Auctions with Signaling Concerns

Olivier Bos and Tom Truyts

Panthéon-Assas University, CEREC, Saint-Louis University - Brussels

2017

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Olivier Bos[†] and Tom Truyts[‡]

Abstract

We study a symmetric private value auction with signaling, in which the auction outcome is used by an outside observer to infer the bidders' types. We elicit conditions under which an essentially unique D1 equilibrium bidding function exists in four auction formats: first-price, second-price, all-pay and the English auction. We obtain a strict ranking in terms of expected revenues: the first-price and all-pay auctions dominate the English auction but are dominated by the second-price auction.

JEL: D44; D82

Keywords: Costly signaling; D1 criterion; social status; art auctions; charity auctions.

*We are grateful to Claude d'Aspremont, Francis Bloch, Damian Damianov, David Ettinger, Françoise Forges, Laurent Lamy, Michel Le Breton, Sander Onderstal, Régis Renault, Jean-Marc Tallon and seminar and conference participants in Brussels, Caen, Louvain-la-Neuve, Paris, Lisbon, Aarhus, Stony Brook, Maastricht, Warsaw, Sankt-Gallen. Olivier Bos gratefully acknowledges financial support from the French National Research Agency (ANR) through the ANR DCPG grant. Tom Truyts gratefully aknowledges financial support from the Belgian French speaking community ARC project n°15/20-072, "Social and Economic Network Formation under Limited Farsightedness: Theory and Applications", Université Saint-Louis - Bruxelles, October 2015 - September 2020.

†Panthéon-Assas University, LEMMA, 4 rue Blaise Desgoffe, 75006 Paris, France. E-mail: olivier.bos@u-paris2.fr, tel.: +33170232005.

‡CEREC, Saint-Louis University - Brussels, Boulevard du Jardin botanique 43, 1000 Brussels, Belgium; Center for Economic Studies, University of Leuven; CORE, Université Catholique de Louvain. E-mail: tom.truyts@usaintlouis.be, tel.: +3222117845.

1 Introduction

Signaling constitutes an important concern in many publicly observable choices of individuals and organizations. Humans tend to care about what others think about them, either because of innate tastes, as humans intrinsically care about others' esteem, or by instrumental reasons, as a higher status often gives access to better mates, partners or resources.[1] Firms care about how other market parties perceive them, because this affects their access to capital and business opportunities. Signaling is also documented to matter in auctions, in art or charity auctions as well as in a more generic industrial or financial context.

Mandel (2009) distinguishes three main motives for buying art: investment, direct consumption and signaling. While art serves as an investment, owners can also enjoy its aesthetic qualities and the prestige derived from showing it to friends and acquaintances. Mandel (2009) suggests that the two latter motives explain an old puzzle: why art systematically seems to underperform as an investment compared to bonds and equity, especially when taking the high variance of its yields into account. The underperformance of art is particularly important for famous masterpieces (Mei and Moses, 2002), which likely have a greater signaling value. Similarly, charities often raise funds by auctioning objects provided to them by celebrities. In recent years, an extensive literature has analyzed charity auctions as auctions in which bidders' preferences are altruistic.[2] However, the predictions of these theoretical contributions were invalidated in a field experiment (Carpenter et al. 2005), while a broad theoretical and empirical literature suggests that signaling and status are important motives for contributions to charities.[3] Moreover,

[1] See Frank (1985, 1999) for a broad introduction to social status in economics, Miller (2000) for an introduction to the biological roots of status concerns, Mason (1998) for a history of economic thought w.r.t. status concerns and Truyts (2010) for a recent survey of the literature.
Cole, Mailath and Postlewaite (1992) derive preferences for status from a two-sided one-to-one matching problem. If the equilibrium matching is assortative, one must appear more attractive than one's peers to secure the best attainable partner.

[2] See for example Engers and McManus (2007) and Goeree et al. (2005).

[3] Glazer and Konrad (1996) and Harbaugh (1998a,b) show that signaling is an important explanation for observed patterns in donations to universities. Kumru and Vesterlund (2010) find that donations are significantly higher if the charity first

the very mechanism of auctioning celebrities' belongings seems to exploit signaling motives. Where altruistic donators can get a warm glow from discreet contributions, it takes a unique object and a public event to make donators shine who (also) care about their public image. Finally, the public behavior of any sizable firm is under constant scrutiny by market analysts and other market parties. A firm's performance in an auction, irrespective of whether it won or lost the auction, is informative to outsiders trying to judge e.g. its profitability, financial situation, strategy or management quality. In these auctions, bidders not only care about their payment and about winning the object, but also about what the auction outcome reveals about their type to outsiders. These inferences about the individual qualities of a bidder depend on the outcome and format of the auction, and in turn affect the equilibrium bidding strategies and thus the outcome of the auction.

We study a symmetric independent private value auction with signaling. A single and indivisible commodity is allocated by means of an auction to the one out of n bidders who submits the highest bid. Each bidder independently draws a private valuation for the auctioned object according to the same distribution and this valuation is her private information. The bidders' payoffs consist of a standard and a signaling component. As in the standard auction model, a winner's *ex post* payoff equals her private valuation for the object minus her payment and a loser's payoff is minus her payment (which is non-zero in the all-pay auction). In addition, we assume that each bidder also cares about the beliefs of an outside party, the receiver, about her type. The receiver is assumed to observe and use the auction outcome, *in casu* the identity and payment of the auction's winner, to form beliefs about the private valuation of all bidders. We study how this form of signaling affects the bidding behavior and auction outcome. How does the payment rule affect the inferences by the receiver and thereby the bidding strategies? Does expected revenue equivalence still apply, or can we strictly rank different auction formats in terms of expected revenues?

collects from high status sponsors because donators like to be associated with higher status groups.

Note that, in general, a bidder's private valuation can reflect e.g. purchasing power, a taste for art, generosity, profitability prospects, expected market penetration or a combination of such factors. In order to keep the model as simple and generic as possible, we disregard how these qualities map into a private valuation, and how the receiver seeks to reverse this mapping to form beliefs about these ultimate qualities from the auction outcome.

Because of the combination of a signaling game and an auction in a single game, such that the beliefs of the receiver directly enter the payoff function of the bidders, a general mechanism design approach to this problem is beyond the current state of the art. For this reason, we analyze the implications of signaling in four well-known auction formats: the first-price auction, the second-price auction, the all-pay auction and the English auction. Auctions with signaling inherit the usual equilibrium multiplicity of signaling games, due to a lack of restrictions on out-of-equilibrium beliefs. Therefore, we restrict out-of-equilibrium beliefs by means of the D1 criterion of Banks and Sobel (1987). The D1 criterion is the most common way of restricting out-of-equilibrium beliefs in signaling games with multiple types, and imposes a monotonicity on out-of-equilibrium beliefs: an out-of-equilibrium bid b is never attributed to a certain bidder type if a higher type bids in equilibrium less than b. We show that only fully separating equilibria survive the D1 criterion if the density function characterizing the *ex ante* distribution of bidders' types is non-increasing. We elicit conditions for the existence of an essentially unique D1 equilibrium bidding function in these four auctions formats, and we show that for a finite number of bidders, the first-price and all-pay auctions outperform the English auction in terms of expected revenues, but are in turn outperformed by the second-price auction. This strict revenue ranking is due to the different amounts of information available to the receiver and the bidders in the different auction formats. In a fully separating equilibrium, the winner's payment allows the receiver to pinpoint the winner's type in the first-price and all-pay auctions, but only imposes a lower bound on the receiver's beliefs about the winner's type in the second-price and English auction.

This will be shown to boost the equilibrium bids of the lowest types in the second-price and English auctions, compared to the other formats, and can induce the highest types to bid strictly more in the first-price auction than in the second-price and English auctions. In addition, the presence of an increasing price(-clock) in the English auction weakens bidders' incentives to overbid their true valuation for signaling purposes, compared to the sealed-bid auctions.

