. The effective action in the continuum approximation is:

$$\exp\left(U^{eff}\right) = \exp\left(-\sum_i \left(\int \left(C_s^{(i)}\right)^2 ds + \frac{1}{T}\int C_{s_1}^{(i)} C_{s_2}^{(i)} ds_1 ds_2\right) + \frac{f}{T}\sum_{i,j}\int\int C_s^{(i)} C_{s_2}^{(j)} ds_1 ds_2\right)$$

whose field theoretic formulation is defined by:

$$\begin{aligned}S(\Psi) &= -\int \Psi^{\dagger}(x)\left[\left(-\nabla^2 + \frac{\alpha}{\sigma^2} + x^2\right)\delta(x-y) + \frac{\left(\frac{x+y}{2}\right)^2}{\sigma^2} + 2\left|\frac{x-y}{\sigma}\right|^2\right]\Psi(y)\,dxdy \\ &\quad -f\int \Psi(x)\Psi^{\dagger}(x)(xy)\Psi(y)\Psi^{\dagger}(y)\,dxdy\end{aligned}$$

Remark that since the variable x stands for $C-\bar{Y}\rightarrow x$, x is not constrained to $x>0$.

$$\begin{aligned}&\int dx\Psi(x)\left(-\nabla^2 + x^2 + \epsilon^2\right)\Psi^{\dagger}(x) + \int dxdy\Psi^{\dagger}(x)\left[\frac{\left(\frac{x+y}{2}\right)^2}{\sigma^2} + 2\left|\frac{x-y}{\sigma}\right|^2\right]\Psi(y) \\ &-f\int \Psi(x)\Psi^{\dagger}(x)(xy)\Psi(y)\Psi^{\dagger}(y)\,dxdy\end{aligned}$$

if no inertia:

$$\begin{aligned}\simeq\ &\int dx\Psi(x)\left(x^2 + \epsilon^2\right)\Psi^{\dagger}(x) + \int dxdy\Psi^{\dagger}(x)\left[\frac{\left(\frac{x+y}{2}\right)^2}{\sigma^2} + 2\left|\frac{x-y}{\sigma}\right|^2\right]\Psi(y) \\ &-f\int \Psi(x)\Psi^{\dagger}(x)(xy)\Psi(y)\Psi^{\dagger}(y)\,dxdy\end{aligned}$$

We show in Appendix 12 that the minimum of $S(\Psi)$ is reached for $\Psi(x)=0$ and that there is no other minimum, even local, so that no phase transition appears. The reason of this vacuum at $\Psi(x)=0$ is the direct consequence of the constraint represented by the term:

$$-\frac{1}{T}f\left[\int \Psi^{\dagger}(x)\,x\Psi(x)\right]\left[\int \Psi^{\dagger}(y)\,y\Psi(y)\right]$$

in the effective action $S(\Psi)$. The minus sign is crucial for preventing any phase transition. Thus the constraints smoothe interactions between agents, which prevents from switching from a symmetric nul equilibrium to an asymmetric one favouring some agents.

As a consequence one can directly consider the graph expansion around $\Psi=0$. Here (149) yields for the two points correlation functions:

$$\begin{aligned}G_K(x,y,\alpha) &= G_K^0(x,y,\alpha) \\ &\quad + \sum_{l>0}(-f)^l G_K^0(x,y_1,\alpha)\,y_1 G_K^0(y_1,y_2,\alpha)\,y_2 ... y_{2l} G_K^0(y_{2l},y,\alpha)\end{aligned} \tag{168}$$

where $G_K^0(x,y,\alpha)$ is the Green function of the operator:

$$\left(-\sigma^2\frac{\nabla^2}{2} + \alpha + x^2\right)\delta(x-y) + \left(\frac{x+y}{2}\right)^2$$

as explained before, the term $\left(\frac{x+y}{2}\right)^2$ induces a smearing in the behavior of the agents, due to the constraint. The contributions in (168) can be resummed so that:

$$G_K^{-1}(x,y,\alpha) = \left(G_K^0\right)^{-1}(x,y,\alpha) + f^2 x G_K^0(x,y,\alpha)\,y$$

and thus $G_K(x,y,\alpha)$ is the Green function of the operator:

$$\left(-\sigma^2\frac{\nabla^2}{2}+\alpha+x^2\right)\delta(x-y)+\left(\frac{x+y}{2}\right)^2+f^2xG_K^0(x,y,\alpha)\,y$$

Now, given that

$$xG_K^0(x,y,\alpha)\,y=G_K^0(x,y,\alpha)\left(\left(\frac{x+y}{2}\right)^2-\left(\frac{x-y}{2}\right)^2\right)$$

and that the term $\left(\frac{x-y}{2}\right)^2$ can be neglected under our basic assumption of low fluctuations, the inclusion of the interaction with other structures modifies the smearing potential $\left(\frac{x+y}{2}\right)^2$ by:

$$\left(\frac{x+y}{2}\right)^2\left(1+f^2G_K^0(x,y,\alpha)\right)$$

Inserting this result in (163), leads to model the apparent behavior of the agent as a brownian path, whose variance is modified from:

$$(\sigma')^2=\frac{\alpha}{\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\sigma^2$$

to

$$(\sigma')^2=\frac{\alpha}{\left(\alpha+\left(1+f^2G_K^0(x,y,\alpha)\right)\right)}\sigma^2$$

In other words, the variance of the movement is reduced by the presence of other agents. The interaction reinforces the effect of the constraint and imposes smaller variations for the individual agents.

9.1.2 Case 2. Several types of agents

If we consider several types of agents denoted by greek indices $\{\alpha...\}$, we can define $C_s^{(i),\alpha}$ as the consumption of agent i belonging to type α. The constraint becomes:

$$C_s^{(i),\alpha}=B_s^{(i),\alpha}+\sum_{i,\beta}\left(f_{\alpha\beta}\left(C_s^{(j),\beta}\right)+\bar{Y}\right)-B_{s+1}^{(i),\alpha}$$

the coefficients $f_{\alpha\beta}$ define the fraction of consumption of an agent β spent in the good produced by agents of type α. They satisfy:

$$\sum_\alpha N_\alpha f_{\alpha\beta}=1$$

where N_α is the number of agents of type α, so that $\sum_\alpha N_\alpha=N$ with N the total number of agents.

As in the previous paragraph, the effective utility for the system becomes:

$$\exp\left(U^{eff}\right)=\exp\left(-\sum_i\left(\int\left(\left(C_s^i\right)^2ds\right)+\frac{1}{T}\left(\int C_s^{(i)}ds\right)^2\right)+\frac{1}{T}\sum_{i,\alpha}\int\int C_s^{(i),\alpha}\left(\sum_{j,\beta}f_{\alpha\beta}C_t^{(j),\beta}\right)dsdt\right)$$

Which leads to the field equivalent description:

$$\begin{aligned}S\left((\Psi_\alpha)\right) &= \sum_\alpha\left(\int dx_\alpha\Psi^\dagger(x_\alpha)\left(-\nabla_\alpha^2+x_\alpha^2+\epsilon^2\right)\Psi_\alpha(x_\alpha)+\int dx_\alpha dy_\alpha\Psi^\dagger(x_\alpha)\left[\frac{\left(\frac{x_\alpha+y_\alpha}{2}\right)^2}{\sigma^2}+2\left|\frac{x_\alpha-y_\alpha}{\sigma}\right|^2\right]\Psi_\alpha(y_\alpha)\right)\\ &\quad -\frac{1}{T}\sum_{\alpha,\beta}f_{\alpha\beta}\left[\int\Psi_\alpha^\dagger(x_\alpha)\,x_\alpha\Psi_\alpha(x_\alpha)\right]\left[\int\Psi_\beta^\dagger(x_\beta)\,x_\beta\Psi_\beta(x_\beta)\right]\end{aligned}$$

Appendix 12 shows that the minimum of $S((\Psi_\alpha))$ is reached for $\Psi_\alpha(x)=0$ and that there is no other minimum, even local. Again this is the direct consequence of the constraint that induces the terms:

$$-\frac{1}{T}\sum_{\alpha,\beta}f_{\alpha\beta}\left[\int\Psi_\alpha^\dagger(x_\alpha)\,x_\alpha\Psi_\alpha(x_\alpha)\right]\left[\int\Psi_\beta^\dagger(x_\beta)\,x_\beta\Psi_\beta(x_\beta)\right]$$

in the effective action. Here again, the constraints smoothe the interactions between agents, and prevents from switching from a symmetric nul equilibrium to an asymmetric one favouring somes groups of agents.

The two points Green functions can be computed similarly to the previous case. In term of graphs, the term $f_{\alpha\beta}\left[\int\Psi_{\alpha}^{\dagger}\left(x_{\alpha}\right)x_{\alpha}\Psi_{\alpha}\left(x_{\alpha}\right)\right]\left[\int\Psi_{\beta}^{\dagger}\left(x_{\beta}\right)x_{\beta}\Psi_{\beta}\left(x_{\beta}\right)\right]$ implies that vertices α with two legs are connected to vertices β, $\alpha\neq\beta$ with two legs through a line labelled $f_{\alpha\beta}$. The factors $f_{\alpha\alpha}$ can be absorbed by $x_{\alpha}\rightarrow\sqrt{f_{\alpha\alpha}}x_{\alpha}$. Keeping only connected graphs, one finds:

$$\begin{aligned}G_{K}\left(x^{(\alpha)},y^{(\alpha)},\alpha\right) &= G_{K}^{0}\left(x^{(\alpha)},y^{(\alpha)},\alpha\right)\\&+\sum_{l>0}\left(-1\right)^{l}f_{l}^{(\alpha)}G_{K}^{0}\left(x^{(\alpha)},y_{1}^{(\alpha)},\alpha\right)y_{1}^{(\alpha)}G_{K}^{0}\left(y_{1}^{(\alpha)},y_{2}^{(\alpha)},\alpha\right)y_{2}^{(\alpha)}...y_{l}^{(\alpha)}G_{K}^{0}\left(y_{l}^{(\alpha)},y^{(\alpha)},\alpha\right)\end{aligned}\tag{169}$$

where $f_{l}^{(\alpha)}$ includes the modifications to $G_{K}^{0}\left(x,y,\alpha\right)$ due to the interactions with all other type of agents:

$$f_{l}^{(\alpha)}=\sum_{k=0}^{E\left(\frac{l}{2}\right)}\left(f_{\alpha\alpha}\right)^{k}\sum_{\beta_{1},...,\beta_{l-k}}f_{\alpha\beta_{1}}...f_{\alpha\beta_{l-k}}\times\int G_{K}^{0(\alpha,l-k)}\prod_{\beta}G_{K}^{0\left(\beta,\sharp_{\beta}\left(\beta_{1},...,\beta_{l-k}\right)\right)}$$

where $\sharp_{\beta}\left(\beta_{1},...,\beta_{l-k}\right)$ is the number of times β appears in the set $\left(\beta_{1},...,\beta_{l-k}\right)$ and with

$$G_{K}^{0(\beta,p)}=\int y_{1}^{(\beta)}G_{K}^{0}\left(y_{1}^{(\beta)},y_{2}^{(\beta)},\alpha\right)y_{2}^{(\beta)}G_{K}^{0}\left(y_{3}^{(\beta)},y_{4}^{(\beta)},\alpha\right)...y_{p-1}^{(\beta)}G_{K}^{0}\left(y_{p-1}^{(\beta)},y_{1}^{(\beta)},\alpha\right)dy_{1}^{(\beta)}...dy_{p}^{(\beta)}$$

For $f_{\alpha\alpha}=1$, and thus $f_{\alpha\beta}=0$ for $\alpha\neq\beta$, so that one recovers the one type of agent case:

$$G_{K}^{-1}\left(x^{(\alpha)},y^{(\alpha)},\alpha\right)=\left(G_{K}^{0}\right)^{-1}\left(x^{(\alpha)},y^{(\alpha)},\alpha\right)+x^{(\alpha)}\left(x^{(\alpha)},y^{(\alpha)},\alpha\right)y^{(\alpha)}$$

For $f_{\alpha\alpha}=0$

$$\begin{aligned}f_{l}^{(\alpha)} &= \sum_{\beta_{1},...,\beta_{l}}f_{\alpha\beta_{1}}...f_{\alpha\beta_{l}}\times\int\prod_{\beta}G_{K}^{0\left(\beta,\sharp_{\beta}\left(\beta_{1},...,\beta_{l}\right)\right)}\\&= \int\left(\sum_{\beta}f_{\alpha\beta}x^{(\beta)}G_{K}^{0}\left(x^{(\beta)},y^{(\beta)},\alpha\right)\right)^{*l}\end{aligned}$$

where $*$ denotes the convolution product, and (169) becomes:

$$\begin{aligned}G_{K}\left(x^{(\alpha)},y^{(\alpha)},\alpha\right) &= G_{K}^{0}\left(x^{(\alpha)},y^{(\alpha)},\alpha\right)\\&+\sum_{l>0}\left(-1\right)^{l}\left(\sum_{\beta}f_{\alpha\beta}G_{K}^{0}\left(x^{(\beta)},y^{(\beta)},\alpha\right)\right)^{l}\\&\times G_{K}^{0}\left(x^{(\alpha)},y_{1}^{(\alpha)},\alpha\right)y_{1}^{(\alpha)}G_{K}^{0}\left(y_{1}^{(\alpha)},y_{2}^{(\alpha)},\alpha\right)y_{2}^{(\alpha)}...y_{l}^{(\alpha)}G_{K}^{0}\left(y_{l}^{(\alpha)},y^{(\alpha)},\alpha\right)\end{aligned}$$

That can be resummed as:

$$G_{K}^{-1}\left(x^{(\alpha)},y^{(\alpha)},\alpha\right)=\bar{G}^{-1}\left(x^{\alpha},y^{(\alpha)},\alpha\right)$$

9.2 A simple business cycle model

In this section we use again the single type of agent model, but we indentify the saving variable B_{s} with the stock of capital involved in the production function, as is usually done standard business cycle models. As a consequence, the budget constraint becomes:

$$C_{i}\left(s\right)=rK_{i}\left(s\right)-\dot{K}_{i}\left(s\right)+Y_{i}\left(s\right)$$

We also endogeneize $Y_i(s)$ and consider this variable as a function of the capital: $Y_i(s) = F_i(K_i(s))$. The budget constraint can thus be written as:

$$C_i(s) = rK_i(s) - \dot{K}_i(s) + F_i(K_i(s))$$

Now, introduce the interest rate r given by the mean productivity of the set of agents:

$$r = \frac{1}{N}\sum_i F_i'(K_i)$$

As before, the effective utility for agent i with constraint writes:

$$U^{eff}(C_i) = \int C_i^2(t)\,dt + \int_{t>s} \exp\left(-\int_s^t r(v)\,dv\right) C_i(s)\,C_i(t)\,dsdt - 2\int_{t>s} C_i(t)\exp\left(-\int_s^t r(v)\,dv\right) Y_i(s)\,dsdt$$

This is computed in Appendix 12., and the result in first approximation in r is:

$$U^{eff}(C_i) = \int C_i^2(t)\,dt - 2\int F_i(K_i(t))\,K_i(t)\,dt + 2\int r(t)\,K_i^2(t)\,dt + 4\int_{t>s} r(s)\,K_i(s)\,F_i(K_i(t))\,dsdt$$

At this point, it is more convenient to switch to a representation in the $K_i(t)$ variable. Replace $\int C_i^2(t)\,dt$ by:

$$\begin{aligned}\int C_i^2(t)\,dt &= \int \left(rK_i(t) - \dot{K}_i(t) + F_i(K_i(t))\right)^2 dt \\ &= \int \dot{K}_i^2(t)\,dt + \int \left(rK_i(t) + F_i(K_i(t))\right)^2 dt \\ &\quad -2\int \dot{K}_i(t)\left(rK_i(t) + F_i(K_i(t))\right) dt\end{aligned}$$

and write the last term as a border contribution (with $F_i = G_i'$):

$$\int \dot{K}_i(t)\left(rK_i(t) + F_i(K_i(t))\right) dt = \left[\frac{1}{2}rK_i^2(t) + G_i(K_i(t))\right]_0^T$$

Since we rule out accumulation of capital at 0 and T, we discard border terms, and this term can be neglected. As a consequence:

$$\begin{aligned}\int C_i^2(t)\,dt &= \int \left(rK_i(t) - \dot{K}_i(t) + F_i(K_i(t))\right)^2 dt \\ &= \int \dot{K}_i^2(t)\,dt + \int \left(rK_i(t) + F_i(K_i(t))\right)^2 dt\end{aligned}$$

and U^{eff} for agent i becomes at the first order in r:

$$\begin{aligned}U^{eff} &= \int \dot{K}_i^2(t)\,dt + \int \left(F_i^2(K_i(t)) - 2F_i(K_i(t))\,K_i(t)\right) dt \\ &\quad +2\int r(t)\left(K_i^2(t) + K_i(t)\,F_i(K_i(t))\right) dt + 4\int_{t>s} r(s)\,K_i(s)\,F_i(K_i(t))\,dsdt\end{aligned}$$

Summing over all agents, the global action for the system is:

$$\begin{aligned}\sum_i U^{eff} &= \sum_i \int \dot{K}_i^2(t)\,dt + \int \left(F_i^2(K_i(t)) - 2F_i(K_i(t))\,K_i(t)\right) dt \\ &\quad +2\int r(t)\left(K_i^2(t) + K_i(t)\,F_i(K_i(t))\right) dt + 4\int_{t>s} r(s)\,K_i(s)\,F_i(K_i(t))\,dsdt \\ &= \sum_i \int \dot{K}_i^2(t)\,dt + \int \left(F_i^2(K_i(t)) - 2F_i(K_i(t))\,K_i(t)\right) dt \\ &\quad +\frac{2}{N}\sum_{i,j}\int F_j'(K_j(t))\left(K_i^2(t) + K_i(t)\,F_i(K_i(t))\right) dt + \frac{4}{N}\sum_{i,j}\int_{t>s} F_j'(K_j(s))\,K_i(s)\,F_i(K_i(t))\,dsdt\end{aligned}$$

And since agents are identical, we can assume that in first approximation the two last terms are:

$$\begin{aligned}\left(K_i^2(t)+K_i(t)F_i(K_i(t))\right) &\simeq \left(K_j^2(t)+K_j(t)F_j(K_j(t))\right)\\ K_i(s) &\simeq K_j(s)\end{aligned}$$

the error in this approximation being of order lower than r. In that approximation:

$$\frac{2}{N}\sum_{i,j}\int F_j'(K_j(t))\left(K_i^2(t)+K_i(t)F_i(K_i(t))\right)dt=2\sum_j\int F_j'(K_j(t))\left(K_j^2(t)+K_j(t)F_j(K_j(t))\right)dt$$

and this individual potential term is of order r, through $F_j'(K_j(t))$. As a consequence, it can be neglected with respect to

$$\int\left(F_i^2(K_i(t))-2F_i(K_i(t))K_i(t)\right)dt$$

We then end up with:

$$\begin{aligned}\sum_i U^{eff} &= \sum_i\int \dot{K}_i^2(t)\,dt+\int\left(F_i^2(K_i(t))-2F_i(K_i(t))K_i(t)\right)dt\\ &\quad+\frac{4}{N}\sum_{i,j}\int_{t>s}F_j'(K_j(s))K_i(s)F_i(K_i(t))\,dsdt\end{aligned}$$

Assuming agents are identical, so that $F_i\equiv F$, such an effective utility for the system has for Field theoretic equivalent:

$$S(\Psi)=\Psi^\dagger(x)\left[\left(-\nabla^2+\left(F^2(x)-2F(x)x\right)\right)\right]\Psi(x)+\frac{4}{N}\int\left(\Psi^\dagger(x)F'(x)x\Psi(x)\,dx\right)\int\Psi^\dagger(y)F(y)\Psi(y)\,dy$$

Depending on the sign of $\left(F^2(x)-2F(x)x\right)$, the action $S(\Psi)$ may present some non trivial saddle point. To inspect this possibility, write the saddle point equation $\frac{\delta}{\delta\Psi}S(\Psi)=0$ as:

$$\begin{aligned}0 &= \left[\left(-\nabla^2+\left(F^2(x)-2F(x)x\right)\right)\right]\Psi(x)\\ &\quad+\frac{4}{N}F'(x)x\left(\int\Psi^\dagger(y)F(y)\Psi(y)\,dy\right)\Psi(x)+\frac{4}{N}\left(\int\Psi^\dagger(y)F'(y)y\Psi(y)\,dy\right)F(x)\Psi(x)\end{aligned}\tag{170}$$

Now, let:

$$\Psi(x)=\sqrt{\eta}\Psi_1(x)$$

with $\|\Psi_1(x)\|=1$, so that (170) can be written in function of $\Psi_1(x)$:

$$\begin{aligned}0 &= \left[\left(-\nabla^2+\left(F^2(x)-2F(x)x\right)\right)\right]\Psi_1(x)\\ &\quad+\eta\frac{4}{N}\left(F'(x)x\int\Psi_1^\dagger(y)F(y)\Psi_1(y)\,dy+F(x)\left(\int\Psi_1^\dagger(y)F'(y)y\Psi_1(y)\,dy\right)\right)\Psi_1(x)\end{aligned}$$

If $\left(F^2(x)-2F(x)x\right)<0$, then, a solution for $\eta\neq0$ may exist. Actually, for such a solution we can compute (170):

$$\begin{aligned}S(\Psi) &= \Psi^\dagger(x)\left[\left(-\nabla^2+\left(F^2(x)-2F(x)x\right)\right)\right]\Psi(x)+\frac{4}{N}\int\left(\Psi^\dagger(x)F'(x)x\Psi(x)\,dx\right)\int\Psi^\dagger(y)F(y)\Psi(y)\,dy\\ &= -\int\eta^2\frac{4}{N}\Psi_1^\dagger(x)\left(F'(x)x\int\Psi_1^\dagger(y)F(y)\Psi_1(y)\,dy+F(x)\left(\int\Psi_1^\dagger(y)F'(y)y\Psi_1(y)\,dy\right)\right)\Psi_1(x)\\ &\quad+\frac{4}{N}\int\left(\Psi^\dagger(x)F'(x)x\Psi(x)\,dx\right)\int\Psi^\dagger(y)F(y)\Psi(y)\,dy\\ &= -\frac{4\eta^2}{N}\int\Psi_1^\dagger(x)\left(F'(x)x\int\Psi_1^\dagger(y)F(y)\Psi_1(y)\,dy+F(x)\left(\int\Psi_1^\dagger(y)F'(y)y\Psi_1(y)\,dy\right)\right)\Psi_1(x)\\ &< 0\end{aligned}$$

Which is below $S(0)$. The solution of (170) may thus present a non trivial minimum, as asserted before. To prove this point, we have to show that among the set of possible solutions of (170), the action $S(\Psi)$ is bounded from below. Moreover, the second order variation of $S(\Psi)$ around the solution with the lowest value of $S(\Psi)$ has to be positive. We write this second order variation $\delta^2 S(\Psi)$. A straightforward computation yields:

$$\begin{aligned}\frac{1}{2}\delta^2 S(\Psi) &= \varphi^\dagger(x)\left[\left(-\nabla^2+\left(F^2(x)-2F(x)x\right)\right)\right]\varphi(x)+\frac{4\eta}{N}\int\left(\varphi^\dagger(x)F'(x)x\varphi(x)dx\right)\int\Psi_1^\dagger(y)F(y)\Psi_1(y)dy \\ &+\frac{4\eta}{N}\int\left(\Psi_1^\dagger(x)F'(x)x\Psi_1(x)dx\right)\int\varphi^\dagger(y)F(y)\varphi(y)dy \\ &+\frac{8\eta}{N}\operatorname{Re}\left(\int\varphi^\dagger(x)F'(x)x\Psi_1(x)dx\right)\operatorname{Re}\left(\int\varphi^\dagger(y)F(y)\Psi_1(y)dy\right)\end{aligned} \tag{171}$$

and we require that $\delta^2 S(\Psi)>0$ at the saddle point. The question of stability may be adressed if a more precise form for $F(x)$ is given, and this will be done below. However, rewriting $\delta^2 S(\Psi)$ in a more compact form will be useful in each case. This rewriting is done in Appendix 12.

To better understand the possibility of a non trivial vacuum, we will assume some particular forms for $F(x)$. The first case we will consider will be:

$$F(x)=c(x-f(x))$$

with $1<c\leqslant 2$, to allow for the possibility of a phase transition, and $f(x)$ slowly increasing with $f(0)=0$. It models a production function with some economies of scale, up to a certain level of capital x to finally reach a constant return to scale when x is large. In that case:

$$\begin{aligned}S(\Psi) &= \int\Psi^\dagger(x)\left[\left(-\nabla^2+\left(F^2(x)-2F(x)x\right)\right)\right]\Psi(x)dx+\frac{4}{N}\int\left(\Psi^\dagger(x)F'(x)x\Psi(x)dx\right)\int\Psi^\dagger(y)F(y)\Psi(y)dy \\ &\simeq \int\Psi^\dagger(x)\left[-\nabla^2+c(x-f(x))((c-2)x-cf(x))\right]\Psi(x)dx \\ &\quad+\frac{4c^2}{N}\left(\int\Psi^\dagger(x)x\Psi(x)dx\right)\left(\int\Psi^\dagger(x)(x-f(x))\Psi(x)dx\right)\end{aligned}$$

where we used that $f'(x)\simeq 0$.

We assume that the integrals are all performed on the range $x>0$, since the variable it represents, the capital stock, is positive. Moreover, we also assume that the parameters are such that our model has a non trivial solution to the saddle point equation. We choose $c=2$ to have a simple example. In that case:

$$S(\Psi)=\int\Psi^\dagger(x)\left[\left(-\nabla^2-4f(x)(x-f(x))\right)\right]\Psi(x)dx+\frac{16}{N}\left(\int\Psi^\dagger(x)x\Psi(x)dx\right)\left(\int\Psi^\dagger(x)(x-f(x))\Psi(x)dx\right) \tag{172}$$

We show in Appendix 12 that a minimum exists for a non trivial value of the field, namely: $\Psi(x)=a\Psi_1(x)$

$$\Psi_1(x)=\alpha Ai\left(\sqrt[3]{\frac{16Aa^2}{N}-4f(x)+\frac{16Ba^2}{N}}\left(x-\frac{\left(\frac{16Aa^2}{N}-4f(x)\right)f(x)}{\left(\frac{16Aa^2}{N}-4f(x)+\frac{16a^2B}{N}\right)}\right)\right)$$

where $Ai(x)$ is the Airy function, α is a normalization constant such that $\|\Psi_1(x)\|=1$, a is a factor satisfying an equation determined by the saddle point equation, and where the constants A and B are defined as:

$$\begin{aligned}\int_{\mathbb{R}^+}\Psi_1^\dagger(x)x\Psi_1(x)dx &= A \\ \int_{\mathbb{R}^+}\Psi_1^\dagger(x)(x-f(x))\Psi_1(x)dx &= B\end{aligned}$$

The second case we consider is:

$$F(x)=x+cx^2 \tag{173}$$

with $0<c<1$. The definition (173) models increasing return to scale. In that case:

$$\begin{aligned}
S(\Psi) &= \eta^2 \int \Psi^{\dagger}(x)\left[\left(-\nabla^2 + \left(F^2(x) - 2F(x)x\right)\right)\right]\Psi(x)\,dx + \frac{4\eta^4}{N}\int \left(\Psi^{\dagger}(x)F'(x)x\Psi(x)\,dx\right)\int \Psi^{\dagger}(y)F(y)\Psi(y)\,dy \\
&= \eta^2 \int \Psi^{\dagger}(x)\left[-\nabla^2 + \left(x+cx^2\right)\left(cx^2-x\right)\right]\Psi(x)\,dx \\
&\quad + \frac{4\eta^4}{N}\left(\int \Psi^{\dagger}(x)(1+2cx)x\Psi(x)\,dx\right)\left(\int \Psi^{\dagger}(x)\left(x+cx^2\right)\Psi(x)\,dx\right) \\
&\simeq \eta^2 \int \Psi^{\dagger}(x)\left[-\nabla^2 + \left(c^2x^4 - x^2\right)\right]\Psi(x)\,dx + \frac{4\eta^4}{N}\left(\int \Psi^{\dagger}(x)\left(x+2cx^2\right)\Psi(x)\,dx\right)\left(\int \Psi^{\dagger}(x)\left(x+cx^2\right)\Psi(x)\,dx\right)
\end{aligned}$$

where $\Psi(x)$ is normalized to 1 and η is a parameter for the norm. The saddle point equation is:

$$\left[-\nabla^2 + c^2x^4 + \left(\frac{4c\eta^2}{N}(A+2B) - 1\right)x^2 + \frac{4x\eta^2}{N}(A+B)\right]\Psi(x) = 0$$

with:

$$\begin{aligned}
A &= \left(\int_{\mathbb{R}^+} \Psi^{\dagger}(x)\left(x+2cx^2\right)\Psi(x)\,dx\right) \\
B &= \left(\int_{\mathbb{R}^+} \Psi^{\dagger}(x)\left(x+cx^2\right)\Psi(x)\,dx\right)
\end{aligned}$$

We show in Appendix 12 that the action $S(\Psi)$ is bounded from below and that it has a minimum obtained as a first order correction in c ot the function :

$$\Psi_0(x) = \eta \exp\left(-\frac{\sqrt{\frac{4c}{N}\eta^2(A+2B)-1}}{2}\left(x + \frac{2}{N}\frac{\eta^2(A+B)}{\frac{4c}{N}\eta^2(A+2B)-1}\right)^2\right)$$

This modified eigenvector $\Psi_0'(x)$ is expressed as a series of c:

$$\begin{aligned}
\left|\Psi_0'(y)\right\rangle &= \left|\Psi_0'(y)\right\rangle - c\sum_{l=1}^{\infty}\frac{\langle\Psi_l(x)|\,x^4\,|\Psi_0(y)\rangle}{2l}\left|\Psi_l(y)\right\rangle \\
&\quad + c^2\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{\langle\Psi_l(x)|\,x^4\,|\Psi_m(y)\rangle\langle\Psi_m(x)|\,x^4\,|\Psi_0(y)\rangle}{4lm}\left|\Psi_l(y)\right\rangle \\
&\quad - c^2\sum_{l=1}^{\infty}\frac{\langle\Psi_0(x)|\,x^4\,|\Psi_0(y)\rangle\langle\Psi_l(x)|\,x^4\,|\Psi_0(y)\rangle}{4l^2}\left|\Psi_l(y)\right\rangle \\
&\quad - \frac{c^2}{2}\sum_{l=1}^{\infty}\frac{\langle\Psi_0(x)|\,x^4\,|\Psi_l(y)\rangle\langle\Psi_l(x)|\,x^4\,|\Psi_0(y)\rangle}{4l^2}\left|\Psi_0(y)\right\rangle
\end{aligned}$$

To conclude, let us stress that we could also introduce interactions between different agents via technology. We neglect the interest rate and the interaction related to it, and suppose $K_i(s)$ is enhanced by a technological factor depending on the accumulated capital, with:

$$F_i(K_i(t)) = \sqrt{G\left(\frac{1}{N}\sum_j \int_0^t K_j(s)\,ds\right)F_i^{(0)}}\,(K_i(t))$$

If we expand $G\left(\frac{1}{N}\sum_j \int_0^t K_j(s)\,ds\right)$ in series:

$$G\left(\frac{1}{N}\sum_j \int_0^t K_j(s)\,ds\right) = \sum_n g_n\left(\frac{1}{N}\sum_j \int_0^t K_j(s)\,ds\right)^n$$

then, given that to our order of approximation:

$$\int \left(\left(\dot{K}_i(t) \right)^2 + \left(F_i(K_i(t)) \right)^2 \right) dt + 2 \int_{t>s} F_i(K_i(s)) F_i(K_i(t)) dt$$

can be replaced with:

$$\int \left(\left(\dot{K}_i(t) \right)^2 + \left(F_i(K_i(t)) \right)^2 \right) dt + 2 \int_{t>s} \frac{F_i^2(K_i(t))}{\sigma^2} dt$$

As a consequence, the technological factor becomes

$$\begin{aligned}
& G\left(\frac{1}{N} \sum_j \int_0^t K_j(s)\, ds \right) \frac{F_i^2(K_i(t))}{\sigma^2} \\
= & \sum_n g_n \left(\frac{1}{N} \sum_j \int_0^t K_j(s)\, ds \right)^n \\
= & \sum_n g_n \sum_i \int dt \left(\frac{1}{N} \sum_j \int_0^t K_j(s)\, ds \right)^n \frac{F_i^2(K_i(t))}{\sigma^2} \\
= & \left(\frac{1}{N} \right)^n \int \left(\sum_{j_1,..j_n} \int_0^t ... \int_0^t K_{j_1}(s_1) K_{j_2}(s_2) ... K_{j_n}(s_n)\, ds_1 ... ds_n \right) \frac{F_i^2(K_i(t))}{\sigma^2} dt \\
= & \sum_n g_n \frac{1}{\sigma^2 (n+1)} \left(\frac{1}{N} \right)^n \int ... \int \sum_{j_1,..j_{n+1}} K_{j_1}(s_1) K_{j_2}(s_2) ... K_{j_n}(s_n) F_{j_{n+1}}^2 \left(K_{j_{n+1}} \left(s_{j_{n+1}} \right) \right) ds_1 ... ds_n ds_{n+1}
\end{aligned}$$

whose field theoretic equivalent is:

$$\begin{aligned}
& \sum_n g_n \frac{1}{\sigma^2 (n+1)} \left(\frac{1}{N} \right)^n \left(\int \Psi^\dagger(x)\, x \Psi(x)\, dx \right)^n \left(\int \Psi^\dagger(x)\, x^2 \Psi(x)\, dx \right) \\
= & \frac{1}{\sigma^2} \left(\int \Psi^\dagger(x)\, x^2 \Psi(x)\, dx \right) \hat{G} \left(\int \Psi^\dagger(x)\, x \Psi(x)\, dx \right)
\end{aligned}$$

for $\hat{G}' = G$. The effective field action is thus:

$$-\Psi^\dagger(x) \left[\left(-\nabla^2 + \left(2 + \frac{\alpha}{\sigma^2} \right) x^2 \hat{G} \left(\int \Psi^\dagger(x)\, x \Psi(x)\, dx \right) \right) \delta(x-y) \right] \Psi(y)$$

The quartic term

$$-\left(2 + \frac{\alpha}{\sigma^2} \right) \int \Psi^\dagger(x) \left[x^2 \hat{G} \delta(x-y) \right] \Psi(y) \left(\int \Psi^\dagger(x)\, x \Psi(x)\, dx \right)$$

models an interaction term resulting from the tchnological factor, as announced. We will not pursue this trail here.

10 Interactions between Fundamental Structures and Phase Transitions. Non trivial Vaccua and integrations of structures

10.1 Interaction between similar Fundamental Structures

In the two previous examples, no phase transition appeared. The constraint implied a single vacuum for any parameter of the system.

In our context, multiple vacua may arise only if the fields considered are defined on a space of at least two dimensions without constraint, that is when agents' actions are multicomponent. Actually, in that case, we saw that the effective global utility functions (see (139), (140) (121)) have the form:

$$\begin{aligned} U_{eff}(X) &= -\frac{1}{2}\dot{X}(t)\left(M^{(S)}+N\right)\dot{X}(t)-\dot{X}(t)M^{(A)}\left(X(t)-\left(\tilde{X}\right)\right) \\ &\quad -\left(X(t)-\left(\tilde{X}\right)\right)\left(N-M^{(S)}\right)\left(X(t)-\left(\tilde{X}\right)\right)+V_{eff}(X(t)) \end{aligned} \tag{174}$$

Recall that in the second section, we noticed that a constant term has to be added to this effective utility. It was discarded when looking at the dynamics of a single system. However, now that we consider a large number of such systems, this constant has to reintroduced. Actually, recall that our model considers interacting copies of the same system, each system interacting over a variable time span s, previously denoted T. For such systems, we sum over the possible time spans through a Laplace transform. In that context, adding a term $sU_{eff}\left(\bar{X}^e\right)$ in $U_{eff}(X)$ leads to shift α by $U_{eff}\left(\bar{X}^e\right)$ after Laplace transform.

Recall (119) that the Laplace transformed Green function becomes, without the potential V_{eff}:

$$\begin{aligned} G^{-1}(x,x_1) &= \left(-\frac{1}{2}\nabla\left(\left(M^{(S)}+N\right)^{-1}\nabla+M^{(A)}\left(x-\left(\tilde{X}\right)\right)\right)\right. \\ &\quad \left.+\left(x-\left(\tilde{X}\right)\right)\left(N-M^{(S)}+M^{(A)}\left(M^{(S)}+N\right)^{-1}M^{(A)}\right)\left(x-\left(\tilde{X}\right)\right)+\alpha\right)\delta(x-x_1) \end{aligned}$$

Then, adding the $U_{eff}\left(\bar{X}^e\right)$ term and letting $y=\left(x-\left(\tilde{X}\right)\right)$ leads to the field action:

$$\begin{aligned} S(\Psi) &= \frac{1}{2}\int\Psi^{\dagger}(y)\left(-\frac{1}{2}\nabla\left(M^{(S)}+N\right)^{-1}\nabla+yM^{(A)}\nabla+m^2\right. \\ &\quad \left.+y\left(N-M^{(S)}+M^{(A)}\left(M^{(S)}+N\right)^{-1}M^{(A)}\right)y+V(y)\right)\Psi(y)\,dy \\ &\quad +\int\sum_{k=2}^{A}V(x_1,...,x_k)\Psi(x_1)\Psi^{\dagger}(x_1)...\Psi(x_k)\Psi^{\dagger}(x_k)\,dx_1...dx_k \end{aligned} \tag{175}$$

where $V(x_1,...,x_k)$ is any interaction potential between the various agents, and where we set:

$$m^2=\left(\alpha+U_{eff}\left(\bar{X}^e\right)\right) \tag{176}$$

As said in the second section, $U_{eff}\left(\bar{X}^e\right)$ can be negative. It is a direct consequence of costly, in utility terms, tensions between the components of the considered structure. Then, depending on the parameters of the system, m^2 can be positive or negative. We nevertheless keep the notation m^2 by reference to the usual mass term in field theory.

The possibility of a non trivial minimum for S arises from two possible mechanisms. To describe this two possibilities we first assume $V(y)=0$, in order to focus on the effect of the interaction term $V(x_1,...,x_k)$. The first part in $S(\Psi)$:

$$\int\frac{1}{2}\left(\Psi^{\dagger}(y)\left(-\frac{1}{2}\nabla\left(M^{(S)}+N\right)^{-1}\nabla+yM^{(A)}\nabla+y\left(N-M^{(S)}+M^{(A)}\left(M^{(S)}+N\right)^{-1}M^{(A)}\right)y+m^2\right)\Psi(y)\right)dy$$

will be expressed in a diagonal form. Consider the concatenated vector $(x,p)^t$ with $p\equiv\nabla$ the "momentum" we can rewrite:

$$\begin{aligned} &\left(-\frac{1}{2}\nabla\left(M^{(S)}+N\right)^{-1}\nabla+yM^{(A)}\nabla+y\left(N-M^{(S)}+M^{(A)}M^{(S)}M^{(A)}\right)y+m^2\right) \\ &= (p,x)^t\left(\left(\begin{array}{cc} M^{(S)}+N & -M^{(A)} \\ -\left(M^{(A)}\right)^t & N-M^{(S)}+M^{(A)}\left(M^{(S)}+N\right)^{-1}M^{(A)} \end{array}\right)\right)\left(\begin{array}{c} p \\ x \end{array}\right)+m^2 \end{aligned}$$

Now, given that we can decompose the matrix

$$\begin{pmatrix} M^{(S)}+N & -M^{(A)} \\ -\left(M^{(A)}\right)^t & \left(N-M^{(S)}+M^{(A)}\left(M^{(S)}+N\right)^{-1}M^{(A)}\right) \end{pmatrix}$$

as:

$$\begin{aligned} & \begin{pmatrix} M^{(S)}+N & -M^{(A)} \\ -\left(M^{(A)}\right)^t & \left(N-M^{(S)}+M^{(A)}\left(M^{(S)}+N\right)^{-1}M^{(A)}\right) \end{pmatrix} \\ = & \begin{pmatrix} 1 & 0 \\ -\left(M^{(A)}\right)^t\left(M^{(S)}+N\right)^{-1} & 1 \end{pmatrix} \\ & \times \begin{pmatrix} M^{(S)}+N & 0 \\ 0 & \left(N-M^{(S)}+M^{(A)}\left(M^{(S)}+N\right)^{-1}M^{(A)}\right)-\left(M^{(A)}\right)^t\left(M^{(S)}+N\right)^{-1}M^{(A)} \end{pmatrix} \\ & \times \begin{pmatrix} 1 & -\left(M^{(S)}+N\right)^{-1}\left(M^{(A)}\right) \\ 0 & 1 \end{pmatrix} \\ = & \begin{pmatrix} 1 & 0 \\ -\left(M^{(A)}\right)^t\left(M^{(S)}+N\right)^{-1} & 1 \end{pmatrix}\begin{pmatrix} M^{(S)}+N & 0 \\ 0 & N-M^{(S)}+2M^{(A)}\left(M^{(S)}+N\right)^{-1}M^{(A)} \end{pmatrix} \\ & \times \begin{pmatrix} 1 & -\left(M^{(S)}+N\right)^{-1}\left(M^{(A)}\right) \\ 0 & 1 \end{pmatrix} \end{aligned}$$

we define the change of variable

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} 1 & -\left(M^{(S)}+N\right)^{-1}\left(M^{(A)}\right) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}$$

which satisfies $[x',p'] = [x',p'] = -1$. We can thus rewrite the differential operator K as:

$$\begin{aligned} K &= \left(-\frac{1}{2}\nabla\left(M^{(S)}+N\right)^{-1}\nabla + yM^{(A)}\nabla + y\left(N-M^{(S)}+M^{(A)}\left(M^{(S)}+N\right)^{-1}M^{(A)}\right)y+m^2\right) \\ &= \left(x',p'\right)^t\begin{pmatrix} M^{(S)}+N & 0 \\ 0 & N-M^{(S)}+2M^{(A)}\left(M^{(S)}+N\right)^{-1}M^{(A)} \end{pmatrix}\begin{pmatrix} x' \\ p' \end{pmatrix}+m^2 \end{aligned}$$

which describes a set of coupled oscillators. A second change of variables allows to diagonalize $M^{(S)}+N = ODO^t$ and to obtain K in a standard form. We let:

$$\begin{aligned} \begin{pmatrix} x_1 \\ p_1 \end{pmatrix} &= \begin{pmatrix} O\sqrt{D}O^t & 0 \\ 0 & O\left(\sqrt{D}\right)^{-1}O^t \end{pmatrix}\begin{pmatrix} x' \\ p' \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{M^{(S)}+N} & 0 \\ 0 & \left(\sqrt{M^{(S)}+N}\right)^{-1} \end{pmatrix}\begin{pmatrix} 1 & -\left(M^{(S)}+N\right)^{-1}\left(M^{(A)}\right) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix} \end{aligned}$$

This change of variable preserves the commutation relations between x and p and leads to the following expression for K :

$$K = \left(x_1,p_1\right)^t\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{M^{(S)}+N}\left(N-M^{(S)}+2M^{(A)}\left(M^{(S)}+N\right)^{-1}M^{(A)}\right)\sqrt{M^{(S)}+N} \end{pmatrix}\begin{pmatrix} x_1 \\ p_1 \end{pmatrix}+m^2 \tag{177}$$

Thus, K may present some states with $\Psi(y) \neq 0$ and $S(\Psi) < 0$ in three cases. First, if $M^{(S)}+N$ has some negative eigenvalues, second if

$$N-M^{(S)}+2M^{(A)}\left(M^{(S)}+N\right)^{-1}M^{(A)}$$

presents some negative eigenvalues, or ultimately, if the term m^2, which represents the internal tension between components of a fundamental structures is negative and large enough to lower the minimum of $S(\Psi)$ to some negative value.

The two first possibilities are similar, and differ from the third one. We will focus on this last possibility. Actually, the two first possiblities represent an unstable system that will quickly break out, and thus no stability can be achieved. The third possibility rather describes a milder instability with a certain persistance in the dynamic system.

However, this latter kind of instability may be turned into a stable minimum, through a mechanism of interaction between similar structures.

Consider for example, that we add to K an interaction potential modelling the simplest form of long term interactions between two fundamental structures:

$$V(y_1, y_2) = U(y_1) U(y_2)$$

where $U(y) > 0$ and such that the minimum for U is reached at $y = 0$. We assume that $m^2 < 0$ and that K has a finite number of negative eigenvalues, which means that the first eigenvalues of the harmonic oscillator are lowered to a negative value by m^2.

We also assume that the matrix elements of $U(y_1)$ between the eigenfunctions of K are positives. This is often the case for standard examples, if we choose $U(y_1) = (y_1)^t C(y_1)$ with C definite and positive. Actually, up to the perturbation term $yM^{(A)}\nabla$, K is of harmonic oscillator type. For such operators, the matrices elements of $(y_1)^t C(y_1)$ are positive.

Given the sign of $U(y_1)$, it models an attractive force between two types of similar structures (note in passing the analogy with neural activity, where neurons, firing together, tend to bind together). The saddle point equation including this potential is then:

$$0 = K\Psi(y) + 2U(y)\Psi(y)\int \left(\Psi(y_2) U(y_2) \Psi^{\dagger}(y_2)\right) dy_2$$

We show in Appendix 13 that for a potential of large enough magnitude and peaked around the minimum of K, the saddle point presents a non trivial solution which is a minimum: $\Psi(x) = \sqrt{\eta}\Psi_1(x)$ where $\Psi_1(x)$ has norm 1 and satisfies:

$$\Psi_1(y) = \frac{\langle\Psi_1| K |\Psi_1\rangle}{\langle\Psi_1| U |\Psi_1\rangle} K^{-1} U(y_1) \Psi_1(y)$$

The vector $|\Psi_1\rangle$ is a combination of the eigenvectors of K with negative eigenvalues, so that $\langle\Psi_1| K |\Psi_1\rangle < 0$. Moreover the norm of $\Psi(x)$ is:

$$\eta = -\frac{1}{2}\frac{\langle\Psi_1| K |\Psi_1\rangle}{\left(\langle\Psi_1| U |\Psi_1\rangle\right)^2} > 0$$

Appendix 13 shows also that the same results hold if internal tensions are modelled by a more general potential $V(y)$ than a simple shift $m^2 < 0$. It is sufficient that the potential $V(y)$ has a negative minimum of large enough magnitude.

10.1.1 Example, the three agent model

In Appendix 6, we show that the effective action for the three agents model is given by:

$$\begin{aligned} U_{eff}(X(t)) &= \left(X(t) - \bar{X}\right)\left(N - M^S\right)\left(X(t) - \bar{X}\right) \\ &\quad + \frac{1}{2}\left(X(t) - X(t-1)\right)\left(N + M^S\right)\left(X(t) - X(t-1)\right) - \left(X(t) - \bar{X}\right) M^A \left(X(t-1) - \bar{X}\right) \\ &\quad + \left(\bar{X} - \bar{X}^e\right)\left(N - (M - M')^S\right)\left(\bar{X} - \bar{X}^e\right) - \left(\bar{X} - \bar{X}_2^{(2)}\right) M'\bar{X} \end{aligned} \tag{178}$$

where the matrices and vectors involved are defined in section 1 and Appendix 6. The vector $\bar{X}$ is computed in Appendix 6, and represents the equilibrium value reached by the three agents' system. The vectors $\bar{X}_2^{(2)}$

and $\bar{X}^e$ represent the goals, i.e. the desired values for $\bar{X}$, for agents 2 and 1 respectively. Due to these competing objectives, the equilibrium $\bar{X}$ is a combination of these two vectors. Appendix 6 shows that:

$$\bar{X} = \bar{X}^e + \left(N - M^S\right)^{-1} \left(\left(N + M^S\right) \bar{X}^e - \frac{1}{2} \left(M'\right)^t \bar{X}_2^{(2)}\right) \tag{179}$$

The term in bracket in (178):

$$U_{eff}\left(\bar{X}\right) = \left\{\left(\bar{X} - \bar{X}^e\right) \left(N + M^S - \left(M - M'\right)^S\right) \left(\bar{X} - \bar{X}^e\right) - \left(\bar{X} - \bar{X}_2^{(2)}\right) M' \bar{X}\right\}$$

represents the loss in utility due to the competing goals between the different elements of the structure. Even if, globally, it is optimal to stabilize around $\bar{X}$, each sub-component experiences a loss from the difference between $\bar{X}$ and it's own goal. As a consequence, at least for some values of the parameter, this term is negative. Actually, assume that, due to its strategic advantage and the magnitude of the stress it can impose to its subcomponents, the third agent is able to drive $\bar{X}$ close to $\bar{X}^e$. Then:

$$U_{eff}\left(\bar{X}\right) \simeq -\left(\bar{X} - \bar{X}_2^{(2)}\right) M' \bar{X} \tag{180}$$

and given the definition of M', this last term measures the loss experienced by the second agent when $\bar{X}$, i.e. the equilibrium value of $X(t)$ is away from $\bar{X}_2^{(2)}$, thus $U_{eff}\left(\bar{X}\right) < 0$. Then, the term (180) induces an instability in the system by lowering the lowest eigenvalue of the Green function. To get more insight about this phenomenon, we computed the matrices involved in $U_{eff}(X(t))$ for $\beta \to 0$:

$$\begin{aligned} U_{eff}(X(t)) \quad = \quad & \left(X(t) - \bar{X}\right) \left(I - M^S\right) \left(X(t) - \bar{X}\right) \\ & + \frac{1}{2} \left(X(t) - X(t-1)\right) \left(I + M^S\right) \left(X(t) - X(t-1)\right) - \left(X(t) - \bar{X}\right) M^A \left(X(t-1) - \bar{X}\right) \\ & + \left((1 - M)^{-1} X_1\right)^t \left(M^t M - 2M\right) \left((1 - M)^{-1} X_1\right) \end{aligned} \tag{181}$$

Appendix 6 shows that the operator appearing in (177), except the mass term:

$$\tilde{K} = (x_1, p_1)^t \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{M^{(S)}} \left(N - M^{(S)} + 2M^{(A)} \left(M^{(S)} + N\right)^{-1} M^{(A)}\right) \sqrt{M^{(S)}} \end{pmatrix} \begin{pmatrix} x_1 \\ p_1 \end{pmatrix}$$

has positive eigenvalues for a range of parameters of relatively small magnitude, so that the stability is preserved. We also show that, as previously said, competing objectives between the components of the system imply the possibility of a constant term

$$\left((1 - M)^{-1} X_1\right)^t \left(M^t M - 2M\right) \left((1 - M)^{-1} X_1\right)$$

of negative sign. The stability may also be impaired by any internal negative potential in the direction of the lowest eigenvalue of $\tilde{K}$. As one could expect, this direction corresponds to a state of maximal strain imposed by agent 3 to agent 2. These states may be more easily turned into an unstable one than others by some perturbation.

However, as explained in the previous paragraph, any positive interaction potential between different structures, and pointing in the direction of instability may restore the stability to produce some composed states. Thus, this is the relative instability of such states that makes possible, in an indirect manner, the aggregation into integrated structures with more degreees of freedom.

10.2 Interaction between different types of fundamental structures

10.2.1 Non trivial saddle point, effective action and integrated structure

The whole procedure of the previous paragraph can be generalized when different types of structures interact. Having chosen a system of coordinates such that the field action ultimately takes the form:

$$\begin{aligned} S(\Psi) &= \int \frac{1}{2}\left(\Psi_k^{\dagger}(y)\left(-\frac{1}{2}(\nabla_i)^2 - yM_k^{(A)}\nabla_i + yD_k y + V(y_k)\right)\Psi_k(y)\right)dy \\ &\quad + \int \sum_{k=2}^{A} V(x_1,...,x_k)\,\Psi_{i_1}(x_1)\,\Psi_{i_1}^{\dagger}(x_1)\,...\Psi_{i_k}(x_k)\,\Psi_{i_k}^{\dagger}(x_k)\,dx_1...dx_k \end{aligned} \tag{182}$$

In (182), operators of the form

$$K_i = \left(-\frac{1}{2}(\nabla_i)^2 - yM_k^{(A)}\nabla_i + yD_k y + V(y_k)\right)$$

appear. If some of them have negative eigenvalues due to a negative minimum of $V(y_k)$, and if the interaction potentials $V(x_1,...,x_k)$ are positive, then the saddle point equations:

$$\begin{aligned} 0 &= \left(-\frac{1}{2}(\nabla_i)^2 - yM_l^{(A)}\nabla_i + yD_l y + V_l(y)\right)\Psi_l(y) \\ &\quad + \left(\frac{\partial}{\partial \Psi_l^{\dagger}(y)} \int \sum_{k=2}^{A} V(x_1,...,x_k)\,\Psi_{i_1}(x_1)\,\Psi_{i_1}^{\dagger}(x_1)\,...\Psi_{i_k}(x_k)\,\Psi_{i_k}^{\dagger}(x_k)\,dx_1...dx_k\right)\Psi_l(y) \end{aligned} \tag{183}$$

may have non trivial minima. This possibility is studied in Appendix 13. We show that for a potential $V(x_1,...,x_k)$ oriented towards the lowest eigenstates of the operators K_i, the whole system has a non trivial minimum with $S(\Psi) < 0$. This minimum is a composed state made of the lowest eigenstates of the K_i along their directions of instability. In the rest of the paragraph we will detail this statement and it's implications, in particular the form of the composed state and it's interpretation in terms of integrated structure.

To do so, we need to precise some notations. In the sequel we will write $\Psi_i^{(0)}(x_i)$ for the lowest eigenstates of the operators K_i and $\Psi_i^{(n_i)}(x_i)$ for the other eigenstates of the K_i. We can write a composed states in the following way: Assume that the potential connects p_1 copies of structure 1, p_2 copies of structure 2 and so on until p_r copies of structure r. Thus, we can write the potential $V\left((x_1)_{p_1},...,(x_k)_{p_r}\right)$ with $p_1 + ... + p_r = k$ where $(x_i)_{p_i}$ represents p_i independent copies of x_i. In other words, $(x_i)_{p_i}$ is a coordinate system for $F_i \times ... \times F_i$ with F_i the manifold of states for structure i. Given these notations, a composed state for the various structures writes as a sum of eigenstates:

$$\sum_{(n_1)_{p_1},(n_2)_{p_2},...,(n_r)_{p_r}} a_{(n_1)_{p_1},(n_2)_{p_2},...,(n_r)_{p_r}} \left[\Psi_1^{(n_1)}(x_1)\right]_{p_1} ... \left[\Psi_l^{(n_l)}(x_l)\right]_{p_l} ... \left[\Psi_r^{(n_r)}(x_r)\right]_{p_r}$$

where $\left[\Psi_i^{(n_i)}(x_i)\right]_{p_i}$ is a product of p_i copies of eigenstates for structure i:

$$\left[\Psi_i^{(n_i)}(x_i)\right]_{p_i} = \Psi_i^{(n_i)_1}((x_i)_1)\,\Psi_i^{(n_i)_2}((x_i)_2)\,...\Psi_i^{(n_i)_{p_i}}\left((x_i)_{p_i}\right)$$

The interaction involves then p_i copies of the i-th structure. We will also denote, as a shortcut for identical copies of the lowest eigenstate:

$$\left[\Psi_i^{(0)}(x_1)\right]_{p_i} = \Psi_i^{(0)}((x_i)_1)\,\Psi_i^{(0)}((x_i)_2)\,...\Psi_i^{(0)}\left((x_i)_{p_i}\right)$$

To precise the condition on the potential that allows for a non trivial saddle point, we write the potential $V\left(x_1\right)_{p_1},...,\left(x_r\right)_{p_r}$ as a kernel in an operator formalism, whose form in the eigenstate basis is:

$$\begin{aligned}
&V\left(\left(x_1\right)_{p_1},...,\left(x_r\right)_{p_r},\left(y_1\right)_{p_1},...,\left(y_r\right)_{p_r}\right) \quad (184)\\
=&\sum_{\left(n_1\right)_{p_1},\left(n_2\right)_{p_2},...,\left(n_r\right)_{p_r}} V_{\left(n_1\right)_{p_1},\left(n_2\right)_{p_2},...,\left(n_r\right)_{p_r}}\left[\Psi_1^{(n_1)}\left(x_1\right)\right]_{p_1}\left[\Psi_1^{(n_1)\dagger}\left(y_1\right)\right]_{p_1}...\left[\Psi_l^{(n_l)}\left(x_l\right)\right]_{p_l}\left[\Psi_l^{(n_l)\dagger}\left(y_l\right)\right]_{p_l}\\
&\times...\left[\Psi_r^{(n_r)}\left(x_r\right)\right]_{p_r}\left[\Psi_r^{(n_r)\dagger}\left(y_r\right)\right]_{p_r}
\end{aligned}$$

where the coefficients $V_{\left(n_1\right)_{p_1},\left(n_2\right)_{p_2},...,\left(n_r\right)_{p_r}}$ are defined by:

$$\begin{aligned}
V_{\left(n_1\right)_{p_1},\left(n_2\right)_{p_2},...,\left(n_r\right)_{p_r}} =& \int\left[\Psi_1^{(n_1)\dagger}\left(x_1\right)\right]_{p_1}...\left[\Psi_l^{(n_l)\dagger}\left(x_l\right)\right]_{p_l}...\left[\Psi_l^{(n_l)\dagger}\left(x_l\right)\right]_{p_l}V\left(\left(x_1\right)_{p_1},...,\left(x_r\right)_{p_r}\right) \quad (185)\\
&\times\left[\Psi_1^{(n_1)}\left(x_1\right)\right]_{p_1}...\left[\Psi_l^{(n_l)}\left(x_l\right)\right]_{p_l}...\left[\Psi_r^{(n_r)}\left(x_r\right)\right]_{p_r}\left[\Psi_l^{(n_l)\dagger}\left(x_l\right)\right]_{p_l}
\end{aligned}$$

Our hypothesis is that the potential localizes around the ground states of each structure. This translates in:

$$V_{\left(n_1\right)_{p_1},\left(n_2\right)_{p_2},...,\left(n_r\right)_{p_r}} << V_{(0)_{p_1},(0)_{p_2},...,(0)_{p_r}} \text{ if some } \left(n_i\right)_{p_i}\neq(0)_{p_i} \quad (186)$$

where $(0)_{p_i}$ denote multi-indices with all their components set to zero. Actually, this condition means that in 184, the terms proportional to the tensor products of ground states:

$$\left[\Psi_1^{(0)}\left(x_1\right)\right]_{p_1}\left[\Psi_1^{(0)\dagger}\left(y_1\right)\right]_{p_1}...\left[\Psi_l^{(0)}\left(x_l\right)\right]_{p_l}\left[\Psi_l^{(0)\dagger}\left(y_l\right)\right]_{p_l}\times...\left[\Psi_r^{(0)}\left(x_r\right)\right]_{p_r}\left[\Psi_r^{(0)\dagger}\left(y_r\right)\right]_{p_r}$$

dominate, as required. As a consequence of the assumption 184, the matrix elements of 185 rewrite:

$$\begin{aligned}
&V\left(\left(x_1\right)_{p_1},...,\left(x_r\right)_{p_r},\left(y_1\right)_{p_1},...,\left(y_r\right)_{p_r}\right)\\
=&V_0\left[\Psi_1^{(0)}\left(x_1\right)\right]_{p_1}\left[\Psi_1^{(0)\dagger}\left(y_1\right)\right]_{p_1}...\left[\Psi_l^{(0)}\left(x_l\right)\right]_{p_l}\left[\Psi_l^{(0)\dagger}\left(y_l\right)\right]_{p_l}...\left[\Psi_r^{(0)}\left(x_r\right)\right]_{p_r}\left[\Psi_r^{(0)\dagger}\left(y_r\right)\right]_{p_r}\\
+&\sum_{\substack{\left(n_1\right)_{p_1},\left(n_2\right)_{p_2},...,\left(n_r\right)_{p_r},\\ \text{not all }\left(n_i\right)_{p_i}\text{ are nul}}} V_{\left(n_1\right)_{p_1},\left(n_2\right)_{p_2},...,\left(n_r\right)_{p_r}}\left[\Psi_1^{(n_1)}\left(x_1\right)\right]_{p_1}\left[\Psi_1^{(n_1)\dagger}\left(y_1\right)\right]_{p_1}...\left[\Psi_l^{(n_l)}\left(x_l\right)\right]_{p_l}\left[\Psi_l^{(n_l)\dagger}\left(y_l\right)\right]_{p_l}\\
&\times...\left[\Psi_r^{(n_r)}\left(x_r\right)\right]_{p_r}\left[\Psi_r^{(n_r)\dagger}\left(y_r\right)\right]_{p_r}\\
\text{with:}\quad&V_0\equiv V_{(0)_{p_1},(0)_{p_2},...,(0)_{p_r}} >> V_{\left(n_1\right)_{p_1},\left(n_2\right)_{p_2},...,\left(n_r\right)_{p_r}}
\end{aligned}$$

Appendix 13 shows that under some conditions on V_0, a non trivial saddle point exists and satisfies:

$$\begin{aligned}
0 =& K_l\Psi_l\left(y_l\right)+\left(\frac{\partial}{\partial\Psi_l^{\dagger}\left(y\right)}\int V\left(\left(x_1\right)_{p_1},...,\left(x_k\right)_{p_k}\right)\left[\Psi_1\left(x_1\right)\Psi_1^{\dagger}\left(x_1\right)\right]_{p_1}...\right.\\
&\left.\left[\Psi_l\left(x_l\right)\Psi_l^{\dagger}\left(x_l\right)\right]_{p_l}...\left[\Psi_k\left(x_k\right)\Psi_k^{\dagger}\left(x_k\right)\right]_{p_k}d\left(x_1\right)_{p_1}...d\left(x_k\right)_{p_k}\right)\\
=& K_l\Psi_l\left(y_l\right)+p_l\left(\int V\left(\left(x_1\right)_{p_1},...,\left(x_k\right)_{p_k}\right)\left[\Psi_1\left(x_1\right)\Psi_1^{\dagger}\left(x_1\right)\right]_{p_1}...\right.\\
&\left.\left[\Psi_l\left(x_l\right)\Psi_l^{\dagger}\left(x_l\right)\right]_{p_l-1}...\left[\Psi_k\left(x_k\right)\Psi_k^{\dagger}\left(x_k\right)\right]_{p_k}d\left(x_1\right)_{p_1}...d\left(x_k\right)_{p_k}\right)\Psi_l\left(y\right) \quad (187)
\end{aligned}$$

Considering the correspondence between the micro and the collective interpretation of the system, we can wonder about the implications of this non trivial saddle point at the individual level of effective utilities.

To do so we consider the second order fluctuations of the field action around the saddle point. Coming back to the system described by (182):

$$\begin{aligned} S(\Psi) &= \int \frac{1}{2}\left(\Psi_k^{\dagger}(y)\left(-\frac{1}{2}(\nabla_i)^2 - yM_k^{(A)}\nabla_i + yD_k y + V(y_k)\right)\Psi_k(y)\right)dy \\ &+ \int \sum_{k=2}^{A} V(x_1,...,x_k)\,\Psi_{i_1}(x_1)\,\Psi_{i_1}^{\dagger}(x_1)...\Psi_{i_k}(x_k)\,\Psi_{i_k}^{\dagger}(x_k)\,dx_1...dx_k \end{aligned}$$

we can describe these fluctuations around the minimum by decomposing:

$$\Psi_l(x_l) = \hat{\Psi}_l(x_l) + \delta\Psi_l(x_l)$$

where $\hat{\Psi}_l(x_l)$ satisfies (187). Let $\left(\hat{\Psi}_l(x_l)\right)$ be the concatenated vector with components $\hat{\Psi}_l(x_l)$. The second order variation for $S(\Psi)$ is then:

$$\begin{aligned} S(\Psi) &= S\left(\left(\hat{\Psi}_l(x_l)\right)\right) + \int \sum_l \delta\Psi_l^{\dagger}(x_l)\,K_l\delta\Psi_l(x_l) \qquad (188) \\ &+ \int \sum_{l,\alpha_l,\beta_l} \delta\Psi_l^{\dagger}\left((x_l)_{\alpha_l}\right)\left\{\sum_{k=2}^{A} V\left((x_1)_{p_1},...,(x_k)_{p_r}\right)\right. \\ &\times \left[\hat{\Psi}_1(x_1)\right]_{p_1}\left[\hat{\Psi}_1^{\dagger}(x_1)\right]_{p_1}...\times\left[\hat{\Psi}_l(x_l)\right]_{p_l-1}\left[\hat{\Psi}_l^{\dagger}(x_l)\right]_{p_l-1}\times \\ &...\times\left[\hat{\Psi}_r(x_r)\right]_{p_r}\left[\hat{\Psi}_r^{\dagger}(y_r)\right]_{p_r} d(x_1)_{p_1}...d(x_l)_{p_l-2}..d(x_r)_{p_r}\left.\right\}\delta\Psi_l\left((x_l)_{\beta_l}\right)d(x_l)_{\alpha_l}\,d(x_l)_{\beta_l} \\ &+ \int \sum_{l,n,l\neq n,\alpha_n,\beta_l} \delta\Psi_n^{\dagger}\left((x_n)_{\alpha_n}\right)\left\{\sum_{k=2}^{A} V\left((x_1)_{p_1},...,(x_k)_{p_r}\right)\right. \\ &\times\left[\hat{\Psi}_1(x_1)\right]_{p_1}\left[\hat{\Psi}_1^{\dagger}(x_1)\right]_{p_1}...\times\left[\hat{\Psi}_l(x_l)\right]_{p_l-1}\left[\hat{\Psi}_l^{\dagger}(x_l)\right]_{p_l}\times...\left[\hat{\Psi}_n(x_n)\right]_{p_n}\left[\hat{\Psi}_n^{\dagger}(x_n)\right]_{p_n-1} \\ &\times\left[\hat{\Psi}_r(x_r)\right]_{p_r}\left[\hat{\Psi}_r^{\dagger}(y_r)\right]_{p_r} d(x_1)_{p_1}...d(x_l)_{p_l-1}..d(x_r)_{p_r}\,d(y_1)_{p_1}...d(y_n)_{p_n-1}...d(y_r)_{p_r}\left.\right\} \\ &\times\delta\Psi_l\left((x_l)_{\beta_l}\right)d(x_n)_{\alpha_n}\,d(x_l)_{\beta_l} \end{aligned}$$

where the indices α_l, β_l and α_n run over the copies of x_l and x_n, so that $\alpha_l = 1,...,p_l$ and the same for β_l, and $\alpha_n = 1,...,p_n$. The implications of the phase tansition at the individual level can be understood starting from the effective action (188). Each term of (188) is the field counterpart of some effective utility term.

The first contribution in (188):

$$\int \sum_l \delta\Psi_l^{\dagger}(x_l)\,K_l\delta\Psi_l(x_l)$$

is simply the action describing a non interacting set of structures - the initial structures from which the model was built. The second term in (188) can be interpreted as a potential for these individual initial structures. Actually, for a given structure l:

$$\begin{aligned} &\sum_{\alpha_l,\beta_l}\int \delta\Psi_l^{\dagger}\left((x_l)_{\alpha_l}\right)\left\{\sum_{k=2}^{A} V\left((x_1)_{p_1},...,(x_k)_{p_r}\right)\times\left[\hat{\Psi}_1(x_1)\right]_{p_1}\left[\hat{\Psi}_1^{\dagger}(x_1)\right]_{p_1}...\times\left[\hat{\Psi}_l(x_l)\right]_{p_l-1}\left[\hat{\Psi}_l^{\dagger}(x_l)\right]_{p_l-1}\right. \\ &...\times\left[\hat{\Psi}_r(x_r)\right]_{p_r}\left[\hat{\Psi}_r^{\dagger}(y_r)\right]_{p_r} d(x_1)_{p_1}...d(x_l)_{p_l-2}..d(x_r)_{p_r}\left.\right\}\delta\Psi_l\left((x_l)_{\beta_l}\right)d(x_l)_{\alpha_l}\,d(x_l)_{\beta_l} \equiv V_l \end{aligned}$$

can be integrated over the $(x_n)_{p_n}$ with $n \neq l$ to yield:

$$V_l = \sum_{\alpha_l,\beta_l}\int \delta\Psi_l^{\dagger}\left((x_l)_{\alpha_l}\right)V_l^{(e)}\left((x_l)_{\alpha_l},(x_l)_{\beta_l}\right)\delta\Psi_l\left((x_l)_{\beta_l}\right)d(x_l)_{\alpha_l}\,d(x_l)_{\beta_l}$$

with:

$$
\begin{aligned}
V_l^{(e)}\left((x_l)_{\alpha_l},(x_l)_{\beta_l}\right) &= \int\left\{\sum_{k=2}^{A} V\left((x_1)_{p_1},...,(x_k)_{p_r}\right)\times\left[\hat{\Psi}_1(x_1)\right]_{p_1}\left[\hat{\Psi}_1^{\dagger}(x_1)\right]_{p_1}...\times\left[\hat{\Psi}_l(x_l)\right]_{p_l-1}\left[\hat{\Psi}_l^{\dagger}(x_l)\right]_{p_l-1}\right.\\
&\left....\times\left[\hat{\Psi}_r(x_r)\right]_{p_r}\left[\hat{\Psi}_r^{\dagger}(y_r)\right]_{p_r} d(x_1)_{p_1}...d(x_l)_{p_l-2}..d(x_r)_{p_r}\right\}
\end{aligned}
$$

Since the two structures $(x_l)_{\alpha_l}$ and $(x_l)_{\beta_l}$ involved in this expressions are identical, the potential can be considered as symetric, and, up to a symetrization factor $p_l(p_l-1)$, reduces to:

$$
V_l=\int\delta\Psi_l^{\dagger}(x)\,V_l^{(e)}(x,y)\,\delta\Psi_l(y)\,dxdy
$$

This term has a straightforward interpretation. It represents a non local auto-interaction of structure l with itself, as the constraints studied in the previous section: interactions with other structures globally sum up and produce this overall binding on structure l. Mathematicaly, it corresponds to modifying the effective utility of structure l by adding a non local potential:

$$
\int U_{eff}(X_l(t))\,dt\rightarrow\int U_{eff}(X_l(t))\,dt+\int\hat{V}_l(X_l(t),X_l(s))\,dtds
$$

This potential $\hat{V}_l(x,y)$ differs from $V_l^{(e)}(x,y)$ because it includes some corrections depending on the characteristics of the system. Interactions involving the copies of the same structures $(x_l)_{\alpha_l}$ can nevertheless be assumed to be "quite" local, so that these copies interact at the same point. In that case:

$$
V_l=\int\delta\Psi_l^{\dagger}(x)\,V_l^{(e)}(x,x)\,\delta\Psi_l(x)\,dx \tag{189}
$$

and this local interaction corresponds, at the individual level, to replacing $\int U_{eff}(X_l(t))\,dt$ by:

$$
\int U_{eff}(X_l(t))\,dt+\int V_l^{(e)}(X_l(t),X_l(s))\,dtds
$$

Some non local corrections can be added if we approximate the non diagonal contributions of $V_l^{(e)}(x,y)$ by some additional derivative terms:

$$
\begin{aligned}
&\int\delta\Psi_l^{\dagger}(x)\,V_l^{(e)}(x,y)\,\delta\Psi_l(y)\,dxdy\\
\simeq\;&\int\delta\Psi_l^{\dagger}(x)\,V_l^{(e)}(x,x)\,\delta\Psi_l(x)\,dx+\int\nabla\delta\Psi_l^{\dagger}(x)\,W_l^{(e)}(x)\,\delta\Psi_l(x)\,dx\\
&+\int\delta\Psi_l^{\dagger}(x)\,W_l^{(e)}(x)\,\nabla\delta\Psi_l(x)\,dx\\
&+\int\nabla\delta\Psi_l^{\dagger}(x)\,Z_l^{(e)}(x)\,\nabla\delta\Psi_l(x)\,dx
\end{aligned}
$$

and those corrections add some inertial term to the effective action. They have the form:

$$
\int W_l^{(e)}(X_l(t))\,\dot{X}_l(t)\,dt+\int\dot{X}_l(t)\,Z_l^{(e)}(X_l(t))\,\dot{X}_l(t)\,dt
$$

Having interpreted the two first contributions in (188), we can turn to the third type of term:

$$
\begin{aligned}
V_{nl}\equiv\;&\int\sum_{\alpha_n,\beta_l}\delta\Psi_n^{\dagger}\left((x_n)_{\alpha_n}\right)\left\{\sum_{k=2}^{A}V\left((x_1)_{p_1},...,(x_k)_{p_r}\right)\right.\\
&\times\left[\hat{\Psi}_1(x_1)\right]_{p_1}\left[\hat{\Psi}_1^{\dagger}(x_1)\right]_{p_1}...\times\left[\hat{\Psi}_l(x_l)\right]_{p_l-1}\left[\hat{\Psi}_l^{\dagger}(x_l)\right]_{p_l}\times...\left[\hat{\Psi}_n(x_n)\right]_{p_n}\left[\hat{\Psi}_n^{\dagger}(x_n)\right]_{p_n-1}\\
&\left.\times\left[\hat{\Psi}_r(x_r)\right]_{p_r}\left[\hat{\Psi}_r^{\dagger}(y_r)\right]_{p_r}d(x_1)_{p_1}...d(x_l)_{p_l-1}..d(x_r)_{p_r}\,d(y_1)_{p_1}...d(y_n)_{p_n-1}...d(y_r)_{p_r}\right\}\\
&\times\delta\Psi_l\left((x_l)_{\beta_l}\right)d(x_n)_{\alpha_n}\,d(x_l)_{\beta_l}
\end{aligned}
$$

for $l \neq n$. As before this can be integrated over all the variables, except $(x_n)_{\alpha_n}$ and $(x_l)_{\beta_l}$ to obtain:

$$V_{nl} = \sum_{\alpha_n, \beta_l} \int \delta\Psi_n^{\dagger} \left((x_l)_{\alpha_l}\right) V_{nl}^{(e)} \left((x_n)_{\alpha_n}, (x_l)_{\beta_l}\right) \delta\Psi_l \left((x_l)_{\beta_l}\right) d(x_n)_{\alpha_n} d(x_l)_{\beta_l}$$

with:

$$\begin{aligned} V_{nl}^{(e)} \left((x_n)_{\alpha_n}, (x_l)_{\beta_l}\right) &= \int \Bigg\{ \sum_{k=2}^{A} V \left((x_1)_{p_1}, ..., (x_k)_{p_r}\right) \\ &\times \left[\hat{\Psi}_1 (x_1)\right]_{p_1} \left[\hat{\Psi}_1^{\dagger} (x_1)\right]_{p_1} ... \times \left[\hat{\Psi}_l (x_l)\right]_{p_l - 1} \left[\hat{\Psi}_l^{\dagger} (x_l)\right]_{p_l} \times ... \left[\hat{\Psi}_n (x_n)\right]_{p_n} \left[\hat{\Psi}_n^{\dagger} (x_n)\right]_{p_n - 1} \\ &... \times \left[\hat{\Psi}_r (x_r)\right]_{p_r} \left[\hat{\Psi}_r^{\dagger} (y_r)\right]_{p_r} d(x_1)_{p_1} ...d(x_l)_{p_l - 1} ..d(x_r)_{p_r} d(y_1)_{p_1} ...d(y_n)_{p_n - 1} ...d(y_r)_{p_r} \Bigg\} \end{aligned} \tag{190}$$

As for the derivation of $V_l^{(e)}$, the symetry existing between the copies $(x_n)_{\alpha_n}$ on one side, and the copies $(x_l)_{\beta_l}$ on the other side, allows to simplify, up to some symetry factor $p_l p_n$:

$$V_{nl} = \int \delta\Psi_n^{\dagger} (x) V_{nl}^{(e)} (x, y) \delta\Psi_l (y) dxdy \tag{191}$$

From the perspective of the individual initial agents, this term has no equivalent. Actually, an interaction between two different structures should involve, at the field theoretic level, a quartic contribution, i.e. having the form:

$$\int \delta\Psi_n (x) \delta\Psi_n^{\dagger} (x) \delta\Psi_l (y) \delta\Psi_l^{\dagger} (y) dxdy$$

Being of order 2, the potential (191) must be interpreted as an individual, non local, contribution to some effective utility. However, the variables involved in (191) belong to the coordinate spaces of 2 different structures, n and l. Consequently (191) can be interpreted as a utility contribution for a single integrated structure "nl", differing from "n" or "l", and absent from the initial model.

The fact that interactions should be non local describe a "non causal" dynamics for the whole set of interacting structures: in the field formulation, the set of structures acts as a global environment for the others. The existence of a non trivial minimum at the field theoretic level, i.e. the fundamental state, translates in the emergence of an integrated structure at the individual level. Its behavior breaks the causal dynamics of the initial structures as individual systems. The integrated structure emerging from the non trivial vacuum has to be understood as some "average" or typical structure, and the system of agents is in fact an assembly of such integrated structures. They interact together, through non local effective potentials. At the individual level, this leads to a non local self interaction for the representative structure of the assembly, the non locality modeling the action of the environment created by the set of structures on the representative one.