Signaling in auctions had been studied by at least two strains of theoretical literature. The closest to our analysis are models of information transmission in auctions in function of an aftermarket. Goeree (2003) studies oligopolists' bidding for a single-license patent on a cost reducing technology. Each oligopolist has private information about the cost reduction which winning the patent would imply for her firm and other oligopolists try to infer the winner's production cost reduction from the auction outcome to determine their strategies in the aftermarket Cournot competition game. However, Goeree's setting and results are in several respects different from ours. First, Goeree (2003) assumes that the receiver (the other oligopolists) observes the winner's bid, rather than her payment. Second, the structure of the aftermarket imposes a single crossing condition on the winning bidder's payoffs in Goeree (2003): higher types benefit more from an improvement in the receiver's beliefs. And third, in Goeree (2003) only the winning bidder cares about the receiver's inference. Together, these three assumptions imply that Goeree obtains an expected revenue equivalence of the first-price, second-price and English auctions.

Giovannoni and Makris' (2014) paper is probably the closest to the present analysis. They study takeovers by a standard IPV auction. A firm's private valuation is interpreted by a post-auction jobmarket for managers as a signal of its manager's ability to extract revenue from an acquisition. Giovannoni and Makris focus on 4 different information treatments: all bids revealed, no bids revealed, the highest bid revealed and the second highest bid revealed, and assume that the receiver observes in addition whether a firm has lost or won the auction. They elicit conditions under which these information treatments can be ranked in terms of expected revenues for a large class of auctions. However, the present analysis differs from Giovannoni and Makris' in that we focus on different auction formats, in our treatment of the outside observer's beliefs, in that we demonstrate the uniqueness of a D1 equilibrium, and in terms of the presented results and intuitions. Unlike Giovannoni and Makris, we obtain a strict revenue ranking of different auction formats with the same information treatment.

In a setting similar to Goeree (2003), Katzman and Rhodes-Kropf (2008) show that the auctioneer's announcement policy of bids can change the auction's revenue and efficiency, while Das Varma (2003) elicits conditions for equilibrium existence for a first-price auction with an aftermarket with linear demand functions and Cournot or Bertrand competition. Liu (2012) analyzes takeovers through an ascending auction, and shows how the winner's bidding strategy can signal the firm's post-takeover value to the market. Marinovic (2014) studies how different accounting rules affect the seller and bidders' incentives to signal their value in a first-price auction asset acquisition context. Molnar and Virag (2008) show how the shape of the bidder's profit function w.r.t. the outside observer's inferences affects an auctioneer's revenue maximizing information treatment. Haile (2003) studies how bidders' incentives for signaling their type in function of a resale auction depend on the auction formats and information assumptions.

A second strain of literature studies signaling to other bidders in dynamic auctions. Avery (1998) shows that bidders may use 'jump bids' in the English auction to signal a high valuation in order to scare away competing bidders, thus decreasing the auction's expected revenue and breaking expected revenue equivalence. Hörner and Sahuguet (2007) compare in a dynamic auction context jump bids and cautious bids as strategic signals about private valuation towards other bidders. Finally, this paper studies how an auctioneer can extract different rents from agents seeking to signal their type to an uninformed party through the auction mechanism, and thus relates to a larger literature about the supply of signaling mechanisms by a profit maximizing monopolist (see e.g. Rayo (2013)).

The paper is organized as follows. Section 2 introduces the formal setting and equilibrium concept. Sections 3, 4 and 5 respectively characterize the D1 perfect Bayesian equilibrium of the first-price and all-pay auctions, the second-price auction and the English auction. The expected revenues of these auctions are compared in Section 6. Section 7 concludes. All proofs are collected in Appendix.

2 Formal Setting

Consider n bidders, indexed i, competing for a single object which is allocated through an auction to the highest bidder. Bidder i's valuation for the object (her 'type'), is denoted V_i, and is assumed i.i.d. and drawn according to a C^2 distribution function F with support on $[\underline{v}, \bar{v}] \subset \mathbb{R}_+$. Let $f \equiv F'$ denote the density function. Bidder i's realization of V_i, denoted v_i, is her private information, but the number of bidders and the distribution F are common knowledge.

To participate in the auction, a bidder submits a non-negative bid. As all bidders share the same beliefs about the other bidder's valuations, they are assumed to follow a symmetric bidding strategy $\beta : [\underline{v}, \bar{v}] \to \mathbb{R}_+$.[4] Let $\boldsymbol{b} = \boldsymbol{\beta}(\boldsymbol{v})$ denote the vector of bids given a vector of valuations $\boldsymbol{v}$, with b_i the effective bid of $i-$th bidder. An auction mechanism maps a vector of bids $\boldsymbol{b}$ to a winner, denoted i^*, and vector of payments $\boldsymbol{p}$. We assume a fair tie breaking in case of multiple highest bids.[5]

Besides the auction's outcome, bidders also care about the beliefs of an uninformed party, the receiver, about their type. This receiver can represent e.g. the general public or press, business contacts or acquaintances of the bidder or experts related to the object sale. The receiver is assumed to observe the auction's winner and her payment (i^*, p_{i^*}). This either represents a scenario in which the winner and her payment are reported in media outlets, or it reflects a distinction between a payment being 'hard' verifiable evidence, and claims of bids being 'soft' informa-

[4] We denote the bidding strategy in any auction format by β, and only add an additional superscript to specify the auction format when comparing bidding functions of different auction formats for an expected revenue comparison.

[5] That is, for all $i \in \{j | b_j = \max \boldsymbol{b}\}$ we have $\Pr(i = i^*) = \frac{1}{|\{j|b_j = \max \boldsymbol{b}\}|}$.

tion, which is difficult to verify. We keep this information assumption constant throughout the different auction formats to ensure comparability.[6] The receiver's beliefs, denoted μ, are a probability distribution over the type space, such that $\mu_i\left(v|\left(i^*,p_{i^*}\right)\right)$ is the probability of bidder i being of valuation type v given $\left(i^*,p_{i^*}\right)$. Let $\mu\left(\mathbf{v}|\left(i^*,p_{i^*}\right)\right)$ then be a probability distribution over vectors of valuations $\mathbf{v}$ given $\left(i^*,p_{i^*}\right)$. The receiver's beliefs are (Bayesian) consistent with a bidding strategy β if

$$\mu\left(\boldsymbol{v}|\left(i^*,p\right)\right)=\frac{\Pr\left(i^*,p_{i^*}|\boldsymbol{\beta}\left(\boldsymbol{v}\right)\right)\prod_i f\left(v_i\right)}{\int\Pr\left(i^*,p_{i^*}|\boldsymbol{\beta}\left(\boldsymbol{v'}\right)\right)\prod_i f\left(v_i'\right)\boldsymbol{dv'}}.^7 \tag{1}$$

The utility of bidder i, given an auction outcome $\left(i^*,\mathbf{p}\right)$, consists of two parts. The first part is standard: the valuation for the object for the winner of the auction, minus the payment (which can be nonzero for all bidders in the all-pay auction). The second part is the expected value of the receiver's beliefs about bidder i's type given $\left(i^*,p_{i^*}\right)$, denoted $E\left(V_i|\mu_i\left(V_i|i^*,p_{i^*}\right)\right)$:

$$u_i(v_i,p_i|\mu_i)=\begin{cases} v_i-p_i+E\left(V_i|\mu_i\left(V_i|i^*,p_{i^*}\right)\right) & \text{for winner } i=i^* \\ -p_i+E\left(V_i|\mu_i\left(V_i|i^*,p_{i^*}\right)\right) & \text{for loser } i\neq i^* \end{cases}$$

This utility function either represents a psychological game, in which bidders care directly about the receiver's beliefs, as humans care about the good opinion of others, or is reduced form of a game in which the receiver chooses an action given her beliefs, while the bidders care about this action.[8] In the latter case, an explicit analysis of the receiver's prob-

[6] In the all-pay auction in particular, an alternative information treatment in which all payments (and thus all bids) are revealed seems equally plausible, especially in light of the above distinction between 'hard' and 'soft' information. However, we prefer to maintain a single information treatment to ensure maximal comparability of equilibrium bidding and expected revenues in the different auction formats.