10.2.2 Effective utility for the integrated structures

To conclude let us model an effective utility for these integrated structures. Looking back to (191) leads us to consider extended coordinate systems to model an integrated structure. Define the concatenated vector:

$$X_{nl}(t) \equiv (X_n(t), X_l(t))$$

This concatenated vector models the extended, or integrated, aspect of the structure "nl". To find an effective utility $U_{eff}(X_{nl}(t))$ for $X_{nl}(t)$ and include the different contributions relative to the structure "nl", we first study its field theoretic counterpart. It gathers contributions proportional to K_l and K_n in (188) plus (189) for n and l, and (191). This results in effective action written S^{nl}:

$$S^{nl} = \int \delta\Psi_l^{\dagger} (x_l) K_l \delta\Psi_l (x_l) + \int \delta\Psi_n^{\dagger} (x_n) K_n \delta\Psi_n (x_n) + V_l + V_n + V_{nl} \tag{192}$$

Since V_{nl} mixes some structures, our aim is to find an action $\bar{S}^{nl}\left(\Psi_{nl}\left(x_l, x_n\right)\right)$ depending on an "extended" field $\Psi_{nl}\left(x_l, x_n\right)$ of the two variables x_l and x_n. This will yield the same Green functions as those computed with S^{nl}. To do so it will be sufficient to compute the four points functions and we will explain why in the sequel. Recall that the four first contributions of S^{nl}, when considered as field actions for two independent structures, yield a Green function that is a product of the two independent Green function:

$$G_l\left(x_l, y_l\right) G_n\left(x_n, y_n\right) \equiv \left(K_l+V_l\right)^{-1}\left(K_n+V_n\right)^{-1}$$

To compute the Green functions of S^{nl} we include V_{nl} in the following way: we first rewrite S^{nl} in a more convenient form:

$$S^{nl}=\left(\begin{array}{c}\delta\Psi_l\left(x_l\right)\\ \delta\Psi_n\left(x_n\right)\end{array}\right)^{\dagger}\left(\begin{array}{cc}K_l+V_l & V_{l,n}^{(e)}\\ V_{n,l} & K_n+V_n\end{array}\right)\left(\begin{array}{c}\delta\Psi_l\left(x_l\right)\\ \delta\Psi_n\left(x_n\right)\end{array}\right) \tag{193}$$

and define:

$$A=\left(\begin{array}{cc}K_l+V_l^{(e)} & V_{l,n}^{(e)}\\ V_{n,l}^{(e)} & K_n+V_n^{(e)}\end{array}\right)$$

From this set up, the computation of the four points Green function is straightforward:

$$G\left(x_l^{(2)}, x_n^{(2)}, x_l^{(1)}, x_n^{(1)}\right) \equiv \left\langle\left(\delta\Psi_l\right)^{\dagger}\left(x_l^{(2)}\right)\left(\delta\Psi_n\right)^{\dagger}\left(x_n^{(2)}\right)\delta\Psi_l\left(x_l^{(1)}\right)\delta\Psi_n\left(x_n^{(1)}\right)\right\rangle$$

For (193), the function $G\left(x_l^{(2)}, x_n^{(2)}, x_l^{(1)}, x_n^{(1)}\right)$ is given by:

$$\begin{aligned}&\left\langle\left(\delta\Psi_l\right)^{\dagger}\left(x_l^{(2)}\right)\left(\delta\Psi_n\right)^{\dagger}\left(x_n^{(2)}\right)\delta\Psi_l\left(x_l^{(1)}\right)\delta\Psi_n\left(x_n^{(1)}\right)\right\rangle\\ =\;& A^{-1}\left(x_l^{(2)}, x_l^{(1)}\right)A^{-1}\left(x_n^{(2)}, x_n^{(1)}\right)+A^{-1}\left(x_n^{(2)}, x_l^{(1)}\right)A^{-1}\left(x_l^{(2)}, x_n^{(1)}\right)\end{aligned}$$

Moreover A^{-1} can be computed to yield:

$$A^{-1}=\left(\begin{array}{cc}X & Y\\ Z & T\end{array}\right)$$

with:

$$\begin{aligned}
X &= \left(\left(K_l+V_l^{(e)}\right)-V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}V_{n,l}^{(e)}\right)^{-1}\\
Y &= -\left(\left(K_l+V_l^{(e)}\right)-V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}V_{n,l}^{(e)}\right)^{-1}V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}\\
Z &= -\left(\left(K_n+V_n^{(e)}\right)-V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}V_{l,n}^{(e)}\right)^{-1}V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}\\
T &= \left(\left(K_n+V_n^{(e)}\right)-V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}V_{l,n}^{(e)}\right)^{-1}
\end{aligned}$$

and one obtains straightforwardly:

$$\begin{aligned}
G\left(x_l^{(2)}, x_n^{(2)}, x_l^{(1)}, x_n^{(1)}\right) &\equiv \left\langle\left(\delta\Psi_l\right)^{\dagger}\left(x_l^{(2)}\right)\left(\delta\Psi_n\right)^{\dagger}\left(x_n^{(2)}\right)\delta\Psi_l\left(x_l^{(1)}\right)\delta\Psi_n\left(x_n^{(1)}\right)\right\rangle\\
&= \left(\left(K_l+V_l^{(e)}\right)-V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}V_{n,l}^{(e)}\right)^{-1}\left(x_l^{(2)}, x_l^{(1)}\right)\\
&\quad\times\left(\left(K_n+V_n^{(e)}\right)-V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}V_{l,n}^{(e)}\right)^{-1}\left(x_n^{(2)}, x_n^{(1)}\right)\\
&\quad+\left(\left(K_l+V_l^{(e)}\right)-V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}V_{n,l}^{(e)}\right)^{-1}V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}\left(x_l^{(2)}x_n^{(1)}\right)\\
&\quad\times\left(\left(K_n+V_n^{(e)}\right)-V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}V_{l,n}^{(e)}\right)^{-1}V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}\left(x_n^{(2)}, x_l^{(1)}\right)
\end{aligned}$$

From this result one can derive the following identity:

$$
\begin{aligned}
&\left(\left(K_l+V_l^{(e)}\right)-V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}V_{n,l}^{(e)}\right)_{x_l^{(2)}}\left(\left(K_n+V_n^{(e)}\right)-V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}V_{l,n}^{(e)}\right)_{x_n^{(2)}}\\
&.\left\langle\left(\delta\Psi_l\right)^{\dagger}\left(x_l^{(2)}\right)\left(\delta\Psi_n\right)^{\dagger}\left(x_n^{(2)}\right)\delta\Psi_l\left(x_l^{(1)}\right)\delta\Psi_n\left(x_n^{(1)}\right)\right\rangle\\
=\;&\delta\left(x_l^{(2)}-x_l^{(1)}\right)\delta\left(x_n^{(2)}-x_n^{(1)}\right)+V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}\left(x_l^{(2)}x_n^{(1)}\right)V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}\left(x_n^{(2)},x_l^{(1)}\right)\\
&+V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}\left(x_n^{(2)},x_l^{(1)}\right)V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}\left(x_l^{(2)}x_n^{(1)}\right)
\end{aligned}
$$

which implies that:

$$
\begin{aligned}
&\int\left\{\left(1+\left(V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}\right)\left(x_l^{(2)},x_n\right)\left(V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}\right)\left(x_n^{(2)},x_l\right)\right)^{-1}\right\} \qquad (194)\\
&\times\left\{\left(\left(K_l+V_l^{(e)}\right)-V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}V_{n,l}^{(e)}\right)\left(x_l,x_l'\right)\left(\left(K_n+V_n^{(e)}\right)-V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}V_{l,n}^{(e)}\right)\left(x_n,x_n'\right)\right\}\\
&G\left(x_l',x_n',x_l^{(1)},x_n^{(1)}\right)dx_l'dx_n'dx_ldx_n\\
=\;&\delta\left(x_l^{(2)}-x_l^{(1)}\right)\delta\left(x_n^{(2)}-x_n^{(1)}\right)
\end{aligned}
$$

which leads ultimately to the expression for the four points Green functions of S^{nl}:

$$
\begin{aligned}
&G^{-1}\left(x_l^{(2)},x_n^{(2)},x_l^{(1)},x_n^{(1)}\right)\\
=\;&\int\left\{\left(1+V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}\right)^{-1}\left(x_l^{(2)},x_n^{(2)},x_l,x_n\right)\right\}\\
&\times\left\{\left(\left(K_l+V_l^{(e)}\right)-V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}V_{n,l}^{(e)}\right)\left(x_l,x_l^{(1)}\right)\right.\\
&\left.\times\left(\left(K_n+V_n^{(e)}\right)-V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}V_{l,n}^{(e)}\right)\left(x_n,x_n^{(1)}\right)\right\}dx_ldx_n
\end{aligned}
$$

If we consider $V_{l,n}^{(e)}$ relatively of small magnitude, a first approximation is:

$$
\begin{aligned}
G^{-1}\left(x_l^{(2)},x_n^{(2)},x_l^{(1)},x_n^{(1)}\right)=\;&\left(\left(K_l+V_l^{(e)}\right)-V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}V_{n,l}^{(e)}\right)\left(x_l^{(2)},x_l^{(1)}\right) \qquad (195)\\
&\times\left(\left(K_n+V_n^{(e)}\right)-V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}V_{l,n}^{(e)}\right)\left(x_n^{(2)},x_n^{(1)}\right)\\
&-\int V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}\left(x_l^{(2)},x_n\right)V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}\left(x_n^{(2)},x_l\right)\\
&\times\left\{\left(\left(K_l+V_l^{(e)}\right)-V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}V_{n,l}^{(e)}\right)\left(x_l,x_l^{(1)}\right)\right.\\
&\left.\times\left(\left(K_n+V_n^{(e)}\right)-V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}V_{l,n}^{(e)}\right)\left(x_n,x_n^{(1)}\right)\right\}dx_ldx_n\\
\simeq\;&\left(K_l+V_l^{(e)}\right)\left(x_l^{(2)},x_l^{(1)}\right)\left(K_n+V_n^{(e)}\right)\left(x_n^{(2)},x_n^{(1)}\right)\\
&-V_{l,n}^{(e)}\left(K_n+V_n^{(e)}\right)^{-1}V_{n,l}^{(e)}\left(x_l^{(2)},x_l^{(1)}\right)\left(K_n+V_n^{(e)}\right)\left(x_n^{(2)},x_n^{(1)}\right)\\
&-\left(K_l+V_l^{(e)}\right)\left(x_l^{(2)},x_l^{(1)}\right)V_{n,l}^{(e)}\left(K_l+V_l^{(e)}\right)^{-1}V_{l,n}^{(e)}\left(x_n^{(2)},x_n^{(1)}\right)\\
&-V_{l,n}^{(e)}\left(x_l^{(2)},x_n^{(1)}\right)V_{n,l}^{(e)}\left(x_n^{(2)},x_l^{(1)}\right)
\end{aligned}
$$

Since the first term does not mix the coordinates x_l and x_n, it describes the two structures independently. The three first terms on the contrary introduce the interactions that induces the integration of the two structures into one.

Having found the four point functions for S^{nl}, let us turn to the interpretation in terms of integrated structures. In this interpretation, the four point Green function

$$G\left(x_l^{(2)}, x_n^{(2)}, x_l^{(1)}, x_n^{(1)}\right)$$

has been defined to satify (194). The form of the Dirac function in the RHS, $\delta\left(x_l^{(2)} - x_l^{(1)}\right)\delta\left(x_n^{(2)} - x_n^{(1)}\right)$ has been chosen so that $G\left(x_l^{(2)}, x_n^{(2)}, x_l^{(1)}, x_n^{(1)}\right)$ is in fact a two points transition function in the coordinate space (x_l, x_n):

$$G\left(x_l^{(2)}, x_n^{(2)}, x_l^{(1)}, x_n^{(1)}\right) \equiv G\left(\left(x_l^{(2)}, x_n^{(2)}\right), \left(x_l^{(1)}, x_n^{(1)}\right)\right)$$

This function should thus be the two points Green function for $\Psi_{nl}(x_l, x_n)$ to recover the same result of the initial effective action S^{nl}. As a consequence, we directly find the required effective action for $\Psi_{nl}(x_l, x_n)$ that computes this Green function:

$$\bar{S}^{nl}(\Psi_{nl}) = \int \Psi_{nl}^{\dagger}\left(x_l^{(2)}, x_n^{(2)}\right) G^{-1}\left(x_l^{(2)}, x_n^{(2)}, x_l^{(1)}, x_n^{(1)}\right) \Psi_{nl}\left(x_l^{(1)}, x_n^{(1)}\right) dx_l^{(2)} dx_n^{(2)} dx_l^{(1)} dx_n^{(1)} \tag{196}$$

This formula a-posteriori justifies, the need for the four point Green functions of S^{nl}. Note that, using the previous approximation (195) we can also write:

$$\begin{aligned}\bar{S}^{nl}(\Psi_{nl}) \simeq & \int \Psi_{nl}^{\dagger}\left(x_l^{(2)}, x_n^{(2)}\right)\left\{\left(K_l + V_l^{(e)}\right)\left(K_n + V_n^{(e)}\right)\right.\\ & -V_{l,n}^{(e)}\left(K_n + V_n^{(e)}\right)^{-1} V_{n,l}^{(e)}\left(K_n + V_n^{(e)}\right) - \left(K_l + V_l^{(e)}\right) V_{n,l}^{(e)}\left(K_l + V_l^{(e)}\right)^{-1} V_{l,n}^{(e)}\\ & \left.-V_{l,n}^{(e)}\left(x_l^{(2)}, x_n^{(1)}\right) V_{n,l}^{(e)}\left(x_n^{(2)}, x_l^{(1)}\right)\right\} \Psi_{nl}\left(x_l^{(1)}, x_n^{(1)}\right) dx_l^{(2)} dx_n^{(2)} dx_l^{(1)} dx_n^{(1)}\end{aligned}$$

This formula defining the effective action for an assembly of integrated structures may be set in a more readable form if we consider some approximations. We assume the following form for the operators $K_l + V_l^{(e)}$ and $K_n + V_n^{(e)}$ (which implies that $V_l^{(e)}$ and $V_n^{(e)}$ are locals)

$$\begin{aligned}K_l + V_l^{(e)} &= -\frac{1}{2}(\nabla_l)^2 - U_l(x_l)\\ K_n + V_n^{(e)} &= -\frac{1}{2}(\nabla_n)^2 - U_n(x_n)\end{aligned}$$

as well as a low inertia:

$$\begin{aligned}(\nabla_l)^2 &< < U_l(x_l)\\ (\nabla n)^2 &< < U_n(x_n)\end{aligned}$$

These further simplifications yield:

$$\begin{aligned}&\bar{S}^{nl}(\Psi_{nl}) \\ \simeq & \int \Psi_{nl}^{\dagger}(x_l, x_n)\left(-\frac{1}{2}U_n(x_n)(\nabla_l)^2 - \frac{1}{2}U_l(x_l)(\nabla_n)^2 - U_l(x_l)U_n(x_n) - 2V_{l,n}^{(e)}(x_l, x_n)V_{n,l}^{(e)}(x_l, x_n)\right)\Psi_{nl}(x_l, x_n)\,dx_l dx_n\\ & -\int \Psi_{nl}^{\dagger}\left(x_l^{(2)}, x_n^{(2)}\right) V_{l,n}^{(e)}\left(x_l^{(2)}, x_n^{(1)}\right) V_{n,l}^{(e)}\left(x_n^{(2)}, x_l^{(1)}\right) \Psi_{nl}\left(x_l^{(2)}, x_n^{(2)}\right) dx_l^{(1)} dx_l^{(2)} dx_n^{(1)} dx_n^{(2)}\end{aligned} \tag{197}$$

Ultimately, it is straightforward to find a first approximation of the effective utility for an integrated structures which corresponds to (197):

$$\begin{aligned}&\int\left(\frac{1}{2}U_n(X_n(s))\dot{X}_l^2(s) + \frac{1}{2}U_l(X_l(s))\dot{X}_n^2(s) + U_l(X_l(s))U_n(X_n(s))\right)ds\\ &-2\int V_{l,n}^{(e)}(X_l(s), X_n(s))V_{n,l}^{(e)}(X_l(s), X_n(s))\,ds - \int \hat{V}_{n,l}(X_l(s_1), X_n(s_2))\,ds_1 ds_2\end{aligned}$$

where $\hat{V}_{n,l}$ is equal to $V_{l,n}^{(e)} V_{n,l}^{(e)}$ plus some inertia corrections, similarly to the derivation of $V_l^{(e)}$ (see (189). This mixed utility presents some local aspect as for a usual utility in the concatenated control variable $(X_l(s), X_n(s))$. It presents also non local contibutions resulting from the constraints this agent imposes on itself through its subcomponents and its environment. Note also that the inertia terms $\dot{X}_l^2(s)$ and $\dot{X}_n^2(s)$ are factored by variable contributions. This models the non trivial trajectory for the structure in the space (X_l, X_n) as a consequence of internal interactions between the structure's subcomponents.

10.3 Extension: Several type agents, effective field action

10.3.1 Principle

In the previous section we have studied the possibility of emergence for an integrated effective structure that was absent from the initial interacting system. The integrated structure includes several previously independent structures and possess characteristics of its own, that were not present in initial ones. However taking an other point of view and studying the aggregation of several different elements can be interesting in some cases. Rather than aggregating all types of structures, one may integrate the behavior of one or some of them. so that its influence only appears as a substratum for the dynamics of other structures.

It amounts to consider a system with one type of agent less, but with a modified action which takes into account the interactions with the suppressed agent as a global modification of the system. This representation fits well for systems with "hidden" agents if, for some purposes, we are interested in the behavior of one (or several) particular types of agents. By integrating out the remaining types of agents, one can focus on the dynamic of a certain class, given an integrated landscape.

The general principle is the following. Consider the computation of the path integral

$$\int \exp\left(-S\left(\left\{\Psi^{(k)}\right\}_{k=1...M}\right)\right)\mathcal{D}\left\{\Psi^{(k)}\right\}_{k=1...M} \tag{198}$$

where we take the most general form of action:

$$\begin{aligned}
&S\left(\left\{\Psi^{(k)}\right\}_{k=1...M}\right) \qquad (199)\\
=\ &\sum_k \int d\hat{X}_k \left(\left(-\frac{1}{2}\Psi^{(k)\dagger}\left(\hat{X}_k\right)\left[\left(\nabla_k\right)\left(\nabla_k - M_k^{(1)}\left(\hat{X}_k - \left(\tilde{X}\right)_k\right)\right) + m_k^2 + V\left(\hat{X}_k\right)\right]\Psi^{(k)}\left(\hat{X}_k\right)\right)\right)\\
&\underbrace{+\sum_k \sum_n V_n\left(\left\{\hat{X}_k^{(i)}\right\}_{1\leqslant i\leqslant n}\right)\prod_{1\leqslant i\leqslant n}\Psi^{(k)\dagger}\left(\hat{X}_k^{(i)}\right)\Psi^{(k)}\left(\hat{X}_k^{(i)}\right)}_{\text{intra species interaction}}\\
&\underbrace{+\sum_m \sum_{k_1...k_m}\sum_{n_1...n_m} V_{n_1...n_m}\left(\left\{\hat{X}_{k_j}^{(i_{n_j})}\right\}_{1\leqslant i_{n_j}\leqslant n_j}\right)\prod_{j=1}^m \prod_{1\leqslant i_{n_j}\leqslant n_j}\Psi^{(k_j)\dagger}\left(\hat{X}_{k_j}^{(i_{n_j})}\right)\Psi^{(k_j)}\left(\hat{X}_{k_j}^{(i_{n_j})}\right)}_{\text{inter species interaction}}
\end{aligned}$$

and partition M as $M = M_1 + M_2$. We aim at getting rid of the structures $M_2 + 1...M$. To do so, the integration over $\left\{\Psi^{(k)}\right\}_{k=M_2+1...M}$ in (198) can be performed using the methods given in the previous paragraphs (by graphs, saddle point approximation, or both). Though it is usually impossible to get an exact result (we will give below examples for which it is), in principle, the integrals over $\left\{\Psi^{(k)}\right\}_{k=M_2+1...M}$, will leave us with:

$$\int \exp\left(-S\left(\left\{\Psi^{(k)}\right\}_{k=1...M_2}\right)\right)\mathcal{D}\left\{\Psi^{(k)}\right\}_{k=1...M_2} \tag{200}$$

where:

$$
\begin{aligned}
& S_{eff}\left(\left\{\Psi^{(k)}\right\}_{k=1\dots M_1}\right) \\
= & \sum_k \int d\hat{X}_k \left(\left(-\frac{1}{2}\Psi^{(k)\dagger}\left(\hat{X}_k\right)\left[\left(\nabla_k\right)\left(\nabla_k - M_k^{(1)}\left(\hat{X}_k - \left(\tilde{X}\right)_k\right)\right) + m_k^2 + V^{eff}\left(\hat{X}_k\right)\right]\Psi^{(k)}\left(\hat{X}_k\right)\right)\right) \\
& + \sum_k \sum_n V_n^{eff}\left(\left\{\hat{X}_k^{(i)}\right\}_{1\leqslant i\leqslant n}\right) \prod_{1\leqslant i\leqslant n} \Psi^{(k)\dagger}\left(\hat{X}_k^{(i)}\right)\Psi^{(k)}\left(\hat{X}_k^{(i)}\right) \\
& + \sum_m \sum_{k_1\dots k_m} \sum_{n_1\dots n_m} V_{n_1\dots n_m}^{eff}\left(\left\{\hat{X}_{k_j}^{(i_{n_j})}\right\}_{1\leqslant i_{n_j}\leqslant n_j}\right) \prod_{j=1}^m \prod_{1\leqslant i_{n_j}\leqslant n_j} \Psi^{(k_j)\dagger}\left(\hat{X}_{k_j}^{(i_{n_j})}\right)\Psi^{(k_j)}\left(\hat{X}_{k_j}^{(i_{n_j})}\right)
\end{aligned}
$$

The individual potential $V^{eff}\left(\hat{X}_k\right)$ is also affected. Actually, in the integration process, interaction terms involving only "integrated structures" plus one "non integrated one" leave us with the modified individual potential $V^{eff}\left(\hat{X}_k\right)$, and implies a modified individual behavior. Besides, the interaction process between remaining structures is itself modified by it's surrounding.

10.3.2 Example: two types of agents

To be more precise, consider a simple two agents model, with a one dimensional space of configuration for each agent: the propagator for a block (i.e. a fundamental structure) k (here k will take two values, i_1 or i_2) is:

$$-\nabla_k^2 + m_k^2 + \left((x_i)_k - \check{Y}_{eff}\right)(\Lambda_i)_k\left((x_i)_k - \check{Y}_{eff}\right)$$

where the matrix $(\Lambda_i)_k$ is $p \times p$ and the mass term being defined by (see (176) for example):

$$m^2 = \left(\alpha + U_{eff}\left(\bar{X}^e\right)\right)$$

Depending on the parameters of the system, m_k^2 can be positive or negative. Moreover, we consider a non reciprocal interaction term:

$$V\left(x_{i_1}, x_{i_2}\right) = \delta\left(x_{i_2} - \hat{x}_{i_2}^{(i_1)}\right)^2 \tag{201}$$

i.e. this models a strain imposed by type 1 agents on type 2 agents. This choice introduces conveniently an asymmetry between agents. Thus, we do not expect the same results for the effective actions of the two agents: the landscape created by the dominating agent is different from the one created by the dominated one. This fact will appear while considering the possibilities of phase transitions.

Gathering the potential (201) with the propagators for each structure yields the action of the two agents system:

$$
\begin{aligned}
S\left(\Psi_{i_2}\left(x_{i_2}\right),\right) = & \Psi_{i_1}\left(x_{i_1}\right)\left(-\nabla^2 + m_{i_1}^2 + \left(x_{i_1} - \left(\check{Y}_{eff}\right)_{i_1}\right)\Lambda_{i_1}\left(x_{i_1} - \left(\check{Y}_{eff}\right)_{i_1}\right)\right)\Psi_{i_1}^{\dagger}\left(x_{i_1}\right) \\
& \Psi_{i_2}\left(x_{i_2}\right)\left(-\nabla^2 + m_{i_2}^2 + \left(x_{i_2} - \left(\check{Y}_{eff}\right)_{i_2}\right)\Lambda_{i_2}\left(x_{i_2} - \left(\check{Y}_{eff}\right)_{i_2}\right)\right)\Psi_{i_2}^{\dagger}\left(x_{i_2}\right) \\
& + \delta \int dx_{i_2}\Psi_{i_2}\left(x_{i_2}\right)\Psi_{i_2}^{\dagger}\left(x_{i_2}\right)\left(x_{i_2} - \hat{x}_{i_2}^{(i_1)}\right)^2 \int dx_{i_1}\Psi_{i_1}\left(x_{i_1}\right)\Psi_{i_1}^{\dagger}\left(x_{i_1}\right)
\end{aligned}
$$

Two possibilities arise from the general method developped in the previous paragraph. One integrate the behavior of one of the two structures, and inspect the implications for the remaining one. We will start by integrating the behavior of the second agent and find an effective action for the first one:

Effective action for the first agent We consider the initial action of the second agent:

$$\begin{aligned}
&\Psi_{i_2}(x_{i_2})\left(-\nabla^2+m_{i_2}^2+\left(x_{i_2}-\left(\check{Y}_{eff}\right)_{i_2}\right)\Lambda_{i_2}\left(x_{i_2}-\left(\check{Y}_{eff}\right)_{i_2}\right)\right)\Psi_{i_2}^{\dagger}(x_{i_2})\\
&+\delta\int dx_{i_2}\Psi_{i_2}(x_{i_2})\Psi_{i_2}^{\dagger}(x_{i_2})\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})\\
=\;&\Psi_{i_2}(x_{i_2})\left(-\nabla^2+m_{i_2}^2+\left(x_{i_2}-\left(\check{Y}_{eff}\right)_{i_2}\right)\Lambda_{i_2}\left(x_{i_2}-\left(\check{Y}_{eff}\right)_{i_2}\right)\right.\\
&\left.+\delta\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})\Psi_{i_2}^{\dagger}(x_{i_2})\right)
\end{aligned}$$

Up to some normalization that we will reintroduced later, the integral for the exponential of this expression is straightforward and yields:

$$\begin{aligned}
&\exp\left(-\Psi_{i_2}(x_{i_2})\left(-\nabla^2+m_{i_2}^2+\left(x_{i_2}-\left(\check{Y}_{eff}\right)_{i_2}\right)\Lambda_{i_2}\left(x_{i_2}-\left(\check{Y}_{eff}\right)_{i_2}\right)\right.\right.\\
&\left.\left.+\delta\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})\Psi_{i_2}^{\dagger}(x_{i_2})\right)\right)\times\mathcal{D}\Psi_{i_2}(x_{i_2})\Psi_{i_2}^{\dagger}(x_{i_2})\\
=\;&\left(\det\left(\left(-\nabla^2+m_{i_2}^2+\left(x_{i_2}-\left(\check{Y}_{eff}\right)_{i_2}\right)\Lambda_{i_2}\left(x_{i_2}-\left(\check{Y}_{eff}\right)_{i_2}\right)+\delta\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})\right)\right)\right)^{-1}\\
=\;&\exp\left(-Tr\left(-\nabla^2+m_{i_2}^2+\left(x_{i_2}-\left(\check{Y}_{eff}\right)_{i_2}\right)\Lambda_{i_2}\left(x_{i_2}-\left(\check{Y}_{eff}\right)_{i_2}\right)+\delta\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})\right)\right)
\end{aligned}\tag{202}$$

This term can thus be reintroduced in the action for the remaining field $\Psi_{i_1}(x_{i_1})$, and thus the integration over the second strucure field leads to an effective action for $\Psi_{i_1}(x_{i_1})$:

$$\begin{aligned}
&S_{ef.}(\Psi_{i_1}(x_{i_1}))\\
=\;&S(\Psi_{i_1}(x_{i_1}))\\
&+Tr\ln\left(-\nabla^2+m_{i_2}^2+\left(x_{i_2}-\left(\check{Y}_{eff}\right)_{i_2}\right)\Lambda_{i_2}\left(x_{i_2}-\left(\check{Y}_{eff}\right)_{i_2}\right)+\delta\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})\right)
\end{aligned}$$

Recall now that the spectrum for the operator:

$$-\nabla^2+m_{i_2}^2+\left(x_{i_2}-\left(\check{Y}_{eff}\right)_{i_2}\right)\Lambda_{i_2}\left(x_{i_2}-\left(\check{Y}_{eff}\right)_{i_2}\right)\tag{203}$$

is given by

$$\left(n+\frac{1}{2}\right)(\Lambda_{i_2})+m_{i_2}^2\tag{204}$$

We assume that the eigenvalues of the diagonal matrix Λ_{i_2} are positive to ensure the stability of the system. The trace of (203) is:

$$\sum_n\left(n+\frac{1}{2}\right)(\Lambda_{i_2})+m_{i_2}^2$$

Here we have used the notation $\left(n+\frac{1}{2}\right)(\Lambda_{i_2})$ to write the product between a vector of m half integers $\left(n_1+\frac{1}{2},...,n_m+\frac{1}{2}\right)$ and the p eigenvalues of Λ_{i_2}.

Actually, as explainded in Appendix 9, the kernel of (203) is:

$$G(x,y)=\sum_n\psi_n(x)\left(m_i^2+\left(n+\frac{1}{2}\right)(\Lambda_i)_k\right)\psi_n^*(y)$$

Then:

$$\begin{aligned}
Tr\left(-\nabla^2+m_{i_2}^2+\left(x_{i_2}-\left(\check{Y}_{eff}\right)_{i_2}\right)\Lambda_{i_2}\left(x_{i_2}-\left(\check{Y}_{eff}\right)_{i_2}\right)\right)&=\int G(x,x)\,dx\\
&=\sum_n\left(m_i^2+\left(n+\frac{1}{2}\right)(\Lambda_i)_k\right)
\end{aligned}$$

due to the orthonormality of the eigenfunctions $\psi_n(x)$, and (204) follows. As a consequence, for an operator:

$$\left(-\nabla^2 + m_{i_2}^2 + \left(x_{i_2} - \left(\check{Y}_{eff}\right)_{i_2}\right)\Lambda_{i_2}\left(x_{i_2} - \left(\check{Y}_{eff}\right)_{i_2}\right) + \delta\left(x_{i_2} - \hat{x}_{i_2}^{(i_1)}\right)^2 \int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})\right)$$

which is quadratic in potential, the spectrum is similar to the spectrum of (203) and can be found by writing:

$$\begin{aligned} & \left(-\nabla^2 + m_{i_2}^2 + \left(x_{i_2} - \left(\check{Y}_{eff}\right)_{i_2}\right)\Lambda_{i_2}\left(x_{i_2} - \left(\check{Y}_{eff}\right)_{i_2}\right) + \delta\left(x_{i_2} - \hat{x}_{i_2}^{(i_1)}\right)^2 \int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})\right) \\ = & -\nabla^2 + m_{i_2}^2 \\ & + \left(x_{i_2} - \frac{\Lambda_{i_2}\left(\check{Y}_{eff}\right)_{i_2} + \hat{x}_{i_2}^{(i_1)}\delta\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})}{\Lambda_{i_2} + \delta\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})}\right) \\ & \times\left(\left(\Lambda_{i_2} + \delta\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})\right)\right)\left(x_{i_2} - \frac{\Lambda_{i_2}\left(\check{Y}_{eff}\right)_{i_2} + \hat{x}_{i_2}^{(i_1)}\delta\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})}{\Lambda_{i_2} + \delta\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})}\right) \\ & + \frac{\delta\Lambda_{i_2}\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})}{\Lambda_{i_2} + \delta\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})}\left(\left(\check{Y}_{eff}\right)_{i_2} - \hat{x}_{i_2}^{(i_1)}\right)^2 \end{aligned} \tag{205}$$

This is again an operator with quadratic potential, with an additional positive constant and a shift of variables. Its trace is then similar to (204):

$$\begin{aligned} & Tr\left(\ln\left(\left(-\nabla^2 + m_{i_2}^2 + \left(x_{i_2} - \left(\check{Y}_{eff}\right)_{i_2}\right)\Lambda_{i_2}\left(x_{i_2} - \left(\check{Y}_{eff}\right)_{i_2}\right) + \delta\left(x_{i_2} - \hat{x}_{i_2}^{(i_1)}\right)^2 \int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})\right)\right)\right) \\ = & \sum_n \ln\left(\left(n+\frac{1}{2}\right)\left(\Lambda_{i_2} + \delta\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})\right) + m_{i_2}^2 + \frac{\delta\Lambda_{i_2}\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})}{\Lambda_{i_2} + \delta\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})}\left(\left(\check{Y}_{eff}\right)_{i_2} - \hat{x}_{i_2}^{(i_1)}\right)^2\right) \end{aligned}$$

for n integers. As a consequence:

$$\begin{aligned} S_{ef.}\left(\Psi_{i_1}(x_{i_1})\right) = & S\left(\Psi_{i_1}(x_{i_1})\right) \\ & + \sum_n \ln\left(\left(n+\frac{1}{2}\right)\left(\Lambda_{i_2} + \delta\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})\right) + m_{i_2}^2\right. \\ & \left. + \frac{\delta\Lambda_{i_2}\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})}{\Lambda_{i_2} + \delta\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})}\left(\left(\check{Y}_{eff}\right)_{i_2} - \hat{x}_{i_2}^{(i_1)}\right)^2\right) \\ = & S\left(\Psi_{i_1}(x_{i_1})\right) \\ & + \sum_n \ln\left(\left(\left(n+\frac{1}{2}\right)\Lambda_{i_2} + m_{i_2}^2\right)\right. \\ & \left. \times\left(1 + \frac{\left(n+\frac{1}{2}\right)\delta\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1}) + \frac{\delta\Lambda_{i_2}\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})}{\Lambda_{i_2}+\delta\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})}\left(\left(\check{Y}_{eff}\right)_{i_2} - \hat{x}_{i_2}^{(i_1)}\right)^2}{\left(n+\frac{1}{2}\right)\Lambda_{i_2} + m_{i_2}^2}\right)\right) \end{aligned}$$

We can now come back to the problem of normalization mentioned before. We showed before that for normalization reasons, (202) has to be divided by its value for a nul interaction potential. As a consequence, we can normalize this sum by substracting it's value for a nul interaction, i.e.:

$$\begin{aligned} & \exp\left(-\Psi_{i_2}(x_{i_2})\left(-\nabla^2 + m_{i_2}^2 + \left(x_{i_2} - \left(\check{Y}_{eff}\right)_{i_2}\right)\Lambda_{i_2}\left(x_{i_2} - \left(\check{Y}_{eff}\right)_{i_2}\right)\right)\Psi_{i_2}^{\dagger}(x_{i_2})\right) \\ & \times\mathcal{D}\Psi_{i_2}(x_{i_2})\Psi_{i_2}^{\dagger}(x_{i_2}) \\ = & \exp\left(-Tr\left(-\nabla^2 + m_{i_2}^2 + \left(x_{i_2} - \left(\check{Y}_{eff}\right)_{i_2}\right)\Lambda_{i_2}\left(x_{i_2} - \left(\check{Y}_{eff}\right)_{i_2}\right)\right)\right) \end{aligned}$$

whose value is:

$$\sum_n \ln\left(\left(n+\frac{1}{2}\right)(\Lambda_{i_2})+m_{i_2}^2\right)$$

by virtue of (202). This value has thus to be substracted to $S_{ef.}(\Psi_{i_1}(x_{i_1}))$, and as a consequence, one has:

$$\begin{aligned}
&S_{ef.}(\Psi_{i_1}(x_{i_1})) \qquad (206)\\
=\ &S(\Psi_{i_1}(x_{i_1}))\\
&+\sum_n \ln\left(1+\frac{\left(n+\frac{1}{2}\right)\delta\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})+\frac{\delta\Lambda_{i_2}\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})}{\Lambda_{i_2}+\delta\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})}\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}{\left(n+\frac{1}{2}\right)\Lambda_{i_2}+m_{i_2}^2}\right)
\end{aligned}$$

However, this sum does not converge in n. This is a standard phenomenom when dealing with infinite degrees of freedom. Several methods exist to rule out this problem, and usually in physical problems, methods of renormalization are used. Nevertheless, we can use here a more simple solution. Actually, for a system in interaction, all frequencies of oscillations need not always be assumed to participate to the dynamics. At least we can assume high frequencies to be quickly dampened. As a consequence, the sum in (206) will be regularized in a realistic way if we introduce a cut off in this sum. It amounts to assume bounded frequencies for the field Ψ_{i_2}. We assume $n\leqslant N$. Moreover, for later purpose we normalize the field, by introducing $\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})=\eta$ and rescale

$$\Psi_{i_1}(x_{i_1})\rightarrow\sqrt{\eta}\Psi_{i_1}(x_{i_1})$$

with now $\Psi_{i_1}(x_{i_1})$ of norm 1.

Ultimately, the effective action for agents of type 1 is thus:

$$S_{ef.}(\Psi_{i_1}(x_{i_1}))=\eta S(\Psi_{i_1}(x_{i_1}))+\sum_{n\leqslant N}\ln\left(1+\frac{\left(n+\frac{1}{2}\right)\delta\eta+\frac{\delta\Lambda_{i_2}\eta}{\Lambda_{i_2}+\delta\eta}\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}{\left(n+\frac{1}{2}\right)\Lambda_{i_2}+m_{i_2}^2}\right) \qquad (207)$$

Effective action for the second agent Reversing the roles and integrating over $\Psi_{i_1}(x_{i_1})$ will yield the effective action for $\Psi_{i_2}(x_{i_2})$. Skipping some details from the previous paragraph procedure, the effective action for $\Psi_{i_2}(x_{i_2})$ is obtained as:

$$\begin{aligned}
S_{ef.}(\Psi_{i_2}(x_{i_2})) &= S(\Psi_{i_2}(x_{i_2}))+Tr\ln\left(\begin{array}{c}-\nabla^2+m_{i_1}^2+\left(x_{i_1}-\left(\check{Y}_{eff}\right)_{i_1}\right)\Lambda_{i_1}\left(x_{i_1}-\left(\check{Y}_{eff}\right)_{i_1}\right)\\+\delta\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\Psi_{i_2}(x_{i_2})\Psi_{i_2}^{\dagger}(x_{i_2})\end{array}\right)\\
&= S(\Psi_{i_2}(x_{i_2}))+\sum_n\int dx_{i_1}\psi_n\left(x_{i_1}-\left(\check{Y}_{eff}\right)_{i_1}\right)\psi_n^*\left(\left(x_{i_1}-\left(\check{Y}_{eff}\right)_{i_1}\right)\right)\\
&\quad\times\ln\left(\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2+\delta\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\Psi_{i_2}(x_{i_2})\Psi_{i_2}^{\dagger}(x_{i_2})\right)\\
&= S(\Psi_{i_2}(x_{i_2}))+\sum_n\ln\left(\begin{array}{c}-\nabla^2+m_{i_1}^2+\left(x_{i_1}-\left(\check{Y}_{eff}\right)_{i_1}\right)\Lambda_{i_1}\left(x_{i_1}-\left(\check{Y}_{eff}\right)_{i_1}\right)\\+\delta\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\Psi_{i_2}(x_{i_2})\Psi_{i_2}^{\dagger}(x_{i_2})\end{array}\right)
\end{aligned}$$

As before, we normalize this expression by substracting some reference quantity:

$$\sum_n\ln\left(\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2\right)$$

which leads us to:

$$S_{ef.}(\Psi_{i_2}(x_{i_2}))=S(\Psi_{i_2}(x_{i_2}))+\sum_n\ln\left(1+\delta\frac{\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\Psi_{i_2}(x_{i_2})\Psi_{i_2}^{\dagger}(x_{i_2})}{\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2}\right)$$

As in the previous paragraph, we regularize this quantity by allowing only a finite number of Fourier components, $n \leq N$. The result is then:

$$S_{ef.}\left(\Psi_{i_2}\left(x_{i_2}\right)\right)=S\left(\Psi_{i_2}\left(x_{i_2}\right)\right)+\sum_{n\leq N}\ln\left(1+\delta\frac{\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\Psi_{i_2}\left(x_{i_2}\right)\Psi_{i_2}^{\dagger}\left(x_{i_2}\right)}{\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2}\right)$$

Normalizing the field $\Psi_{i_2}\left(x_{i_2}\right)$:

$$\Psi_{i_2}\left(x_{i_2}\right)\rightarrow\sqrt{\eta}\Psi_{i_2}\left(x_{i_2}\right)$$

will yield ultimately:

$$S_{ef.}\left(\Psi_{i_2}\left(x_{i_2}\right)\right)=\eta S\left(\Psi_{i_2}\left(x_{i_2}\right)\right)+\sum_{n\leq N}\ln\left(1+\eta\delta\frac{\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\Psi_{i_2}\left(x_{i_2}\right)\Psi_{i_2}^{\dagger}\left(x_{i_2}\right)}{\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2}\right)$$

10.3.3 Possibility of phase transition

The interesting differences between the effective actions $S_{ef.}\left(\Psi_{i_1}\left(x_{i_1}\right)\right)$ and $S_{ef.}\left(\Psi_{i_2}\left(x_{i_2}\right)\right)$, are embedded in the possibility of phase transition in each case. This requires to study the possibility of a minimum for $S_{ef.}\left(\Psi_{i_1}\left(x_{i_1}\right)\right)$ and then for $S_{ef.}\left(\Psi_{i_2}\left(x_{i_2}\right)\right)$ with $\eta>0$. This possibility depends on the parameters involved in each effective utility. A detailled study is performed in Appendix 14, and the results are the following:

First agent: We consider several cases, depending on the parameters of the system.

For $\delta>0$, if:

$$\begin{aligned}\frac{1}{2}\left(\Lambda_{i_2}\right)-m_{i_1}^2 &< 0\\ \frac{1}{2}\left(\Lambda_{i_1}\right)-m_{i_1}^2+\delta\sum_{n\leqslant N}\frac{\left(n+\frac{1}{2}\right)+\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}{\left(n+\frac{1}{2}\right)\left(\Lambda_{i_2}\right)+m_{i_2}^2} &< 0\end{aligned}$$

and if there is an η_0 such that

$$\frac{1}{2}\left(\Lambda_{i_1}\right)-m_{i_1}^2+\sum_{n\leqslant N}\frac{\left(n+\frac{1}{2}\right)\delta+\frac{\left(\Lambda_{i_2}\right)^2\delta}{\left(\Lambda_{i_2}+\delta\eta_0\right)^2}\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}{\left(n+\frac{1}{2}\right)\left(\Lambda_{i_2}+\delta\eta_0\right)+m_{i_2}^2+\frac{\delta\Lambda_{i_2}\eta_0}{\Lambda_{i_2}+\delta\eta_0}\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}>0 \tag{208}$$

then there exists $\eta_1\neq 0$ such that

$$\Psi_{i_1}^{(0)}\left(x_{i_1}\right)=\sqrt{\eta_1}\left(\frac{\sqrt{a}}{\pi}\right)^{\frac{1}{4}}\exp\left(-\frac{\sqrt{a}}{2}x_{i_1}^2\right)$$

is a minimum for the action $S_{ef.}$. Remark that, since the RHS of (208) is increasing for $\eta_0>0$, the condition (208) is fulfilled for some values of the parameters. As a consequence a non trivial vaccum exists. Implications of this result have been explained earlier.

For $\delta<0$ the conditions are simpler. If:

$$\begin{aligned}\frac{1}{2}\left(\Lambda_{i_2}\right)-m_{i_1}^2 &> 0\\ \frac{1}{2}\left(\Lambda_{i_1}\right)-m_{i_1}^2+\delta\sum_{n\leqslant N}\frac{\left(n+\frac{1}{2}\right)+\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}{\left(n+\frac{1}{2}\right)\Lambda_{i_2}+m_{i_2}^2} &< 0\end{aligned}$$

then there is $\eta\neq 0$, such that $\sqrt{\eta}\Psi_{i_1}^{(0)}\left(x_{i_1}\right)$ is the minimum of $S_{ef.}\left(\Psi_{i_1}\left(x_{i_1}\right)\right)$.

For all other cases the minimum for $S_{ef.}\left(\Psi_{i_1}\left(x_{i_1}\right)\right)$ is reached for $\eta=0$ and no phase transition occurs.

The possibillity for several minima is the consequence of the strain imposed by the first agent on the second. Given the values of the parameters, the first agent may adapt to the behavior of the second one and lead the system toward an other equilibrium.

Effective action for the second agent Appendix 14 shows that the expectation value $\langle \Psi_{i_2}(x_{i_2})\rangle$ is nul, and then, that no phase transition occurs. This is the consequence of the asymmetric potential of interaction between the agents. If actions of the first type of agents are integrated out, for reasons such as different time scale for these actions, or large fluctuations among the first type of agents, the second type of agents integrate these behaviors as an external medium and only one equilibrium is reached. The characteristic of this equilibrium is studied in the next paragraph.

10.3.4 Consequence of phase transition

We can now inspect the consequences of the phase transition for the two agents. For the first agent, two phases may appear. Consider its effective action where the notation $\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})=\eta$ has been reintroduced.

$$\begin{aligned}S_{ef.}(\Psi_{i_1}(x_{i_1})) &= S(\Psi_{i_1}(x_{i_1})) \\ &+\sum_{n\leqslant N}\ln\left(1+\frac{\left(n+\frac{1}{2}\right)\delta\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})+\frac{\delta\Lambda_{i_2}\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})}{\Lambda_{i_2}+\delta\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})}\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}{\left(n+\frac{1}{2}\right)\Lambda_{i_2}+m_{i_2}^2}\right)\end{aligned} \tag{209}$$

We will compute the second order approximation of (209) for of each of these phases.

In the case of a trivial background expectation $\Psi_{i_1}(x_{i_1})=0$, the second order expansion of $S_{ef.}(\Psi_{i_1}(x_{i_1}))$ is:

$$S_{ef.}(\Psi_{i_1}(x_{i_1}))=S(\Psi_{i_1}(x_{i_1}))+\sum_{n\leqslant N}\frac{\left(n+\frac{1}{2}+\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\right)}{\left(n+\frac{1}{2}\right)\Lambda_{i_2}+m_{i_2}^2}\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})$$

In the phase where the minimum of $S_{ef.}$ is reached for a field $\Psi_{i_1}^0(x_{i_1})\neq 0$, we shift $\Psi_{i_1}(x_{i_1})\rightarrow\Psi_{i_1}(x_{i_1})+\Psi_{i_1}^0(x_{i_1})$, which leads to:

$$\begin{aligned}S_{ef.}(\Psi_{i_1}(x_{i_1})) &= S_{ef.}\left(\Psi_{i_1}^0(x_{i_1})\right)+S(\Psi_{i_1}(x_{i_1})) \\ &+\sum_{n\leqslant N}\frac{\left(\delta\left(n+\frac{1}{2}\right)+\delta\frac{\Lambda_{i_2}^2}{\left(\Lambda_{i_2}+\eta^2\delta\right)^2}\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\right)}{\left(m_{i_2}^2+\Lambda_{i_2}\left(n+\frac{1}{2}\right)\right)\left(\frac{\delta(\eta^2)\left(n+\frac{1}{2}\right)+\delta\Lambda_{i_2}\frac{\eta^2}{\delta(\eta^2)+\Lambda_{i_2}}\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}{m_{i_2}^2+\Lambda_{i_2}\left(n+\frac{1}{2}\right)}+1\right)}\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1}) \\ &+\sum_{n\leqslant N}\int dx_{i_1}\left(\Psi_{i_1}(x_{i_1})\right)^{\dagger}\Psi_{i_1}(x_{i_1}) \\ &\times\left(\frac{\left(\delta\left(n+\frac{1}{2}\right)+\delta\Lambda_{i_2}\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\frac{\Lambda_{i_2}}{\left(\Lambda_{i_2}+\eta^2\delta\right)^2}\right)^2}{2\left(\left(\eta^2\delta\left(n+\frac{1}{2}\right)+\eta^2\delta\Lambda_{i_2}\frac{\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}{\delta\eta^2+\Lambda_{i_2}}\right)+\left(m_{i_2}^2+\Lambda_{i_2}\left(n+\frac{1}{2}\right)\right)\right)^2}\right. \\ &\left.+\delta^2\frac{\frac{\Lambda_{i_2}^2}{\left(\Lambda_{i_2}+\eta^2\delta\right)^3}\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}{\left(\eta^2\delta\left(n+\frac{1}{2}\right)+\eta^2\delta\Lambda_{i_2}\frac{\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}{\delta\eta^2+\Lambda_{i_2}}\right)+m_{i_2}^2+\Lambda_{i_2}\left(n+\frac{1}{2}\right)}\right)\end{aligned}$$

The interpretation is the following. In both cases, the effective action for the first type of agent is shifted by a quadratic term in $\Psi_{i_1}(x_{i_1})$ of the type:

$$\lambda\int dx_{i_1}\Psi_{i_1}(x_{i_1})\Psi_{i_1}^{\dagger}(x_{i_1})+\mu\left(\int dx_{i_1}\left(\left(\Psi_{i_1}^0(x_{i_1})\right)\left(\Psi_{i_1}(x_{i_1})\right)^{\dagger}+\left(\Psi_{i_1}(x_{i_1})\right)\left(\Psi_{i_1}^0(x_{i_1})\right)^{\dagger}\right)\right)^2$$

Coming back to the individual behaviors, we have seen in the previous paragraph that it amounts to modify the utility of an individual agent by a constant term. In other words, the introduction of surrounding type 2 agents does not change the equilibrium value. However the introduction of this constant quadratic term dampens the oscillations around he equilibrium. In fact, this shift in the action corresponds to a shift in $m_{i_1}^2$, or, equivalently, a shift in α, the parameter which measures the inverse of interaction duration for type 1 agents. It means that integrating the behavior of second-type agents is equivalent to reduce the duration for the interaction process of type 1 agents. Type 1 agents spend time controling type 2 agents, which is a loss of time/energy. As a consequence type 2 agents act as stabilizers. The dampening effect in the oscillation depends on the phase of the system.

Now, switching to the effective action for the second agent:

$$S_{ef.}\left(\Psi_{i_2}\left(x_{i_2}\right)\right)=S\left(\Psi_{i_2}\left(x_{i_2}\right)\right)+\sum_{n\leq N}\ln\left(1+\delta\frac{\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\Psi_{i_2}\left(x_{i_2}\right)\Psi_{i_2}^{\dagger}\left(x_{i_2}\right)}{\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2}\right)$$

We have seen that there is no phase transition (i.e. the minimum of $S_{ef.}\left(\Psi_{i_2}\left(x_{i_2}\right)\right)$ is for $\Psi_{i_2}\left(x_{i_2}\right)=0$). At the second order approximation, the effective action $S_{ef.}\left(\Psi_{i_2}\left(x_{i_2}\right)\right)$ is then:

$$\begin{aligned}S_{ef.}\left(\Psi_{i_2}\left(x_{i_2}\right)\right) &= S\left(\Psi_{i_2}\left(x_{i_2}\right)\right)+\sum_{n\leq N}\delta\frac{\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\Psi_{i_2}\left(x_{i_2}\right)\Psi_{i_2}^{\dagger}\left(x_{i_2}\right)}{\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2}\\ &= S\left(\Psi_{i_2}\left(x_{i_2}\right)\right)+\left(\sum_{n\leq N}\frac{\delta}{\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2}\right)\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\Psi_{i_2}\left(x_{i_2}\right)\Psi_{i_2}^{\dagger}\left(x_{i_2}\right)\end{aligned}$$

Here, the situation is different with respect to type 1 agents. Coming back to the individual utilities corresponding to this colllective field, the first order correction due to agent 1 is to shift the effective action by a term $\delta\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2$: the attractive (for $\delta>0$) or repulsive (for $\delta<0$) potential. The only consequence for the second agent is thus a shift hat for the second species is shifted is the frequencies of oscillations

$$\Lambda_{i_2}\rightarrow\Lambda_{i_2}+\left(\sum_{n\leq N}\frac{\delta}{\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2}\right)$$

depending on the sign of δ, fasten ou dampened. The center of oscillation is also shifted as a combination of $\left(\check{Y}_{eff}\right)_{i_1}$ and $\hat{x}_{i_2}^{(i_1)}$. In terms of effective utility it means that the equilibrium value of the second type of agent is shifted. By a computation analog to (205), the shift in the equilibrium value is:

$$\left(\check{Y}_{eff}\right)_{i_2}\rightarrow\frac{\Lambda_{i_2}\left(\check{Y}_{eff}\right)_{i_2}+\hat{x}_{i_2}^{(i_1)}\left(\sum_{n\leq N}\frac{\delta}{\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2}\right)}{\Lambda_{i_2}+\left(\sum_{n\leq N}\frac{\delta}{\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2}\right)}$$

As a consequence, the background initiated by the first type of agents modifies both the system's equilibrium, which is shifted towards the goals of the first agents, and the frequencies of oscillations around the equilibrium.

11 Introducing macro time scale and aggregated quantities

This section studies the possibility to define aggregated quantities in a system with a large number of agents. These quantities have to be relevant at the scale of the whole system. The field formalism is appropriate for this since it allows to connect micro and macro scales.

To do so, we start with the probabilisitic description of a system with N agents whose effective statistical weights is (127):

$$\sum_N \frac{1}{N!}\prod_{i=1}^N \int\int \mathcal{D}x_i(t)\exp\left(-\sum_i\int_0^s\left(\frac{\dot{x}_i^2}{2}(t)+K(x_i(t))\,dt\right)-\sum_{k=2}^A\sum_{i_1,\dots i_k}\int_0^s V_k\left(x_{i_1}(t_1)\dots x_{i_k}(t_k)\right)dt_1\dots dt_k\right) \tag{210}$$

The expectation of $\int_0^s\sum_i x_i(t)\,dt$ can then be computed by adding a linear potential $J_i x_i(t)$ to $K(x_i(t))$ and by taking the derivative at $J=0$ of (210):

$$\begin{aligned}&\int\exp(-\alpha s)\left\langle\int_0^s\sum_i x_i(t)\,dt\right\rangle ds \\ =&\left(\frac{\partial}{\partial J}\left(\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp(-\alpha s_i)\int\mathcal{D}x_i(t)\exp\left(\begin{array}{c}-\sum_i\int_0^{s_i}\left(\frac{\dot{x}_i^2}{2}(t)+(K(x_i(t))+Jx_i(t))\,dt\right)\\-\sum_{k=2}^A\sum_{i_1,\dots i_k}\int_0^{s_i}V_k\left(x_{i_1}(t_1)\dots x_{i_k}(t_k)\right)dt_1\dots dt_k\end{array}\right)\right)\right)_{J=0}\end{aligned}\tag{211}$$

The quantity $\sum_i x_i(t)$ is the agregated value of the quantities $x_i(t)$ over the set of agents,. We will explain later why we integrate this quantity over the whole duration of the interaction by integrating for t between 0 and s.

To switch to the field representation, we need to compute the Laplace transform of $\left\langle\int_0^s\sum_i x_i(t)\,dt\right\rangle$:

$$\begin{aligned}&\int\exp(-\alpha s)\left\langle\int_0^s\sum_i x_i(t)\,dt\right\rangle ds \\ =&\left(\frac{\partial}{\partial J}\left(\sum_N\frac{1}{N!}\prod_{i=1}^N\int\exp(-\alpha s)\int\mathcal{D}x_i(t)\exp\left(\begin{array}{c}-\sum_i\int_0^{s_i}\left(\frac{\dot{x}_i^2}{2}(t)+(K(x_i(t))+Jx_i(t))\,dt\right)\\-\sum_{k=2}^A\sum_{i_1,\dots i_k}\int_0^{s_i}V_k\left(x_{i_1}(t_1)\dots x_{i_k}(t_k)\right)dt_1\dots dt_k\end{array}\right)\right)ds\right)_{J=0}\end{aligned}\tag{212}$$

Which amounts to compute the average, over time, of $\left\langle\int_0^s\sum_i x_i(t)\,dt\right\rangle$ with a mean duration process of $\frac{1}{\alpha}$.

Now, switching to the field formalism, the RHS of (212) can be computed by using the field theoretic action:(143):

$$\begin{aligned}&S\left(\left\{\Psi^{(k)}\right\}_{k=1\dots M}\right)\\ =&\sum_k\int d\hat{X}_k\left(\left(-\frac{1}{2}\Psi^{(k)\dagger}\left(\hat{X}_k\right)\left[(\nabla_k)\left(\nabla_k-M_k^{(1)}\left(\hat{X}_k-\left(\tilde{X}\right)_k\right)\right)+m_k^2+V\left(\hat{X}_k\right)\right]\Psi^{(k)}\left(\hat{X}_k\right)\right)\right)\\ &\underbrace{+\sum_k\sum_n V_n\left(\left\{\hat{X}_k^{(i)}\right\}_{1\leqslant i\leqslant n}\right)\prod_{1\leqslant i\leqslant n}\Psi^{(k)\dagger}\left(\hat{X}_k^{(i)}\right)\Psi^{(k)}\left(\hat{X}_k^{(i)}\right)}_{\text{intra species interaction}}\\ &\underbrace{+\sum_m\sum_{k_1\dots k_m}\sum_{n_1\dots n_m}V_{n_1\dots n_m}\left(\left\{\hat{X}_{k_j}^{(i_{n_j})}\right\}_{1\leqslant i_{n_j}\leqslant n_j}\right)\prod_{j=1}^m\prod_{1\leqslant i_{n_j}\leqslant n_j}\Psi^{(k_j)\dagger}\left(\hat{X}_{k_j}^{(i_{n_j})}\right)\Psi^{(k_j)}\left(\hat{X}_{k_j}^{(i_{n_j})}\right)}_{\text{inter species interaction}}\end{aligned}\tag{213}$$

In the line of our general formalism, different types of agents have been introduced. To replicate the result of (212) we have to modify the potential in the field formalism. Actually, the introduction of $J\int_0^s\sum_i x_i(t)\,dt$ in (212), or more generally of $J\int_0^s\sum_{i_k}x_{i_k}(t)\,dt$ if several types of agents are considered, translates at the field level by replacing $V\left(\hat{X}_k\right)$ by $V\left(\hat{X}_k\right)+J\hat{X}_k$ in (213). As a consequence, the aggregated quantity $\int \exp(-\alpha s)\left\langle\int_0^s \sum_i x_i(t)\,dt\right\rangle ds$ is directly given by:

$$\begin{aligned}
&\int \exp(-\alpha s)\left\langle\int_0^s \sum_i x_i(t)\,dt\right\rangle ds \\
=\;&\left(\frac{\partial}{\partial J}\left(\int \exp\left(-S\left(\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)+\sum_k J_k\int \Psi^{(k)\dagger}\left(\hat{X}_k\right)\hat{X}_k\Psi^{(k)}\left(\hat{X}_k\right)d\hat{X}_k\right)\mathcal{D}\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)\right)_{J=0}
\end{aligned}$$

And the quantity $\left\langle\int_0^s \sum_i x_i(t)\,dt\right\rangle$ can be recovered by the inverse laplace transform of the previous quantity:

$$\begin{aligned}
&\left\langle\int_0^s \sum_i x_i(t)\,dt\right\rangle \\
=\;&\mathcal{L}^{-1}\left(\left(\frac{\partial}{\partial J}\int \exp\left(-S\left(\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)\right.\right.\right. \\
&\left.\left.\left.+\sum_k J_k\int \Psi^{(k)\dagger}\left(\hat{X}_k\right)\hat{X}_k\Psi^{(k)}\left(\hat{X}_k\right)d\hat{X}_k\right)\right)\mathcal{D}\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)_{J=0}
\end{aligned}$$

Remark that the field formalism allows to compute an average quantity over the all duration process, $\left\langle\int_0^s \sum_i x_i(t)\,dt\right\rangle$ but that we cannot differentiate the quantity $\left\langle\int_0^s \sum_i x_i(t)\,dt\right\rangle$ with respect to s to get:

$$\begin{aligned}
\left\langle\sum_i x_i(s)\right\rangle=\mathcal{L}^{-1}&\left(\alpha\frac{\partial}{\partial J}\left(\int \exp\left(-S\left(\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)+\sum_k J_k\int \Psi^{(k)\dagger}\left(\hat{X}_k\right)\hat{X}_k\Psi^{(k)}\left(\hat{X}_k\right)d\hat{X}_k\right)\right.\right. \\
&\left.\left.\times\mathcal{D}\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)\right)_{J=0}
\end{aligned}$$

Actually, in $\left\langle\int_0^s \sum_i x_i(t)\,dt\right\rangle$ the bracket term, the expectation over the path depends itself on s through the weight appearing in (210). Thus, the field formalism

However, remind that $T=\frac{1}{\alpha}$ can be seen as the mean time for the process of interaction between the agents of the system, one can interpret $\int \exp(-\alpha s)\left\langle\int_0^s \sum_i x_i(t)\,dt\right\rangle ds$ as the mean quantity $\bar{X}=\sum_i x_i(t)$ aggregated over a period T. This a static view however, since nothing in the interaction process makes a difference between two different time span, T and T' except the fact that a different length of the process will yield a different result.

Three different and non exclusive ways connect our formalism with a dynamic evolution of the macro quantities $\bar{X}(T)$. The first is to assume that all parameters in (213) depend exogenously on T. It represents the evolution of interactions, technology, or any quantity external to the system. The evolution of the parameters may imply some phase transitions in the system. The second way is to consider each individual agent's equilibrium values as given. This comes as an external condition: $\left(\tilde{X}\right)_k=\frac{1}{NT}\left(\bar{X}(T-1)\right)_k$ (rewritten also $\left(\bar{X}(T-1)\right)_k$ for the sake of simplicity). The third way, which is also the more usual and more direct, comes from the inclusion of constraints that encompass some exogenous quantities. For example, a budget constraint includes the average endowment $\bar{Y}$ at time t for an agent. We can consider this average endowment to be given by some past average accumulated quantities, say $\bar{X}(T-1)$ and to replace it by $\bar{Y}\rightarrow\frac{1}{NT}\left(\bar{X}(T-1)\right)$ which is proportionnal to $\bar{X}(T-1)$. For several types of agents, the average endowment $\bar{Y}_k$ for the type k can thus replaced by $\bar{X}_k(T-1)$. The contributions of these terms, that are like $\bar{Y}\hat{X}_k$, are linear terms and can by themselves be integrated in the constants $\left(\tilde{X}\right)_k$.

The first of these ways is exogenous, the two others are endogenous. Combining these posibilities allows to reintroduce some macro time dependence leads to consider the effective action:

$$\begin{aligned}
& S\left(\left\{\Psi^{(k)}\right\}_{k=1\ldots M}\right) \\
= & \sum_k \int d\hat{X}_k \left(\left(-\frac{1}{2}\Psi^{(k)\dagger}\left(\hat{X}_k\right)\left[\left(\nabla_k\right)\left(\nabla_k - M_k^{(1)}\left(\hat{X}_k - \left(\bar{X}\left(T-1\right)\right)_k\right)\right) + m_k^2\left(T\right) + V\left(\hat{X}_k, T\right)\right]\Psi^{(k)}\left(\hat{X}_k\right)\right)\right) \\
& + \sum_k \sum_n V_n\left(\left\{\hat{X}_k^{(i)}\right\}_{1\leqslant i\leqslant n}, T\right)\prod_{1\leqslant i\leqslant n}\Psi^{(k)\dagger}\left(\hat{X}_k^{(i)}\right)\Psi^{(k)}\left(\hat{X}_k^{(i)}\right) \\
& + \sum_m \sum_{k_1\ldots k_m}\sum_{n_1\ldots n_m} V_{n_1\ldots n_m}\left(\left\{\hat{X}_{k_j}^{\left(i_{n_j}\right)}\right\}_{1\leqslant i_{n_j}\leqslant n_j}, T\right)\prod_{j=1}^m \prod_{1\leqslant i_{n_j}\leqslant n_j}\Psi^{(k_j)\dagger}\left(\hat{X}_{k_j}^{\left(i_{n_j}\right)}\right)\Psi^{(k_j)}\left(\hat{X}_{k_j}^{\left(i_{n_j}\right)}\right) \\
& + \sum_k J_k \int \Psi^{(k)\dagger}\left(\hat{X}_k\right)\hat{X}_k \Psi^{(k)}\left(\hat{X}_k\right) d\hat{X}_k
\end{aligned} \tag{214}$$

where now some exogenous dependencies in T have been introduced in the interaction parameters, through the interaction potentials and in $m_k^2\left(T\right)$. To point the relation with more usual models of statistical physics, these exogenous variations are usually responsible for phase transition of a system. As explained above the macro quantity $\left(\bar{X}\left(T-1\right)\right)_k$ satisfies a recursive equation:

$$\begin{aligned}
\left(\bar{X}\left(T\right)\right)_k = & \left(\frac{\partial}{\partial J_k}\left(\int \exp\left(-S\left(\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right) + \sum_k J_k \int \Psi^{(k)\dagger}\left(\hat{X}_k\right)\hat{X}_k \Psi^{(k)}\left(\hat{X}_k\right) d\hat{X}_k\right)\right.\right. \\
& \left.\left.\times \mathcal{D}\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)\right)_{J_k=0}
\end{aligned} \tag{215}$$

The exploration of such recursive system is left for future works.

11.1 From micro to macro relations

Would some micro relations between some quantities be stable when switching to the macro scale ? Consider at the micro level a quantity that can be written:

$$z_{i_k}\left(t\right) = h\left(x_{i_k}\left(t\right)\right)$$

where $x_{i_k}\left(t\right)$ is the control variable at time t for an agent of type k, and compute its aggregated version over the duration of the interaction process:

$$\bar{Z}_k = \int_0^s \sum_i h\left(x_{i_k}\left(t\right)\right) dt$$

Then, similarly to (215):

$$\begin{aligned}
\left(\bar{Z}\right)_k = & \left(\frac{\partial}{\partial J_k}\left(\int \exp\left(-S\left(\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right) + \sum_k J_k \int \Psi^{(k)\dagger}\left(\hat{X}_k\right) h\left(\hat{X}_k\right) \Psi^{(k)}\left(\hat{X}_k\right) d\hat{X}_k\right)\right.\right. \\
& \left.\left.\times \mathcal{D}\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)\right)_{J_k=0}
\end{aligned} \tag{216}$$

We also need the aggregated quantity corresponding to the $x_{i_k}\left(t\right)$:

$$\bar{Z}_k = \int_0^s \sum_i x_{i_k}\left(t\right) dt$$

$$\begin{aligned}
\left(\bar{X}\right)_k = & \left(\frac{\partial}{\partial J_k}\left(\int \exp\left(-S\left(\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right) + \sum_k J_k \int \Psi^{(k)\dagger}\left(\hat{X}_k\right) h\left(\hat{X}_k\right) \Psi^{(k)}\left(\hat{X}_k\right) d\hat{X}_k\right)\right.\right. \\
& \left.\left.\times \mathcal{D}\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)\right)_{J_k=0}
\end{aligned} \tag{217}$$

A consequence of (216) is that if h is linear, $h\left(\hat{X}_k\right) = \gamma \hat{X}_k$ with γ an arbitrary constant, then after agregation $\left(\bar{Z}\right)_k = h\left(\bar{X}_k\right)$ and the micro relation is preserved at the macro level. However for a more general relation this is not the case: Actually, computing the derivative in (216):

$$\left(\bar{Z}\right)_k = \left(\int\left(\int \Psi^{(k)\dagger}\left(\hat{X}_k\right) h\left(\hat{X}_k\right) \Psi^{(k)}\left(\hat{X}_k\right) d\hat{X}_k\right) \exp\left(-S\left(\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)\right) \mathcal{D}\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)$$

and using the definition of the interaction Green function:

$$G(x,y) = \left(\int \Psi^{(k)\dagger}(x)\Psi^{(k)}(y) \exp\left(-S\left(\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)\right) \mathcal{D}\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)$$

yields:

$$\left(\bar{Z}\right)_k = \int h(x) G(x,x) dx \tag{218}$$

To compare with $\left(\bar{X}\right)_k$ we can specialize to $h(x) = x$ to write:

$$\left(\bar{X}\right)_k = \int x G(x,x) dx \tag{219}$$

and the comparison between (218)and (219) shows that the relation

$$\left(\bar{Z}\right)_k = h\left(\left(\bar{X}\right)_k\right)$$

is not valid for a general function h. Only if translation invariance is present in the model, that is $G(x,y) = G(y-x)$ and thus $G(x,x) = G(0,0)$, then some simple macro relations can be found (normalizing $G(0,0)$ to 1). Actually, in that case:

$$\begin{aligned}\left(\bar{Z}\right)_k &= \int h(x) dx \\ \left(\bar{X}\right)_k &= \int x dx\end{aligned}$$

Assuming that the lower bound is equal to 0 in both integrals, we change the variable $u = \frac{x^2}{2}$ in the first integral to get:

$$\left(\bar{Z}\right)_k = \int^{\sqrt{2\left(\bar{X}\right)_k}} h(x) dx$$

However, in the models at stake in this work, involving effective utility of harmonic oscillators plus interaction terms, the translation invariance is not preserved, and no simple macro relation can be found.

11.2 Effect of phase transition on aggregated quantities

Aggregated quantities are given by average quantities in the field formalism, and as such, they should be affected by phase transitions occuring with the parameters evolution. To inspect this phenomenon, we start with the expression for an aggregated quantity:

$$\left(\bar{X}(T)\right)_k = \left(\frac{\partial}{\partial J_k}\left(\int \exp\left(-S\left(\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right) + \sum_k J_k \int \Psi^{(k)\dagger}\left(\hat{X}_k\right) \hat{X}_k \Psi^{(k)}\left(\hat{X}_k\right) d\hat{X}_k\right) \mathcal{D}\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right) \tag{220}$$

Assume a non zero vacuum expectation value for the $\Psi^{(k)}\left(\hat{X}_k\right)$, denoted $\Psi_0^{(k)}\left(\hat{X}_k\right)$, write $\Psi^{(k)}\left(\hat{X}_k\right)=\Psi_0^{(k)}\left(\hat{X}_k\right)+\delta\Psi^{(k)}\left(\hat{X}_k\right)$ and expand the exponential in (220) to the second order in $\delta\Psi^{(k)}$:

$$\begin{aligned}
&S\left(\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)+\sum_k J_k\int\Psi^{(k)\dagger}\left(\hat{X}_k\right)\hat{X}_k\Psi^{(k)}\left(\hat{X}_k\right)d\hat{X}_k \qquad (221)\\
=\;&S\left(\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)+\sum_k J_k\int\Psi_0^{(k)\dagger}\left(\hat{X}_k\right)\hat{X}_k\Psi_0^{(k)}\left(\hat{X}_k\right)d\hat{X}_k\\
&+\sum_{k_1,k_2}\int\left(\delta\Psi^{(k_2)}\right)^\dagger\left(\hat{X}_{k_2}\right)\frac{\delta^2 S\left(\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)}{\delta\Psi_0^{(k_1)}\left(\hat{X}_{k_1}\right)\delta\left(\Psi_0^{(k_2)}\right)^\dagger\left(\hat{X}_{k_2}\right)}\delta\Psi^{(k_1)}\left(\hat{X}_{k_1}\right)d\hat{X}_{k_1}d\hat{X}_{k_2}\\
&+\sum_k J_k\int\delta\Psi^{(k)\dagger}\left(\hat{X}_k\right)\hat{X}_k\delta\Psi^{(k)}\left(\hat{X}_k\right)d\hat{X}_k\\
&+\text{higher order terms in }\delta\Psi^{(k)}\left(\hat{X}_k\right)
\end{aligned}$$

In (221) $\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}$ depends implicitely on the J_k through the first order condition that defines the saddle point:

$$\frac{\delta S\left(\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)}{\delta\left(\Psi_0^{(k)}\right)^\dagger\left(\hat{X}_k\right)}+J_k\hat{X}_k\Psi_0^{(k)}\left(\hat{X}_k\right)=0 \qquad (222)$$

The first order condition (222) can be used to compute the J dependency of the two first terms in the right hand side of (221). Actually:

$$\begin{aligned}
&\frac{\partial}{\partial J_k}\left(S\left(\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)+\sum_k J_k\int\Psi_0^{(k)\dagger}\left(\hat{X}_k\right)\hat{X}_k\Psi_0^{(k)}\left(\hat{X}_k\right)d\hat{X}_k\right)\\
=\;&\left(\frac{\delta S\left(\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)}{\delta\left(\Psi_0^{(k)}\right)^\dagger\left(\hat{X}_k\right)}+J_k\hat{X}_k\Psi_0^{(k)}\left(\hat{X}_k\right)\right)\frac{\partial\Psi_0^{(k)}\left(\hat{X}_k\right)}{\partial J_k}\\
&+\int\Psi_0^{(k)\dagger}\left(\hat{X}_k\right)\hat{X}_k\Psi_0^{(k)}\left(\hat{X}_k\right)d\hat{X}_k\\
=\;&\int\Psi_0^{(k)\dagger}\left(\hat{X}_k\right)\hat{X}_k\Psi_0^{(k)}\left(\hat{X}_k\right)d\hat{X}_k
\end{aligned}$$

and (220) becomes at the second order approximation:

$$\begin{aligned}
&\left(\bar{X}(T)\right)_k \qquad (223)\\
=\;&\int\Psi_0^{(k)\dagger}\left(\hat{X}_k\right)\hat{X}_k\Psi_0^{(k)}\left(\hat{X}_k\right)d\hat{X}_k\\
&+\left\langle\int\delta\Psi^{(k)\dagger}\left(\hat{X}_k\right)\hat{X}_k\delta\Psi^{(k)}\left(\hat{X}_k\right)d\hat{X}_k\right\rangle_{\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}}\\
&+\left\langle\int\left(\delta\Psi^{(k_2)}\right)^\dagger\left(\hat{X}_{k_2}\right)\left(\frac{\partial}{\partial J_k}\left(\frac{\delta^2 S\left(\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)}{\delta\Psi_0^{(k_1)}\left(\hat{X}_{k_1}\right)\delta\left(\Psi_0^{(k_2)}\right)^\dagger\left(\hat{X}_{k_2}\right)}\right)\right)_{J_k=0}\delta\Psi^{(k_1)}\left(\hat{X}_{k_1}\right)d\hat{X}_{k_1}d\hat{X}_{k_2}\right\rangle_{\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}}
\end{aligned}$$

where we define for any field dependent quantity $A\left(\left\{\delta\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)$:

$$\begin{aligned}
&\left\langle A\left(\left\{\delta\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)\right\rangle_{\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}} \\
= &\int A\left(\left\{\delta\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right) \\
&\times\exp\left(-\sum_{k_1,k_2}\int\left(\delta\Psi^{(k_2)}\right)^{\dagger}\left(\hat{X}_{k_2}\right)\frac{\delta^2 S\left(\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)}{\delta\Psi_0^{(k_1)}\left(\hat{X}_{k_1}\right)\delta\left(\Psi_0^{(k_2)}\right)^{\dagger}\left(\hat{X}_{k_2}\right)}\delta\Psi^{(k_1)}\left(\hat{X}_{k_1}\right)d\hat{X}_{k_1}d\hat{X}_{k_2}\right)\mathcal{D}\left\{\Psi^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}
\end{aligned}$$

Equation (223) can be further simplified, since the quantity

$$\left(\frac{\partial}{\partial J_k}\left(\frac{\delta^2 S\left(\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)}{\delta\Psi_0^{(k_1)}\left(\hat{X}_{k_1}\right)\delta\left(\Psi_0^{(k_2)}\right)^{\dagger}\left(\hat{X}_{k_2}\right)}\right)\right)_{J_k=0}\tag{224}$$

can also be found by using again the first order condition (222). Actually, differentiating (222) with respect to J_k and then letting $J_k=0$ yields:

$$\frac{\delta^2 S\left(\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)}{\delta\left(\Psi_0^{(k)}\right)^{\dagger}\left(\hat{X}_k\right)\delta\left(\Psi_0^{(k)}\right)\left(\hat{X}_k\right)}\left(\frac{\partial\Psi_0^{(k)}\left(\hat{X}_k\right)}{\partial J_k}\right)_{J_k=0}+\hat{X}_k\Psi_0^{(k)}\left(\hat{X}_k\right)=0$$

which implies:

$$\left(\frac{\partial\Psi_0^{(k)}\left(\hat{X}_k\right)}{\partial J_k}\right)_{J_k=0}=-\left(\frac{\delta^2 S\left(\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)}{\delta\left(\Psi_0^{(k)}\right)^{\dagger}\left(\hat{X}_k\right)\delta\left(\Psi_0^{(k)}\right)\left(\hat{X}_k\right)}\right)^{-1}\hat{X}_k\Psi_0^{(k)}\left(\hat{X}_k\right)\tag{225}$$

so that (224) can be expressed as:

$$\begin{aligned}
&\left(\frac{\partial}{\partial J_k}\left(\frac{\delta^2 S\left(\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)}{\delta\Psi_0^{(k_1)}\left(\hat{X}_{k_1}\right)\delta\left(\Psi_0^{(k_2)}\right)^{\dagger}\left(\hat{X}_{k_2}\right)}\right)\right)_{J_k=0} \\
= &\left(\frac{\delta^3 S\left(\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)}{\delta\Psi_0^{(k_0)}\left(\hat{X}_{k_0}\right)\delta\Psi_0^{(k_1)}\left(\hat{X}_{k_1}\right)\delta\left(\Psi_0^{(k_2)}\right)^{\dagger}\left(\hat{X}_{k_2}\right)}\right)\left(\frac{\partial\Psi_0^{(k_0)}\left(\hat{X}_{k_0}\right)}{\partial J_k}\right)_{J_k=0} \\
= &-\left(\frac{\delta^3 S\left(\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)}{\delta\Psi_0^{(k_0)}\left(\hat{X}_{k_0}\right)\delta\Psi_0^{(k_1)}\left(\hat{X}_{k_1}\right)\delta\left(\Psi_0^{(k_2)}\right)^{\dagger}\left(\hat{X}_{k_2}\right)}\right)\left(\frac{\delta^2 S\left(\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)}{\delta\left(\Psi_0^{(k_0)}\right)^{\dagger}\left(\hat{X}_{k_0}\right)\delta\left(\Psi_0^{(k_0)}\right)\left(\hat{X}_{k_0}\right)}\right)^{-1}\hat{X}_{k_0}\Psi_0^{(k_0)}\left(\hat{X}_{k_0}\right)
\end{aligned}$$

From this relation, one can deduce that, in the quadratic approximation, i.e. if interactions terms in $S\left(\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}\right)$ are relatively low compared to the quadratic contributions, the last term in (223) can be neglected, and one is left with:

$$\left(\bar{X}\left(T\right)\right)_k=\int\Psi_0^{(k)\dagger}\left(\hat{X}_k\right)\hat{X}_k\Psi_0^{(k)}\left(\hat{X}_k\right)d\hat{X}_k+\left\langle\int\delta\Psi^{(k)\dagger}\left(\hat{X}_k\right)\hat{X}_k\delta\Psi^{(k)}\left(\hat{X}_k\right)d\hat{X}_k\right\rangle_{\left\{\Psi_0^{(k)}\left(\hat{X}_k\right)\right\}_{k=1\ldots M}}\tag{226}$$

In most cases the second term in (226) is centered around the equilibrium value $\left(\tilde{X}\right)_k$ (see (213)). The first term in (226) is the macro quantity $\left(\bar{X}\left(T\right)\right)_k$ evaluated in the phase defined by the state $\Psi_0^{(k_0)}$. In other words, the aggregated value $\left(\bar{X}\left(T\right)\right)_k$ depends on the phase of the environment. When there is no phase transition we get:

$$\left(\bar{X}\left(T\right)\right)_k = \left(\tilde{X}\right)_k$$

and the aggregated value matches with the micro equilibrium value. But when a phase transition occurs, we rather have:

$$\left(\bar{X}\left(T\right)\right)_k = \left(\tilde{X}\right)_k + \int \Psi_0^{(k)\dagger}\left(\hat{X}_k\right)\hat{X}_k\Psi_0^{(k)}\left(\hat{X}_k\right)d\hat{X}_k$$

and the system's interactions have moved the system to an other equilibrium.

If we were to consider the last contribution to (223), this term would represent a correction due to the fluctuations of the environment, that depend themselves on the phase of the system.

12 Conclusion

This work has investigated the dynamical patterns of a system with N heterogenous economic agents. For a small number of agents, relaxing the optimizing behavior for a probabilist description centered around the optimal path allows to deal with some otherwise untractable systems. The classical optimization solution can be retrieved, in some cases, as the average dynamics of our formalism. Moreover, this probabilitic treatment can conveniently describe the fluctuation patterns of agents' behaviors. The transition functions of the system are computed by path integrals. They describe the system as a random process, whose fluctuations are deviations from the classical path. For large N, collective behaviors are better studied by switching to a field formalism, as usually done in statistical physics. Techniques of perturbation expansion, non trivial vacuua and phase transitions yield some insights about the relevant quantities of the system. Some aggregate or effective structures absent in the initial micro description, may appear, and become relevant at the collective level. A phenomenon of emergence is thus possible.

Moreover, our formalism allows to interpret the influence of the dynamics of the system as a whole at the individual level. This approach presents some circular features. On the one hand, while resulting from the individual relations, the macro scale cannot be reduced to a sum of individual systems. On the other hand, individual behaviors are shaped by the environment.

Our work ends with a short inspection of the aggregation issue in our context. We show that some agregated quantities can be retrieved from the field formalism. We introduce a macro time scale that should allow to derive an approximate dynamics for the macro quantities, based on the field formalism. This extension is left for future researches.

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Appendix 1

We show that, as claimed in the first section, that our probabilistic definition of the agents behavior encompasses the usual optimization behavior in the limit of no uncertainty. For $\sigma_j^2 \to 0$ and then $\sigma_i^2 \to 0$, we aim at showing that (9)

$$\exp\left(U_{eff}\left(X_i\left(t\right)\right)\right) = \int \exp\left(\frac{U_t^{(i)}}{\sigma_i^2}\right)$$
$$\times \exp\left(\sum_k \sum_{j\neq i} \frac{\hat{U}_{eff}^{t_i}\left(X_j\left(t+k\right)\right)}{\sigma_j^2}\right) d\left\{X_j\left(t+k\right)\right\}_{j\neq i} d\left\{X_i\left(t+k\right)\right\}$$

is peaked around the classical optimization solution, where:

$$U_t^{(i)} = \sum_{n\geqslant 0} \beta^n u_{t+n}^{(i)} \left(X_i\left(t+n\right), \left(X_j\left(t+n-1\right)\right)_{j\neq i}\right)$$

is the intertemporal utility of agent j and

$$\hat{U}_{eff}^{t_i}\left(X_j\left(t\right), \left(X_k\left(t-1\right)\right)\right) = -\frac{1}{2}\left(X_j\left(t\right) - X_j\left[\left(X_k\left(t-1\right)\right)\right]\right)^t A_{jj}\left(X_j\left(t\right) - X_j\left[\left(X_k\left(t-1\right)\right)\right]\right)$$

with $X_j\left[\left(X_k\left(t-1\right)\right)\right]$ is the solution for X_j of

$$0 = \left(\frac{\partial}{\partial X_j\left(t\right)} U_{eff}^{t_i}\left(X_j\left(t\right), \left(X_k\left(t-1\right)\right)\right)\right)_{X_j(t)=X_j[(X_k(t-1))]}$$

for a given $\left(X_k\left(t-1\right)\right)$. The function $U_{eff}^{t_i}\left(X_j\left(t\right), \left(X_i\left(t-1\right)\right)\right)$ has been defined in the first section as the i-th truncated effective utility for agent j.

To do so, recall first that in the classical set up, agent i optimizes:

$$U_t^{(i)} = E_i^t \sum_{n\geqslant 0} \beta^n u_{t+n}^{(i)} \left(X_i\left(t+n\right), \left(X_j\left(t+n-1\right)\right)_{j\neq i}\right)$$

knowing the impact of $X_i\left(t\right)$ on $\left(X_j\left(t+n-1\right)\right)$. Then, agent i optimizes $U_t^{(i)}$, taking into account that the agents j about which agent i has the knowledge of their behavior, act by optimizing a certain utility function $U_{eff}^{(j)}\left(X_j\left(t\right), \left(X_i\left(t-1\right)\right)\right)$. Thus, the $\left(X_j\left(t+n-1\right)\right)_{j\neq i}$ are not independent variables, but depends on $X_i\left(t-1\right)$ through agent j first order condition:

$$\frac{\partial}{\partial X_j\left(t\right)} U_{eff}^{(j)}\left(X_j\left(t\right), \left(X_i\left(t-1\right)\right)\right) = 0 \tag{227}$$

The classical solution of optimization problem for agent i:

$$\frac{\partial}{\partial X_i\left(t\right)} U_t^{(i)} = \frac{\partial}{\partial X_i\left(t\right)} E_i^t \sum_{n\geqslant 0} \beta^n u_{t+n}^{(i)} \left(X_i\left(t+n\right), \left(X_j\left(t+n-1\right)\right)_{j\neq i}\right) = 0$$

becomes, using (227):

$$\begin{aligned} 0 &= \frac{\partial}{\partial X_i\left(t\right)} u_t^{(i)} \left(X_i\left(t\right), \left(X_j\left(t-1\right)\right)_{j\neq i}\right) \\ &\quad + E_i^t \sum_{n\geqslant 2} \sum_j \frac{\partial X_j\left(t+n-1\right)}{\partial X_i\left(t\right)} \beta^n \frac{\partial}{\partial X_j\left(t+n-1\right)} u_{t+n}^{(i)} \left(X_i\left(t+n\right), \left(X_j\left(t+n-1\right)\right)_{j\neq i}\right) \end{aligned} \tag{228}$$

and the $X_j\left(t+n-1\right)$ satisfy:

$$\begin{aligned} 0 &= E_i^t \frac{\partial}{\partial X_j\left(t\right)} U_{eff}^{(j)}\left(X_j\left(t\right), \left(X_i\left(t-1\right)\right)\right) \\ &= \frac{\partial}{\partial X_j\left(t\right)} U_{eff}^{t_i}\left(X_j\left(t\right), \left(X_i\left(t-1\right)\right)\right) \end{aligned} \tag{229}$$

One can find $\left(\frac{\partial X_j(t)}{\partial X_k(t-1)}\right)$ from this relation by differentiation:

$$\begin{aligned}0 &= \sum_k \left(\frac{\partial X_j(t)}{\partial X_k(t-1)}\right)\frac{\partial}{\partial X_k(t-1)}\frac{\partial}{\partial X_j(t)}\left(U_{eff}^{(j)}\left(X_j(t),\left(X_i(t-1)\right)\right)\right)\\ &\quad+\frac{\partial}{\partial X_j(t)}\frac{\partial}{\partial X_j(t)}\left(U_{eff}^{(j)}\left(X_j(t),\left(X_i(t-1)\right)\right)\right)\end{aligned}$$

which yields:

$$\begin{aligned}\left(\frac{\partial X_j(t)}{\partial X_k(t-1)}\right) &= -\sum_k\left(\frac{\partial^2}{\partial X_k(t-1)\,\partial X_j(t)}\left(U_{eff}^{(j)}\left(X_j(t),\left(X_i(t-1)\right)\right)\right)\right)^{-1}\\ &\quad\times\frac{\partial^2}{\partial X_j(t)\,\partial X_j(t)}\left(U_{eff}^{(j)}\left(X_j(t),\left(X_i(t-1)\right)\right)\right)\end{aligned}$$

and $\frac{\partial X_j(t+n)}{\partial X_i(t)}$ is found recursively:

$$\frac{\partial X_j(t+n)}{\partial X_i(t)} = \sum_{l\neq i}\frac{\partial X_j(t+n)}{\partial X_l(t+n-1)}\frac{\partial X_l(t+n-1)}{\partial X_i(t)}$$

the sum is for $l \neq i$ since the $X_i(t)$, $X_i(t')$ are independent variables on which agent i optimizes.