[7] Note that then $\mu_i\left(v|\left(i^*,p_{i^*}\right)\right)=\int_{\{\mathbf{v}|v_i=v\}}\mu\left(\mathbf{v}|\left(i^*,p_{i^*}\right)\right)d\mathbf{v}$

[8] It can strike readers as counterintuitive that losing bidders seem to win something in terms of the receiver's inference. However, this is only true if non-participants receive payoff zero. Rather, we assume that the receiver always forms beliefs about the bidders. Under exogenous participation, a non-participating bidder obtains payoff $E\left(V\right)$, because the auction reveals no information about her. In this case, losing bidders lose in equilibrium compared to their non-participation payoff. We return to this issue in Section 7, where we discuss endogenous entry.

lem is easily integrated, but does not add much. Although somewhat restrictive, this linear payoff structure is the most natural benchmark case to study the role of signaling in auctions, because it guarantees a tractable solution and ensures that the auction formats under consideration are expected revenue equivalent without signaling.[9]

We study the symmetric perfect Bayesian equilibria (PBE) of this auction game with signaling. A PBE is then described by a pair bidding strategy and beliefs (β, μ) such that:

1. The bidding function β maximizes expected utility for all v, given that all other bidders play β and given the receiver's beliefs μ

2. The receiver's beliefs μ are Bayesian consistent with the bidding function β, as in (1).

Because this equilibrium concept imposes no restrictions on out-of-equilibrium beliefs, i.e. how the receiver interprets auction outcomes which should never occur on the equilibrium path, we face the usual equilibrium multiplicity of signaling games. Therefore, we use the D1 criterion of Banks and Sobel (1987), which refines the set of equilibria by restricting out-of-equilibrium beliefs. The D1 criterion restricts out-of-equilibrium beliefs by considering which bidder types are more likely to gain from an out-of-equilibrium bid, compared to their equilibrium expected utility. More precisely, if the set of beliefs for which a bidder gains from a deviation to an out-of equilibrium bid b (w.r.t. her equilibrium expected utility) is larger for one bidder type than for another, then the

[9] Note that Goeree (2003) can allow for a more general implicit payoff function. However, the receiver's equilibrium beliefs are necessarily degenerate in his setting. In the present setting with nondegenerate equilibrium beliefs, such payoff functions entail far more complications and a loss of some tractable solutions and results. Moreover, the main inuitions are more clearly presented with a linear structure. One can also conceive a payoff function

$$\begin{cases} v_i - p_i + \gamma E\left(V_i | \mu_i\left(V_i | i^*, p_{i^*}\right)\right) & \text{for winner } i = i^* \\ -p_i + \gamma E\left(V_i | \mu_i\left(V_i | i^*, p_{i^*}\right)\right) & \text{for loser } i \neq i^*, \end{cases}$$

with $0 < \gamma < \infty$ measuring the relative importance of signaling. This would not change our results qualitatively, but merely complicate the analysis.

D1 criterion requires out-of-equilibrium beliefs to attribute zero probability the latter type having deviated to b.[10] The D1 criterion imposes a certain monotonicity on out-of-equilibrium beliefs, which excludes many implausible equilibria, as illustrated below.

3 First-price and all-pay auctions

In this Section, we derive the essentially unique D1 perfect Bayesian equilibrium bidding strategies for the first-price auction and all-pay auction. In the first-price auction, the winner pays her own bid. Because the receiver observes the identity of the winner and her payment, she observes the winner's bid. Thus, if $\beta'(.) > 0$, the winner's type is fully revealed in equilibrium. The receiver is not able to distinguish among the different losers. The following simple example demonstrates that without imposing the D1 criterion, a multiplicity of equilibria can be supported by often implausible out-of-equilibrium beliefs.

Example 1 (Zero revenue auction) *Let $n = 2$ and F the uniform distribution on $[0,1]$. Then a PBE exists in which all bidder types bid zero, $\beta(.) = 0$, and $\mu_{i.}(v|(.,0)) = 1$ for all v and i, while for any $p_{i^*} > 0$ beliefs about the winner are degenerate at $v = 0$, i.e. $\mu_{i^*}(v'|(i^*, p_{i^*})) = 0$ for all $v' > 0$. In this case, the expected utility of a v type in equilibrium is $\frac{v}{2} + \frac{1}{2}$, i.e. both winner and loser are inferred by the receiver as $E(V) = \frac{1}{2}$, and both bidders win the auction with a probability of $\frac{1}{2}$. A bidder deviating to a bid $\varepsilon > 0$ wins with certainty, pays ε and is inferred as a zero valuation type, which implies expected utility $v - \varepsilon$, which is strictly*

[10] As outlined in Appendix, the exact formal implementation of the D1 criterion depends on the auction format. Formally, for types v', v'' and out-of-equilibrium message m, beliefs μ, a utility function $u(m, \mu|v)$ and equilibrium utility levels $u^*(v)$, define the following two sets of beliefs which make a type v sending m resp. strictly better off than in equilibrium and equally well off as in equilibrium:

$$M^+(m,v) = \{\mu | u(m,\mu|v) > u^*(v)\}$$
$$M^0(m,v) = \{\mu | u(m,\mu|v) = u^*(v)\}.$$

Then the D1-criterion requires

$$M^+(m,v') \cup M^0(m,v') \subset M^+(m,v'') \Longrightarrow \mu(v'|m) = 0.$$

below $\frac{v}{2}+\frac{1}{2}$ for all $v \in [0,1]$. As no bidder makes a strictly positive bid, the illustrated beliefs are consistent with the PBE bidding strategies.

Note that, among the commonly used equilibrium selection criteria for signaling games, the less restrictive Intuitive criterion of Cho and Kreps (1987) cannot rule out the (admittedly extreme) equilibrium in example 1.[11] This motivates our use of the D1 criterion to restrict out-of-equilibrium beliefs. Although the D1 criterion typically excludes (semi)pooling PBE in monotonic signaling games at the one hand, and although (semi)pooling strategies are normally easily excluded in canonical auction games with the present preference structure at the other hand, the exercise of excluding (semi)pooling equilibria by means of the D1 criterion is less obvious when both games are combined into an auction with signaling. The reason is that bidders cannot be excluded to bid above their valuation for the object (and typically do so in equilibrium). As usual, the D1 criterion ensures that the receiver puts zero probability on all types lower than the maximal type in a pool when observing a bid marginally above the common bid in this pool. In monotonic signaling games this implies that a marginal increase above the pool's signal is rewarded by a discrete jump in terms of inference by the receiver, which immediately excludes (semi)pooling equilibria. In the present setting, however, such a marginal increase in bid also increases a deviating bidder's chances of winning the auction, and thereby her expected payment, by a discrete amount.

In what follows, we restrict F to be concave, i.e. $f'(.) \leq 0$. This condition is (amply) sufficient to ensure that the D1 criterion has enough

[11] The Intuitive criterion requires that no bidder type can gain from deviating to an out-of-equilibrium bid for all possible beliefs that assign zero probability to types who can never gain from such a deviation (w.r.t. their equilibrium payoff), i.e. not for any beliefs by the receiver. In example 1, first, bidders with a valuation strictly lower than $\hat{v}(b) = 2b - 1$ can never gain from a deviation to out-of-equilibrium bid $b \in \left[\frac{1}{2}, 1\right]$, and are excluded from the support of out-of-equilibrium beliefs. Second, in this case, no bidder type can gain from a deviation to b for all remaining out-of-equilibrium beliefs. The worst out-of-equilibrium belief puts all probability mass on type $\hat{v}(b)$, such that a deviator gets $v + 2b - 1 - b$, which is strictly below the equilibrium payoff $\frac{v+1}{2}$ for all v. For $b < \frac{1}{2}$, no types can be excluded, such that the worst possible out-of-equilibrium beliefs are better than the equilibrium beliefs (which put all mass on the 0 type). No bidder can ever gain from a deviation to a bid $b > 1$.

bite in the present setting to exclude complicated (semi)pooling in the D1 PBE, keeping our analysis tractable. Although restrictive, this condition is likely satisfied if we believe that only the top end of e.g. the income distribution participates in art or charity auctions. More importantly, however, this condition will prove close to a necessary condition for the existence of a fully separating PBE in the second-price auction and English auction. Because we seek to compare the equilibrium bidding and expected revenues of the first-price auction with the second-price and English auctions, we impose this condition from the start. Note that this condition implies the common log-concavity of F or the non-decreasing hazard rate condition. It neither implies, nor is implied by the log-concave density condition imposed by Goeree (2003).[12] A similar condition is found in Segev and Sela (2014). This condition implies that only fully separating equilibria survive the D1 criterion.[13]

Lemma 1 *If $f'(.) \leq 0$, all D1 PBE are fully separating, with $\beta'(.) > 0$.*

If the D1 PBE of the first-price auction is fully separating, then the winner's type is fully revealed to receiver, as $\beta^{-1}(\beta(v_{i^*})) = v_{i^*}$. If the winner's type is known to be v_{i^*}, the expectation of a loser's valuation is $\frac{1}{F(v_{i^*})} \int_{\underline{v}}^{v_{i^*}} x dF(x)$. However, if a $\tilde{v}$ type does not win, she *ex ante* does not know the winner's valuation (except that it is above $\tilde{v}$), such that her expectation of the receiver's inference about a loser is

$$\frac{\int_{\tilde{v}}^{\bar{v}} \frac{1}{F(y)} \int_{\underline{v}}^{y} x dF(x) \, dF^{n-1}(y)}{1 - \left(F^{n-1}(\tilde{v})\right)}.$$

Moreover, if β is strictly increasing and valuations are drawn independently, the probability of winning for a bidder with a valuation v

[12] Clearly, Goeree's (2003) condition allows for a broader class of type distributions. However, Goeree ensures the existence of a fully separating equilibrium by imposing that the winning bidder's payoffs are strictly convex w.r.t. the uninformed party's inference. In the present setting with linear payoffs, this is achieved by a stronger restriction on the type distribution.

[13] We use 'fully separating' to indicate that no bid is chosen by different types in equilibrium. In the present setting, this does not imply that the receiver's equilibrium beliefs are degenerate, which is sometimes used as an alternative definition of 'fully separating' equilibrium.

is $F^{n-1}(v)$. Given an equilibrium bidding function β and corresponding beliefs, we understand a type v bidder's problem as maximizing her expected utility by choosing which type $\tilde{v}$ to mimic, by submitting the latter's equilibrium bid $\beta(\tilde{v})$ to obtain her expected inference by the receiver and probability of winning. As such, the v type bidder's problem is

$$\max_{\tilde{v}} \left(F^{n-1}(\tilde{v})\right)(v-\beta(\tilde{v})+\tilde{v})+\left(1-\left(F^{n-1}(\tilde{v})\right)\right)\frac{\int_{\tilde{v}}^{\bar{v}}\frac{1}{F(y)}\int_{\underline{v}}^{y}xdF(x)\,dF^{n-1}(y)}{\left(1-\left(F^{n-1}(\tilde{v})\right)\right)}.$$

The first order condition can be written

$$\frac{\partial}{\partial\tilde{v}}\left(F^{n-1}(\tilde{v})\beta(\tilde{v})\right)=\left(F^{n-1}(\tilde{v})\right)'(v+\tilde{v})+\left(F^{n-1}(\tilde{v})\right)-\frac{1}{F(\tilde{v})}\int_{\underline{v}}^{\tilde{v}}xdF(x)\left(F^{n-1}(\tilde{v})\right)', \tag{2}$$

Of course, in equilibrium β must be such that each bidder strictly prefers her own type's equilibrium bid. Therefore, we impose $\tilde{v}=v$, solve the differential equation in (2) and simplify the bidding function. Finally, we verify the second order condition for each v to ensure that each type maximizes her expected utility by choosing her equilibrium bid $\beta(v)$. Let $E\left(V_1^{(n-1)}|V\leq v\right)$ denote the expected value of the highest order statistic out of $n-1$ draws, for a distribution truncated at the right at v, and let $E(V|V\leq v)$ denote the expected value of a single draw, for F truncated at the right at v. The essentially unique D1 PBE bidding function for the first-price auction is then characterized in the following Proposition.

Proposition 1 *For $n\geq 3$ and $f'(.)\leq 0$, an essentially unique first-price auction D1 PBE exists, and its bidding strategy is*

$$\beta(v)=v+\frac{n-1}{n-2}\left(E\left(V_1^{(n-1)}|V\leq v\right)-E(V|V\leq v)\right) \tag{3}$$

with $\lim_{v\to\underline{v}^+}\beta(v)=\underline{v}$, $\beta(\bar{v})=\bar{v}+\frac{n-1}{n-2}\left(E\left(V_1^{(n-1)}\right)-E(V)\right)$ *and finally* $\beta'(.)>1$.

Remark that for arbitrary F there is no fully separating equilibrium in the first-price auction with two bidders. The equilibrium bidding strategy in (3) is only essentially unique because the equilibrium bid of the $\underline{v}$ valuation type is undetermined: because she has zero probability of winning the auction in equilibrium, any bid in the interval $[0, \underline{v}]$ is payoff equivalent. In the limit, however, the lowest valuation types bid $\underline{v}$. In terms of inference by the receiver, the lowest valuation types have little to gain from winning. Winning reveals them as lowest types, while they are better off in terms of inference by losing against a higher type. Yet, if an interval of lowest valuation types would bid weakly below $\underline{v}$ (while respecting $\beta'(.) > 0$), then the $\underline{v}$ type can profitably deviate to a bid $\underline{v}$ to win with non-zero probability, pay $\underline{v}$ for an object valued $\underline{v}$ and be inferred by the receiver to have a valuation strictly above $\underline{v}$. As suggested above, all bidders with a valuation strictly higher than $\underline{v}$ bid above their valuation of the object. The difference between a bidder's valuation for the object and her equilibrium bid strictly increases with the bidder's valuation. For $n \to +\infty$, the highest valuation types bid $\beta(\bar{v}) = 2\bar{v} - E(V)$, which equals their valuation for the object plus the difference in the receiver's inference about them if winning ($\bar{v}$) rather than losing ($E(V)$) the auction.

Example 2 (Uniform on $[0,1]$) *In this case, the bidders' problem is*

$$\max_{\tilde{v}} (v - \beta(\tilde{v}) + \tilde{v}) \tilde{v}^{n-1} + \left(1 - \tilde{v}^{n-1}\right) \frac{n-1}{2n} \frac{1 - \tilde{v}^n}{1 - \tilde{v}^{n-1}}.$$

The D1 PBE bidding function is

$$\beta(v) = \frac{3n-1}{2n} v.$$

We now proceed to the all-pay auction. Contrary to the first-price auction, all the losers pay their own bid in the all-pay auction. As before, the receiver observes only the identity and payment of the winner. The all-pay auction suffers from the same equilibrium multiplicity due to out-of equilibrium beliefs as the first-price auction, and imposing the D1-criterion excludes all (semi)pooling equilibria if f is non-increasing. The proof is technically identical to that of the first-price auction (Lemma 1). As such, the winner's type is fully revealed to the receiver, and the latter's expected inference is equivalent in the first-price and the all-pay auction. In the absence of signaling, the expected payoff in the all-pay auction equals the expected payoff in the first-price auction minus $\left(1 - F^{n-1}\left(\tilde{v}\right)\right)\left(\beta(\tilde{v})\right)$ (such that bidders pay their bid with probability 1 instead of $F^{n-1}\left(\tilde{v}\right)$).

In the absence of signaling, both these auctions are revenue equivalent. The addition of an identical term to the expected payoffs of both auction formats, i.e. the expected inference of the receiver, affects the equilibrium bidding functions and the expected payments is the same way in both auctions. As a result, the equilibrium bidding function of the all-pay auction can be obtained by an adaptation of the usual proof of the revenue equivalence theorem (see e.g. Krishna (2009)).