Now, we show that we recover these optimization equations when the uncertainty in our description goes to 0. In the weight:

$$\begin{aligned}\exp\left(U_{eff}\left(X_i(t)\right)\right) &= \int \exp\left(\frac{U_t^{(i)}}{\sigma_i^2}\right)\exp \\ &\quad\times\exp\left(\sum_k\sum_{j\neq i}\frac{\hat{U}_{eff}^{t_i}\left(X_j(t+k)\right)}{\sigma_j^2}\right)d\left\{X_j(t+k)\right\}_{j\neq i}d\left\{X_i(t+k)\right\}\end{aligned} \tag{230}$$

$$\exp\left(U_{eff}\left(X_i(t)\right)\right) = \int \exp\left(\frac{U_t^{(i)}}{\sigma_i^2}\right)\times\exp\left(\sum_k\sum_{j\neq i}\frac{\hat{U}_{eff}^{t_i}\left(X_j(t+k)\right)}{\sigma_j^2}\right)d\left\{X_j(t+k)\right\}_{j\neq i}d\left\{X_i(t+k)\right\}$$

Given that $\hat{U}_{eff}^{t_i}\left(X_j(t+k)\right)$ are positive, for $\sigma_j^2 \to 0$, the path localizes around the maximum of $\hat{U}_{eff}^{t_i}$, solution of:

$$\hat{U}_{eff}^{t_i}\left(X_j(t+k)\right) = 0$$

so that $X_j(t+k)$ is set to $X_j\left[\left(X_k(t+k-1)\right)\right]$ which is solution of the saddle point equation for $U_{eff}^{t_i}$.

$$0 = \left(\frac{\partial}{\partial X_j(t+k)}U_{eff}^{t_i}\left(X_j(t+k),\left(X_k(t+k-1)\right)\right)\right)_{X_j(t)=X_j\left[\left(X_k(t-1)\right)\right]}$$

That is the value of $X_j(t)$ that are solutions of:

$$\frac{\partial}{\partial X_j(s)}U_{eff}^{t_i}\left(X_j(s),\left(X_i(s-1)\right)\right) = 0 \text{ for } s \geqslant t$$

Solving for the $X_j(s)$, $j \neq i$ allows to express recursively all the $X_j(s)$, $j \neq i$ as functions of $X_i(t)$, $X_i(s)$, $s > t$ and $X_j(t-1)$, $j \neq i$, then, the integrations reduce to a sequence of integrals on the $X_i(s)$, $s > t$. Ultimately, for $\sigma_i^2 \to 0$, the path localizes around the solutions of:

$$0 = \frac{\partial}{\partial X_i(t+k)}U_t^{(i)} \text{ for } k \geqslant 0$$

where the $X_j(s)$, $j \neq i$ for $s \geqslant t$ have been replaced as functions of $X_i(t)$, $X_i(s)$, $s > t$ and $X_j(t-1)$, $j \neq i$, which yields for $k \geqslant 0$:

$$\begin{aligned} 0 &= \frac{\partial}{\partial X_i(t+k)} u_t^{(i)} \left(X_i(t+k), (X_j(t-1))_{j \neq i} \right) \\ &+ \sum_{n \geqslant k+2} \sum_j \frac{\partial X_j(t+n-1)}{\partial X_i(t+k)} \beta^n \frac{\partial}{\partial X_j(t+n-1)} u_{t+n}^{(i)} \left(X_i(t+n), (X_j(t+n-1))_{j \neq i} \right) \end{aligned}$$

This is the sequence of optimization equations, as planned by agent i at time t with $X_j(t+k)$ satifying

$$0 = \frac{\partial}{\partial X_j(t+k)} U_{eff}^{t_i} \left(X_j(t+k), (X_k(t+k-1)) \right) \text{ for } k \geqslant 0$$

as needed. As a consequence, the result is proved.

Note that for quadratic utilities:

$$\left(\frac{\partial X_j(t)}{\partial X_k(t-1)} \right) = (A_{jj})^{-1} A_{jk}$$

and

$$\begin{aligned} U_{eff}(X_j(t), (X_i(t-1))) &= -\frac{1}{2} \left(X_j(t) + (A_{jj})^{-1} A_{jk} (X_k(t-1)) \right)^t A_{jj} \left(X_j(t) + (A_{jj})^{-1} A_{jk} (X_k(t-1)) \right) \\ &= \hat{U}_{eff}(X_j(t), (X_i(t-1))) \end{aligned}$$

and the result rewrites as:

$$\begin{aligned} &\exp(U_{eff}(X_i(t))) = \int \exp\left(\frac{U_t^{(i)}}{\sigma_i^2} \right) \\ &\times \exp\left(\sum_k \sum_{j \neq i} \frac{U_{eff}^{t_i}(X_j(t+k))}{\sigma_j^2} \right) d\{X_j(t+k)\}_{j \neq i} \, d\{X_i(t+k)\} \end{aligned}$$

which peaks on the optimization solution for $\sigma_j^2 \to 0$ and then $\sigma_i^2 \to 0$, as claimed in section 1.

Appendix 2

As recorded in the text, we rewrite the utilities in terms of the variables $Y_i(t)$

$$
\begin{aligned}
U_t^{(i)} &= \sum_t \beta^t \left(\sum_{j<i} \begin{pmatrix} X_i(t) A_{ii}^{(i)} X_i(t) + \left(\left(X_j(t-1) - \bar{X}_j^{(i)} \right) A_{jj}^{(i)} \left(X_j(t-1) - \bar{X}_j^{(i)} \right) \right) \\ +2X_i(t) A_{ij}^{(i)} \left(X_j(t-1) \right) - \bar{X}_j^{(i)} \end{pmatrix} \right. \\
&\quad \left. + \sum_{j>i} 2X_i(t) A_{ij}^{(i)} \left(X_j(t-1) \right) \right) \\
&= \sum_t Y_i(t) \begin{pmatrix} A_{ii}^{(i)} & 0 \\ 0 & 0 \end{pmatrix} Y_i(t) + Y_i(t-1) \begin{pmatrix} 0 & 0 \\ 0 & \beta A_{\{jj\}}^{(i)} \end{pmatrix} Y_i(t-1) \\
&\quad + Y_i(t) \begin{pmatrix} 0 & \beta^{\frac{1}{2}} A_{ij}^{(i)} \\ \beta^{\frac{1}{2}} A_{ji}^{(i)} & 0 \end{pmatrix} Y_i(t-1) + \sum_{j>i} 2X_i(t) A_{ij}^{(i)} \left(X_j(t-1) \right)
\end{aligned}
$$

expected utilty at t.We will also add possibility for an inertia term:

$$
-X_i(t)\, \epsilon_{ii}^{(i)} X_i(t-1)
$$

Each agent j behaves at time t with a so called effective utility $U_{eff}(X_j(t)) \equiv U_{eff}(X_j)$ whose recursive form for the non normalized $U_{eff}(X_j)$ is assumed to be:

$$
\begin{aligned}
U_{eff}(Y_j(s)) &= Y_j^{(e)}(s) \begin{pmatrix} \left(A_{jj}^{(j)} \right)_{eff} & 0 \\ 0 & 0 \end{pmatrix} Y_j^{(e)}(s) - 2Y_j^{(e)}(s) \begin{pmatrix} \left(\epsilon_{jj}^{(j)} \right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)} \right)_{eff} \\ 0 & 0 \end{pmatrix} Y_j^{(e)}(s-1) \\
&\quad + \sum_{i \geqslant k > j} 2X_j(t) A_{jk}^{(j)} \left(X_k(t-1) \right)
\end{aligned}
$$

where $Y_j^{(e)}$ has been defined in (34):

$$
Y_j^{(e)}(t+k) = \left(\beta^{\frac{k}{2}} \left(X_k(t+k) - \bar{X}_k^{(j)e} \right)_{k \leq j} \right)
$$

The normalization of $\exp\left(U_{eff}(Y_j(t)) \right)$ is obtained by letting (we omit temporarily the upperscript (e)):

$$
C \int \exp\left(U_{eff}(Y_j(t)) \right) \left(d\left(Y_j(t) \right) \right) = 1
$$

writing:

$$
\begin{aligned}
U_{eff}(Y_j(t)) &= \left(Y_j^{(e)}(t) + \left(A_{jj}^{(j)} \right)_{eff}^{-1} \left(\sum_{i \geqslant k > j} A_{jk}^{(j)} X_k(t-1) - \begin{pmatrix} \left(\epsilon_{jj}^{(j)} \right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)} \right)_{eff} \end{pmatrix} Y_j(t-1) \right) \right)^t \left(A_{jj}^{(j)} \right)_{eff} \\
&\quad \times \left(Y_j^{(e)}(t) + \left(A_{jj}^{(j)} \right)_{eff}^{-1} \left(\sum_{i \geqslant k > j} A_{jk}^{(j)} X_k(t-1) - \begin{pmatrix} \left(\epsilon_{jj}^{(j)} \right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)} \right)_{eff} \end{pmatrix} Y_j(t-1) \right) \right) \\
&\quad - \left(\sum_{i \geqslant k > j} A_{jk}^{(j)} X_k(t-1) - \begin{pmatrix} \left(\epsilon_{jj}^{(j)} \right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)} \right)_{eff} \end{pmatrix} Y_j(t-1) \right)^t \\
&\quad \times \left(A_{jj}^{(j)} \right)_{eff}^{-1} \left(\sum_{i \geqslant k > j} A_{jk}^{(j)} X_k(t-1) - \begin{pmatrix} \left(\epsilon_{jj}^{(j)} \right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)} \right)_{eff} \end{pmatrix} Y_j(t-1) \right)
\end{aligned}
$$

yields the normalization factor (introducing again the upperscript (e)):

$$\frac{1}{\mathcal{N}} = \exp\left(\sum_{i\geqslant k>j} A_{jk}^{(j)} X_k(t-1) + \left(\begin{array}{cc} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{array} \right) Y_j^{(e)}(t-1)\right)^t$$
$$\times \left(A_{jj}^{(j)}\right)_{eff}^{-1} \left(\sum_{i\geqslant k>j} A_{jk}^{(j)} X_k(t-1) + \left(\begin{array}{cc} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{array} \right) Y_j^{(e)}(t-1)\right)$$

and the normalized effective utility becomes:

$$U_{eff}^{(n)}(Y_j(t)) = \left(Y_j^{(e)}(t) + \left(A_{jj}^{(j)}\right)_{eff}^{-1} \left(\sum_{i\geqslant k>j} A_{jk}^{(j)} X_k(t-1) - \left(\begin{array}{cc} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{array} \right) Y_j(t-1)\right)\right)^t \left(A_{jj}^{(j)}\right)_{eff}$$
$$\times \left(Y_j^{(e)}(t) + \left(A_{jj}^{(j)}\right)_{eff}^{-1} \left(\sum_{i\geqslant k>j} A_{jk}^{(j)} X_k(t-1) - \left(\begin{array}{cc} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{array} \right) Y_j(t-1)\right)\right)$$

Given the definition of $Y_j^{(e)}(s)$ one can concatenate all the vectors $Y_j^{(e)}(s)$ for $i<j$ to form a vector $\left(Y_j^{(e)}(s)\right)_{j<i}$ and given the definition of $Y_i(s)$ one can write:

$$\left(Y_j^{(e)}(s)\right)_{j<i} = (Y_i(s))_{j<i} + \beta^{s-t}\left(\bar{X}_j^{(i)} - \bar{X}_j^{(j)e}\right)_{j<i}$$

where the subscript $j<i$ means that we only concatenate the component vectors of $Y_i(s)$ for $j<i$. This is $Y_i(s)$ without its component along i. Concatenate this vector with $(Y_i(s))_i$, that is adding the component along i one obtains a composed vector:

$$\hat{Y}_i(s) = \left((Y_i(s))_i, \left(Y_j^{(e)}(s)\right)_{j<i}\right)$$

We will also need to define:

$$\tilde{Y}_i(s) = \left((Y_i(s))_i, (Y_i(s))_{j<i} + \beta^{s-t}\left(\bar{X}_j^{(i)}\right)_{j<i}\right)$$

The normalization factor has to be added to the global weight (i.e. the normalized effective utility) to be taken into account for agent i is then (in the sequel, the sum over $j<i$ is always understood):

$$U_{eff}(Y_i(t)) = \sum_{t>0}\beta^t\left(\sum_{j<i}\left(\begin{array}{c} X_i(t) A_{ii}^{(i)} X_i(t) - X_i(t)\epsilon_{ii}^{(i)} X_i(t-1) + X_j(t-1) A_{jj}^{(i)} X_j(t-1) \\ +2X_i(t) A_{ij}^{(i)}(X_j(t-1)) \end{array} \right) + U_{eff}^{(n)}(Y_j)\right)$$
$$+\sum_{j>i} 2X_i(t) A_{ij}^{(i)}(X_j(t-1))$$
$$= \sum_{t>0} Y_i(t)\left(\begin{array}{cc} A_{ii}^{(i)} & 0 \\ 0 & \beta A_{jj}^{(i)} \end{array} \right) Y_i(t) + \beta^{\frac{1}{2}} Y_i(t)\left(\begin{array}{cc} -\epsilon_{ii}^{(i)} & 2A_{ij}^{(i)} \\ 0 & 0 \end{array} \right) Y_i(t-1)$$
$$+\sum_{t>0} Y_j^{(e)}(t)\left(\begin{array}{cc} 0 & 0 \\ 0 & \left(A_{jj}^{(j)}\right)_{eff} \end{array} \right) Y_j^{(e)}(t) + \beta^{\frac{1}{2}} Y_j^{(e)}(t)\left(\begin{array}{cc} 0 & 0 \\ 0 & \left\{\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}\right\} \end{array} \right) Y_j^{(e)}(t-1)$$
$$+\beta_i^{\frac{1}{2}} \tilde{Y}_i(t)\left(\begin{array}{cc} 0 & 0 \\ 2A_{ji}^{(j)} & \left\{2A_{\{kj\}i>k>j}^{(j)}\right\} \end{array} \right) \tilde{Y}_i(t-1) + \sum_{j>i} 2X_i(t) A_{ij}^{(i)}(X_j(t-1))$$
$$+\tilde{Y}_i(t)\left(\begin{array}{cc} B_{11} & B'_{12} \\ (B'_{12})^t & B'_{22} \end{array} \right) \tilde{Y}_i(t) + Y_j^{(e)}(t)\left(\begin{array}{cc} 0 & 0 \\ 0 & B"_{22} \end{array} \right) Y_j^{(e)}(t) + \tilde{Y}_i(t)\left(\begin{array}{cc} 0 & B_{12}^{(3)} \\ \left(B_{12}^{(3)}\right)^t & B_{22}^{(3)} \end{array} \right) Y_j^{(e)}(t)$$

where, by convention $Y_j^{(e)}(t)$ has been extended with a nul component in the coordinate i, that is: $Y_j^{(e)}(t) \rightarrow \begin{pmatrix} 0 \\ Y_j^{(e)}(t) \end{pmatrix}$. Then, $U_{eff}(Y_i(t))$ can be written:

$$\begin{aligned} U_{eff}(Y_i(t)) &= \sum_{t>0} Y_i(t) \begin{pmatrix} A_{ii}^{(i)} & 0 \\ 0 & \beta A_{jj}^{(i)} \end{pmatrix} Y_i(t) + \sum_{t>0} Y_j^{(e)}(t) \begin{pmatrix} 0 & 0 \\ 0 & \left(A_{jj}^{(j)}\right)_{eff} + B"_{22} \end{pmatrix} Y_j^{(e)}(t) \\ &+ \beta_i^{\frac{1}{2}} \tilde{Y}_i(t) \begin{pmatrix} B_{11} & B'_{12} \\ (B'_{12})^t & B'_{22} \end{pmatrix} \tilde{Y}_i(t) + \tilde{Y}_i(t) \begin{pmatrix} 0 & B_{12}^{(3)} \\ \left(B_{12}^{(3)}\right)^t & B_{22}^{(3)} \end{pmatrix} Y_j^{(e)}(t) \\ &+ \beta^{\frac{1}{2}} Y_i(t) \begin{pmatrix} -\epsilon_{ii}^{(i)} & 2A_{ij}^{(i)} \\ 0 & 0 \end{pmatrix} Y_i(t-1) + \beta_i^{\frac{1}{2}} \tilde{Y}_i(t) \begin{pmatrix} 0 & 0 \\ 2A_{ji}^{(j)} & \left\{2A_{\{kj\}i>k>j}^{(j)}\right\} \end{pmatrix} \tilde{Y}_i(t-1) \\ &+ \beta^{\frac{1}{2}} Y_j^{(e)}(t) \begin{pmatrix} 0 & 0 \\ 0 & \left\{\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}\right\} \end{pmatrix} Y_j^{(e)}(t-1) + \sum_{j>i} 2X_i(t) A_{ij}^{(i)} (X_j(t-1)) \end{aligned} \tag{231}$$

We aim at writing $U_{eff}(Y_i(t))$ under the form:

$$\begin{aligned} U_{eff}(Y_i(t)) &= \sum_{t>0} Y_i^{(e)}(t) \begin{pmatrix} A_{ii}^{(i)} + B_{11} & B_{12} \\ B_{12}^t & \left\{\left(A_{jj}^{(j)}\right)_{eff} + \beta A_{jj}^{(i)}, B_{22}\right\} \end{pmatrix} Y_i^{(e)}(t) \\ &+ 2\beta^{\frac{1}{2}} Y_i^{(e)}(t) \begin{pmatrix} -\epsilon_{ii}^{(i)} & A_{ij}^{(i)} \\ A_{ji}^{(j)} & \left\{\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}, A_{\{kj\}i>k>j}^{(j)}\right\} \end{pmatrix} Y_i^{(e)}(t-1) \\ &+ \sum_{j>i} 2X_i(t) A_{ij}^{(i)} (X_j(t-1)) \end{aligned} \tag{232}$$

where:

$$\begin{aligned} B_{11} &= \beta A_{ij}^{(j)} \left(A_{jj}^{(j)}\right)_{eff}^{-1} A_{ji}^{(j)} \\ B_{12} &= \left\{\beta A_{ij}^{(j)} \left(A_{jj}^{(j)}\right)_{eff}^{-1} A_{jk}^{(j)}, \beta\left(A_{ij}^{(j)} \left(A_{jj}^{(j)}\right)_{eff}^{-1} \begin{pmatrix} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{pmatrix}\right)\right\} \\ B_{22} &= \left\{ \begin{array}{c} \beta A_{lj}^{(j)} \left(A_{jj}^{(j)}\right)_{eff}^{-1} A_{jk}^{(j)}, \\ \beta \begin{pmatrix} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{pmatrix}^t \left(A_{jj}^{(j)}\right)_{eff}^{-1} \begin{pmatrix} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{pmatrix} \\ \beta\left(A_{kj}^{(j)} \left(A_{jj}^{(j)}\right)_{eff}^{-1} \begin{pmatrix} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{pmatrix}\right)^S \end{array} \right\} \end{aligned} \tag{233}$$

$$\begin{aligned} B'_{12} &= \beta A_{ij}^{(j)} \left(A_{jj}^{(j)}\right)_{eff}^{-1} A_{jk}^{(j)} \\ B'_{22} &= \beta A_{lj}^{(j)} \left(A_{jj}^{(j)}\right)_{eff}^{-1} A_{jk}^{(j)} \\ B"_{22} &= \beta \begin{pmatrix} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{pmatrix}^t \left(A_{jj}^{(j)}\right)_{eff}^{-1} \begin{pmatrix} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{pmatrix} \\ B_{12}^{(3)} &= \beta\left(A_{ij}^{(j)} \left(A_{jj}^{(j)}\right)_{eff}^{-1} \begin{pmatrix} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{pmatrix}\right) \\ B_{22}^{(3)} &= \beta\left(A_{kj}^{(j)} \left(A_{jj}^{(j)}\right)_{eff}^{-1} \begin{pmatrix} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{pmatrix}\right)^S \end{aligned} \tag{234}$$

with $M^S = \frac{1}{2}(M + M^t)$ for any matrix M, and where we have defined:

$$Y_i^{(e)}(s) = \beta^{\frac{s-t}{2}} \left(X_j(s) - \left(\bar{X}_j^{(i)e} \right) \right)_{j \leqslant i}$$

For a vector $\left(\bar{X}_j^{(i)e}\right)$ to be determined. Given the form of (232), is the equilibrium value of (231) when $X_j(t-1) = 0$ for $j > i$. Thus, $\bar{X}_j^{(i)e}$ is found as the solution of the first order condition $\frac{\partial}{\partial Y_i(t)} U_{eff}(Y_i(t)) = 0$ when $X_j(t-1) = 0$ for $j > i$. This equation yields:

$$\begin{aligned}
&\begin{pmatrix} A_{ii}^{(i)} & 0 \\ 0 & \beta A_{jj}^{(i)} \end{pmatrix} \left(\bar{X}_j^{(i)e} - \bar{X}_j^{(i)} \right) + \begin{pmatrix} 0 & 0 \\ 0 & \left(A_{jj}^{(j)}\right)_{eff} + B"_{22} \end{pmatrix} \left(\bar{X}_j^{(i)e} - \begin{pmatrix} 0 \\ \left(\bar{X}_j^{(j)e}\right) \end{pmatrix} \right) \\
&+ \beta_i^{\frac{1}{2}} \begin{pmatrix} B_{11} & B'_{12} \\ (B'_{12})^t & B'_{22} \end{pmatrix} \bar{X}_j^{(i)e} + \frac{1}{2} \begin{pmatrix} 0 & B_{12}^{(3)} \\ \left(B_{12}^{(3)}\right)^t & B_{22}^{(3)} \end{pmatrix}^S \bar{X}_j^{(i)e} \\
&+ \frac{1}{2} \begin{pmatrix} 0 & B_{12}^{(3)} \\ \left(B_{12}^{(3)}\right)^t & B_{22}^{(3)} \end{pmatrix}^S \left(\bar{X}_j^{(i)e} - \begin{pmatrix} 0 \\ \left(\bar{X}_j^{(j)e}\right) \end{pmatrix} \right) \\
&+ \beta^{\frac{1}{2}} \begin{pmatrix} -\epsilon_{ii}^{(i)} & 2A_{ij}^{(i)} \\ 0 & 0 \end{pmatrix}^S \left(\bar{X}_j^{(i)e} - \bar{X}_j^{(i)} \right) + \beta_i^{\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 2A_{ji}^{(j)} & \left\{ 2A_{\{kj\} i > k > j}^{(j)} \right\} \end{pmatrix}^S \bar{X}_j^{(i)e} \\
&+ \beta^{\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 0 & \left\{ \left(\epsilon_{\{kj\} k \leqslant j}^{(j)} \right)_{eff} \right\} \end{pmatrix}^S \left(\bar{X}_j^{(i)e} - \begin{pmatrix} 0 \\ \left(\bar{X}_j^{(j)e}\right) \end{pmatrix} \right) = 0
\end{aligned}$$

The constant terms in this equation are

$$\begin{aligned}
&\begin{pmatrix} A_{ii}^{(i)} & 0 \\ 0 & \beta A_{jj}^{(i)} \end{pmatrix} \bar{X}_j^{(i)} + \begin{pmatrix} 0 & 0 \\ 0 & \left(A_{jj}^{(j)}\right)_{eff} + B"_{22} \end{pmatrix} \begin{pmatrix} 0 \\ \left(\bar{X}_j^{(j)e}\right) \end{pmatrix} \\
&+ \frac{1}{2} \begin{pmatrix} 0 & B_{12}^{(3)} \\ \left(B_{12}^{(3)}\right)^t & B_{22}^{(3)} \end{pmatrix}^S \begin{pmatrix} 0 \\ \left(\bar{X}_j^{(j)e}\right) \end{pmatrix} \\
&+ \beta^{\frac{1}{2}} \begin{pmatrix} -\epsilon_{ii}^{(i)} & 2A_{ij}^{(i)} \\ 0 & 0 \end{pmatrix}^S \bar{X}_j^{(i)} + \beta^{\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 0 & \left\{ \left(\epsilon_{\{kj\} k \leqslant j}^{(j)} \right)_{eff} \right\} \end{pmatrix}^S \begin{pmatrix} 0 \\ \left(\bar{X}_j^{(j)e}\right) \end{pmatrix}
\end{aligned}$$

and the equation for $\bar{X}_j^{(i)e}$ becomes:

$$\begin{aligned}
&\left(\begin{pmatrix} A_{ii}^{(i)} + B_{11} & B_{12} \\ B_{12}^t & \left\{ \left(A_{jj}^{(j)}\right)_{eff} + \beta A_{jj}^{(i)}, B_{22} \right\} \end{pmatrix} \right. \\
&\left. + \beta^{\frac{1}{2}} \begin{pmatrix} -2\epsilon_{ii}^{(i)} & A_{ij}^{(i)} + A_{ij}^{(j)} \\ A_{ji}^{(j)} + A_{ji}^{(i)} & 2 \left(\left\{ \left(\epsilon_{\{kj\} k \leqslant j}^{(j)} \right)_{eff}, A_{\{kj\} i > k > j}^{(j)} \right\} \right)^S \end{pmatrix} \right) \left(\bar{X}_j^{(i)e} \right) \\
= \; &\left(\begin{pmatrix} A_{ii}^{(i)} & 0 \\ 0 & \beta A_{jj}^{(i)} \end{pmatrix} + \beta^{\frac{1}{2}} \begin{pmatrix} -\epsilon_{ii}^{(i)} & A_{ij}^{(i)} \\ A_{ji}^{(i)} & 0 \end{pmatrix} \right) \left(\bar{X}_j^{(i)} \right) \\
&+ \left(\begin{pmatrix} 0 & 0 \\ 0 & \left(A_{jj}^{(j)}\right)_{eff} + B"_{22} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & B_{12}^{(3)} \\ \left(B_{12}^{(3)}\right)^t & B_{22}^{(3)} \end{pmatrix} + \beta^{\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ 0 & \left(\left\{ \left(\epsilon_{\{kj\} k \leqslant j}^{(j)} \right)_{eff} \right\} \right)^S \end{pmatrix} \right) \begin{pmatrix} 0 \\ \left(\bar{X}_j^{(j)e}\right) \end{pmatrix}
\end{aligned}$$

with solution:

$$\bar{X}_j^{(i)e} = \begin{pmatrix} A_{ii}^{(i)} + B_{11} - \sqrt{\beta}\epsilon_{ii}^{(i)} & \left\{B_{12}, 2\sqrt{\beta}\left(A_{ij}^{(i)}\right)^S\right\} \\ \left\{B_{12}^t, 2\sqrt{\beta}\left(A_{ji}^{(j)}\right)^S\right\} & \left\{\begin{array}{c} \left(A_{jj}^{(j)}\right)_{eff} + \beta A_{jj}^{(i)}, B_{22}, \\ \left(2\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}, 2A_{\{kj\}i>k>j}^{(j)}\right)^S \end{array}\right\} \end{pmatrix}^{-1} \tag{235}$$
$$\times\left\{\begin{pmatrix} A_{ii}^{(i)} - \sqrt{\beta}\frac{\epsilon_{ii}^{(i)}}{2} & \left\{\frac{B_{12}^{(3)}}{2}, \sqrt{\beta}A_{ij}^{(i)}\right\} \\ \frac{\left(B_{12}^{(3)}\right)^t}{2} & \beta A_{jj}^{(i)} + \frac{B_{22}^{(3)}}{2} \end{pmatrix}\left(\bar{X}_j^{(i)}\right)\right.$$
$$\left.+\begin{pmatrix} 0 & \frac{B_{12}^{(3)}}{2} \\ \frac{\left(B_{12}^{(3)}\right)^t}{2} & \left\{\left(A_{jj}^{(j)}\right)_{eff}, B"_{22}, \frac{B_{22}^{(3)}}{2}, \sqrt{\beta}\left(\frac{\epsilon_{\{kj\}k\leqslant j}^{(j)}}{2}\right)_{eff}\right\} \end{pmatrix}\begin{pmatrix} 0 \\ \left(\bar{X}_j^{(j)e}\right) \end{pmatrix}\right\}$$

Including the terms $X_i(t)A_{ii}^{(i)}X_i(t)$, $X_i(t)A_{ij}^{(i)}(X_j(t-1))$ and $U_{eff}(Y_j)$ at t. Using $Y_j(t-1) \hookrightarrow Y_i(t-1)$, by extension of notation $\left(\hat{Y}_j\right)_{eff} \hookrightarrow \left(0, ..., \left(\hat{Y}_j\right)_{eff}, ...0\right)$ in the sum

$$\begin{aligned}
&\sum_{t\geqslant 0}\beta^t\left(\sum_{j<i}\left(X_i(t)A_{ii}^{(i)}X_i(t) - X_i(t)\epsilon_{ii}^{(i)}X_i(t-1) + X_j(t-1)A_{jj}^{(i)}X_j(t-1) + 2X_i(t)A_{ij}^{(i)}(X_j(t-1))\right) + U_{eff}(Y_j)\right) \\
&+\sum_{j>i}2X_i(t)A_{ij}^{(i)}(X_j(t-1)) \\
=\ &\sum_{t\geqslant 0}Y_i(t)\begin{pmatrix} A_{ii}^{(i)} + B_{11} & B_{12} \\ B_{12}^t & \left\{\left(A_{jj}^{(j)}\right)_{eff} + \beta A_{jj}^{(i)}, B_{22}\right\} \end{pmatrix}Y_i(t) \\
&+\beta^{\frac{1}{2}}Y_i(t)\begin{pmatrix} -\epsilon_{ii}^{(i)} & 2A_{ij}^{(i)} \\ 2A_{ji}^{(j)} & \left\{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}, 2A_{\{kj\}i>k>j}^{(j)}\right\} \end{pmatrix}Y_i(t-1) \\
&-\sum_{t>0}2Y_i(t-1).\left(\hat{Y}_j\right)_{eff} + \sum_{j>i}2X_i(t)A_{ij}^{(i)}(X_j(t-1)) \\
&+\sum_{t>0}\left(\sum_{i\geqslant k>j}A_{jk}^{(j)}X_k(t) - \begin{pmatrix} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{kj\}k<j}^{(j)}\right)_{eff} \end{pmatrix}Y_j(t-1)\right)^t \\
&\times\left(A_{jj}^{(j)}\right)_{eff}^{-1}\left(\sum_{i\geqslant k>j}A_{jk}^{(j)}X_k(t) - \begin{pmatrix} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{kj\}k<j}^{(j)}\right)_{eff} \end{pmatrix}Y_j(t-1)\right)
\end{aligned} \tag{236}$$

The second lower part of $Y_i(t)$ includes all substructures of $X_i(t)$. Then $A_{\{jj\}}^{(i)}$ (written latter as $A_{jj}^{(i)}$ for the sake of implicity) is a Block matrix including all interaction between j and k for j and $k<i$.

$\left(A_{\{jj\}}^{(i)} + \left(A_{jj}^{(j)}\right)_{eff}, \left(A_{\{kk\}k<j}^{(j)}\right)_{eff}\right)$ matrix obtained by letting $A_{\{jj\}}^{(i)} + \left(A_{jj}^{(j)}\right)_{eff}$ in place (j,j) and $\left(A_{\{kk\}k<j}^{(j)}\right)_{eff}$ in place (k,k). The bracket denotes this operation for the all collection of j substrctrs. Same operation for $\left\{\left(\left(A_{\{jk\}k<j}^{(j)}, A_{\{kj\}k<j}^{(j)}\right)_{eff}, A_{\{kj\}i>k>j}^{(j)}\right)\right\}$.

Define also

$$\begin{aligned} A_{ji}^{(i)} &= \left(A_{ij}^{(i)}\right)^t \\ \epsilon_{\{jk\}j\geqslant k}^{(j)} &= \left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)^t \end{aligned}$$

and rewrite:

$$\begin{aligned} \begin{pmatrix} -\epsilon_{ii}^{(i)} & 2A_{ij}^{(i)} \\ 2A_{ji}^{(j)} & \left\{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}, 2A_{\{kj\}i>k>j}^{(j)}\right\} \end{pmatrix} &= \begin{pmatrix} -\epsilon_{ii}^{(i)} & A_{ij}^{(i)}+A_{ij}^{(j)} \\ A_{ji}^{(i)}+A_{ji}^{(j)} & \left\{ \begin{array}{c} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ A_{\{kj\}i>k>j}^{(j)}, A_{\{jk\}i>k>j}^{(j)} \end{array} \right\} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & A_{ij}^{(i)}-A_{ij}^{(j)} \\ -\left(A_{ji}^{(i)}-A_{ji}^{(j)}\right) & \left\{ \begin{array}{c} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ A_{\{kj\}i>k>j}^{(j)}, -A_{\{jk\}i>k>j}^{(j)} \end{array} \right\} \end{pmatrix} \end{aligned}$$

The two first terms in (236) can thus be rewritten as:

$$\begin{aligned} & Y_i(t) \begin{pmatrix} A_{ii}^{(i)}+B_{11} & B_{12} \\ B_{12}^t & \left\{\left(A_{jj}^{(j)}\right)_{eff}+\beta A_{jj}^{(i)}, B_{22}\right\} \end{pmatrix} Y_i(t) \\ & +\sqrt{\beta} Y_i(t) \begin{pmatrix} -\epsilon_{ii}^{(i)} & 2A_{ij}^{(i)} \\ 2A_{ji}^{(j)} & \left\{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}, 2A_{\{kj\}i>k>j}^{(j)}\right\} \end{pmatrix} Y_i(t-1) \\ = \; & Y_i(t) \begin{pmatrix} A_{ii}^{(i)}+B_{11} & B_{12} \\ B_{12}^t & \left\{\left(A_{jj}^{(j)}\right)_{eff}+\beta A_{jj}^{(i)}, B_{22}\right\} \end{pmatrix} Y_i(t) \\ & -\frac{\sqrt{\beta}}{2}\left(Y_i(t)-Y_i(t-1)\right) \begin{pmatrix} -\epsilon_{ii}^{(i)} & A_{ij}^{(i)}+A_{ij}^{(j)} \\ A_{ji}^{(i)}+A_{ji}^{(j)} & \left\{ \begin{array}{c} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ A_{\{kj\}i>k>j}^{(j)}, A_{\{jk\}i>k>j}^{(j)} \end{array} \right\} \end{pmatrix} \left(Y_i(t)-Y_i(t-1)\right) \\ & +\frac{\sqrt{\beta}}{2} Y_i(t) \begin{pmatrix} -\epsilon_{ii}^{(i)} & A_{ij}^{(i)}+A_{ij}^{(j)} \\ A_{ji}^{(i)}+A_{ji}^{(j)} & \left\{ \begin{array}{c} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ A_{\{kj\}i>k>j}^{(j)}, A_{\{jk\}i>k>j}^{(j)} \end{array} \right\} \end{pmatrix} Y_i(t) \\ & +\frac{\sqrt{\beta}}{2} Y_i(t-1) \begin{pmatrix} -\epsilon_{ii}^{(i)} & A_{ij}^{(i)}+A_{ij}^{(j)} \\ A_{ji}^{(i)}+A_{ji}^{(j)} & \left\{ \begin{array}{c} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ A_{\{kj\}i>k>j}^{(j)}, A_{\{jk\}i>k>j}^{(j)} \end{array} \right\} \end{pmatrix} Y_i(t-1) \\ & +\sqrt{\beta}\left(Y_i(t)-Y_i(t-1)\right) \begin{pmatrix} 0 & A_{ij}^{(i)}-A_{ij}^{(j)} \\ -\left(A_{ji}^{(i)}-A_{ji}^{(j)}\right) & \left\{ \begin{array}{c} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ A_{\{kj\}i>k>j}^{(j)}, -A_{\{jk\}i>k>j}^{(j)} \end{array} \right\} \end{pmatrix} Y_i(t) \end{aligned}$$

As a consequence, discarding the terms quadratic or linear in $Y_i(t-1)$ since they are absorbed in the normalization at time t, the sum in (236) starting from $t+1$ is then:

$$\begin{aligned}
& \sum_{s>t} Y_i(s) \begin{pmatrix} A_{ii}^{(i)} + B_{11} - \sqrt{\beta}\epsilon_{ii}^{(i)} & \left\{\sqrt{\beta}\left(A_{ij}^{(i)} + A_{ij}^{(j)}\right), B_{12}\right\} \\ \left\{\sqrt{\beta}\left(A_{ji}^{(i)} + A_{ji}^{(j)}\right), B_{12}^{t}\right\} & \left\{\begin{matrix} \beta A_{jj}^{(i)} + \left(A_{jj}^{(j)}\right)_{eff}, B_{22} \\ \sqrt{\beta}\left\{\begin{matrix} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2} \\ , A_{\{kj\}i>k>j}^{(j)}, A_{\{jk\}i>k>j}^{(j)} \end{matrix}\right\} \end{matrix}\right\} \end{pmatrix} Y_i(s) \\
& -\frac{\sqrt{\beta}}{2}\left(Y_i(s) - Y_i(s-1)\right) \begin{pmatrix} -\epsilon_{ii}^{(i)} & A_{ij}^{(i)} + A_{ij}^{(j)} \\ A_{ji}^{(i)} + A_{ji}^{(j)} & \left\{\begin{matrix} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ A_{\{kj\}i>k>j}^{(j)}, A_{\{jk\}i>k>j}^{(j)} \end{matrix}\right\} \end{pmatrix} \left(Y_i(s) - Y_i(s-1)\right) \\
& +\sum_{s>t} \sqrt{\beta}\left(Y_i(s) - Y_i(s-1)\right) \begin{pmatrix} 0 & A_{ij}^{(i)} - A_{ij}^{(j)} \\ -\left(A_{ji}^{(i)} - A_{ji}^{(j)}\right) & \left\{\begin{matrix} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ A_{\{kj\}i>k>j}^{(j)}, -A_{\{jk\}i>k>j}^{(j)} \end{matrix}\right\} \end{pmatrix} Y_i(s-1) \\
& -2\sqrt{\beta} Y_i(s) . \left(\hat{Y}_i + \left(\hat{Y}_j\right)_{eff}\right) \\
& +\frac{\sqrt{\beta}}{2} Y_i(t) \begin{pmatrix} -\epsilon_{ii}^{(i)} & A_{ij}^{(i)} + A_{ij}^{(j)} \\ A_{ji}^{(i)} + A_{ji}^{(j)} & \left\{\begin{matrix} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ A_{\{kj\}i>k>j}^{(j)}, A_{\{jk\}i>k>j}^{(j)} \end{matrix}\right\} \end{pmatrix} Y_i(t) \\
= & \sum_{s>t} -\frac{1}{2}\left(Y_i(s) - Y_i(s-1)\right) A \left(Y_i(s) - Y_i(s-1)\right) + Y_i(s) B Y_i(s) + \left(Y_i(s) - Y_i(s-1)\right) C Y_i(s-1) \\
& +\frac{1}{2} Y_i(t) A Y_i(t) \\
\sim & \sum_{s>t} -\frac{1}{2}\left(Y_i(s) - Y_i(s-1)\right) A \left(Y_i(s) - Y_i(s-1)\right) + \left(Y_i(s) - \hat{Y}_i^{(1)}\right) B \left(Y_i(s) - \hat{Y}_i^{(1)}\right) \\
& + \left(Y_i(s) - Y_i(s-1)\right) C Y_i(s-1) + Y_i(t) B Y_i(t) \\
& +\frac{1}{2} Y_i(t) A Y_i(t)
\end{aligned} \tag{237}$$

$$A = \sqrt{\beta} \begin{pmatrix} -\epsilon_{ii}^{(i)} & A_{ij}^{(i)} + A_{ij}^{(j)} \\ A_{ji}^{(i)} + A_{ji}^{(j)} & \left\{\begin{matrix} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ A_{\{kj\}i>k>j}^{(j)}, A_{\{jk\}i>k>j}^{(j)} \end{matrix}\right\} \end{pmatrix} \tag{238}$$

$$B = \begin{pmatrix} A_{ii}^{(i)} + B_{11} - \sqrt{\beta}\epsilon_{ii}^{(i)} & \left\{\sqrt{\beta}\left(A_{ij}^{(i)} + A_{ij}^{(j)}\right), B_{12}\right\} \\ \left\{\sqrt{\beta}\left(A_{ji}^{(i)} + A_{ji}^{(j)}\right), B_{12}^{t}\right\} & \left\{\begin{matrix} \beta A_{jj}^{(i)} + \left(A_{jj}^{(j)}\right)_{eff}, B_{22} \\ \sqrt{\beta}\left\{\begin{matrix} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2} \\ , A_{\{kj\}i>k>j}^{(j)}, A_{\{jk\}i>k>j}^{(j)} \end{matrix}\right\} \end{matrix}\right\} \end{pmatrix}$$

$$C = \sqrt{\beta} \begin{pmatrix} 0 & A_{ij}^{(i)} - A_{ij}^{(j)} \\ -\left(A_{ji}^{(i)} - A_{ji}^{(j)}\right) & \left\{\begin{matrix} \frac{\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ -A_{\{kj\}i>k>j}^{(j)}, A_{\{jk\}i>k>j}^{(j)} \end{matrix}\right\} \end{pmatrix}$$

The sum includes the potential at time t but not the inertial term.

The effective action for $Y_i(t)$ is computed in the following way: it is know ([**?**]) that for a quadratic weight as the one obtained in (237), the integral over future variables $Y_i(s)$ localizes around the classical solution of motion starting at $Y_i(t)$ and such that $Y_i(s) \to 0$ for $s \to \infty$. That is, to compute the integrals of (237) on $Y_i(s)$ it is enough to minimize (237) on the $Y_i(s)$, $s > t$ with $Y_i(t)$ as initial condition and to compute (237) for this solution.

The equation for the classical solution of (237):

$$\begin{aligned} \sim \quad & \sum_{s>t} -\frac{1}{2}\left(Y_i(s) - Y_i(s-1)\right) A \left(Y_i(s) - Y_i(s-1)\right) + Y_i(s) BY_i(s) + \left(Y_i(s) - Y_i(s-1)\right) CY_i(s-1) \\ & + Y_i(t) BY_i(t) + \frac{1}{2} Y_i(t) AY_i(t) \end{aligned} \tag{239}$$

is of the usual Euler Lagrange type:

$$\left(Y_i(s) - \hat{Y}_i^{(1)}\right) A \left(Y_i(s+1) - 2Y_i(s) + Y_i(s-1)\right) + 2Y_i(s) BY_i(s) - \left(Y_i(s) - \hat{Y}_i^{(1)}\right) C \left(Y_i(s+1) - Y_i(s-1)\right) = 0 \tag{240}$$

and it's solution is of the kind:

$$Y_i(s) = D^{t-s} Y_i(t) \tag{241}$$

We show in Appendix 1.b. that the matrix D satifies:

$$(A - C) D^2 + 2(B - A) D + (A + C) = 0 \tag{242}$$

We also give a recursive equation for D in this appendix.

We now compute each term of the action

$$\sum_{s \geqslant t} -\frac{1}{2}\left(Y_i(s+1) - Y_i(s)\right) A \left(Y_i(s+1) - Y_i(s)\right) + Y_i(s) BY_i(s) + Y_i(s+1) CY_i(s) \tag{243}$$

along this classical solution to find our effective utility. We to first rewrite the first term in (243) as a discrete version of the integration by part:

$$\begin{aligned} & \sum_{s \geqslant t} -\frac{1}{2}\left(Y_i(s+1) - Y_i(s)\right) A \left(Y_i(s+1) - Y_i(s)\right) \\ = \quad & \sum_{s \geqslant t} -\frac{1}{2}\left(Y_i(s+1) - Y_i(s)\right) A \left(Y_i(s+1) - Y_i(s)\right) \\ = \quad & \frac{1}{2} Y_i(t) A \left(Y_i(t+1) - Y_i(t)\right) + \frac{1}{2} \sum_{s \geqslant t} Y_i(s) A \left(Y_i(s+1) - 2Y_i(s) + Y_i(s-1)\right) \end{aligned}$$

We gather all these contributions with the second term in the classical action (243) and use (240) as well as (241) to find:.

$$\begin{aligned}
&\sum_{s\geqslant t}-\frac{1}{2}\left(Y_i\left(s+1\right)-Y_i\left(s\right)\right)A\left(Y_i\left(s+1\right)-Y_i\left(s\right)\right)+Y_i\left(s\right)BY_i\left(s\right)+Y_i\left(s+1\right)CY_i\left(s\right)+\frac{1}{2}Y_i\left(t\right)AY_i\left(t\right)\\
=&\sum_{s\geqslant t}\frac{1}{2}Y_i\left(s\right)A\left(Y_i\left(s+1\right)-2Y_i\left(s\right)+Y_i\left(s-1\right)\right)+Y_i\left(s\right)BY_i\left(s\right)+Y_i\left(s+1\right)CY_i\left(s\right)\\
&+\frac{1}{2}Y_i\left(t\right)A\left(Y_i\left(t+1\right)-Y_i\left(t\right)\right)+\frac{1}{2}Y_i\left(t\right)AY_i\left(t\right)\\
=&\sum_{s>t}\frac{1}{2}Y_i\left(s\right)A\left(Y_i\left(s+1\right)-2Y_i\left(s\right)+Y_i\left(s-1\right)\right)+Y_i\left(s\right)BY_i\left(s\right)-\frac{1}{2}Y_i\left(s+1\right)CY_i\left(s\right)\\
&+\frac{1}{2}Y_i\left(t\right)A\left(Y_i\left(t+1\right)-Y_i\left(t\right)\right)+\frac{1}{2}Y_i\left(t+1\right)CY_i\left(t\right)+\frac{1}{2}Y_i\left(t\right)AY_i\left(t\right)\\
\sim&\ \frac{1}{2}Y_i\left(t\right)A\left(Y_i\left(t+1\right)-Y_i\left(t\right)\right)+\frac{1}{2}Y_i\left(t+1\right)CY_i\left(t\right)+\frac{1}{2}Y_i\left(t\right)AY_i\left(t\right)\\
=&\ \frac{1}{2}Y_i\left(t\right)A\left(Y_i\left(t+1\right)-Y_i\left(t\right)\right)+\frac{1}{2}\left(Y_i\left(t+1\right)-Y_i\left(t\right)\right)CY_i\left(t\right)+\frac{1}{2}Y_i\left(t\right)AY_i\left(t\right)\\
=&\ \frac{1}{2}Y_i\left(t\right)\left(\left(A-C\right)\left(D-1\right)\right)Y_i\left(t\right)+\frac{1}{2}Y_i\left(t\right)AY_i\left(t\right)
\end{aligned}\tag{244}$$

To find the effective utility for agent i, that is $U_{eff}\left(Y_i\left(t\right)\right)$, we also include the time t contribution that was first discarded in our computation and consider the intermediate effective utility:

$$\begin{aligned}
&U_{eff}^{int}\left(Y_i\left(t\right)\right)=\frac{1}{2}Y_i\left(t\right)\left(\left(A-C\right)\left(D-1\right)\right)Y_i\left(t\right)+\frac{1}{2}Y_i\left(t\right)AY_i\left(t\right)\\
&+Y_i\left(t\right)\begin{pmatrix}A_{ii}^{(i)} & 0\\ 0 & \beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}\end{pmatrix}Y_i\left(t\right)+\sqrt{\beta}Y_i\left(t\right)\begin{pmatrix}-\epsilon_{ii}^{(i)} & 2A_{ij}^{(i)}\\ 2A_{ji}^{(j)} & \left\{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff},2A_{\{kj\}i>k>j}^{(j)}\right\}\end{pmatrix}Y_i\left(t-1\right)\\
&+\sum_{j>i}2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right)
\end{aligned}$$

This is still not $U_{eff}\left(X_i\left(t\right)\right)$ since it depends on the $X_j\left(t\right)$ that should also be integrated out.

Before doing so, we can simplify $U_{eff}^{int}\left(Y_i\left(t\right)\right)$, by neglecting the contributions depending on $t-1$ only (we will use the notation $\sim$ each time we neglect such terms):

$$\begin{aligned}
&U_{eff}^{int}\left(Y_i\left(t\right)\right)=\frac{1}{2}Y_i\left(t\right)\left(\left(A-C\right)\left(D-1\right)\right)Y_i\left(t\right)+\frac{1}{2}Y_i\left(t\right)AY_i\left(t\right)\\
&+Y_i\left(t\right)\begin{pmatrix}A_{ii}^{(i)} & 0\\ 0 & \beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}\end{pmatrix}Y_i\left(t\right)+\sqrt{\beta}Y_i\left(t\right)\begin{pmatrix}-\epsilon_{ii}^{(i)} & 2A_{ij}^{(i)}\\ 2A_{ji}^{(j)} & \left\{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff},2A_{\{kj\}i>k>j}^{(j)}\right\}\end{pmatrix}Y_i\left(t-1\right)\\
&+\sum_{j>i}2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right)\\
=&\ \frac{1}{2}Y_i\left(t\right)\left(\left(A-C\right)\left(D-1\right)+2B\right)Y_i\left(t\right)-\frac{1}{2}Y_i\left(t\right)AY_i\left(t\right)+Y_i\left(t\right)\left(A+C\right)Y_i\left(t-1\right)+\sum_{j>i}2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right)
\end{aligned}\tag{245}$$

Since C is antisymetric, this is also equal to:

$$\begin{aligned}
U_{eff}^{int}\left(Y_i\left(t\right)\right) &= \frac{1}{2}Y_i\left(t\right)\left(\left(A-C\right)\left(D-1\right)+2B\right)Y_i\left(t\right)-\frac{1}{2}Y_i\left(t\right)\left(A-C\right)Y_i\left(t\right)+Y_i\left(t\right)\left(A+C\right)Y_i\left(t-1\right) \\
&+\sum_{j>i}2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right) \\
&= \frac{1}{2}Y_i\left(t\right)\left(\left(A-C\right)\left(D-2\right)+2B\right)Y_i\left(t\right)+Y_i\left(t\right)\left(A+C\right)Y_i\left(t-1\right)-Y_i\left(t\right)\hat{Y}_i+\sum_{j>i}2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right) \\
&= \frac{1}{2}Y_i\left(t\right)\left(\left(A-C\right)\left(D-1\right)+2B\right)Y_i\left(t\right)-\frac{1}{2}\left(Y_i\left(t\right)-Y_i\left(t-1\right)\right)A\left(Y_i\left(t\right)-Y_i\left(t-1\right)\right)+Y_i\left(t\right)CY_i\left(t-1\right) \\
&+\sum_{j>i}2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right)
\end{aligned} \tag{246}$$

and then:

$$\begin{aligned}
U_{eff}^{int}\left(Y_i\left(t\right)\right) &\sim \frac{1}{2}Y_i\left(t\right)\left(\left(A-C\right)\left(D-2\right)+2B\right)Y_i\left(t\right)+Y_i\left(t\right)AY_i\left(t-1\right)+Y_i\left(t\right)CY_i\left(t-1\right)+\sum_{j>i}2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right) \\
&\sim \frac{1}{2}\left(Y_i\left(t\right)+\left(\left(A-C\right)\left(D-2\right)+2B\right)^{-1}\left(\left(A+C\right)\left(Y_i\left(t-1\right)\right)\right)\right) \\
&\times\left(\left(A-C\right)\left(D-2\right)+2B\right)\left(Y_i\left(t\right)+\left(\left(A-C\right)\left(D-2\right)+2B\right)^{-1}\left(\left(A+C\right)\left(Y_i\left(t-1\right)\right)\right)\right) \\
&+\sum_{j>i}2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right)
\end{aligned}$$

Now, the Integration on $X_j\left(t\right)$ for $j<i$ yields:

$$\begin{aligned}
&U_{eff}\left(X_i\left(t\right)\right)=\frac{1}{2}\left(Y_i\left(t\right)+\left(\left(A-C\right)\left(D-2\right)+2B\right)^{-1}\left(\left(A+C\right)\left(Y_i\left(t-1\right)\right)\right)\right)_i \\
&\times\left(N_{ii}\right)\left(Y_i\left(t\right)+\left(\left(A-C\right)\left(D-2\right)+2B\right)^{-1}\left(\left(A+C\right)\left(Y_i\left(t-1\right)\right)\right)\right)_i \\
&+\sum_{j>i}2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right) \\
\sim\; &-\frac{1}{2}\left(\left(Y_i\left(t\right)\right)_iM_{ii}\left(Y_i\left(t-1\right)\right)_i+T\right)-\frac{1}{2}\left(\left(Y_i\left(t\right)\right)M_{ij}\left(Y_i\left(t-1\right)\right)_j+T\right) \\
&+\frac{1}{2}\left(Y_i\left(t\right)\right)_i\left(N_{ii}\right)\left(Y_i\left(t\right)\right)_i+\sum_{j>i}2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right)
\end{aligned} \tag{247}$$

where the matrices used in the previous expression are given by:

$$\begin{aligned}
N_{ii} &= \left(\left(A-C\right)\left(D-2\right)+2B\right)_{ii}-\left(\left(A-C\right)\left(D-2\right)+2B\right)_{ij}\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)_{jj}\right)^{-1}\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)_{ji}\right) \\
M_{ii} &= \left(N_{ii}\right)\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)^{-1}\left(A+C\right)\right)_{ii} \\
M_{ij} &= \left(N_{ii}\right)\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)^{-1}\left(A+C\right)\right)_{ij}
\end{aligned}$$

and where the "T" means the transpose of the expression in the same parenthesis.

It can also be written in a form reminding the continuous time description:

$$\begin{aligned}
U_{eff}\left(X_i\left(t\right)\right) &= -\frac{1}{2}\dot{X}_i\left(t\right)\hat{M}_{ii}\dot{X}_i\left(t\right)-\left(X_i\left(t\right)-\left(\hat{Y}_i^{(1)}\right)_t\right)M_{ij}\left(\frac{1}{\sqrt{\beta}}X_j\left(t-1\right)-\left(\hat{Y}_i^{(1)}\right)_j\right) \\
&+\frac{1}{2}\left(X_i\left(t\right)-\left(\hat{Y}_i^{(1)}\right)_i\right)\left(\hat{N}_{ii}\right)\left(X_i\left(t\right)-\left(\hat{Y}_i^{(1)}\right)_i\right)+\sum_{j>i}2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right)
\end{aligned} \tag{248}$$

where we defined:

$$\dot{X}_i(t) = (X_i(t) - X_i(t-1))$$

and where the matrices used in the previous expression are given by:

$$\begin{aligned}
N_{ii} &= ((A-C)(D-2)+2B)_{ii} - ((A-C)(D-2)+2B)_{ij}\left(((A-C)(D-2)+2B)_{jj}\right)^{-1}\left(((A-C)(D-2)+2B)_{ji}\right) \\
\hat{M}_{ii} &= \left(((A-C)(D-2)+2B)^{-1}(A+C)\right)_{ii}(N_{ii})\left(((A-C)(D-2)+2B)^{-1}(A+C)\right)_{ii} \\
M_{ij} &= (N_{ii})\left(((A-C)(D-2)+2B)^{-1}(A+C)\right)_{ij} \\
\hat{N}_{ii} &= N_{ii} + M_{ii}
\end{aligned}$$

Adding up all effctv weight for all structres leads to consider the term

$$\begin{aligned}
&\sum_i \sum_{j>i} 2X_i(t) A_{ij}^{(i)} (X_j(t-1)) \\
=\ & 2\sum_{i,j} X_i(t) \hat{A}_{ij} X_j(t-1)
\end{aligned}$$

with $\hat{A}_{ij} = A_{ij}^{(i)}$ if $j < i$, 0 otherwise.

By the same trick as before it leads in the continuum to the result:

$$\begin{aligned}
&\sum_{i,j} X_i(t)\left(\hat{A}_{ij} + \left(\hat{A}_{ji}\right)^t\right) X_j(t-1) \\
&+\sum_{i,j} X_i(t)\left(\hat{A}_{ij} - \left(\hat{A}_{ji}\right)^t\right) X_j(t-1) \\
=\ & -\frac{1}{2}(X_i(t) - X_i(t-1))\left(\hat{A}_{ij} + \left(\hat{A}_{ji}\right)^t\right)(X_j(t) - X_j(t-1)) \\
&+\frac{1}{2}X_i(t)\left(\hat{A}_{ij} + \left(\hat{A}_{ji}\right)^t\right)X_j(t) + \frac{1}{2}X_i(t-1)\left(\hat{A}_{ij} + \left(\hat{A}_{ji}\right)^t\right)X_j(t-1) \\
&-\frac{1}{2}\sum_{i,j} X_i(t)\left(\hat{A}_{ij} - \left(\hat{A}_{ji}\right)^t\right)(X_j(t) - X_j(t-1))
\end{aligned}$$

Later in the sum on t, $\frac{1}{2}X_i(t)\left(\hat{A}_{ij} + \left(\hat{A}_{ji}\right)^t\right)X_j(t) + \frac{1}{2}X_i(t-1)\left(\hat{A}_{ij} + \left(\hat{A}_{ji}\right)^t\right)X_j(t-1)$ will replaced by $X_i(t)\left(\hat{A}_{ij} + \left(\hat{A}_{ji}\right)^t\right)X_j(t)$ for an overall weight:

$$\begin{aligned}
&-\frac{1}{2}\sum_{i,j}\dot{X}_i(t)\left(\hat{A}_{ij} + \left(\hat{A}_{ji}\right)^t\right)\dot{X}_j(t) \\
&+\sum_{i,j} X_i(t)\left(\hat{A}_{ij} + \left(\hat{A}_{ji}\right)^t\right)X_j(t) - \frac{1}{2}\sum_{i,j} X_i(t)\left(\hat{A}_{ij} - \left(\hat{A}_{ji}\right)^t\right)\dot{X}_j(t) \\
=\ & -\frac{1}{2}\sum_{i,j}\dot{X}_i(t)\hat{A}_{ij}^{(s)}\dot{X}_j(t) + \sum_{i,j} X_i(t)\hat{A}_{ij}^{(s)}X_j(t) - \frac{1}{2}\sum_{i,j} X_i(t)\hat{A}_{ij}^{(a)}\dot{X}_j(t)
\end{aligned}$$

The total effective action is then:

$$\begin{aligned}
&-\frac{1}{2}\dot{X}_i(t)\hat{M}_{ii}\dot{X}_i(t) - \left(X_i(t) - \hat{Y}_i^{(1)}\right)M_{ij}\left(X_j(t-1) - \left(\hat{Y}_i^{(1)}\right)_j\right) + \frac{1}{2}\left(X_i(t) - \left(\hat{Y}_i^{(1)}\right)_i\right)\left(\hat{N}_{ii}\right)\left(X_i(t) - \left(\hat{Y}_i^{(1)}\right)_i\right) \\
&-\frac{1}{2}\sum_{i,j}\dot{X}_i(t)\hat{A}_{ij}^{(s)}\dot{X}_j(t) + \sum_{i,j} X_i(t)\hat{A}_{ij}^{(s)}X_j(t) - \frac{1}{2}\sum_{i,j} X_i(t)\hat{A}_{ij}^{(a)}\dot{X}_j(t)
\end{aligned}$$

We want to rewrite the quadratic terms in a form that will be useful when looking at the continuous approximatio. Introduce:

$$X\left(t\right)=\left(X_{i}\left(t\right)\right)\text{ and }\left(\hat{Y}^{(1)}\right)=\left(\left(\hat{Y}^{(1)}\right)_{i}\right)$$

and rewrite the various terms in the previous form:

$$\begin{aligned}
&-\left(X_{i}\left(t\right)-\hat{Y}_{i}^{(1)}\right)M_{ij}\left(X_{j}\left(t-1\right)-\left(\hat{Y}_{i}^{(1)}\right)_{j}\right)\\
\sim\quad&-X_{i}\left(t\right)M_{ij}\left(X_{j}\left(t-1\right)-\left(\hat{Y}_{i}^{(1)}\right)_{j}\right)\\
\sim\quad&-\left(X_{i}\left(t\right)-X_{i}\left(t-1\right)\right)M_{ij}\left(X_{j}\left(t-1\right)-\left(\hat{Y}_{i}^{(1)}\right)_{j}\right)\\
=\quad&-\left(X_{i}\left(t\right)-X_{i}\left(t-1\right)\right)M_{ij}\left(X_{j}\left(t-1\right)\right)\\
&+\left(X_{i}\left(t\right)-X_{i}\left(t-1\right)\right)M_{ij}\left(\hat{Y}_{i}^{(1)}\right)_{j}
\end{aligned}$$

The second term is a derivative that will cancel when integrating on t. We are then led to:

$$\sim-\frac{1}{2}\dot{X}\left(t\right)M\dot{X}\left(t\right)+\frac{1}{2}\left(X\left(t\right)-\left(\hat{Y}^{(1)}\right)\right)\left(\hat{N}\right)\left(X\left(t\right)-\left(\hat{Y}^{(1)}\right)\right)+X\left(t\right)\hat{A}^{(s)}X\left(t\right)-\dot{X}\left(t\right)\tilde{M}X\left(t\right)$$

where:

$$\hat{M}=\left(\hat{M}_{ii}+\hat{A}_{ij}^{(s)}\right),\ \hat{N}=\left(\hat{N}_{ii}\right),\ \tilde{M}=\left(\tilde{M}_{ij}-\frac{1}{2}\hat{A}_{ij}^{(s)}\right)$$

Since the symetric part of $\tilde{M}$ cancels when integrating over t, $\tilde{M}$ can be considered as antisymetric, and M and A symetric. We can write:

$$\begin{aligned}
&-\frac{1}{2}\dot{X}\left(t\right)\hat{M}\dot{X}\left(t\right)+\frac{1}{2}\left(X\left(t\right)-\left(\hat{Y}^{(1)}\right)\right)\left(\hat{N}\right)\left(X\left(t\right)-\left(\hat{Y}^{(1)}\right)\right)+X\left(t\right)\hat{A}^{(s)}X\left(t\right)-\dot{X}\left(t\right)\tilde{M}X\left(t\right)\\
=\quad&-\frac{1}{2}\left(\dot{X}\left(t\right)-\tilde{M}'X\left(t\right)\right)\hat{M}\left(\dot{X}\left(t\right)-\tilde{M}'X\left(t\right)\right)+\frac{1}{2}\left(X\left(t\right)-\left(\hat{Y}^{(1)}\right)\right)\left(\hat{N}\right)\left(X\left(t\right)-\left(\hat{Y}^{(1)}\right)\right)+X\left(t\right)\left(\hat{N}'\right)X\left(t\right)
\end{aligned}$$

where:

$$\begin{aligned}
\hat{N}'&=\hat{A}^{(s)}+\tilde{M}\hat{M}^{-1}\tilde{M}\\
\tilde{M}'&=\hat{M}^{-1}\tilde{M}
\end{aligned}$$

Appendix 3

The quadratic action has to a classical solution whose Equation is:

$$A\left(Y_i\left(s+1\right)-2Y_i\left(s\right)+Y_i\left(s-1\right)\right)+2B\left(Y_i\left(s\right)-\hat{Y}_i^{(1)}\right)-C\left(Y_i\left(s+1\right)-Y_i\left(s-1\right)\right)=0$$

The solution of this second order difference equation with initial condition $Y_i\left(t\right)$ is:

$$\left(Y_i\left(s\right)-\hat{Y}_i^{(1)}\right)=D^{t-s}\left(Y_i\left(t\right)-\hat{Y}_i^{(1)}\right) \tag{249}$$

where the matrix D satisfies:

$$\begin{aligned} A\left(D^2-2D+1\right)+2BD-C\left(D^2-1\right) &= 0 \qquad (250)\\ \left(A-C\right)\left(D-1\right)^2+2\left(B-C\right)\left(D-1\right)+2B &= 0 \end{aligned}$$

$$\left(A-C\right)D^2+2\left(B-A\right)D+\left(A+C\right)=0 \tag{251}$$

Writing $B=A+\delta$ one obtains:

$$\left(A-C\right)D^2+2\delta D+\left(A+C\right)=0 \tag{252}$$

The unicity of D is granted by the problem at hand. We look for a solution whose β expansion is obtained recursively, and whose first term is identical to the one obtained for $\beta=0$ in the initial problem. To do so, we can find, at least, a recursive solution to this equation. Rescaling $A\rightarrow\frac{A}{\sqrt{\beta}}$, $C\rightarrow\frac{C}{\sqrt{\beta}}$, D can be obtained as a series expansion in $\sqrt{\beta}$, $\sum\left(\sqrt{\beta}\right)^n D_n$. Equation (252) becomes:

$$\begin{aligned} \sqrt{\beta}\left(A-C\right)\left(1-\sum_{n=1}^{\infty}\left(\sqrt{\beta}\right)^n D_n\right)^2-2\left(\delta+\sqrt{\beta}\left(A-C\right)\right)\left(1-\sum_{n=1}^{\infty}\left(\sqrt{\beta}\right)^n D_n\right)+2\left(\delta+\sqrt{\beta}A\right) &= 0\\ \left(\sqrt{\beta}\left(A-C\right)+\sum_{n=2}^{\infty}\left(\sqrt{\beta}\right)^n\left(\left(A-C\right)\left(\sum_{k=1}^{n}D_kD_{n-1-k}\right)+2\delta D_n\right)\right)-2\left(\delta I+\sqrt{\beta}\left(A-C\right)\right)+2\left(\delta+\sqrt{\beta}A\right) &= 0\\ \left(\sqrt{\beta}\left(A-C\right)+\sum_{n=2}^{\infty}\left(\sqrt{\beta}\right)^n\left(\left(A-C\right)\left(\sum_{k=1}^{n}D_kD_{n-1-k}\right)+2\delta D_n\right)+2\sqrt{\beta}\delta D_1\right)+2\sqrt{\beta}C &= 0 \end{aligned}$$

$$\left(A+C\right)+\sum_{n=1}^{\infty}\left(\sqrt{\beta}\right)^n\left(\left(A-C\right)\left(\sum_{k=1}^{n}D_kD_{n-k}\right)+2\delta D_{n+1}\right)+2\delta D_1=0$$

As a consequence, the first term is

$$D_1=-\delta^{-1}\frac{A+C}{2}$$

and

$$D_{n+1}=-\delta^{-1}\frac{\left(A-C\right)}{2}\left(\sum_{k=1}^{n}D_kD_{n-k}\right)$$

Appendix 4

To solve the class of models presented in the text, the equation (252) can be cast into the block form:

$$0 = \sqrt{\beta}\begin{pmatrix} \frac{-\epsilon_{ii}^{(i)}}{2} & A_{ij}^{(j)} \\ A_{ji}^{(i)} & \left\{\frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}\end{pmatrix} D^2 + \begin{pmatrix} A_{ii}^{(i)} + B_{11} & B_{12} \\ B_{12}^t & \left\{\beta A_{jj}^{(i)} + \left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}\end{pmatrix} D$$
$$+\sqrt{\beta}\begin{pmatrix} \frac{-\epsilon_{ii}^{(i)}}{2} & A_{ij}^{(i)} \\ A_{ji}^{(j)} & \left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}\end{pmatrix}$$

with:

$$D = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

is the block decomposition of D imposed by the matrices $\epsilon_{ii}^{(i)}$, $A_{ij}^{(j)}$...

In most systems, the "per se" inertia $\epsilon_{ii}^{(i)}$ is nul. If moreover $A_{ji}^{(i)} = 0$, that is agent i is sensitive to his substructures goals, but not directly to their actions, one can find E and F as functions of the other matrix blocks. Actually, given that in that case (252) writes as:

$$0 = \sqrt{\beta}\begin{pmatrix} 0 & A_{ij}^{(j)} \\ 0 & \left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}\end{pmatrix} D^2 + \begin{pmatrix} A_{ii}^{(i)} + B_{11} & B_{12} \\ B_{12}^t & \left\{\beta A_{jj}^{(i)} + \left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}\end{pmatrix} D$$
$$+\sqrt{\beta}\begin{pmatrix} 0 & 0 \\ A_{ji}^{(j)} & \left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}\end{pmatrix}$$

one can divide the equation (252) in two blocks:

$$0 = \sqrt{\beta}\begin{pmatrix} A_{ij}^{(j)} \\ \left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}\end{pmatrix} (GE + HG) \tag{253}$$
$$+ \begin{pmatrix} A_{ii}^{(i)} + B_{11} & B_{12} \\ B_{12}^t & \left\{\beta A_{jj}^{(i)} + \left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}\end{pmatrix}\begin{pmatrix} E \\ G\end{pmatrix} + \sqrt{\beta}\begin{pmatrix} 0 \\ A_{ji}^{(j)}\end{pmatrix}$$

and:

$$0 = \sqrt{\beta}\begin{pmatrix} A_{ij}^{(j)} \\ \left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}\end{pmatrix} (GF + H^2) \tag{254}$$
$$+ \begin{pmatrix} A_{ii}^{(i)} + B_{11} & B_{12} \\ B_{12}^t & \left\{\beta A_{jj}^{(i)} + \left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}\end{pmatrix}\begin{pmatrix} F \\ H\end{pmatrix} + \sqrt{\beta}\begin{pmatrix} 0 \\ \left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}\end{pmatrix}$$

The first one (253) allows to find E. Actually, the two equations of (253) yield:

$$(GE + HG) = -\left(\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\left(B_{12}^t E + \left\{\beta A_{jj}^{(i)} + \left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\} G + \sqrt{\beta} A_{ji}^{(j)}\right) \tag{255}$$

and

$$
\begin{aligned}
0 &= \left(A_{ii}^{(i)}+B_{11}\right)E+B_{12}G-\sqrt{\beta}A_{ij}^{(j)}\left(\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\\
&\quad\times\left(B_{12}^{t}E+\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}G+\sqrt{\beta}A_{ji}^{(j)}\right)
\end{aligned}
$$

so that:

$$
\begin{aligned}
E &= \left(\left(A_{ii}^{(i)}+B_{11}\right)-\sqrt{\beta}A_{ij}^{(j)}\left(\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}B_{12}^{t}\right)^{-1} \qquad (256)\\
&\quad\times\left(\sqrt{\beta}A_{ij}^{(j)}\left(\left(\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}G+\sqrt{\beta}A_{ji}^{(j)}\right)\right)-B_{12}G\right)
\end{aligned}
$$

normalize $A_{ii}^{(i)}=1$, and use that $\left(A_{jj}^{(j)}\right)_{eff}$ can be considered as symetric.

$$
\begin{aligned}
B_{12} &= \beta A_{ij}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\left(\left\{\frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)\\
B_{11} &= \beta A_{ij}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{ji}^{(j)}\\
B_{22} &= \left\{\begin{array}{c}
\beta A_{lj}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{jk}^{(j)},\\
\beta\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)\left(A_{jj}^{(j)}\right)_{eff}^{-1}\left(\left\{\frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)\\
\frac{\beta}{2}\left(A_{kj}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\left(\left\{\frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}+\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)\right)
\end{array}\right\}
\end{aligned}
$$

$$
\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}=\left\{\begin{array}{c}
\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},\beta A_{lj}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{jk}^{(j)},\\
\beta\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)\left(A_{jj}^{(j)}\right)_{eff}^{-1}\left(\left\{\frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right),\\
\frac{\beta}{2}\left(A_{kj}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\left(\left\{\frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}+\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)\right)
\end{array}\right\}
$$

$$
\begin{aligned}
&\left(\left(A_{ii}^{(i)}+B_{11}\right)-A_{ij}^{(j)}\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}B_{12}^{t}\right)^{-1}\\
&= \left(1-\beta A_{ij}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{ji}^{(j)}+B_{11}\right)^{-1}=1
\end{aligned}
$$

Thus, the expressions for E simplify as:

$$
E=\sqrt{\beta}A_{ij}^{(j)}\left(\left(\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}G+\sqrt{\beta}A_{ji}^{(j)}\right)\right)-B_{12}G \qquad (257)
$$

Similarly, the second block (254) leads to:

$$0 = \sqrt{\beta}\left\{\frac{-\left(\epsilon^{(j)}_{\{kj\}j\geqslant k}\right)_{eff}}{2}, A^{(j)}_{\{kj\}i>k>j}\right\}\left(GF+H^2\right)+B_{21}F+\left\{\beta A^{(i)}_{jj}+\left(A^{(j)}_{jj}\right)_{eff}, B_{22}\right\}H \quad (258)$$
$$+\sqrt{\beta}\left\{\frac{-\left(\epsilon^{(j)}_{\{jk\}k\leqslant j}\right)_{eff}}{2}, A^{(j)}_{\{jk\}i>k>j}\right\}$$

yielding $\left(GF+H^2\right)$:

$$\left(GF+H^2\right) = -\left(\sqrt{\beta}\left\{\frac{-\left(\epsilon^{(j)}_{\{kj\}j\geqslant k}\right)_{eff}}{2}, A^{(j)}_{\{kj\}i>k>j}\right\}\right)^{-1} \quad (259)$$
$$\times\left(B^t_{12}F+\left\{\beta A^{(i)}_{jj}+\left(A^{(j)}_{jj}\right)_{eff}, B_{22}\right\}H+\sqrt{\beta}\left\{\frac{-\left(\epsilon^{(j)}_{\{jk\}k\leqslant j}\right)_{eff}}{2}, A^{(j)}_{\{jk\}i>k>j}\right\}\right)$$

and after coming back to (254), the expression for F:

$$F = \sqrt{\beta}A^{(j)}_{ij}\left(\sqrt{\beta}\left\{\frac{-\left(\epsilon^{(j)}_{\{kj\}j\geqslant k}\right)_{eff}}{2}, A^{(j)}_{\{kj\}i>k>j}\right\}\right)^{-1} \quad (260)$$
$$\times\left(\left\{\beta A^{(i)}_{jj}+\left(A^{(j)}_{jj}\right)_{eff}, B_{22}\right\}H+\sqrt{\beta}\left\{\frac{-\left(\epsilon^{(j)}_{\{jk\}k\leqslant j}\right)_{eff}}{2}, A^{(j)}_{\{jk\}i>k>j}\right\}\right)-B_{12}H$$

The resolution of the problem is thus reduced to a system of two remaining equations:

$$\left(\sqrt{\beta}A^{(j)}_{ij}G+\left(A^{(i)}_{ii}+B_{11}\right)\right)E+\left(B_{12}+\sqrt{\beta}A^{(j)}_{ij}H\right)G=0 \quad (261)$$

$$\left(\sqrt{\beta}A^{(j)}_{ij}G+\left(A^{(i)}_{ii}+B_{11}\right)\right)F+\left(B_{12}+\sqrt{\beta}A^{(j)}_{ij}H\right)H=0 \quad (262)$$

where E and F are given in (256) and (53).

Multiply the second equation (262) by $H^{-1}G$ and compare with (261) one obtains:

$$FH^{-1}G=E$$

This can used to write that:

$$\left(GF+H^2\right)H^{-1}G=(GE+HG)$$

and, using (255) (259), one is led to:

$$\left(B^t_{12}F+\left\{\beta A^{(i)}_{jj}+\left(A^{(j)}_{jj}\right)_{eff}, B_{22}\right\}H+\sqrt{\beta}\left\{\frac{-\left(\epsilon^{(j)}_{\{jk\}k\leqslant j}\right)_{eff}}{2}, A^{(j)}_{\{jk\}i>k>j}\right\}\right)H^{-1}G$$
$$=\left(B^t_{12}E+\left\{\beta A^{(i)}_{jj}+\left(A^{(j)}_{jj}\right)_{eff}, B_{22}\right\}G+\sqrt{\beta}A^{(j)}_{ji}\right)$$

and using again that $FH^{-1}G=E$:

$$\left(\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)H^{-1}G=\sqrt{\beta}A_{ji}^{(j)}$$

One can thus express G as a function of H:

$$G=H\left(\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}A_{ji}^{(j)} \tag{263}$$

The all problem then reduces to find H. To do so, one uses (258):

$$\begin{aligned}0 &= \sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\left(GF+H^2\right)+B_{21}F+\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}H \\ &\quad+\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\end{aligned} \tag{264}$$

which is, after expanding the terms involved in this equation:

$$\begin{aligned}&\left(\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}H\left(\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}A_{ji}^{(j)}+B_{21}\right) \\ &\times\left(A_{ij}^{(j)}\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)\right)^{-1} \\ &\times\left(\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}H+\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)-B_{12}H\right) \\ &+\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}H^2 \\ &+\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}H+\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\end{aligned} \tag{265}$$

This equation completes the resolution by yielding H. However it is simpler to solve if we cast it into an other form through a change of variable. Actually, using (263) and (53), equation (258) can be organized in the following way. Regroup the terms proportional to F and let:

$$H'=H\left(\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}+\sqrt{\beta}\left(A_{jj}^{(j)}\right)_{eff}^{-1}$$

then:

$$
\begin{aligned}
&\sqrt{\beta}\left\{\frac{-\left(\epsilon^{(j)}_{\{kj\}j\geqslant k}\right)_{eff}}{2}, A^{(j)}_{\{kj\}i>k>j}\right\}(G+B_{21})\,F \\
=\quad&\left(\sqrt{\beta}\left\{\frac{-\left(\epsilon^{(j)}_{\{kj\}j\geqslant k}\right)_{eff}}{2}, A^{(j)}_{\{kj\}i>k>j}\right\}H\left(\left\{\frac{-\left(\epsilon^{(j)}_{\{jk\}k\leqslant j}\right)_{eff}}{2}, A^{(j)}_{\{jk\}i>k>j}\right\}\right)^{-1}A^{(j)}_{ji}\right. \\
&\left.+\,\beta\left(\left\{\frac{-\left(\epsilon^{(j)}_{\{kj\}j\geqslant k}\right)_{eff}}{2}, A^{(j)}_{\{kj\}i>k>j}\right\}\right)\left(A^{(j)}_{jj}\right)^{-1}_{eff}A^{(j)}_{ji}\right)F \\
=\quad&\sqrt{\beta}\left\{\frac{-\left(\epsilon^{(j)}_{\{kj\}j\geqslant k}\right)_{eff}}{2}, A^{(j)}_{\{kj\}i>k>j}\right\}\left(H\left(\left\{\frac{-\left(\epsilon^{(j)}_{\{jk\}k\leqslant j}\right)_{eff}}{2}, A^{(j)}_{\{jk\}i>k>j}\right\}\right)^{-1}+\sqrt{\beta}\left(A^{(j)}_{jj}\right)^{-1}_{eff}\right)A^{(j)}_{ji}F \\
=\quad&\sqrt{\beta}\left\{\frac{-\left(\epsilon^{(j)}_{\{kj\}j\geqslant k}\right)_{eff}}{2}, A^{(j)}_{\{kj\}i>k>j}\right\}H'A^{(j)}_{ji}F \\
=\quad&\sqrt{\beta}\left\{\frac{-\left(\epsilon^{(j)}_{\{kj\}j\geqslant k}\right)_{eff}}{2}, A^{(j)}_{\{kj\}i>k>j}\right\}H'A^{(j)}_{ji}A^{(j)}_{ij}\left(\left\{\frac{-\left(\epsilon^{(j)}_{\{kj\}j\geqslant k}\right)_{eff}}{2}, A^{(j)}_{\{kj\}i>k>j}\right\}\right)^{-1} \\
&\left(\left(\left\{\beta A^{(i)}_{jj}+\left(A^{(j)}_{jj}\right)_{eff}, B_{22}\right\}\left(H'-\left(A^{(j)}_{jj}\right)^{-1}_{eff}\right)+\sqrt{\beta}\right)\right. \\
&-\,\beta\left\{\frac{-\left(\epsilon^{(j)}_{\{kj\}j\geqslant k}\right)_{eff}}{2}, A^{(j)}_{\{kj\}i>k>j}\right\}\left(A^{(j)}_{jj}\right)^{-1}_{eff}\left(\left\{\frac{-\left(\epsilon^{(j)}_{\{jk\}j\geqslant k}\right)_{eff}}{2}, A^{(j)}_{\{jk\}i>k>j}\right\}\right)\left(H'-\left(A^{(j)}_{jj}\right)^{-1}_{eff}\right) \\
&\times\left\{\frac{-\left(\epsilon^{(j)}_{\{jk\}k\leqslant j}\right)_{eff}}{2}, A^{(j)}_{\{jk\}i>k>j}\right\}
\end{aligned}
$$

The remaining terms

$$
\sqrt{\beta}\left\{\frac{-\left(\epsilon^{(j)}_{\{kj\}j\geqslant k}\right)_{eff}}{2}, A^{(j)}_{\{kj\}i>k>j}\right\}H^2+\left\{\beta A^{(i)}_{jj}+\left(A^{(j)}_{jj}\right)_{eff}, B_{22}\right\}H+\sqrt{\beta}\left\{\frac{-\left(\epsilon^{(j)}_{\{jk\}k\leqslant j}\right)_{eff}}{2}, A^{(j)}_{\{jk\}i>k>j}\right\}
$$

can also be factored:

$$\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}H^2+\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}H \tag{266}$$

$$+\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}$$

$$= \left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}\left(\sqrt{\beta}H\left(H'-\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)+\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\right.$$

$$\left.\times\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}\left(H'-\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)+\sqrt{\beta}\right)\right)$$

$$\times\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}$$

And (266) becomes:

$$0=\sqrt{\beta}\qquad H'A_{ji}^{(j)}A_{ij}^{(j)}\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}$$

$$\times\left(\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}\left(H'-\sqrt{\beta}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)+\sqrt{\beta}\right)\right.$$

$$-\beta\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\left(\left\{\frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}\right)\left(H'-\sqrt{\beta}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)\Bigg)$$

$$+\sqrt{\beta}H\left(H'-\sqrt{\beta}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)+\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}$$

$$\times\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}\left(H'-\sqrt{\beta}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)+\sqrt{\beta}\right)$$

or equivalently:

$$0=\left(\sqrt{\beta}H'A_{ji}^{(j)}A_{ij}^{(j)}+1\right)\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}$$

$$\times\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}\left(H'-\sqrt{\beta}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)+\sqrt{\beta}\right)$$

$$-\sqrt{\beta}\beta H'A_{ji}^{(j)}A_{ij}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\left(\left\{\frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}\right)\left(H'-\sqrt{\beta}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)$$

$$+\sqrt{\beta}\left(H'-\sqrt{\beta}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)\left(\left\{\frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}\right)\left(H'-\sqrt{\beta}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)$$

factor by $\left(\left\{\frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}\right)\left(H'-\sqrt{\beta}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)$ on the right, multiply by $(H')^{-1}$ and factor by $\left(\sqrt{\beta}A_{ji}^{(j)}A_{ij}^{(j)}+(H')^{-1}\right)$ on the left yields:

$$\begin{aligned}
0 &= \left(\sqrt{\beta}A_{ji}^{(j)}A_{ij}^{(j)}+(H')^{-1}\right)\\
&\times\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\\
&\times\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}+\sqrt{\beta}\left(H'-\sqrt{\beta}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)^{-1}\right)\left(\left\{\frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}\\
&-\beta\left(\sqrt{\beta}A_{ji}^{(j)}A_{ij}^{(j)}+(H')^{-1}\right)\left(A_{jj}^{(j)}\right)_{eff}^{-1}+\sqrt{\beta}
\end{aligned}$$

or, which is equivalent:

$$\begin{aligned}
&\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}+\sqrt{\beta}\left(H'-\sqrt{\beta}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)^{-1}\right)\\
&\times\left(\left\{\frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}\\
=\;&\beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}-\sqrt{\beta}\left(\sqrt{\beta}A_{ji}^{(j)}A_{ij}^{(j)}+(H')^{-1}\right)^{-1} \qquad (267)
\end{aligned}$$

For later purpose, note that the transpose of this equation shows that $(H')^t$ is solution for the same equation. Given the unicity of solution when $\beta\rightarrow 0$, $(H')^t=H'$, thus H' is symetric.