Proposition 2 *If $f'(.) \leq 0$ and $n \geq 3$, then a unique all-pay auction D1 PBE exists, and its bidding function is*

$$\beta\left(v\right) = F^{n-1}(v)\beta^{I}\left(v\right),$$

with $\beta^{I}(.)$ the the first-price auction D1 PBE bidding function, $\lim\limits_{v \to \underline{v}^{+}} \beta\left(v\right) = 0$ and $\beta\left(\bar{v}\right) = \beta^{I}(\bar{v})$.

As for the first-price auction, no D1 PBE exists in general for two bidders.

4 Second-price auction

In the second-price sealed bid auction, the winner pays the second highest bid. Because the receiver only observes the identity and payment of the winner, the latter only allows her to bound the set of possible bids of the winner from below and the set of possible bids of the losers from above. This difference in information available to the receiver considerably alters the bidders' expected payoff and equilibrium bidding.

The second-price auction also suffers from a multiplicity of equilibria due to insufficient restrictions on out-of-equilibrium beliefs, which is equally remedied by imposing the D1 criterion. However, the role of out-of-equilibrium beliefs slightly differs between the first- and second-price auctions. A bidder deviating unilaterally to a bid above the highest equilibrium bid always wins the auction. But such a deviation will not be revealed, because the winner only pays the second highest bid, which has an equilibrium interpretation. Therefore, bids cannot be constrained from above by possibly implausible out-of-equilibrium beliefs in the second-price auction, and a zero revenue auction as in Example 1 is impossible for the second-price auction. Out-of-equilibrium beliefs for bids below the minimal equilibrium bid affect bidding in neither the first- nor the second-price auction because such deviations are never observed, such that implausible out-of-equilibrium beliefs can never constrain equilibrium bidding from below. However, discontinuities in the bidding function at intermediate valuations can be supported by particular out-of-equilibrium beliefs. Such deviations are revealed to the receiver if they constitute the second highest bid, in which case they fix the inference about all losing bidders, including the deviator. Similar to Lemma 1, the following Lemma demonstrates that any D1 PBE bidding function is strictly increasing for non-increasing density functions.

Lemma 2 *If $f'(.) \leq 0$ and $n \geq 3$, then $\beta'(.) > 0$ in any D1 PBE of the second-price auction.*

We now proceed step by step to construct the problem of a v type bidder choosing which type $\tilde{v}$ to mimic in the second-price auction. As before, a strictly increasing bidding function implies that a type v bidder choosing the $\tilde{v}$ type's equilibrium bid wins with probability $F^{n-1}(\tilde{v})$. In this case, her payoff is:

$$v - \frac{1}{F^{n-1}(\tilde{v})} \int_{\underline{v}}^{\tilde{v}} \beta(x)\, dF^{n-1}(x) + \frac{1}{F^{n-1}(\tilde{v})} \int_{\underline{v}}^{\tilde{v}} \frac{\int_x^{\overline{v}} y dF(y)}{1 - F(x)} dF^{n-1}(x).$$

The second term is the expected payment if $\beta(\tilde{v})$ is the winning bid and the third term is the receiver's expected inference about a winner of valuation $\tilde{v}$. If the second highest bidder is of type x, then the inference about the winner is $\frac{\int_x^{\overline{v}} ydF(y)}{1-F(x)}$. But because the second highest bid is unknown to the bidder, the third term takes the expectation over the second highest bid.

Second, with probability $(n-1)\,F^{n-2}(\tilde{v})\,(1-F(\tilde{v}))$ bid $\beta(\tilde{v})$ is the second highest bid. In this case, the receiver's inference about any losing bidder is

$$\frac{\tilde{v}}{n-1}+\frac{n-2}{n-1}\frac{\int_{\underline{v}}^{\tilde{v}} x dF(x)}{F(\tilde{v})},$$

as one of the $n-1$ losers has valuation $\tilde{v}$ while the $n-2$ others' valuations are weakly lower than $\tilde{v}$. Finally, with probability $1-F^{n-1}(\tilde{v})-(n-1)\,F^{n-2}(\tilde{v})\,(1-F(\tilde{v}))$, a type $\tilde{v}$ bidder is neither the highest nor second highest bidder. For this case, a bidder forms an expectation over the second highest bid to asses the receiver's expected inference about the losing bidders.[14]

The expected utility of a valuation v bidder choosing type $\tilde{v}$'s bidding strategy is then:

$$\int_{\underline{v}}^{\tilde{v}}(v-\beta(x))\,dF^{n-1}(x)+\int_{\underline{v}}^{\tilde{v}}\frac{\int_{x}^{\bar{v}} y dF(y)}{1-F(x)}dF^{n-1}(x)$$
$$+(n-1)\,F^{n-2}(\tilde{v})\,(1-F(\tilde{v}))\left(\frac{\tilde{v}}{n-1}+\frac{n-2}{n-1}\frac{\int_{\underline{v}}^{\tilde{v}} x dF(x)}{F(\tilde{v})}\right)$$
$$+\int_{\tilde{v}}^{\bar{v}}\left(\frac{y}{n-1}+\frac{n-2}{n-1}\frac{\int_{\underline{v}}^{y} x dF(x)}{F(y)}\right)d\left((n-1)\,F^{n-2}(y)-(n-2)\,F^{n-1}(y)\right).$$

[14]Note that these inferences differ significantly from those presented in Giovannoni and Makris (2014) for the same case.

The first order condition is

$$\beta(\tilde{v})\left(F^{n-1}(\tilde{v})\right)' = v\left(F^{n-1}(\tilde{v})\right)' + \frac{\int_{\tilde{v}}^{\bar{v}} x dF(x)}{1-F(\tilde{v})}\left(F^{n-1}(\tilde{v})\right)'$$
$$+F^{n-2}(\tilde{v})(1-F(\tilde{v}))\left(1+(n-2)\frac{\tilde{v}f(\tilde{v})F(\tilde{v}) - f(\tilde{v})\int_{\underline{v}}^{\tilde{v}} x dF(x)}{F^2(\tilde{v})}\right)$$
$$+\left((n-2)f(\tilde{v})F^{n-3}(\tilde{v}) - (n-1)f(\tilde{v})F^{n-2}(\tilde{v})\right)\left(\tilde{v}+(n-2)\frac{\int_{\underline{v}}^{\tilde{v}} x dF(x)}{F(\tilde{v})}\right)$$
$$-\left((n-2)f(\tilde{v})F^{n-3}(\tilde{v}) - (n-2)f(\tilde{v})F^{n-2}(\tilde{v})\right)\left(\tilde{v}+(n-2)\frac{\int_{\underline{v}}^{\tilde{v}} x dF(x)}{F(\tilde{v})}\right).$$

After dividing both sides by $\left(F^{n-1}(\tilde{v})\right)' = (n-1)F^{n-2}(\tilde{v})f(\tilde{v})$, imposing $\tilde{v} = v$ and simplifying, we obtain

$$\beta(v) = v + \frac{\int_{\tilde{v}}^{\bar{v}} x dF(x)}{1-F(\tilde{v})} + \frac{1-F(v)}{f(v)(n-1)}\left(1+\frac{n-2}{F(v)}\left(v - \frac{\int_{\underline{v}}^{v} x dF(x)}{F(v)}\right)\right)$$
$$-\frac{1}{n-1}\left(v+(n-2)\frac{\int_{\underline{v}}^{v} x dF(x)}{F(v)}\right). \qquad (4)$$

The essentially unique D1 PBE bidding function for the second-price auction is then characterized by the following Proposition.

Proposition 3 *If either $n \geq 4$ and $f'(.) \leq 0$ or $n = 3$ and $f'(.) < 0$, then an essentially unique second-price auction D1 PBE exists, and its bidding strategy is*

$$\beta(v) = \frac{n-2}{n-1}\frac{v - E(V|V \leq v)}{F(v)} + E(V|V \geq v) + \frac{1-F(v)}{(n-1)f(v)},$$

with $\lim_{v\to\underline{v}} \beta(v) = E(V) + \frac{n}{n-1}\frac{1}{2f(\underline{v})}$ *and* $\lim_{v\to\bar{v}} \beta(v) = \bar{v} + \frac{n-2}{n-1}(\bar{v} - E(V))$.