This equation, once solved, allows to find E, F, G by (275), (53) and (263), and then the dynamical matrix D from which we derive the effective action, as explained in appendix 1. The dynamical matrix D is then:

$$D=\frac{1}{\sqrt{\beta}}\begin{pmatrix} E & F \\ G & F\end{pmatrix}$$

We now include the coefficient $\frac{1}{\sqrt{\beta}}$ in the definition of E, F, G, H.

Having found D, we recover the matrices needed to compute the effective action, by finding an expression for $\frac{1}{2}\left((A-C)D+2B\right)^S$. However, since,

$$\begin{aligned}
&\left((A-C)(D-2)+2B\right)\\
=\;&\left((A-C)D+2(B-A)\right)+2C
\end{aligned}$$

and C is antisymetric,

$$\left((A-C)(D-2)+2B\right)^S=\left((A-C)D+2(B-A)\right)^S$$

Which can be rewritten:

$$\begin{aligned}
\frac{1}{2}\left((A-C)D+2B\right)^S &= \left(\begin{pmatrix} 0 & \Gamma \\ 0 & \Theta\end{pmatrix}\begin{pmatrix} E & F \\ G & H\end{pmatrix}+\begin{pmatrix} \Delta_1 & B_{12} \\ B_{21} & \Delta_2\end{pmatrix}\right)^S\\
&= \begin{pmatrix} \Gamma G+\Delta_1 & \Gamma H+B_{12} \\ \Theta G+B_{21} & \Theta H+\Delta_2\end{pmatrix}^S
\end{aligned}$$

with:

$$\begin{aligned}
\Gamma &= \sqrt{\beta} A_{ij}^{(j)} \\
\Theta &= \sqrt{\beta} \left\{ \frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)} \right\} \\
\Delta_1 &= A_{ii}^{(i)} + B_{11} \\
\Delta_2 &= \left\{ \beta A_{jj}^{(i)} + \left(A_{jj}^{(j)}\right)_{eff}, B_{22} \right\}
\end{aligned}$$

and $A_{ii}^{(i)}$ normalized to 1. By construction, Δ_1 and Δ_2 are symetric matrices. Given (275), (53), (54) (265) and (267) it yields:

$$\begin{aligned}
\Gamma G + \Delta_1 &= \sqrt{\beta} A_{ij}^{(j)} H \left(\left\{ \frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)} \right\} \right)^{-1} A_{ji}^{(j)} + \Delta_1 \qquad (268) \\
&= A_{ij}^{(j)} \left(H' - \sqrt{\beta} \left(A_{jj}^{(j)}\right)_{eff}^{-1} \right) A_{ji}^{(j)} + \Delta_1 \\
\Gamma H &= A_{ij}^{(j)} \left(H' - \sqrt{\beta} \left(A_{jj}^{(j)}\right)_{eff}^{-1} \right) \left\{ \frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)} \right\} \\
\Theta G &= \left\{ \frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)} \right\} H \left(\left\{ \frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)} \right\} \right)^{-1} A_{ji}^{(j)} \\
&= \left\{ \frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)} \right\} \left(H' - \sqrt{\beta} \left(A_{jj}^{(j)}\right)_{eff}^{-1} \right) A_{ji}^{(j)}
\end{aligned}$$

Since H' is symetric, as explained before, and since $\left(A_{jj}^{(j)}\right)_{eff}^{-1}$ is symetric by construction, then $\Gamma G + 1$ is symetric and moreover $\Gamma H = (\Theta G)^t$. Moreover,

$$\Theta H = \left\{ \frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)} \right\} \left(H' - \sqrt{\beta} \left(A_{jj}^{(j)}\right)_{eff}^{-1} \right) \left\{ \frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)} \right\}$$

is also symetric. As a consequence:

$$\frac{1}{2}\left((A-C)D + 2B\right)^S = \frac{1}{2}\left((A-C)D + 2B\right) = \begin{pmatrix} \Gamma G + \Delta_1 & \Gamma H + B_{12} \\ \Theta G + B_{21} & \Theta H + \Delta_2 \end{pmatrix}$$

and:

$$\begin{aligned}
&\left(\left((A-C)(D-2) + 2B\right)^S\right)^{-1} (A+C) \\
&= \begin{pmatrix} \Upsilon_1 & -(\Gamma G + \Delta_1)^{-1} (\Gamma H + B_{12}) \times \Upsilon_2 \\ -(\Theta H + \Delta_2)^{-1} (\Theta G + B_{21}) \times \Upsilon_1 & \Upsilon_2 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ \Phi & \Psi \end{pmatrix} \\
&= \begin{pmatrix} -(\Gamma G + \Delta_1)^{-1} (\Gamma H + B_{12}) \Upsilon_2 \Phi & -(\Gamma G + \Delta_1)^{-1} (\Gamma H + B_{12}) \Upsilon_2 \Psi \\ \Upsilon_2 \Phi & \Upsilon_2 \Psi \end{pmatrix}
\end{aligned}$$

where:

$$\Upsilon_1 = \left((\Gamma G+\Delta_1)-(\Gamma H+B_{12})(\Theta H+\Delta_2)^{-1}(\Theta G+B_{21})\right)^{-1}$$
$$\Upsilon_2 = \left(\Theta H+\Delta_2-(\Theta G+B_{21})(\Gamma G+\Delta_1)^{-1}(\Gamma H+B_{12})\right)^{-1}$$

The matrices intervening in the effective action (247)

$$\begin{aligned}
N_{ii} &= ((A-C)(D-2)+2B)_{ii}-((A-C)(D-2)+2B)_{ij}\left(((A-C)(D-2)+2B)_{jj}\right)^{-1}\left(((A-C)(D-2)+2B)_{ji}\right) \\
&= \Gamma G+\Delta_1-(\Gamma H+B_{12})(\Theta H+\Delta_2)^{-1}(\Theta G+B_{21}) \\
M_{ii} &= (N_{ii})\left(\left(((A-C)(D-2)+2B)^{S}\right)^{-1}(A+C)\right)_{ii} \\
&= -(N_{ii})(\Gamma G+\Delta_1)^{-1}(\Gamma H+B_{12})\left(\Theta H+\Delta_2-(\Theta G+B_{21})(\Gamma G+\Delta_1)^{-1}(\Gamma H+B_{12})\right)^{-1}\Gamma^t \\
M_{ij} &= (N_{ii})\left(((A-C)(D-2)+2B)^{-1}(A+C)\right)_{ij} \\
&= -(N_{ii})(\Gamma G+\Delta_1)^{-1}(\Gamma H+B_{12})\left(\Theta H+\Delta_2-(\Theta G+B_{21})(\Gamma G+\Delta_1)^{-1}(\Gamma H+B_{12})\right)^{-1}\Theta^t
\end{aligned}$$

Where the various matrices are given by (268).

When $A_{ij}^{(j)}$ is $(m+k)\times m$ (that is, $A_{ij}^{(j)}$ has more rows than columns), one can go further in the resolution and obtain more tractable relation than (265). The reason is that in that case, the dominating agent has a number of action variables greater or equal to the number of substructures. This over determination creates some symetries (possibilities of switching the way of action to get equivalent results).

These symetries reflect in the following way: Consider k matrices V_l $l=1...k$ where $\dim(V_l)=\dim\left(A_{ji}^{(i)}\right)$ which is $m\times(m+k)$. Each V_l is filled with 1 in m places and 0 elsewhere, such that $rank(V_l)=m$.

Coming back to (261) and (262), we multiply the first equation (261) by $(V_l)^t$ on the right allows for expressing H as a function of G.

$$\left(\sqrt{\beta}A_{ij}^{(j)}G+\left(A_{ii}^{(i)}+B_{11}\right)\right)E(V_l)^t+\left(B_{12}+\sqrt{\beta}A_{ij}^{(j)}H\right)G(V_l)^t=0 \tag{269}$$

$$\left(\sqrt{\beta}A_{ij}^{(j)}G+\left(A_{ii}^{(i)}+B_{11}\right)\right)F+\left(B_{12}+\sqrt{\beta}A_{ij}^{(j)}H\right)H=0 \tag{270}$$

Then multiply the first equation by $\left(G(V_l)^t\right)^{-1}$ and (262) by H^{-1}. Then, since $\left(\sqrt{\beta}A_{ij}^{(j)}G+\left(A_{ii}^{(i)}+B_{11}\right)\right)$ is a square matrix one obtains:

$$E(V_l)^t\left(G(V_l)^t\right)^{-1}=FH^{-1}$$

Given (52) and (53) it is equivalent to:

$$\begin{aligned}
&\left(\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}-B_{12}\right)+\sqrt{\beta}\left(V_lA_{ij}^{(j)}\right)^t\left(G(V_l)^t\right)^{-1}\right) \\
=&\left(\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}-B_{12}\right)+\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}H^{-1}\right)
\end{aligned}$$

that is:

$$\left(V_lA_{ij}^{(j)}\right)^t\left(G(V_l)^t\right)^{-1}=\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}H^{-1}$$

That expresses H as a function of G:

$$H = \left(G\left(V_l\right)^t\right)\left(\left(V_l A_{ij}^{(j)}\right)^t\right)^{-1}\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}$$

With E, F, H expressed as functions of G, the all problem consists now in finding G. However, given (263):

$$G = H\left(\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1} A_{ji}^{(j)}$$

one obtains:

$$\begin{aligned} G &= \left(G\left(V_l\right)^t\right)\left(\left(V_l A_{ij}^{(j)}\right)^t\right)^{-1} A_{ji}^{(j)} \\ &\equiv X^{-1}\left(A_{ij}^{(j)}\right)^t \end{aligned}$$

with $X = \left(V_l A_{ij}^{(j)}\right)^t \left(G\left(V_l\right)^t\right)^{-1}$. Then, the all system reduces to find $\left(G\left(V_l\right)^t\right)$, or equivalently X which appears to be a more convenient variable. With that choice of variables, H rewrites:

$$H = X^{-1}\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}$$

However, the independence of H in l yields $k-1$ constraint equations, that ultimately reduce the free parameters to $\left(G\left(V_l\right)^t\right)$. Actually when $l \neq m$:

$$\begin{aligned} H &= \left(G\left(V_l\right)^t\right)\left(\left(V_l A_{ij}^{(j)}\right)^t\right)^{-1}\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\} \\ &= \left(G\left(V_m\right)^t\right)\left(\left(V_m A_{ij}^{(j)}\right)^t\right)^{-1}\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\} \end{aligned}$$

that is:

$$\left(G\left(V_m\right)^t\right) = \left(G\left(V_l\right)^t\right)\left(\left(V_l A_{ij}^{(j)}\right)^t\right)^{-1}\left(\left(V_m A_{ij}^{(j)}\right)^t\right)$$

If V_m is partitionned in two matrices:

$$V_m = \left(V_m^{(1)}, V_m^{(2)}\right)$$

and $V_m^{(1)}$ is transverse to V_l (by transverse we mean that the 1 of the submatrix $V_m^{(1)}$ are not in the same columns as the 1 of V_l), the constraint allows to express $rank\left(V_m^{(1)}\right)$ parameters of V_l in function of $\left(G\left(V_l\right)^t\right)$ that remain the parameters to determine:

$$\left(G\left(V_m^{(1)}, 0\right)^t\right) = \left(G\left(V_l\right)^t\right)\left(\left(V_l A_{ij}^{(j)}\right)^t\right)^{-1}\left(\left(\left(V_m^{(1)}, 0\right) A_{ij}^{(j)}\right)^t\right)$$

This allows to find G as a function of $\left(G\left(V_l\right)^t\right)$.

Actually,

$$\left(G\left(V_m^{(1)},0\right)^t\right)=\left(G\left(V_l\right)^t\right)\left(\left(V_lA_{ij}^{(j)}\right)^t\right)^{-1}\left(\left(\left(V_m^{(1)},0\right)A_{ij}^{(j)}\right)^t\right)$$

or:

$$G\left(V_m^{(1)}\right)^t=G\left(V_l\right)^t\left(\left(V_lA_{ij}^{(j)}\right)^t\right)^{-1}\left(\left(V_m^{(1)}\right)A_{ij}^{(j)}\right)^t$$

allows to compute $G\left(A_{ii}^{(i)}\right)^{-1}\sqrt{\beta}A_{ij}^{(j)}$ in the following way:

Partition G and $\left(A_{ii}^{(i)}\right)^{-1}\sqrt{\beta}A_{ij}^{(j)}$ along $V_m^{(1)}$ and V_l:

$$\begin{aligned}G&=\left(G\left(\left(V_m^{(1)},0\right)\right)^t,G\left(\left(0,V_l^{(1)}\right)\right)^t\right)\\&=\left(G\left(V_m^{(1)}\right)^t,\left(G\left(V_l\right)^t\right)\right)\end{aligned}$$

$$\sqrt{\beta}A_{ij}^{(j)}=\begin{pmatrix}V_m^{(1)}\\V_l\end{pmatrix}\sqrt{\beta}A_{ij}^{(j)}$$

where $V_l^{(1)}$ is defined by $V_l=\left(0,V_l^{(1)}\right)$.

As a consequence:

$$\left(\sqrt{\beta}A_{ij}^{(j)}G+\left(A_{ii}^{(i)}+B_{11}\right)\right)=\left(\sqrt{\beta}A_{ij}^{(j)}X^{-1}\left(A_{ij}^{(j)}\right)^t+\left(A_{ii}^{(i)}+B_{11}\right)\right)$$

and the equation (261) becomes:

$$\left(\sqrt{\beta}A_{ij}^{(j)}X^{-1}\left(A_{ij}^{(j)}\right)^t+\left(A_{ii}^{(i)}+B_{11}\right)\right)E\left(V_l\right)^t+\left(B_{12}+\sqrt{\beta}A_{ij}^{(j)}H\right)G\left(V_l\right)^t=0$$

which is equivalent to:

$$\begin{aligned}&\left(\sqrt{\beta}A_{ij}^{(j)}X^{-1}\left(A_{ij}^{(j)}\right)^t+\left(A_{ii}^{(i)}+B_{11}\right)\right)E\left(V_l\right)^t\left(G\left(V_l\right)^t\right)^{-1}\\=&-\left(B_{12}+\sqrt{\beta}A_{ij}^{(j)}H\right)\\=&-\left(B_{12}+\sqrt{\beta}A_{ij}^{(j)}X^{-1}\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)\\=&-\left(\beta A_{ij}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}+\sqrt{\beta}A_{ij}^{(j)}X^{-1}\right)\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\end{aligned}\tag{271}$$

Use the expression for B_{11} multiply by V_l and simplify by $\left(V_lA_{ij}^{(j)}\right)$:

$$\begin{aligned}&\left(\sqrt{\beta}X^{-1}\left(A_{ij}^{(j)}\right)^t+\left(V_lA_{ij}^{(j)}\right)^{-1}V_l\left(1+\beta A_{ij}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{ji}^{(j)}\right)\right)E\left(V_l\right)^t\left(G\left(V_l\right)^t\right)^{-1}\\=&-\left(\beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}+\sqrt{\beta}X^{-1}\right)\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\end{aligned}\tag{272}$$

given that:

$$
\begin{aligned}
E\left(V_l\right)^t\left(G\left(V_l\right)^t\right)^{-1} &= \left(A_{ij}^{(j)}\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}\right)\right)^{-1} \\
&\times\left(\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}\right)G\left(V_l\right)^t+\sqrt{\beta}A_{ji}^{(j)}\left(V_l\right)^t\right)-B_{12}G\left(V_l\right)^t\right)\left(G\left(V_l\right)^t\right)^{-1} \\
&= \left(A_{ij}^{(j)}\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}\right)\right)^{-1} \\
&\times\left(\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}\right)+\sqrt{\beta}X\right) \\
&-\left(\beta A_{ij}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\} \\
&= A_{ij}^{(j)}\left(\left(\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}\right)\right)^{-1}\left(\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}\right)+\sqrt{\beta}X\right)\right. \\
&\times\left.\left(\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}-\beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right) \\
&\times\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}
\end{aligned}
$$

Equation (272) becomes:

$$
\begin{aligned}
&-\left(\beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}+\sqrt{\beta}X^{-1}\right) \\
=\ &\left(\sqrt{\beta}X^{-1}\left(A_{ij}^{(j)}\right)^tA_{ij}^{(j)}+\left(1+\beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{ji}^{(j)}A_{ij}^{(j)}\right)\right) \\
&\times\left(\left(\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2}, A_{\{kj\}i>k>j}^{(j)}\right\}\right)\right)^{-1}\left(\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}\right)+\sqrt{\beta}X\right)\right. \\
&\times\left.\left(\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2}, A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}-\beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)
\end{aligned}
$$

that is:

$$\begin{aligned}
& -\left(\beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}+\sqrt{\beta}X^{-1}\right) && (273)\\
= & \left(\left(\sqrt{\beta}X^{-1}+\beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)A_{ji}^{(j)}A_{ij}^{(j)}+1\right)\\
& \times\left(\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\left(\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}\right)+\sqrt{\beta}X\right)\right.\\
& \left.\times\left(\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}-\beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}\right)
\end{aligned}$$

This quadratic equation for X, when solved for X, allows to find the all matrix D. Actually, collecting our previous results:

$$\begin{aligned}
\left(G\left(V_l\right)^t\right) &= X^{-1}\left(V_l A_{ij}^{(j)}\right)^t\\
G &= X^{-1}\left(A_{ij}^{(j)}\right)^t\\
H &= X^{-1}\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\\
E &= \left(\sqrt{\beta}A_{ij}^{(j)}\left(\left(\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}G+\sqrt{\beta}A_{ji}^{(j)}\right)\right)-B_{12}G\right)\\
F &= \sqrt{\beta}A_{ij}^{(j)}\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\\
&\quad\times\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}H+\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)-B_{12}H
\end{aligned}$$

Note for the sequel that since the equation for X can be rewriten in a symetric form. Actually, set

$$Y=\sqrt{\beta}X^{-1}+\beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}$$

the equation (273) is turned to:

$$\left(YA_{ji}^{(j)}A_{ij}^{(j)}+1\right)\left(\begin{array}{c}\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\left(\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}\right)+\sqrt{\beta}\left(\frac{Y}{\sqrt{\beta}}-\sqrt{\beta}\right)^{-1}\right)\\ \left(\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}-\beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}\end{array}\right)=-Y$$

or, simplifying by Y:

$$\left(A_{ji}^{(j)}A_{ij}^{(j)}+Y^{-1}\right)\left(\begin{array}{c}\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\left(\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}\right)+\sqrt{\beta}\left(\frac{Y}{\sqrt{\beta}}-\sqrt{\beta}\right)^{-1}\right)\\ \left(\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}-\beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}\end{array}\right)=-1$$

which means that the two matrics in the left hand side are each other inverse (up to a minus sign). One thus also have:

$$\left(\begin{array}{c}\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\left(\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}\right)+\sqrt{\beta}\left(\frac{Y}{\sqrt{\beta}}-\sqrt{\beta}\right)^{-1}\right)\\ \left(\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}-\beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}\end{array}\right)\left(A_{ji}^{(j)}A_{ij}^{(j)}+Y^{-1}\right)=-1$$

whose transpose is (we recall here that B_{22} is symetric by construction, as well as $A_{jj}^{(i)}$ by assumption of the model, and $\left(A_{jj}^{(j)}\right)_{eff}$ by construction):

$$\left(A_{ji}^{(j)}A_{ij}^{(j)}+\left(Y^{t}\right)^{-1}\right)\left(\begin{array}{c}\left(\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\left(\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}\right)+\sqrt{\beta}\left(\frac{Y^{t}}{\sqrt{\beta}}-\sqrt{\beta}\right)^{-1}\right)\\ \left(\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}-\beta\left(A_{jj}^{(j)}\right)_{eff}^{-1}\end{array}\right)=-1$$

and then Y^t is also solution of the problem, which in turn implies that X^t is also solution of (273). However, since we look for the unique solution X corresponding to the perturbative solution in powers of β, one deduce that X is symetric, $X^t = X$. Moreover, since $X = \left(V_l A_{ij}^{(j)}\right)^t \left(G\left(V_l\right)^t\right)^{-1}$, one can also say that $X^{-1} = \left(G\left(V_l\right)^t\right)\left(\left(V_l A_{ij}^{(j)}\right)^t\right)^{-1}$ is symetric. This is useful below.

Having found D, we recover the matrices needed to compute the effective action, by finding an expression for $\frac{1}{2}\left(\left(A-C\right)D+2B\right)^S$. However, since,

$$\begin{aligned}&\left(\left(A-C\right)\left(D-2\right)+2B\right)\\=&\left(\left(A-C\right)D+2\left(B-A\right)\right)+2C\end{aligned}$$

and C is antisymetric,

$$\left(\left(A-C\right)\left(D-2\right)+2B\right)^S=\left(\left(A-C\right)D+2\left(B-A\right)\right)^S$$

Which can be rewritten:

$$\begin{aligned}\frac{1}{2}\left(\left(A-C\right)D+2B\right)^S &= \left(\left(\begin{array}{cc}0&\Gamma\\0&\Theta\end{array}\right)\left(\begin{array}{cc}E&F\\G&H\end{array}\right)+\left(\begin{array}{cc}1&0\\0&\Delta\end{array}\right)\right)^S\\ &= \left(\begin{array}{cc}\Gamma G+1&\Gamma H\\ \Theta G&\Theta H+\Delta\end{array}\right)^S\end{aligned}$$

with:

$$\begin{aligned}\Gamma &= \sqrt{\beta}A_{ij}^{(j)}\\ \Theta &= \sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\end{aligned}$$

and $A_{ii}^{(i)}$ normalized to 1.

$$\begin{aligned}
\Gamma G+1 &= \sqrt{\beta}A_{ij}^{(j)}X^{-1}\left(A_{ij}^{(j)}\right)^t+1\\
\Gamma H &= \sqrt{\beta}A_{ij}^{(j)}X^{-1}\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}\\
\Theta G &= \sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}X^{-1}\left(A_{ij}^{(j)}\right)^t
\end{aligned}$$

Since X^{-1} is symetric, as explained before, then $\Gamma G+1$ is symetric and moreover $\Gamma H=(\Theta G)^t$.
Moreover,

$$\Theta H=\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}X^{-1}\left\{\frac{-\left(\epsilon_{\{jk\}k\leqslant j}^{(j)}\right)_{eff}}{2},A_{\{jk\}i>k>j}^{(j)}\right\}$$

As a consequence:

$$\frac{1}{2}\left((A-C)D+2B\right)^S=\frac{1}{2}\left((A-C)D+2B\right)=\begin{pmatrix}\Gamma G+1 & \Gamma H\\ \Theta G & \Theta H+\Delta\end{pmatrix}$$

and:

$$\begin{aligned}
&\left(\left((A-C)(D-2)+2B\right)^S\right)^{-1}(A+C)\\
&=\begin{pmatrix}\left((\Gamma G+1)-\Gamma H(\Theta H+\Delta)^{-1}\Theta G\right)^{-1} & \begin{array}{c}-(\Gamma G+1)^{-1}\Gamma H\\ \times\left(\Theta H+\Delta-\Theta G(\Gamma G+1)^{-1}\Gamma H\right)^{-1}\end{array}\\ \begin{array}{c}-(\Theta H+\Delta)^{-1}\Theta G\\ \times\left((\Gamma G+1)-\Gamma H(\Theta H+\Delta)^{-1}\Theta G\right)^{-1}\end{array} & \left(\Theta H+\Delta-\Theta G(\Gamma G+1)^{-1}\Gamma H\right)^{-1}\end{pmatrix}\\
&\times\begin{pmatrix}0 & 0\\ \Phi & \Psi\end{pmatrix}\\
&=\begin{pmatrix}\begin{array}{c}-(\Gamma G+1)^{-1}\Gamma H\\ \times\left(\Theta H+\Delta-\Theta G(\Gamma G+1)^{-1}\Gamma H\right)^{-1}\Phi\end{array} & \begin{array}{c}-(\Gamma G+1)^{-1}\Gamma H\\ \times\left(\Theta H+\Delta-\Theta G(\Gamma G+1)^{-1}\Gamma H\right)^{-1}\Psi\end{array}\\ \left(\Theta H+\Delta-\Theta G(\Gamma G+1)^{-1}\Gamma H\right)^{-1}\Phi & \left(\Theta H+\Delta-\Theta G(\Gamma G+1)^{-1}\Gamma H\right)^{-1}\Psi\end{pmatrix}
\end{aligned}$$

The previous expression can be concatenated again.

$$\begin{aligned}
H &= GV^t(V\Gamma)^{-1}\Theta^t\\
&= X^{-1}\Theta^t
\end{aligned}$$

$$\left(\Theta H+\Delta-\Theta G(\Gamma G+1)^{-1}\Gamma H\right)$$

$$\left(\sqrt{\beta}A_{ij}^{(j)}G+\left(A_{ii}^{(i)}+B_{11}\right)\right)E(V_l)^t+\left(B_{12}+\sqrt{\beta}A_{ij}^{(j)}H\right)G(V_l)^t=0 \tag{274}$$

$$E=\left(\sqrt{\beta}A_{ij}^{(j)}\left(\left(\left(\sqrt{\beta}\left\{\frac{-\left(\epsilon_{\{kj\}j\geqslant k}^{(j)}\right)_{eff}}{2},A_{\{kj\}i>k>j}^{(j)}\right\}\right)\right)^{-1}\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}G+\sqrt{\beta}A_{ji}^{(j)}\right)\right)-B_{12}G\right) \tag{275}$$

$$\left(\Gamma G+\Delta_1\right)\left(\Gamma\Theta^{-1}\left(\Delta_2 G+\Gamma^t\right)-B_{12}G\right)=-\left(B_{12}+\Gamma H\right)G$$

$$V_l\left(\Gamma G+\Delta_1\right)\left(\Gamma\Theta^{-1}\left(\Delta_2 G+\Gamma^t\right)-B_{12}G\right)\left(V_l\right)^t+V_l\left(B_{12}+\Gamma H\right)G\left(V_l\right)^t=0$$

$$\left(V_l\Gamma\right)\left(G\Gamma+1+B_{21}'\Gamma\right)\left(\Theta^{-1}\left(\Delta_2 G+\Gamma^t\right)-B_{11}'G\right)\left(V_l\right)^t+V_l\left(B_{12}+\Gamma H\right)G\left(V_l\right)^t=0$$

$$\left(G\Gamma+1+B_{21}'\Gamma\right)\left(\Theta^{-1}\left(\Delta_2 G\left(V_l\right)^t+\left(V_l\Gamma\right)^t\right)-B_{11}'G\left(V_l\right)^t\right)+\left(\left(V_l\Gamma\right)^{-1}\left(V_l\Gamma B_{11}'\right)+H\right)G\left(V_l\right)^t=0$$

$$\left(G\Gamma+1+B_{21}'\Gamma\right)\left(\Theta^{-1}\left(\Delta_2+\left(V_l\Gamma\right)^t\left(G\left(V_l\right)^t\right)^{-1}\right)-B_{11}'\right)+B_{11}'=-H$$

$$\begin{aligned}
& \left(\Gamma G+\Delta_1\right)^{-1}\left(\Gamma H+B_{12}\right)\\
= & -\left(\Gamma G+\Delta_1\right)^{-1}\left(\left(\Gamma G+\Delta_1\right)\Gamma\left(\Theta^{-1}\left(\Delta_2+\left(V_l\Gamma\right)^t\left(G\left(V_l\right)^t\right)^{-1}\right)-B_{11}'\right)\right)\\
= & -\Gamma\left(\Theta^{-1}\left(\Delta_2+\left(V_l\Gamma\right)^t\left(G\left(V_l\right)^t\right)^{-1}\right)-B_{11}'\right)\\
= & -\left(\Gamma\Theta^{-1}\left(\Delta_2 G+\Gamma^t\right)-B_{12}G\right)
\end{aligned}$$

$$\begin{aligned}
& \Theta H+\Delta_2-\left(\Theta G+B_{21}\right)\left(\Gamma G+\Delta_1\right)^{-1}\left(\Gamma H+B_{12}\right)\\
= & -\Theta\left(\left(G\Gamma+1+B_{21}'\Gamma\right)\left(\Theta^{-1}\left(\Delta_2+\left(V_l\Gamma\right)^t\left(G\left(V_l\right)^t\right)^{-1}\right)-B_{11}'\right)+B_{11}'\right)\\
& +\left(\Theta G+B_{21}\right)\Gamma\left(\Theta^{-1}\left(\Delta_2+\left(V_l\Gamma\right)^t\left(G\left(V_l\right)^t\right)^{-1}\right)-B_{11}'\right)+\Delta_2\\
= & -\Theta\left(\left(G\Gamma+1+B_{21}'\Gamma\right)\left(\Theta^{-1}\left(\Delta_2+\left(V_l\Gamma\right)^t\left(G\left(V_l\right)^t\right)^{-1}\right)-B_{11}'\right)+B_{11}'\right)\\
& +\Theta\left(G\Gamma+B_{21}'\Gamma\right)\left(\Theta^{-1}\left(\Delta_2+\left(V_l\Gamma\right)^t\left(G\left(V_l\right)^t\right)^{-1}\right)-B_{11}'\right)+\Delta_2\\
= & -\Theta\left(\Theta^{-1}\left(\Delta_2+\left(V_l\Gamma\right)^t\left(G\left(V_l\right)^t\right)^{-1}\right)-B_{11}'+B_{11}'\right)+\Delta_2\\
= & -\left(V_l\Gamma\right)^t\left(G\left(V_l\right)^t\right)^{-1}\\
= & -\Theta^t H^{-1}
\end{aligned}$$

$$B_{21}=\Theta\left(A_{jj}^{(j)}\right)_{eff}^{-1}\Gamma^t$$

$$\left(\Gamma G+\Delta_1\right)\Gamma\left(\Theta^{-1}\left(\Delta_2 G+\Gamma^t\right)-B_{12}'G\right)\left(V_l\right)^t+\left(B_{12}'+\Gamma H\right)G\left(V_l\right)^t=0$$

$$\left(\Gamma G+\Delta_1\right)\Gamma\left(\Theta^{-1}\left(\Delta_2+\Gamma^t\left(G\left(V_l\right)^t\right)^{-1}\right)-B_{12}'\right)+\Gamma\left(B_{12}'+H\right)=0$$

$$\begin{aligned}
& \Theta H+\Delta_2-\left(\Theta G+B_{21}\right)\left(\Gamma G+\Delta_1\right)^{-1}\left(\Gamma H+B_{12}\right)\\
& \Theta H+\Delta_2+\left(\Theta G+B_{21}\right)\Gamma\left(\Theta^{-1}\left(\Delta_2+\Gamma^t\left(G\left(V_l\right)^t\right)^{-1}\right)-B_{12}'\right)
\end{aligned}$$

$$-\Theta\left[\left(G\Gamma+\Delta_1\right)\left(\Theta^{-1}\left(\Delta_2+\left(V_l\Gamma\right)^t\left(G\left(V_l\right)^t\right)^{-1}\right)-B_{12}\right)+\left(V_l\Gamma\right)^{-1}\left(V_lB_{12}\right)\right]$$
$$+\left(\Theta G+B_{21}\right)\Gamma\left(\Theta^{-1}\left(\Delta_2+\Gamma^t\left(G\left(V_l\right)^t\right)^{-1}\right)-B_{12}\right)+\Delta_2$$

$$\Theta H+\Delta_2-\left(\Theta G+B_{21}\right)\left(\Gamma G+\Delta_1\right)^{-1}\left(\Gamma H+B_{12}\right)=-H^{-1}$$

$$\begin{aligned}
&\left(\Gamma G+\Delta_1\right)^{-1}\left(\Gamma H+B_{12}\right)\left(\Theta H+\Delta_2-\left(\Theta G+B_{21}\right)\left(\Gamma G+\Delta_1\right)^{-1}\left(\Gamma H+B_{12}\right)\right)^{-1}\\
=\;&-\left(\Gamma G+\Delta_1\right)^{-1}\left(\Gamma H+B_{12}\right)H\left(\Theta^t\right)^{-1}\\
=\;&\left(\Gamma\Theta^{-1}\left(\Delta_2G+\Gamma^t\right)-B_{12}G\right)H\left(\Theta^t\right)^{-1}\\
=\;&-\Gamma\left(\Theta^{-1}\left(\Delta_2+\left(V_l\Gamma\right)^t\left(G\left(V_l\right)^t\right)^{-1}\right)-B'_{11}\right)H\left(\Theta^t\right)^{-1}\\
=\;&-\Gamma\left(\Theta^{-1}\left(\Delta_2+\Theta^tH^{-1}\right)-B'_{11}\right)H\left(\Theta^t\right)^{-1}\\
=\;&-\Gamma\left(\Theta^{-1}\left(\Delta_2H\left(\Theta^t\right)^{-1}+1\right)-B'_{11}H\left(\Theta^t\right)^{-1}\right)
\end{aligned}$$

And ultimately the matrices involved in (247) become:

$$\begin{aligned}
N_{ii} &= \left(\left(A-C\right)\left(D-2\right)+2B\right)_{ii}-\left(\left(A-C\right)\left(D-2\right)+2B\right)_{ij}\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)_{jj}\right)^{-1}\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)_{ji}\right)\\
&= \Gamma G+\Delta_1-\left(\Gamma H+B_{12}\right)\left(\Theta H+\Delta_2\right)^{-1}\left(\Theta G+B_{21}\right)\\
M_{ii} &= \left(N_{ii}\right)\left(\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)^S\right)^{-1}\left(A+C\right)\right)_{ii}\\
&= \left(N_{ii}\right)\Gamma\left(\Theta^{-1}\left(\Delta_2H\left(\Theta^t\right)^{-1}+1\right)-B'_{11}H\left(\Theta^t\right)^{-1}\right)\Gamma^t\\
M_{ij} &= \left(N_{ii}\right)\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)^{-1}\left(A+C\right)\right)_{ij}\\
&= \left(N_{ii}\right)\Gamma\left(\Theta^{-1}\left(\Delta_2H\left(\Theta^t\right)^{-1}+1\right)-B'_{11}H\left(\Theta^t\right)^{-1}\right)\Theta^t
\end{aligned}$$

Appendix 5

For the strategic agent, the matrices defining the effective utility are given by 36, with, in this case:

$$A = \sqrt{\beta}\begin{pmatrix} 0 & A_{1j}^{(1)}+A_{1j}^{(j)} \\ A_{j1}^{(1)}+A_{j1}^{(j)} & \left\{A_{kj}^{(j)},A_{jk}^{(j)}\right\} \end{pmatrix}$$

$$B = \begin{pmatrix} Id^{(1)}+\beta A_{1j}^{(j)}\left(A_{jj}^{(j)}\right)^{-1}A_{j1}^{(j)} & \sqrt{\beta}\left(A_{1j}^{(1)}+A_{1j}^{(j)}\right)+\left\{\beta A_{ij}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{jk}^{(j)}\right\} \\ \sqrt{\beta}\left(A_{j1}^{(1)}+A_{j1}^{(j)}\right)+\left(\left\{\beta A_{ij}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{jk}^{(j)}\right\}\right)^{t} & \left\{\begin{matrix}\left(\beta A_{jj}^{(1)}+A_{jj}^{(j)}\right), \\ \sqrt{\beta}\left\{A_{kj}^{(j)},A_{jk}^{(j)}\right\}\end{matrix}\right\} \end{pmatrix}$$

$$C = \sqrt{\beta}\begin{pmatrix} 0 & A_{1j}^{(1)}-A_{1j}^{(j)} \\ -\left(A_{j1}^{(1)}-A_{j1}^{(j)}\right) & \left\{A_{kj}^{(j)},-A_{jk}^{(j)}\right\} \end{pmatrix}$$

As described in the text, we need to find the expression for the matrix D, and the effective utility for the dominant agent will be deduced from it's expression. The matrix D satifies the equation:

$$\sqrt{\beta}\begin{pmatrix} 0 & A_{1j}^{(j)} \\ 0 & \left\{A_{jk}^{(j)}\right\} \end{pmatrix}D^2+\begin{pmatrix} 1+\beta A_{ij}^{(j)}A_{ji}^{(j)} & \beta A_{ij}^{(j)}A_{jk}^{(j)} \\ \left(\beta A_{ij}^{(j)}A_{jk}^{(j)}\right)^{t} & \left\{\begin{matrix}\left(\beta A_{jj}^{(1)}+A_{jj}^{(j)}\right),\sqrt{\beta}\left\{A_{kj}^{(j)},A_{jk}^{(j)}\right\} \\ \beta A_{lj}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{jk}^{(j)}\end{matrix}\right\} \end{pmatrix}D+\sqrt{\beta}\begin{pmatrix} 0 & 0 \\ A_{j1}^{(j)} & \left\{A_{kj}^{(j)}\right\} \end{pmatrix}=0 \tag{276}$$

To solve this equation, we partition this matrix as:

$$D=\begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

and applying (256), (53) appendix 1.b allows to find all the parameters as a function of H:

$$E=\left(\sqrt{\beta}A_{1j}^{(j)}\left(\left(\sqrt{\beta}\left\{A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\left(\left\{\beta A_{jj}^{(1)}+A_{jj}^{(j)},\beta A_{lj}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{jk}^{(j)}\right\}G+\sqrt{\beta}A_{j1}^{(j)}\right)\right)-\beta A_{1j}^{(j)}A_{jk}^{(j)}G\right)$$

$$F=\left(\sqrt{\beta}A_{1j}^{(j)}\left(\sqrt{\beta}\left\{A_{\{kj\}1>k>j}^{(j)}\right\}\right)^{-1}\left(\left\{\beta A_{jj}^{(i)}+A_{jj}^{(j)},\beta A_{lj}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{jk}^{(j)}\right\}H+\sqrt{\beta}\left\{A_{\{jk\}1>k>j}^{(j)}\right\}\right)-\beta A_{1j}^{(j)}A_{jk}^{(j)}H\right)$$

$$G=H\left(\left\{A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}A_{ji}^{(j)}$$

The problem reduces to find H and H satisfies (265), whose expression, given our assumptions about the parameters $A_{jj}^{(j)}=A_{jj}^{(1)}=1$ in this particular case:

$$\begin{aligned}0=&\left(\sqrt{\beta}\left\{A_{\{kj\}i>k>j}^{(j)}\right\}H\left(\left\{A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}A_{ji}^{(j)}+B_{21}\right)\\&\times\left(\left(A_{ij}^{(j)}\left(\left\{A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}H+\sqrt{\beta}\left\{A_{\{jk\}i>k>j}^{(j)}\right\}\right)-B_{12}H\right)\right)\\&+\sqrt{\beta}\left\{A_{\{kj\}i>k>j}^{(j)}\right\}H^2\\&+\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22}\right\}H+\sqrt{\beta}\left\{A_{\{jk\}i>k>j}^{(j)}\right\}\end{aligned} \tag{277}$$

Given our hypothesis concerning the agent's interactions, we can use the following normalizations $A_{j1}^{(j)}=\alpha$, $A_{\{jk\}i>k>j}^{(j)}=(1)-\delta_{jk}$ where we denote by (1) the matrix filled with 1 in every row. As a consequence, one can find the inverse of $\left\{A_{\{kj\}i>k>j}^{(j)}\right\}$:

$$\left(\left\{A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}=\frac{1}{N-1}\left(1-\delta_{jk}\right)-\frac{N-2}{N-1}\delta_{jk}=\frac{1}{N-1}(1)-\left(\delta_{jk}\right)$$

$A_{j1}^{(j)}A_{1j}^{(j)} = \alpha^2(1)$, N number of agts. $A_{lj}^{(j)}A_{jk}^{(j)} = ((N-2)(1)+\delta_{jk})$. As a consequence we compute some intermediate quantities involved in (277):

$$\left\{\beta A_{jj}^{(i)} + \left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\} = \left\{1+\beta, \beta A_{lj}^{(j)}A_{jk}^{(j)}\right\} = 1+2\beta+\beta(N-2)(1)$$

$$\left(\left\{A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1} = \frac{1}{N-1}(1)-(\delta_{jk})$$

and

$$\begin{aligned}\left(\left\{A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}A_{j1}^{(j)} &= \left(\frac{1}{N-1}(1)-(\delta_{jk})\right)\alpha(1,..,1)^t \\ &= \alpha\left(\frac{N}{N-1}-1\right)(1,..,1)^t = \frac{\alpha}{N-1}(1,..,1)^t\end{aligned}$$

$$\beta A_{1j}^{(j)}\left(\left\{A_{\{jk\}i>k>j}^{(j)}\right\}\right) = \beta\alpha(1)((1)-\delta_{jk}) = \beta\alpha(N-1)(1)$$

We look for a solution for (277) of form:

$$H = \sqrt{\beta}V(1)+\sqrt{\beta}W$$

We first solve the case for $N>1$ and consider $N=1$ as a particular case.

Using first that all the matrices involved in (277) commute leads to:

$$\begin{aligned}0 &= \left(\sqrt{\beta}H+\beta\left\{A_{\{jk\}i>k>j}^{(j)}\right\}\right)A_{ji}^{(j)}A_{ij}^{(j)} \\ &\times\left(\left(\left(\left\{A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}H+\sqrt{\beta}\left\{A_{\{jk\}i>k>j}^{(j)}\right\}\right)-\beta\left\{A_{\{jk\}i>k>j}^{(j)}\right\}H\right)\right) \\ &+\sqrt{\beta}\left\{A_{\{kj\}i>k>j}^{(j)}\right\}H^2 \\ &+\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}H+\sqrt{\beta}\left\{A_{\{jk\}i>k>j}^{(j)}\right\}\end{aligned}$$

And this expression can be factored ultimately as:

$$\begin{aligned}0 = &\left(\left(\sqrt{\beta}H\left(\left\{A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}+\beta\right)A_{ji}^{(j)}A_{ij}^{(j)}+1\right)\left(\left\{\beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22}\right\}H+\sqrt{\beta}\left\{A_{\{jk\}i>k>j}^{(j)}\right\}\right) \\ &+\sqrt{\beta}\left\{A_{\{kj\}i>k>j}^{(j)}\right\}H^2-\beta\left(\sqrt{\beta}H+\beta\left\{A_{\{jk\}i>k>j}^{(j)}\right\}\right)A_{ji}^{(j)}A_{ij}^{(j)}\left\{A_{\{jk\}i>k>j}^{(j)}\right\}H\end{aligned} \tag{278}$$

Replacing then for the various expressions involved in (278) yields:

$$\begin{aligned}0 &= \left(\left(\beta(V(1)+W)\left(\frac{1}{N-1}(1)-(\delta_{jk})\right)+\beta\right)\alpha^2(1)+1\right) \\ &\times(((1+2\beta+\beta((N-2)(1)))(V(1)+W)+((1)-\delta_{jk}))) \\ &+\beta((1)-\delta_{jk})(V(1)+W)^2-\beta^2((V(1)+W)+((1)-\delta_{jk}))\alpha^2(1)((1)-\delta_{jk})(V(1)+W)\end{aligned}$$

and this leads to a system of equations:

$$
\begin{aligned}
0 &= \left(\left(2\beta+1\right)W-\beta W^2-1\right)\\
0 &= V^2\alpha^2\frac{\beta^2}{N-1}\left(N-2\right)N^4+\begin{pmatrix} V\alpha^2\frac{\beta}{N-1}\left(V\left(2\beta+1\right)+\beta W\left(N-2\right)+1\right)-V\alpha^2\beta^2\left(V+1\right)\\ -V\alpha^2\beta^2\left(V-\frac{W}{N-1}\right)\left(N-2\right)\end{pmatrix}N^3\\
&+\begin{pmatrix} V^2\beta+\alpha^2\beta^2\left(\left(V-W\right)\left(V+1\right)-V\left(W-1\right)\right)\\ -\alpha^2\beta\left(V-\frac{W}{N-1}\right)\left(V\left(2\beta+1\right)+\beta W\left(N-2\right)+1\right)+V\alpha^2\beta\left(\beta-\beta W\right)\left(N-2\right)+V\alpha^2\beta\frac{W(2\beta+1)-1}{N-1}\end{pmatrix}N^2\\
&+\begin{pmatrix} \alpha^2\beta^2\left(W\left(V+1\right)+\left(W-1\right)\left(V-W\right)\right)-\beta\left(V^2-2VW\right)\\ +\alpha^2\left(\beta-\beta W\right)\left(V\left(2\beta+1\right)+\beta W\left(N-2\right)+1\right)+V\beta\left(N-2\right)-\alpha^2\beta\left(V-\frac{W}{N-1}\right)\left(W\left(2\beta+1\right)-1\right)\end{pmatrix}N\\
&+\left(V\left(2\beta+1\right)+\beta\left(W^2-2VW\right)+\alpha^2\left(\beta-\beta W\right)\left(W\left(2\beta+1\right)-1\right)+\beta W\left(N-2\right)+\alpha^2\beta^2W\left(W-1\right)+1\right)
\end{aligned}
$$

which reduces to:

$$
W=\frac{1}{\beta}\left(1+2\beta-\sqrt{4\beta^2+1}\right)
$$

and:

$$
\begin{aligned}
0 &= N\beta\frac{\left(N-1\right)^2+N\alpha^2\left(1+\beta\right)}{N-1}V^2\\
&+\frac{2\beta\left(\left(N-1\right)^2+N\alpha^2\left(1+\beta\right)\right)+\left(\left(2+\beta\right)N\left(N-1\right)\alpha^2\beta+\left(N-3\right)N^2\beta+\left(4N-2\right)\beta+N-1\right)}{N-1}V\\
&+\frac{\left(\left(N-1\right)\beta+\left(1+\beta\right)\alpha^2\beta\right)W^2+\beta\left(N-1\right)\left(N+2\alpha^2+\alpha^2\beta-2\right)W+\left(N-1\right)\left(\left(N-1\right)\alpha^2\beta+1\right)}{N-1}
\end{aligned}
$$

Once V, W, H are recovered, one can ultimately find the other matrices that determine the dynamics of the system. For G, one has directly:

$$
\begin{aligned}
G &= H\left(\left\{A^{(j)}_{\{jk\}i>k>j}\right\}\right)^{-1}A^{(j)}_{ji}\\
&\alpha\sqrt{\beta}\frac{NV+W}{N-1}\left(1,..,1\right)^t
\end{aligned}
$$

For E and F we use need the expressions for the matrices involved in the problem:

$$
\begin{aligned}
A^{(j)}_{1j}\left(\left\{A^{(j)}_{\{kj\}i>k>j}\right\}\right)^{-1} &= \alpha\left(1,...,1\right)\left(\frac{1}{N-1}\left(1\right)-\left(\delta_{jk}\right)\right)\\
&= \alpha\left(1,...,1\right)\left(\frac{N}{N-1}-1\right)=\frac{\alpha}{N-1}\left(1,...,1\right)
\end{aligned}
$$

and

$$
\beta A^{(j)}_{1j}\left(\left\{A^{(j)}_{\{jk\}i>k>j}\right\}\right)=\beta\alpha\left(1\right)\left(\left(1\right)-\delta_{jk}\right)=\beta\alpha\left(N-1\right)\left(1\right)
$$

Then E and F are given by:

$$
\begin{aligned}
E &= \sqrt{\beta}A^{(j)}_{1j}\left(\left(\sqrt{\beta}\left\{A^{(j)}_{\{kj\}i>k>j}\right\}\right)^{-1}\left(\left\{\beta A^{(1)}_{jj}+A^{(j)}_{jj},\beta A^{(j)}_{lj}\left(A^{(j)}_{jj}\right)^{-1}_{eff}A^{(j)}_{jk}\right\}G+\sqrt{\beta}A^{(j)}_{j1}\right)\right)-\beta A^{(j)}_{1j}A^{(j)}_{jk}G\\
&= \frac{\alpha}{N-1}\left(1,...,1\right)\left(\left(\left(1+2\beta+\beta\left(\left(N-2\right)\left(1\right)\right)\right)\alpha\sqrt{\beta}\frac{VN}{N-1}\left(1,...,1\right)^t\right)+\sqrt{\beta}\alpha\left(1,...,1\right)^t\right)\\
&\quad-\beta\alpha\left(1,...,1\right)\left(\left(1\right)-\delta_{jk}\right)\alpha\sqrt{\beta}\frac{VN}{N-1}\left(1,...,1\right)^t\\
&= \frac{\alpha N}{N-1}\left(\left(\left(1+2\beta+\beta\left(\left(N-2\right)N\right)\right)\alpha\sqrt{\beta}\frac{VN}{N-1}\right)+\sqrt{\beta}\alpha\right)-\beta\alpha^2\sqrt{\beta}VN^2\\
&= N\sqrt{\beta}\alpha^2\frac{N-1+NV\left(1+\beta\right)}{\left(N-1\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
F &= \left(\sqrt{\beta}A_{1j}^{(j)}\left(\sqrt{\beta}\left\{A_{\{kj\}1>k>j}^{(j)}\right\}\right)^{-1}\left(\left\{\beta A_{jj}^{(i)}+A_{jj}^{(j)},\beta A_{lj}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{jk}^{(j)}\right\}H+\sqrt{\beta}\left\{A_{\{jk\}1>k>j}^{(j)}\right\}\right)-\beta A_{1j}^{(j)}A_{jk}^{(j)}H\right) \\
&= \frac{\alpha}{N-1}\left(1,...,1\right)\Big(\left(1+2\beta+\beta\left(\left(N-2\right)\left(1\right)\right)\right)\sqrt{\beta}V\left(1\right)+\sqrt{\beta}\left(\left(1\right)-\delta_{jk}\right)\Big)-\beta\alpha\left(1,...,1\right)\left(\left(1\right)-\delta_{jk}\right)\sqrt{\beta}V\left(1\right) \\
&= \left(1,...,1\right)\left(\frac{\alpha}{N-1}\Big(\left(1+2\beta+\beta\left(\left(N-2\right)N\right)\right)\sqrt{\beta}VN+\sqrt{\beta}\left(N-1\right)\Big)-\beta\alpha\left(N-1\right)\sqrt{\beta}VN\right) \\
&= \left(1,...,1\right)\sqrt{\beta}\alpha\frac{N-1+NV\left(1+\beta\right)}{N-1}
\end{aligned}
$$

The case $N=1$ has to be considered separately, since for $N=1$, $\left\{A_{\{kj\}i>k>j}^{(j)}\right\}=0$. We can however recover the solution by letting $\left\{A_{\{kj\}i>k>j}^{(j)}\right\}=\epsilon$, and considering the limit $\epsilon\rightarrow 0$. We look for a solution: $H=\sqrt{\beta}W$. (278) becomes:

$$
\left(\left(\beta W\left(\epsilon\right)^{-1}+\beta\right)\alpha^{2}+1\right)\left(\left(1+\beta\right)W+\epsilon\right)+\beta\epsilon W^{2}-\beta^{2}\left(W+\epsilon\right)\alpha^{2}\epsilon W
$$

or, when reorganized in W.

$$
\left(\beta\epsilon-\alpha^{2}\beta^{2}\epsilon+\alpha^{2}\frac{\beta}{\epsilon}\left(\beta+1\right)\right)W^{2}+\left(\left(\beta+1\right)\left(\beta\alpha^{2}+1\right)+\alpha^{2}\beta-\alpha^{2}\beta^{2}\epsilon^{2}\right)W+\epsilon\left(\beta\alpha^{2}+1\right)=0
$$

Looking for a solution $W=\epsilon w$, yields by a first order expansion in ϵ:

$$
0=\epsilon\left(w\left(\left(\beta+1\right)\left(\beta\alpha^{2}+1\right)+\alpha^{2}\beta\right)+\alpha^{2}\beta+w^{2}\alpha^{2}\beta\left(\beta+1\right)+1\right)
$$

and the solution $w=-\frac{1}{\beta+1}$, which allows to recover the solution obtained by solving directly (276):

$$
H=-\frac{\sqrt{\beta}}{\beta+1}\epsilon\rightarrow 0
$$

$$
\begin{aligned}
G &= H\left(\left\{A_{\{jk\}i>k>j}^{(j)}\right\}\right)^{-1}A_{ji}^{(j)} \\
&= H\left(\epsilon\right)^{-1}\alpha \\
&= -\frac{\sqrt{\beta}}{\beta+1}\epsilon\left(\epsilon\right)^{-1}\alpha=-\frac{\sqrt{\beta}}{\beta+1}\alpha
\end{aligned}
$$

$$
\begin{aligned}
E &= \left(\sqrt{\beta}A_{1j}^{(j)}\left(\left(\sqrt{\beta}\left\{A_{\{kj\}i>k>j}^{(j)}\right\}\right)^{-1}\left(\left\{\beta A_{jj}^{(1)}+A_{jj}^{(j)},\beta A_{lj}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{jk}^{(j)}\right\}G+\sqrt{\beta}A_{j1}^{(j)}\right)\right)-\beta A_{1j}^{(j)}A_{jk}^{(j)}G\right) \\
&= \left(\alpha\left(\left(\epsilon\right)^{-1}\left(\left(1+\beta\right)G+\sqrt{\beta}\alpha\right)\right)-\beta\alpha\epsilon G\right)\rightarrow 0
\end{aligned}
$$

$$
\begin{aligned}
F &= \left(\sqrt{\beta}A_{1j}^{(j)}\left(\sqrt{\beta}\left\{A_{\{kj\}1>k>j}^{(j)}\right\}\right)^{-1}\left(\left\{\beta A_{jj}^{(i)}+A_{jj}^{(j)},\beta A_{lj}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{jk}^{(j)}\right\}H+\sqrt{\beta}\left\{A_{\{jk\}1>k>j}^{(j)}\right\}\right)-\beta A_{1j}^{(j)}A_{jk}^{(j)}H\right) \\
&= \left(\alpha\left(\epsilon\right)^{-1}\left(\left(1+\beta\right)H+\sqrt{\beta}\epsilon\right)-\beta\alpha\epsilon H\right)\rightarrow 0
\end{aligned}
$$

Having found the matrices E, F , G and H so that the dynamic matrix D for the first agent is known, one can find the effective action. We use the general formula (248) developped in the the previous section:

$$
\begin{aligned}
U_{eff}\left(X_{i}\left(t\right)\right) &= \frac{1}{2}\left(X_{i}\left(t\right)-\bar{X}_{i}^{(i)e}\right)N_{ii}\left(X_{i}\left(t\right)-\bar{X}_{i}^{(i)e}\right)-\left(X_{i}\left(t\right)-\bar{X}_{i}^{(i)e}\right)\frac{M_{ii}}{\sqrt{\beta}}\left(X_{i}\left(t-1\right)-\bar{X}_{i}^{(i)e}\right) \\
&-\sum_{j<i}\left(X_{i}\left(t\right)-\bar{X}_{i}^{(i)e}\right)\frac{M_{ij}}{\sqrt{\beta}}\left(X_{j}\left(t-1\right)-\bar{X}_{j}^{(i)e}\right)+\sum_{j>i}2X_{i}\left(t\right)A_{ij}^{(i)}\left(X_{j}\left(t-1\right)\right)
\end{aligned} \quad (279)
$$

where we have introduced some objectives $\bar{X}_i^{(i)e}$, $\bar{X}_j^{(i)e}$ for the first agent. In the text, these objectives are set to 0, since we want to focus on the dynamical pattern of the system rather than on it's equilibrium. The matrices M_{ii}, M_{ij}, N_{ii} are computed in Appendix 1, (36). They are:

$$\begin{aligned}
N_{ii} &= ((A-C)(D-2)+2B)_{ii}^S - ((A-C)(D-2)+2B)_{ij}^S \left(((A-C)(D-2)+2B)_{jj}^S\right)^{-1} \left(((A-C)(D-2)+2B)_{ji}^S\right) \\
M_{ii} &= (N_{ii})\left(\left(((A-C)(D-2)+2B)^S\right)^{-1}(A+C)\right)_{ii} \\
M_{ij} &= (N_{ii})\left(\left(((A-C)(D-2)+2B)^S\right)^{-1}(A+C)\right)_{ij}
\end{aligned}$$

where the upperscript S denotes the symetrization of a matrix. We first need to compute the symetrized matrix $((A-C)D+2(B-A))^S$. Since C is antisymetric,

$$((A-C)(D-2)+2B)^S = ((A-C)D+2(B-A))^S$$

As before, we start with the case $N>1$, and we will consider the case $N=1$ later. For $N>1$, the relevant matrices are:

$$\begin{aligned}
(A-C) &= \sqrt{\beta}\begin{pmatrix} 0 & A_{1j}^{(j)} \\ 0 & \left\{A_{jk}^{(j)}\right\} \end{pmatrix} = \sqrt{\beta}\begin{pmatrix} 0 & (1,...,1)\,\alpha \\ 0 & (1)-1 \end{pmatrix} \qquad (280) \\
(A+C) &= \sqrt{\beta}\begin{pmatrix} 0 & 0 \\ \alpha(1,..,1)^t & (1)-1 \end{pmatrix} \\
D &= \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \sqrt{\beta}\begin{pmatrix} N\alpha^2\frac{N-1+NV(1+\beta)}{(N-1)^2} & (1,...,1)\,\alpha\frac{N-1+NV(1+\beta)}{N-1} \\ \alpha\frac{NV+W}{N-1}(1,..,1)^t & V(1)+W \end{pmatrix} \\
B-A &= \begin{pmatrix} 1+\beta A_{ij}^{(j)}A_{ji}^{(j)} & \beta A_{ij}^{(j)}A_{jk}^{(j)} \\ \left(\beta A_{ij}^{(j)}A_{jk}^{(j)}\right)^t & \left\{\begin{matrix} \left(\beta A_{jj}^{(1)}+A_{jj}^{(j)}\right) \\ \beta A_{lj}^{(j)}\left(A_{jj}^{(j)}\right)_{eff}^{-1}A_{jk}^{(j)} \end{matrix}\right\} \end{pmatrix} \\
&= \begin{pmatrix} 1+\beta\alpha^2 & \beta(N-1)(1,...,1) \\ (\beta(N-1)(1,...,1))^t & (1+2\beta)+\beta((N-2)(1)) \end{pmatrix}
\end{aligned}$$

And we find:

$$\begin{aligned}
&((A-C)D+2(B-A)) \\
= &\begin{pmatrix} \left(1+\beta\alpha^2\right)+\beta\alpha^2N\frac{NV+W}{N-1} & \beta(1,...,1)(\alpha(VN+W)+(N-1)) \\ \beta(\alpha(NV+W)+(N-1))(1,..,1)^t & \beta(V(N-1)+W+(N-2))(1)-\beta W+(1+2\beta) \end{pmatrix}
\end{aligned}$$

The inverse of this block matrix is given by:

$$((A-C)D+2(B-A))^{-1} = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$$

with:

$$X = \left(\alpha^2 N\frac{NV+W}{N-1} + \left(1+\beta\alpha^2\right) - \frac{N\beta^2\left(\alpha\left(VN+W\right)+\left(N-1\right)\right)^2\left(2\beta-\beta W+1\right)}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)\right)}\right)^{-1}$$

$$Y = -\left(1,..,1\right)\left(\left(\begin{array}{c}\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\left(1\right)\\ -\beta W+\left(1+2\beta\right)\end{array}\right)^{-1}\beta\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)$$

$$Z = -\left(\begin{array}{c}\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\left(1\right)\\ -\beta W+\left(1+2\beta\right)\end{array}\right)^{-1}\beta\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\left(1,..,1\right)^t$$

$$T = \left(\begin{array}{c}\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\beta\alpha^2 N\frac{NV+W}{N-1}+\left(1+\beta\alpha^2\right)}\right)\left(1\right)\\ -\beta W+\left(1+2\beta\right)\end{array}\right)^{-1}$$

These terms involve the following quantity:

$$T = \left(\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\left(1\right)+\left(1+2\beta\right)-\beta W\right)^{-1}$$

$$= -\frac{\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\left(1\right)}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\right)}$$
$$-\frac{\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\right)}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\right)}$$

One can compute $\left(\left(\left(A-C\right)D+2\left(B-A\right)\right)^{-1}\left(A+C\right)\right)$ by using (280). Some blocks are involved in the computation, that are:

$$\begin{aligned}
& -\left(1,...,1\right)\\
& \times\beta\left(\alpha\left(VN+W\right)+\left(N-1\right)\right)\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)\left(1\right)-\beta W+\left(1+2\beta\right)\right)^{-1}\beta\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)\\
& \times\left(1,..,1\right)^t\\
= \; & \left(1,...,1\right)N\beta^2\left(\alpha\left(VN+W\right)+\left(N-1\right)\right)^2\\
& \times\frac{\left[\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)\left(1\right)-\left(\left(1+2\beta\right)-\beta W+N\beta\left(V\left(N-1\right)+W+\beta\left(N-2\right)\right)\right)\right]}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\beta\left(V\left(N-1\right)+W+\beta\left(N-2\right)\right)\right)}\\
& \times\left(1,..,1\right)^t
\end{aligned}$$

$$\begin{aligned}
& -\left(1,...,1\right)\left(\beta\alpha\left(VN+W\right)+\left(N-1\right)\right)\\
& \times\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)\left(1\right)-\beta W+\left(1+2\beta\right)\right)^{-1}\beta\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)\left(1,..,1\right)^t\\
= \; & \frac{\beta^2\left(\alpha\left(VN+W\right)+\left(N-1\right)\right)^2}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)\right)}\\
& \times\left[\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)N^2-\left(\left(1+2\beta\right)-\beta W+N\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)\right)N\right]\\
= \; & -\frac{N\beta^2\left(\alpha\left(VN+W\right)+\left(N-1\right)\right)^2\left(2\beta-\beta W+1\right)}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)\right)}
\end{aligned}$$

$$
\begin{aligned}
& -(1,..,1)^t\,\beta\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)\left(\beta\alpha^2 N\frac{NV+W}{N-1}+\left(1+\beta\alpha^2\right)\right)^{-1}\beta\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)(1,..,1)\\
= & -\frac{\beta^2\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\alpha^2\beta N\frac{NV+W}{N-1}}\,(1)
\end{aligned}
$$

And as a consequence, the blocks involved in $\left(\left(\left(A-C\right)D+2\left(B-A\right)\right)^{-1}\left(A+C\right)\right)$ are:

$$
\begin{aligned}
& \left(\left(\left(A-C\right)D+2\left(B-A\right)\right)^{-1}\left(A+C\right)\right)_{11}\\
= & \frac{\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)N^2}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\right)}\\
& -\frac{\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\right)N}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\right)}\\
& \times\sqrt{\beta}\alpha\frac{\beta\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\\
= & -\frac{\alpha\sqrt{\beta}\beta N\left(1+2\beta-\beta W\right)\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\right)}
\end{aligned}
$$

$$
\begin{aligned}
& \left(\left(\left(A-C\right)D+2\left(B-A\right)\right)^{-1}\left(A+C\right)\right)_{1j}\\
= & (1,..,1)\frac{\left(N-1\right)}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\right)}\\
& \times\left(\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)N\right.\\
& \left.-\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\right)\right)\\
& \times\sqrt{\beta}\beta\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\\
= & -(1,..,1)\,\beta\sqrt{\beta}\frac{\left(N-1\right)\left(1+2\beta-\beta W\right)\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2}{\left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\right)}
\end{aligned}
$$

The matrices involved in (279) are then ultimately obtained as:

$$\begin{aligned}
N_{11} &= \left(1+\beta\alpha^2\right)+\beta\alpha^2 N\frac{NV+W}{N-1} \\
&-\beta^2\left(\alpha\left(NV+W\right)+\left(N-1\right)\right)^2 \\
&\times\left(\frac{\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha(NV+W)+(N-1)\right)^2}{(1+\beta\alpha^2)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)N^2}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha(NV+W)+(N-1)\right)^2}{(1+\beta\alpha^2)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\right)}\right. \\
&\left.-\frac{N\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha(NV+W)+(N-1)\right)^2}{(1+\beta\alpha^2)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\right)}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha(NV+W)+(N-1)\right)^2}{(1+\beta\alpha^2)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\right)}\right) \\
M_{11} &= -\left(N_{11}\right)\frac{\alpha\sqrt{\beta}\beta N\left(1+2\beta-\beta W\right)\frac{\left(\alpha(NV+W)+(N-1)\right)}{(1+\beta\alpha^2)+\beta\alpha^2 N\frac{NV+W}{N-1}}}{\left(\left(1+2\beta\right)-\beta W\right)\left(\left(1+2\beta\right)-\beta W+N\left(\beta\left(V\left(N-1\right)+W+\left(N-2\right)\right)-\beta^2\frac{\left(\alpha(NV+W)+(N-1)\right)^2}{(1+\beta\alpha^2)+\beta\alpha^2 N\frac{NV+W}{N-1}}\right)\right)} \\
M_{1j} &= -\left(N_{11}\right)\left(1,..,1\right)\frac{\sqrt{\beta}\left(N-1\right)\beta\left(1+2\beta-\beta W\right)\frac{\left(\alpha(NV+W)+(N-1)\right)}{(1+\beta\alpha^2)+\beta\alpha^2 N\frac{NV+W}{N-1}}}{\left(\left(1+2\beta\right)-W\right)\left(\left(1+2\beta\right)-W+N\left(\left(V\left(N-1\right)+W+\beta\left(N-2\right)\right)-\frac{\left(\alpha(NV+W)+\beta(N-1)\right)^2}{(1+\beta\alpha^2)+\alpha^2 N\frac{NV+W}{N-1}}\right)\right)}
\end{aligned}$$

Having found the matrices N_{11}, M_{11} and M_{1j}, the full action for the system of agents is:

$$\begin{aligned}
U_{eff}\left(X_j\left(t\right)\right)+U_{eff}\left(X_i\left(t\right)\right) &= \sum_{j<1}\left(-X_j\left(t\right)A_{jj}^{(j)}X_j\left(t\right)+2X_j\left(t\right)A_{jk}^{(j)}\left(X_k\left(t-1\right)\right)+2X_j\left(t\right)A_{j1}^{(j)}\left(X_1\left(t-1\right)\right)\right) \\
&+\frac{1}{2}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)N_{ii}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)-\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)\frac{M_{ii}}{\sqrt{\beta}}\left(X_i\left(t-1\right)-\bar{X}_i^{(i)e}\right) \\
&-\sum_{j<i}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)\frac{M_{ij}}{\sqrt{\beta}}\left(X_j\left(t-1\right)-\bar{X}_j^{(i)e}\right)
\end{aligned}$$

Then, the mean dynamic, saddle point of the previous global effective utility, is given by the dynamic evolution:

$$\begin{pmatrix} X_i\left(t\right)-\bar{X}_i^{(i)e} \\ X_j\left(t\right) \end{pmatrix}=M_1\begin{pmatrix} X_i\left(t-1\right)-\bar{X}_i^{(i)e} \\ X_j\left(t-1\right)-\bar{X}_j^{(i)e} \end{pmatrix}+M_2\begin{pmatrix} X_i\left(t-1\right) \\ X_j\left(t-1\right) \end{pmatrix}$$

with:

$$\begin{aligned}
M_1 &= \begin{pmatrix} \left(N_{11}\right)^{-1}M_{11} & \left(N_{11}\right)^{-1}M_{1j} \\ 0 & 0 \end{pmatrix} \\
M_2 &= \begin{pmatrix} 0 & 0 \\ \alpha\left(1,..,1\right)^t & \left(1\right)-1 \end{pmatrix}
\end{aligned}$$

On one hand, the previous equation leads to an equilibrium defined by:

$$\begin{pmatrix} \bar{X}_i-\bar{X}_i^{(i)e} \\ \bar{X}_j \end{pmatrix}=M_1\begin{pmatrix} \bar{X}_i-\bar{X}_i^{(i)e} \\ \bar{X}_j-\bar{X}_j^{(i)e} \end{pmatrix}+M_2\begin{pmatrix} \bar{X}_i \\ \bar{X}_j \end{pmatrix}$$

that is:

$$\begin{pmatrix} \bar{X}_i \\ \bar{X}_j \end{pmatrix}=\left(1-M\right)^{-1}\left(\begin{pmatrix} \bar{X}_i^{(i)e} \\ 0 \end{pmatrix}-M_1\begin{pmatrix} \bar{X}_i^{(i)e} \\ \bar{X}_j^{(i)e} \end{pmatrix}\right)$$

with:

$$M = M_1 + M_2 = \begin{pmatrix} (N_{11})^{-1} M_{11} & (N_{11})^{-1} M_{1j} \\ \alpha & (1) - 1 \end{pmatrix}$$

On the other hand, the matrix M and it's eigenvalues yield the dynamical pattern of the system.

$$\begin{aligned} M &= \begin{pmatrix} (N_{11})^{-1} M_{11} & (N_{11})^{-1} M_{1j} \\ \alpha & (1) - 1 \end{pmatrix} \\ &= \begin{pmatrix} -N\alpha \frac{\sqrt{\beta}}{N-1}\Omega & -(1,..,1)\,\Omega \\ \alpha\,(1,..,1)^t & (1) - 1 \end{pmatrix} \end{aligned}$$

with:

$$\Omega = \frac{(N-1)(2\beta - W + 1)\frac{(\alpha(NV+W)+\beta(N-1))}{\alpha^2 N \frac{NV+W}{N-1} + (1+\beta\alpha^2)}}{((1+2\beta) - W)\left((1+2\beta) - W + N\left((V(N-1) + W + \beta(N-2)) - \frac{(\alpha(NV+W)+\beta(N-1))^2}{(1+\beta\alpha^2)+\alpha^2 N \frac{NV+W}{N-1}}\right)\right)}$$

whose eigenvalues are:

$$-1, \frac{1}{2}(a+1) \pm \frac{1}{2}\sqrt{a^2 - 2(N-1)a + 4Nb}$$

with:

$$\begin{aligned} a &= -\frac{\alpha\sqrt{\beta}N(2\beta - W + 1)\frac{(\alpha(NV+W)+\beta(N-1))}{\alpha^2 N \frac{NV+W}{N-1} + (1+\beta\alpha^2)}}{((1+2\beta) - W)\left((1+2\beta) - W + N\left((V(N-1) + W + \beta(N-2)) - \frac{(\alpha(NV+W)+\beta(N-1))^2}{(1+\beta\alpha^2)+\alpha^2 N \frac{NV+W}{N-1}}\right)\right)} \\ b &= -\frac{\alpha N(N-1)(2\beta - W + 1)\frac{(\alpha(NV+W)+\beta(N-1))}{\alpha^2 N \frac{NV+W}{N-1} + (1+\beta\alpha^2)}}{((1+2\beta) - W)\left((1+2\beta) - W + N\left((V(N-1) + W + \beta(N-2)) - \frac{(\alpha(NV+W)+\beta(N-1))^2}{(1+\beta\alpha^2)+\alpha^2 N \frac{NV+W}{N-1}}\right)\right)} \end{aligned}$$

Having found the dynamical pattern for $N > 1$, we can focus on the case $N = 1$. For $N = 1$ the formula reduce to:

$$\begin{aligned} (A - C) &= \sqrt{\beta}\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \\ (A + C) &= \sqrt{\beta}\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} \\ D &= \begin{pmatrix} \frac{1}{\sqrt{\beta}}E & \frac{1}{\sqrt{\beta}}F \\ \frac{1}{\sqrt{\beta}}G & \frac{1}{\sqrt{\beta}}H \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\frac{\alpha}{1+\beta} & 0 \end{pmatrix} \\ B - A &= \begin{pmatrix} 1+\beta\alpha^2 & \beta\alpha \\ \beta\alpha & 1+\beta \end{pmatrix} \end{aligned}$$

leading directly to:

$$\begin{aligned} ((A-C)(D-2) + 2B)^S &= ((A-C)D + 2(B-A))^S \\ &= \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -\frac{\alpha}{1+\beta} & 0 \end{pmatrix} + \begin{pmatrix} 1+\beta\alpha^2 & \beta\alpha \\ \beta\alpha & 1+\beta \end{pmatrix} \\ &= \begin{pmatrix} 1 + \alpha^2\beta - \frac{\alpha^2}{\beta+1} & \alpha\beta \\ \alpha\beta & 1+\beta \end{pmatrix} \end{aligned}$$

and we find:

$$\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)^{S}\right)^{-1}\left(A+C\right)=\left(\begin{array}{cc} -\frac{\beta\alpha^{2}}{\beta-\alpha^{2}+\alpha^{2}\beta+1} & 0\\ \alpha^{2}\frac{\beta\left(\alpha^{2}\beta^{2}+\alpha^{2}\beta-\alpha^{2}+\beta+1\right)}{\left(\alpha\beta^{2}+\alpha\beta\right)\left(\beta-\alpha^{2}+\alpha^{2}\beta+1\right)} & 0 \end{array}\right)$$

As a consequence, the coefficient for the effective utilities are:

$$\begin{aligned} N_{ii} &= 1+\alpha^{2}\beta-\frac{\alpha^{2}}{\beta+1}-\alpha\left(\alpha\beta\right)^{2}\frac{\beta\left(\alpha^{2}\beta^{2}+\alpha^{2}\beta-\alpha^{2}+\beta+1\right)}{\left(\alpha\beta^{2}+\alpha\beta\right)\left(\beta-\alpha^{2}+\alpha^{2}\beta+1\right)}\\ M_{ii} &= \left(N_{ii}\right)\frac{\beta\alpha^{2}}{\beta-\alpha^{2}+\alpha^{2}\beta+1}\\ M_{ij} &= 0 \end{aligned}$$

The previous formula for the equilibrium and the dynamic matrix are still valid. The matrix M is:

$$M=\left(\begin{array}{cc} \frac{\beta\alpha^{2}}{\beta-\alpha^{2}+\alpha^{2}\beta+1} & 0\\ \alpha & 0 \end{array}\right)$$

with eigenvalues $\alpha^{2}\frac{\beta}{\beta-\alpha^{2}+\alpha^{2}\beta+1},0$.

Appendix 6

Recal that the model described above starts with utilities of the kind:

$$-\frac{1}{2}\left(Y_i\left(t\right)-Y_i\left(t-1\right)\right)A\left(Y_i\left(t\right)-Y_i\left(t-1\right)\right)-\frac{1}{2}\left(Y_i\left(t\right)-\hat{Y}_i^{(1)}\right)AD\left(Y_i\left(t\right)-\hat{Y}_i^{(1)}\right)+Y_i\left(t\right)C\left(Y_i\left(t-1\right)-\hat{Y}_i^{(1)}\right) \tag{281}$$

where:

$$\begin{aligned}
A &= \sqrt{\beta}\begin{pmatrix} -\epsilon_{ii}^{(i)} & A_{ij}^{(i)}+A_{ij}^{(j)} \\ A_{ji}^{(i)}+A_{ji}^{(j)} & \left\{\begin{array}{c} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2},\frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ A_{\{kj\}i>k>j}^{(j)},A_{\{jk\}i>k>j}^{(j)} \end{array}\right\} \end{pmatrix} \\
B &= \begin{pmatrix} A_{ii}^{(i)}+B_{11}-\sqrt{\beta}\epsilon_{ii}^{(i)} & \left\{\sqrt{\beta}\left(A_{ij}^{(i)}+A_{ij}^{(j)}\right),B_{12}\right\} \\ \left\{\sqrt{\beta}\left(A_{ji}^{(i)}+A_{ji}^{(j)}\right),B_{12}^{t}\right\} & \left\{\begin{array}{c} \beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff},B_{22} \\ \sqrt{\beta}\left\{\begin{array}{c} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2},\frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2} \\ ,A_{\{kj\}i>k>j}^{(j)},A_{\{jk\}i>k>j}^{(j)} \end{array}\right\} \end{array}\right\} \end{pmatrix} \\
C &= \sqrt{\beta}\begin{pmatrix} 0 & A_{ij}^{(i)}-A_{ij}^{(j)} \\ -\left(A_{ji}^{(i)}-A_{ji}^{(j)}\right) & \left\{\begin{array}{c} \frac{\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2},\frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ -A_{\{kj\}i>k>j}^{(j)},A_{\{jk\}i>k>j}^{(j)} \end{array}\right\} \end{pmatrix}
\end{aligned}$$

Start with the utilities of the three agents:

$$\begin{aligned}
&-\frac{1}{2}\left(n\left(t\right)+1-w\left(t-1\right)\right)^2-\alpha n\left(t\right)s_n\left(t-1\right) \\
&-\frac{1}{2}\rho\left(1-w\left(t-1\right)-\tilde{f}\right)^2-\frac{1}{2}\gamma\left(w\left(t-1\right)-\tilde{w}\right)^2-\frac{1}{2}s_n^2\left(t\right)-\frac{1}{2}s_f^2\left(t\right)-\frac{1}{2}s_w^2\left(t\right) \\
&-\frac{1}{2}\left(w\left(t\right)-w_0\right)^2-\frac{1}{2}\delta n^2\left(t-1\right)-\nu n\left(t-1\right)w\left(t\right)-\kappa s_f\left(t-1\right)\left(1-w\left(t\right)-\tilde{f}\right)-\eta s_w\left(t-1\right)\left(w\left(t\right)-\tilde{t}\right)
\end{aligned}$$

and put them in the following form corresponding to our general model:

$$\begin{aligned}
& \left(n\left(t\right)+1\right)^2+2\alpha n\left(t\right)s_n\left(t-1\right)-2n\left(t\right)w\left(t-1\right) \\
= & \left(n\left(t\right)+1\right)^2+2\alpha n\left(t\right)\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}s\left(t-1\right)-2n\left(t\right)w\left(t-1\right)
\end{aligned}$$

$$\begin{aligned}
& s\left(t\right)\left(Id\right)s\left(t\right)+\rho\left(1-w\left(t-1\right)-\tilde{f}\right)^2+\gamma\left(w\left(t-1\right)-\tilde{w}\right)^2 \\
= & s\left(t\right)\left(Id\right)s\left(t\right)+\left(\rho+\gamma\right)\left(w\left(t-1\right)-\frac{\rho}{\left(\rho+\gamma\right)}\left(1-\tilde{f}\right)-\frac{\gamma}{\left(\rho+\gamma\right)}\tilde{w}\right)^2
\end{aligned}$$

$$\begin{aligned}
& \left(w\left(t\right)-w_0\right)^2+\delta n^2\left(t-1\right)+2\nu n\left(t-1\right)w\left(t\right)+2\kappa s_f\left(t-1\right)\left(1-w\left(t\right)-\tilde{f}\right)+2\eta s_w\left(t-1\right)\left(w\left(t\right)-\tilde{t}\right) \\
= & \left(w\left(t\right)-w_0\right)^2+\delta n^2\left(t-1\right)+2\nu n\left(t-1\right)w\left(t\right)-2\kappa s_f\left(t-1\right)\left(w\left(t\right)-\left(1-\tilde{f}\right)\right)+2\eta s_w\left(t-1\right)\left(w\left(t\right)-\tilde{t}\right)
\end{aligned}$$

Effective action for the first agent:

Starting with the less strategic agent utility

$$(n(t))(1)(n(t)) - 2n(t)(1)w(t-1) + 2\alpha n(t)\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} s(t-1) + 2n(t)$$

we add some inertia in this agent's behavior:

$$(n(t))(1)(n(t)) - \epsilon_1 n(t) n(t-1) - 2n(t)(1)w(t-1) + 2\alpha n(t)\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} s(t-1) + 2n(t)$$

and the matrices defined in (281) are:

$$\begin{aligned} A &= -\epsilon_1\sqrt{\beta} \\ B &= 1-\epsilon_1\sqrt{\beta} \\ C &= 0 \end{aligned}$$

The equation for the dynamic matrix (282)

$$(A-C)D^2 + 2(B-A)D + (A+C) = 0$$

reduces to:

$$AD^2 + 2(B-A)D + AD = 0$$

with solution

$$D = 1 - \sqrt{-2A^{-1}B}$$

$$\begin{aligned} \frac{-\epsilon_1\sqrt{\beta}}{2}D^2 + 1 - \epsilon_1\sqrt{\beta} &= 0 \\ D &= 1 - \sqrt{\frac{2\left(1-\epsilon_1\sqrt{\beta}\right)}{\epsilon_1\sqrt{\beta}}} \\ \hat{Y}_i^{(1)} &= -\frac{1}{1-\epsilon_1\sqrt{\beta}} \end{aligned}$$

in the limit $\epsilon_1 \to 0$

$$\begin{aligned} &((A-C)(D-2)+2B) \\ = \;& \epsilon_1\sqrt{\beta}\left(\sqrt{\frac{2\left(1-\epsilon_1\sqrt{\beta}\right)}{\epsilon_1\sqrt{\beta}}} + 1\right) + 2\left(1-\epsilon_1\sqrt{\beta}\right) \\ \to \;& 2 \end{aligned}$$

and the effective utility (which in this case is also the intermediate effective utility)

$$\begin{aligned} &\left(Y_i(t) - \hat{Y}_i^{(1)} + ((A-C)(D-2)+2B)^{-1}\left((A+C)\left(Y_i(t-1) - \hat{Y}_i^{(1)}\right)\right)\right) \\ &\times ((A-C)(D-2)+2B)\left(Y_i(t) - \hat{Y}_i^{(1)} + ((A-C)(D-2)+2B)^{-1}\left((A+C)\left(Y_i(t-1) - \hat{Y}_i^{(1)}\right)\right)\right) \end{aligned}$$

is in this limit $\epsilon_1 \to 0$:

$$\left(n(t) - \left(n^{(1)}\right)_{eff}\right) 2\left(n(t) - \left(n^{(1)}\right)_{eff}\right) - 2n(t)(1)w(t-1) + 2\alpha n(t)\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} s(t-1)$$

with

$$\left(n^{(1)}\right)_{eff} = 0$$

so that ultimately:

$$U_{eff}(n(t)) = 2(n(t))^2 - 2n(t)(1)w(t-1) + 2\alpha n(t)\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} s(t-1)$$

Using (235), the equilibrium value for this agent is just: $\bar{X}_j^{(i)e} = \bar{n}_1^{(1)e} = 0$.