For the second-price auction, the qualification 'essential' reflects that the equilibrium bidding function is undetermined at both extremes of the typespace. If $\beta'(.) > 0$, then a $\underline{v}$ type has the highest or second-highest bid with zero probability, such that all bids in $\left[0, E(V) + \frac{n}{n-1}\frac{1}{2f(\underline{v})}\right]$ are in equilibrium payoff equivalent. At the other hand, for finite n a $\bar{v}$ type wins with probability 1 and does not pay her own bid, such that all bids weakly above $\lim_{v\to\bar{v}}\beta(v)$ are in equilibrium payoff equivalent. However, the D1 equilibrium bidding function is uniquely determined on $(\underline{v},\bar{v})$. The limit of the equilibrium bidding function at $\underline{v}$ lies strictly above the average valuation for the object, and thus well above the equilibrium bid in the first-price auction. The reason is that if a very low valuation type wins the auction, the winner is inferred as slightly higher than a $E(V)$ type by the receiver, while all losers are inferred almost as $\underline{v}$ types, because the second highest bidder's type is below the winner's valuation. Therefore, the lowest types bid at least their valuation $\underline{v}$ plus the difference in inference by the receiver $E(V)-\underline{v}$ in equilibrium. A further comparison with the equilibrium bidding function of the English auction in the next Section will provide more intuition for the second-price D1 PBE bidding function. [15]

Note that in the second-price auction, there is also no fully separating equilibrium with two bidders, and even not with three bidders if the density f is constant over some interval of the support. In the following example with a uniform distribution on $[0,1]$, we comment on this non-existence of an equilibrium with two or three bidders.

Example 3 (Uniform on $[0,1]$) *For F uniform on $[0,1,]$, the expected payoff of a v type bidder imitating a $\tilde{v}$ type is*

$$\begin{aligned}
&\tilde{v}^{n-1}\left(v_i+\frac{1+\frac{n-1}{n}\tilde{v}}{2}\right)-(n-1)\int_0^{\tilde{v}}x^{n-2}\beta(x)\,dx \\
&+(n-1)\,\tilde{v}^{n-2}(1-\tilde{v})\left(\frac{n-2}{n-1}\frac{\tilde{v}}{2}+\frac{\tilde{v}}{n-1}\right) \\
&+\left(1-\tilde{v}^{n-1}-(n-1)\,\tilde{v}^{n-2}(1-\tilde{v})\right)\frac{\int_{\tilde{v}}^1\left(\frac{x}{n-1}+\frac{n-2}{n-1}\frac{x}{2}\right)d\left((n-1)\,x^{n-2}-(n-2)\,x^{n-1}\right)}{\left(1-\tilde{v}^{n-1}-(n-1)\,\tilde{v}^{n-2}(1-\tilde{v})\right)}.
\end{aligned}$$

The D1 PBE bidding function is

$$\beta(v)=\frac{2n-1+(n-3)\,v}{2(n-1)}.$$

[15] Note that the above equilibrium bidding function differs from that in Giovannoni and Makris (2014), presumably because of the difference in the receiver's inferences.

If $n = 2$, a losing bidder is always identified by her true valuation v, while winners are identified only as the average between the valuation of the loser (in expectation half of her own valuation) and the maximum valuation 1, *i.e. the expected inference for $n = 2$ is*

$$\tilde{v}\left(\frac{1}{2}+\frac{\tilde{v}}{4}\right)+(1-\tilde{v})\,\tilde{v}=\frac{3}{4}\tilde{v}\,(2-\tilde{v})\,.$$

For two bidders, the receiver's inference increases more with $\tilde{v}$ if a bidder loses, but the probability of losing decreases with $\tilde{v}$, such that the marginal effect of $\tilde{v}$ on the receiver's expected inference, i.e. $\frac{3}{2}(1-\tilde{v})$, decreases with $\tilde{v}$ at a constant rate $\frac{3}{2}$. This decrease more than offsets the higher valuation types' incentives to bid strictly more than lower types, which inhibits the existence of a D1 PBE At $n = 3$, both these effects cancel out exactly. Thus, for $n \leq 3$, we have no D.1. equilibrium bidding function.

A similar logic applies if f is constant over an interval in the support of a more general distribution function, such that Proposition 3 requires either that $n \geq 4$ and $f'(.) \leq 0$ or that $n = 3$ and $f'(.) < 0$.

5 English auction

An important reason for the popularity of the second-price auction among auction theorists is its common strategic equivalence with the English auction, which is more frequently used in reality. However, this equivalence ceases to exist in the presence of signaling. This result can be surprising, because the introduction of other externalities, such as financial externalities in charity auctions (e.g. Engers and McManus (2007)), did not break up the strategic equivalence.

The English auction can be studied in various formalizations. We consider a minimal information "button auction" (see e.g. Milgrom (2004)), in which the auctioneer lets the price continuously increase on a price clock. Each bidder chooses when to exit the auction by releasing a button, and such exit is irrevocable. The last bidder holding her button wins, and fixes the price by releasing her button. Bidders only observe whether two or more bidders are still pushing their button or not, and the latter implies that the auction has a winner. This minimal information setting remains closest to the second-price auction, as bidders can learn little about the other bidders' valuations during the auction. We maintain the assumption that the receiver only observes the identity and the payment of the winner.[16]

In this auction, each bidder has to decide on each moment (or price) whether to stay in or to exit. Note then that in equilibrium, the exit price is increasing with v because the prospects in terms of inference by the receiver at a certain price are identical for different types, while the lower type values winning the auction strictly less. Again, we restrict out-of-equilibrium beliefs by means of the D1 criterion to avoid the multiplicity of equilibria, and establish that any D1 PBE is fully separating.

Lemma 3 *In any D1 PBE, the exit rule β is a continuous and strictly increasing function of v.*

In the usual English auction, the winner drops out immediately after the second last bidder's exit. An inspection of the payoff function shows that once a bidder has won the auction, our setting does not provide her with means to credibly reveal a higher valuation to the receiver (contrary e.g. to Goeree (2003)). In the present setting, the winner's problem would be to choose an exit price b_{i^*} to maximize $v_{i^*} - b_{i^*} + E\left(V|\mu_{i^*}\left(V|i^*, b_{i^*}\right)\right)$. The lack of single crossing property, due to the additive structure of the payoff function, implies that if the receiver would interpret a higher bid in such way that the winner prefers to bid strictly above the second highest bid, then all types of winners would strictly prefer to do so. Therefore, if the penultimate quitter has valuation v', then the receiver must have an expectation $E\left(V|V \geq v'\right)$ of the winner's valuation for any payment above $\beta\left(v'\right)$, and the winner must exit immediately at $\beta\left(v'\right)$.

[16] Obviously other information regimes, e.g. the receiver observing all bids, are equally plausible in this setting. The plausibility of these different scenarios depends on the specific context and the identity of the receiver (e.g. another bidder or the general public reading media outlets). We prefer the present assumption because it keeps the kind of information the receiver disposes of constant throughout the different auction formats.

If the bidding strategy (i.e. exit price) is strictly increasing with type and if the winner exits at the second highest bid, then the second highest bidder fixes the payoff of all bidders. Since bidders do not observe previous exits by lower valuation bidder's, the latter's strategy does not affect equilibrium bidding. Of course, a bidder does not know whether she has the second highest valuation, but she optimizes her strategy as if this were the case. A type v bidder then leaves the auction when the price hits the bid of a $\tilde{v}$ type, such that

$$v - \beta(\tilde{v}) + \frac{1}{1 - F(\tilde{v})} \int_{\tilde{v}}^{\bar{v}} x dF(x) = \frac{\tilde{v}}{n-1} + \frac{n-2}{n-1} \frac{\int_{\underline{v}}^{\tilde{v}} x dF(x)}{F(\tilde{v})}. \tag{5}$$

The left hand side of (5) is the payoff a type v bidder gets if she wins at price $\beta(\tilde{v})$, while the right hand side is a loser's payoff, if she releases the button at price $\beta(\tilde{v})$ with only two bidders left. This exit rule defines a unique equilibrium bidding function of the second highest valuation type, which determines the auction price. This is equivalent to having at each price b type $\beta^{-1}(b)$ leaving the auction, such that the optimal exit price of type v satisfies

$$v - b + \frac{1}{1 - F\left(\beta^{-1}(b)\right)} \int_{\beta^{-1}(b)}^{\bar{v}} x dF(x) = \frac{\beta^{-1}(b)}{n-1} + \frac{n-2}{n-1} \frac{\int_{\underline{v}}^{\beta^{-1}(b)} x dF(x)}{F\left(\beta^{-1}(b)\right)} \tag{6}$$

Note in (6) that the receiver's inference about the winner and about all losers increases with b (or $\tilde{v}$). However, the following proposition establishes that in equilibrium the costs of mimicking a higher type in terms of payment increase faster than the benefits in terms of inference, such that this equality establishes the essentially unique D1 equilibrium exit rule for the English auction.

Proposition 4 *If $n \geq 3$ and $f'(.) \leq 0$, then an essentially unique D1 PBE exists for the (minimal information, button) English auction, and its exit rule is*

$$\beta(v) = \frac{n-2}{n-1} \left(v - \frac{\int_{\underline{v}}^{v} x dF(x)}{F(v)} \right) + \frac{\int_{v}^{\bar{v}} x dF(x)}{1 - F(v)}, \tag{7}$$

with $\lim_{v \to \underline{v}^+} \beta(v) = E(V)$ *and* $\lim_{v \to \bar{v}} \beta(v) = \bar{v} + \frac{n-2}{n-1}(\bar{v} - E(V))$.

Given the optimal exit strategy of a winner in the English auction, the second-price and English auctions are equivalent in terms of information for the receiver. A closer comparison of equilibrium bidding in both auctions can therefore also further clarify the equilibrium in the second-price auction. When comparing the equilibrium bidding functions of the second-price and English auctions, we note both are identical up to the two following additional terms in the former:

$$\frac{1 - F(v)}{F(v)} \frac{n-2}{n-1} \left(v - \frac{\int_{\underline{v}}^{v} x dF(x)}{F(v)} \right) + \frac{(1 - F(v))}{(n-1) f(v)} > 0,$$

which vanish for $v \to \bar{v}$. A closer inspection of (4) shows that these two additional terms, the third right hand side term in (4), reflect the effect on the receiver's expected inference about all the losing bidders of a marginal increased bid for a given probability of being the second highest bidder.

The main difference between the second-price and English auctions is that in the latter, the set of possible second highest bids is bounded from below by the increasing price clock. If the English auction has no winner at price b, then all active bidders can take it as a given that the second highest bid is at least b, and that the receiver's expected inference about the winner will be bounded from below by $\beta^{-1}(b)$. This lower bound on the second highest bid also bounds the receiver's expected inference about the losers from below. Therefore, each bidder just compares her payoff as a winner and as a loser with the second higher bid and quits if both are equal. If she turns out not being the second highest bidder, then the payoff of losing certainly exceeds her payoff of winning. As such, (7) means that an active bidder exits when the price equals her valuation plus the difference between the receiver's inference about the winner and a loser if this exit price were the second highest bid.

In the second-price auction, no increasing price clock bounds the second highest bid. First, in case of winning, a high valuation bidder must consider the possibility of paying the bid of a very low valuation bidder when winning, consequently being inferred as the expected value of any type above the latter by the receiver. The benefits of the potentially lower payment are compensated by the low inference by the receiver. In the case of losing the auction, a bidder can bound the receiver's inference about her type from below by means of her own bid. Compared to the English auction, this provides an additional marginal benefit to bidding in the second-price auction, which disappears as v approaches $\bar{v}$ (for which the probability of losing goes to zero).

Example 4 (Uniform on $[0, 1]$) *For F the uniform distribution, equality* (5) *becomes*

$$v - \beta(\tilde{v}) + \frac{1 + \tilde{v}}{2} = \frac{\tilde{v}}{n-1} + \frac{n-2}{n-1}\frac{\tilde{v}}{2},$$

which implies the D1 PBE exit rule

$$\beta(v) = \frac{1}{2} + \frac{2n-3}{2(n-1)}v.$$

6 Expected revenue comparison

We now compare the expected revenue of the four auction designs analyzed so far. We denote the expected revenue by ER^k, with $k = I, II, E, A$ indicating respectively first-price auction, the second-price auction, the English auction and the all-pay auction. As pointed out in Section 3, the all-pay and first-price auctions are equivalent in terms of expected payments, such that $ER^I = ER^A$. The following proposition shows that for finite n, we obtain a strict ranking in term of expected revenue of the English, first-price and second-price auctions.

Proposition 5 (Expected revenue ranking) *If $f'(.) \leq 0$ and $n \geq 4$, and if n is finite, then in the D1 PBE:*

$$ER^{II} > ER^I = ER^A > ER^E.$$

The following example illustrates this strict expected revenue ranking for F being the uniform distribution.

Example 5 (Uniform on $[0,1]$) *For the uniform distribution on* $[0,1]$, *Figure 6 represents the D1 PBE bidding functions for the auction formats studied so far. The expected revenue of the first-price, second-price and English auctions is then:*

$$ER^I = ER^A = \frac{3n-1}{2n}\int_0^1 v dv^n = \frac{3n-1}{2(n+1)},$$

$$\begin{aligned} ER^{II} &= \frac{n(n-1)}{2(n-1)}\int_0^1 (2n-1+(n-3)v)\left(v^{n-2}-v^{n-1}\right)dv \\ &= \frac{3(n-1)n+2}{2(n^2-1)} \end{aligned}$$

and

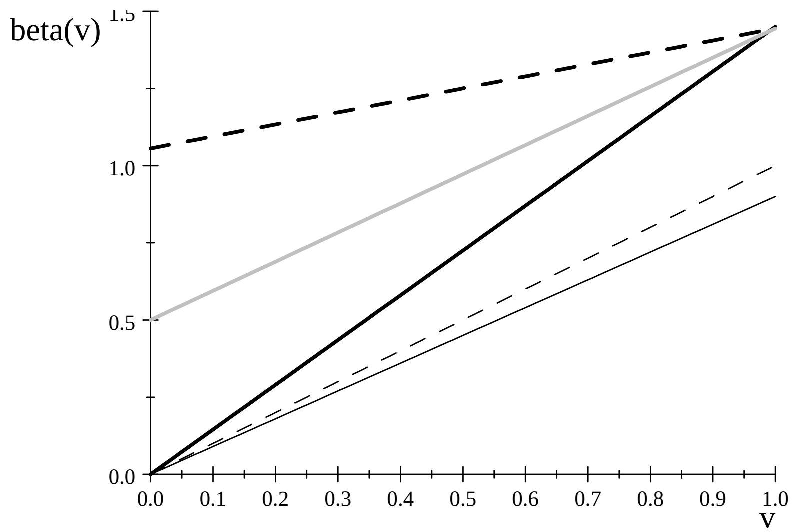
$$ER^E = \frac{1}{2}+\frac{n(2n-3)}{2}\left(\int_0^1 v^{n-1}dv-\int_0^1 v^n dv\right) = \frac{1}{2}+\frac{(2n-3)}{2(n+1)}$$

such that the first-price auction outperforms the English auction,

$$ER^E - ER^I = \frac{1}{2}+\frac{(2n-3)}{2(n+1)}-\frac{3(n-1)n+2}{2(n^2-1)} = -\frac{n}{n^2-1}<0,$$

and but is outperformed by the second-price auction:

$$ER^{II}-ER^I = \frac{3(n-1)n+2}{2(n^2-1)}-\frac{3n-1}{2(n+1)} = \frac{1}{2(n-1)}>0.$$



The D1 PBE bidding for $U[0,1]$ with $n = 10$, for the first-price (solid), second-price (dashed) and English (grey) auctions, with (bold) and without (thin) prestige motives.