Effective action for the second agent:

Here again, we can identify the utility for the second agent. The action for the first agent:

$$U_{eff}(n(t)) = 2(n(t))^2 - 2n(t)(1)w(t-1) + 2\alpha n(t)\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} s(t-1)$$

leads to consider the additional quadratic weight $\frac{\beta w^2(t-1)}{\sigma^2}$. Starting with the intertemporal utility for agent 2:

$$w(t)^2 + \delta n^2(t-1) + 2\nu n(t-1)w(t) - 2\kappa s_f(t-1)\left(w(t) - \left(1-\tilde{f}\right)\right) + 2\eta s_w(t-1)\left(w(t)-\tilde{t}\right) - 2w(t)w_0$$

the identification of the affective utility in (281) starts by setting:

$$\begin{aligned}
A &= \sqrt{\beta}\begin{pmatrix} 0 & -\frac{1}{\sigma^2}-\nu \\ -\frac{1}{\sigma^2}-\nu & 0 \end{pmatrix} \\
B &= \begin{pmatrix} 1+\frac{\beta}{\sigma^2} & \sqrt{\beta}\left(-\frac{1}{\sigma^2}-\nu\right) \\ \sqrt{\beta}\left(-\frac{1}{\sigma^2}-\nu\right) & \frac{1}{\sigma^2}+\beta\delta \end{pmatrix} \\
C &= \sqrt{\beta}\begin{pmatrix} 0 & \left(-\nu+\frac{1}{\sigma^2}\right) \\ \left(\nu-\frac{1}{\sigma^2}\right) & 0 \end{pmatrix}
\end{aligned}$$

The equation for the dynamic matrix D

$$(A-C)D^2 + 2(B-A)D + (A+C) = 0 \tag{282}$$

since

$$\begin{aligned}
A-C &= 2\sqrt{\beta}\begin{pmatrix} 0 & -\frac{1}{\sigma^2} \\ -\nu & 0 \end{pmatrix} \\
B-A &= \begin{pmatrix} 1+\frac{\beta}{\sigma^2} & 0 \\ 0 & \frac{1}{\sigma^2}+\beta\delta \end{pmatrix} \\
A+C &= 2\sqrt{\beta}\begin{pmatrix} 0 & -\nu \\ -\frac{1}{\sigma^2} & 0 \end{pmatrix}
\end{aligned}$$

One looks for a solution $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for (282):

$$\sqrt{\beta}\begin{pmatrix} 0 & -\frac{1}{\sigma^2} \\ -\nu & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 + \begin{pmatrix} 1+\frac{\beta}{\sigma^2} & 0 \\ 0 & \frac{1}{\sigma^2}+\beta\delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \sqrt{\beta}\begin{pmatrix} 0 & -\nu \\ -\frac{1}{\sigma^2} & 0 \end{pmatrix} = 0$$

and:

$$D = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$$

$$\sqrt{\beta}\begin{pmatrix} 0 & -\frac{1}{\sigma^2} \\ -\nu & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^2 + \begin{pmatrix} 1+\frac{\beta}{\sigma^2} & 0 \\ 0 & \frac{1}{\sigma^2}+\beta\delta \end{pmatrix}\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + \sqrt{\beta}\begin{pmatrix} 0 & -\nu \\ -\frac{1}{\sigma^2} & 0 \end{pmatrix}$$

$$\begin{aligned}
b\left(\frac{1}{\sigma^2}\beta+1\right) - \sqrt{\beta}\nu - \frac{1}{\sigma^2}bc\sqrt{\beta} &= 0 \\
c\left(\beta\delta+\frac{1}{\sigma^2}\right) - \frac{1}{\sigma^2}\sqrt{\beta} - bc\sqrt{\beta}\nu &= 0
\end{aligned}$$

$$b = \frac{1}{\sigma^2 c\sqrt{\beta}\nu}\left(c\left(1+\sigma^2\beta\delta\right) - \sqrt{\beta}\right)$$

$$\left(1+\sigma^2\beta\delta\right)\sqrt{\beta}c^2-\left(\delta\sigma^4\beta-\sigma^4\beta\nu^2+\delta\sigma^2\beta^2+\sigma^2+2\beta\right)c+\left(\beta+\sigma^2\right)\sqrt{\beta}=0$$

$$c=\frac{\left(\delta\sigma^2\beta\left(\sigma^2+\beta\right)-\left(\sigma^2\right)^2\beta\nu^2+\sigma^2+2\beta\right)-\sqrt{\left(\delta\sigma^2\beta\left(\sigma^2+\beta\right)-\left(\sigma^2\right)^2\beta\nu^2+\sigma^2+2\beta\right)^2-4\left(\beta+\sigma^2\right)\left(1+\sigma^2\beta\delta\right)\beta}}{2\left(1+\sigma^2\beta\delta\right)\sqrt{\beta}}$$

$$\begin{aligned}
b &= \frac{\sigma^2+\sigma^2\beta\left(\sigma^2\nu^2+\beta\delta+\sigma^2\delta\right)-\sqrt{\left(\delta\sigma^2\beta\left(\sigma^2+\beta\right)-\left(\sigma^2\right)^2\beta\nu^2+\sigma^2+2\beta\right)^2-4\left(\beta+\sigma^2\right)\left(1+\sigma^2\beta\delta\right)\beta}}{2\left(\sigma^2+\beta\right)\sigma^2\sqrt{\beta}\nu}\\
&= 1-\sigma^2\left(\delta-\nu^2\right)\beta\sqrt{\beta}+O\left(\sqrt{\beta}\beta^2\right)
\end{aligned}$$

since C is antisymetric:

$$\frac{1}{2}\left(\left(A-C\right)\left(D-2\right)+2B\right)^S=\left(A-C\right)D+2\left(B-A\right)$$

$$\begin{aligned}
\frac{1}{2}\left(\left(A-C\right)\left(D-2\right)+2B\right)^S &= \sqrt{\beta}\begin{pmatrix}0 & -\frac{1}{\sigma^2}\\ -\nu & 0\end{pmatrix}\begin{pmatrix}0 & b\\ c & 0\end{pmatrix}+\begin{pmatrix}1+\frac{\beta}{\sigma^2} & 0\\ 0 & \frac{1}{\sigma^2}+\beta\delta\end{pmatrix}\\
&= \begin{pmatrix}\frac{1}{\sigma^2}\beta-\frac{1}{\sigma^2}c\sqrt{\beta}+1 & 0\\ 0 & \beta\delta+\frac{1}{\sigma^2}-b\sqrt{\beta}\nu\end{pmatrix}
\end{aligned}$$

From now on the upperscript S will be omitted.As a consequence, the intermediate effective utility (see appendix 1) is:

$$\begin{aligned}
&\left(Y_i\left(t\right)-\hat{Y}_i^{(1)}+\left(\left(A-C\right)\left(D-2\right)+2B\right)^{-1}\left(\left(A+C\right)\left(Y_i\left(t-1\right)-\hat{Y}_i^{(1)}\right)\right)\right)\\
&\times\left(\left(A-C\right)\left(D-2\right)+2B\right)\left(Y_i\left(t\right)-\hat{Y}_i^{(1)}+\left(\left(A-C\right)\left(D-2\right)+2B\right)^{-1}\left(\left(A+C\right)\left(Y_i\left(t-1\right)-\hat{Y}_i^{(1)}\right)\right)\right)
\end{aligned}$$

The relevant matrices are then:

$$\begin{aligned}
\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)^S\right)^{-1}\left(A+C\right) &= \left(2\begin{pmatrix}\frac{1}{\sigma^2}\beta-\frac{1}{\sigma^2}c\sqrt{\beta}+1 & 0\\ 0 & \beta\delta+\frac{1}{\sigma^2}-b\sqrt{\beta}\nu\end{pmatrix}\right)^{-1}2\sqrt{\beta}\begin{pmatrix}0 & -\nu\\ -\frac{1}{\sigma^2} & 0\end{pmatrix}\\
&= \begin{pmatrix}0 & -\frac{\sigma^2\sqrt{\beta}\nu}{\sigma^2+\beta-c\sqrt{\beta}}\\ -\frac{\sqrt{\beta}}{\sigma^2\beta\delta-\sigma^2b\sqrt{\beta}\nu+1} & 0\end{pmatrix}
\end{aligned}$$

The matrices needed to compute the effective action are then:

$$\begin{aligned}
N_{ii} &= \left(\left(A-C\right)\left(D-2\right)+2B\right)_{ii}-\left(\left(A-C\right)\left(D-2\right)+2B\right)_{ij}\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)_{jj}\right)^{-1}\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)_{ji}\right)\\
&= 1+\frac{1}{\sigma^2}\beta-\frac{c}{\sigma^2}\sqrt{\beta}\\
&= 1+\frac{1}{\sigma^2}\beta\\
&\quad-\frac{\left(\delta\sigma^2\beta\left(\sigma^2+\beta\right)-\left(\sigma^2\right)^2\beta\nu^2+\sigma^2+2\beta\right)-\sqrt{\left(\delta\sigma^2\beta\left(\sigma^2+\beta\right)-\left(\sigma^2\right)^2\beta\nu^2+\sigma^2+2\beta\right)^2-4\left(\beta+\sigma^2\right)\left(1+\sigma^2\beta\delta\right)\beta}}{2\sigma^2\left(1+\sigma^2\beta\delta\right)}\\
M_{ii} &= \left(N_{ii}\right)\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)^{-1}\left(A+C\right)\right)_{ii}\\
&= 0\\
M_{ij} &= \left(N_{ii}\right)\frac{1}{\sqrt{\beta}}\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)^{-1}\left(A+C\right)\right)_{ij}=-\left(1+\frac{1}{\sigma^2}\beta-c\sqrt{\beta}\right)\frac{\sigma^2\nu}{\sigma^2+\beta-\sigma^2c\sqrt{\beta}}=-\nu\\
\hat{N}_{ii} &= N_{ii}+M_{ii}=N_{ii}=1+\frac{1}{\sigma^2}\beta-c\sqrt{\beta}
\end{aligned}$$

To find the equilibrium values $\bar{X}_j^{(2)e}$, we use (235) and our previous result that $\bar{X}_1^{(1)e} = 0$. Moreover, given the utility of the second agent, it's optimal goal would be $w = 0$.Then, $\bar{X}_j^{(i)} = 0$. In that case (235) becomes: $\bar{X}_j^{(i)e} = 0$. As a consequence, the effective action:

$$\begin{aligned}
U_{eff}\left(X_i\left(t\right)\right) &= -\frac{1}{2}\left(X_i\left(t\right)-\left(\hat{Y}_i^{(1)}\right)_i\right)M_{ii}\left(X_j\left(t-1\right)-\left(\hat{Y}_i^{(1)}\right)_j\right) \\
&\quad -\left(X_i\left(t\right)-\left(\hat{Y}_i^{(1)}\right)_j\right)M_{ij}\left(\frac{1}{\sqrt{\beta}}\left(X_j\left(t-1\right)-\left(\hat{Y}_i^{(1)}\right)_j\right)\right) \\
&\quad +\frac{1}{2}\left(X_i\left(t\right)-\left(\hat{Y}_i^{(1)}\right)_i\right)\left(N_{ii}\right)\left(X_i\left(t\right)-\left(\hat{Y}_i^{(1)}\right)_i\right)+\sum_{j>i}2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right)
\end{aligned}$$

becomes for the second agent:

$$U_{eff}\left(w\left(t\right)\right)=\left(1+\frac{1}{\sigma^2}\beta-c\sqrt{\beta}\right)w^2\left(t\right)-2\nu w\left(t\right)n\left(t-1\right)+\kappa s_f\left(t-1\right)\left(1-w\left(t\right)-\tilde{f}\right)+\eta s_w\left(t-1\right)\left(w\left(t\right)-\tilde{t}\right)$$

which implies the inertia:

$$\epsilon_{12}=\nu$$

Effective Action for the third agent

Starting with the utility for the third agent at time t,

$$s\left(t\right)\left(Id\right)s\left(t\right)+\left(\rho+\gamma\right)\left(w\left(t-1\right)-\frac{\rho}{\left(\rho+\gamma\right)}\left(1-\tilde{f}\right)-\frac{\gamma}{\left(\rho+\gamma\right)}\tilde{w}\right)^2 \tag{283}$$

and including the additional normalization factor for agents 1 and 2 effective utility:

$$\begin{aligned}
&\frac{\beta}{\sigma^2}\left(-w\left(t-1\right)+\alpha\begin{pmatrix}1&0&0\end{pmatrix}s\left(t-1\right)\right)^2+\frac{\beta}{\sigma^2}\frac{1}{\left(1+\frac{1}{\sigma^2}\beta-c\sqrt{\beta}\right)}\left(\nu n\left(t-1\right)-\kappa s_f\left(t-1\right)+\eta s_w\left(t-1\right)\right)^2 \\
=\;&\frac{\beta}{\sigma^2}\begin{pmatrix} w^2\left(t-1\right)+\alpha^2 s_n^2\left(t-1\right)-2\alpha w\left(t-1\right)s_n\left(t-1\right)+\nu^2 n^2\left(t-1\right) \\ +\kappa^2 s_f^2\left(t-1\right)+\eta^2 s_w^2\left(t-1\right)-2\nu\kappa n\left(t-1\right)s_f\left(t-1\right)+2\nu\eta n\left(t-1\right)s_w\left(t-1\right)-2\kappa\eta s_f\left(t-1\right)s_w\left(t-1\right)\end{pmatrix}
\end{aligned}$$

and defining as before: η, κ, $\alpha \to \times\frac{1}{\beta}$

$$\begin{aligned}
A &= \frac{\sqrt{\beta}}{\sigma^2}\begin{pmatrix} 0&0&0&\eta&0\\ 0&0&0&-\kappa&0\\ 0&0&0&0&\alpha\\ \eta&-\kappa&0&0&-1-\epsilon_{12}\\ 0&0&\alpha&-1-\epsilon_{12}&0 \end{pmatrix} \\
B &= \frac{1}{\sigma^2}\begin{pmatrix} \sigma^2+\beta\frac{\eta^2}{d} & -\beta\frac{\kappa\eta}{d} & 0 & \sqrt{\beta}\eta & -\beta\frac{\nu\eta}{d}\\ -\beta\frac{\kappa\eta}{d} & \sigma^2+\beta\frac{\kappa^2}{d} & 0 & -\sqrt{\beta}\kappa & \beta\frac{\nu\kappa}{d}\\ 0&0&\sigma^2+\beta\frac{\alpha^2}{d} & -\beta\alpha & \sqrt{\beta}\alpha\\ \sqrt{\beta}\eta & -\sqrt{\beta}\kappa & -\beta\alpha & d+\beta+\sigma^2\beta\omega & \sqrt{\beta}\left(-1-\nu\right)\\ -\beta\frac{\nu\eta}{d} & \beta\frac{\nu\kappa}{d} & \sqrt{\beta}\alpha & \sqrt{\beta}\left(-1-\nu\right) & 1+\beta\frac{\nu^2}{d} \end{pmatrix} \\
C &= \frac{\sqrt{\beta}}{\sigma^2}\begin{pmatrix} 0&0&0&-\eta&0\\ 0&0&0&\kappa&0\\ 0&0&0&0&-\alpha\\ \eta&-\kappa&0&0&1-\epsilon_{12}\\ 0&0&\alpha&-1+\epsilon_{12}&0 \end{pmatrix}
\end{aligned}$$

where we set

$$d = 1 + \frac{1}{\sigma^2}\beta - \frac{c}{\sigma^2}\sqrt{\beta}$$
$$\omega = (\rho + \gamma)$$

The equation for the dynamic matrix D is then:

$$(A - C) D^2 + 2 (B - A) D + (A + C) = 0 \tag{284}$$

$$0 = \sqrt{\beta} \begin{pmatrix} 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & -\kappa & 0 \\ 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -\epsilon_{12} & 0 \end{pmatrix} D^2 + \begin{pmatrix} \sigma^2 + \beta\frac{\eta^2}{d} & -\beta\frac{\kappa\eta}{d} & 0 & 0 & -\beta\frac{\nu\eta}{d} \\ -\beta\frac{\kappa\eta}{d} & \sigma^2 + \beta\frac{\kappa^2}{d} & 0 & 0 & \beta\frac{\nu\kappa}{d} \\ 0 & 0 & \sigma^2 + \beta\frac{\alpha^2}{d} & -\beta\alpha & 0 \\ 0 & 0 & -\beta\alpha & d + \beta + \sigma^2\beta\omega & 0 \\ -\beta\frac{\nu\eta}{d} & \beta\frac{\nu\kappa}{d} & 0 & 0 & 1 + \beta\frac{\nu^2}{d} \end{pmatrix} D$$
$$+\sqrt{\beta} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \eta & -\kappa & 0 & 0 & -\epsilon_{12} \\ 0 & 0 & \alpha & -1 & 0 \end{pmatrix} \tag{285}$$

$$\left(B_{11} - \Gamma\Theta^{-1}B_{21}\right)^{-1} \left(\Gamma\Theta^{-1}\left(B_{22}G + \Phi\right) - B_{12}G\right)$$
$$\left(\begin{pmatrix} 1 + \frac{\beta}{\sigma^2}\eta^2 & -\frac{\beta}{\sigma^2}\kappa\eta & 0 \\ -\frac{\beta}{\sigma^2}\kappa\eta & 1 + \frac{\beta}{\sigma^2}\kappa^2 & 0 \\ 0 & 0 & 1 + \frac{\beta}{\sigma^2}\alpha^2 \end{pmatrix} - \begin{pmatrix} \eta & 0 \\ -\kappa & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -\epsilon_{12} & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & -\frac{\beta}{\sigma^2}\alpha \\ \frac{\beta}{\sigma^2}\nu\eta & -\frac{\beta}{\sigma^2}\nu\kappa & 0 \end{pmatrix}\right)^{-1}$$
$$\times \left(\begin{pmatrix} \eta & 0 \\ -\kappa & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -\epsilon_{12} & 0 \end{pmatrix}^{-1} \left(\begin{pmatrix} d + \frac{\beta}{\sigma^2} & 0 \\ 0 & 1 + \nu^2\frac{\beta}{\sigma^2} \end{pmatrix} G + \beta \begin{pmatrix} \eta & -\kappa & 0 \\ 0 & 0 & \alpha \end{pmatrix}\right) - \begin{pmatrix} 0 & \frac{\beta}{\sigma^2}\nu\eta \\ 0 & -\frac{\beta}{\sigma^2}\nu\kappa \\ -\frac{\beta}{\sigma^2}\alpha & 0 \end{pmatrix} G\right)$$

Of the type:

$$\begin{pmatrix} 0 & \Gamma \\ 0 & \Theta \end{pmatrix} D^2 + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} D + \begin{pmatrix} 0 & 0 \\ \Phi & \Psi \end{pmatrix} = 0$$
$$D = \frac{D}{\sqrt{\beta}}$$

with:

$$\Gamma = \begin{pmatrix} \eta & 0 \\ -\kappa & 0 \\ 0 & \alpha \end{pmatrix}$$
$$\Theta = \begin{pmatrix} 0 & -1 \\ -\epsilon_{12} & 0 \end{pmatrix}$$
$$\Delta = \begin{pmatrix} \beta(\gamma + \rho) + N_{22} & 0 \\ 0 & 1 \end{pmatrix}$$
$$\Phi = \beta \begin{pmatrix} \eta & -\kappa & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$
$$\Psi = \beta \begin{pmatrix} 0 & -\epsilon_{12} \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \eta & 0 \\ -\kappa & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -\epsilon_{12} & 0 \end{pmatrix}^{-1}$$

$$\begin{aligned} D &= \begin{pmatrix} E & F \\ G & H \end{pmatrix} \\ D^2 &= \begin{pmatrix} E^2 + FG & EF + FH \\ GE + HG & GF + H^2 \end{pmatrix} \end{aligned}$$

The equation for D:

$$\begin{pmatrix} 0 & \Gamma \\ 0 & \Theta \end{pmatrix} D^2 + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} D + \begin{pmatrix} 0 & 0 \\ \Phi & \Psi \end{pmatrix} = 0$$

can be decomposed in blocks:

$$\begin{pmatrix} \Gamma (GE + HG) & \Gamma \left(GF + H^2\right) \\ \Theta (GE + HG) & \Theta \left(GF + H^2\right) \end{pmatrix} + \begin{pmatrix} B_{11}E + B_{12}G & B_{11}F + B_{12}H \\ B_{21}E + B_{22}G & B_{21}F + B_{22}H \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \Phi & \Psi \end{pmatrix} = 0$$

leading to two systems:

$$\begin{cases} \Gamma (GE + HG) + (B_{11}E + B_{12}G) = 0 \\ \Theta (GE + HG) + (B_{21}E + B_{22}G) + \Phi = 0 \end{cases}$$

$$\begin{aligned} (GE + HG) &= -\Theta^{-1} ((B_{21}E + B_{22}G) + \Phi) \\ \Gamma\Theta^{-1} ((B_{21}E + B_{22}G) + \Phi) &= (B_{11}E + B_{12}G) \\ E &= \left(B_{11} - \Gamma\Theta^{-1}B_{21}\right)^{-1} \left(\Gamma\Theta^{-1} (B_{22}G + \Phi) - B_{12}G\right) \end{aligned}$$

and

$$\begin{cases} \Gamma \left(GF + H^2\right) + (B_{11}F + B_{12}H) = 0 \\ \Theta \left(GF + H^2\right) + (B_{21}F + B_{22}H) + \Psi = 0 \end{cases}$$

$$F = \left(B_{11} - \Gamma\Theta^{-1}B_{21}\right)^{-1} \left(\Gamma\Theta^{-1} (B_{22}H + \Psi) - B_{12}H\right)$$

The two remaining equations:

$$\begin{cases} \Gamma (GE + HG) + (B_{11}E + B_{12}G) = 0 \\ \Theta (GE + HG) + (B_{21}E + B_{22}G) + \Phi = 0 \end{cases}$$

$$\begin{cases} \Gamma \left(GF + H^2\right) + (B_{11}F + B_{12}H) = 0 \\ \Theta \left(GF + H^2\right) + (B_{21}F + B_{22}H) + \Psi = 0 \end{cases}$$

$$\begin{aligned} (\Gamma G + B_{11}) E + (\Gamma HG + B_{12}G) &= 0 \\ (\Gamma G + B_{11}) F + \left(\Gamma H^2 + B_{12}H\right) &= 0 \end{aligned}$$

allow to find a relation between G and H. Let:

$$V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

multiply the first equation by V^t.

Multiply the first equation by $\left(G (V)^t\right)^{-1}$ and the second one by H^{-1}. It yields:

$$(\Gamma G + B_{11}) E (V)^t \left(G (V)^t\right)^{-1} = (\Gamma G + B_{11}) FH^{-1}$$

and then, since $(\Gamma G + B_{11})$ is square:

$$E\left(V\right)^{t}\left(G\left(V\right)^{t}\right)^{-1} = FH^{-1}$$

Using the equation for E and F gives:

$$\begin{aligned} E\left(V\right)^{t}\left(G\left(V\right)^{t}\right)^{-1} &= \left(B_{11} - \Gamma\Theta^{-1}B_{21}\right)^{-1}\left(\Gamma\Theta^{-1}\left(B_{22} + \Phi\left(V\right)^{t}\left(G\left(V\right)^{t}\right)^{-1}\right) - B_{12}\right) \\ FH^{-1} &= \left(B_{11} - \Gamma\Theta^{-1}B_{21}\right)^{-1}\left(\Gamma\Theta^{-1}\left(B_{22} + \Psi H^{-1}\right) - B_{12}\right) \end{aligned}$$

$$\Gamma\Theta^{-1}\left(B_{22} + \Phi\left(V\right)^{t}\left(G\left(V\right)^{t}\right)^{-1}\right) = \Gamma\Theta^{-1}\left(B_{22} + \Psi H^{-1}\right)$$

multiply by $\left(V\right)^{t}$ on the left and simplify by $\left(V\right)^{t}\Gamma\Theta^{-1}$:

$$\Phi\left(V\right)^{t}\left(G\left(V\right)^{t}\right)^{-1} = \Psi H^{-1}$$

$$\Gamma^{t}\left(V\right)^{t}\left(G\left(V\right)^{t}\right)^{-1} = \Theta^{t}H^{-1}$$

since $\left(V\Gamma\right)^{t} = \left(V\Gamma\right)$, it leads ultimately to:

$$H = \left(G\left(V\right)^{t}\right)\left(V\Gamma\right)^{-1}\Theta^{t}$$

This last equation allows to reduce the problem to find (GV^{t}). Actually, we can take benefit from the arbitraryness of the matrix V to make an other choice. Let

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

one also have:

$$H = \left(G\left(W\right)^{t}\right)\left(W\Gamma\right)^{-1}\Theta^{t}$$

and the two identities for H yield:

$$\left(G\left(V\right)^{t}\right)\left(V\Gamma\right)^{-1} = \left(G\left(W\right)^{t}\right)\left(W\Gamma\right)^{-1}$$

Writing

$$G = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

the previous equation leads directly to:

$$\begin{aligned} a &= -\frac{\eta}{\kappa}b \\ d &= -\frac{\eta}{\kappa}e \end{aligned}$$

Thus, it remains to determine

$$GV^{t} = \begin{pmatrix} b & c \\ e & f \end{pmatrix}$$

To do so, recall that:

$$\begin{aligned} E &= \left(B_{11} - \Gamma\Theta^{-1}B_{21}\right)^{-1}\left(\Gamma\Theta^{-1}\left(B_{22}G + \Phi\right) - B_{12}G\right) \\ &= \begin{pmatrix} 0 & 0 & -\eta\frac{f+\alpha\beta}{\sigma^{2}\nu} \\ 0 & 0 & \kappa\frac{f+\alpha\beta}{\sigma^{2}\nu} \\ d\alpha\eta\frac{-\kappa\beta+bd+\sigma^{2}b\beta\omega}{\kappa(\alpha^{2}\beta+\sigma^{2}d-d\alpha^{2}\beta)} & d\alpha\frac{\kappa\beta-bd-\sigma^{2}b\beta\omega}{\alpha^{2}\beta+\sigma^{2}d-d\alpha^{2}\beta} & 0 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
F &= \left(B_{11}-\Gamma\Theta^{-1}B_{21}\right)^{-1}\left(\Gamma\Theta^{-1}\left(B_{22}H+\Psi\right)-B_{12}H\right)\\
&= \begin{pmatrix} \eta\frac{f+\alpha\beta}{\sigma^2\nu\alpha} & 0 \\ -\kappa\frac{f+\alpha\beta}{\sigma^2\nu\alpha} & 0 \\ 0 & d\alpha\nu\frac{\kappa\beta-bd-\sigma^2 b\beta\omega}{\kappa(\alpha^2\beta+\sigma^2 d-d\alpha^2\beta)} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
H &= \left(G\left(W\right)^t\right)\left(W\Gamma\right)^{-1}\Theta^t\\
&= \begin{pmatrix} 0 & \frac{b}{\kappa}\nu \\ -\frac{f}{\alpha} & 0 \end{pmatrix}
\end{aligned}
$$

and insert these relations in (285), the equation for D to find:

$$
D=\frac{1}{\sqrt{\beta}}\begin{pmatrix} 0 & 0 & -\eta\frac{f+\alpha\beta}{\sigma^2\nu} & \eta\frac{f+\alpha\beta}{\sigma^2\nu\alpha} & 0 \\ 0 & 0 & \kappa\frac{f+\alpha\beta}{\sigma^2\nu} & -\kappa\frac{f+\alpha\beta}{\sigma^2\nu\alpha} & 0 \\ d\alpha\eta\frac{-\kappa\beta+bd+\sigma^2 b\beta\omega}{\kappa(\alpha^2\beta+\sigma^2 d-d\alpha^2\beta)} & d\alpha\frac{\kappa\beta-bd-\sigma^2 b\beta\omega}{\alpha^2\beta+\sigma^2 d-d\alpha^2\beta} & 0 & 0 & d\alpha\nu\frac{\kappa\beta-bd-\sigma^2 b\beta\omega}{\kappa(\alpha^2\beta+\sigma^2 d-d\alpha^2\beta)} \\ -\frac{\eta}{\kappa}b & b & 0 & 0 & \frac{b}{\kappa}\nu \\ 0 & 0 & f & -\frac{f}{\alpha} & 0 \end{pmatrix}
$$

replace b by $b\kappa$ and set $r^2=\kappa^2+\eta^2$, then the equation for b and f are:

$$
\begin{aligned}
0 &= b\alpha^3\beta^2-\alpha^3\beta^2+bd^2f\alpha^2-bd\alpha^3\beta^2+\sigma^2bd^2\alpha+bd\alpha^3\beta+bf\alpha^2\beta+\sigma^2bdf-df\alpha^2\beta\\
&\quad+\sigma^2bd\alpha\beta+\sigma^2b\alpha^3\beta^2\omega-\sigma^2d\alpha\beta-bdf\alpha^2\beta+\sigma^4bd\alpha\beta\omega+\sigma^2bdf\alpha^2\beta\omega\\
0 &= -f\beta r^2-\sigma^2df+bdfr^2-\alpha\beta^2r^2-\sigma^2f\beta\nu^2+bd\alpha r^2\beta-\sigma^2d\alpha\beta+\sigma^2bdf\nu^2
\end{aligned}
$$

with:

$$
\begin{aligned}
d &= 1+\frac{1}{\sigma^2}\beta\\
&\quad-\frac{1}{\sigma^2}\frac{\left(\delta\sigma^2\beta\left(\sigma^2+\beta\right)-\sigma^4\beta\nu^2+\sigma^2+2\beta\right)-\sqrt{\left(\delta\sigma^2\beta\left(\sigma^2+\beta\right)-\left(\sigma^2\right)^2\beta\nu^2+\sigma^2+2\beta\right)^2-4\left(\beta+\sigma^2\right)\left(1+\sigma^2\beta\delta\right)\beta}}{2\left(1+\sigma^2\beta\delta\right)}
\end{aligned}
$$

And the relevant matrices for our problem become:

$$
\begin{aligned}
&\sqrt{\beta}\sigma^2\left(\left(A-C\right)D+2\left(B-A\right)\right)\\
&= \begin{pmatrix} \frac{\beta\eta^2+\sigma^2d-bd\eta^2}{d} & \eta\kappa\frac{-\beta+bd}{d} & 0 & 0 & \nu\eta\frac{-\beta+bd}{d} \\ \eta\kappa\frac{-\beta+bd}{d} & \frac{\kappa^2\beta+\sigma^2d-bd\kappa^2}{d} & 0 & 0 & \nu\kappa\frac{\beta-bd}{d} \\ 0 & 0 & \frac{\alpha^2\beta+\sigma^2d+df\alpha}{d} & -\left(f+\alpha\beta\right) & 0 \\ 0 & 0 & -\left(f+\alpha\beta\right) & \frac{f+d\alpha+\alpha\beta+\sigma^2\alpha\beta\omega}{\alpha} & 0 \\ \nu\eta\frac{-\beta+bd}{d} & \nu\kappa\frac{\beta-bd}{d} & 0 & 0 & \frac{d+\beta\nu^2-bd\nu^2}{d} \end{pmatrix}
\end{aligned}
$$

$$\left(\left(\left(A-C\right)D+2\left(B-A\right)\right)^{S}\right)^{-1}\left(A+C\right)$$

$$= \begin{pmatrix} \frac{\beta\eta^2+\sigma^2 d-bd\eta^2}{d} & \eta\kappa\frac{-\beta+bd}{d} & 0 & 0 & \nu\eta\frac{-\beta+bd}{d} \\ \eta\kappa\frac{-\beta+bd}{d} & \frac{\kappa^2\beta+\sigma^2 d-bd\kappa^2}{d} & 0 & 0 & \nu\kappa\frac{\beta-bd}{d} \\ 0 & 0 & \frac{\alpha^2\beta+\sigma^2 d+df\alpha}{d} & -\left(f+\alpha\beta\right) & 0 \\ 0 & 0 & -\left(f+\alpha\beta\right) & \frac{f+d\alpha+\alpha\beta+\sigma^2\alpha\beta\omega}{\alpha} & 0 \\ \nu\eta\frac{-\beta+bd}{d} & \nu\kappa\frac{\beta-bd}{d} & 0 & 0 & \frac{d+\beta\nu^2-bd\nu^2}{d} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \eta & -\kappa & 0 & 0 & -\nu \\ 0 & 0 & \alpha & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \alpha\nu\eta\frac{-\beta+bd}{\varkappa} & \nu\eta\frac{\beta-bd}{\varkappa} & 0 \\ 0 & 0 & \alpha\kappa\nu\frac{\beta-bd}{\varkappa} & \kappa\nu\frac{-\beta+bd}{\varkappa} & 0 \\ \eta d\alpha\frac{f+\alpha\beta}{\chi} & -\kappa d\alpha\frac{f+\alpha\beta}{\chi} & 0 & 0 & -\nu d\alpha\frac{f+\alpha\beta}{\chi} \\ \eta\alpha\frac{\alpha^2\beta+\sigma^2 d+df\alpha}{\chi} & -\kappa\alpha\frac{\alpha^2\beta+\sigma^2 d+df\alpha}{\chi} & 0 & 0 & -\nu\alpha\frac{\alpha^2\beta+\sigma^2 d+df\alpha}{\chi} \\ 0 & 0 & \alpha\frac{\kappa^2\beta+\beta\eta^2+\sigma^2 d-bd\kappa^2-bd\eta^2}{\varkappa} & \frac{\kappa^2\beta+\beta\eta^2+\sigma^2 d-bd\kappa^2-bd\eta^2}{\varkappa} & 0 \end{pmatrix}$$

with:

$$\chi = \alpha^3\beta^2+d^2f\alpha^2-d\alpha^3\beta^2+\sigma^2 d^2\alpha+d\alpha^3\beta+f\alpha^2\beta+\sigma^2 df+\sigma^2 d\alpha\beta+\sigma^2\alpha^3\beta^2\omega-df\alpha^2\beta+\left(\sigma^2\right)^2 d\alpha\beta\omega+\sigma^2 df\alpha^2\beta\omega$$

$$\varkappa = -\kappa^2\beta-\beta\eta^2-\sigma^2 d+bd\kappa^2+bd\eta^2-\sigma^2\beta\nu^2+\sigma^2 bd\nu^2$$

which leads to the expression for N_{ii}, M_{ii}, M_{ij}:

$$\begin{aligned} \sigma_{ii}^2 &= \left(\left(A-C\right)\left(D-2\right)+2B\right)_{ii} \\ &\quad -\left(\left(A-C\right)\left(D-2\right)+2B\right)_{ij}\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)_{jj}\right)^{-1}\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)_{ji}\right) \\ &= \begin{pmatrix} \frac{\beta\eta^2+\sigma^2 d-bd\eta^2}{d} & \eta\kappa\frac{-\beta+bd}{d} & 0 \\ \eta\kappa\frac{-\beta+bd}{d} & \frac{\kappa^2\beta+\sigma^2 d-bd\kappa^2}{d} & 0 \\ 0 & 0 & \frac{\alpha^2\beta+\sigma^2 d+df\alpha}{d} \end{pmatrix} \\ &\quad -\begin{pmatrix} 0 & \nu\eta\frac{-\beta+bd}{d} \\ 0 & \nu\kappa\frac{\beta-bd}{d} \\ -\left(f+\alpha\beta\right) & 0 \end{pmatrix}\begin{pmatrix} \frac{f+d\alpha+\alpha\beta+\sigma^2\alpha\beta\omega}{\alpha} & 0 \\ 0 & \frac{d+\beta\nu^2-bd\nu^2}{d} \end{pmatrix}^{-1} \\ &\quad \times\begin{pmatrix} 0 & 0 & -\left(f+\alpha\beta\right) \\ \nu\eta\frac{-\beta+bd}{d} & \nu\kappa\frac{\beta-bd}{d} & 0 \end{pmatrix} \\ &\quad \times\begin{pmatrix} \frac{\beta\eta^2+\sigma^2 d-bd\eta^2}{d} & \eta\kappa\frac{-\beta+bd}{d} & 0 & 0 & \nu\eta\frac{-\beta+bd}{d} \\ \eta\kappa\frac{-\beta+bd}{d} & \frac{\kappa^2\beta+\sigma^2 d-bd\kappa^2}{d} & 0 & 0 & \nu\kappa\frac{\beta-bd}{d} \\ 0 & 0 & \frac{\alpha^2\beta+\sigma^2 d+df\alpha}{d} & -\left(f+\alpha\beta\right) & 0 \\ 0 & 0 & -\left(f+\alpha\beta\right) & \frac{f+d\alpha+\alpha\beta+\sigma^2\alpha\beta\omega}{\alpha} & 0 \\ \nu\eta\frac{-\beta+bd}{d} & \nu\kappa\frac{\beta-bd}{d} & 0 & 0 & \frac{d+\beta\nu^2-bd\nu^2}{d} \end{pmatrix} \\ &= \begin{pmatrix} \frac{-\beta\eta^2-\sigma^2 d+bd\eta^2-\sigma^2\beta\nu^2+\sigma^2 bd\nu^2}{-d-\beta\nu^2+bd\nu^2} & \kappa\eta\frac{\beta-bd}{-d-\beta\nu^2+bd\nu^2} & 0 \\ \kappa\eta\frac{\beta-bd}{-d-\beta\nu^2+bd\nu^2} & \frac{-\kappa^2\beta-\sigma^2 d+bd\kappa^2-\sigma^2\beta\nu^2+\sigma^2 bd\nu^2}{-d-\beta\nu^2+bd\nu^2} & 0 \\ 0 & 0 & \frac{\chi}{d\left(f+d\alpha+\alpha\beta+\sigma^2\alpha\beta\omega\right)} \end{pmatrix} \\ M_{ii} &= N_{ii}\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)^{-1}\left(A+C\right)\right)_{ii} = N_{ii}\begin{pmatrix} 0 & 0 & \alpha\nu\eta\frac{-\beta+bd}{\varkappa} \\ 0 & 0 & \alpha\kappa\nu\frac{\beta-bd}{\varkappa} \\ \eta d\alpha\frac{f+\alpha\beta}{\chi} & -\kappa d\alpha\frac{f+\alpha\beta}{\chi} & 0 \end{pmatrix} \\ M_{ij} &= N_{ii}\left(\left(\left(A-C\right)\left(D-2\right)+2B\right)^{-1}\left(A+C\right)\right)_{ij} = N_{ii}\begin{pmatrix} \nu\eta\frac{\beta-bd}{\varkappa} & 0 \\ \kappa\nu\frac{-\beta+bd}{\varkappa} & 0 \\ 0 & -\nu d\alpha\frac{f+\alpha\beta}{\chi} \end{pmatrix} \end{aligned}$$

and the effective utility for the third agent is:

$$U_{eff}\left(X_i\left(t\right)\right)=\frac{1}{2}\left(X_i\left(t\right)-\left(\hat{Y}_i^{(1)}\right)_i\right)N_{ii}\left(X_i\left(t\right)-\left(\hat{Y}_i^{(1)}\right)_i\right)-\left(X_i\left(t\right)-\left(\hat{Y}_i^{(1)}\right)_j\right)M_{ij}\left(X_j\left(t-1\right)-\left(\hat{Y}_i^{(1)}\right)_j\right)$$

Or, rexpressed in the variables $s\left(t\right)$:

$$U_{eff}\left(s\left(t\right)\right)=\left(s\left(t\right)-\left(s^{(3)}\right)_{eff}\right)N_{ii}\left(s\left(t\right)-\left(s^{(3)}\right)_{eff}\right)-\left(s\left(t\right)-\left(s^{(3)}\right)_{eff}\right)M_{ij}\begin{pmatrix} w\left(t-1\right)-\left(w^{(3)}\right)_{eff} \\ n\left(t-1\right)-\left(n^{(3)}\right)_{eff}\end{pmatrix}$$

where the constants

$$\begin{pmatrix} \left(s^{(3)}\right)_{eff} \\ \left(w^{(3)}\right)_{eff} \\ \left(n^{(3)}\right)_{eff}\end{pmatrix}=\bar{X}^{(3)e}$$

form a 5 dimensional vector. The vector $\bar{X}^{(3)e}$ satisfy (235), which reduces to:

$$\begin{aligned}\bar{X}^{(3)e} &= \begin{pmatrix} A_{ii}^{(i)}+B_{11}-\sqrt{\beta}\epsilon_{ii}^{(i)} & \left\{B_{12},2\sqrt{\beta}\left(A_{ij}^{(i)}\right)^S\right\} \\ \left\{B_{12}^t,2\sqrt{\beta}\left(A_{ji}^{(j)}\right)^S\right\} & \left\{\begin{matrix}\left(A_{jj}^{(j)}\right)_{eff}+\beta A_{jj}^{(i)},B_{22}, \\ \left(2\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff},2A_{\{kj\}i>k>j}^{(j)}\right)^S\end{matrix}\right\}\end{pmatrix}^{-1} \\ &\times\left\{\begin{pmatrix} A_{ii}^{(i)}-\sqrt{\beta}\frac{\epsilon_{ii}^{(i)}}{2} & \left\{\frac{B_{12}^{(3)}}{2},\sqrt{\beta}A_{ij}^{(i)}\right\} \\ \frac{\left(B_{12}^{(3)}\right)^t}{2} & \beta A_{jj}^{(i)}+\frac{B_{22}^{(3)}}{2}\end{pmatrix}\left(\bar{X}_j^{(i)}\right)\right\}\end{aligned} \tag{286}$$

given that $\left(\bar{X}_j^{(j)e}\right)=0$, as shown in the previous computations for the first two agents. Moreover (283) shows that:

$$\left(\bar{X}_j^{(i)}\right)=\left(\frac{\rho}{(\rho+\gamma)}\left(1-\tilde{f}\right)+\frac{\gamma}{(\rho+\gamma)}\tilde{w}\right)\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}$$

and then (286) simplifies as:

$$\begin{aligned}&\frac{\bar{X}^{(3)e}}{\left(\frac{\rho}{(\rho+\gamma)}\left(1-\tilde{f}\right)+\frac{\gamma}{(\rho+\gamma)}\tilde{w}\right)} \\ &= \begin{pmatrix} A_{ii}^{(i)}+B_{11}-\sqrt{\beta}\epsilon_{ii}^{(i)} & \left\{B_{12},\sqrt{\beta}\left(A_{ij}^{(i)}+A_{ij}^{(j)}\right)\right\} \\ \left\{B_{12}^t,\sqrt{\beta}\left(A_{ji}^{(j)}+A_{ji}^{(j)}\right)\right\} & \left\{\begin{matrix}\left(A_{jj}^{(j)}\right)_{eff}+\beta A_{jj}^{(i)},B_{22}, \\ \left(2\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff},2A_{\{kj\}i>k>j}^{(j)}\right)^S\end{matrix}\right\}\end{pmatrix}^{-1}\times\begin{pmatrix}\left\{\frac{B_{12}^{(3)}}{2},\sqrt{\beta}A_{ij}^{(i)}\right\}\begin{pmatrix}1\\0\end{pmatrix} \\ \left(\beta A_{jj}^{(i)}+\frac{B_{22}^{(3)}}{2}\right)\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} \\ &= B^{-1}\times\begin{pmatrix}\left\{\frac{B_{12}^{(3)}}{2},\sqrt{\beta}A_{ij}^{(i)}\right\}\begin{pmatrix}1\\0\end{pmatrix} \\ \left(\beta A_{jj}^{(i)}+\frac{B_{22}^{(3)}}{2}\right)\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix}\end{aligned}$$

Using again (283) yields $A_{ij}^{(i)} = 0$, $A_{jj}^{(i)} = \begin{pmatrix} \rho+\gamma & 0 \\ 0 & 0 \end{pmatrix}$, whereas (234) gives $B_{12}^{(3)}$ and $B_{22}^{(3)}$:

$$\begin{aligned}
B_{12}^{(3)} &= \beta \left(A_{ij}^{(j)} \left(A_{jj}^{(j)} \right)_{eff}^{-1} \begin{pmatrix} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{pmatrix} \right) \\
B_{22}^{(3)} &= \beta \left(A_{kj}^{(j)} \left(A_{jj}^{(j)} \right)_{eff}^{-1} \begin{pmatrix} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{pmatrix} \right)^{S}
\end{aligned}$$

Given the effective action for the two first agents, one has:

$$\begin{aligned}
\left(\epsilon_{jj}^{(j)}\right)_{eff} &= 0 \\
\left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} &= \begin{pmatrix} 0 & -\sqrt{\beta}\nu \\ 0 & 0 \end{pmatrix} \\
\left(A_{jj}^{(j)}\right)_{eff} &= \begin{pmatrix} 1+\frac{1}{\sigma^2}\beta - c\sqrt{\beta} & 0 \\ 0 & 1 \end{pmatrix} \\
A_{ij}^{(j)} &= \begin{pmatrix} \sqrt{\beta}\eta & 0 \\ -\sqrt{\beta}\kappa & 0 \\ 0 & \sqrt{\beta}\alpha \end{pmatrix} \\
A_{kj}^{(j)} &= \begin{pmatrix} 0 & -\sqrt{\beta}\nu \\ -\sqrt{\beta} & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
B_{12}^{(3)} &= \beta \left(\begin{pmatrix} \sqrt{\beta}\eta & 0 \\ -\sqrt{\beta}\kappa & 0 \\ 0 & \sqrt{\beta}\alpha \end{pmatrix} \begin{pmatrix} 1+\frac{1}{\sigma^2}\beta - c\sqrt{\beta} & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -\sqrt{\beta}\nu \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & -\sigma^2\beta^2\nu\frac{\eta}{\sigma^2+\beta-\sigma^2 c\sqrt{\beta}} \\ 0 & \sigma^2\kappa\beta^2\frac{\nu}{\sigma^2+\beta-\sigma^2 c\sqrt{\beta}} \\ 0 & 0 \end{pmatrix} \\
B_{22}^{(3)} &= \beta \left(\begin{pmatrix} 0 & -\sqrt{\beta}\nu \\ -\sqrt{\beta} & 0 \end{pmatrix} \begin{pmatrix} 1+\frac{1}{\sigma^2}\beta - c\sqrt{\beta} & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -\sqrt{\beta}\nu \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2\beta^2\frac{\nu}{\sigma^2+\beta-\sigma^2 c\sqrt{\beta}} \end{pmatrix}
\end{aligned}$$

and then:

$$\begin{pmatrix} \left\{\frac{B_{12}^{(3)}}{2}, \sqrt{\beta}A_{ij}^{(i)}\right\} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \left(\beta A_{jj}^{(i)} + \frac{B_{22}^{(3)}}{2}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \frac{B_{12}^{(3)}}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \left(\beta A_{jj}^{(i)} + \frac{B_{22}^{(3)}}{2}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \beta\left(\rho+\gamma\right)$$

so that:

$$\begin{aligned}
\bar{X}^{(3)e} &= \beta\left(\rho\left(1-\tilde{f}\right)+\gamma\tilde{w}\right) B^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sqrt{\beta}\eta\left(d-\beta\nu\right)\frac{\alpha^2\beta+\sigma^2 d - d\alpha^2\beta}{C} \\ \kappa\sqrt{\beta}\left(d-\beta\nu\right)\frac{\alpha^2\beta+\sigma^2 d - d\alpha^2\beta}{C} \\ -\sigma^2 d\alpha\beta\nu\frac{d-\beta\nu}{C} \\ \frac{\alpha^2\left((1-d)r^2+\sigma^2\nu^2-d\alpha^2 r^2\right)\beta^2 + d\left(\left(\sigma^2\right)^2\nu^2 - \sigma^2 d\alpha^2 + \sigma^2\alpha^2 + \sigma^2 r^2\right)\beta + \left(\sigma^2\right)^2 d^2}{C} \\ \sqrt{\beta}\frac{\alpha^2(1-d)r^2\beta^2 + \sigma^2 d\left((1-d+\nu)\alpha^2 + r^2\right)\beta + d^2\left(\sigma^2\right)^2(1+\nu)}{C} \end{pmatrix}
\end{aligned}$$

with:

$$\begin{aligned}
C &= \alpha^2\left(\nu^2 - d\nu^2 + \kappa^2\omega + \eta^2\omega + \sigma^2\nu^2\omega - d\kappa^2\omega - d\eta^2\omega\right)\beta^3 \\
&\quad + d\left(\sigma^2\nu^2 - 2\alpha^2\nu + \left(\sigma^2\right)^2\nu^2\omega + \sigma^2\alpha^2\omega + \sigma^2\kappa^2\omega + \sigma^2\eta^2\omega + 2d\alpha^2\nu - \sigma^2 d\alpha^2\omega\right)\beta^2 \\
&\quad + d^2\left(-2\sigma^2\nu + \alpha^2 + \left(\sigma^2\right)^2\omega - d\alpha^2\right)\beta + \sigma^2 d^3
\end{aligned}$$

Global action for the system

Gathering the previous result, one can gather all effective utilities into the global system utility:

$$\begin{aligned} U_{eff} &= \left(s(t)-\left(s^{(3)}\right)_{eff}\right) N_{ii}\left(s(t)-\left(s^{(3)}\right)_{eff}\right)-\left(s(t)-\left(s^{(3)}\right)_{eff}\right) M_{ij}\left(\begin{array}{c} w(t-1)-\left(w^{(3)}\right)_{eff} \\ n(t-1)-\left(n^{(3)}\right)_{eff} \end{array}\right) \\ &\quad +(1-c)\, w^2(t)+2\nu w(t)\, n(t-1)+\kappa s_f(t-1)\left(1-w(t)-\tilde{f}\right)+\eta s_w(t-1)\left(w(t)-\tilde{t}\right) \\ &\quad +(n(t))^2-2n(t)\, w(t-1)+2\alpha n(t)\left(\begin{array}{ccc} 1 & 0 & 0 \end{array}\right) s(t-1) \end{aligned}$$

We can first study the stabilty of the system by having a look on the clasical system associated to this effective utility. Discarding the equilibrium value, the first order condition can expressed by:

$$\left(\begin{array}{ccc} N_{11} & 0 & 0 \\ 0 & N_{22} & 0 \\ 0 & 0 & 1 \end{array}\right) X(t)=\left(\begin{array}{ccc} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & 0 \end{array}\right) X(t-1)$$

where $X(t)$ concatenates in column the vectors $s(t)$, $w(t)$ and $n(t)$. The solution of the system is then:

$$\begin{aligned} X(t) &= \left(\begin{array}{ccc} (N_{11})^{-1} M_{11} & (N_{11})^{-1} M_{12} & (N_{11})^{-1} M_{13} \\ (N_{22})^{-1} M_{21} & (N_{22})^{-1} M_{22} & (N_{22})^{-1} M_{23} \\ M_{31} & M_{32} & 0 \end{array}\right) X(t-1) \\ &\equiv M X(t-1) \end{aligned} \tag{287}$$

Recall that

$$M_{ij}=(N_{ii})\left(((A-C)(D-2)+2B)^{-1}(A+C)\right)_{ij}$$

and as a consequence one obtains the various matrices involved in the dynamics:

$$\begin{aligned} (N_{11})^{-1} M_{11} &= \left(((A-C)(D-2)+2B)^{-1}(A+C)\right)_{11} \\ &= \left(\begin{array}{ccc} 0 & 0 & \alpha\frac{\beta\nu\eta+\sigma^2 b\eta\epsilon_{12}}{\varrho} \\ 0 & 0 & -\alpha\frac{\kappa\beta\nu+\sigma^2 b\kappa\epsilon_{12}}{\varrho} \\ \frac{1}{\beta}\eta(\alpha\beta-bd\alpha) & -\frac{\kappa}{\beta}(\alpha\beta-bd\alpha) & 0 \end{array}\right) \\ (N_{11})^{-1}(M_{12}, M_{13}) &= \left(\left(((A-C)(D-2)+2B)^{-1}(A+C)\right)_{12}, \left(((A-C)(D-2)+2B)^{-1}(A+C)\right)_{13}\right) \\ &= \left(\begin{array}{cc} -\frac{\beta\nu\eta+\sigma^2 b\eta\epsilon_{12}}{\varrho} & 0 \\ \frac{\kappa\beta\nu+\sigma^2 b\kappa\epsilon_{12}}{\varrho} & 0 \\ 0 & -\frac{1}{\beta}\epsilon_{12}(\alpha\beta-bd\alpha) \end{array}\right) \end{aligned}$$

with:

$$\begin{aligned} \varrho &= \sigma^2 b\kappa^2-\kappa^2\beta-\beta\nu^2-\beta\eta^2-\sigma^2+\sigma^2 b\eta^2+\sigma^2 b\epsilon_{12}^2+b\kappa^2\beta\nu^2+b\beta\nu^2\eta^2+b\kappa^2\beta\epsilon_{12}^2+b\beta\eta^2\epsilon_{12}^2 \\ &\quad +2b\kappa^2\beta\nu\epsilon_{12}+2b\beta\nu\eta^2\epsilon_{12} \end{aligned}$$

and:

$$\left(\begin{array}{ccc} (N_{22})^{-1} M_{21} & (N_{22})^{-1} M_{22} & (N_{22})^{-1} M_{23} \\ M_{31} & M_{32} & 0 \end{array}\right)=\left(\begin{array}{ccccc} \frac{\eta}{d} & -\frac{\kappa}{d} & 0 & 0 & -\frac{\nu}{d} \\ 0 & 0 & \alpha & -1 & 0 \end{array}\right)$$

The determinant has three nul eigenvalues, and the two last ones satisfy:

$$\begin{aligned} \lambda &= \pm\sqrt{\sigma^2\nu\left(d+\beta\nu^2-bd\nu^2\right)} \\ &\quad \times\sqrt{\frac{\chi-d^2\alpha^2(f+\alpha\beta)}{d\left(-\sigma^2 d+(bd-\beta)r^2-\sigma^2\beta\nu^2+\sigma^2 bd\nu^2\right)\chi}} \end{aligned}$$

with $r^2 = \eta^2 + \kappa^2$. Then one can study these eigenvalues numerically as functions of the system parameters. This will be the goal of next paragraph.

The effective utility allows also to study the stability of the all structure in interaction with a large set of similar structures. We rewrite:

$$\begin{aligned} U_{eff} &= \left(X(t) - \bar{X}^e\right) \begin{pmatrix} N_{11} & 0 & 0 \\ 0 & N_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(X(t) - \bar{X}^e\right) - \left(X(t) - \bar{X}^e\right) \begin{pmatrix} 0 & M_{12} & M_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(X(t-1) - \bar{X}^e\right) \\ &\quad - \left(X(t) - \bar{X}_2^{(2)}\right) \begin{pmatrix} M_{11} & 0 & 0 \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & 0 \end{pmatrix} X(t-1) \end{aligned} \tag{288}$$

where:

$$\bar{X}^e = \left(\bar{X}^{(1)e}, 0, 0\right)^t$$

and:

$$\bar{X}_2^{(2)} = \left(\frac{\rho}{(\rho+\gamma)}\left(1 - \tilde{f}\right) + \frac{\gamma}{(\rho+\gamma)}\tilde{w}\right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Then, the saddle point equation for the equilibrium value $\bar{X}$, derived from (288):

$$\begin{aligned} 0 &= 2\begin{pmatrix} N_{11} & 0 & 0 \\ 0 & N_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\bar{X} - \bar{X}^e\right) - 2\begin{pmatrix} 0 & M_{12} & M_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^S \left(\bar{X} - \bar{X}^e\right) \\ &\quad - \begin{pmatrix} M_{11} & 0 & 0 \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & 0 \end{pmatrix} \bar{X} - \begin{pmatrix} M_{11} & 0 & 0 \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & 0 \end{pmatrix}^t \left(\bar{X} - \bar{X}_2^{(2)}\right) \end{aligned}$$

or, which is equivalent:

$$\begin{aligned} 2\left(N - M^S\right)\left(\bar{X} - \bar{X}^e\right) &= \begin{pmatrix} M_{11} & 0 & 0 \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & 0 \end{pmatrix} \bar{X}^e + \begin{pmatrix} M_{11} & 0 & 0 \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & 0 \end{pmatrix}^t \left(\bar{X}^e - \bar{X}_2^{(2)}\right) \\ &\equiv 2M^S \bar{X}^e - \left(M'\right)^t \bar{X}_2^{(2)} \end{aligned} \tag{289}$$

whose solution is:

$$\bar{X} = \bar{X}^e + \left(N - M^S\right)^{-1} \left(M^S \bar{X}^e - \frac{1}{2}\left(M'\right)^t \bar{X}_2^{(2)}\right) \tag{290}$$

We can now express $U_{eff}(X(t))$ as:

$$\begin{aligned} U_{eff}\left(\bar{X}\right) &= \left(\bar{X} - \bar{X}^e\right)\left(N - \left(M - M'\right)^S\right)\left(\bar{X} - \bar{X}^e\right) - \left(\bar{X} - \bar{X}_2^{(2)}\right) M'\bar{X} \\ &= \left(M^S \bar{X}^e - \frac{1}{2}\left(M'\right)^t \bar{X}_2^{(2)}\right)^t \left(N - M^S\right)^{-1}\left(N - \left(M - M'\right)^S\right)\left(N - M^S\right)^{-1}\left(M^S \bar{X}^e - \frac{1}{2}\left(M'\right)^t \bar{X}_2^{(2)}\right) \\ &\quad - \left(\bar{X} - \bar{X}_2^{(2)}\right) M'\bar{X} \end{aligned}$$

$$\begin{aligned} U_{eff}(X(t)) &= \left(X(t)-\bar{X}\right)N\left(X(t)-\bar{X}\right)-\left(X(t)-\bar{X}\right)NM\left(X(t-1)-\bar{X}\right) \\ &\quad +U_{eff}\left(\bar{X}\right) \\ &= \left(X(t)-\bar{X}\right)N\left(X(t)-\bar{X}\right)-\left(X(t)-\bar{X}\right)M\left(X(t-1)-\bar{X}\right) \\ &\quad +\left(\bar{X}-\bar{X}^{e}\right)\left(N-\left(M-M'\right)^{S}\right)\left(\bar{X}-\bar{X}^{e}\right)-\left(\bar{X}-\bar{X}_{2}^{(2)}\right)M'\bar{X} \end{aligned}$$

An other convenient form for $U_{eff}(X(t))$ in the sequel is obtained by writing it's continuous time approximation (174), plus its constant term:

$$\begin{aligned} U_{eff}(X(t)) &= \left(X(t)-\bar{X}\right)\left(N-M^{S}\right)\left(X(t)-\bar{X}\right)+\frac{1}{2}\left(X(t)-X(t-1)\right)\left(M^{S}+N\right)\left(X(t)-X(t-1)\right) \\ &\quad -\left(X(t)-\bar{X}\right)M^{A}\left(X(t-1)-\bar{X}\right) \\ &\quad +\left(\bar{X}-\bar{X}^{e}\right)\left(N-\left(M-M'\right)^{S}\right)\left(\bar{X}-\bar{X}^{e}\right)-\left(\bar{X}-\bar{X}_{2}^{(2)}\right)M'\bar{X} \end{aligned} \tag{291}$$

One can also consider that some externalities produce an inertia term of the form: with $\epsilon>0$, that will seen below as stabilizing the system, so that ultimately:

$$\begin{aligned} U_{eff}(X(t)) &= \left(X(t)-\bar{X}\right)\left(N-M^{S}\right)\left(X(t)-\bar{X}\right) \\ &\quad +\frac{1}{2}\left(X(t)-X(t-1)\right)\left(N+M^{S}\right)\left(X(t)-X(t-1)\right)-\left(X(t)-\bar{X}\right)M^{A}\left(X(t-1)-\bar{X}\right) \\ &\quad +\left(\bar{X}-\bar{X}^{e}\right)\left(N-\left(M-M'\right)^{S}\right)\left(\bar{X}-\bar{X}^{e}\right)-\left(\bar{X}-\bar{X}_{2}^{(2)}\right)M'\bar{X} \end{aligned} \tag{292}$$

For the purpose of some applications, we record the particular results for $\beta\rightarrow 0$. As explained before, in that case, the effective utility simplifies to the initial utility:

$$\begin{aligned} & (n(t)+1)^{2}+2\alpha n(t)s_{n}(t-1)-2n(t)w(t-1) \\ = \; & (n(t)+1)^{2}+2\alpha n(t)\begin{pmatrix}1 & 0 & 0\end{pmatrix}s(t-1)-2n(t)w(t-1) \end{aligned}$$

$$\begin{aligned} & s(t)(Id)s(t)+\rho\left(1-w(t-1)-\tilde{f}\right)^{2}+\gamma\left(w(t-1)-\tilde{w}\right)^{2} \\ = \; & s(t)(Id)s(t)+(\rho+\gamma)\left(w(t-1)-\frac{\rho}{(\rho+\gamma)}\left(1-\tilde{f}\right)-\frac{\gamma}{(\rho+\gamma)}\tilde{w}\right)^{2} \end{aligned}$$

$$\begin{aligned} & (w(t)-w_{0})^{2}+\delta n^{2}(t-1)+2\nu n(t-1)w(t)+2\kappa s_{f}(t-1)\left(1-w(t)-\tilde{f}\right)+2\eta s_{w}(t-1)\left(w(t)-\tilde{t}\right) \\ = \; & (w(t)-w_{0})^{2}+\delta n^{2}(t-1)+2\nu n(t-1)w(t)-2\kappa s_{f}(t-1)\left(w(t)-\left(1-\tilde{f}\right)\right)+2\eta s_{w}(t-1)\left(w(t)-\tilde{t}\right) \end{aligned}$$

That can be gathered in a matricial expression:

$$U_{eff}(X(t))=(X(t)-X_{1})I(X(t)-X_{1})-2X(t)MX(t-1)$$

with:

$$M=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\\eta&-\kappa&0&0&-\nu\\0&0&\alpha&-1&0\end{pmatrix}=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\r\cos(\theta)&-r\sin(\theta)&0&0&-\nu\\0&0&\alpha&-1&0\end{pmatrix}\text{ and }X_{1}=\begin{pmatrix}0\\0\\0\\w_{0}\\-1\end{pmatrix}$$

The saddle point equation:

$$\bar{X}=(1-M)^{-1}X_{1}$$

yields the "constant term":

$$\begin{aligned}
U_{eff}\left(\bar{X}\right) &= \left(\bar{X}-X_1\right)\left(\bar{X}-X_1\right)-2\bar{X}M\bar{X} \\
&= \left(M\left(1-M\right)^{-1}X_1\right)^t\left(M\left(1-M\right)^{-1}X_1\right)-2\left(\left(1-M\right)^{-1}X_1\right)^t M\left(1-M\right)^{-1}X_1 \\
&= \left(\left(1-M\right)^{-1}X_1\right)^t\left(M^tM-2M\right)\left(\left(1-M\right)^{-1}X_1\right)
\end{aligned}$$

and we can gather these results:

$$U_{eff}\left(X\left(t\right)\right)=\left(X\left(t\right)-\bar{X}\right)I\left(X\left(t\right)-\bar{X}\right)-2X\left(t\right)MX\left(t-1\right)+\left(\left(1-M\right)^{-1}X_1\right)^t\left(M^tM-2M\right)\left(\left(1-M\right)^{-1}X_1\right)$$

that can be rewritten as in (292):

$$\begin{aligned}
U_{eff}\left(X\left(t\right)\right) &= \left(X\left(t\right)-\bar{X}\right)\left(I-M^S\right)\left(X\left(t\right)-\bar{X}\right)+\frac{1}{2}\left(X\left(t\right)-X\left(t-1\right)\right)\left(\epsilon^2+M^S\right)\left(X\left(t\right)-X\left(t-1\right)\right)-\left(X\left(t\right)-\bar{X}\right)M \\
&\quad +\left(\left(1-M\right)^{-1}X_1\right)^t\left(M^tM-2M\right)\left(\left(1-M\right)^{-1}X_1\right)
\end{aligned}$$

The matrices involved in the previous expression are:

$$\begin{aligned}
\left(M^tM-2M\right) &= \begin{pmatrix} 0 & 0 & 0 & r\cos\left(\theta\right) & 0 \\ 0 & 0 & 0 & -r\sin\left(\theta\right) & 0 \\ 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -\nu & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ r\cos\left(\theta\right) & -r\sin\left(\theta\right) & 0 & 0 & -\nu \\ 0 & 0 & \alpha & -1 & 0 \end{pmatrix} \\
&\quad -2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ r\cos\left(\theta\right) & -r\sin\left(\theta\right) & 0 & 0 & -\nu \\ 0 & 0 & \alpha & -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} r^2\cos^2\theta & -r^2\cos\theta\sin\theta & 0 & 0 & -r\nu\cos\theta \\ -r^2\cos\theta\sin\theta & r^2\sin^2\theta & 0 & 0 & r\nu\sin\theta \\ 0 & 0 & \alpha^2 & -\alpha & 0 \\ -2r\cos\theta & 2r\sin\theta & -\alpha & 1 & 2\nu \\ -r\nu\cos\theta & r\nu\sin\theta & -2\alpha & 2 & \nu^2 \end{pmatrix}
\end{aligned}$$

$$\left(1-M\right)^{-1}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -r\frac{\cos\theta}{\nu-1} & r\frac{\sin\theta}{\nu-1} & \alpha\frac{\nu}{\nu-1} & -\frac{1}{\nu-1} & \frac{\nu}{\nu-1} \\ r\frac{\cos\theta}{\nu-1} & -r\frac{\sin\theta}{\nu-1} & -\frac{\alpha}{\nu-1} & \frac{1}{\nu-1} & -\frac{1}{\nu-1} \end{pmatrix}$$

$$\left(1-M\right)^{-1}X_1=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -r\frac{\cos\theta}{\nu-1} & r\frac{\sin\theta}{\nu-1} & \alpha\frac{\nu}{\nu-1} & -\frac{1}{\nu-1} & \frac{\nu}{\nu-1} \\ r\frac{\cos\theta}{\nu-1} & -r\frac{\sin\theta}{\nu-1} & -\frac{\alpha}{\nu-1} & \frac{1}{\nu-1} & -\frac{1}{\nu-1} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ w_0 \\ -1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\nu-1}\left(\nu+w_0\right) \\ \frac{1}{\nu-1}\left(w_0+1\right) \end{pmatrix}$$

$$
\begin{aligned}
& \left((1-M)^{-1} X_1\right)^t \left(M^t M - 2M\right) \left((1-M)^{-1} X_1\right) \\
= & \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\nu-1}(\nu+w_0) & \frac{1}{\nu-1}(w_0+1) \end{pmatrix} \\
& \times \begin{pmatrix} r^2\cos^2\theta & -r^2\cos\theta\sin\theta & 0 & 0 & -r\nu\cos\theta \\ -r^2\cos\theta\sin\theta & -\frac{1}{2}r^2(\cos 2\theta - 1) & 0 & 0 & r\nu\sin\theta \\ 0 & 0 & \alpha^2 & -\alpha & 0 \\ -2r\cos\theta & 2r\sin\theta & -\alpha & 1 & 2\nu \\ -r\nu\cos\theta & r\nu\sin\theta & -2\alpha & 2 & \nu^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\nu-1}(\nu+w_0) \\ \frac{1}{\nu-1}(w_0+1) \end{pmatrix} \\
= & -\frac{1}{(\nu-1)^2}\left(-\nu^2 w_0^2 + 2\nu w_0^2 + 2\nu w_0 + 2\nu + w_0^2 + 2w_0\right)
\end{aligned}
$$

$$
N + M^S = I + M^S = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & r\cos\theta & 0 \\ 0 & 2 & 0 & -r\sin\theta & 0 \\ 0 & 0 & 2 & 0 & \alpha \\ r\cos\theta & -r\sin\theta & 0 & 2 & -\nu-1 \\ 0 & 0 & \alpha & -\nu-1 & 2 \end{pmatrix}
$$

$$
I - M^S = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & -r\cos(\theta) & 0 \\ 0 & 2 & 0 & r\sin(\theta) & 0 \\ 0 & 0 & 2 & 0 & -\alpha \\ -r\cos(\theta) & r\sin(\theta) & 0 & 2 & \nu+1 \\ 0 & 0 & -\alpha & \nu+1 & 2 \end{pmatrix}
$$

with eigenvalues $1 \pm \frac{1}{2\sqrt{2}}\sqrt{\left(\alpha^2 + (1+\nu)^2 + r^2\right) \pm \sqrt{\left(\alpha^2 + (1+\nu)^2 + r^2\right)^2 - 4r^2\alpha^2}}$

$$
M^A = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2}r\cos\theta & 0 \\ 0 & 0 & 0 & \frac{1}{2}r\sin\theta & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}\alpha \\ \frac{1}{2}r\cos\theta & -\frac{1}{2}r\sin\theta & 0 & 0 & -\frac{1}{2}(\nu-1) \\ 0 & 0 & \frac{1}{2}\alpha & \frac{1}{2}(\nu-1) & 0 \end{pmatrix}
$$

For some values of the parameters, the eigenvalues of $I \pm M^S$ are positives.

For the purpose of section 9, we need to find a matrix relevant to the computation of (177):

$$
\sqrt{N + M^{(S)}}\left(N - M^{(S)} + 2M^{(A)}\left(M^{(S)} + N\right)^{-1} M^{(A)}\right)\sqrt{N + M^{(S)}}
$$

The eigenvalues of this matrix will tell if the field theoretic version of the three agents model, which describes the interaction of a large number of copies of the three agents system, will present some stable pattern (if the eigenvalues are positive), or some unstable ones (for negative eigenvalues). To compute $\sqrt{N + M^{(S)}}$ we rewrite $I + M^S$ by using the previous change of variable. One has:

$$
I + M^S = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & R\sin(v)\cos\theta & 0 \\ 0 & 2 & 0 & -R\sin(v)\sin\theta & 0 \\ 0 & 0 & 2 & 0 & R\cos(v)\cos u \\ R\sin(v)\cos\theta & -R\sin(v)\sin\theta & 0 & 2 & -R\cos(v)\sin u \\ 0 & 0 & R\cos(v)\cos u & -R\cos(v)\sin u & 2 \end{pmatrix}
$$

The parameters R inserted in $I+M^S$ are defined by:

$$\begin{aligned} s^2 &= \alpha^2+(1+\nu)^2 \\ R^2 &= \left(s^2+r^2\right) \\ s &= R\cos(v),\ r=R\sin(v) \\ \alpha &= s\cos u,\ (1+\nu)=s\sin u \end{aligned}$$

We will restrict to the $\theta=\frac{\pi}{4}=u$. By setting the internal parameters to the same value it reduces the problem to compare relative strength of these parameters to N, which is equal to I and to the magnitude of it's action, which is read through R. Then:

$$I+M^S = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & \frac{R}{2} & 0 \\ 0 & 2 & 0 & -\frac{R}{2} & 0 \\ 0 & 0 & 2 & 0 & \frac{R}{2} \\ \frac{R}{2} & -\frac{R}{2} & 0 & 2 & -\frac{R}{2} \\ 0 & 0 & \frac{R}{2} & -\frac{R}{2} & 2 \end{pmatrix},\ I-M^S=\frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & -\frac{R}{2} & 0 \\ 0 & 2 & 0 & \frac{R}{2} & 0 \\ 0 & 0 & 2 & 0 & -\frac{R}{2} \\ -\frac{R}{2} & \frac{R}{2} & 0 & 2 & \frac{R}{2} \\ 0 & 0 & -\frac{R}{2} & \frac{R}{2} & 2 \end{pmatrix}$$

$$M^A = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -\frac{R}{2} & 0 \\ 0 & 0 & 0 & \frac{R}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{R}{2} \\ \frac{R}{2} & -\frac{R}{2} & 0 & 0 & \frac{R}{2} \\ 0 & 0 & \frac{R}{2} & -\frac{R}{2} & 0 \end{pmatrix}$$

$$\sqrt{N+M^{(S)}}\left(N-M^{(S)}+2M^{(A)}\left(M^{(S)}+N\right)^{-1}M^{(A)}\right)\sqrt{N+M^{(S)}}$$

The matrix $\sqrt{N+M^{(S)}}$ is computed by the diagonalization of $N+M^{(S)}$ whose eigenvectors and eigenvalues are:

$$\begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & -\sqrt{2}\sqrt{\frac{1}{2}-\frac{1}{4}\sqrt{2}}-2\sqrt{\frac{1}{2}-\frac{1}{4}\sqrt{2}} & -\sqrt{2}\sqrt{\frac{1}{2}-\frac{1}{4}\sqrt{2}} & 1 \end{pmatrix} \text{ for } 1-\frac{1}{2}R\sqrt{\frac{1}{2}-\frac{1}{4}\sqrt{2}}$$

$$\begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & \sqrt{2}\sqrt{\frac{1}{2}-\frac{1}{4}\sqrt{2}}+2\sqrt{\frac{1}{2}-\frac{1}{4}\sqrt{2}} & \sqrt{2}\sqrt{\frac{1}{2}-\frac{1}{4}\sqrt{2}} & 1 \end{pmatrix} \text{ for } 1+\frac{1}{2}R\sqrt{\frac{1}{2}-\frac{1}{4}\sqrt{2}}$$

$$\begin{pmatrix} -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & \sqrt{2}\sqrt{\frac{1}{4}\sqrt{2}+\frac{1}{2}}-2\sqrt{\frac{1}{4}\sqrt{2}+\frac{1}{2}} & \sqrt{2}\sqrt{\frac{1}{4}\sqrt{2}+\frac{1}{2}} & 1 \end{pmatrix} \text{ for } 1-\frac{1}{2}R\sqrt{\frac{1}{4}\sqrt{2}+\frac{1}{2}}$$

$$\begin{pmatrix} -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 2\sqrt{\frac{1}{4}\sqrt{2}+\frac{1}{2}}-\sqrt{2}\sqrt{\frac{1}{4}\sqrt{2}+\frac{1}{2}} & -\sqrt{2}\sqrt{\frac{1}{4}\sqrt{2}+\frac{1}{2}} & 1 \end{pmatrix} \text{ for } 1+\frac{1}{2}R\sqrt{\frac{1}{4}\sqrt{2}+\frac{1}{2}}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \end{pmatrix} \text{ for } 1$$

Moreover one computes directly that:

$$N-M^{(S)}+2M^{(A)}\left(M^{(S)}+N\right)^{-1}M^{(A)}$$

$$= \begin{pmatrix} \frac{1}{8}R^2\frac{8R^2-128}{R^4-32R^2+128}+1 & -\frac{1}{8}R^2\frac{8R^2-128}{R^4-32R^2+128} & -4\frac{R^3}{R^4-32R^2+128} & \frac{1}{2}R^3\frac{R^2-8}{R^4-32R^2+128}-\frac{1}{4}R & -16\frac{R^2}{R^4-32R^2+128} \\ -\frac{1}{8}R^2\frac{8R^2-128}{R^4-32R^2+128} & \frac{1}{8}R^2\frac{8R^2-128}{R^4-32R^2+128}+1 & 4\frac{R^3}{R^4-32R^2+128} & \frac{1}{4}R-\frac{1}{2}R^3\frac{R^2-8}{R^4-32R^2+128} & 16\frac{R^2}{R^4-32R^2+128} \\ -4\frac{R^3}{R^4-32R^2+128} & 4\frac{R^3}{R^4-32R^2+128} & 2R^2\frac{R^2-8}{R^4-32R^2+128}+1 & -4R^2\frac{R^2-4}{R^4-32R^2+128} & \frac{1}{2}R^3\frac{R^2-16}{R^4-32R^2+128} \\ R^3\frac{R^2-8}{2R^4-64R^2+256}-\frac{1}{4}R & \frac{1}{4}R-R^3\frac{R^2-8}{2R^4-64R^2+256} & -4R^2\frac{R^2-4}{R^4-32R^2+128} & 2R^2\frac{5R^2-24}{R^4-32R^2+128}+1 & \frac{1}{4}R-\frac{1}{2}\frac{R^5}{R^4-32R^2+128} \\ -16\frac{R^2}{R^4-32R^2+128} & 16\frac{R^2}{R^4-32R^2+128} & R^3\frac{R^2-16}{2R^4-64R^2+256}-\frac{1}{4}R & \frac{1}{4}R-\frac{1}{2}\frac{R^5}{R^4-32R^2+128} & 2R^2\frac{R^2-16}{R^4-32R^2+128} \end{pmatrix}$$

These formula allow to compute the eigenvalues and eigenvectors of

$$\sqrt{M^{(S)}}\left(N-2M^{(S)}+M^{(A)}\left(M^{(S)}+\left(M^{(S)}\right)^{-1}\right)M^{(A)}\right)\sqrt{M^{(S)}}$$

For large r, the eigenvalues become negative since the magnitude of the parameters induces an instability. For $R \leqslant 1$ one finds a stable dynamics, and we report the eigenvalues for $R = 0.5$ and 1 as examples.

For $R = 0.5$, the eigenvalues are: $(0.835\,29, 0.874\,28, 0.929\,90, 0.973\,32, 1)$.

For $R = 0.3$, the eigenvalues are: $(0.286\,17, 0.350\,47, 0.735\,61, 0.900\,95, 1)$.

For $R = 2$, the eigenvalues are: $(-1.431\,8, -0.811\,19, 0.452\,54, 0.798\,76, 1)$.

Moreover the matrices of eigenvectors, multiplied by $\sqrt{I + M^{(S)}}$ yields the eigenvectors, in the initial coordinates corresponding to these eigenvalues. The results are:

$$\begin{pmatrix} 1.553\,6 & 1.007\,4 & 0.969\,08 & 0.729\,91 & 1.0 \\ -1.553\,6 & -1.007\,4 & -0.969\,08 & -0.729\,91 & 1.0 \\ -3.752\,8 & 0.417\,13 & 6.874\,5 \times 10^{-2} & -10.278 & 0 \\ 9.059\,7 & 0.173\,09 & -0.166\,22 & -4.256\,7 & 0 \\ -2.199\,0 & 1.424\,7 & -1.370\,3 & 1.033\,1 & 0 \end{pmatrix} \text{for } R = 0.5$$

$$\begin{pmatrix} 1.566\,1 & 1.027\,3 & 0.876\,21 & 0.705\,86 & 1.000\,00 \\ -1.566\,1 & -1.027\,3 & -0.876\,21 & -0.705\,86 & 1.000\,00 \\ -2.041\,8 & 0.788\,05 & 0.115\,32 & -5.362\,2 & 0 \\ 4.928\,7 & 0.326\,79 & -0.278\,44 & -2.221\,1 & 0 \\ -2.216 & 1.452\,5 & -1.239\,1 & 0.998\,3 & 0 \end{pmatrix} \text{for } R = 1$$

Then one can check from the eigenvectors matrices that the more stable directions are the one for which the system moves maximaly towards the directions of the substructures. In that case this direction of motion relaxes the stress imposed by the dominating structure. The more stable solution is mainly driven toward the second, intermediate agent, which acts as a pivotal point in the stability. Other modes are alternatively driven mainly into the direction of one of the subtructures.

The eigenvalue 1 and its eigenvector is a particular case. Due to the exceeding number of parameters compared to the directions of oscillations, this eigenvalue corresponds to an internal oscillation of the third agent, and does not involve the two others.

On the other hand, for $R = 2$ the relevant matrix of eigenvectors is:

$$\begin{pmatrix} 1.055\,4 & 1.569\,9 & 0.710\,46 & 0.670\,57 & 1.0 \\ -1.055\,4 & -1.569\,9 & -0.710\,46 & -0.670\,57 & 1.0 \\ 1.099\,6 & -1.506\,8 & 0.126\,98 & -3.751\,3 & 0 \\ 0.455\,46 & 3.637\,8 & -0.306\,65 & -1.553\,8 & 0 \\ 1.492\,6 & -2.220\,2 & -1.004\,7 & 0.948\,35 & 0 \end{pmatrix}$$

one has a reversed result. The two unstables directions correspond to a motion mainly in the direction of the substructures. Actually, for $R = 2$ the parameters of the interactions are strong enough, so that the coupled oscillations between the two substructures present an unstable pattern.

Results for various types of uncertainty

We compare the results for the classical dynamics for various degree of uncertainty σ^2 in agents behaviors. We look at three examples, mild uncertainty $\sigma^2 = 1$, full uncertainty, $\sigma^2 \to \infty$, no uncertainty $\sigma^2 \to 0$, which converges to the classical case.

The most interesting case for us will be $\sigma^2 = 1$, the two others one being bechmarks cases. Some interpretations will be given in the text, in section 2. Here, we give the relevant parameters for each of these cases, but the interpretations will rely on the eigenvalues of the dynamic system, since these eigenvalues describe the pattern of behavior of the structure as a whole. Recall that these eigenvalues are given by:

$$\lambda = \pm\sqrt{\sigma^2 \nu \left(d + \beta\nu^2 - bd\nu^2\right)} \times \sqrt{\frac{N}{D}}$$

with:

$$\begin{aligned}
N &= \alpha^3\beta^2 - d\alpha^3\beta^2 - d^2\alpha^3\beta + \sigma^2 d^2\alpha + d\alpha^3\beta + f\alpha^2\beta + \sigma^2 df + \sigma^2 d\alpha\beta + \sigma^2\alpha^3\beta^2\omega - df\alpha^2\beta + \left(\sigma^2\right)^2 d\alpha\beta\omega + \sigma^2 df\alpha^2\beta\omega \\
D &= d\left(-\sigma^2 d + (bd-\beta) r^2 - \sigma^2\beta\nu^2 + \sigma^2 bd\nu^2\right) \\
&\quad \times \left(\alpha^3\beta^2 + d^2 f\alpha^2 - d\alpha^3\beta^2 + \sigma^2 d^2\alpha + d\alpha^3\beta + f\alpha^2\beta + \sigma^2 df + \sigma^2 d\alpha\beta + \sigma^2\alpha^3\beta^2\omega - df\alpha^2\beta + (N)^2 d\alpha\beta\omega + \sigma^2 df\alpha^2\beta\omega\right)
\end{aligned}$$

For $\sigma^2 \to 0$, one finds for the parameters of the system and it's eigenvalues, to the second order in β:

$$\begin{aligned}
d &= \frac{1}{2}\left(1 + \beta^2\delta + \sqrt{\left(\delta\beta^2+1\right)^2 - 4\beta^2\nu^2}\right) \\
b &= \frac{\beta}{d} \\
f &= -\alpha\beta
\end{aligned}$$

$$\lambda = \pm\sqrt{-\frac{\nu}{d}} = \sqrt{\frac{2\nu}{\left(1 + \beta^2\delta + \sqrt{\left(\delta\beta^2+1\right)^2 - 4\beta^2\nu^2}\right)}} = \sqrt{-\nu}\left(1 - \frac{\beta^2}{2}\left(\delta - \nu^2\right)\right) + O\left(\beta^3\right)$$

and we recover the classical results as needed.

For $\sigma^2 \to \infty$, one obtains:

$$\begin{aligned}
d &= 1 + \frac{1}{\sigma^2}\beta \\
&\quad - \frac{1}{\sigma^2}\frac{\left(\delta\sigma^2\beta\left(\sigma^2+\beta\right) - \left(\sigma^2\right)^2\beta\nu^2 + \sigma^2 + 2\beta\right) - \sqrt{\left(\delta\sigma^2\beta\left(\sigma^2+\beta\right) - \left(\sigma^2\right)^2\beta\nu^2 + \sigma^2 + 2\beta\right)^2 - 4\left(\beta+\sigma^2\right)\left(1+\sigma^2\beta\delta\right)\beta}}{2\left(1+\sigma^2\beta\delta\right)} \\
&= 1 + o\left(\frac{1}{\left(\sigma^2\right)^2}\right) \\
b &= o\left(\frac{1}{\left(\sigma^2\right)^2}\right) \\
f &= -\alpha\frac{\beta}{\beta\nu^2+1}
\end{aligned}$$

and the eigenvalues are:

$$\begin{aligned}
\lambda &= \pm\sqrt{\sigma^2\nu\left(1+\beta\nu^2\right)\frac{\left(\sigma^2\right)^2\alpha\beta\omega}{\left(-\sigma^2-\sigma^2\beta\nu^2\right)\left(\left(\sigma^2\right)^2\alpha\beta\omega\right)}} \\
&= \pm\sqrt{-\nu}
\end{aligned}$$

As said before, the case for $\sigma^2 = 1$ is the most interestng for us, since in gneral it corresponds to what we aim at modeling: agents anticipating other agents, but taking into account for uncertain intrinsic behaviors. The computations to the second order in β, simplify to yield the following values for the parameters:

$$\begin{aligned}
b &= \beta - \omega\beta^2 \\
f &= -\alpha\beta \\
d &= 1 - \beta^2\left(\nu^2 - \delta\right)
\end{aligned}$$

$$\lambda = \pm\sqrt{-\nu} - \frac{1}{2}\beta^2\sqrt{-\nu}\left(\omega r^2 + \delta - \nu^2\right) + O\left(\beta^3\right)$$

Appendix 7

We start again with the postulated effective action

$$\begin{aligned} U_{eff}\left(X_j\left(t\right)\right) &= Y_j\left(t\right)\begin{pmatrix} N_{ii} & 0 \\ 0 & 0 \end{pmatrix} Y_j\left(t\right) - 2Y_j\left(t\right)\begin{pmatrix} M_{ii} & M_{ij} \\ 0 & 0 \end{pmatrix} Y_j\left(t-1\right) \\ &+ \sum_{i\geqslant k>j} 2X_j\left(t\right) A_{jk}^{(j)}\left(X_k\left(t-1\right)\right) + V_{eff}^{(j)}\left(Y_j\left(t\right)\right) \end{aligned}$$

$$\begin{aligned} U_{eff}\left(Y_j\left(t\right)\right) &= Y_j\left(t\right)\begin{pmatrix} \left(A_{jj}^{(j)}\right)_{eff} & 0 \\ 0 & 0 \end{pmatrix} Y_j\left(t\right) - 2Y_j\left(t\right)\begin{pmatrix} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \\ 0 & 0 \end{pmatrix} Y_j\left(t-1\right) \\ &+ \sum_{i\geqslant k>j} 2X_j\left(t\right) A_{jk}^{(j)}\left(X_k\left(t-1\right)\right) + V_{eff}^{(j)}\left(Y_j\left(t\right)\right) \end{aligned}$$

with $V_{eff}^{(j)}\left(X_j\left(t\right)\right)$ a certain function of $X_j\left(t\right)$, that depends on the potentials $V_i^{(i)}\left(X_i\left(t\right)\right)$ and $V_j^{(i)}\left(X_j\left(t-1\right)\right)$.

Note that for the sake of the exposition we discard all the constants $\bar{X}_j^{(i)}$,... $\bar{X}_j^{(i)e}$ but that they can be reintroduced at the end of the computation.

Recall that (11) allows to find recursively the utilities $U_{eff}\left(X_j\left(t\right)\right)$:

$$\exp\left(U_{eff}\left(X_i\left(t\right)\right)\right) = \int \exp\left(U_t^{(i)}\right) \prod_{rk(j)<rk(i)} \prod_{s\geqslant t} \exp\left(\sum_{s\geqslant t} \frac{U_{eff}\left(X_j\left(s\right)\right)}{N}\right) dX_j\left(s\right) \tag{294}$$

As recorded in the text, we rewrite the utilities in terms of the variables $Y_i\left(t\right)$. We use the general form (44)

$$\begin{aligned} U_t^{(i)} &= \sum_{k\geqslant 0} Y_i\left(t+k\right)\begin{pmatrix} A_{ii}^{(i)} & 0 \\ 0 & 0 \end{pmatrix} Y_i\left(t+k\right) + Y_i\left(t+k-1\right)\begin{pmatrix} -\epsilon_{ii}^{(i)} & 0 \\ 0 & \beta A_{\{jj\}}^{(i)} \end{pmatrix} Y_i\left(t+k-1\right) \\ &+ Y_i\left(t+k\right)\begin{pmatrix} 0 & \beta^{\frac{1}{2}} A_{ij}^{(i)} \\ \beta^{\frac{1}{2}} A_{ji}^{(i)} & 0 \end{pmatrix} Y_i\left(t+k-1\right) \\ &+ \sum_{j>i} 2X_i\left(t\right) A_{ij}^{(i)}\left(X_j\left(t-1\right)\right) + \sum_{k\geqslant 0} \beta^k V_i^{(i)}\left(\frac{X_i\left(t+k\right)}{\beta^{\frac{k}{2}}}\right) + \sum_{j<i}\left(V_j^{(i)}\left(\frac{Y_j\left(\left(t+k\right)-1\right)}{\beta^{\frac{k-1}{2}}}\right)\right) \end{aligned}$$

The normalization of $\exp\left(U_{eff}\left(Y_j\left(t\right)\right)\right)$ is obtained by letting:

$$C\int \exp\left(U_{eff}\left(Y_j\left(t\right)\right)\right)\left(d\left(Y_j\left(t\right)\right)\right) = 1$$

writing:

$$\begin{aligned} U_{eff}\left(Y_j\left(t\right)\right) &= Y_j\left(t\right)\left(A_{jj}^{(j)}\right)_{eff} Y_j\left(t\right) - 2Y_j\left(t\right)\left(\sum_{i\geqslant k>j} A_{jk}^{(j)} X_k\left(t-1\right) - \left(\begin{array}{cc} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{array}\right) Y_j\left(t-1\right)\right) \\ &+ V_{eff}^{(j)}\left(Y_j\left(t\right)\right) \\ &\equiv U_{eff}^{quad}\left(Y_j\left(t\right)\right) + V_{eff}^{(j)}\left(Y_j\left(t\right)\right) \end{aligned}$$

then

$$\begin{aligned} \int \exp\left(U_{eff}\left(Y_j\left(t\right)\right)\right)\left(d\left(Y_j\left(t\right)\right)\right) &= \int \exp\left(U_{eff}^{quad}\left(Y_j\left(t\right)\right) + V_{eff}^{(j)}\left(Y_j\left(t\right)\right)\right)\left(d\left(Y_j\left(t\right)\right)\right) \\ &= \exp\left(V_{eff}^{(j)}\left(\frac{\partial}{\partial U_j}\right)\right) \int \exp\left(U_{eff}^{quad}\left(Y_j\left(t\right)\right)\right)\left(d\left(Y_j\left(t\right)\right)\right) \end{aligned}$$

with:

$$U_j = \left(\sum_{i \geqslant k > j} A_{jk}^{(j)} X_k (t-1) - \left(\begin{array}{cc} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{array} \right) Y_j (t-1) \right)$$

To compute

$$\int \exp \left(U_{eff}^{quad} (Y_j (t)) \right) (d (Y_j (t)))$$

we use that:

$$\begin{aligned}
U_{eff}^{quad} (Y_j (t)) &= \left(Y_j (t) + \left(A_{jj}^{(j)} \right)_{eff}^{-1} \left(\sum_{i \geqslant k > j} A_{jk}^{(j)} X_k (t-1) - \left(\begin{array}{cc} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{array} \right) Y_j (t-1) \right) \right)^t \left(A_{jj}^{(j)} \right)_{eff} \\
&\times \left(Y_j (t) + \left(A_{jj}^{(j)} \right)_{eff}^{-1} \left(\sum_{i \geqslant k > j} A_{jk}^{(j)} X_k (t-1) - \left(\begin{array}{cc} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{array} \right) Y_j (t-1) \right) \right) \\
&- \left(\sum_{i \geqslant k > j} A_{jk}^{(j)} X_k (t-1) - \left(\begin{array}{cc} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{array} \right) Y_j (t-1) \right)^t \\
&\times \left(A_{jj}^{(j)} \right)_{eff}^{-1} \left(\sum_{i \geqslant k > j} A_{jk}^{(j)} X_k (t-1) - \left(\begin{array}{cc} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{array} \right) Y_j (t-1) \right)
\end{aligned}$$

which yields, up to an irrelevant constant:

$$\begin{aligned}
& \int \exp \left(U_{eff}^{quad} (Y_j (t)) \right) (d (Y_j (t))) \\
= \; & \exp \left(- \left(\sum_{i \geqslant k > j} A_{jk}^{(j)} X_k (t-1) - \left(\begin{array}{cc} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{array} \right) Y_j (t-1) \right)^t \right. \\
& \left. \times \left(A_{jj}^{(j)} \right)_{eff}^{-1} \left(\sum_{i \geqslant k > j} A_{jk}^{(j)} X_k (t-1) - \left(\begin{array}{cc} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{array} \right) Y_j (t-1) \right) \right) \\
= \; & \exp \left(- (U_j)^t \left(A_{jj}^{(j)} \right)_{eff}^{-1} U_j \right)
\end{aligned}$$

and

$$\begin{aligned}
& \exp \left(V_{eff}^{(j)} \left(\frac{\partial}{\partial U_j} \right) \right) \int \exp \left(U_{eff}^{quad} (Y_j (t)) \right) (d (Y_j (t))) \\
= \; & \exp \left(V_{eff}^{(j)} \left(\frac{\partial}{\partial U_j} \right) \right) \exp \left(- (U_j)^t \left(A_{jj}^{(j)} \right)_{eff}^{-1} U_j \right)
\end{aligned}$$

and the normalization factor:

$$\left(\exp \left(V_{eff}^{(j)} \left(\frac{\partial}{\partial U_j} \right) \right) \exp \left(- (U_j)^t \left(A_{jj}^{(j)} \right)_{eff}^{-1} U_j \right) \right)^{-1} \equiv \exp \left((U_j)^t \left(A_{jj}^{(j)} \right)_{eff}^{-1} U_j + \hat{V}_{eff}^{(j)} (U_j) \right)$$

this choice of decomposition being justified by the fact that for $V_{eff}^{(j)} = 0$, one recovers a normalization of $\exp \left((U_j)^t \left(A_{jj}^{(j)} \right)_{eff}^{-1} U_j \right)$, as in the quadratic case.

This normalization factor has to be added to the global weight (i.e. the normalized effective utility) to be taken into account for agent i is then, similarly to Appendix 1:

$$U_{eff}\left(Y_j(t)\right) = \sum_{t>0} Y_i(t) \begin{pmatrix} A_{ii}^{(i)} + B_{11} & B_{12} \\ B_{12}^t & \left\{ \left(A_{jj}^{(j)}\right)_{eff} + \beta A_{jj}^{(i)}, B_{22} \right\} \end{pmatrix} Y_i(t) + 2\beta^{\frac{1}{2}} Y_i(t) \begin{pmatrix} -\epsilon_{ii}^{(i)} & A_{ij}^{(i)} \\ A_{ji}^{(j)} & \left\{ \left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}, 2A_{\{kj\}i>k>j}^{(j)} \right\} \end{pmatrix} Y_i(t-1) + \hat{V}_{eff}^{(j)}(U)$$

where $M^S = \frac{1}{2}\left(M + M^t\right)$ for any matrix M, and

$$B_{11} = \beta A_{ij}^{(j)} \left(A_{jj}^{(j)}\right)_{eff}^{-1} A_{ji}^{(j)}$$

$$B_{12} = \left\{ \beta A_{ij}^{(j)} \left(A_{jj}^{(j)}\right)_{eff}^{-1} A_{jk}^{(j)}, \beta \left(A_{ij}^{(j)} \left(A_{jj}^{(j)}\right)_{eff}^{-1} \left(\left(\epsilon_{jj}^{(j)}\right)_{eff} \quad \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \right) \right) \right\}$$

$$B_{22} = \left\{ \begin{array}{c} \beta A_{lj}^{(j)} \left(A_{jj}^{(j)}\right)_{eff}^{-1} A_{jk}^{(j)}, \\ \beta \left(\left(\epsilon_{jj}^{(j)}\right)_{eff} \quad \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \right)^t \left(A_{jj}^{(j)}\right)_{eff}^{-1} \left(\left(\epsilon_{jj}^{(j)}\right)_{eff} \quad \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \right) \\ \beta \left(A_{kj}^{(j)} \left(A_{jj}^{(j)}\right)_{eff}^{-1} \left(\left(\epsilon_{jj}^{(j)}\right)_{eff} \quad \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \right) \right)^S \end{array} \right\}$$

As a consequence, the total weight appearing in (294) is the same as in appendix 1, plus the non quadratic contributions due to $V_i^{(i)}\left(X_i(t)\right)$, $V_j^{(i)}\left(X_j(t-1)\right)$ and $V_{eff}^{(j)}\left(Y_j(t)\right)$. The same operations can be thus performed and in the end the total weight to integrate in the R.H.S. of (294) is

$$\begin{aligned} W &= \sum_{s>t} \left(-\frac{1}{2}\left(Y_i(s) - Y_i(s-1)\right) A \left(Y_i(s) - Y_i(s-1)\right) + Y_i(s) B Y_i(s) + \left(Y_i(s) - Y_i(s-1)\right) C Y_i(s-1) \right) \qquad (295) \\ &+ Y_i(t) B Y_i(t) + \frac{1}{2} Y_i(t) A Y_i(t) \\ &+ \sum_{k\geqslant 0} \beta^k \left(V_i^{(i)}\left(\frac{\left(Y_i(t+k)\right)_i}{\beta^{\frac{k}{2}}} \right) + \sum_{j<i} \left(V_j^{(i)}\left(\frac{Y_j((t+k)-1)}{\beta^{\frac{k-1}{2}}} \right) \right) + \frac{1}{N} \hat{V}_{eff}^{(j)}\left(\frac{U_j(t+k-1)}{\beta^{\frac{k-1}{2}}} \right) \right) \\ &= \sum_{s>t} -\frac{1}{2}\left(Y_i(s) - Y_i(s-1)\right) A \left(Y_i(s) - Y_i(s-1)\right) + Y_i(s) B Y_i(s) + \left(Y_i(s) - Y_i(s-1)\right) C Y_i(s-1) \\ &+ Y_i(t) B Y_i(t) + \frac{1}{2} Y_i(t) A Y_i(t) - \sum_{k\geqslant 0} \beta^{(2-l)k} \left(\sum_{l\geqslant 3} \sum_{n_1,\dots n_l} B_{n_1,\dots n_l}^{(i)} \left(Y_i(t+k)\right)_{n_1} \dots \left(Y_i(t+k)\right)_{n_l} \right) \end{aligned}$$

with:

$$A = \sqrt{\beta}\begin{pmatrix} -\epsilon_{ii}^{(i)} & A_{ij}^{(i)}+A_{ij}^{(j)} \\ A_{ji}^{(i)}+A_{ji}^{(j)} & \left\{ \begin{array}{c} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ A_{\{kj\}i>k>j}^{(j)}, A_{\{jk\}i>k>j}^{(j)} \end{array} \right\} \end{pmatrix}$$

$$B = \begin{pmatrix} A_{ii}^{(i)}+B_{11}-\sqrt{\beta}\epsilon_{ii}^{(i)} & \left\{\sqrt{\beta}\left(A_{ij}^{(i)}+A_{ij}^{(j)}\right), B_{12}\right\} \\ \left\{\sqrt{\beta}\left(A_{ji}^{(i)}+A_{ji}^{(j)}\right), B_{12}^{t}\right\} & \left\{ \begin{array}{c} \beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}, B_{22} \\ \sqrt{\beta}\left\{ \begin{array}{c} \frac{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2} \\ , A_{\{kj\}i>k>j}^{(j)}, A_{\{jk\}i>k>j}^{(j)} \end{array} \right\} \end{array} \right\} \end{pmatrix}$$

$$C = \sqrt{\beta}\begin{pmatrix} 0 & A_{ij}^{(i)}-A_{ij}^{(j)} \\ -\left(A_{ji}^{(i)}-A_{ji}^{(j)}\right) & \left\{ \begin{array}{c} \frac{\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}}{2}, \frac{-\left(\epsilon_{\{jk\}j\geqslant k}^{(j)}\right)_{eff}}{2}, \\ -A_{\{kj\}i>k>j}^{(j)}, A_{\{jk\}i>k>j}^{(j)} \end{array} \right\} \end{pmatrix}$$

and

$$U_j(t-1) = \left(\sum_{i\geqslant k>j} A_{jk}^{(j)} Y_k(t-1) - \left(\begin{array}{cc} \left(\epsilon_{jj}^{(j)}\right)_{eff} & \left(\epsilon_{\{jk\}k<j}^{(j)}\right)_{eff} \end{array} \right) Y_j(t-1)\right)$$

The potential:

$$V_i^{(i)}\left(\frac{(Y_i(t+k))_i}{\beta^{\frac{k}{2}}}\right)+\sum_{j<i}\left(V_j^{(i)}\left(\frac{Y_j((t+k)-1)}{\beta^{\frac{k-1}{2}}}\right)\right)+\frac{1}{N}\hat{V}_{eff}^{(j)}\left(\frac{U_j(t+k-1)}{\beta^{\frac{k-1}{2}}}\right)$$

depends only on $Y_i(t+k)$ and will be denoted $\hat{V}^{(i)}\left(\frac{Y_i(t+k)}{\beta^{\frac{k}{2}}}\right)$.

Then the integral in (11) is computed in the following way. Write:

$$\begin{aligned} \exp(W) &= \exp\Bigg(\sum_{s>t} -\frac{1}{2}(Y_i(s)-Y_i(s-1))A(Y_i(s)-Y_i(s-1))+Y_i(s)BY_i(s) \\ &\quad +(Y_i(s)-Y_i(s-1))CY_i(s-1)+Y_i(t)BY_i(t)+\frac{1}{2}Y_i(t)AY_i(t)\Bigg)-\sum_{s>t}\beta^{s-tk}\hat{V}^{(i)}\left(\frac{Y_i(s)}{\beta^{\frac{s-t}{2}}}\right) \\ &= \Bigg\{\exp\left(-\sum_{s>t}\beta^{s-tk}\hat{V}^{(i)}\left(\frac{1}{\beta^{\frac{s-t}{2}}}\frac{\partial}{\partial(J_i(s))}\right)\right) \\ &\quad \exp\Bigg(\sum_{s>t}-\frac{1}{2}(Y_i(s)-Y_i(s-1))A(Y_i(s)-Y_i(s-1))+Y_i(s)BY_i(s) \\ &\quad +(Y_i(s)-Y_i(s-1))CY_i(s-1)+J_i(s)Y_i(s)+Y_i(t)BY_i(t)+\frac{1}{2}Y_i(t)AY_i(t)\Bigg)\Bigg\}_{J_i(s)=0} \end{aligned}$$

where $J_i(s)$ is an external source term. Then we have to compute in the first place the integral of a very similar weight as in the quadratic case. The only difference is the appearance of the source term. However, it is known that such a term does not modify the fact that the successive gaussian integrals can be evaluated at the saddle point.

The action we have to consider is then:

$$\sum_{s>t}\left(-\frac{1}{2}(Y_i(s)-Y_i(s-1))A(Y_i(s)-Y_i(s-1))+Y_i(s)BY_i(s)+(Y_i(s)-Y_i(s-1))CY_i(s-1)+J_i(s)Y_i(s)\right) \tag{296}$$

and the equation for the classical solution of (295) is then of the usual Euler Lagrange type and quite similar to (240):

$$A\left(Y_i(s+1)-2Y_i(s)+Y_i(s-1)\right)+2BY_i(s)-C\left(Y_i(s+1)-Y_i(s-1)\right)+J_i(s)=0 \tag{297}$$

and it's solution is of the kind:

$$Y_i(s)=D^{s-t}\left(Y_i(t)+F_i(s)\right) \tag{298}$$

with $F_i(t)=0$ and where the equation for the matrix D is given in Appendix 4:

$$(A-C)D^2+2(B-A)D+(A+C)=0$$

we insert (298) in (297) which leads to:

$$A\left(D^2\left(F_i(s+1)-F_i(s)\right)-\left(F_i(s)-F_i(s-1)\right)\right)-C\left(D^2\left(F_i(s+1)-F_i(s)\right)+\left(F_i(s)-F_i(s-1)\right)\right)+J_i(s)=0$$

Let:

$$G_i(s)=\left(F_i(s)-F_i(s-1)\right)$$

the equation for G_i is

$$A\left(D^2G_i(s+1)-G_i(s)\right)-C\left(D^2G_i(s+1)+G_i(s)\right)+J_i(s)=0$$

or:

$$(A-C)D^2G_i(s+1)-(A+C)G_i(s)+J_i(s)=0$$
$$-\left(2(B-A)D+(A+C)\right)G_i(s+1)-(A+C)G_i(s)+J_i(s)=0$$

and it's solution is:

$$G_i(s)=(A+C)^{-1}\sum_{n\geqslant 0}\left((A-C)D^2\right)^n J_i(s+n)$$

and then:

$$F_i(s)=(A+C)^{-1}\sum_{t<u\leq s}\sum_{n\geqslant 0}\left((A-C)D^2\right)^n J_i(u+n) \tag{299}$$

to satisfy the initial condition $F_i(t)=0$.

Replacing the solution (241) in (296), this last quantity can be evaluated in the same way as in appendix 1. One find a quadratic term, as in appendix 1:

$$\frac{1}{2}Y_i(t)A\left(Y_i(t+1)-Y_i(t)\right)+\frac{1}{2}\left(Y_i(t+1)-Y_i(t)\right)CY_i(t)+\frac{1}{2}Y_i(t)AY_i(t)$$

and an additional term coming from the source term. It appears to be an infinite sum

$$\frac{1}{2}\sum_{s>t}J_i(s)Y_i(s)=\frac{1}{2}\sum_{s>t}J_i(s)D^{s-t}\left(Y_i(t)+F_i(s)\right)$$

using (241) and (299) it yields an overall contribution:

$$\begin{aligned}
&\frac{1}{2}Y_i(t)A(D(Y_i(t)+F_i(t+1))-Y_i(t))+\frac{1}{2}(D(Y_i(t)+F_i(t+1))-Y_i(t))CY_i(t)+\frac{1}{2}Y_i(t)AY_i(t)\\
&+\frac{1}{2}\sum_{s>t}J_i(s)Y_i(s)\\
=\;&\frac{1}{2}Y_i(t)((A-C)(D-1))Y_i(t)+\frac{1}{2}Y_i(t)AY_i(t)+\frac{1}{2}Y_i(t)(A-C)DF_i(t+1)\\
&+\frac{1}{2}\sum_{s>t}J_i(s)D^{s-t}Y_i(t)+\frac{1}{2}\sum_{s>t}\sum_{t<u\leq s}\sum_{n\geqslant 0}J_i(s)(A+C)^{-1}\left((A-C)D^2\right)^n J_i(u+n)\\
=\;&\frac{1}{2}Y_i(t)((A-C)(D-1))Y_i(t)+\frac{1}{2}Y_i(t)AY_i(t)\\
&+\frac{1}{2}Y_i(t)(A-C)D(A+C)^{-1}\sum_{n\geqslant 0}\left((A-C)D^2\right)^n J_i(t+n+1)\\
&+\frac{1}{2}\sum_{s>t}J_i(s)D^{s-t}Y_i(t)+\frac{1}{2}\sum_{s>t}\sum_{t<u\leq s}\sum_{n\geqslant 0}J_i(s)(A+C)^{-1}\left((A-C)D^2\right)^n J_i(u+n)\\
=\;&\frac{1}{2}Y_i(t)((A-C)(D-1))Y_i(t)+\frac{1}{2}Y_i(t)AY_i(t)\\
&+\frac{1}{2}\sum_{s>t}Y_i(t)\left(\left(D^{s-t}\right)^t+(A-C)D(A+C)^{-1}\left((A-C)D^2\right)^n\right)J_i(s)\\
&+\frac{1}{2}\sum_{s>t}\sum_{t<u\leq s}\sum_{n\geqslant 0}J_i(s)(A+C)^{-1}\left((A-C)D^2\right)^n J_i(u+n)
\end{aligned}$$

Then, adding the time t contributions leads to:

$$\begin{aligned}
&Y_i(t)\begin{pmatrix} A_{ii}^{(i)} & 0 \\ 0 & \beta A_{jj}^{(i)}+\left(A_{jj}^{(j)}\right)_{eff}\end{pmatrix}Y_i(t)+\sqrt{\beta}Y_i(t)\begin{pmatrix} -\epsilon_{ii}^{(i)} & 2A_{ij}^{(i)} \\ 2A_{ji}^{(j)} & \left\{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff},2A_{\{kj\}i>k>j}^{(j)}\right\}\end{pmatrix}Y_i(t-1)\\
&+\sum_{j>i}2X_i(t)A_{ij}^{(i)}(X_j(t-1))+V_i^{(i)}(X_i(t))\\
&+\frac{1}{2}Y_i(t)\sum_{s>t}\left(\left(D^{s-t}\right)^t+(A-C)D(A+C)^{-1}\left((A-C)D^2\right)^n\right)J_i(s)\\
&+\frac{1}{2}\sum_{s>t}\sum_{t<u\leq s}\sum_{n\geqslant 0}J_i(s)(A+C)^{-1}\left((A-C)D^2\right)^n J_i(u+n)
\end{aligned}$$

As before, the term $V_j^{(i)}(X_j(t-1))$ has been discarded, since it depends only on $t-1$ and will be cancelled by the nomalization.

After computations similar to that of appendix 1, the integral over $Y_i(t)$, $j<i$ yields the effective action:

$$\begin{aligned}A &= -\frac{1}{2}(Y_i(t))_i M_{ii}\left((Y_i(t-1))_i + \left(\left(\left(D^{s-t}\right)^t + (A-C)D(A+C)^{-1}\left((A-C)D^2\right)^n\right)J_i(s)\right)_i\right)\\
&-\frac{1}{2}(Y_i(t))M_{ij}\left((Y_i(t-1))_j + \left(\left(\left(D^{s-t}\right)^t + (A-C)D(A+C)^{-1}\left((A-C)D^2\right)^n\right)J_i(s)\right)_j\right)\\
&+\frac{1}{2}(Y_i(t))_i(N_{ii})(Y_i(t))_i + \sum_{j>i}2X_i(t)A_{ij}^{(i)}(X_j(t-1))\\
&+\frac{1}{2}\sum_{s>t}^{t}\left(\left(\left(D^{s-t}\right)^t + (A-C)D(A+C)^{-1}\left((A-C)D^2\right)^n\right)J_i(s)\right)^t\\
&\qquad\times(A+C)\sum_{s>t}\left(\left(D^{s-t}\right)^t + (A-C)D(A+C)^{-1}\left((A-C)D^2\right)^n\right)J_i(s)\\
&+\frac{1}{2}\sum_{s>t}\sum_{t<u\leq s}\sum_{n\geqslant 0}J_i(s)(A+C)^{-1}\left((A-C)D^2\right)^n J_i(u+n)\end{aligned}$$

where the matrices used in the previous expression are given by:

$$\begin{aligned}N_{ii} &= ((A-C)(D-2)+2B)_{ii} - ((A-C)(D-2)+2B)_{ij}\left(((A-C)(D-2)+2B)_{jj}\right)^{-1}\left(((A-C)(D-2)+2B)_{ji}\right)\\
M_{ii} &= (N_{ii})\left(((A-C)(D-2)+2B)^{-1}(A+C)\right)_{ii}\\
M_{ij} &= (N_{ii})\left(((A-C)(D-2)+2B)^{-1}(A+C)\right)_{ij}\end{aligned}$$

Remark that applying $\frac{1}{\beta^{\frac{s-t}{2}}}\frac{\partial}{\partial(J_i(s))}$ to $\exp(A)$ produces a term:

$$((Y_i(t))_i + F(J_i(s)))\exp(A)$$

where $F(J_i(s))$ is a linear function of $FJ_i(s)$. As a consequence, one shows recursively that:

$$\frac{\partial}{\partial(J_i(s_1))}\cdots\frac{\partial}{\partial(J_i(s_n))}\exp(A) = F_{(n)}\left((Y_i(t))_i, J_i(s_1)\ldots J_i(s_1)\right)\exp(A)$$

for some function $F_{(n)}\left((Y_i(t))_i, J_i(s_1)\ldots J_i(s_1)\right)$. As a consequence:

$$\begin{aligned}&\ln\left(\left\{\exp\left(-\sum_{s>t}\beta^{s-tk}\hat{V}^{(i)}\left(\frac{1}{\beta^{\frac{s-t}{2}}}\frac{\partial}{\partial(J_i(s))}\right)\right)\exp(A)\right\}_{J_i(s)=0}\right)\\
&= -\frac{1}{2}(Y_i(t))_i M_{ii}(Y_i(t-1))_i - \frac{1}{2}(Y_i(t))M_{ij}(Y_i(t-1))_j\\
&\quad+\frac{1}{2}(Y_i(t))_i(N_{ii})(Y_i(t))_i + \sum_{j>i}2X_i(t)A_{ij}^{(i)}(X_j(t-1)) + V_{eff}^{(i)}(Y_i(t))\end{aligned}$$

where $V_{eff}^{(i)}(Y_i(t))$ is some function obtained by the application of the derivatives $\frac{\partial}{\partial(J_i(s_1))}$ appearing in the series expnsion of $\exp\left(-\sum_{s>t}\beta^{s-tk}\hat{V}^{(i)}\left(\frac{1}{\beta^{\frac{s-t}{2}}}\frac{\partial}{\partial(J_i(s))}\right)\right)$ and then setting $J_i(s)=0$. The previous expression is then the expected formula for $U_{eff}(X_i(t))$.

Appendix 8

One applies the method of appendix 1, but using the recursive form for the agents effective utility:

$$\begin{aligned}U_{eff}\left(X_i\left(t\right)\right) &= \frac{1}{2}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)N_{ii}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)-\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)\frac{M_{ii}}{\sqrt{\beta}}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right) \quad (300)\\ &-\sum_{j<i}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)\frac{M_{ij}}{\sqrt{\beta}}\left(X_j\left(t-1\right)-\bar{X}_j^{(i)e}\right)+\sum_{j>i}2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right)\\ &+\sum_{j\leqslant i}X_i\left(t\right)K_{ij}^{(i)}\left(E_t^{(i)}\sum_s Z_j\left(s\right)\right)\end{aligned}$$

$E_t^{(i)}\sum_s Z_j\left(s\right)=Z_j\left(s\right)$ for $s\leqslant t$. It can be reduced to the form of appendix 1:

$$\begin{aligned}U_{eff}\left(X_i\left(t\right)\right) &= \frac{1}{2}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)N_{ii}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)-\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)\frac{M_{ii}}{\sqrt{\beta}}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)\\ &-\sum_{j<i}\left(X_i\left(t\right)-\bar{X}_i^{(i)e}\right)\frac{M_{ij}}{\sqrt{\beta}}\left(X_j\left(t-1\right)-\bar{X}_j^{(i)e}\right)+\sum_{j>i}2X_i\left(t\right)A_{ij}^{(i)}\left(X_j\left(t-1\right)\right)\end{aligned}$$

by the shift $\bar{X}_i^{(i)e}\rightarrow\bar{X}_i^{(i)e}+\left(N_{ii}\right)^{-1}\sum_{j\leqslant i}K_{ij}^{(i)}\left(E_t^{(i)}\sum_s Z_j\left(s\right)\right)$

Then, since one considers the computation of $U_{eff}\left(X_i\left(t\right)\right)$ for an agent i, all effective actions $U_{eff}\left(X_j\left(t\right)\right)$ for $j<i$ have to be modified by this shift: $\bar{X}_j^{(j)e}\rightarrow\bar{X}_j^{(j)e}+\left(N_{jj}\right)^{-1}\sum_{k\leqslant j}K_{jk}^{(i)}\left(E_t^{(j)}\sum_s Z_k\left(s\right)\right)$. It is known that the saddle point computation to obtain the integrals over $X_i\left(s\right)$ and $X_j\left(s\right)$ is still valid when the $\bar{X}_j^{(j)e}$ depend on t (which is the case here after the shift), then the all method of appendix 1 applies.

Before integration, one then arrives to the intermediate effective utility (237):

$$\begin{aligned}\sim\;&\sum_{s>t}-\frac{1}{2}\left(Y_i\left(s\right)-Y_i\left(s-1\right)\right)A\left(Y_i\left(s\right)-Y_i\left(s-1\right)\right)+\left(Y_i\left(s\right)-\hat{Y}_i^{(1)}\right)B\left(Y_i\left(s\right)-\hat{Y}_i^{(1)}\right)\\ &+\left(Y_i\left(s\right)-Y_i\left(s-1\right)\right)CY_i\left(s-1\right)+Y_i\left(t\right)BY_i\left(t\right)\\ &+\frac{1}{2}Y_i\left(t\right)AY_i\left(t\right)\end{aligned}$$

Integration over $Y_i\left(s\right)$, $s>t$ would lead to (246), but recall that for $X_i\left(s\right)=\left(Y_i\left(s\right)\right)_i$ one has to impose the constraint $X_i\left(s\right)=B_i\left(s\right)+E_t^{(i)}Z_i\left(s\right)-B_i\left(s+1\right)$ for all s, as well as the transversality condition $B_i\left(s\right)\rightarrow 0$, $t\rightarrow T$. For a matter of convenience, in the sequel, we will write $Z_i\left(s\right)$ for $E_t^{(i)}Z_i\left(s\right)$ and restore this notation in the end.

One can thus integrate over the vector which is the concatenation of $B_i\left(s\right)+Z_i\left(s\right)-B_i\left(s+1\right)$ and $\left(Y_i\left(s\right)\right)_j$ for $j<i$ and $s>t$. One changes the variables $B_i\left(s\right)=B_i'\left(s\right)-\sum_{i\geqslant 0}Z_i\left(s+i\right)$, so that $B_i\left(s\right)+Z_i\left(s\right)-B_i\left(s+1\right)=B_i'\left(s\right)-B_i'\left(s+1\right)$ and the transversality condition is $B_i'\left(s\right)\rightarrow 0$, $t\rightarrow T$. Then the integrals over $B_i'\left(s\right)$ can be changed by change of variables as integrals over $B_i'\left(s\right)-B_i'\left(s+1\right)$

The result of the integration is thus (244):

$$\frac{1}{2}Y_i\left(t\right)\left(\left(A-C\right)\left(D-1\right)\right)Y_i\left(t\right)+\frac{1}{2}Y_i\left(t\right)AY_i\left(t\right)$$

with a difference with case studied in Appendix 1: as in the simple exemple presented in the text, the series of integrals over $B_i\left(s\right)$ results in replacing in (246) $X_i\left(t\right)$ by $B_i'\left(s\right)-B_i'\left(s+1\right)$. The result for the integration is thus:

$$\frac{1}{2}\hat{Y}_i\left(t\right)\left(\left(A-C\right)\left(D-1\right)\right)\hat{Y}_i\left(t\right)+\frac{1}{2}\hat{Y}_i\left(t\right)A\hat{Y}_i\left(t\right)$$

with:

$$\hat{Y}_i\left(t\right)=\begin{pmatrix}B_i'\left(t\right)-B_i'\left(t+1\right)\\ \left(Y_i\left(s\right)\right)_j\text{ for }j<i\end{pmatrix}$$

and where $B_i'(s) - B_i'(s+1)$ satisfies:

$$\left(B_i'(s) - B_i'(s+1)\right) = D\left(B_i'(s-1) - B_i'(s)\right)$$

(the matrix D is the dynamic matrix (242)). This relation alongside with the transversality condition allows to rewrite the sum:

$$\begin{aligned} B_i'(t+1) &= \left(B_i'(t+1) - B_i'(t+2)\right) + ... + \left(B_i'(T-1) - B_i'(T)\right) \\ &= \left(1 + D^2 + ... + D^T\right)\left(B_i'(t+1) - B_i'(t+2)\right) \\ &= \frac{D\left(1-D^T\right)}{1-D}\left(B_i'(t) - B_i'(t+1)\right) \end{aligned}$$

where we used $B_i'(T) = 0$. As a consequence:

$$\left(B_i'(t) - B_i'(t+1)\right) = \frac{(1-D)}{D\left(1-D^T\right)} B_i'(t+1)$$

and we are left with:

$$\frac{1}{2}\hat{Y}_i(t)\left((A-C)(D-1)\right)\hat{Y}_i(t) + \frac{1}{2}\hat{Y}_i(t) A\hat{Y}_i(t)$$

with:

$$\begin{aligned} \hat{Y}_i(t) &= \begin{pmatrix} \frac{(1-D)}{D(1-D^T)} B_i'(t+1) \\ \left(Y_i(s)\right)_j \text{ for } j<i \end{pmatrix} \\ &= \begin{pmatrix} \frac{(1-D)}{D(1-D^T)}\left(B_i(t+1) + \sum_{i\geqslant 0} Z_i(s+i)\right) \\ \left(Y_i(s)\right)_j \text{ for } j<i \end{pmatrix} \end{aligned}$$

then we use the constraint recursively to write:

$$B_i(t+1) + \sum_{s>t} Z_i(s) = -\sum_{s\leqslant t} X_i(s) + \sum_s Z_i(s)$$

and then:

$$\hat{Y}_i(t) = \begin{pmatrix} \frac{(1-D)}{D(1-D^T)}\left(-\sum_{s\leqslant t} X_i(s) + \sum_s Z_i(s)\right) \\ \left(Y_i(s)\right)_j \text{ for } j<i \end{pmatrix}$$

Thus, as in appendix 1 formula (246), one adds contributions due to specific (i.e. non effective) time t utility (we also change the sign of the first component of $\hat{Y}_i(t)$, using the fact that the utiliy is quadratic) to obtain a non integrated effective utility:

$$\begin{aligned} \hat{U}_{eff}^{int}\left(Y_i(t)\right) &= \frac{1}{2}\begin{pmatrix} \left(\frac{(1-D)}{D(1-D^T)}\right)\left(\sum_{s\leqslant t} X_i(s) - \sum_s Z_i(s)\right) \\ \left(Y_i(s)\right)_j \text{ for } j<i \end{pmatrix}\left((A-C)(D-1)+A\right) \\ &\times \begin{pmatrix} \left(\frac{(1-D)}{D(1-D^T)}\right)\left(\sum_{s\leqslant t} X_i(s) - \sum_s Z_i(s)\right) \\ \left(Y_i(s)\right)_j \text{ for } j<i \end{pmatrix} \\ &+ Y_i(t)\begin{pmatrix} A_{ii}^{(i)} & 0 \\ 0 & \beta A_{jj}^{(i)} + \left(A_{jj}^{(j)}\right)_{eff} \end{pmatrix} Y_i(t) + \sqrt{\beta} Y_i(t)\begin{pmatrix} -\epsilon_{ii}^{(i)} & 2A_{ij}^{(i)} \\ 2A_{ji}^{(j)} & \left\{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}, 2A_{\{kj\}i>k>j}^{(j)}\right\} \end{pmatrix} Y_i(t-1) \\ &+ \sum_{j>i} 2X_i(t) A_{ij}^{(i)}\left(X_j(t-1)\right) \end{aligned} \tag{301}$$

Using that, see (245) and (246):

$$\begin{aligned} & Y_i(t)\begin{pmatrix} A_{ii}^{(i)} & 0 \\ 0 & \beta A_{jj}^{(i)} + \left(A_{jj}^{(j)}\right)_{eff} \end{pmatrix} Y_i(t) + \sqrt{\beta} Y_i(t)\begin{pmatrix} -\epsilon_{ii}^{(i)} & 2A_{ij}^{(i)} \\ 2A_{ji}^{(j)} & \left\{-\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}, 2A_{\{kj\}i>k>j}^{(j)}\right\} \end{pmatrix} Y_i(t-1) \\ = \; & Y_i(t)(B-A)Y_i(t) + Y_i(t)(A+C)Y_i(t-1) \end{aligned}$$

and developping in (301) the quadratic terms in $\sum_{s\leqslant t} X_i(s) - \sum_s Z_i(s)$ as $X_i(t) + \sum_{s<t} X_i(s) - \sum_s Z_i(s)$ as well as discarding terms that do not depend on $X_i(t)$ and $(Y_i(t))_j$ for $j < i$ yields:

$$
\begin{aligned}
\hat{U}_{eff}^{int}(Y_i(t)) = &\ \frac{1}{2}\begin{pmatrix} \left(\frac{(1-D)}{D(1-D^T)}\right) X_i(t) \\ (Y_i(t))_j \text{ for } j<i \end{pmatrix} ((A-C)D) \begin{pmatrix} \left(\frac{(1-D)}{D(1-D^T)}\right) X_i(t) \\ (Y_i(t))_j \text{ for } j<i \end{pmatrix} \\
&+ Y_i(t)(B-A)Y_i(t) + Y_i(t)(A+C)Y_i(t-1) + \sum_{j>i} 2X_i(t) A_{ij}^{(i)} (X_j(t-1)) \\
&+ \begin{pmatrix} \left(\frac{(1-D)}{D(1-D^T)}\right)\left(\sum_{s<t} X_i(s) - \sum_s Z_i(s)\right) \\ 0 \end{pmatrix} ((A-C)D) \begin{pmatrix} \left(\frac{(1-D)}{D(1-D^T)}\right) X_i(t) \\ (Y_i(t))_j \text{ for } j<i \end{pmatrix} \\
= &\ \frac{1}{2} Y_i(t) P^t ((A-C)D) P Y_i(t) \\
&+ Y_i(t)(B-A)Y_i(t) + Y_i(t)(A+C)Y_i(t-1) + \sum_{j>i} 2X_i(t) A_{ij}^{(i)} (X_j(t-1)) \\
&+ \left(\sum_{s<t} X_i(s) - \sum_s Z_i(s)\right) P_i^t ((A-C)D)(Y_i(t))_j \\
&+ \left(\sum_{s<t} X_i(s) - \sum_s Z_i(s)\right) \left(\frac{(1-D)}{D(1-D^T)}\right)^t ((A-C)D) \left(\frac{(1-D)}{D(1-D^T)}\right) X_i(t) \\
= &\ \frac{1}{2} Y_i(t) \left(P^t ((A-C)D) P + 2(B-A)\right) Y_i(t) + Y_i(t)(A+C)Y_i(t-1) \\
&+ \left(\sum_{s<t} X_i(s) - \sum_s Z_i(s)\right) P_i^t ((A-C)D) \tilde{P} Y_i(t) + \sum_{j>i} 2X_i(t) A_{ij}^{(i)} (X_j(t-1)) \\
&+ \left(\sum_{s<t} X_i(s) - \sum_s Z_i(s)\right) P_i^t ((A-C)D) P_i X_i(t)
\end{aligned}
$$

where:

$$
\begin{aligned}
P &= \begin{pmatrix} P_i & 0 \\ 0 & 1_j \end{pmatrix}, \tilde{P} = \begin{pmatrix} 0 & 0 \\ 0 & 1_j \end{pmatrix} \\
P_i &= \left(\frac{(1-D)}{D(1-D^T)}\right) \\
1_j &= \text{identity matrix for the block } j<i
\end{aligned}
$$

Then, to obtain the effective utility for $X_i(t)$ one can integrate over the $(Y_i(t))_j$ for $j < i$.

$$
\begin{aligned}
& U_{eff}\left(X_i(t)\right)=\frac{1}{2}\Big(Y_i(t)+\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1} \\
& \times(A+C)\left(Y_i(t-1)\right)+\tilde{P}^t\left((A-C)D\right)P_i\left(\sum_{s<t}X_i(s)-\sum_s Z_i(s)\right)\Big)_i \\
& \times\left(N_{ii}\right)\Big(Y_i(t)+\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1} \\
& (A+C)\left(Y_i(t-1)\right)+\tilde{P}^t\left((A-C)D\right)P_i\left(\sum_{s<t}X_i(s)-\sum_s Z_i(s)\right)\Big)_i \\
& +\sum_{j>i}2X_i(t)A_{ij}^{(i)}\left(X_j(t-1)\right) \\
& +\left(\sum_{s<t}X_i(s)-\sum_s Z_i(s)\right)P_i^t\left((A-C)D\right)P_iX_i(t) \\
\sim\ & -\frac{1}{2}\left(\left(Y_i(t)\right)_i M_{ii}\left(Y_i(t-1)\right)_i+T\right)-\frac{1}{2}\Big(\left(Y_i(t)\right)M_{ij}\left(Y_i(t-1)\right)_j+T\Big) \\
& +\frac{1}{2}\left(Y_i(t)\right)_i\left(N_{ii}\right)\left(Y_i(t)\right)_i+\sum_{j>i}2X_i(t)A_{ij}^{(i)}\left(X_j(t-1)\right) \\
& +\left(\sum_{s<t}X_i(s)-\sum_s Z_i(s)\right)\Big(P_i^t\left((A-C)D\right)\tilde{P}\left(N_{ii}\right)\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1}+P_i^t\left((A-C)D\right)P_i\Big)X_i(t)
\end{aligned}
\tag{302}
$$

where the matrices used in the previous expression are given by:

$$
\begin{aligned}
N_{ii} &= \left(P^t\left((A-C)D\right)P+2(B-A)\right)_{ii} \\
&\quad -\left(P^t\left((A-C)D\right)P+2(B-A)\right)_{ij}^{-1}\Big(\left(P^t\left((A-C)D\right)P+2(B-A)\right)_{jj}\Big)\Big(\left(P^t\left((A-C)D\right)P+2(B-A)\right)_{ji}\Big) \\
M_{ii} &= \left(N_{ii}\right)\Big(\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1}(A+C)\Big)_{ii} \\
M_{ij} &= \left(N_{ii}\right)\Big(\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1}(A+C)\Big)_{ij}
\end{aligned}
\tag{303}
$$

and where the "T" means the transpose of the expression in the same parenthesis. Then, as explained in the text, the terms

$$
\left(\sum_{s<t}X_i(s)\right)\Big(P_i^t\left((A-C)D\right)\tilde{P}\left(N_{ii}\right)\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1}+P_i^t\left((A-C)D\right)P_i\Big)X_i(t)
\tag{304}
$$

may be approximated by:

$$
\left(X_i(t)+X_i(t-1)\right)\Big(P_i^t\left((A-C)D\right)\tilde{P}\left(N_{ii}\right)\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1}+P_i^t\left((A-C)D\right)P_i\Big)X_i(t)
$$

and these terms may be included in the quadratic terms of the effective utility to produce the result announced in (300) with (and restoring $\sum_s Z_i(s) \rightarrow \left(E_t^{(i)} \sum_s Z_i(s)\right)$):

$$\begin{aligned}
N_{ii} &= \left(P^t\left((A-C)D\right)P+2\left(B-A\right)\right)_{ii} \\
&\quad -\left(P^t\left((A-C)D\right)P+2\left(B-A\right)\right)_{ij}^{-1}\left(\left(P^t\left((A-C)D\right)P+2\left(B-A\right)\right)_{jj}\right)\left(\left(P^t\left((A-C)D\right)P+2\left(B-A\right)\right)_{ji}\right) \\
&\quad +\left(P_i^t\left((A-C)D\right)\tilde{P}\left(N_{ii}\right)\left(P^t\left((A-C)D\right)P+2\left(B-A\right)\right)^{-1}+P_i^t\left((A-C)D\right)P_i\right) \\
M_{ii} &= \left(N_{ii}\right)\left(\left(P^t\left((A-C)D\right)P+2\left(B-A\right)\right)^{-1}\left(A+C\right)\right)_{ii} \\
&\quad +\left(P_i^t\left((A-C)D\right)\tilde{P}\left(N_{ii}\right)\left(P^t\left((A-C)D\right)P+2\left(B-A\right)\right)^{-1}+P_i^t\left((A-C)D\right)P_i\right) \\
M_{ij} &= \left(N_{ii}\right)\left(\left(P^t\left((A-C)D\right)P+2\left(B-A\right)\right)^{-1}\left(A+C\right)\right)_{ij} \\
K_{ii}^{(i)} &= \left(P_i^t\left((A-C)D\right)\tilde{P}\left(N_{ii}\right)\left(P^t\left((A-C)D\right)P+2\left(B-A\right)\right)^{-1}+P_i^t\left((A-C)D\right)P_i\right)
\end{aligned} \tag{305}$$

Note also that in (300), the terms $\bar{X}_j^{(j)e}$ have to be shifted by $\bar{X}_j^{(j)e} \rightarrow \bar{X}_j^{(j)e}+\left(N_{jj}\right)^{-1}\sum_{k\leqslant j} K_{jk}^{(i)}\left(E_t^{(j)}\sum_s Z_k(s)\right)$ and as a consequence, (235) implies that $\bar{X}_i^{(i)e}$ is shifted by

$$\begin{aligned}
\delta\bar{X}_j^{(i)e} &= \begin{pmatrix} A_{ii}^{(i)}+B_{11}-\sqrt{\beta}\epsilon_{ii}^{(i)} & \left\{B_{12}, 2\sqrt{\beta}\left(A_{ij}^{(i)}\right)^S\right\} \\ \left\{B_{12}^t, 2\sqrt{\beta}\left(A_{ji}^{(j)}\right)^S\right\} & \left\{\begin{array}{c}\left(A_{jj}^{(j)}\right)_{eff}+\beta A_{jj}^{(i)}, B_{22}, \\ \left(2\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}, 2A_{\{kj\}i>k>j}^{(j)}\right)^S\end{array}\right\}\end{pmatrix}^{-1} \\
&\quad \times\begin{pmatrix} 0 & \frac{B_{12}^{(3)}}{2} \\ \frac{\left(B_{12}^{(3)}\right)^t}{2} & \left\{\left(A_{jj}^{(j)}\right)_{eff}, B"_{22}, \frac{B_{22}^{(3)}}{2}, \sqrt{\beta}\left(\frac{\epsilon_{\{kj\}k\leqslant j}^{(j)}}{2}\right)_{eff}\right\}\end{pmatrix}\begin{pmatrix} 0 \\ \left(N_{jj}\right)^{-1}\sum_{k\leqslant j} K_{jk}^{(i)}\left(E_t^{(j)}\sum_s Z_k(s)\right)\end{pmatrix}
\end{aligned} \tag{306}$$

Keeping the i th coordinate of this shift and computing the expansion of the terms including $\delta\bar{X}_j^{(i)e}$ in the effective utility (300) yields a contribution:

$$\begin{aligned}
&\sum_{j<i} X_i(t) K_{ij}^{(i)} \left(E_t^{(i)} \sum_s Z_j(s) \right) \\
&= X_i(t) \begin{pmatrix} N_{ii} & 0 \end{pmatrix} \begin{pmatrix} A_{ii}^{(i)} + B_{11} - \sqrt{\beta}\epsilon_{ii}^{(i)} & \left\{ B_{12}, 2\sqrt{\beta}\left(A_{ij}^{(i)}\right)^S \right\} \\ \left\{ B_{12}^t, 2\sqrt{\beta}\left(A_{ji}^{(j)}\right)^S \right\} & \left\{ \begin{array}{c} \left(A_{jj}^{(j)}\right)_{eff} + \beta A_{jj}^{(i)}, B_{22}, \\ \left(2\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}, 2A_{\{kj\}i>k>j}^{(j)} \right)^S \end{array} \right\} \end{pmatrix}^{-1} \\
&\times \begin{pmatrix} 0 & \frac{B_{12}^{(3)}}{2} \\ \frac{\left(B_{12}^{(3)}\right)^t}{2} & \left\{ \left(A_{jj}^{(j)}\right)_{eff}, B"_{22}, \frac{B_{22}^{(3)}}{2}, \sqrt{\beta}\left(\frac{\epsilon_{\{kj\}k\leqslant j}^{(j)}}{2}\right)_{eff} \right\} \end{pmatrix} \\
&\times \begin{pmatrix} 0 \\ \left(N_{jj}\right)^{-1} K_{ij}^{(i)} \left(\sum_{j\leqslant i} E_t^{(i)} \sum_s Z_j(s) \right) \end{pmatrix} \\
&+ X_i(t) \begin{pmatrix} M_{ii} & M_{ij} \end{pmatrix} \begin{pmatrix} A_{ii}^{(i)} + B_{11} - \sqrt{\beta}\epsilon_{ii}^{(i)} & \left\{ B_{12}, 2\sqrt{\beta}\left(A_{ij}^{(i)}\right)^S \right\} \\ \left\{ B_{12}^t, 2\sqrt{\beta}\left(A_{ji}^{(j)}\right)^S \right\} & \left\{ \begin{array}{c} \left(A_{jj}^{(j)}\right)_{eff} + \beta A_{jj}^{(i)}, B_{22}, \\ \left(2\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}, 2A_{\{kj\}i>k>j}^{(j)} \right)^S \end{array} \right\} \end{pmatrix}^{-1} \\
&\times \begin{pmatrix} 0 & \frac{B_{12}^{(3)}}{2} \\ \frac{\left(B_{12}^{(3)}\right)^t}{2} & \left\{ \left(A_{jj}^{(j)}\right)_{eff}, B"_{22}, \frac{B_{22}^{(3)}}{2}, \sqrt{\beta}\left(\frac{\epsilon_{\{kj\}k\leqslant j}^{(j)}}{2}\right)_{eff} \right\} \end{pmatrix} \\
&\times \begin{pmatrix} 0 \\ \left(N_{jj}\right)^{-1} K_{ij}^{(i)} \left(\sum_{j\leqslant i} E_t^{(i)} \sum_s Z_j(s-1) \right) \end{pmatrix}
\end{aligned}$$

In the approximation of the continuous limit

$$\left(\sum_{j\leqslant i} E_t^{(i)} \sum_s Z_j(s-1) \right) \simeq \left(\sum_{j\leqslant i} E_t^{(i)} \sum_s Z_j(s) \right)$$

and the all contribution due to the constraint reduces to:

$$\begin{aligned}
&\sum_{j<i} X_i(t) K_{ij}^{(i)} \left(E_t^{(i)} \sum_s Z_j(s) \right) \\
&= X_i(t) \begin{pmatrix} N_{ii} + M_{ii} & M_{ij} \end{pmatrix} \begin{pmatrix} A_{ii}^{(i)} + B_{11} - \sqrt{\beta}\epsilon_{ii}^{(i)} & \left\{ B_{12}, 2\sqrt{\beta}\left(A_{ij}^{(i)}\right)^S \right\} \\ \left\{ B_{12}^t, 2\sqrt{\beta}\left(A_{ji}^{(j)}\right)^S \right\} & \left\{ \begin{array}{c} \left(A_{jj}^{(j)}\right)_{eff} + \beta A_{jj}^{(i)}, B_{22}, \\ \left(2\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}, 2A_{\{kj\}i>k>j}^{(j)} \right)^S \end{array} \right\} \end{pmatrix}^{-1} \\
&\times \begin{pmatrix} 0 & \frac{B_{12}^{(3)}}{2} \\ \frac{\left(B_{12}^{(3)}\right)^t}{2} & \left\{ \left(A_{jj}^{(j)}\right)_{eff}, B"_{22}, \frac{B_{22}^{(3)}}{2}, \sqrt{\beta}\left(\frac{\epsilon_{\{kj\}k\leqslant j}^{(j)}}{2}\right)_{eff} \right\} \end{pmatrix} \begin{pmatrix} 0 \\ \left(N_{jj}\right)^{-1} K_{ij}^{(i)} \left(\sum_{j\leqslant i} E_t^{(i)} \sum_s Z_j(s) \right) \end{pmatrix}
\end{aligned}$$

which implies that:

$$K_{ij}^{(i)} = \begin{pmatrix} N_{ii}+M_{ii} & M_{ij} \end{pmatrix} \begin{pmatrix} A_{ii}^{(i)}+B_{11}-\sqrt{\beta}\epsilon_{ii}^{(i)} & \left\{B_{12}, 2\sqrt{\beta}\left(A_{ij}^{(i)}\right)^{S}\right\} \\ \left\{B_{12}^{t}, 2\sqrt{\beta}\left(A_{ji}^{(j)}\right)^{S}\right\} & \left\{ \begin{matrix} \left(A_{jj}^{(j)}\right)_{eff}+\beta A_{jj}^{(i)}, B_{22}, \\ \left(2\left(\epsilon_{\{kj\}k\leqslant j}^{(j)}\right)_{eff}, 2A_{\{kj\}i>k>j}^{(i)}\right)^{S} \end{matrix} \right\} \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} \frac{B_{12}^{(3)}}{2}\left(N_{jj}\right)^{-1} K_{ij}^{(i)}\left(\sum_{j\leqslant i} E_{t}^{(i)} \sum_{s} Z_{j}(s)\right) \\ \left\{\left(A_{jj}^{(j)}\right)_{eff}, B"_{22}, \frac{B_{22}^{(3)}}{2}, \sqrt{\beta}\left(\frac{\epsilon_{\{kj\}k\leqslant j}^{(j)}}{2}\right)_{eff}\right\}\left(N_{jj}\right)^{-1} K_{ij}^{(i)}\left(\sum_{j\leqslant i} E_{t}^{(i)} \sum_{s} Z_{j}(s)\right) \end{pmatrix}$$

Now, if we were to keep the terms (304) without approximation, the effective utility (300) should be modified from the begining to include some additional lag terms:

$$\begin{aligned} U_{eff}\left(X_i(t)\right) &= \frac{1}{2}\left(X_i(t)-\bar{X}_i^{(i)e}\right) N_{ii}^{(0)}\left(X_i(t)-\bar{X}_i^{(i)e}\right)-\left(X_i(t)-\bar{X}_i^{(i)e}\right)\frac{M_{ii}^{(0)}}{\sqrt{\beta}}\left(X_i(t)-\bar{X}_i^{(i)e}\right) \quad \text{(Ueffinibsbs)} \\ &\quad -\sum_{j<i}\left(X_i(t)-\bar{X}_i^{(i)e}\right)\frac{M_{ij}^{(0)}}{\sqrt{\beta}}\left(X_j(t-1)-\bar{X}_j^{(i)e}\right)+\sum_{j>i} 2X_i(t) A_{ij}^{(i)}\left(X_j(t-1)\right) \\ &\quad +X_i(t) M_i\left(E_t^{(i)}\sum_s Z_i(s)\right)+\sum_{j\leqslant i}\sum_{s<t} X_j(s)\epsilon_{ij}^{(i,n)} X_i(t) \end{aligned}$$

These terms modify the matrices A, B, and C in (238) by modifying the inertia terms $\epsilon_{\{kj\}k\leqslant j}^{(j)}$ as a sum

$$\epsilon_{\{kj\}k\leqslant j}^{(j)} \to \epsilon_{\{kj\}k\leqslant j}^{(j)}+\sum_{n\geqslant 2}\epsilon_{\{kj\}k\leqslant j}^{(j,n)} L^{n-1}$$

and as well for their transpose:

$$\epsilon_{\{jk\}j\geqslant k}^{(j)} \to \epsilon_{\{jk\}j\geqslant k}^{(j)}+\sum_{n\geqslant 2}\epsilon_{\{jk\}j\geqslant k}^{(j,n)} L^{n-1}$$

and these operators are included in the computations that are similar to the previous one. Now, the saddle point equation (240) is still valid, as well as it's solution (241). However two modifications have to be included. First, Given that the saddle point equation is derived from (239):

$$\begin{aligned} \sim \quad & \sum_{s>t} -\frac{1}{2}\left(Y_i(s)-Y_i(s-1)\right) A\left(Y_i(s)-Y_i(s-1)\right)+Y_i(s) BY_i(s)+\left(Y_i(s)-Y_i(s-1)\right) CY_i(s-1) \\ & +Y_i(t) BY_i(t)+\frac{1}{2} Y_i(t) AY_i(t) \end{aligned}$$

and since this equation includes coupling between $Y_i(t)$ and $Y_i(t\pm n)$, due to the inertia terms $\epsilon_{\{kj\}k\leqslant j}^{(j,n)} L^{n-1}$, then in (240):

$$\left(Y_i(s)-\hat{Y}_i^{(1)}\right) A\left(Y_i(s+1)-2Y_i(s)+Y_i(s-1)\right)+2Y_i(s) BY_i(s)-\left(Y_i(s)-\hat{Y}_i^{(1)}\right) C\left(Y_i(s+1)-Y_i(s-1)\right)=0$$

this fact is taken into account by replacing in A, B, C the terms $\epsilon_{\{kj\}k\leqslant j}^{(j,n)} L^{n-1}$ by $\epsilon_{\{kj\}k\leqslant j}^{(j,n)}\left(L^{n-1}+L^{-(n-1)}\right)$ (this is the analog of the symetrization process appearing in this kind of equations but translated to the lag operators level $L \to L^{-1}$ in this transposition), and as well for their transpose $\epsilon_{\{jk\}j\geqslant k}^{(j,n)} L^{n-1}$ that have to be replaced by $\epsilon_{\{jk\}j\geqslant k}^{(j,n)}\left(L^{n-1}+L^{-(n-1)}\right)$.

Second, one can first solve formaly for D as in (241) by letting:

$$Y_i(s) = D^{t-s} Y_i(t)$$

and the solution is formally the same as if no inertia was present. But this equation, solves D as a function $D\left(\epsilon_{\{kj\}k\leqslant j}^{(j,n)}\left(L^{n-1}+L^{-(n-1)}\right), \epsilon_{\{jk\}j\geqslant k}^{(j,n)}\left(L^{n-1}+L^{-(n-1)}\right)\right)$. Let us call $\tilde{D}$ this solution. To find the "true" matrix D as a function of the parameters, one replaces in $Y_i(s+1) = DY_i(s)$, and in that case:

$$DY_i(s) = \tilde{D}\left(\epsilon_{\{kj\}k\leqslant j}^{(j,n)}\left(L^{n-1}+L^{-(n-1)}\right), \epsilon_{\{jk\}j\geqslant k}^{(j,n)}\left(L^{n-1}+L^{-(n-1)}\right)\right) Y_i(s)$$

and given the solution $Y_i(t)$, $LY_i(s) = DY_i(s)$, the previous relation can also be written:

$$DY_i(s) = \tilde{D}\left(\epsilon_{\{kj\}k\leqslant j}^{(j,n)}\left(D^{n-1}+D^{-(n-1)}\right), \epsilon_{\{jk\}j\geqslant k}^{(j,n)}\left(D^{n-1}+D^{-(n-1)}\right)\right) Y_i(s)$$

which yields the equation for D:

$$D = \tilde{D}\left(\epsilon_{\{kj\}k\leqslant j}^{(j,n)}\left(D^{n-1}+D^{-(n-1)}\right), \epsilon_{\{jk\}j\geqslant k}^{(j,n)}\left(D^{n-1}+D^{-(n-1)}\right)\right)$$

Then this equation can be solved as a series expansion in the $\epsilon_{\{kj\}k\leqslant j}^{(j,n)}$. In fact, as seen in the text, the inertial term are of order $\frac{1}{T}$ where T is the characteristic length of the interaction process. As such, T is the "largest" parameter in the system, and the series expansion can be stopped at the first order.

Once D has been found, the resolution is the same as before. One arrives at the effective action given in (302), which yields ultimately the required form (102). Then, one expands the coefficients involved in (302) as series $\epsilon_{\{kj\}k\leqslant j}^{(j,n)}$ to obtain:

$$\begin{aligned}
N_{ii}^{(0)} &= \Big[\left(P^t\left((A-C)D\right)P+2(B-A)\right)_{ii} \\
&\quad -\left(P^t\left((A-C)D\right)P+2(B-A)\right)_{ij}^{-1}\left(\left(P^t\left((A-C)D\right)P+2(B-A)\right)_{jj}\right)\left(\left(P^t\left((A-C)D\right)P+2(B-A)\right)_{ji}\right) \\
&\quad +\left(P_i^t\left((A-C)D\right)\tilde{P}(N_{ii})\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1}+P_i^t\left((A-C)D\right)P_i\right)\Big]_{\text{zeroth}} \\
M_{ii}^{(0)} &= \Big[(N_{ii})\left(\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1}(A+C)\right)_{ii} \\
&\quad +\left(P_i^t\left((A-C)D\right)\tilde{P}(N_{ii})\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1}+P_i^t\left((A-C)D\right)P_i\right)\Big]_{\text{zeroth}} \\
M_{ij}^{(0)} &= \left[(N_{ii})\left(\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1}(A+C)\right)_{ij}\right]_{\text{zeroth}} \\
M_i &= \left(P_i^t\left((A-C)D\right)\tilde{P}(N_{ii})\left(P^t\left((A-C)D\right)P+2(B-A)\right)^{-1}+P_i^t\left((A-C)D\right)P_i\right)
\end{aligned}$$

the subscript zeroth standing for the zeroth order expansion in the $\epsilon_{\{kj\}k\leqslant j}^{(j,n)}$. The expression for the matrices $N_{ii}^{(0)}$, $M_{ii}^{(0)}$ and $M_{ij}^{(0)}$ are the same as the one presented in (305) since $N_{ii}^{(0)}$, $M_{ii}^{(0)}$ and $M_{ij}^{(0)}$ are obtained by the zeroth order expansion of $U_{eff}(X_i(t))$ in $\epsilon_{\{kj\}k\leqslant j}^{(j,n)}$ and thus their expression is similar to the case without constraint. The higher order terms in the expansion, in fact the first order being sufficient, will be gathered to yield the terms $\sum_{j\leqslant i}\sum_{s<t} X_j(s) N_i X_i(t)$. We do not present any detailled formula here, since it depends for each particular case on the form of $\tilde{D}$ as a function of the $\epsilon_{\{kj\}k\leqslant j}^{(j,n)}$.

Appendix 9

An operator of the form

$$-\nabla^2 + ax^2 + \alpha = \left(-\nabla + \sqrt{a}x\right)\left(\nabla + \sqrt{a}x\right) + \alpha + \sqrt{a}$$

has eigenvalues

$$n\sqrt{a} + \alpha + \sqrt{a} = (n+1)\sqrt{a} + \alpha$$

with eigenvectors:

$$\varphi_n(x) = \left(\frac{\sqrt{a}}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{1}{2^n n!}} H_n\left(a^{\frac{1}{4}}x\right) \exp\left(-\frac{\sqrt{a}}{2}x^2\right)$$

where the $H_n\left(a^{\frac{1}{4}}x\right)$ are the Hermite polynomials. The Green function

$$G = \left(-\nabla^2 + ax^2 + \alpha\right)^{-1}$$

which is equal to the propagator:

$$G(x,y) = \langle \Psi(x) \Psi(y) \rangle$$

is given by:

$$G(x,y) = \sum_n \frac{\varphi_n(x)\varphi_m^{\dagger}(x)}{(n+1)\sqrt{a} + \alpha}$$

Applying this results to our problem yields $G(x,y)$:

$$\begin{aligned} G(x,y) &= \langle x| \frac{1}{-\nabla_k^2 + m_i^2 + \left((x_i)_k - \left(\check{Y}_{eff}\right)_k\right)(\Lambda_i)_k \left((x_i)_k - \left(\check{Y}_{eff}\right)_k\right)} |y\rangle \\ &= \sum_n \psi_n(x) \frac{1}{m_i^2 + \left(n + \frac{1}{2}\right)(\Lambda_i)_k} \psi_n^*(y) \end{aligned}$$

Appendix 10

As explained in the text, we have to compute the Green function under the following form:

$$\begin{aligned}\bar{G}(x,y) &= P(0,s,x_i,y_i)\left\langle \exp\left(-\frac{1}{T}\left(\int_0^s X(u)\,du\right)\left(\int_0^T X(u)\,du\right)\right)\right\rangle \\ &= \frac{\exp\left(-\frac{(x-y)^2}{\sigma^2 s}\right)}{\sqrt{s}}\left\langle \exp\left(-\frac{1}{T}\left(\int_0^s X(u)\,du\right)\left(\int_0^T X(u)\,du\right)\right)\right\rangle\end{aligned}$$

Where $X(u)$ a brownian motion starting at x_i at time 0 and reaching y_i at time s and

$$\left\langle \exp\left(-\frac{1}{s}\left(\int_0^s X(u)\,du\right)\left(\int_0^s X(u)\,du\right)\right)\right\rangle$$

is the expectation value of $\exp\left(\frac{1}{s}\left(\int_0^s X(u)\,du\right)^2\right)$ given the process $X(u)$.

The appearance of the factor $P(0,s,x_i,y_i)$ in (160a) comes from the fact that in (159) the measure is not normalized, and (159) is computed for the measure of a free Brownian motion. The global weight for the path starting at x_i at time 0 and reaching y_i at time s is thus not equal to 1 but to $P(0,s,x_i,y_i)$. We then decompose $X(u)$ as:

$$X(u) = \left(\frac{t}{s}x + \frac{s-t}{s}y\right) + \left(B(u) - \left(\frac{u}{s}\right)B(s)\right)$$

where $B(u)$ is a free brownian motion. Then, the use of Ito formula yields:

$$\begin{aligned}\left(\int_0^s X(u)\,du\right) &= \int_0^t \left(\frac{t}{s}x + \frac{s-t}{s}y\right)dt + \left(\int_0^s \left(B(u) - \left(\frac{u}{s}\right)B(s)\right)du\right) \\ &= s\left(\frac{x+y}{2}\right) + \left(sB(s) - \int_0^s u\,dB(u)\right) - \frac{s}{2}B(s) \\ &= s\left(\frac{x+y}{2}\right) + \left(\frac{s}{2}B(s) - \int_0^s u\,dB(u)\right) \\ &= s\left(\frac{x+y}{2}\right) + \left(\int_0^s \left(\frac{s}{2}-u\right)dB(u)\right)\end{aligned}$$

and one obtains:

$$\begin{aligned}\exp\left(-\frac{1}{s}\left(\int_0^s X(u)\,du\right)^2\right) &= \exp\left(-\frac{1}{s}\left(s\left(\frac{x+y}{2}\right) + \left(\int_0^s \left(\frac{s}{2}-u\right)dB(u)\right)\right)^2\right) \\ &= \exp\left(-\frac{1}{s}\left(\begin{array}{c} s^2\left(\frac{x+y}{2}\right)^2 + 2s\left(\frac{x+y}{2}\right)\left(\int_0^s\left(\frac{s}{2}-u\right)dB(u)\right) \\ +2\int_0^s\int_0^{u_1}\left(\frac{s}{2}-u_1\right)\left(\frac{s}{2}-u_2\right)dB(u_2)\,dB(u_1) + \sigma^2\int_0^s\left(\frac{s}{2}-u\right)^2du \end{array}\right)\right) \\ &= \exp\left(-\frac{1}{s}\left(\begin{array}{c} s^2\left(\frac{x+y}{2}\right)^2 + 2s\left(\frac{x+y}{2}\right)\left(\int_0^s\left(\frac{s}{2}-u\right)dB(u)\right) \\ +2\int_0^s\int_0^{u_1}\left(\frac{s}{2}-u_1\right)\left(\frac{s}{2}-u_2\right)dB(u_2)\,dB(u_1) + \sigma^2\frac{1}{12}s^3 \end{array}\right)\right)\end{aligned}$$

so that:

$$\begin{aligned}&\left\langle \exp\left(-\frac{1}{s}\left(\int_0^s X(u)\,du\right)^2\right)\right\rangle \\ = \;& \exp\left(-\frac{1}{s}\left(\begin{array}{c} s^2\left(\frac{x+y}{2}\right)^2 + \sigma^2\frac{1}{12}s^3 + 2s\left(\frac{x+y}{2}\right)\left(\int_0^s\left(\frac{s}{2}-u\right)dB(u)\right) \\ +2\int_0^s\int_0^{u_1}\left(\frac{s}{2}-u_1\right)\left(\frac{s}{2}-u_2\right)dB(u_2)\,dB(u_1) \end{array}\right)\right) \\ &\left\langle \exp\left(-\frac{1}{s}\left(2s\left(\frac{x+y}{2}\right)\left(\int_0^s\left(\frac{s}{2}-u\right)dB(u)\right) + 2\int_0^s\int_0^{u_1}\left(\frac{s}{2}-u_1\right)\left(\frac{s}{2}-u_2\right)dB(u_2)\,dB(u_1)\right)\right)\right\rangle\end{aligned}$$

using classical methods of differential stochastic equations:

$$\begin{aligned}
&\left\langle \exp\left(-\frac{1}{s}\left(\left(\int_0^s f(u)\,dB(u)\right)+\int_0^s\int_0^{u_1} g(u_1,u_2)\,dB(u_2)\,dB(u_1)\right)\right)\right\rangle \\
=\;& \exp\left(-\frac{1}{2s^2}\int_0^s f^2(u)\,du-\frac{1}{2s^2}\int_0^s\int_0^{u_1} g^2(u_1,u_2)\,du_1du_2\right)
\end{aligned}$$

and then:

$$\begin{aligned}
&\left\langle \exp\left(-\alpha\left(\int_0^s X(u)\,du\right)^2\right)\right\rangle \\
=\;& \exp\left(-\left(\begin{array}{c}\left(s\left(\frac{x+y}{2}\right)^2+\sigma^2\frac{1}{12}s^2\right)+2\sigma^2\left(\frac{x+y}{2}\right)^2\left(\int_0^s\left(\frac{s}{2}-u\right)^2du\right)\\ +2\frac{\sigma^4}{s^2}\int_0^s\int_0^{u_1}\left(\frac{s}{2}-u_1\right)^2\left(\frac{s}{2}-u_2\right)^2du_2du_1\end{array}\right)\right)\frac{\exp\left(-\frac{(x-y)^2}{\sigma^2 s}\right)}{\sqrt{s}} \\
=\;& \exp\left(-\left(s\left(\frac{x+y}{2}\right)^2+\frac{\sigma^2}{12}s^2+\frac{\sigma^2 s^3}{6}\left(\frac{x+y}{2}\right)^2+\frac{\sigma^4 s^4}{144}\right)\right)\frac{\exp\left(-\frac{(x-y)^2}{\sigma^2 s}\right)}{\sqrt{s}}
\end{aligned}$$

In the text, we consider several approximation that yield a simplified form of the Green function. These hypothesis are justified in the text. We assume first that $\sigma<\alpha$, and $s\simeq\frac{1}{\alpha}$. Moreover the individual fluctuations $|x-y|$, which are of order $\sigma\sqrt{s}\simeq\frac{\sigma}{\sqrt{\alpha}}$, will be neglected with respect to the mean path $\frac{x+y}{2}$ over the all duration of interaction. It translates in $\frac{x+y}{2}>>|x-y|$ and since $|x-y|$ is of order $\sigma\sqrt{s}\simeq\frac{\sigma}{\sqrt{\alpha}}$, $\left(\frac{x+y}{2}\right)^2>>\sigma^2 s$ and $\left(\frac{x+y}{2}\right)^2>>\frac{\sigma^2}{\alpha}$.

Then one can rewrite some contributions:

$$\frac{\sigma^2 s^3}{6}\left(\frac{x+y}{2}\right)^2+\frac{\sigma^4}{144}s^4\simeq\left(\frac{\sigma^2}{\alpha^2}\right)\frac{\alpha}{6}s^2\left(\frac{x+y}{2}\right)^2+\frac{1}{144}\left(\frac{\sigma^4}{\alpha^4}\right)$$

and:

$$\begin{aligned}
& s\left(\frac{x+y}{2}\right)^2+\frac{\sigma^2}{12}s^2+\frac{\sigma^2 s^3}{6}\left(\frac{x+y}{2}\right)^2+\frac{\sigma^4}{144}s^4 \\
\simeq\;& s\left(\frac{x+y}{2}\right)^2+\frac{\sigma^2}{12}s^2
\end{aligned}$$

the error made in neglecting $\frac{\sigma^2 s^3}{6}\left(\frac{x+y}{2}\right)^2$ being lower than $\frac{1}{6}$. One has, in first approximation:

$$\begin{aligned}
\bar{G}(\alpha,x,y) &= \mathcal{L}\left[\left\langle\exp\left(-\frac{1}{s}\left(\int_0^s X(u)\,du\right)\left(\int_0^s X(u)\,du\right)\right)\right\rangle\frac{\exp\left(-\frac{(x-y)^2}{\sigma^2 s}\right)}{\sqrt{s}}\right] \\
&= \mathcal{L}\left[\exp\left(-\left(s\left(\frac{x+y}{2}\right)^2+\frac{\sigma^2}{12}s^2+\frac{\sigma^2 s^3}{6}\left(\frac{x+y}{2}\right)^2+\frac{\sigma^4}{144}s^4\right)\right)\frac{\exp\left(-\frac{(x-y)^2}{\sigma^2 s}\right)}{\sqrt{s}}\right] \\
&\simeq \mathcal{L}\left[\exp\left(-s\left(\frac{x+y}{2}\right)^2\right)\frac{\exp\left(-\frac{(x-y)^2}{\sigma^2 s}\right)}{\sqrt{s}}\right] \\
&= \exp\left(\left(\left(\frac{x+y}{2}\right)^2\right)\frac{\partial}{\partial\alpha}\right)\frac{\exp\left(-\sqrt{2\alpha}\left|\frac{x-y}{\sigma}\right|\right)}{\sqrt{2\alpha}} \\
&= \frac{\exp\left(-\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\left|\frac{x-y}{\sigma}\right|\right)}{\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}}
\end{aligned}$$

To find a field formalism including $\bar{G}(\alpha, x, y)$, the green function modified by the constraints one has to find a differential equation satisfied by $\bar{G}(\alpha, x, y)$, and one does so by first computing:

$$\left(\nabla^2 - 2\frac{\alpha + \left(\frac{x+y}{2}\right)^2}{\sigma^2}\right)\bar{G}(\alpha, x, y)$$

Using that:

$$\frac{\sigma^2}{2}\left(\nabla^2 - 2\frac{\alpha}{\sigma^2}\right)\frac{\exp\left(-\sqrt{2\alpha}\left|\frac{x-y}{\sigma}\right|\right)}{\left(\frac{\sqrt{2\alpha}}{\sigma}\right)} = \delta(x-y)$$

one obtains:

$$\begin{aligned}
&\left(\nabla^2 - 2\frac{\alpha + \left(\frac{x+y}{2}\right)^2}{\sigma^2}\right)\bar{G}(\alpha, x, y) \\
= \;& \delta(x-y) + (x+y)^2\frac{\partial^2}{\partial\alpha^2}\frac{\exp\left(-\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\left|\frac{x-y}{\sigma}\right|\right)}{\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}} + \frac{\partial}{\partial\alpha}\frac{\exp\left(-\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\left|\frac{x-y}{\sigma}\right|\right)}{\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}} \\
& -2\left((x+y)\frac{\partial}{\partial\alpha}\frac{\exp\left(-\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\left|\frac{x-y}{\sigma}\right|\right)}{\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}}\right)\left(\left(\frac{\partial}{\partial x}\left|\frac{x-y}{\sigma}\right|\right)\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\right) \\
= \;& \delta(x-y) + (x+y)^2\left(\frac{3}{\left(\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\right)^5} + 3\frac{\left|\frac{x-y}{\sigma}\right|}{\left(2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)\right)^2} + \frac{\left|\frac{x-y}{\sigma}\right|^2}{\left(\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\right)^3}\right) \\
& \times\exp\left(-\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\left|\frac{x-y}{\sigma}\right|\right) \\
& -\left(\left|\frac{x-y}{\sigma}\right| + \frac{1}{\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}}\right)\frac{\exp\left(-\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\left|\frac{x-y}{\sigma}\right|\right)}{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)} \\
& +2(x+y)\left(\left|\frac{x-y}{\sigma}\right| + \frac{1}{\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}}\right)\frac{\exp\left(-\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\left|\frac{x-y}{\sigma}\right|\right)}{\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}}\frac{H(x-y)-H(y-x)}{\sigma}
\end{aligned}$$

As a consequence $\bar{G}(\alpha, x, y)$ satisfies the following differential equation:

$$\begin{aligned}
\delta(x-y) \;=\; & \frac{\sigma^2}{2}\left[\left(\nabla^2 - 2\frac{\alpha+\left(\frac{x+y}{2}\right)^2}{\sigma^2}\right) - \frac{(x+y)^2}{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\left(\left(\frac{3}{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)} + \frac{3\left|\frac{x-y}{\sigma}\right|}{\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}} + \left|\frac{x-y}{\sigma}\right|^2\right)\right)\right. \\
& \left. -\frac{1+\left|\frac{x-y}{\sigma}\right|\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}}{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}\left(\left(2(x+y)\frac{H(x-y)-H(y-x)}{\sigma}\right)\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)} - 1\right)\right]\bar{G}(\alpha, x, y)
\end{aligned}$$

Then use our assumptions about the parameters to obtain:

$$\begin{aligned}
\frac{3\sigma^2}{\left(2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)\right)} &< \frac{3\sigma^2}{\alpha} << \left(\frac{x+y}{2}\right)^2 \\
\frac{3\sigma^2\left|\frac{x-y}{\sigma}\right|}{\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}} &< \frac{3\sigma\left|x-y\right|}{\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}} << \sigma < \alpha \\
\sigma^2\left|\frac{x-y}{\sigma}\right|^2 &= \left|x-y\right|^2 << \left(\frac{x+y}{2}\right)^2 \\
\sigma^2\frac{1+\left|\frac{x-y}{\sigma}\right|\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}}{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)} &< \frac{\sigma^2}{\sqrt{\alpha}\left(\frac{x+y}{2}\right)} + \frac{\sigma\left|x-y\right|}{\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}} \\
&< \; < \sigma + \sigma < 2\alpha
\end{aligned}$$

and the differential equation reduces to:

$$\delta\left(x-y\right) = \left(\frac{\sigma^2}{2}\nabla^2 - 2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)\right)\bar{G}\left(\alpha,x,y\right)$$

Appendix 11

When some discount rate is introduced, we go back to the initial individual agent formulation and modify it accordingly. Recall that the transition probabilities between two consecutive state variables of the system are defined by (89):

$$\langle B_{s+1} | | B_s \rangle = \int \prod_{i=2}^{T} dB_{s+i} \exp \left(U(C_s) + \sum_{i>0} U(C_{s+i}) \right)$$

but now, the constraint rewrites:

$$B_{s+1} = (1+r)(B_s + Y_s - C_s)$$

or equivalently:

$$C_s = B_s + Y_s - \frac{B_{s+1}}{(1+r)}$$

Then, the integral over the B_{s+i} is similar to the previous one, since one can change the variables: $\frac{B_{s+i}}{(1+r)^i} \to B_{s+i}$ for $i > 1$.

$$\begin{aligned}
&= \int \prod_{i=2}^{T} dB_{s+i} \exp \left(-(C_s - \bar{C})^2 - \sum_{i>0} (C_{s+i} - \bar{C})^2 \right) \\
&= \int \prod_{i=2}^{T} dB_{s+i} \exp \left(-\left(B_s + Y_s - \frac{B_{s+1}}{(1+r)} - \bar{C} \right)^2 - \sum_{i>0} \left(B_{s+i} + Y_{s+i} - \frac{B_{s+i+1}}{(1+r)} - \bar{C} \right)^2 \right) \\
&= \int \prod_{i=2}^{T} (1+r)^i dB'_{s+i} \exp \left(-\left(B_s + Y_s - \frac{B_{s+1}}{(1+r)} - \bar{C} \right)^2 - \sum_{i>0} (1+r)^i \left(B'_{s+i} - B'_{s+i+1} + \frac{Y_{s+i} - \bar{C}}{(1+r)^i} \right)^2 \right) \\
&= \left(\prod_{i=2}^{T} (1+r)^i \right) \exp \left(-\left(B_s + Y_s - \frac{B_{s+1}}{(1+r)} - \bar{C} \right)^2 - \frac{1}{\sum_{i>0} (1+r)^{-i}} \left(\frac{B_{s+1}}{(1+r)} + \sum_{i>0} \frac{Y_{s+i} - \bar{C}}{(1+r)^i} \right)^2 \right) \\
&= \left(\prod_{i=2}^{T} (1+r)^i \right) \exp \left(-\left(B_s + Y_s - \frac{B_{s+1}}{(1+r)} - \bar{C} \right)^2 - \frac{r}{(1+r)^T - 1} \left(\frac{B_{s+1}}{(1+r)} + \sum_{i>0} \frac{Y_{s+i} - \bar{C}}{(1+r)^i} \right)^2 \right) \qquad (308)
\end{aligned}$$

where the sum has been performed up to T where T is the time horizon defined previously.

The factor $\prod_{i=2}^{T} (1+r)^i$ can be included in the normalization factor, as explained before, and then we are left with:

$$\begin{aligned}
\langle B_{s+1} | | B_s \rangle &= \int \prod_{i=2}^{T} dB_{s+i} \exp \left(U(C_s) + \sum_{i>0} U(C_{s+i}) \right) \\
&= \exp \left(-\left(B_s + Y_s - \frac{B_{s+1}}{(1+r)} - \bar{C} \right)^2 - r \left(\frac{B_{s+1}}{(1+r)} + \sum_{i>0} \frac{Y_{s+i} - \bar{C}}{(1+r)^i} \right)^2 \right) \qquad (309)
\end{aligned}$$

which is similar to (89), except the $\frac{1}{(1+r)}$ factor in front of B_{s+1} and the $(1+r)^i$ multiplying $(Y_{s+i} - \bar{C})$. One also replaces T by $\frac{1}{r}$. Then the previous analysis following (89) applies, except that, writing B_{s+1} as a function of the past is now:

$$B_{s+1} = \sum_{i \leqslant 0} \frac{Y_{s+i}}{(1+r)^i} - \sum_{i \leqslant 0} \frac{C_{s+i}}{(1+r)^i} \qquad (310)$$

that will lead directly to the weight, after normalization:

$$\exp\left(-\left(C_s-\frac{\sigma}{\frac{1}{r}+\sigma}\bar{C}\right)^2-\frac{2r}{(1+r)^T-1+\sigma}C_s\left(\sum_{i<0}\frac{C_{s+i}}{(1+r)^i}\right)-\frac{2r}{(1+r)^T-1+\sigma}C_s\left(\sum_{i\leqslant 0}\frac{Y_{s+i}}{(1+r)^i}+\sum_{i>0}\frac{\bar{Y}}{(1+r)^i}\right)\right)$$

The global weight, over all periods is then:

$$\begin{aligned}\exp\left(-\sum_s\left(C_s-\frac{\sigma}{\frac{1}{r}+\sigma}\bar{C}\right)^2-\frac{2r}{(1+r)^T-1+r\sigma}\sum_{s_1,s_2}C_{s_1}(1+r)^{|s_2-s_1|}C_{s_2}\right. \\ \left.-\frac{2r}{(1+r)^T-1+r\sigma}\sum_{s_1,s_2}C_{s_1}\left((1+r)^{s_1-s_2}E_{s_1}(Y_{s_2})\right)\right)\end{aligned} \tag{311}$$

with $E_{s_1}Y_{s_2}=(1+r)^{s_2-s_1}Y_{s_2}$ if $s_2\leqslant s_1$ $E_{s_1}Y_{s_2}=(1+r)^{s_2-s_1}\bar{Y}$ if $s_1\leqslant s_2$.

Now, switching to an endogenous expression for Y_{s_2}, we introduce an index i to describe a set of N agents. Each of them is described by an action $C_s^{(i)}$ and has an endowment $Y_s^{(i)}=\alpha\sum_j C_s^{(j)}$. The global weight for the set of agents is then:

$$\begin{aligned}\exp\left(-\sum_s\left(C_s^{(i)}-\frac{\sigma}{\frac{1}{r}+\sigma}\bar{C}\right)^2-\frac{2r}{(1+r)^T-1+r\sigma}\sum_i\sum_{s_1,s_2}C_{s_1}^{(i)}(1+r)^{|s_2-s_1|}C_{s_2}^{(i)}\right. \\ \left.-\frac{2r}{(1+r)^T-1+r\sigma}\sum_{s_1,s_2}\sum_{i,j}C_{s_1}^{(i)}\left((1+r)^{s_2-s_1}C_{s_1}^{(j)}\right)\right)\end{aligned} \tag{312}$$

To understand the field theoretic equivalent of the two last terms in (312), one proceeds as follows. First, neglecting as before the term proportional to σ, we turn to a continuous representation:

$$\begin{aligned}\exp\left(-\int_0^s ds_1\left(C_{s_1}^{(i)}-\frac{\sigma}{\frac{1}{r}+\sigma}\bar{C}\right)^2-\frac{2r}{\exp(rT)-1}\sum_i\int_0^s ds_2\int_0^{s_2}ds_1 C_{s_1}^{(i)}\exp(r(s_2-s_1))C_{s_2}^{(i)}\right. \\ \left.-\frac{2r}{\exp(rT)-1}\int_0^s ds_1\int_0^s ds_2\sum_{i,j}C_{s_1}^{(i)}\left(\exp(r(s_2-s_1))C_{s_2}^{(j)}\right)\right)\end{aligned} \tag{313}$$

The second term of (313):

$$-\frac{2r}{\exp(rT)-1}\sum_i\int_0^s ds_2\int_0^{s_2}ds_1 C_{s_1}^{(i)}\exp(r(s_2-s_1))C_{s_2}^{(i)}$$

can be introduced in very similar way to the case $r=0$, in (313), but now, terms of the form $\exp(r(s_2-s_1))$ are inserted:

$$\begin{aligned}&\int\mathcal{D}x_i(t)\exp\left(-\sum_i\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}(t)+K(x_i(t))\,dt\right)-\frac{2r}{\exp(rT)-1}\sum_i\int_0^s ds_2\int_0^{s_2}ds_1 C_{s_1}^{(i)}\exp(r(s_2-s_1))C_{s_2}^{(i)}\right) \\ =&\left\langle\exp\left(-\frac{2r}{\exp(rT)-1}\sum_i\int_0^s ds_2\int_0^{s_2}ds_1 C_{s_1}^{(i)}\exp(r(s_2-s_1))C_{s_2}^{(i)}\right)\right\rangle\end{aligned}$$

where the Brackets denote the expectation for a Brownian path moving between x and y during a time s. As before, describing the estimated interaction duration time T by s, one is left with:

$$\begin{aligned}&\int\mathcal{D}x_i(t)\exp\left(-\sum_i\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}(t)+K(x_i(t))\,dt\right)-\frac{2r}{\exp(rT)-1}\sum_i\int_0^s ds_2\int_0^{s_2}ds_1 C_{s_1}^{(i)}\exp(r(s_2-s_1))C_{s_2}^{(i)}\right) \\ =&\left\langle\exp\left(-\frac{2r}{\exp(rs)-1}\sum_i\int_0^s ds_2\int_0^{s_2}ds_1 C_{s_1}^{(i)}\exp(r(s_2-s_1))C_{s_2}^{(i)}\right)\right\rangle\end{aligned}$$

We write:

$$X(u) = \left(\frac{u}{s}x + \frac{s-u}{s}y\right) + \left(B(u) - \left(\frac{u}{s}\right)B(s)\right)$$

where $B(u)$ is a free brownian motion. As before, to our order of approximation, in integrals of the kind:

$$\left(\int_0^s \exp(ru)\, X(u)\, du\right) = \int_0^s \exp(ru)\left(\frac{u}{s}x + \frac{s-u}{s}y\right)du + \left(\int_0^s \left(B(u) - \left(\frac{u}{s}\right)B(s)\right)du\right)$$

we can neglect contribution due to the ito integrals and approximate $X(u)$ by $\left(\frac{u}{s}x + \frac{s-u}{s}y\right)$. Then:

$$\begin{aligned}
&\sum_i \int_0^s ds_2 \int_0^{s_2} ds_1 X_{s_1} \exp(r(s_2 - s_1)) X_{s_2} \\
= &\int_0^s \left(\frac{v}{s}x + \frac{s-v}{s}y\right)\exp(rv)\left(\frac{1}{r}\left(y - \exp(-rv)\left(\frac{v}{s}x + \frac{s-v}{s}y\right)\right) + \frac{1}{sr^2}(1 - \exp(-rv))(x-y)\right)dv \\
= &\int_0^s \left(\frac{1}{r}\left(\left(\frac{v}{s}x + \frac{s-v}{s}y\right)\exp(rv)\, y - \left(\frac{v}{s}x + \frac{s-v}{s}y\right)^2\right) + \frac{\left(\frac{v}{s}x + \frac{s-v}{s}y\right)}{sr^2}(\exp(rv) - 1)(x-y)\right)dv
\end{aligned}$$

Each term in the previous integral can be computed directly:

$$\begin{aligned}
&\int_0^s \left(\frac{1}{r}\left(\left(\frac{v}{s}x + \frac{s-v}{s}y\right)\exp(rv)\, y - \left(\frac{v}{s}x + \frac{s-v}{s}y\right)^2\right)\right)dv \\
= &\frac{1}{r^2}(x\exp(rs) - y)\, y - \frac{1}{sr^3}y(x-y)(\exp(rs) - 1) - \frac{1}{3r}s\left(x^2 + xy + y^2\right)
\end{aligned}$$

$$\begin{aligned}
&\int_0^s \left(\frac{\left(\frac{v}{s}x + \frac{s-v}{s}y\right)}{sr^2}(\exp(rv) - 1)(x-y)\right)dv \\
= &\frac{1}{r^3 s^2}(x-y)\left(s\left(e^{rs}x - y\right) - \frac{1}{r}\left(e^{rs} - 1\right)(x-y)\right) - \frac{1}{2r^2}(x-y)(x+y)
\end{aligned}$$

So that one finds:

$$\begin{aligned}
&\sum_i \int_0^s ds_2 \int_0^{s_2} ds_1 X_{s_1} \exp(r(s_2 - s_1)) X_{s_2} \\
= &\frac{1}{r^2}(x\exp(rs) - y)\, y - \frac{1}{sr^3}y(x-y)(\exp(rs) - 1) - \frac{1}{3r}s\left(x^2 + xy + y^2\right) \\
&+ \frac{1}{r^3 s^2}(x-y)\left(s\left(e^{rs}x - y\right) - \frac{1}{r}\left(e^{rs} - 1\right)(x-y)\right) - \frac{1}{2r^2}(x-y)(x+y)
\end{aligned}$$

One can simplify this result for two different regimes. In the first one, the interaction duration is relatively short so that $(rs) << 1$, or, which is equivalent, $\left(\frac{r}{\alpha}\right) << 1$ since $\frac{1}{\alpha}$ is the mean duration, and in that case, in first approximation:

$$2\sum_i \int_0^s ds_2 \int_0^{s_2} ds_1 X_{s_1} \exp(r(s_2 - s_1)) X_{s_2} = \frac{1}{4}s^2(x+y)^2 + \frac{1}{15}(rs)\, s^2\left(x^2 + 3xy + y^2\right)$$

The second term appears as a correction with respect to the case with no discount rate in the constraint. Since $rs << 1$ one can approximate s by it's mean $\frac{1}{\alpha}$, then $rs \simeq \frac{r}{\alpha}$ and:

$$\begin{aligned}
2\sum_i \int_0^s ds_2 \int_0^{s_2} ds_1 X_{s_1} \exp(r(s_2 - s_1)) X_{s_2} &= s^2\left[\left(\frac{x+y}{2}\right)^2 + \frac{1}{15}\frac{r}{\alpha}\left(5\left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2\right)\right] \quad (314) \\
&= s^2\left[\left(\frac{x+y}{2}\right)^2\left(1 + \frac{r}{3\alpha}\right) - \frac{1}{15}\frac{r}{\alpha}\left(\frac{x-y}{2}\right)^2\right]
\end{aligned}$$

then including the discount rate keeps the global form of the green function, but reduces the binding tendency to set $x = y$. Due to the discount rate, the various periods are no more equivalent, which is reducing the smoothing behavior. This reduction reflects in the introduction of the term $--\frac{1}{15}\frac{r}{\alpha}\left(\frac{x-y}{2}\right)^2$. Actually, (314) implies that in the approximation $rs << 1$:

$$\begin{aligned}
&\frac{2r}{\exp(rs)-1}\sum_i \int_0^s ds_2 \int_0^{s_2} ds_1 C_{s_1}^{(i)} \exp\left(r\left(s_2 - s_1\right)\right) C_{s_2}^{(i)} \\
\simeq\ & \frac{2}{s}\sum_i \int_0^s ds_2 \int_0^{s_2} ds_1 C_{s_1}^{(i)} \exp\left(r\left(s_2 - s_1\right)\right) C_{s_2}^{(i)} \\
=\ & s\left[\left(\frac{x+y}{2}\right)^2\left(1+\frac{r}{3\alpha}\right) - \frac{1}{15}\frac{r}{\alpha}\left(\frac{x-y}{2}\right)^2\right]
\end{aligned}$$

which leads, as in the text to the following Green function:

$$\begin{aligned}
\bar{G}(\alpha,x,y) &= \mathcal{L}\left[\left\langle \exp\left(\frac{1}{s}\left(\int_0^s X(u)\,du\right)\left(\int_0^s X(u)\,du\right)\right)\right\rangle \frac{\exp\left(-\frac{(x-y)^2}{\sigma^2 s}\right)}{\sqrt{s}}\right] \\
&= \exp\left(\left[\left(\frac{x+y}{2}\right)^2\left(1+\frac{r}{3\alpha}\right) - \frac{1}{15}\frac{r}{\alpha}\left(\frac{x-y}{2}\right)^2\right]\frac{\partial}{\partial\alpha}\right)\frac{\exp\left(-\sqrt{2\alpha}\left|\frac{x-y}{\sigma}\right|\right)}{\sqrt{2\alpha}} \\
&= \frac{\exp\left(-\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\left(1+\frac{r}{3\alpha}\right)-\frac{1}{15}\frac{r}{\alpha}\left(\frac{x-y}{2}\right)^2\right)}\left|\frac{x-y}{\sigma}\right|\right)}{\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\left(1+\frac{r}{3\alpha}\right)-\frac{1}{15}\frac{r}{\alpha}\left(\frac{x-y}{2}\right)^2\right)}}
\end{aligned}$$

$\bar{G}(\alpha,x,y)$ which satisfies:

$$\begin{aligned}
\delta(x-y) &= \left[\left(\nabla^2 - \frac{\alpha+q(x,y)}{\sigma^2}\right) - \frac{1}{2(\alpha+q(x,y))}\left(\left(\frac{6q(x,y)}{(\alpha+q(x,y))} + 12\frac{\left|\frac{x-y}{\sigma}\right| q(x,y)}{\sqrt{2\left(\alpha+\left(\frac{x+y}{2}\right)^2\right)}} + 4q(x,y)\left|\frac{x-y}{\sigma}\right|^2\right)\right)\right. \\
&\quad -\frac{1+\sqrt{2(\alpha+q(x,y))}}{2(\alpha+q(x,y))} \\
&\quad \times\left.\left(\left(\left(2(x+y)\left(1+\frac{r}{3\alpha}\right) - \frac{2}{15}\frac{r}{\alpha}(x-y)\right)\frac{H(x-y)-H(y-x)}{\sigma}\right)\sqrt{2(\alpha+q(x,y))}-1\right)\right]\bar{G}(\alpha,x,y)
\end{aligned}$$

with:

$$q(x,y) = \left(\frac{x+y}{2}\right)^2\left(1+\frac{r}{3\alpha}\right) - \frac{1}{15}\frac{r}{\alpha}\left(\frac{x-y}{2}\right)^2$$

In the limit $\sigma << 1$, one finds then a quadratic term in the action:

$$\begin{aligned}
&\Psi^\dagger(x)\left[\nabla^2 - \frac{\alpha+\left(\frac{x+y}{2}\right)^2\left(1+\frac{r}{3\alpha}\right)-\frac{1}{15}\frac{r}{\alpha}\left(\frac{x-y}{2}\right)^2}{\sigma^2} - 2\left|\frac{x-y}{\sigma}\right|^2\right]\Psi(y) \qquad (315) \\
=\ &\Psi^\dagger(x)\left[\nabla^2 - \frac{\alpha+\left(\frac{x+y}{2}\right)^2\left(1+\frac{r}{3\alpha}\right)}{\sigma^2} - \left(2-\frac{1}{15}\frac{r}{\alpha}\right)\left|\frac{x-y}{\sigma}\right|^2\right]\Psi(y)
\end{aligned}$$

and the presence of $r \neq 0$ reduces, as announced, the second smoothing term $\left|\frac{x-y}{\sigma}\right|^2$ that constrains $x-y$ to oscillate around 0.

The second regime is more appropriate, since it focuses on the long run effect of a discount rate. Actually, for $rs > 1$, the interaction process is sufficiently long to allow the discount rate r to impact the dynamic system. In that case:

$$\begin{aligned}
& \frac{2r}{\exp(rs)-1}\sum_i \int_0^s ds_2 \int_0^{s_2} ds_1 C_{s_1}^{(i)} \exp\left(r\left(s_2-s_1\right)\right) C_{s_2}^{(i)} \\
\simeq & \frac{2r}{\exp(rs)}\sum_i \int_0^s ds_2 \int_0^{s_2} ds_1 C_{s_1}^{(i)} \exp\left(r\left(s_2-s_1\right)\right) C_{s_2}^{(i)} \\
\simeq & 2r\frac{\left(y-x\left(1-rs\right)\right)\left(x-y\left(1-rs\right)\right)}{r^4 s^2} \\
\simeq & \frac{2xy}{r}
\end{aligned}$$

and this modifies the Green function as:

$$\bar{G}(\alpha,x,y)=\frac{\exp\left(-\sqrt{2\alpha}\left|\frac{x-y}{\sigma}\right|-\frac{2xy}{r}\right)}{\sqrt{2\alpha}}$$

which satisfies:

$$\left(\nabla^2-\frac{\alpha}{\sigma^2}\right)\bar{G}(\alpha,x,y)=\delta(x-y)+\left(\frac{4y^2}{r^2}\bar{G}(\alpha,x,y)+4\frac{xy}{r\sqrt{2\alpha}}\left|\frac{x-y}{\sigma}\right|\right)\bar{G}(\alpha,x,y)$$

which leads to the quadratic term:

$$\Psi^{\dagger}(x)\left[\nabla^2-\frac{\alpha}{\sigma^2}-2\frac{x^2+y^2}{r^2}-4\frac{\sqrt{2\alpha}xy}{r\sigma}\left(H(x-y)-H(y-x)\right)\right]\Psi(y) \tag{316}$$

The the third term in (313) can also be written

$$\exp\left(-\frac{2\alpha}{\frac{1}{r}+\sigma}\sum_{i,j}\int_0^s ds_1 \exp\left(-rs_1\right)C_{s_1}^{(i)}\int_0^s ds_2\exp\left(rs_2\right)C_{s_2}^{(j)}\right) \tag{317}$$

We have seen previously how to introduce the field theoretic counterpart of such a product. One has to find the counterpart of each term $\exp\left(\int_0^s ds_1 \exp\left(-rs_1\right)C_{s_1}^{(i)}\right)$ and $\exp\left(\int_0^s ds_2 \exp\left(rs_2\right)C_{s_2}^{(j)}\right)$, and then to take simply the product of the field equivalent quantities. We then focus only on $\int_0^s ds_2 \exp\left(rs_2\right)C_{s_2}^{(i)}$, and compute it's expectation in the path integral to find it's field theoretic formulation.

$$\begin{aligned}
& \int \exp\left(-\alpha s_i\right)\int \mathcal{D}x_i(t)\exp\left(-\sum_i\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}(t)+K\left(x_i(t)\right)dt\right)\right)\exp\left(\int_0^s ds_1\exp\left(-rs_1\right)C_{s_1}^{(i)}\right) \\
& \left(\int_0^s ds_1 ... \int_0^{s_n} ds_{2n} C_{s_1}^{(i)}\exp\left(rs_1\right)C_{s_2}^{(i)}\exp\left(rs_2\right)...C_{s_{n-1}}^{(i)}\exp\left(rs_{n-1}\right)C_{s_n}^{(i)}\exp\left(rs_n\right)\right) \\
= & \sum_n\left\langle\int_0^s ds_1 ... \int_0^{s_n} ds_{2n} C_{s_1}^{(i)}\exp\left(rs_1\right)C_{s_2}^{(i)}\exp\left(rs_2\right)...C_{s_{n-1}}^{(i)}\exp\left(rs_{n-1}\right)C_{s_n}^{(i)}\exp\left(rs_n\right)\right\rangle
\end{aligned}$$

where the expectation $\langle A\rangle$ of any expression A is computed for the weight

$$\int \mathcal{D}x_i(t)\exp\left(-\sum_i\int_{x_i(0)=x_i}^{x_i(s)=y_i}\left(\frac{\dot{x}_i^2}{2}(t)+K\left(x_i(t)\right)dt\right)\right)$$

and the path integral leads to contributions:

$$\left\langle \int_0^s ds_1 ... \int_0^{s_n} ds_{2n} C_{s_1}^{(i)} \exp(rs_1) C_{s_2}^{(i)} \exp(rs_2) ... C_{s_{n-1}}^{(i)} \exp(rs_{n-1}) C_{s_n}^{(i)} \exp(rs_n) \right\rangle \tag{318}$$
$$= \int \int_0^s ds_1 ... \int_0^{s_{2n}} ds_{2n} P(0, s_1, x_i, X_1) X_1 \exp(rs_1) P(s_1, s_2, X_1, X_2) X_2 ... X_n \exp(rs_n)$$
$$\times P(s_{n-1}, s_n, X_{n-1}, X_{2n}) P(s_n, s, X_n, y_i) dX_1 ... dX_n$$

Writing:

$$\begin{aligned} & \exp(rs_1) ... \exp(rs_n) \\ = & \exp(nrs_1) ... \exp(2r(s_{n-1} - s_{n-2})) \exp(r(s_n - s_{n-1})) \end{aligned}$$

(318) can be transformed as:

$$\left\langle \int_0^s ds_1 ... \int_0^{s_n} ds_{2n} C_{s_1}^{(i)} \exp(rs_1) C_{s_2}^{(i)} \exp(rs_2) ... C_{s_{n-1}}^{(i)} \exp(rs_{n-1}) C_{s_n}^{(i)} \exp(rs_n) \right\rangle$$
$$= \int \int_0^s ds_1 ... \int_0^{s_{2n}} ds_{2n} P(0, s_1, x_i, X_1) X_1 \exp(nrs_1) P(s_1, s_2, X_1, X_2) X_2 ... X_n \exp(r(s_n - s_{n-1}))$$
$$\times P(s_{n-1}, s_n, X_{2n-1}, X_{2n}) P(s_n, s, X_{2n}, y_i) dX_1 ... dX_{2n}$$

whose Laplace transform is:

$$G_{+nr} * X * G_{+(n-1)r} * X * ... G_{+r} * X * G G_{+nr} * X * G_{+(n-1)r} * X * ... G_{+r} * X * G \tag{319}$$

One can find an approximation for such contributions by the following trick. Actually, write the convolution of the Green functions, without the interacting term X as:

$$G_{+nr} * G_{+(n-1)r} * ... * G_{+r} * G$$

as a product of operators:

$$\left(G^{-1} + nr\right)\left(G^{-1} + (n-1)r\right) ... \left(G^{-1} + r\right) G^{-1}$$

And this product is formally a product series

$$\begin{aligned} & \prod_{k=1}^{n} \left((rG)^{-1} + k\right) \\ = & \frac{N! \Gamma\left((rG)^{-1} + 1\right)}{r^N \Gamma\left((rG)^{-1} + N + 1\right)} \end{aligned}$$

Using asymptotic expansion for $\Gamma\left((rG)^{-1} + 1\right)$ and $\Gamma\left((rG)^{-1} + N + 1\right)$, assuming r small, yields:

$$\frac{N! \Gamma\left((rG)^{-1} + 1\right)}{r^N \Gamma\left((rG)^{-1} + N + 1\right)} = \frac{N!}{r^N} \exp\left(-\left(\left((rG)^{-1} + N + 1\right) \ln\left((rG)^{-1} + N + 1\right) - \left((rG)^{-1} + 1\right) \ln\left((rG)^{-1} + 1\right)\right)\right)$$

Factor the first term in the exponential by $(rG)^{-1} + 1$ leads to a first order expansion:

$$\begin{aligned} \left((rG)^{-1} + N + 1\right) \ln\left((rG)^{-1} + N + 1\right) = & \left((rG)^{-1} + 1\right) \ln\left((rG)^{-1} + 1\right) \\ & + N\left(\ln\left((rG)^{-1} + 1\right) + 1\right) \\ & + \frac{1}{2} \frac{N^2}{\left((rG)^{-1} + 1\right)} \end{aligned}$$

and:

$$\begin{aligned}
\frac{N!\Gamma\left(\left(rG\right)^{-1}+1\right)}{r^{N}\Gamma\left(\left(rG\right)^{-1}+N+1\right)} &= \frac{N!}{r^{N}}\exp\left(-N\left(\ln\left(\left(rG\right)^{-1}+1\right)+1\right)-\frac{1}{2}\frac{N^{2}}{\left(\left(rG\right)^{-1}+1\right)}\right) \\
&= \frac{N!}{r^{N}}\left(\left(\left(rG\right)^{-1}+1\right)\right)^{-N}\exp\left(-N\right)\exp\left(-\frac{1}{2}\frac{N^{2}}{\left(\left(rG\right)^{-1}+1\right)}\right) \\
&= N!\left(\left(G\right)^{-1}+r\right)^{-N}\exp\left(-N\right)\exp\left(-\frac{1}{2}\frac{N^{2}r}{\left(\left(G\right)^{-1}+r\right)}\right) \\
&\simeq N!\left(\left(G\right)^{N}\right)\exp\left(-N\right)\exp\left(-NrG-\frac{1}{2}\frac{N^{2}r}{\left(\left(G\right)^{-1}+r\right)}\right)
\end{aligned}$$

The terms in the series expansion becomes negligible for a value of N, denoted $\bar{N}$ that is proportional $\frac{1}{1-a_{ij}}$ and then the previous contributions are approximated by:

$$G_{+nr}*X*G_{+(n-1)r}*X*...G_{+r}*X*G$$

$$\begin{aligned}
&\exp\left(-\left(r+\frac{1}{2}\frac{\bar{N}r}{(1+rG)}\right)G\right) \\
\simeq\ &\exp\left(-\left(r+\frac{1}{2}\bar{N}r\right)G\right)
\end{aligned}$$

and then, (319): rewrites

$$\begin{aligned}
&G_{+nr}*X*G_{+(n-1)r}*X*...G_{+r}*X*G \\
\simeq\ &G*\left\{\exp\left(-\left(r+\frac{1}{2}\bar{N}r\right)G\right)*X\right\}*G*\left\{\exp\left(-\left(r+\frac{1}{2}\bar{N}r\right)G\right)*X\right\}* \\
&...G*\left\{\exp\left(-\left(r+\frac{1}{2}\bar{N}r\right)G\right)*X\right\}*G
\end{aligned} \tag{320}$$

where:

$$\left\{\exp\left(-\left(r+\frac{1}{2}\bar{N}r\right)G\right)*X\right\}=\frac{1}{2}\left(\exp\left(-\left(r+\frac{1}{2}\bar{N}r\right)G\right)X+X\exp\left(-\left(r+\frac{1}{2}\bar{N}r\right)G\right)\right)$$

This leads to an interaction potential:

$$\left\{\exp\left(-\left(r+\frac{1}{2}\bar{N}r\right)G\right)*X\right\}$$

and a field contribution:

$$\int\Psi^{\dagger}(x)\left(\exp\left(-\left(r+\frac{1}{2}\bar{N}r\right)G(x,y)\right)\frac{x+y}{2}\right)\Psi^{\dagger}(y)\,dxdy \tag{321}$$

We can come back to our problem and find the field counterpart of: (317).

Similarly to (321), the term

$$\exp\left(\int_{0}^{s}ds_{1}\exp\left(-rs_{1}\right)C_{s_{1}}^{(i)}\right)$$

induces a field counterpart:

$$\int \Psi^{\dagger}(x)\left(\exp\left(\left(r+\frac{1}{2}\bar{N}r\right)G(x,y)\right)\frac{x+y}{2}\right)\Psi^{\dagger}(y)\,dxdy \tag{322}$$

Then the expansion of

$$\exp\left(\sum_{i,j}\int_0^s ds_1\exp(-rs_1)\,C_{s_1}^{(i)}\int_0^s ds_2\exp(rs_2)\,C_{s_2}^{(j)}\right)$$

yields contributions of the form:

$$\begin{aligned}&\int\int_0^s ds_1...\int_0^{s_{2n}} ds_{2n}P(0,s_1,x_i,X_1)\,X_1\exp(rs_1)\,P(s_1,s_2,X_1,X_2)\,X_2...X_n\exp(rs_n)\\&\times P(s_{n-1},s_n,X_{n-1},X_{2n})\,P(s_n,s,X_n,y_i)\,dX_1...dX_n\\&\times\int\int_0^s ds_1'...\int_0^{s_{2n}} ds_{2n}'P(0,s_1',x_i,X_1')\,X_1'\exp(-rs_1')\,P(s_1',s_2',X_1',X_2')\,X_2'...X_n'\exp(-rs_n')\\&\times P\left(s_{n-1}',s_n',X_{n-1}',X_{2n}'\right)P(s_n',s',X_n',y_i)\,dX_1'...dX_{2n}'\end{aligned}$$

and as previously, using (321) and (322), in field theoretic formalism it leads to the potential:

$$\begin{aligned}&\left[\int\Psi^{\dagger}(x)\left(\exp\left(-\left(r+\frac{1}{2}\bar{N}r\right)G(x,y)\right)\frac{x+y}{2}\right)\Psi^{\dagger}(y)\,dxdy\right]\\&\times\left[\int\Psi^{\dagger}(w)\left(\exp\left(\left(r+\frac{1}{2}\bar{N}r\right)G(w,z)\right)\frac{w+y}{2}\right)\Psi^{\dagger}(z)\,dwdz\right]\end{aligned}$$

The Green function introduced here are similar to (162) and includes the constraint at the individual level. Ultimately, gathering this result and (316) we are left with the following action with constraint and discount rate in the case $\frac{r}{\alpha}>>1$:

$$\begin{aligned}&S\left(\left\{\Psi^{(k)}\right\}_{k=1...M}\right) \qquad (323)\\
=\;&\frac{1}{2}\sum_k\int d\hat{X}_k^{(1)}d\hat{X}_k^{(2)}\Psi^{(k)\dagger}\left(\hat{X}_k^{(2)}\right)\left[\left[\left(\nabla_{\hat{X}_k^{(1)}}\right)\left(\nabla_{\hat{X}_k^{(1)}}-M_k^{(1)}\left(\hat{X}_k^{(1)}-\left(\tilde{X}\right)_k\right)\right)+m_k^2+V\left(\hat{X}_k^{(1)}\right)\right]\delta\left(\hat{X}_k^{(1)}-\hat{X}_k^{(2)}\right)\right.\\
&\left.+\underbrace{2\frac{\left(\hat{X}_k^{(1)}\right)^2+\left(\hat{X}_k^{(2)}\right)^2}{r^2}-4\frac{\sqrt{2\alpha}\hat{X}_k^{(1)}\hat{X}_k^{(2)}}{r\sigma}\left(H\left(\hat{X}_k^{(1)}-\hat{X}_k^{(2)}\right)-H\left(\hat{X}_k^{(2)}-\hat{X}_k^{(1)}\right)\right)}_{\text{constraint, individual level}}\right]\Psi^{(k)\dagger}\left(\hat{X}_k^{(2)}\right)\\
&+\underbrace{\sum_k\sum_n V_n\left(\left\{\hat{X}_k^{(i)}\right\}_{1\leqslant i\leqslant n}\right)\prod_{1\leqslant i\leqslant n}\Psi^{(k)\dagger}\left(\hat{X}_k^{(i)}\right)\Psi^{(k)}\left(\hat{X}_k^{(i)}\right)}_{\text{intra species interaction}}\\
&+\underbrace{\sum_m\sum_{k_1...k_m}\sum_{n_1...n_m}V_{n_1...n_m}\left(\left\{\hat{X}_{k_j}^{(i_{n_j})}\right\}_{1\leqslant i_{n_j}\leqslant n_j}\right)\prod_{j=1}^m\prod_{1\leqslant i_{n_j}\leqslant n_j}\Psi^{(k_j)\dagger}\left(\hat{X}_{k_j}^{(i_{n_j})}\right)\Psi^{(k_j)}\left(\hat{X}_{k_j}^{(i_{n_j})}\right)}_{\text{inter species interaction}}\\
&+\underbrace{\begin{array}{c}\sum_{k_1,k_2}a_{k_1,k_2}\int\int\left(\Psi^{(k_1)\dagger}\left(\hat{X}_{k_1}^{(1)}\right)\left(\exp\left(-\left(r+\frac{1}{2}\bar{N}r\right)G\left(\hat{X}_{k_1}^{(1)},\hat{X}_{k_1}^{(2)}\right)\right)\frac{\hat{X}_{k_1}^{(1)}+\hat{X}_{k_1}^{(2)}}{2}\right)\Psi^{(k_1)}\left(\hat{X}_{k_1}^{(2)}\right)\right)d\hat{X}_{k_1}^{(1)}d\hat{X}_{k_1}^{(2)}\\ \times\int\int\left(\Psi^{(k_2)\dagger}\left(\hat{X}_{k_2}^{(1)}\right)\left(\exp\left(\left(r+\frac{1}{2}\bar{N}r\right)G\left(\hat{X}_{k_2}^{(1)},\hat{X}_{k_2}^{(2)}\right)\right)\frac{\hat{X}_{k_2}^{(1)}+\hat{X}_{k_2}^{(2)}}{2}\right)\Psi^{(k_2)}\left(\hat{X}_{k_2}^{(2)}\right)\right)d\hat{X}_{k_2}^{(1)}d\hat{X}_{k_2}^{(2)}\end{array}}_{\text{constraint, collective level}}\end{aligned}$$

Appendix 12

Case 1: one type of agents

As said in the text, we compute the sadlle point for:

$$\begin{aligned} S(\Psi) &= -\int \Psi^{\dagger}(x)\left[\left(-\nabla^2+\frac{\alpha}{\sigma^2}+x^2\right)\delta(x-y)+\frac{\left(\frac{x+y}{2}\right)^2}{\sigma^2}+2\left|\frac{x-y}{\sigma}\right|^2\right]\Psi(y)\,dxdy \\ & -f\int \Psi(x)\Psi^{\dagger}(x)(xy)\Psi(y)\Psi^{\dagger}(y)\,dxdy \end{aligned}$$

and show that the minimum solution is for $\Psi(x)=0$.

The saddle point equation is:

$$\left(x^2+\epsilon^2\right)\Psi(x)+\int\left[\frac{\left(\frac{x+y}{2}\right)^2}{\sigma^2}+2\left|\frac{x-y}{\sigma}\right|^2\right]\Psi(y)\,dy-2fx\Psi(x)\int y\Psi(y)\Psi^{\dagger}(y)\,dy=0$$

and this implies that the only absolute minimum is reached for $\Psi(x)=0$. Actually, for a non nul solution of this saddle point equation, the effective action rewrites:

$$\begin{aligned} S(\Psi) &= \left(-f\int \Psi(x)\Psi^{\dagger}(x)\left(x\left(y+\bar{Y}\right)\right)\Psi(y)\Psi^{\dagger}(y)\,dxdy\right. \\ &\left. -\int \Psi^{\dagger}(x)\left(2fx\Psi(x)\int\left(y+\frac{\bar{Y}}{2}\right)\Psi(y)\Psi^{\dagger}(y)\,dy+f\bar{Y}\Psi(x)\int y\Psi(y)\Psi^{\dagger}(y)\,dy\right)dx\right) \\ &= f\int \Psi^{\dagger}(x)\Psi(x)x\left(y+\bar{Y}\right)\Psi(y)\Psi^{\dagger}(y)\,dy \\ &= f\left(\int \Psi^{\dagger}(x)x\Psi(x)\,dx\right)^2 \end{aligned}$$

Since $f>0$, the last term is positive and the only minimum is for $\Psi(x)=0$.

Moreover, any solution to the saddle point equation $\Psi(x)\neq 0$ is not even a local minimum. Actually, the computation of $\frac{\partial^2 S}{\partial\Psi(x)\partial\Psi(y)}$ for this solution yields:

$$\begin{aligned} &\left(x^2+\epsilon^2\right)\delta(x-y)+\left[\frac{\left(\frac{x+y}{2}\right)^2}{\sigma^2}+2\left|\frac{x-y}{\sigma}\right|^2\right]-2fx\left[\int y\Psi(y)\Psi^{\dagger}(y)\,dy\right]\delta(x-y)-2fxy\Psi(x)\Psi^{\dagger}(y) \\ =\;& -2fxy\Psi(x)\Psi^{\dagger}(y) \end{aligned}$$

This corresponds to a local maximum since $f>0$ and :

$$\begin{aligned} &\int \varphi^{\dagger}(x)\frac{\partial^2 S}{\partial\Psi(x)\partial\Psi(y)}\varphi(y)\,dxdy \\ =\;& 2f\left(\int \varphi^{\dagger}(x)x\Psi(x)\,dx\right)^2 \end{aligned}$$

Case 2: several types of agents

We proceed in a similar way as for the case of a single type of agents. Given the action functional:

$$\begin{aligned} S\left(\left(\Psi_{\alpha}\right)\right)= & \sum_{\alpha}\left(\int dx_{\alpha}\Psi^{\dagger}\left(x_{\alpha}\right)\left(x_{\alpha}^2+\epsilon^2\right)\Psi_{\alpha}\left(x_{\alpha}\right)+\int dx_{\alpha}dy_{\alpha}\Psi^{\dagger}\left(x_{\alpha}\right)\left[\frac{\left(\frac{x_{\alpha}+y_{\alpha}}{2}\right)^2}{\sigma^2}+2\left|\frac{x_{\alpha}-y_{\alpha}}{\sigma}\right|^2\right]\Psi_{\alpha}\left(y_{\alpha}\right)\right) \\ & -\frac{1}{T}\sum_{\alpha,\beta}f_{\alpha\beta}\left[\int \Psi_{\alpha}^{\dagger}\left(x_{\alpha}\right)x_{\alpha}\Psi_{\alpha}\left(x_{\alpha}\right)\right]\left[\int \Psi_{\beta}^{\dagger}\left(x_{\beta}\right)x_{\beta}\Psi_{\beta}\left(x_{\beta}\right)\right] \end{aligned}$$

the saddle point equation becomes:

$$\begin{aligned} 0 &= \left(\left(x_{\alpha}^{2}+\epsilon^{2}\right)\Psi_{\alpha}\left(x_{\alpha}\right)+\int dy_{\alpha}\left[\frac{\left(\frac{x_{\alpha}+y_{\alpha}}{2}\right)^{2}}{\sigma^{2}}+2\left|\frac{x_{\alpha}-y_{\alpha}}{\sigma}\right|^{2}\right]\Psi_{\alpha}\left(y_{\alpha}\right)\right) \\ & -\frac{1}{T}\sum_{\beta,\ \alpha\neq\beta}\left(f_{\alpha\beta}+f_{\beta\alpha}\right)x_{\alpha}\Psi_{\alpha}\left(x_{\alpha}\right)\left[\int\Psi_{\beta}^{\dagger}\left(x_{\beta}\right)x_{\beta}\Psi_{\beta}\left(x_{\beta}\right)\right] \\ & -\frac{2}{T}f_{\alpha\alpha}x_{\alpha}\Psi_{\alpha}\left(x_{\alpha}\right)\left[\int\Psi_{\alpha}^{\dagger}\left(x_{\alpha}\right)x_{\alpha}\Psi_{\alpha}\left(x_{\alpha}\right)\right] \end{aligned}$$

and the action for this solution is:

$$\begin{aligned} S\left(\left(\Psi_{\alpha}\right)\right) &= \sum_{\alpha}\left(\int dx_{\alpha}\Psi^{\dagger}\left(x_{\alpha}\right)\left(x_{\alpha}^{2}+\epsilon^{2}\right)\Psi_{\alpha}\left(x_{\alpha}\right)+\int dx_{\alpha}dy_{\alpha}\Psi^{\dagger}\left(x_{\alpha}\right)\left[\frac{\left(\frac{x_{\alpha}+y_{\alpha}}{2}\right)^{2}}{\sigma^{2}}+2\left|\frac{x_{\alpha}-y_{\alpha}}{\sigma}\right|^{2}\right]\Psi_{\alpha}\left(y_{\alpha}\right)\right) \\ & -\frac{1}{T}\sum_{\alpha,\beta}f_{\alpha\beta}\left[\int\Psi_{\alpha}^{\dagger}\left(x_{\alpha}\right)x_{\alpha}\Psi_{\alpha}\left(x_{\alpha}\right)\right]\left[\int\Psi_{\beta}^{\dagger}\left(x_{\beta}\right)x_{\beta}\Psi_{\beta}\left(x_{\beta}\right)\right] \\ &= \frac{1}{T}\sum_{\beta,\ \alpha\neq\beta}\left(f_{\alpha\beta}+f_{\beta\alpha}\right)\Psi_{\alpha}^{\dagger}\left(x_{\alpha}\right)x_{\alpha}\Psi_{\alpha}\left(x_{\alpha}\right)\left[\int\Psi_{\beta}^{\dagger}\left(x_{\beta}\right)x_{\beta}\Psi_{\beta}\left(x_{\beta}\right)\right] \\ & +\frac{2}{T}f_{\alpha\alpha}x_{\alpha}\Psi_{\alpha}\left(x_{\alpha}\right)\left[\int\Psi_{\alpha}^{\dagger}\left(x_{\alpha}\right)x_{\alpha}\Psi_{\alpha}\left(x_{\alpha}\right)\right] \\ & -\frac{1}{T}\sum_{\alpha,\beta}f_{\alpha\beta}\left[\int\Psi_{\alpha}^{\dagger}\left(x_{\alpha}\right)x_{\alpha}\Psi_{\alpha}\left(x_{\alpha}\right)\right]\left[\int\Psi_{\beta}^{\dagger}\left(x_{\beta}\right)x_{\beta}\Psi_{\beta}\left(x_{\beta}\right)\right] \end{aligned}$$

which simplifies as:

$$S\left(\left(\Psi_{\alpha}\right)\right)=\frac{1}{T}\sum_{\alpha,\beta}f_{\alpha\beta}\left[\int\Psi_{\alpha}^{\dagger}\left(x_{\alpha}\right)x_{\alpha}\Psi_{\alpha}\left(x_{\alpha}\right)\right]\left[\int\Psi_{\beta}^{\dagger}\left(x_{\beta}\right)x_{\beta}\Psi_{\beta}\left(x_{\beta}\right)\right]$$

confirming that $\Psi_{\alpha}\left(x_{\alpha}\right)=0$ is the absolute minimum. The reason of this vacuum at $\Psi_{\alpha}\left(x_{\alpha}\right)=0$ is the direct consequence of the constraint that induces the terms:

$$-\frac{1}{T}\sum_{\alpha,\beta}f_{\alpha\beta}\left[\int\Psi_{\alpha}^{\dagger}\left(x_{\alpha}\right)x_{\alpha}\Psi_{\alpha}\left(x_{\alpha}\right)\right]\left[\int\Psi_{\beta}^{\dagger}\left(x_{\beta}\right)x_{\beta}\Psi_{\beta}\left(x_{\beta}\right)\right]$$

in the effective action. The minus sign is crucial for preventing any phase transition. Thus the constraints smoothes the interaction between agents. It prevents from switching from a symetric (nul) equilibrium to an asymetric one favouring somes groups of agents.Assume that there is a solution $\Psi\left(x\right)\neq 0$ for this equation.

As before one can check that any other solution of the saddle point equation is not a minimum by studying the stability of this solution. One computes the second order matrix elements:

$$\begin{aligned} & \frac{\partial^{2}S}{\partial\Psi_{\alpha}\left(x_{\alpha}\right)\partial\Psi_{\alpha}\left(y_{\alpha}\right)} \\ &= \left(x_{\alpha}^{2}+\epsilon^{2}-x_{\alpha}\left(\frac{1}{T}\sum_{\beta,\ \alpha\neq\beta}\left(f_{\alpha\beta}+f_{\beta\alpha}\right)\left[\int\Psi_{\beta}^{\dagger}\left(x_{\beta}\right)x_{\beta}\Psi_{\beta}\left(x_{\beta}\right)\right]-\frac{2}{T}f_{\alpha\alpha}\left[\int\Psi_{\alpha}^{\dagger}\left(x_{\alpha}\right)x_{\alpha}\Psi_{\alpha}\left(x_{\alpha}\right)\right]\right)\right)\delta\left(x_{\alpha}-y_{\alpha}\right) \\ & +\left[\frac{\left(\frac{x_{\alpha}+y_{\alpha}}{2}\right)^{2}}{\sigma^{2}}+2\left|\frac{x_{\alpha}-y_{\alpha}}{\sigma}\right|^{2}\right]-\frac{2}{T}f_{\alpha\alpha}x_{\alpha}\Psi_{\alpha}\left(x_{\alpha}\right)\Psi_{\alpha}^{\dagger}\left(y_{\alpha}\right)y_{\alpha} \\ &= -\frac{2}{T}f_{\alpha\alpha}x_{\alpha}\Psi_{\alpha}\left(x_{\alpha}\right)\Psi_{\alpha}^{\dagger}\left(y_{\alpha}\right)y_{\alpha} \end{aligned}$$

and:

$$\frac{\partial^{2}S}{\partial\Psi_{\alpha}\left(x_{\alpha}\right)\partial\Psi_{\beta}\left(y_{\beta}\right)}=-\frac{1}{T}\left(f_{\alpha\beta}+f_{\beta\alpha}\right)x_{\alpha}\Psi_{\alpha}\left(x_{\alpha}\right)\Psi_{\beta}^{\dagger}\left(x_{\beta}\right)x_{\beta}$$

So that the second order variation, for an arbitrary $\varphi_\alpha(x_\alpha)$ becomes:

$$\begin{aligned} & -\frac{1}{T}\sum_{\alpha,\beta}\int \varphi_\alpha(x_\alpha)\frac{\partial^2 S}{\partial\Psi_\alpha(x_\alpha)\partial\Psi_\beta(y_\beta)}\varphi_\beta(y_\beta)\,dx_\alpha dy_\beta \\ = & -\frac{1}{T}\sum_{\alpha,\beta}(f_{\alpha\beta}+f_{\beta\alpha})\left(\int \varphi_\alpha(x_\alpha)\,x_\alpha\Psi_\alpha^\dagger(x_\alpha)\,dx_\alpha\right)\left(\int \varphi_\beta(y_\beta)\,x_\beta\Psi_\beta^\dagger(x_\beta)\,dx_\beta\right)\end{aligned}$$

which is negative if we choose the perturbation in a single direction $\varphi_\alpha(x_\alpha)$.

Case 3: endogenous interest rates

We start with the following function

$$U^{eff}(C_i)=\int C_i^2(t)\,dt+2\int_{t>s}\exp\left(-\int_s^t r(v)\,dv\right)C_i(s)\,C_i(t)\,dsdt-2\int_{t>s}C_i(t)\exp\left(-\int_s^t r(v)\,dv\right)Y_i(s)\,dsdt \tag{324}$$

and transform the last two terms. One first obtains:

$$\begin{aligned} & 2\int_{t>s}\exp\left(-\int_s^t r(v)\,dv\right)C_i(s)\,C_i(t)\,dsdt-2\int_{t>s}C_i(t)\exp\left(-\int_s^t r(v)\,dv\right)Y_i(s)\,dsdt \quad (325)\\ = & \; 2\int_{t>s}\left(rK_i(s)-\dot K_i(s)+F_i(K_i(s))\right)\exp\left(-\int_s^t r(v)\right)\left(rK_i(t)-\dot K_i(t)+F_i(K_i(t))\right)dsdt \\ & -2\int_{t>s}F_i(K_i(s))\exp\left(-\int_s^t r(v)\,dv\right)\left(rK_i(t)-\dot K_i(t)+F_i(K_i(t))\right)dsdt \\ = & \; 2\int_{t>s}\left(rK_i(s)-\dot K_i(s)\right)\exp\left(-\int_s^t r(v)\right)F_i(K_i(t))\,dsdt \\ & +2\int_{t>s}\left(rK_i(s)-\dot K_i(s)\right)\exp\left(-\int_s^t r(v)\right)\left(rK_i(t)-\dot K_i(t)\right)dsdt\end{aligned}$$

We compute separately these two expressions. The last term can be decomposed as:

$$\begin{aligned} & \int_{t>s}\left(rK_i(s)-\dot K_i(s)\right)\exp\left(-\int_s^t r(v)\right)\left(rK_i(t)-\dot K_i(t)\right)dsdt \quad (326)\\ = & \int_{t>s}\dot K_i(s)\exp\left(-\int_s^t r(v)\right)\dot K_i(t)\,dsdt-\int_{t>s}r(s)\,K_i(s)\exp\left(-\int_s^t r(v)\right)\dot K_i(t)\,dsdt \\ & -\int_{t>s}\dot K_i(s)\exp\left(-\int_s^t r(v)\right)rK_i(t)\,dsdt \\ & +\int_{t>s}r(s)\,K_i(s)\exp\left(-\int_s^t r(v)\right)r(t)\,K_i(t)\,dsdt\end{aligned}$$

The first term in (326) is:

$$\begin{aligned} & \int_{t>s}\dot K_i(s)\exp\left(-\int_s^t r(v)\right)\dot K_i(t)\,dsdt \\ = & \int K_i(t)\,\dot K_i(t)\,dt-\int_{t>s}r(s)\,K_i(s)\exp\left(-\int_s^t r(v)\right)\dot K_i(t)\,dsdt \\ = & \frac{1}{2}\left[K_i^2(t)\right]_0^T-\int_{t>s}r(s)\,K_i(s)\exp\left(-\int_s^t r(v)\right)\dot K_i(t)\,dsdt\end{aligned}$$

The border terms that can be neglected (no accumulation at 0 and T), so that to the first order in r (326) simplifies:

$$\begin{aligned}
& \int_{t>s} \left(r(s) K_i(s) - \dot{K}_i(s)\right) \exp\left(-\int_s^t r(v)\right) \left(r(t) K_i(t) - \dot{K}_i(t)\right) dsdt \\
= & -\int_{t>s} \dot{K}_i(s) r(t) K_i(t) dsdt \\
& -2\int_{t>s} r(s) K_i(s) \dot{K}_i(t) dsdt \\
= & -\int r(t) K_i^2(t) dt + 2\int r(s) K_i^2(s) ds \\
= & \int r(t) K_i^2(t) dt
\end{aligned}$$

by assuming again that $K_i(0) = K_i(T) = 0$. The first term in (325) can also be simplified at the first order in r:

$$\begin{aligned}
& \int_{t>s} \left(r(s) K_i(s) - \dot{K}_i(s)\right) \exp\left(-\int_s^t r(v)\right) F_i(K_i(t)) dsdt \\
= & -\int_{t>s} \left(r(s) K_i(s) + \dot{K}_i(s)\right) \exp\left(-\int_s^t r(v)\right) F_i(K_i(t)) dsdt \\
& +2\int_{t>s} r(s) K_i(s) \exp\left(-\int_s^t r(v)\right) F_i(K_i(t)) dsdt \\
= & -\int_{t>s} \frac{d}{ds}\left(K_i(s) \exp\left(-\int_s^t r(v)\right)\right) F_i(K_i(t)) dsdt \\
& +2\int_{t>s} r(s) K_i(s) \exp\left(-\int_s^t r(v)\right) F_i(K_i(t)) dsdt \\
= & -\int K_i(t) F_i(K_i(t)) dt + 2\int_{t>s} r(s) K_i(s) F_i(K_i(t)) dsdt
\end{aligned}$$

Then, using (324) and (325) $U^{eff}(C_i)$ can be written:

$$U^{eff}(C_i) = \int C_i^2(t) dt - 2\int F_i(K_i(t)) K_i(t) dt + 2\int r(t) K_i^2(t) dt + 4\int_{t>s} r(s) K_i(s) F_i(K_i(t)) dsdt$$

Case 3: Saddle points and stability: general form of the second order variation

We start by writing the second order variation in a convenient way. A straightforward computation yields:

$$\begin{aligned}
\frac{1}{2}\delta^2 S(\Psi) = & \varphi^\dagger(x)\left[\left(-\nabla^2 + \left(F^2(x) - 2F(x)x\right)\right)\right]\varphi(x) + \frac{4\eta}{N}\int\left(\varphi^\dagger(x) F'(x) x\varphi(x) dx\right)\int \Psi_1^\dagger(y) F(y)\Psi_1(y) dy \\
& +\frac{4\eta}{N}\int\left(\Psi_1^\dagger(x) F'(x) x\Psi_1(x) dx\right)\int\varphi^\dagger(y) F(y)\varphi(y) dy \\
& +\frac{8\eta}{N}\operatorname{Re}\left(\int\varphi^\dagger(x) F'(x) x\Psi_1(x) dx\right)\operatorname{Re}\left(\int\varphi^\dagger(y) F(y)\Psi_1(y) dy\right)
\end{aligned} \tag{327}$$

and the last term can be decomposed in a useful way for the sequel. Assume that $F(x) - F'(x)x > 0$ (if the reverse is true the role of $F'(x)x$ and $F(x)$ are exchanged), then

$$\begin{aligned}
& \frac{8\eta}{N}\operatorname{Re}\left(\int \varphi^{\dagger}(x) F'(x) x \Psi_1(x) dx\right) \operatorname{Re}\left(\int \varphi^{\dagger}(x) F(x) \Psi_1(x) dx\right) \\
= & \frac{8\eta}{N}\operatorname{Re}\left(\int \varphi^{\dagger}(x) F'(x) x \Psi_1(x) dx\right) \operatorname{Re}\left(\int \varphi^{\dagger}(x) F'(x) x \Psi_1(x) dx\right) \\
& + \frac{8\eta}{N}\operatorname{Re}\left(\int \varphi^{\dagger}(x) F'(x) x \Psi_1(x) dx\right) \operatorname{Re}\left(\int \varphi^{\dagger}(y) \left(F(x) - F'(x)x\right) \Psi_1(x) dx\right)
\end{aligned}$$

The last expression can be estimated by the Cauchy Schwarz inequality as:

$$\begin{aligned}
& \left|\operatorname{Re}\left(\int \varphi^{\dagger}(x) F'(x) x \Psi_1(x) dx\right) \operatorname{Re}\left(\int \varphi^{\dagger}(y) \left(F(x) - F'(x)x\right) \Psi_1(x) dx\right)\right| \\
< & \sqrt{\int \left(\varphi^{\dagger}(x) F'(x) x \varphi(x) dx\right)} \sqrt{\int \Psi_1^{\dagger}(x) F'(x) x \Psi_1(x) dx} \\
& \times \sqrt{\int \left(\Psi_1^{\dagger}(x) \left(F(x) - F'(x)x\right) \Psi_1(x) dx\right)} \sqrt{\int \varphi^{\dagger}(x) \left(F(x) - F'(x)x\right) \varphi(x) dx}
\end{aligned}$$

Letting then

$$\begin{aligned}
A &= \int \left(\Psi_1^{\dagger}(x) F'(x) x \Psi_1(x) dx\right) \\
B &= \int \Psi_1^{\dagger}(y) F(y) \Psi_1(y) dy
\end{aligned}$$

the three last terms in (327) can be regrouped and estimated in the following way:

$$\begin{aligned}
&\frac{4\eta}{N}\int\left(\varphi^{\dagger}(x)F'(x)x\varphi(x)dx\right)\int\Psi_1^{\dagger}(x)F(x)\Psi_1(x)dx+\frac{4\eta}{N}\int\left(\Psi_1^{\dagger}(x)F'(x)x\Psi_1(x)dx\right)\int\left(\varphi^{\dagger}(x)F(x)\varphi(x)dx\right)\\
&+\frac{8\eta}{N}\operatorname{Re}\left(\int\varphi^{\dagger}(x)F'(x)x\Psi_1(x)dx\int\varphi^{\dagger}(x)F(x)\Psi_1(x)dx\right)\\
=\;&\frac{4\eta}{N}\int\left(\varphi^{\dagger}(x)F'(x)x\varphi(x)dx\right)\int\Psi_1^{\dagger}(x)F(x)\Psi_1(x)dx+\frac{4\eta}{N}\int\left(\Psi_1^{\dagger}(x)F'(x)x\Psi_1(x)dx\right)\int\left(\varphi^{\dagger}(x)F(x)\varphi(x)dx\right)\\
&+\frac{8\eta}{N}\operatorname{Re}\left(\int\varphi^{\dagger}(x)F'(x)x\Psi_1(x)dx\right)\operatorname{Re}\left(\int\varphi^{\dagger}(x)F'(x)x\Psi_1(x)dx\right)\\
&+\operatorname{Re}\left(\int\varphi^{\dagger}(x)F'(x)x\Psi_1(x)dx\right)\operatorname{Re}\left(\int\varphi^{\dagger}(y)\left(F(x)-F'(x)x\right)\Psi_1(x)dx\right)\\
>\;&\frac{4\eta}{N}\int\left(\varphi^{\dagger}(x)F'(x)x\varphi(x)dx\right)\int\Psi_1^{\dagger}(x)F(x)\Psi_1(x)dx+\frac{4\eta}{N}\int\left(\Psi_1^{\dagger}(x)F'(x)x\Psi_1(x)dx\right)\int\left(\varphi^{\dagger}(x)F(x)\varphi(x)dx\right)\\
&+\operatorname{Re}\left(\int\varphi^{\dagger}(x)F'(x)x\Psi_1(x)dx\right)\operatorname{Re}\left(\int\varphi^{\dagger}(y)\left(F(x)-F'(x)x\right)\Psi_1(x)dx\right)\\
>\;&\frac{4\eta}{N}\int\left(\varphi^{\dagger}(x)F'(x)x\varphi(x)dx\right)\int\Psi_1^{\dagger}(x)F(x)\Psi_1(x)dx+\frac{4\eta}{N}\int\left(\Psi_1^{\dagger}(x)F'(x)x\Psi_1(x)dx\right)\int\left(\varphi^{\dagger}(x)F(x)\varphi(x)dx\right)\\
&-\frac{8\eta}{N}\sqrt{\int\left(\varphi^{\dagger}(x)F'(x)x\varphi(x)dx\right)}\sqrt{\int\Psi_1^{\dagger}(x)F'(x)x\Psi_1(x)dx}\\
&\times\sqrt{\int\left(\Psi_1^{\dagger}(x)\left(F(x)-F'(x)x\right)\Psi_1(x)dx\right)}\sqrt{\int\varphi^{\dagger}(x)\left(F(x)-F'(x)x\right)\varphi(x)dx}\\
=\;&\frac{8\eta}{N}\int\left(\varphi^{\dagger}(x)F'(x)x\varphi(x)dx\right)\int\left(\Psi_1^{\dagger}(x)F'(x)x\Psi_1(x)dx\right)\\
&+\frac{4\eta}{N}\int\left(\varphi^{\dagger}(x)F'(x)x\varphi(x)dx\right)\int\Psi_1^{\dagger}(x)\left(F(x)-F'(x)x\right)\Psi_1(x)dx\\
&+\frac{4\eta}{N}\int\left(\Psi_1^{\dagger}(x)F'(x)x\Psi_1(x)dx\right)\int\left(\varphi^{\dagger}(x)\left(F(x)-F'(x)x\right)\varphi(x)dx\right)\\
&-\frac{8\eta}{N}\sqrt{\int\left(\varphi^{\dagger}(x)F'(x)x\varphi(x)dx\right)}\sqrt{\int\Psi_1^{\dagger}(x)F'(x)x\Psi_1(x)dx}\\
&\times\sqrt{\int\left(\Psi_1^{\dagger}(x)\left(F(x)-F'(x)x\right)\Psi_1(x)dx\right)}\sqrt{\int\varphi^{\dagger}(x)\left(F(x)-F'(x)x\right)\varphi(x)dx}\\
=\;&\frac{8\eta}{N}\int\left(\varphi^{\dagger}(x)F'(x)x\varphi(x)dx\right)\int\left(\Psi_1^{\dagger}(x)F'(x)x\Psi_1(x)dx\right)\\
&+\frac{4\eta}{N}\left(\sqrt{\int\left(\varphi^{\dagger}(x)F'(x)x\varphi(x)dx\right)}\sqrt{\int\left(\Psi_1^{\dagger}(x)\left(F(x)-F'(x)x\right)\Psi_1(x)dx\right)}\right.\\
&\left.-\sqrt{\int\varphi^{\dagger}(x)\left(F(x)-F'(x)x\right)\varphi(x)dx}\sqrt{\int\Psi_1^{\dagger}(x)F'(x)x\Psi_1(x)dx}\right)^2\\
>\;&\frac{8\eta}{N}\int\left(\varphi^{\dagger}(x)F'(x)x\varphi(x)dx\right)\int\left(\Psi_1^{\dagger}(x)F'(x)x\Psi_1(x)dx\right)
\end{aligned}$$

and inserting this in (327) yields:

$$\begin{aligned}\frac{1}{2}\delta^2 S(\Psi) &> \varphi^{\dagger}(x)\left[\left(-\nabla^2+\left(F^2(x)-2F(x)x\right)\right]\varphi(x)\right. \\ &\quad +\frac{8\eta}{N}\int\left(\varphi^{\dagger}(x)F'(x)x\varphi(x)dx\right)\int\left(\Psi_1^{\dagger}(x)F'(x)x\Psi_1(x)dx\right) \\ &= \varphi^{\dagger}(x)\left[\left(-\nabla^2+\left(F^2(x)-2F(x)x\right)\right)+\frac{8\eta}{N}AF'(x)x\right]\varphi(x)\end{aligned} \tag{328}$$

If $F'(x)x-F(x)>0$ we have rather:

$$\begin{aligned}\frac{1}{2}\delta^2 S(\Psi) &> \varphi^{\dagger}(x)\left[\left(-\nabla^2+\left(F^2(x)-2F(x)x\right)\right]\varphi(x)\right. \\ &\quad +\frac{8\eta}{N}\int\left(\varphi^{\dagger}(x)F(x)\varphi(x)dx\right)\int\left(\Psi_1^{\dagger}(x)F(x)\Psi_1(x)dx\right) \\ &= \varphi^{\dagger}(x)\left[\left(-\nabla^2+\left(F^2(x)-2F(x)x\right)\right)+\frac{8\eta}{N}BF(x)\right]\varphi(x)\end{aligned} \tag{329}$$

This can be positive depending on the parameters of the model. The sign of $\delta^2 S(\Psi)$ will be studied for each case in the next paragraphs of this section.

Case 3: Saddle points and stability: example of scale economy

Now, to understand further the non trivial vacuum in the model of this paragraph, we will assume some particular forms for $F(x)$. We start with the action of the first case where

$$F(x)=c(x-f(x))$$

and the action given by (172):

$$S(\Psi)=\int\Psi^{\dagger}(x)\left[\left(-\nabla^2-4f(x)(x-f(x))\right)\right]\Psi(x)dx+\frac{16}{N}\left(\int\Psi^{\dagger}(x)x\Psi(x)dx\right)\left(\int\Psi^{\dagger}(x)(x-f(x))\Psi(x)dx\right)$$

The saddle point equation is:

$$0=\left(-\nabla^2-4f(x)(x-f(x))\right)\Psi(x)+\frac{16}{N}\left(A(x-f(x))+Bx\right)\Psi(x)$$

with:

$$\begin{aligned}\int_{\mathbb{R}^+}\Psi^{\dagger}(x)x\Psi(x)dx &= A \\ \int_{\mathbb{R}^+}\Psi^{\dagger}(x)(x-f(x))\Psi(x)dx &= B\end{aligned}$$

and this equation can be reorganized:

$$0=-\nabla^2\Psi(x)+\left(\left(\frac{16A}{N}-4f(x)\right)(x-f(x))+\frac{16B}{N}x\right)\Psi(x)$$

For $\frac{16A}{N}-4f(x)+\frac{16B}{N}>0$, a square integrable solution on $\mathbb{R}^+$ exists. Given that $f(x)$ is slowly varying, a first approximation for the saddle point equation is:

$$0=\left(-\nabla^2-4f(x)(x-f(x))\right)\Psi(x)+\frac{16}{N}\left(A(x-f(x))+Bx\right)\Psi(x) \tag{330}$$

Now, we factor $\Psi(x) = a\Psi_1(x)$ with $\Psi_1(x)$ of norm 1. Then (330) becomes:

$$\begin{aligned}
0 &= \left(\left(-\nabla^2 - 4f(x)(x - f(x))\right) + \frac{16a^2}{N}\left(A(x - f(x)) + Bx\right)\right)\Psi_1(x) \\
&= \left(\left(-\nabla^2 + \left(\frac{16a^2}{N}(A+B) - 4f(x)\right)\left(x - \frac{\left(\frac{16Aa^2}{N} - 4f(x)\right)f(x)}{\left(\frac{16Aa^2}{N} - 4f(x) + \frac{16a^2B}{N}\right)}\right)\right)\right)\Psi_1(x)
\end{aligned}$$

where the constants A and B have been redefined as:

$$\begin{aligned}
\int_{\mathbb{R}^+} \Psi_1^\dagger(x)\, x\Psi_1(x)\, dx &= A \\
\int_{\mathbb{R}^+} \Psi_1^\dagger(x)(x - f(x))\Psi_1(x)\, dx &= B
\end{aligned} \tag{331}$$

The solution $\Psi_1(x)$ is proportionnal to an Airy function:

$$\Psi_1(x) = \alpha Ai\left(\sqrt[3]{\frac{16Aa^2}{N} - 4f(x) + \frac{16Ba^2}{N}}\left(x - \frac{\left(\frac{16Aa^2}{N} - 4f(x)\right)f(x)}{\left(\frac{16Aa^2}{N} - 4f(x) + \frac{16a^2B}{N}\right)}\right)\right)$$

with the following normalization condition:

$$\int_{\mathbb{R}^+} \alpha^2\left(Ai\left(\sqrt[3]{\frac{16Aa^2}{N} - 4f(x) + \frac{16Ba^2}{N}}\left(x - \frac{\left(\frac{16Aa^2}{N} - 4f(x)\right)f(x)}{\left(\frac{16Aa^2}{N} - 4f(x) + \frac{16a^2B}{N}\right)}\right)\right)\right)^2 dx = 1$$

To have a minimum, one needs to show that the action $S(\Psi)$ is bounded from below. Note that given the saddle point equation:

$$\begin{aligned}
S(\Psi) &= \int \Psi^\dagger(x)\left[\left(-\nabla^2 - 4f(x)(x - f(x))\right)\right]\Psi(x)\, dx + \frac{16}{N}\left(\int \Psi^\dagger(x)\, x\Psi(x)\, dx\right)\left(\int \Psi^\dagger(x)(x - f(x))\Psi(x)\, dx\right) \\
&= -\frac{16a^4}{N}\int \Psi_1^\dagger(x)\left(A(x - f(x)) + Bx\right)\Psi_1(x) + \frac{16a^2}{N}\left(\int \Psi_1^\dagger(x)\, x\Psi_1(x)\, dx\right)\left(\int \Psi_1^\dagger(x)(x - f(x))\Psi_1(x)\, dx\right) \\
&= -\frac{32}{N}ABa^4 + \frac{16}{N}ABa^4 = -\frac{16}{N}ABa^4 < 0
\end{aligned}$$

One thus has to show that Aa^2 and are Ba^2 bounded.

To do so, one uses the normalization equation and the defining equations for A and B rewritten as:

$$\begin{aligned}
\int_{\mathbb{R}^+} \alpha^2\left(Ai\left(\sqrt[3]{\frac{16Aa^2}{N} - 4f(x) + \frac{16Ba^2}{N}}\left(x - \frac{\left(\frac{16Aa^2}{N} - 4f(x)\right)f(x)}{\left(\frac{16Aa^2}{N} - 4f(x) + \frac{16a^2B}{N}\right)}\right)\right)\right)^2 dx &= 1 \\
\alpha^2\int_{\mathbb{R}^+} x\left(Ai\left(\sqrt[3]{\frac{16Aa^2}{N} - 4f(x) + \frac{16Ba^2}{N}}\left(x - \frac{\left(\frac{16Aa^2}{N} - 4f(x)\right)f(x)}{\left(\frac{16Aa^2}{N} - 4f(x) + \frac{16Ba^2}{N}\right)}\right)\right)\right)^2 dx &= A \\
\alpha^2\int_{\mathbb{R}^+} (x - f(x))\left(Ai\left(\sqrt[3]{\frac{16Aa^2}{N} - 4f(x) + \frac{16Ba^2}{N}}\left(x - \frac{\left(\frac{16Aa^2}{N} - 4f(x)\right)f(x)}{\left(\frac{16Aa^2}{N} - 4f(x) + \frac{16Ba^2}{N}\right)}\right)\right)\right)^2 dx &= B
\end{aligned}$$

By a change of variable

$$u = \sqrt[3]{\frac{16Aa^2}{N} - 4f(x) + \frac{16Ba^2}{N}}\left(x - \frac{\left(\frac{16Aa^2}{N} - 4f(x)\right)f(x)}{\left(\frac{16Aa^2}{N} - 4f(x) + \frac{16a^2B}{N}\right)}\right)$$

and using that $f(x)$ is slowly varying, and that $f(0)=0$, so that

$$x-\frac{\left(\frac{16Aa^2}{N}-4f(x)\right)f(x)}{\left(\frac{16Aa^2}{N}-4f(x)+\frac{16a^2B}{N}\right)}$$

is increasing from 0 to $+\infty$, one gets, in first approximation ($f(x)$ is considered as constant and can be replaced by its mean $\bar{f}$):

$$\begin{aligned}
\int_{\mathbb{R}^+}\alpha^2(Ai(u))^2\frac{du}{\sqrt[3]{\frac{16Aa^2}{N}-4f(x)+\frac{16Ba^2}{N}}} &= 1\\
\alpha^2\int_{\mathbb{R}^+}\left(\frac{u}{\sqrt[3]{\frac{16Aa^2}{N}-4f(x)+\frac{16Ba^2}{N}}}+\frac{\left(\frac{16Aa^2}{N}-4f(x)\right)f(x)}{\left(\frac{16Aa^2}{N}-4f(x)+\frac{16Ba^2}{N}\right)}\right)(Ai(u))^2\frac{du}{\sqrt[3]{\frac{16Aa^2}{N}-4f(x)+\frac{16Ba^2}{N}}} &= A\\
\alpha^2\int_{\mathbb{R}^+}\left(\frac{u}{\sqrt[3]{\frac{16Aa^2}{N}-4f(x)+\frac{16Ba^2}{N}}}+\frac{\left(\frac{16Aa^2}{N}-4f(x)\right)f(x)}{\left(\frac{16Aa^2}{N}-4f(x)+\frac{16Ba^2}{N}\right)}-f(x)\right)(Ai(u))^2\frac{du}{\sqrt[3]{\frac{16Aa^2}{N}-4f(x)+\frac{16Ba^2}{N}}} &= B
\end{aligned}$$

As a consequence α^2 is of order $\sqrt[3]{\frac{16Aa^2}{N}-4f(x)+\frac{16Ba^2}{N}}$, and then:

$$\begin{aligned}
\int_{\mathbb{R}^+}\left(\frac{u}{\sqrt[3]{\frac{16Aa^2}{N}-4f(x)+\frac{16Ba^2}{N}}}+\frac{\left(\frac{16Aa^2}{N}-4f(x)\right)f(x)}{\left(\frac{16Aa^2}{N}-4f(x)+\frac{16Ba^2}{N}\right)}\right)(Ai(u))^2du &= dA\\
\int_{\mathbb{R}^+}\left(\frac{u}{\sqrt[3]{\frac{16Aa^2}{N}-4f(x)+\frac{16Ba^2}{N}}}+\frac{\left(\frac{16Aa^2}{N}-4f(x)\right)f(x)}{\left(\frac{16Aa^2}{N}-4f(x)+\frac{16Ba^2}{N}\right)}-f(x)\right)(Ai(u))^2\frac{du}{\sqrt[3]{\frac{16Aa^2}{N}-4f(x)+\frac{16Ba^2}{N}}} &= dB
\end{aligned}$$

with $d=\int_{\mathbb{R}^+}(Ai(u))^2du$. In first approximation $A=B$ and one is reduced to the following relation between A and a:

$$\int_{\mathbb{R}^+}\left(\frac{u}{\sqrt[3]{\frac{32Aa^2}{N}-4f(x)}}+\frac{\left(\frac{16Aa^2}{N}-4f(x)\right)f(x)}{\frac{32Aa^2}{N}-4f(x)}\right)(Ai(u))^2du=dA$$

We replace $f(x)$ by $\bar{f}$, so that this relation becomes

$$\int_{\mathbb{R}^+}\left(\frac{u}{\sqrt[3]{\frac{32Aa^2}{N}-4f(x)}}+\frac{\bar{f}}{2}\right)(Ai(u))^2du=dA$$

or, in a more compact form:

$$\frac{e}{\sqrt[3]{\frac{32Aa^2}{N}-4\bar{f}}}+\frac{\bar{f}}{2}d=dA$$

where we defined $e=\int_{\mathbb{R}^+}u(Ai(u))^2du$. The relation between A and a reduces to:

$$e^3=\left(dA-\frac{\bar{f}}{2}d\right)^3\left(\frac{32Aa^2}{N}-4\bar{f}\right) \tag{332}$$

In most cases, depending on $\bar{f}$, this equation has a positive solution with $\frac{32Aa^2}{N}-4\bar{f}>0$ as needed. Now, for $a\to\infty$

$$A=\frac{N}{32a^2}\left(4\bar{f}+\frac{e^3}{\left(\frac{\bar{f}}{2}d\right)^3}\right)$$

As a consequence Aa^2 is then bounded and has a maximum, so that $S(\Psi)$ is bounded from below and has a minimum for the value of a that maximizes Aa^2.

The second order variation simplifies in that particular case as follows. Here, $F'(x)x > F(x)$ and (329) applies:

$$\begin{aligned}
\frac{1}{2}\delta^2 S(\Psi) &> \varphi^\dagger(x)\left[\left(-\nabla^2 + \left(F^2(x) - 2F(x)x\right)\right)\right]\varphi(x) \\
&\quad + \frac{8\eta}{N}\int\left(\varphi^\dagger(x)F(x)\varphi(x)dx\right)\int\left(\Psi_1^\dagger(x)F(x)\Psi_1(x)dx\right) \\
&= \varphi^\dagger(x)\left[\left(-\nabla^2 + \left(F^2(x) - 2F(x)x\right)\right) + \frac{8\eta}{N}BF(x)\right]\varphi(x)
\end{aligned}$$

and here it writes:

$$\begin{aligned}
&\varphi^\dagger(x)\left[-\nabla^2 + \left(F^2(x) - 2F(x)x\right) + \frac{32\eta}{N}AF(x)\right]\varphi(x) \\
&= \varphi^\dagger(x)\left[-\nabla^2 - 4f(x)(x - f(x)) + \frac{32\eta}{N}B(x - f(x))\right]\varphi^\dagger(x) \\
&= \varphi^\dagger(x)\left[-\nabla^2 + \left(\frac{32\eta}{N}B - 4f(x)\right)(x - f(x))\right]\varphi^\dagger(x)
\end{aligned}$$

$$\left(\frac{8\eta}{N}B - 4f(x)\right) \simeq \left(\frac{32a^2}{N}A - 4f(x)\right) > 0$$

in our assumptions, and thus:

$$\frac{1}{2}\delta^2 S(\Psi) > 0$$

Case 3: Saddle points and stability: example of increasing return to scale

The second case we consider is:

$$F(x) = x + cx^2$$

with $0 < c < 1$ which measures an increasing return to scale. In that case:

$$\begin{aligned}
S(\Psi) &= \eta^2\int\Psi^\dagger(x)\left[\left(-\nabla^2 + \left(F^2(x) - 2F(x)x\right)\right)\right]\Psi(x)dx \\
&\quad + \frac{4\eta^4}{N}\int\left(\Psi^\dagger(x)F'(x)x\Psi(x)dx\right)\int\Psi^\dagger(y)F(y)\Psi(y)dy \\
&= \eta^2\int\Psi^\dagger(x)\left[-\nabla^2 + \left(x + cx^2\right)\left(cx^2 - x\right)\right]\Psi(x)dx \\
&\quad + \frac{4\eta^4}{N}\left(\int\Psi^\dagger(x)(1 + 2cx)x\Psi(x)dx\right)\left(\int\Psi^\dagger(x)\left(x + cx^2\right)\Psi(x)dx\right) \\
&\simeq \eta^2\int\Psi^\dagger(x)\left[-\nabla^2 + \left(c^2x^4 - x^2\right)\right]\Psi(x)dx \\
&\quad + \frac{4\eta^4}{N}\left(\int\Psi^\dagger(x)\left(x + 2cx^2\right)\Psi(x)dx\right)\left(\int\Psi^\dagger(x)\left(x + cx^2\right)\Psi(x)dx\right)
\end{aligned}$$

and the saddle point equation is:

$$\left[-\nabla^2 + c^2x^4 + \left(\frac{4c}{N}(A + 2B) - 1\right)x^2 + \frac{4x}{N}(A + B)\right]\Psi(x) = 0$$

with:

$$\begin{aligned} A &= \eta^2 \left(\int_{\mathbb{R}^+} \Psi^\dagger (x) \left(x + 2cx^2 \right) \Psi (x) \, dx \right) \\ B &= \eta^2 \left(\int_{\mathbb{R}^+} \Psi^\dagger (x) \left(x + cx^2 \right) \Psi (x) \, dx \right) \end{aligned}$$

One shows in appendiw 6.a that the action $S(\Psi)$ is bounded from below and that it has a minimum obtained as a first order correction in c ot the function :

$$\Psi_0 (x) = \eta \exp \left(- \frac{\sqrt{\frac{4c}{N} \eta^2 (A + 2B) - 1}}{2} \left(x + \frac{2}{N} \frac{\eta^2 (A + B)}{\frac{4c}{N} \eta^2 (A + 2B) - 1} \right)^2 \right)$$

This can be reorganized as:

$$\left[-\nabla^2 + c^2 x^4 + \left(\frac{4c}{N} (A + 2B) - 1 \right) \left(x + \frac{2}{N} \frac{(A + B)}{\frac{4c}{N} (A + 2B) - 1} \right)^2 - \left(\frac{2}{N} \frac{(A + B)}{\frac{4c}{N} (A + 2B) - 1} \right)^2 \right] \Psi (x) = 0$$

We write the solution $\Psi(x) = \eta \Psi_1 (x)$, and $\Psi_1 (x)$ has a norm equal to 1. The saddle point equation becomes ultimately:

$$\left[-\nabla^2 + c^2 x^4 + \left(\frac{4c}{N} \eta^2 (A + 2B) - 1 \right) \left(x + \frac{2}{N} \frac{\eta^2 (A + B)}{\frac{4c}{N} \eta^2 (A + 2B) - 1} \right)^2 - \left(\frac{2}{N} \frac{\eta^2 (A + B)}{\frac{4c}{N} \eta^2 (A + 2B) - 1} \right)^2 \right] \Psi (x) = 0$$

with:

$$\begin{aligned} A &= \left(\int_{\mathbb{R}^+} \Psi_1^\dagger (x) \left(x + 2cx^2 \right) \Psi_1 (x) \, dx \right) \\ B &= \left(\int_{\mathbb{R}^+} \Psi_1^\dagger (x) \left(x + cx^2 \right) \Psi_1 (x) \, dx \right) \end{aligned}$$

Note that the action at the saddle point solution is equal to:

$$\begin{aligned} S(\Psi) &= \int \Psi^\dagger (x) \left[-\nabla^2 + \left(c^2 x^4 - x^2 \right) \right] \Psi (x) \, dx + \frac{4}{N} \left(\int \Psi^\dagger (x) \left(x + 2cx^2 \right) \Psi (x) \, dx \right) \left(\int \Psi^\dagger (x) \left(x + cx^2 \right) \Psi (x) \, dx \right) \\ &= -\eta^2 \int \Psi^\dagger (x) \left(\frac{4}{N} \left(A \left(x + cx^2 \right) + \left(x + 2cx^2 \right) B \right) \right) \Psi (x) \, dx \\ &\quad + \frac{4}{N} \left(\int \Psi^\dagger (x) \left(x + 2cx^2 \right) \Psi (x) \, dx \right) \left(\int \Psi^\dagger (x) \left(x + cx^2 \right) \Psi (x) \, dx \right) \\ &= - \left(\frac{4\eta^4}{N} (AB + AB) \right) + \frac{4\eta^4}{N} AB = -\frac{4\eta^4}{N} AB < 0 \end{aligned}$$

As before, one has to show that this is bounded from below.

We start first by soving the saddle point equation. Since $c << 1$, the term $c^2 x^4$ can be treated perturbatively and one rather solves:

$$\left[-\nabla^2 + \left(\frac{4c}{N} \eta^2 (A + 2B) - 1 \right) \left(x + \frac{2}{N} \frac{\eta^2 (A + B)}{\frac{4c}{N} \eta^2 (A + 2B) - 1} \right)^2 - \left(\frac{2}{N} \frac{\eta^2 (A + B)}{\frac{4c}{N} \eta^2 (A + 2B) - 1} \right)^2 \right] \Psi (x) = 0$$

adding some corrections due to $c^2 x^4$ later. The change of variable

$$x' = \sqrt[4]{\frac{4c}{N} \eta^2 (A + 2B) - 1} \left(x + \frac{2}{N} \frac{\eta^2 (A + B)}{\frac{4c}{N} \eta^2 (A + 2B) - 1} \right)$$

for

$$\frac{4c}{N}\eta^2 (A+2B) - 1 > 0$$

yields the saddle point in a normalized form:

$$\left[-\nabla_{x'}^2 + (x')^2 - \epsilon\right] \Psi(x) = 0 \tag{333}$$

with:

$$\epsilon = \frac{\left(\frac{2}{N}\frac{\eta^2(A+B)}{\frac{4c}{N}\eta^2(A+2B)-1}\right)^2}{\sqrt{\frac{4c}{N}\eta^2(A+2B)-1}} = \frac{\left(\frac{2}{N}\eta^2(A+B)\right)^2}{\left(\frac{4c}{N}\eta^2(A+2B)-1\right)^{\frac{5}{2}}}$$

$$X^4 - (3X-1)^5$$

, Solution is: $\{[X = 0.535\,62]\}$The equation (333) has a bounded solution only if $\epsilon = 2n+1$ with n a non negative integer. The solution of norm 1 in that case is:

$$\Psi_n(x') = H_n(x') \exp\left(-\frac{(x')^2}{2}\right)$$

with $H_n(x')$ the n-th Hermite polynomial. The condition to find a solution of norm 1 is thus:

$$\frac{\left(\frac{2}{N}\eta^2(A+B)\right)^2}{\left(\frac{4c}{N}\eta^2(A+2B)-1\right)^{\frac{5}{2}}} = 2n+1 \tag{334}$$

where

$$\begin{aligned} A+B &= \left(\int_{\mathbb{R}^+} \Psi_n(x')\left(2x+3cx^2\right)\Psi_n(x')\,dx\right) \\ (A+2B) &= \left(\int_{\mathbb{R}^+} \Psi_n(x')\left(3x+4cx^2\right)\Psi_n(x')\,dx\right) \end{aligned}$$

and:

$$x = \frac{x'}{\sqrt[4]{\frac{4c}{N}\eta^2(A+2B)-1}} - \frac{2}{N}\frac{\eta^2(A+B)}{\frac{4c}{N}\eta^2(A+2B)-1}$$

These equations are show that $\eta^2 A$ and $\eta^2 B$ are of same order, and then (334) yields that

$$\frac{2}{N}\eta^2(A+B) \sim \frac{1}{(2n+1)^2} \tag{335}$$

so that

$$S(\Psi) = -\eta^4 \frac{4}{N} AB < 0$$

has its minimum for $n=0$. More precisely, for $n=0$, (334) gives:

$$\frac{\left(\frac{2}{N}\eta^2(A+B)\right)^2}{\left(\frac{4c}{N}\eta^2(A+2B)-1\right)^{\frac{5}{2}}} = 1$$

and thus in the first approximation $A=B$, $\frac{4c}{N}\eta^2 A = 1$ and $\frac{4c}{N}\eta^2(A+2B) - 1 = 2$.

As a consequence, one has shown that the action $S(\Psi)$ is bounded from below and that its minimum is obtained for:

$$\Psi_0(x') = \exp\left(-\frac{(x')^2}{2}\right)$$

or, coming back to the initial variable:

$$\Psi_0(x) = \exp\left(-\frac{\sqrt{\frac{4c}{N}\eta^2(A+2B)-1}}{2}\left(x+\frac{2}{N}\frac{\eta^2(A+B)}{\frac{4c}{N}\eta^2(A+2B)-1}\right)^2\right)$$

The inclusion of the corrective term c^2x^4 can be done perturbatively. To the second order, the eigenvalues of the operator in the left hand side of (333) are transformed as:

$$E'_n = E_n + c\langle\Psi_0(x)|x^4|\Psi_0(x)\rangle + c^2\sum_{l=1,l\neq n}^{\infty}\frac{|\langle\Psi_n(x)|x^4|\Psi_l(x)\rangle|^2}{E_n-E_l}$$

with $E_n = 2n+1$ and (334) is modified as:

$$\begin{aligned} &\frac{\left(\frac{2}{N}\eta^2(A+B)\right)^2}{\left(\frac{4c}{N}\eta^2(A+2B)-1\right)^{\frac{5}{2}}} \\ = \; & 2n+1+c\langle\Psi_0(x)|x^4|\Psi_0(x)\rangle - c^2\sum_{l=1}^{\infty}\frac{|\langle\Psi_0(x)|x^4|\Psi_l(x)\rangle|^2}{2l} \end{aligned} \tag{336}$$

Moreover the eigenvector $\Psi_0(x)$ is also modified:

$$\begin{aligned} |\Psi'_0(y)\rangle &= |\Psi'_0(y)\rangle - c\sum_{l=1}^{\infty}\frac{\langle\Psi_l(x)|x^4|\Psi_0(y)\rangle}{2l}|\Psi_l(y)\rangle \\ &+c^2\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{\langle\Psi_l(x)|x^4|\Psi_m(y)\rangle\langle\Psi_m(x)|x^4|\Psi_0(y)\rangle}{4lm}|\Psi_l(y)\rangle \\ &-c^2\sum_{l=1}^{\infty}\frac{\langle\Psi_0(x)|x^4|\Psi_0(y)\rangle\langle\Psi_l(x)|x^4|\Psi_0(y)\rangle}{4l^2}|\Psi_l(y)\rangle \\ &-\frac{c^2}{2}\sum_{l=1}^{\infty}\frac{\langle\Psi_0(x)|x^4|\Psi_l(y)\rangle\langle\Psi_l(x)|x^4|\Psi_0(y)\rangle}{4l^2}|\Psi_0(y)\rangle \end{aligned}$$

These relations modifies to the second order the values of A, B and η. However, as $\eta^2(A+B)$ remains of the same order as $\eta^2(A+2B)$ and since the corrections to the right hand side of (334) given by (336) are finite (only few elements of matrices $\langle\Psi_0(x)|x^4|\Psi_l(x)\rangle$ are non nul), then the asymptotic behavior:

$$\eta^2 A \sim \frac{1}{(2n+1)^2}$$

remains valid. As a consequence, $S(\Psi)$ is bounded from below and $\Psi'_0(x)$ is the minimum of $S(\Psi)$.

Now, to study the stability we have to compute $\delta^2 S(\Psi)$. Here

$$\begin{aligned} F'(x)x - F(x) &= x + 2cx^2 - \left(x+cx^2\right) \\ &= cx^2 > 0 \end{aligned}$$

and thus (328) applies:

$$\begin{aligned} \frac{1}{2}\delta^2 S(\Psi) &> \varphi^{\dagger}(x)\left[\left(-\nabla^2 + \left(F^2(x) - 2F(x)x\right)\right)\right]\varphi(x) \\ &+\frac{8\eta}{N}\int\left(\varphi^{\dagger}(x)F'(x)x\varphi(x)dx\right)\int\left(\Psi_1^{\dagger}(x)F'(x)x\Psi_1(x)dx\right) \\ &= \varphi^{\dagger}(x)\left[\left(-\nabla^2 + \left(F^2(x)-2F(x)x\right)\right) + \frac{8\eta}{N}AF'(x)x\right]\varphi(x) \end{aligned}$$

which, in this particular case, becomes:

$$\begin{aligned}
\frac{1}{2}\delta^2 S(\Psi) &> \varphi^{\dagger}(x)\left[\left(-\nabla^2+\left(F^2(x)-2F(x)\,x\right)\right)\right]\varphi(x) \\
&\quad +\frac{8\eta}{N}\int\left(\varphi^{\dagger}(x)\,F(x)\,\varphi(x)\,dx\right)\int\left(\Psi_1^{\dagger}(x)\,F(x)\,\Psi_1(x)\,dx\right) \\
&= \varphi^{\dagger}(x)\left[\left(-\nabla^2+\left(F^2(x)-2F(x)\,x\right)\right)+\frac{8\eta^2}{N}BF(x)\right]\varphi(x) \\
&= \varphi^{\dagger}(x)\left[-\nabla^2+\left(c^2x^4-x^2\right)+\frac{8\eta^2}{N}B\left(x+cx^2\right)\right]\varphi(x) \\
&= \varphi^{\dagger}(x)\left[-\nabla^2+\left(x+cx^2\right)\left(cx^2-x+\frac{8\eta^2}{N}B\right)\right]\varphi(x)
\end{aligned}$$

Given (335) and $n=0$ for the minimum, $\frac{2}{N}\eta^2(A+B)\sim\frac{4}{N}\eta^2B\sim\frac{1}{(2n+1)^2}=1$

$$cx^2-x+\frac{8\eta^2}{N}B\sim cx^2-x+2$$

and this is positive for $c>\frac{1}{8}$. In this range $\frac{1}{2}\delta^2S(\Psi)>0$.

Appendix 13

Stabilization of a finite number of negative eigenvalues by an interaction potential

We start with the saddle point equation described in the text.

$$0 = K\Psi(y) + 2U(y)\Psi(y)\int \left(\Psi(y_2)U(y_2)\Psi^{\dagger}(y_2)\right)dy_2 \tag{337}$$

with:

$$K = \left(-\frac{1}{2}\nabla\left(M^{(S)}\right)^{-1}\nabla + yM^{(A)}\nabla + yNy\right) + m^2$$

Normalize $\Psi(x) = \sqrt{\eta}\Psi_1(x)$ where $\eta = \int \Psi_1^{\dagger}(y)\Psi_1(y)\,dy$. The saddle point equation including this potential can also be written:

$$0 = K\Psi_1(y) + 2\eta U(y_1)\Psi_1(y)\int \left(\Psi_1(y_2)U(y_2)\Psi_1^{\dagger}(y_2)\right)dy_2 \tag{338}$$

If, as assumed before, K has a negative lowest eigenvalue λ_0, with eigenvector $\Psi^{(n)}(y)$ then, one can find a solution $(\Psi_1(y), \eta > 0)$ of (338).

Then, expand

$$\Psi_1(y) = \sum_{n\geqslant 0} a_n\Psi^{(n)}(y)$$

with $\sum_{n\geqslant 0}|a_n|^2 = 1$, where $\Psi^{(n)}(y)$ are norm one eigenvectors of K with eigenvalues λ_n. Then, take the scalar product of (338) with $\Psi_1^{\dagger}(y_1)$:

$$\begin{aligned} 0 &= \int \Psi_1^{\dagger}(y_1)K\Psi_1(y)\,dy \\ &\quad +2\eta\int \Psi_1^{\dagger}(y_1)U(y_1)\Psi_1(y_1)\,dy_1\int\left(\Psi_1(y_2)U(y_2)\Psi_1^{\dagger}(y_2)\right)dy_2 \end{aligned}$$

which allows to find η:

$$\eta = -\frac{1}{2}\frac{\langle\Psi_1|K|\Psi_1\rangle}{\left(\langle\Psi_1|U|\Psi_1\rangle\right)^2}$$

Thus, if we find a solution with $\eta > 0$, this solution $|\Psi_1\rangle$ is mainly a combination of negative eigenstates of K, so that $\langle\Psi_1|K|\Psi_1\rangle < 0$.

Given that:

$$\left(-\frac{1}{2}\nabla\left(M^{(S)}\right)^{-1}\nabla + yM^{(A)}\nabla + yNy\right)\Psi_1(y) = \sum_n a_n\lambda_n\Psi^{(n)}(y)$$

(338) rewrites:

$$\begin{aligned} 0 &= K\Psi_1(y) - \frac{\langle\Psi_1|K|\Psi_1\rangle}{\left(\langle\Psi_1|U|\Psi_1\rangle\right)^2}U(y_1)\Psi_1(y)\int\left(\Psi_1(y_2)U(y_2)\Psi_1^{\dagger}(y_2)\right)dy_2 \\ &= K\Psi_1(y) - \frac{\langle\Psi_1|K|\Psi_1\rangle}{\langle\Psi_1|U|\Psi_1\rangle}U(y_1)\Psi_1(y) \end{aligned}$$

or, equivalently:

$$\Psi_1(y) = \frac{\langle\Psi_1|K|\Psi_1\rangle}{\langle\Psi_1|U|\Psi_1\rangle}K^{-1}U(y_1)\Psi_1(y) \tag{339}$$

This relation can be written in any orthonormal basis. Using $\Psi_1(y) = \sum_{n\geqslant 0} a_n \Psi^{(n)}(y)$ with $\sum_{n\geqslant 0} |a_n|^2 = 1$ and $\Psi^{(n)}(y)$ the eigenvectors of K for the value λ_k and to order them by increasing eigenvalue, so that $\Psi^{(0)}(y)$ is the state with the lowest eigenvalue $\lambda_0 < 0$ by assumption.

Using the norm condition $\sum_{n\geqslant 0} |a_n|^2 = 1$, and that the previous systems of equation has one relation of dependence, one can get rid of a_0 (up to an irrelevant phase) and reduce the system to:

$$a_n = \frac{\langle\Psi_1|K|\Psi_1\rangle}{(\langle\Psi_1|U|\Psi_1\rangle)}\left(\sum_{m\geqslant 0}\left\langle\Psi^{(n)}(y_1)\right|K^{-1}U(y_1)\left|\Psi^{(m)}(y_1)\right\rangle a_m\right) \text{ for } n\geqslant 1$$

with $\sum_{n\geqslant 1} |a_n|^2 \leqslant 1$ and where a_0 is replaced in the sums in the numerator and denominator by $\sqrt{1-\sum_{n\geqslant 1}|a_n|^2}$. As a consequence the system has a solution (a_n), if the application:

$$(a_n)_{n\geqslant 1} \mapsto \left(\frac{\langle\Psi|K|\Psi\rangle}{(\langle\Psi|U|\Psi\rangle)}\left(\sum_{m\geqslant 1}\left\langle\Psi^{(n)}(y_1)\right|K^{-1}U(y_1)\left|\Psi^{(m)}(y_1)\right\rangle a_m + \left\langle\Psi^{(n)}(y_1)\right|U(y_1)\left|\Psi^{(0)}(y_1)\right\rangle\sqrt{1-\sum_{n\geqslant 1}|a_n|^2}\right)\right)_{n\geqslant 1}$$

has a fixed point. This possibility arises depending on the properties of the potential U. To get a more precise account for this point write the application as:

$$\Gamma : |\Psi\rangle_{n\geqslant 1} \mapsto \left(\frac{K^{-1}U}{\left(\frac{\langle\Psi|U|\Psi\rangle}{\langle\Psi|K|\Psi\rangle}\right)}|\Psi\rangle\right)_{n\geqslant 1}$$

where $()_{n\geqslant 1}$ denotes the projection on the space of eigenvalues $n\geqslant 1$. Let $0 < c < 1$, any arbitrary constant.

Assume that U preserves the space V generated by the negative eigenstates, so that Γ defines an application $V \to V$. We also assume that if 0 is eigenstates of K, it is an isolated point.

A fixed point exists in the ball $B \subset V$ of radius c, $\sum_{n\geqslant 1}^{<0} |a_n|^2 \leqslant c$, where $\sum_{n\geqslant 1}^{<0}$ runs over the negative eigenstates, if for any state $|\Psi\rangle$ of B

$$|\Psi\rangle = \sum_{n\geqslant 1}^{<0} a_n \Psi^{(n)}(y)$$

such that $\sum_{n\geqslant 1}^{<0} |a_n|^2 \leqslant c$ (and thus $|a_0|^2 > 1-c$):

$$\left\|\left(\frac{K^{-1}U}{\left(\frac{\langle\Psi|U|\Psi\rangle}{\langle\Psi|K|\Psi\rangle}\right)}|\Psi\rangle\right)_{n\geqslant 1}\right\|^2 = \frac{\langle\Psi|\left(U\left(K^{-1}\right)^2U\right)|\Psi\rangle'}{\frac{|\langle\Psi|U|\Psi\rangle|^2}{|\langle\Psi|K|\Psi\rangle|^2}} \leqslant c$$

with

$$\begin{aligned}
\langle\Psi|\left(U\left(K^{-1}\right)^2U\right)|\Psi\rangle' &= \sum_{n\geqslant 1}\langle\Psi|UK^{-1}\left|\Psi^{(n)}\right\rangle\left\langle\Psi^{(n)}\right|K^{-1}U|\Psi\rangle \\
&= \sum_{n\geqslant 0}\left|\langle\Psi|UK^{-1}\left|\Psi^{(n)}\right\rangle\right|^2 - \langle\Psi|UK^{-1}\left|\Psi^{(0)}\right\rangle\left\langle\Psi^{(0)}\right|K^{-1}U|\Psi\rangle \\
&= \langle\Psi|\left(U\left(K^{-1}\right)^2U\right)|\Psi\rangle - \langle\Psi|UK^{-1}\left|\Psi^{(0)}\right\rangle\left\langle\Psi^{(0)}\right|K^{-1}U|\Psi\rangle
\end{aligned}$$

so that:

$$\left\|\left(\frac{K^{-1}U}{\left(\frac{\langle\Psi|U|\Psi\rangle}{\langle\Psi|K|\Psi\rangle}\right)}|\Psi\rangle\right)_{n\geqslant 1}\right\|^2 = \frac{|\langle\Psi|K|\Psi\rangle|^2}{|\langle\Psi|U|\Psi\rangle|^2}\left(\langle\Psi|\left(U\left(K^{-1}\right)^2U\right)|\Psi\rangle - \langle\Psi|UK^{-1}\left|\Psi^{(0)}\right\rangle\left\langle\Psi^{(0)}\right|K^{-1}U|\Psi\rangle\right)$$

Then, a sufficient condition to have a fixed point, and thus a non trivial solution to (338) is that

$$\frac{|\langle\Psi|K|\Psi\rangle|^2}{|\langle\Psi|U|\Psi\rangle|^2}\left(\langle\Psi|\left(U\left(K^{-1}\right)^2U\right)|\Psi\rangle-\langle\Psi|UK^{-1}\left|\Psi^{(0)}\right\rangle\left\langle\Psi^{(0)}\right|K^{-1}U|\Psi\rangle\right)<1-c$$

for all $|\Psi\rangle$. This is achieved for example if U develops along the negative eigenstates of K and if the overlap of U and K is concentrated around $\left|\Psi^{(0)}\right\rangle$, that is:

$$U=\left|\Psi^{(0)}\right\rangle U_0\left\langle\Psi^{(0)}\right|+\sum_{i,j\geqslant 0,i+j\neq 0}^{<0}\left|\Psi^{(i)}\right\rangle U_{ij}\left\langle\Psi^{(j)}\right|+...$$

with $\frac{U_i}{U_0}<<1$. Actually, in that case: $|\langle\Psi|U|\Psi\rangle|^2\geqslant(U_0(1-c))^2$ and $|\langle\Psi|K|\Psi\rangle|^2<\lambda_0^2$ and then:

$$\begin{aligned}&\frac{|\langle\Psi|K|\Psi\rangle|^2}{|\langle\Psi|U|\Psi\rangle|^2}\left(\langle\Psi|\left(U\left(K^{-1}\right)^2U\right)|\Psi\rangle-\langle\Psi|UK^{-1}\left|\Psi^{(0)}\right\rangle\left\langle\Psi^{(0)}\right|K^{-1}U|\Psi\rangle\right)\\ \leqslant\ &\frac{\lambda_0^2}{(U_0(1-c))^2}\left(\langle\Psi|\left(U\left(K^{-1}\right)^2U\right)|\Psi\rangle-\langle\Psi|UK^{-1}\left|\Psi^{(0)}\right\rangle\left\langle\Psi^{(0)}\right|K^{-1}U|\Psi\rangle\right)\\ \leqslant\ &\frac{\lambda_0^2}{(U_0(1-c))^2}\sum_{i\geqslant 0,k\geqslant 0,,j>0}^{<0}U_{ij}\frac{1}{\lambda_j^2}U_{jk}\end{aligned}$$

and this is lower than c for $\frac{U_{ij}}{U_0}$ small enough. As a consequence $|\Psi_1\rangle$ is also peaked around $\left|\Psi^{(0)}\right\rangle$, and $\eta=-\frac{1}{2}\frac{\langle\Psi_1|K|\Psi_1\rangle}{(\langle\Psi_1|U|\Psi_1\rangle)}$ is positive as needed. Then, a fixed point exists in B, and thus on the space of all states, for the type of potential considered. The minimum of $S(\Psi)$ is reached for the fixed point with lowest $S(\Psi)$.

The interpretation of this case is clear. A positive potential of interaction counter balances the direction of instability and allow the composed system of two structure two stabilize around a composite extremum.

$$\begin{aligned}&\left(-\frac{1}{2}\nabla\left(M^{(S)}\right)^{-1}\nabla+yM^{(A)}\nabla+yNy+m^2\right)\delta(y-y_1)\\ &+2U(y)\delta(y-y_1)\int\left(\Psi(y_2)U(y_2)\Psi^{\dagger}(y_2)\right)dy_2+2\Psi(y)U(y)U(y_1)\Psi^{\dagger}(y_1)\end{aligned}$$

To inspect if the solution we found is a minimum, one has to compute the second order variation $\langle\varphi|\frac{\partial^2 S}{\partial\Psi_1(x)\partial\Psi_1(x)}|\varphi\rangle$. The variation $\varphi(y)$ is arbitary but can be considered of norm 1, since this norm can be factored from the second order variation, and that only the sign of this variation matters. If one finds condtions on the potential to have

$$\langle\varphi|\frac{\partial^2 S}{\partial\Psi_1(x)\partial\Psi_1(x)}|\varphi\rangle>0$$

we will have found a lower minimum than $\Psi(x)\,0$, since, in that case:

$$\begin{aligned}S(\Psi)&=\int\Psi(y)\left(-\frac{1}{2}\nabla\left(M^{(S)}\right)^{-1}\nabla+yM^{(A)}\nabla+yNy+m^2\right)\Psi^{\dagger}(y)\,dy\\ &\quad+\left(\int\left(\Psi(y_2)U(y_2)\Psi^{\dagger}(y_2)\right)dy_2\right)^2\end{aligned}$$

which is equal, given (337):

$$S(\Psi)=-\left(\int\left(\Psi(y_2)U(y_2)\Psi^{\dagger}(y_2)\right)dy_2\right)^2$$

Now, the second order variation $\langle\varphi|\frac{\partial^2 S}{\partial\Psi_1(x)\partial\Psi_1(x)}|\varphi\rangle$ is computed as:

$$\begin{aligned}
&\int\varphi(y)K\varphi^{\dagger}(y)\,dy\\
&+2\eta\left(\int\varphi(y)U(y)\varphi^{\dagger}(y)\,dy\right)\int\left(\Psi_1(y_2)U(y_2)\Psi_1^{\dagger}(y_2)\right)dy_2+2\eta\left|\int\Psi_1U(y)\varphi^{\dagger}(y)\,dy\right|^2\\
=&\ \langle\varphi|K|\varphi\rangle-\frac{\langle\Psi_1|K|\Psi_1\rangle}{\langle\Psi_1|U|\Psi_1\rangle}\left[\langle\varphi|U|\varphi\rangle+\frac{|\langle\varphi|U|\Psi_1\rangle|^2}{\langle\Psi_1|U|\Psi_1\rangle}\right]\\
=&\ \frac{1}{\langle\Psi_1|U|\Psi_1\rangle}\left(\langle\varphi|K|\varphi\rangle\langle\Psi_1|U|\Psi_1\rangle-\langle\Psi_1|K|\Psi_1\rangle\left(\langle\varphi|U|\varphi\rangle+\frac{|\langle\varphi|U|\Psi_1\rangle|^2}{\langle\Psi_1|U|\Psi_1\rangle}\right)\right)\\
\geqslant&\ \frac{1}{\langle\Psi_1|U|\Psi_1\rangle}\left(\langle\varphi|K|\varphi\rangle\langle\Psi_1|U|\Psi_1\rangle-\langle\Psi_1|K|\Psi_1\rangle\langle\varphi|U|\varphi\rangle\right)
\end{aligned}$$

Given that the saddle point solution satisfies (339):

$$\begin{aligned}
\Psi_1(y)&=\frac{\langle\Psi_1|K|\Psi_1\rangle}{\langle\Psi_1|U|\Psi_1\rangle}K^{-1}U(y_1)\Psi_1(y)\\
1&=\frac{\langle\Psi_1|K|\Psi_1\rangle}{\langle\Psi_1|U|\Psi_1\rangle}\langle\Psi_1|K^{-1}U|\Psi_1\rangle
\end{aligned}$$

one can write:

$$\begin{aligned}
&\langle\varphi|K|\varphi\rangle\langle\Psi_1|U|\Psi_1\rangle-\langle\Psi_1|K|\Psi_1\rangle\langle\varphi|U|\varphi\rangle\\
=&\ \langle\Psi_1|U|\Psi_1\rangle\left(\langle\varphi|K|\varphi\rangle-\frac{1}{\langle\Psi_1|K^{-1}U|\Psi_1\rangle}\langle\varphi|U|\varphi\rangle\right)
\end{aligned}$$

$$\begin{aligned}
&\langle\varphi|\frac{\partial^2 S}{\partial\Psi_1(x)\partial\Psi_1(x)}|\varphi\rangle\\
=&\ \langle\varphi|K|\varphi\rangle-\frac{\langle\Psi_1|K|\Psi_1\rangle}{\langle\Psi_1|U|\Psi_1\rangle}\left[\langle\varphi|U|\varphi\rangle+\frac{|\langle\varphi|U|\Psi_1\rangle|^2}{\langle\Psi_1|U|\Psi_1\rangle}\right]\\
=&\ \langle\varphi|K|\varphi\rangle-\frac{1}{\langle\Psi_1|K^{-1}U|\Psi_1\rangle}\left[\langle\varphi|U|\varphi\rangle+\frac{|\langle\varphi|U|\Psi_1\rangle|^2}{\langle\Psi_1|U|\Psi_1\rangle}\right]\\
\geqslant&\ \lambda_0-\frac{\langle\varphi|U|\varphi\rangle}{\langle\Psi_1|K^{-1}U|\Psi_1\rangle}\\
\geqslant&\ \lambda_0-\frac{U_0}{\langle\Psi_1|K^{-1}U|\Psi_1\rangle}
\end{aligned}$$

where λ_0 is the lowest eigenvalue of and U_0 (which is negative by assumption), the minimum eigenvalue of U_0. Then:

$$\langle\varphi|\frac{\partial^2 S}{\partial\Psi_1(x)\partial\Psi_1(x)}|\varphi\rangle\geqslant\lambda_0-\frac{U_0}{\langle\Psi_1|K^{-1}U|\Psi_1\rangle}$$

and this is positive if

$$U_0>\lambda_0\langle\Psi_1|K^{-1}U|\Psi_1\rangle$$

that is if the potential is strong enough to compensate for the instability of the system.

Instability due to non linear terms

We generalize the previous paragraph by considering the instability introduced by a more general term than m^2. Assume that the operator K (for $m^2=0$) has been set in a basis such that it rewrites in a diagonal form

$$K=-\frac{1}{2}\nabla^2+yD'y \tag{340}$$

where D is a diagonal matrix with eigenvalues $\lambda_i>0$ that ensure the stability of each fundamental structure without the perturbation . Assume that due to the interactions among the structures components, a potential $V(y)$ with a negative minimum is added to K to yield:

$$K_1=-\frac{1}{2}\nabla^2+yD'y+V(y) \tag{341}$$

Operator (341) has the form of an harmonic oscillator plus a perturbation term.

The eigenvalues of K are $E_{n_1,...,n_k}=\sum_{i=1}^k n_{i_k}\sqrt{\lambda_i}+\frac{k}{2}$ where k is the number of components of y, $y=(y_i)$ and the $n_1,...,n_k$ are natural integers.

The eigenfunctions $\Psi_1^{(n_1,...,n_k)}(x)$ corresponding to these eigenvalues of Harmonic oscillators are:

$$\Psi_1^{(n_1,...,n_k)}(y)=\prod_{i=1}^k\varphi_{n_i}(y_i)$$

$$\varphi_n(x)=\left(\frac{\sqrt{a}}{\pi}\right)^{\frac{1}{4}}\sqrt{\frac{1}{2^n n!}}H_n\left(a^{\frac{1}{4}}x\right)\exp\left(-\frac{\sqrt{a}}{2}x^2\right)$$

where the H_n are the Hermite polynomials. Then, introducing the eigenvalues modify both the eigenvalues and eigenfunctions as series expansion of C. We choose a perturbation that shifts essentially the lowest eigenstates of K, that is quadratic and antisymetric (the quadratic and symetric part being included in D' by a series expansion and we assume that this part does not affect the sign of D''s eigenvalues). We choose for the potential the particular form

$$-yV^A(y,\nabla)\nabla$$

which describes, as $-yM^A\nabla$ the internal interaction inside the structure, but taking into account non linear terms (as resulting for non linear utilities for example).

The perturbation can be rewriten, using the usual creation and destruction operators as:

$$\begin{aligned}-yV\nabla &= -\left(a^++a^-\right)V^A\left(a^+-a^-\right)\\ &= 2a^+V^{(A)}a^-\end{aligned}$$

since $V^{(A)}$ is antisymetric. Note that a^+ and a^- have $\dim y=k$ components: $a^+\equiv\left(a_i^+\right)$ and $a^-\equiv\left(a_i^-\right)$ and $y=(a^++a^-)$, $\nabla=(a^+-a^-)$. To model that this potential modifies mainly the lowest eigenstates of K, we choose:

$$\left\langle\Psi_1^{(n_1',...,n_k')}(y)\middle|V^{(A)}\middle|\Psi_1^{(n_1,...,n_k)}(y)\right\rangle=\delta_{(n_1',...,n_k'),(n_1,...,n_k)}f((n_1,...,n_k))$$

with $f((n_1,...,n_k))$ is a quickly decreasing function of $n_1^2+...+n_k^2$.

Since $V^{(A)}$ is antisymetric, and for the ground state $\Psi_1^{(0,...,0)}(y)$, $a^+V^{(A)}a^-\Psi_1^{(0,...,0)}(y)=0$ and one can then deduce that the series expansion for the perturbed ground state is nul. Thus one still have a state $\Psi_1^{(0,...,0)}(y)$ with eigenvalue $\frac{k}{2}$.

As a consequence:

$$\begin{aligned}&\left(-\frac{1}{2}\nabla^2-yV^{(A)}\nabla+yD'y\right)\Psi_1^{(0,...,0)}(y)\\ =&\left(-\frac{1}{2}\nabla^2+yD'y\right)\Psi_1^{(0,...,0)}(y)\\ =&\frac{k}{2}\Psi_1^{(0,...,0)}(y)\end{aligned}$$

and the saddle point equation is not satisfied.

This situation changes for the first excited states. Consider $\Psi_1^{(1,i)}(y) = \Psi_1^{(0,.,1,..,0)}(y)$ with the 1 set in the i th position. These are the first excited energy levels, the closest to $\Psi_1^{(0,...,0)}(y)$ with energy $E_i = \sqrt{\lambda_i} + \frac{k}{2}$. The perturbation expansion for the eigenvalue of

$$\left(-\frac{1}{2}\nabla^2 - yV^{(A)}\nabla + yD'y\right)$$

to the second order is then:

$$\begin{aligned} E_i' &= E_i + 4\sum_{j=1,j\neq i}^{k} \frac{\left|\left\langle \Psi_1^{(1,i)}(y)\right| a^+ V_{ij}^{(A)} a^- \left|\Psi_1^{(1,j)}(y)\right\rangle\right|^2}{\sqrt{\lambda_i} - \sqrt{\lambda_j}} \\ &= E_i + \sum_{j=1,j\neq i}^{k} \frac{4\left(V_{ij}^{(A)}\right)^2}{\sqrt{\lambda_i} - \sqrt{\lambda_j}} \\ &= \sqrt{\lambda_i} + \frac{k}{2} + \sum_{j=1,j\neq i}^{k} \frac{4\left(V_{ij}^{(A)}\right)^2}{\sqrt{\lambda_i} - \sqrt{\lambda_j}} \end{aligned} \tag{342}$$

Note that, due to the hypothesis on f, the shift $E'_{n_1,...,n_k}$ in $E_{n_1,...,n_k}$ can be neglected for $n_1^2 + ... + n_k^2 >> 1$. One can thus focus on the first eigenstates.

We call $\left|\Psi_1'^{(1,i)}(y)\right\rangle$ the corresponding eigenvector to E_i':

$$\begin{aligned} \left|\Psi_1'^{(1,i)}(y)\right\rangle &= \left|\Psi_1^{(1,i)}(y)\right\rangle + \sum_{j=1,j\neq i}^{k} \frac{2V_{ij}^{(A)}}{\lambda_i(\eta) - \lambda_j(\eta)} \left|\Psi_1^{(1,j)}(y)\right\rangle \\ &+ \sum_{j=1,j\neq i}^{k} \sum_{l=1,j\neq i}^{k} \frac{4V_{jl}^{(A)} V_{li}^{(A)}}{\left(\sqrt{\lambda_i} - \sqrt{\lambda_l}\right)\left(\sqrt{\lambda_i} - \sqrt{\lambda_j}\right)} \left|\Psi_1^{(1,j)}(y)\right\rangle \\ &+2\sum_{j=1,j\neq i}^{k} \frac{\left(V_{ij}^{(A)}\right)^2}{\left(\sqrt{\lambda_i} - \sqrt{\lambda_j}\right)^2} \left|\Psi_1^{(1,i)}(y)\right\rangle \end{aligned}$$

This approximation is valid if we assume that $V^{(A)}$ is relatively small with respect to the λ_i and this assumption is necessary if the fundamental structures are assumed to have a certain stability. If we rank the λ_i in increasing order, equation (342) shows that the eigenvalue E_1' is driven below E_1. It means that the equilibrium of the system is reduced by it's internal interactions/tensions. For a sufficient magnitude of the perturbation, one may have $E_1' < 0$ and the previous analysis concerning the stabilization of the system by the interaction between structures apply. Remark, that some other first excited states may be also driven below 0, by the perturbation, but the number of such eigenstates remains finite given our assumptions about the potential V. Higher order excited states have eigenvalues increasing with $n_1^2 + ... + n_k^2 = a$, whereas, f decreases with a.

Generalization to several types of interacting structures

We consider k fields in interacting, characterized independently by an operator:

$$K_l = \left(-\frac{1}{2}(\nabla_l)^2 - y_l M_l^{(A)}\nabla_l + y_l D_l y_l + V_l(y_l)\right)$$

for $l = 1, ..., k$, where the $V_l(y_l)$ have a negative minimal eigenvalue. The saddle point equations for the fields with interaction are then

$$\begin{aligned}
0 &= K_l\Psi_l(y_l) + \left(\frac{\partial}{\partial\Psi_l^\dagger(y)}\int V\left((x_1)_{p_1},...,(x_k)_{p_k}\right)\left[\Psi_1(x_1)\Psi_1^\dagger(x_1)\right]_{p_1}...\right.\\
&\qquad \left.\left[\Psi_l(x_l)\Psi_l^\dagger(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\Psi_k^\dagger(x_k)\right]_{p_k} d(x_1)_{p_1}...d(x_k)_{p_k}\right)\\
&= K_l\Psi_l(y_l) + p_l\left(\int V\left((x_1)_{p_1},...,(x_k)_{p_k}\right)\left[\Psi_1(x_1)\Psi_1^\dagger(x_1)\right]_{p_1}...\right.\\
&\qquad \left.\left[\Psi_l(x_l)\Psi_l^\dagger(x_l)\right]_{p_l-1}...\left[\Psi_k(x_k)\Psi_k^\dagger(x_k)\right]_{p_k} d(x_1)_{p_1}...d(x_k)_{p_k}\right)\Psi_l(y)
\end{aligned}$$

where $(x_l)_{p_l}$ represents p_l copies of the coordinates x_l and $\left[\Psi_l(x_l)\Psi_l^\dagger(x_l)\right]_{p_l}$ indicates a product of p_l independent copies of $\Psi_l(x_l)\Psi_l^\dagger(x_l)$. The interaction involves then p_l copies of the l-th structure. Then, one normalizes

$$\Psi_l(x_l) = \sqrt{\eta_l}\Psi_l^{(1)}(x_l)$$

where $\Psi_l^{(1)}(x_l)$ is of norm 1 and the saddle point equations rewrites:

$$\begin{aligned}
0 &= K_l\Psi_l^{(1)}(y_l) + p_l\frac{\left(\prod_{i=1}^k(\eta_i)^{p_i}\right)}{\eta_l} \qquad (343)\\
&\quad\times\left(\int V\left((x_1)_{p_1},...,(x_k)_{p_k}\right)\left[\Psi_1^{(1)}(x_1)\Psi_1^{(1)\dagger}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\Psi_l^\dagger(x_l)\right]_{p_l-1}\right.\\
&\qquad \left....\left[\Psi_k(x_k)\Psi_k^\dagger(x_k)\right]_{p_k} dx_1...dx_k\right)\Psi_l^{(1)}(y)
\end{aligned}$$

As in the previous case of simililar structures interaction, one can multiply by $\Psi_l^{(1)\dagger}(x_l)$ and integrate to find:

$$\begin{aligned}
0 &= \left\langle\Psi_l^{(1)}(y_l)\right| K_l\left|\Psi_l^{(1)}(y_l)\right\rangle + p_l\frac{\left(\prod_{i=1}^k(\eta_i)^{p_i}\right)}{\eta_l} \qquad (344)\\
&\quad\times\left\langle\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\right]_{p_k}\right|\\
&\qquad \left(V\left((x_1)_{p_1},...,(x_k)_{p_k}\right)\right)\left|\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l-1}...\left[\Psi_k(x_k)\right]_{p_k}\right\rangle
\end{aligned}$$

Where we defined

$$\left|\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l-1}...\left[\Psi_k(x_k)\right]_{p_k}\right\rangle \in (H_1)^{\otimes p_1}\otimes...(H_1)^{\otimes p_l-1}...\otimes(H_k)^{\otimes p_k}$$

the state corresponding to the product of fields $\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\right]_{p_k}$ where the H_l are the state spaces for the structures $l=1,,,k$. Similarly, the individual fields $\Psi_l^{(1)}(y_l)$ are know seen as vector on a tensor product space:

$$\Psi_l^{(1)}(y_l) \equiv \left|\Psi_l^{(1)}(y_l)\right\rangle\otimes 1\otimes...\otimes 1\in(H_1)^{\otimes p_1}\otimes...\otimes(H_k)^{\otimes p_k}$$

The value of the η_l are found to satisfy:

$$\eta_l = -p_l\left(\prod_{i=1}^{k}(\eta_i)^{p_i}\right) \times \frac{\left\langle\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\right]_{p_k}\right|\left(V\left((x_1)_{p_1},...,(x_k)_{p_k}\right)\right)\left|\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\right]_{p_k}\right\rangle}{\left\langle\Psi_l^{(1)}(y_l)\right|K_l\left|\Psi_l^{(1)}(y_l)\right\rangle}$$

for $l = 1...k$, and $\left(\prod_{i=1}^{k}(\eta_i)^{p_i}\right)$ is computed by the product of the k previous relations:

$$\left(\prod_{i=1}^{k}(\eta_i)^{p_i}\right)^{1-\sum_{l=1}^{k}p_l} = \frac{\left(\left\langle\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\right]_{p_k}\right|(V)\left|\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\right]_{p_k}\right\rangle\right)^{\sum_{l=1}^{k}p_l}}{\prod_{l=1}^{k}\left(-\frac{1}{p_l}\left\langle\Psi_l^{(1)}(y_l)\right|K_l\left|\Psi_l^{(1)}(y_l)\right\rangle\right)^{p_l}}$$

where V stands for $V\left((x_1)_{p_1},...,(x_k)_{p_k}\right)$, so that one finds:

$$\left(\prod_{i=1}^{k}(\eta_i)^{p_i}\right) = \left[\frac{\left(\left\langle\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l^{(1)}(x_l)\right]_{p_l}...\left[\Psi_k^{(1)}(x_k)\right]_{p_k}\right|V\left|\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l^{(1)}(x_l)\right]_{p_l}...\left[\Psi_k^{(1)}(x_k)\right]_{p_k}\right\rangle\right)^{\sum_{l=1}^{k}p_l}}{\prod_{l=1}^{k}\left(-\frac{1}{p_l}\left\langle\Psi_l^{(1)}(y_l)\right|K_l\left|\Psi_l^{(1)}(y_l)\right\rangle\right)^{p_l}}\right]^{\frac{1}{1-\sum_{l=1}^{k}p_l}}$$

and:

$$\eta_l = -\frac{1}{p_l}\frac{\left(\left\langle\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l^{(1)}(x_l)\right]_{p_l}...\left[\Psi_k^{(1)}(x_k)\right]_{p_k}\right|V\left|\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l^{(1)}(x_l)\right]_{p_l}...\left[\Psi_k^{(1)}(x_k)\right]_{p_k}\right\rangle\right)^{\frac{1}{1-\sum_{l=1}^{k}p_l}}}{\left(\prod_{l=1}^{k}\left(-p_l\left\langle\Psi_l^{(1)}(y_l)\right|K_l\left|\Psi_l^{(1)}(y_l)\right\rangle\right)^{p_l}\right)^{\frac{1}{1-\sum_{l=1}^{k}p_l}}\left\langle\Psi_l^{(1)}(y_l)\right|K_l\left|\Psi_l^{(1)}(y_l)\right\rangle}$$

As before we ssume that $V\left((x_1)_{p_1},...,(x_k)_{p_k}\right)$ preserves the eigenstates of the K_l (our results would be preserved if they are only preserved in first approximation).

As in the one field case, replacing the values of the η_l in (343), leads to a fixed point equation:

$$K_l \left|\Psi_l^{(1)}(y_l)\right\rangle = -p_l \frac{\left(\prod_{i=1}^{k} (\eta_i)^{p_i}\right)}{\eta_l} \tag{345}$$

$$\times \left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l(x_l)\right]_{p_l-1} ... \left[\Psi_k(x_k)\right]_{p_k} \right| V \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l(x_l)\right]_{p_l-1} ... \left[\Psi_k(x_k)\right]_{p_k} \right\rangle \Psi_l^{(1)}(y)$$

$$= \left\langle \Psi_l^{(1)}(y_l)\right| K_l \left|\Psi_l^{(1)}(y_l)\right\rangle$$

$$\times \frac{\left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l(x_l)\right]_{p_l-1} ... \left[\Psi_k(x_k)\right]_{p_k} \right| V \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l(x_l)\right]_{p_l-1} ... \left[\Psi_k(x_k)\right]_{p_k} \right\rangle \Psi_l^{(1)}(y)}{\left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l(x_l)\right]_{p_l} ... \left[\Psi_k(x_k)\right]_{p_k} \right| V \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l(x_l)\right]_{p_l} ... \left[\Psi_k(x_k)\right]_{p_k} \right\rangle}$$

for $l = 1...k$, where $V\left((x_1)_{p_1}, ..., (x_k)_{p_k}\right)$ is now seen as an operator V on $(H_1)^{\otimes p_1} \otimes ... \otimes (H_k)^{\otimes p_k}$. The partial amplitude:

$$\left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l(x_l)\right]_{p_l-1} ... \left[\Psi_k(x_k)\right]_{p_k} \right| V \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l(x_l)\right]_{p_l-1} ... \left[\Psi_k(x_k)\right]_{p_k} \right\rangle$$

is then an operator on V_l.

As in the one field case, one can developp the fields $\Psi_l^{(1)}(y_l)$ in a basis of eigenvectors of K_l, and since V preserves the negative eigenstates, we can restrict the sum on these states (this will be implicit in the sequel)

$$\Psi_l^{(1)}(y_l) = \sum_{n\geqslant 0} a_{n,l} \Psi_l^{(n)}(y_l) \text{ with} \sum_{n\geqslant 0} |a_n|^2 = 1$$

and $\Psi_l^{(n)}(y_l)$ are eigenvectors of K_l with negative eigenvalues ordered such that $\Psi_l^{(0)}(y)$ is the eigenvector for the lowest eigenvalue $\lambda_{0,l}$.

The equations (345) are not independent for the coefficients $a_{n,l}$ This can be seen by multiplying both sides of (345) by $\Psi_l^{\dagger}(x_l)$ and to integrate to obtain a trivial relation. Actually,

$$0 = \left\langle \Psi_l^{(1)}(y_l)\right| K_l \left|\Psi_l^{(1)}(y_l)\right\rangle \tag{346}$$

$$+p_l \frac{\left(\prod_{i=1}^{k} (\eta_i)^{p_i}\right)}{\eta_l}$$

$$\times \left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l(x_l)\right]_{p_l} ... \left[\Psi_k(x_k)\right]_{p_k} \right| \left(V\left((x_1)_{p_1}, ..., (x_k)_{p_k}\right)\right) \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l(x_l)\right]_{p_l} ... \left[\Psi_k(x_k)\right]_{p_k} \right\rangle$$

is trivial given the definition of $\frac{\left(\prod_{i=1}^{k} (\eta_i)^{p_i}\right)}{\eta_l}$. Thus, one can look for a solution of (345) by choosing the coefficients $a_{0,l}$ with

$$a_{0,l}^2 = c_l \neq 0 \tag{347}$$

, so that the solution we are looking for is a perturbative expansion around the minimum of the K_l. Rewrite first

$$K_l \left|\Psi_l^{(1)}(y_l)\right\rangle \tag{348}$$

$$= \frac{\left\langle \Psi_l^{(1)}(y_l)\right| K_l \left|\Psi_l^{(1)}(y_l)\right\rangle (K_l)^{-1} V_l \Psi_l^{(1)}(y)}{\left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l(x_l)\right]_{p_l} ... \left[\Psi_k(x_k)\right]_{p_k} \right| V \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l(x_l)\right]_{p_l} ... \left[\Psi_k(x_k)\right]_{p_k} \right\rangle}$$

with:

$$V_l=\left\langle\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l-1}...\left[\Psi_k(x_k)\right]_{p_k}\right|V\left|\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l-1}...\left[\Psi_k(x_k)\right]_{p_k}\right\rangle$$

is an operator on V_l. It can be written more compactly as:

$$\begin{aligned}&\left(\left|\Psi_l^{(1)}(y_l)\right\rangle\right) \\ =& \frac{\left(\left\langle\Psi_l^{(1)}(y_l)\right|K_l\left|\Psi_l^{(1)}(y_l)\right\rangle(K)^{-1}V\left(\Psi_l^{(1)}(y)\right)\right)}{\left\langle\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\right]_{p_k}\right|V\left|\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\right]_{p_k}\right\rangle}\end{aligned}\tag{349}$$

where $\left(\left|\Psi_l^{(1)}(y_l)\right\rangle\right)$ is the vector with l components $\left|\Psi_l^{(1)}(y_l)\right\rangle$, and K, $(K)^{-1}V$ are the diagonal matrices with components K_l, $(K_l)^{-1}V_l$ on the diagonal. The vector $\left(\left\langle\Psi_l^{(1)}(y_l)\right|K_l\left|\Psi_l^{(1)}(y_l)\right\rangle(K)^{-1}V\left(\Psi_l^{(1)}(y)\right)\right)$ has l components $\left\langle\Psi_l^{(1)}(y_l)\right|K_l\left|\Psi_l^{(1)}(y_l)\right\rangle(K)^{-1}V\left(\Psi_l^{(1)}(y)\right)$.

Then, replacing for $a_{0,l}$ in (349) implies that

$$\varphi:\left(\left|\Psi_l^{(1)}(y_l)\right\rangle\right)\rightarrow\frac{\left(\left\langle\Psi_l^{(1)}(y_l)\right|(K)\left|\Psi_l^{(1)}(y_l)\right\rangle(K)^{-1}V\left(\left|\Psi_l^{(1)}(y_l)\right\rangle\right)\right)}{\left\langle\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\right]_{p_k}\right|V\left|\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\right]_{p_k}\right\rangle}$$

defines an application from $V=V_1\times...\times V_k$ where the V_l are the negative eigenstates of the K_l. Moreover, using the condition (347) for the norm implies that solving $\cdot$ is equivalent to find a fixed point for this application on the ball of radius c in the finite dimensional space $V_1^{(0)}\times...\times V_k^{(0)}$ where $V_1^{(0)}$ is the orthogonal of the lowest eigenstate in V_1. Given the definition of φ:

$$\begin{aligned}&\left\|\frac{\left(\left\langle\Psi_l^{(1)}(y_l)\right|(K)\left|\Psi_l^{(1)}(y_l)\right\rangle(K)^{-1}V\left|\Psi_l^{(1)}(y_l)\right\rangle\right)}{\left\langle\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\right]_{p_k}\right|V\left|\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\right]_{p_k}\right\rangle}\right\|^2 \\ \leqslant&\left(\frac{Max_l\left|\left\langle\Psi_l^{(1)}(y_l)\right|(K)\left|\Psi_l^{(1)}(y_l)\right\rangle\right|}{\left\langle\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\right]_{p_k}\right|V\left|\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\right]_{p_k}\right\rangle}\right)^2 \\ &\times\left\|\left((K)^{-1}V\left|\Psi_l^{(1)}(y_l)\right\rangle\right)\right\|^2_{V_1^{(0)}\times...\times V_k^{(0)}}\end{aligned}$$

and that:

$$\begin{aligned}&\left\|\left((K)^{-1}V\left(\Psi_l^{(1)}(y)\right)\right)\right\|^2_{V_1^{(0)}\times...\times V_k^{(0)}} \\ =&\sum_{l,m}\left(\left\langle\Psi_m^{(1)}(y_m)\right|V_{ml}(K_l)^{-2}V_{lm}\left|\Psi_m^{(1)}(y_m)\right\rangle-\left\langle\Psi_m^{(1)}(y_m)\right|V_{ml}(K_l)^{-1}\left|\Psi_l^{(0)}(y_l)\right\rangle\left\langle\Psi_l^{(0)}(y_l)\right|(K_l)^{-1}V_{lm}\left|\Psi_m^{(1)}(y_m)\right\rangle\right)\end{aligned}$$

then:

$$\begin{aligned}&\left\|\frac{\left\langle\Psi_l^{(1)}(y_l)\right|K_l\left|\Psi_l^{(1)}(y_l)\right\rangle(K)^{-1}V\left|\Psi_l^{(1)}(y_l)\right\rangle}{\left\langle\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\right]_{p_k}\right|V\left|\left[\Psi_1^{(1)}(x_1)\right]_{p_1}...\left[\Psi_l(x_l)\right]_{p_l}...\left[\Psi_k(x_k)\right]_{p_k}\right\rangle}\right\|^2 \\ \leqslant&\frac{k\lambda_{0,\sup}^2}{V_0^2}\sum_{l,m}\left(\left\langle\Psi_m^{(1)}(y_m)\right|V_{ml}(K_l)^{-2}V_{lm}\left|\Psi_m^{(1)}(y_m)\right\rangle\right. \\ &\left.-\left\langle\Psi_m^{(1)}(y_m)\right|V_{ml}(K_l)^{-1}\left|\Psi_l^{(0)}(y_l)\right\rangle\left\langle\Psi_l^{(0)}(y_l)\right|(K_l)^{-1}V_{lm}\left|\Psi_m^{(1)}(y_m)\right\rangle\right)\end{aligned}$$

where V_0 is the minimum of the potential, and $\lambda_{0,\sup}$ the lowest eigenvalue among the $\lambda_{0,l}$, $l = 1..., k$. Then we arrive to similar conclusion as in the one structure case. A fixed point exist, and then a solution to the saddle point equation (345) if the minimum of the potential is strong enough, and if the potential is mainly localized oriented in the directions of instabilty to compensate them. Actually, in that case:

$$\begin{aligned}
&\sum_{l,m}\left(\left\langle \Psi_m^{(1)}(y_m)\right| V_{ml}(K_l)^{-2} V_{lm}\left|\Psi_m^{(1)}(y_m)\right\rangle - \left\langle \Psi_m^{(1)}(y_m)\right| V_{ml}(K_l)^{-1}\left|\Psi_l^{(0)}(y_l)\right\rangle \left\langle \Psi_l^{(0)}(y_l)\right|(K_l)^{-1} V_{lm}\left|\Psi_m^{(1)}(y_m)\right\rangle\right)\\
=&\sum_{l,m}\left(\sum_q \left\langle \Psi_m^{(1)}(y_m)\right| V_{ml}(K_l)^{-1}\left|\Psi_l^{(k)}(y_l)\right\rangle \left\langle \Psi_l^{(k)}(y_l)\right|(K_l)^{-1} V_{lm}\left|\Psi_m^{(1)}(y_m)\right\rangle\right.\\
&\left.\qquad - \left\langle \Psi_m^{(1)}(y_m)\right| V_{ml}(K_l)^{-1}\left|\Psi_l^{(0)}(y_l)\right\rangle \left\langle \Psi_l^{(0)}(y_l)\right|(K_l)^{-1} V_{lm}\left|\Psi_m^{(1)}(y_m)\right\rangle\right)\\
=&\sum_{l,m}\left(\sum_{q\neq 0} \left\langle \Psi_m^{(1)}(y_m)\right| V_{ml}(K_l)^{-1}\left|\Psi_l^{(k)}(y_l)\right\rangle \left\langle \Psi_l^{(k)}(y_l)\right|(K_l)^{-1} V_{lm}\left|\Psi_m^{(1)}(y_m)\right\rangle\right)
\end{aligned}$$

and given our hypothesis of a potential which is mainly non nul around the $\left|\Psi_l^{(0)}(y_l)\right\rangle$

$$\sum_{l,m}\sum_{q\neq 0} \left\langle \Psi_m^{(1)}(y_m)\right| V_{ml}(K_l)^{-1}\left|\Psi_l^{(k)}(y_l)\right\rangle \left\langle \Psi_l^{(k)}(y_l)\right|(K_l)^{-1} V_{lm}\left|\Psi_m^{(1)}(y_m)\right\rangle < c$$

for a certain constant depending on V. Then

$$\frac{k\lambda_{0,\sup}^2}{V_0}\sum_{l,m}\left(\left\langle \Psi_m^{(1)}(y_m)\right| V_{ml}(K_l)^{-2} V_{lm}\left|\Psi_m^{(1)}(y_m)\right\rangle - \left\langle \Psi_m^{(1)}(y_m)\right| V_{ml}(K_l)^{-1}\left|\Psi_l^{(0)}(y_l)\right\rangle \left\langle \Psi_l^{(0)}(y_l)\right|(K_l)^{-1} V_{lm}\left|\Psi_m^{(1)}(y_m)\right\rangle\right) \cdot$$

and $\frac{ck\lambda_{0,\sup}^2}{V_0} < 1$ is realized for $V_0 >> ck\lambda_{0,\sup}^2$.

Once a saddle point is found, the stability is studied through the second order variation:

$$\sum_{l,m} b_l b_m^* \left\langle \varphi_m\right| \frac{\partial^2 S}{\partial \Psi_l^{(1)}(x)\,\partial \Psi_m^{(1)\dagger}(y)}\left|\varphi_l\right\rangle$$

where, in $\sum_l b_l \left|\varphi_l\right\rangle$, the $\left|\varphi_l\right\rangle$ are normalized to 1, as well as $\sum_l b_l \left|\varphi_l\right\rangle$. Given that the potential V can be considered as being symetric with respect to the identical copies of the structure with coordinates (x_l), one obtains:

$$\begin{aligned}
&\sum_{l,m} b_l b_m^* \langle \varphi_m| \frac{\partial^2 S}{\partial \Psi_l^{(1)}(x_l)\,\partial \Psi_m^{(1)\dagger}(y_m)} |\varphi_l\rangle \\
= \quad &\sum_l |b_l|^2 \langle \varphi_l| K |\varphi_l\rangle + p_l \frac{\left(\prod_{i=1}^k (\eta_i)^{p_i}\right)}{\eta_l} \langle \varphi_l(x_l)| \left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... [\Psi_l(x_l)]_{p_l-1} ... [\Psi_k(x_k)]_{p_k} \right| \\
&\qquad V \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... [\Psi_l(x_l)]_{p_l-1} ... [\Psi_k(x_k)]_{p_k} \right\rangle |\varphi_l(x_l)\rangle \\
&+\sum_l |b_l|^2 p_l(p_l-1) \,\mathrm{Re}\, \frac{\left(\prod_{i=1}^k (\eta_i)^{p_i}\right)}{\eta_l} \langle \varphi_l(x_l)| \left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l^{(1)}(x_l)\right]_{p_l-1} ... \left[\Psi_k^{(1)}(x_k)\right]_{p_k} \right| \\
&\qquad V \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l^{(1)}(x_l)\right]_{p_l-1} ... \left[\Psi_k^{(1)}(x_k)\right]_{p_k} \right\rangle |\varphi_l(y_l)\rangle \\
&+2 \sum_{l,m,l\neq m} p_l p_m \frac{\left(\prod_{i=1}^k (\eta_i)^{p_i}\right)}{\sqrt{\eta_l \eta_m}} \mathrm{Re}\, \langle \varphi_l(x)| \left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l^{(1)}(x_l)\right]_{p_l-1} ... \left[\Psi_k^{(1)}(x_k)\right]_{p_k} \right| \\
&\qquad V \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_m^{(1)}(x_l)\right]_{p_m-1} ... \left[\Psi_k^{(1)}(x_k)\right]_{p_k} \right\rangle |\varphi_m(y)\rangle
\end{aligned}$$

The terms:

$$\langle \varphi_l(x_l)| \left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l^{(1)}(x_l)\right]_{p_l-1} ... \left[\Psi_k^{(1)}(x_k)\right]_{p_k} \right| V \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l^{(1)}(x_l)\right]_{p_l-2} ... \left[\Psi_k^{(1)}(x_k)\right]_{p_k} \right\rangle |\varphi_l(y_l)\rangle$$

$$\mathrm{Re}\, \langle \varphi_l(x)| \left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l^{(1)}(x_l)\right]_{p_l-1} ... \left[\Psi_k^{(1)}(x_k)\right]_{p_k} \right| V \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_m^{(1)}(x_l)\right]_{p_m-1} ... \left[\Psi_k^{(1)}(x_k)\right]_{p_k} \right\rangle |\varphi_m(y)\rangle$$

represent the matrix element between two fields configurations, and this is assumed to be positive since we look for a binding interaction. This is satisfied for a potentil with separate variable, as the one designed in the one field case. Then:

$$\begin{aligned}
&\sum_{l,m} b_l b_m^* \langle \varphi_m| \frac{\partial^2 S}{\partial \Psi_l^{(1)}(x_l)\,\partial \Psi_m^{(1)\dagger}(y_m)} |\varphi_l\rangle \\
\geqslant \quad &\sum_l |b_l|^2 \langle \varphi_l| K |\varphi_l\rangle \delta_{lm} \\
&+\sum_l |b_l|^2 p_l \frac{\left(\prod_{i=1}^k (\eta_i)^{p_i}\right)}{\eta_l} \langle \varphi_l(x_l)| \\
&\left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... [\Psi_l(x_l)]_{p_l-1} ... [\Psi_k(x_k)]_{p_k} \right| V \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... [\Psi_l(x_l)]_{p_l-1} ... [\Psi_k(x_k)]_{p_k} \right\rangle |\varphi_l(x_l)\rangle
\end{aligned}$$

using (344), it reduces to:

$$\begin{aligned}
&\sum_{l,m} b_l b_m^* \langle\varphi_m| \frac{\partial^2 S}{\partial \Psi_l^{(1)}(x_l)\,\partial \Psi_m^{(1)\dagger}(y_m)} |\varphi_l\rangle \\
\geqslant\ & \sum_l |b_l|^2 \Bigg(\langle\varphi_l| K |\varphi_l\rangle \\
&-\frac{\langle\varphi_l(x_l)| \left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... [\Psi_l(x_l)]_{p_l-1} ... [\Psi_k(x_k)]_{p_k} \right| V \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... [\Psi_l(x_l)]_{p_l-1} ... [\Psi_k(x_k)]_{p_k} \right\rangle |\varphi_l(x_l)\rangle}{\left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... [\Psi_l(x_l)]_{p_l-1} ... [\Psi_k(x_k)]_{p_k} \right| V \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... [\Psi_l(x_l)]_{p_l-1} ... [\Psi_k(x_k)]_{p_k} \right\rangle} \\
&\qquad \times \left\langle \Psi_l^{(1)}(y_l) \right| K_l \left| \Psi_l^{(1)}(y_l) \right\rangle \Bigg)
\end{aligned}$$

Multiplying equation (345) on the left by $\left(\left\langle \Psi_l^{(1)}(y_l) \right|\right)$allows to write

$$k = \sum_l \frac{\left\langle \Psi_l^{(1)}(y_l) \right| K_l \left| \Psi_l^{(1)}(y_l) \right\rangle \left\langle \Psi_l^{(1)}(y_l) \right| K_l^{-1} V_l \left| \Psi_l^{(1)}(y_l) \right\rangle}{\left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... [\Psi_l(x_l)]_{p_l} ... [\Psi_k(x_k)]_{p_k} \right| V \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... [\Psi_l(x_l)]_{p_l} ... [\Psi_k(x_k)]_{p_k} \right\rangle}$$

and:

$$\begin{aligned}
&\sum_{l,m} b_l b_m^* \langle\varphi_m| \frac{\partial^2 S}{\partial \Psi_l^{(1)}(x_l)\,\partial \Psi_m^{(1)\dagger}(y_m)} |\varphi_l\rangle \\
\geqslant\ & \sum_l |b_l|^2 \Bigg(\langle\varphi_l| K |\varphi_l\rangle \\
&-\frac{\langle\varphi_l(x_l)| \left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... [\Psi_l(x_l)]_{p_l-1} ... [\Psi_k(x_k)]_{p_k} \right| V \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... [\Psi_l(x_l)]_{p_l-1} ... [\Psi_k(x_k)]_{p_k} \right\rangle |\varphi_l(x_l)\rangle}{k \sum_l \left\langle \Psi_l^{(1)}(y_l) \right| K_l \left| \Psi_l^{(1)}(y_l) \right\rangle \left\langle \Psi_l^{(1)}(y_l) \right| K_l^{-1} V_l \left| \Psi_l^{(1)}(y_l) \right\rangle} \\
&\qquad \times \left\langle \Psi_l^{(1)}(y_l) \right| K_l \left| \Psi_l^{(1)}(y_l) \right\rangle \Bigg) \lambda_{0,\sup} \\
&\qquad - \sum_l |b_l|^2 \frac{\left\langle \Psi_l^{(1)}(y_l) \right| K_l \left| \Psi_l^{(1)}(y_l) \right\rangle}{k \sum_l \left\langle \Psi_l^{(1)}(y_l) \right| K_l \left| \Psi_l^{(1)}(y_l) \right\rangle \left\langle \Psi_l^{(1)}(y_l) \right| K_l^{-1} V_l \left| \Psi_l^{(1)}(y_l) \right\rangle} U_0
\end{aligned}$$

Assuming as before that if some of the K_l have 0 as eigenvalue, this eigenvalue is an isolated point one obtains:

$$\begin{aligned}
&\sum_{l,m} b_l b_m^* \langle\varphi_m| \frac{\partial^2 S}{\partial \Psi_l^{(1)}(x_l)\,\partial \Psi_m^{(1)\dagger}(y_m)} |\varphi_l\rangle \\
\geqslant\ & \lambda_{0,\sup} - \frac{\lambda_{0,\inf}^2}{k \sum_l \lambda_{0,\sup} \left\langle \Psi_l^{(1)}(y_l) \right| V_l \left| \Psi_l^{(1)}(y_l) \right\rangle} U_0
\end{aligned}$$

where $\lambda_{0,\inf}$ is the closest to 0 negative eigenvalue of the operators K_l. Then the saddle point is a minimum for a large enough potential, set along the negative eigenvlues.

Note that a larger k makes stability more difficult to achieve. At this minimum one has:

$$\begin{aligned}
S(\Psi_l(y_l)) &= \sum_l (\eta_l) \left\langle \Psi_l^{(1)}(y_l) \right| K_l \left| \Psi_l^{(1)}(y_l) \right\rangle + \left(\prod_{i=1}^k (\eta_i)^{p_i} \right) \times \\
&\quad \times \left\langle \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l^{(1)}(x_l)\right]_{p_l} ... \left[\Psi_k^{(1)}(x_k)\right]_{p_k} \right| V \left| \left[\Psi_1^{(1)}(x_1)\right]_{p_1} ... \left[\Psi_l^{(1)}(x_l)\right]_{p_l} ... \left[\Psi_k^{(1)}(x_k)\right]_{p_k} \right\rangle
\end{aligned}$$

and using (344):

$$\begin{aligned}
S\left(\Psi_l\left(y_l\right)\right) &= -\left(p_l-1\right)\left(\prod_{i=1}^{k}\left(\eta_i\right)^{p_i}\right) \\
&\times\left\langle\left[\Psi_1^{(1)}\left(x_1\right)\right]_{p_1}...\left[\Psi_l^{(1)}\left(x_l\right)\right]_{p_l}...\left[\Psi_k^{(1)}\left(x_k\right)\right]_{p_k}\right|V\left|\left[\Psi_1^{(1)}\left(x_1\right)\right]_{p_1}...\left[\Psi_l^{(1)}\left(x_l\right)\right]_{p_l}...\left[\Psi_k^{(1)}\left(x_k\right)\right]_{p_k}\right\rangle \\
&\leqslant 0
\end{aligned}$$

Then, for $p_l=1$, the minimum is $S\left(\Psi_l\left(y_l\right)\right)=0$, and we have two states corresponding to this level, the saddle point solution $\Psi_l\left(y_l\right)$ and 0.

For $p_l\geqslant 2$, $S\left(\Psi_l\left(y_l\right)\right)<0$ and the non trivial saddle point $\Psi_l\left(y_l\right)$ is the only minimum.

Appendix 14

Effective action for the first field:

We start with effective action $S_{ef.}\left(\Psi_{i_1}\left(x_{i_1}\right)\right)$

$$S_{ef.}\left(\Psi_{i_1}\left(x_{i_1}\right)\right)=\eta S\left(\Psi_{i_1}\left(x_{i_1}\right)\right)+\sum_{n\leqslant N}\ln\left(1+\frac{\left(n+\frac{1}{2}\right)\delta\eta+\frac{\delta\Lambda_{i_2}\eta}{\Lambda_{i_2}+\delta\eta}\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}{\left(n+\frac{1}{2}\right)\Lambda_{i_2}+m_{i_2}^2}\right)$$

and study the possibility for a non nul minimum, i.e. a minimum with $\eta\neq 0$.

We first consider the case $\delta>0$.

If $\delta>0$ remark that, if $S\left(\Psi_{i_1}\left(x_{i_1}\right)\right)>0$, The function $S_{ef.}\left(\Psi_{i_1}\left(x_{i_1}\right)\right)$ is an increasing function in η and the only minimum of $S_{ef.}\left(\Psi_{i_1}\left(x_{i_1}\right)\right)$ is for $\eta=0$. Then if $\frac{1}{2}\left(\Lambda_{i_1}\right)+m_{i_1}^2>0$, the only solution is $\left\langle\Psi_{i_1}\left(x_{i_1}\right)\right\rangle=0$.

The case $S\left(\Psi_{i_1}\left(x_{i_1}\right)\right)<0$, requires $m_{i_1}^2<0$, so that one replaces $m_{i_2}^2\rightarrow -m_{i_2}^2$ with $m_{i_2}^2>0$.

Then, $S\left(\Psi_{i_1}\left(x_{i_1}\right)\right)<0$ implies that $\frac{1}{2}\left(\Lambda_{i_2}\right)-m_{i_1}^2<0$. The minimum for $S\left(\Psi_{i_1}\left(x_{i_1}\right)\right)$ is obtained if $\Psi_{i_1}\left(x_{i_1}\right)$ is in the fundamental state $\Psi_{i_1}^{(0)}\left(x_{i_1}\right)$ that is the eigenstate of (203) for $n=0$.

$$\begin{aligned}\Psi_{i_1}^{(0)}\left(x_{i_1}\right)&=\left(\frac{\sqrt{a}}{\pi}\right)^{\frac{1}{4}}H_0\left(a^{\frac{1}{4}}x\right)\exp\left(-\frac{\sqrt{a}}{2}x^2\right)\\&=\left(\frac{\sqrt{a}}{\pi}\right)^{\frac{1}{4}}\exp\left(-\frac{\sqrt{a}}{2}x^2\right)\end{aligned}$$

For $n=0$ one then has:

$$S\left(\Psi_{i_1}^{(0)}\left(x_{i_1}\right)\right)=\frac{1}{2}\left(\Lambda_{i_1}\right)-m_{i_1}^2$$

The derivative in η leads to:

$$\frac{\partial S_{ef.}\left(\Psi_{i_1}\left(x_{i_1}\right)\right)}{\partial\eta}=\frac{1}{2}\left(\Lambda_{i_1}\right)-m_{i_1}^2+\sum_{n\leqslant N}\frac{\left(n+\frac{1}{2}\right)\delta+\frac{\left(\Lambda_{i_2}\right)^2\delta}{\left(\Lambda_{i_2}+\delta\eta\right)^2}\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}{\left(n+\frac{1}{2}\right)\left(\Lambda_{i_2}+\delta\eta\right)+m_{i_2}^2+\frac{\delta\Lambda_{i_2}\eta}{\Lambda_{i_2}+\delta\eta}\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}$$

This is increasing for η close to 0 and decreasing for η large.

Then, one can find the conditions for a minimum with $\eta\neq 0$. Actually, since $\frac{\partial S_{ef.}\left(\Psi_{i_1}\left(x_{i_1}\right)\right)}{\partial\eta}\rightarrow\frac{1}{2}\left(\Lambda_{i_1}\right)-m_{i_1}^2<0$ for large η, then if $\frac{\partial S_{ef.}\left(\Psi_{i_1}\left(x_{i_1}\right)\right)}{\partial\eta}\mid_{\eta=0}<0$ and if there exists an $\eta_0>0$ such that $\frac{\partial S_{ef.}\left(\Psi_{i_1}\left(x_{i_1}\right)\right)}{\partial\eta}\mid_{\eta=\eta_0}>0$, then there is an $\eta_1\neq 0$ such that $S_{ef.}\left(\Psi_{i_1}\left(x_{i_1}\right)\right)$ is a minimum. In that case we have a phase transition $\left\langle\Psi_{i_1}\left(x_{i_1}\right)\right\rangle\neq 0$.

For $\delta>0$, the conditions for a phase transition are then:

$$\begin{aligned}\frac{1}{2}\left(\Lambda_{i_2}\right)-m_{i_1}^2&<0\\\frac{\partial S_{ef.}\left(\Psi_{i_1}\left(x_{i_1}\right)\right)}{\partial\eta}&\mid_{\eta=0}=\frac{1}{2}\left(\Lambda_{i_1}\right)-m_{i_1}^2+\delta\sum_{n\leqslant N}\frac{\left(n+\frac{1}{2}\right)+\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}{\left(n+\frac{1}{2}\right)\left(\Lambda_{i_2}\right)+m_{i_2}^2}<0\\\frac{\partial S_{ef.}\left(\Psi_{i_1}\left(x_{i_1}\right)\right)}{\partial\eta}&\mid_{\eta=\eta_0}=\frac{1}{2}\left(\Lambda_{i_1}\right)-m_{i_1}^2+\sum_{n\leqslant N}\frac{\left(n+\frac{1}{2}\right)\delta+\frac{\left(\Lambda_{i_2}\right)^2\delta}{\left(\Lambda_{i_2}+\delta\eta_0\right)^2}\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}{\left(n+\frac{1}{2}\right)\left(\Lambda_{i_2}+\delta\eta_0\right)+m_{i_2}^2+\frac{\delta\Lambda_{i_2}\eta_0}{\Lambda_{i_2}+\delta\eta_0}\left(\left(\check{Y}_{eff}\right)_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2}>0\end{aligned}$$

The case $\delta < 0$ is studied in a similar way.

If $\frac{1}{2}(\Lambda_{i_1})+m_{i_1}^2 < 0$ (that is $m_{i_1}^2$ is negative), $\frac{\partial S_{ef.}(\Psi_{i_1}(x_{i_1}))}{\partial \eta} < 0$, there is no minimum, the action decreases with η which is the (squared) norm of $\Psi_{i_1}(x_{i_1})$. That case means that $S_{ef.}(\Psi_{i_1}(x_{i_1}))$ is unbounded from below, which is meaningless. The model breaks out for this values of the parameters and this case has to be ruled out.

If $\frac{1}{2}(\Lambda_{i_1})+m_{i_1}^2 > 0$, then

$$\frac{\partial S_{ef.}(\Psi_{i_1}(x_{i_1}))}{\partial \eta} = \frac{1}{2}(\Lambda_{i_1})+m_{i_1}^2 + \sum_{n\leqslant N} \frac{\left(n+\frac{1}{2}\right)\delta + \frac{(\Lambda_{i_2})^2\delta}{(\Lambda_{i_2}+\delta\eta)^2}\left(\left(\check{Y}_{eff}\right)_{i_2} - \hat{x}_{i_2}^{(i_1)}\right)^2}{\left(n+\frac{1}{2}\right)(\Lambda_{i_2}+\delta\eta)+m_{i_2}^2 + \frac{\delta\Lambda_{i_2}\eta}{\Lambda_{i_2}+\delta\eta}\left(\left(\check{Y}_{eff}\right)_{i_2} - \hat{x}_{i_2}^{(i_1)}\right)^2}$$

is increasing for η close to 0 and decreasing for η large. Since $\frac{\partial S_{ef.}(\Psi_{i_1}(x_{i_1}))}{\partial \eta} \rightarrow \frac{1}{2}(\Lambda_{i_1})-m_{i_1}^2 > 0$ for large η, then $\frac{\partial S_{ef.}(\Psi_{i_1}(x_{i_1}))}{\partial \eta}|_{\eta=0} > 0$ is the condition for a solution $\eta_1 \neq 0$ to $\frac{\partial S_{ef.}(\Psi_{i_1}(x_{i_1}))}{\partial \eta} = 0$. In other words: If

$$\begin{aligned} \frac{1}{2}(\Lambda_{i_1})+m_{i_1}^2 &> 0 \\ \frac{1}{2}(\Lambda_{i_1})+m_{i_1}^2 + \delta\sum_{n\leqslant N} \frac{\left(n+\frac{1}{2}\right)+\left(\left(\check{Y}_{eff}\right)_{i_2} - \hat{x}_{i_2}^{(i_1)}\right)^2}{\left(n+\frac{1}{2}\right)\Lambda_{i_2}+m_{i_2}^2} &< 0 \end{aligned}$$

then $\frac{\partial S_{ef.}(\Psi_{i_1}(x_{i_1}))}{\partial \eta}$ is nul for a value $\eta \neq 0$, and this value correspond to the minimum of $S_{ef.}(\Psi_{i_1}(x_{i_1}))$.

In that case, there is a phase transition $\langle \Psi_{i_1}(x_{i_1})\rangle \neq 0$.

if, on the contrary

$$\frac{1}{2}(\Lambda_{i_1})-m_{i_1}^2 + \delta\sum_{n\leqslant N} \frac{\left(n+\frac{1}{2}\right)+\left(\left(\check{Y}_{eff}\right)_{i_2} - \hat{x}_{i_2}^{(i_1)}\right)^2}{\left(n+\frac{1}{2}\right)\Lambda_{i_2}+m_{i_2}^2} > 0$$

the minimum is for $\langle \Psi_{i_1}(x_{i_1})\rangle = 0$.

Effective action for the second field:

As explained in the core of the text, the integration of the action for the first agent yields the effective action for the second one:

$$S_{ef.}(\Psi_{i_2}(x_{i_2})) = \eta S(\Psi_{i_2}(x_{i_2})) + \sum_{n\leq N} \ln\left(1+\eta\delta\frac{\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2 \Psi_{i_2}(x_{i_2})\Psi_{i_2}^{\dagger}(x_{i_2})}{\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2}\right) \tag{350}$$

Remark first that for $\delta > 0$ and $\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2 < 0$ or $\delta < 0$ and $\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2 > 0$ one can find $\eta > 0$ such that, whatever $\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2 \Psi_{i_2}(x_{i_2})\Psi_{i_2}^{\dagger}(x_{i_2})$ one has:

$$1+\eta\delta\frac{\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2 \Psi_{i_2}(x_{i_2})\Psi_{i_2}^{\dagger}(x_{i_2})}{\frac{1}{2}\Lambda_{i_1}+m_{i_1}^2} \rightarrow 0^+$$

Thus, being unbouded from below, the model breaks down ($S_{ef.}(\Psi_{i_2}(x_{i_2}))$ being unbouded, one cannot define a probability $\exp(-S_{ef.}(\Psi_{i_2}(x_{i_2})))$).

In other words, for $\delta\left(\frac{1}{2}\Lambda_{i_1}+m_{i_1}^2\right)<0$, it is not possible to define an effective action for $\Psi_{i_2}\left(x_{i_2}\right)$.

For $\delta>0$ and $\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2>0$, the first order condition for η is:

$$S\left(\Psi_{i_2}\left(x_{i_2}\right)\right)+\sum_{n\leq N}\frac{\delta\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\Psi_{i_2}\left(x_{i_2}\right)\Psi_{i_2}^{\dagger}\left(x_{i_2}\right)}{\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2+\eta\delta\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\Psi_{i_2}\left(x_{i_2}\right)\Psi_{i_2}^{\dagger}\left(x_{i_2}\right)}=0 \tag{351}$$

If $\frac{1}{2}\Lambda_{i_2}+m_{i_2}^2>0$, then

$$\begin{aligned}
S\left(\Psi_{i_2}\left(x_{i_2}\right)\right) &\geqslant 0\\
\delta\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\Psi_{i_2}\left(x_{i_2}\right)\Psi_{i_2}^{\dagger}\left(x_{i_2}\right) &\geqslant 0\\
S\left(\Psi_{i_2}\left(x_{i_2}\right)\right)+\sum_{n\leq N}\frac{\delta\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\Psi_{i_2}\left(x_{i_2}\right)\Psi_{i_2}^{\dagger}\left(x_{i_2}\right)}{\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2+\eta\delta\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\Psi_{i_2}\left(x_{i_2}\right)\Psi_{i_2}^{\dagger}\left(x_{i_2}\right)} &> 0
\end{aligned}$$

There is no solution to

$$\frac{\partial}{\partial\eta}S_{ef.}\left(\Psi_{i_2}\left(x_{i_2}\right)\right)$$

and this derivative is positive. As a consequence, the minimum for $S_{ef.}\left(\Psi_{i_2}\left(x_{i_2}\right)\right)$ is reached at $\eta=0$.

If $\frac{1}{2}\Lambda_{i_2}+m_{i_2}^2<0$, (351) may have a solution, but in that case, the second derivative $\frac{\partial}{\partial\eta}S_{ef.}\left(\Psi_{i_2}\left(x_{i_2}\right)\right)$ is negative, the extremum is thus a maximum, and the minimum for $S_{ef.}\left(\Psi_{i_2}\left(x_{i_2}\right)\right)$ is reached at $\eta=0$.

The case $\delta<0$ and $\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2<0$ is treated similarly: The first order condition can be written

$$S\left(\Psi_{i_2}\left(x_{i_2}\right)\right)+\sum_{n\leq N}\frac{(-\delta)\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\Psi_{i_2}\left(x_{i_2}\right)\Psi_{i_2}^{\dagger}\left(x_{i_2}\right)}{-\left(\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2\right)-\eta\delta\int dx_{i_2}\left(x_{i_2}-\hat{x}_{i_2}^{(i_1)}\right)^2\Psi_{i_2}\left(x_{i_2}\right)\Psi_{i_2}^{\dagger}\left(x_{i_2}\right)}=0 \tag{352}$$

and we come back to the case $\delta>0$ and $\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2>0$ by the change of variable:

$$\delta\rightarrow-\delta,\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2\rightarrow-\left(\left(n+\frac{1}{2}\right)\Lambda_{i_1}+m_{i_1}^2\right)$$

Then, gathering all the results of this paragraph, the only vacuum of $S_{ef.}\left(\Psi_{i_2}\left(x_{i_2}\right)\right)$ is $\eta=0$, as announced in the text.