This strict ranking in terms of expected revenues reflects the different amounts of information which are available to the receiver and the bidders in the different auction formats. The absence of a price clock in the sealed bid second-price auction implies an additional marginal benefit of a higher bid in comparison with the English auction: one's bid constrains the receiver's expected inference in case of losing the auction from below. Because of this additional effect, the equilibrium bids are strictly lower in the English auction than in the second–price auction for all bidders with a valuation strictly below the upper bound $\bar{v}$. Since the winner pays the bid of the second highest bidder in both auctions, the second-price auction dominates the English auction.

At the other hand, the uniform example shows that the equilibrium bidding function of the first-price auction can be strictly above that of the English auction near the upper bound $\bar{v}$. The reason is that the gap in terms of the receiver's expected inference between winning and losing is smaller in the English auction. At the one hand, when quitting at $\lim_{v\to\bar{v}} \beta(v)$ losing $\bar{v}$ types are interpreted as $\frac{\bar{v}}{n-1} + \frac{n-2}{n-1}E(V)$ in the English auction rather than as $E(V)$ in the first-price auction. At the other hand, when staying at $\lim_{v\to\bar{v}} \beta(v)$ in the English auction or bidding $\lim_{v \to \bar{v}} \beta(v)$ in the first-price auction, a winning $\bar{v}$ type is in both auctions inferred to be a $\bar{v}$ type. Moreover, bidders pay their own bid in the first-price auction, and that of the second highest bidder in the English auction. This is sufficient for the first-price auction to outperform the English auction in expectation, but insufficient for it to dominate the second-price auction.

Note that this strict ranking of the second-price and English auctions in terms of expected revenue contrasts with Giovannoni and Makris' (2014) finding that the expected revenue depends only on the information revealed to the receiver, and not on the actual auction format, as long as the auction is such that the highest bidder wins, no information about other bids is available during the auction and the expected payment of the lowest bidder type is zero. In the present case, the receiver observes the same information $-$ the winner's identity and the second highest bid $-$ in both the second-price and English auctions, and yet these auctions generate a different expected revenue.

However, note that expected revenue equivalence is restored asymptotically for n going to infinity. In the limit, the bid of the $\bar{v}$ type is identical in all auctions:

$$\lim_{n \to +\infty} \lim_{v \to \bar{v}} \beta^k(v) = 2\bar{v} - E(V)$$

for $k = I, II, E, A$. If $n \to \infty$, both the winner of the auction and the second highest bidder have type $\bar{v}$ with probability 1. As such, the $\bar{v}$ type winner pays her own bid in all auctions. In addition, the $\bar{v}$ type's winning bid must make her indifferent between winning and losing, because another bidder with a valuation of almost $\bar{v}$ type would otherwise benefit from outbidding her.

7 Discussion

We have studied auctions with signaling, in which all bidders care about the expected value of the beliefs about their type of an outside party, who observes the identity and payment of the auction's winner. We characterize the bidding equilibrium and expected revenues in 4 well-known auction formats: the first-price, second-price, all-pay and English auctions. We show that if the outside party's beliefs satisfy the common refinement criterion (D1) and if the type distribution function is concave, then any equilibrium bidding function must be fully separating. Moreover, we obtain a strict ranking of the expected revenues of these auction formats for a finite number of bidders. The first-price and all-pay auctions, which are equivalent in terms of expected payments, do strictly better than the English auction and strictly worse than the second-price auction. Revenue equivalence is only restored asymptotically, if the number of bidders goes to infinity.

These differences in expected revenues stem from the differences in information for the receiver and the bidders in the different auction formats. First, in the second-price and English auctions, the winner does not pay her own bid, such that the winner's payment only imposes a lower bound on the receiver's expected beliefs about the winner's type. This incites the lowest valuation types to bid considerably above their valuation. The reason is that if they win the auction, they pay the bid of an even lower type, while the receiver's expected inference about the winner is just above the average valuation and the expected inference about the losers is close to the lowest possible valuation. In the first-price auction, in contrast, a winning low valuation bidder reveals her true low type.

Second, the highest types tend to bid higher in the first-price auction than in the English auction, because the gap in terms of expected inferences by the receiver between winning and losing the auction is larger in the former. Moreover, the winner has to pay her own bid in the first-price auction. This explains the superiority of the first-price over the English auction.

Third, the increasing price clock in the English auction constrains the set of potential second highest bidders at each moment. If the auction has no winner at a certain price, then the second highest bidder in the auction is at least willing to pay this price. In the sealed bid auctions, such a constraint is absent and a bidder can only depend on her own bid to constrain the expected inference of the receiver about her in the case of her losing the auction. This additional return to bidding in sealed bid auctions explains the superiority of the second-price auction over the English auction, and over the first-price auction.

In short, we show that the auction's format affects its expected revenues if bidders care about a receiver's inferences about their type. This disaccords with Goeree's (2003) finding of revenue equivalence in the presence of signaling incentives. This difference originates from three crucial differences between Goeree's setting and the present model. First, Goeree assumes that the uninformed party observes the winner's bid, rather than her payment. In the second-price auction, this implies a full revelation of the winner's type in Goeree's setting. In the present setting, the incomplete revelation of the winner's type causes the low types to bid significantly above their valuation in the second-price auction. Second, higher bidder types care more about the receiver's inferences in Goeree's setting, and this single crossing condition allows winning bidders to fully reveal their type in the English auction. This equally constrains the equilibrium bids of the lowest types in comparison to the present setting. Third, unlike in Goeree (2003), losing bidders also care about the receiver's inferences in the present setting, and this increases in particular the equilibrium bidding the sealed bid auctions.

In Giovannoni and Makris (2014), losing bidders care about the receiver's inferences as well, and Giovannoni and Makris equally obtain a strict expected revenue ranking of the first-price and second-price auctions (albeit with a different bidding function for the latter). Moreover, Giovannoni and Makris (2014) show that the expected revenue depends only on the information revealed to the receiver, and not on the actual auction format, if three conditions are satisfied: if the auction is such that the highest bidder wins, if no information about the other bidders' behavior is available during the auction and if the expected payment of the lowest bidder type is zero. The fact that both the first-price and the second-price auctions outperform the English auction in terms of expected revenues in the present setting, shows how strict these three conditions apply. In equilibrium, the receiver obtains the same information in the second-price and the English auctions, and during the English auction bidders have no information about the other bidders' behavior except whether or not they have all left the auction, at which point the auction ends. And yet, this last bit of information proves sufficient to reverse the expected revenue ranking, and put the English auction below the first-price auction. This shows that the effects of signaling in auctions depend heavily on small institutional details, and calls for more research on this issue.

The assumption that losing bidders equally care about the receiver's inferences reflects a situation in which the receiver is interested in bidders' valuations as a reflection of individual characteristics, such that the auction's outcome is informative about these characteristics, irrespective of whether a bidder won or lost. If participation is exogenously given, then non-participating bidders get inference $E(V)$ in absence of information transmission, and are in equilibrium generically strictly better off than losing participating bidders. Does that mean that bidders prefer to stay out if entry were endogenous? Not really. If entry were endogenous and if the receiver observes the entry decision of each bidder besides the winner's identity and payment, then the participation decision becomes informative. Consider then a cutoff type $\hat{v}$, with $\hat{v} < \bar{v}$, such that higher valuation types participate, while lower types stay out. If the $\hat{v}$ type bidder stays out, she gets $E(V|V \leq \hat{v})$. If the cutoff type $\hat{v}$ participates, she wins with probability zero if $\beta'(.) > 0$, and gets a receiver's inference between $\hat{v}$ and the winner's valuation. Hence, the cutoff type $\hat{v}$ is strictly better off participating, and endogenous participation implies full participation. Moreover, this means that the auctioneer can and should ask for a strictly positive entry fee. The auctioneer can guarantee full participation while asking all bidders to pay an entry fee which makes the type $\underline{v}$ bidder indifferent between participation and non-participation (and payoff $\underline{v}$). Unfortunately, a characterization of the optimal entry fee is in this setting, in particular for the second-price and English auctions, not a trivial excercise, and is as such outside the scope of this paper.

The present analysis illustrates that the auction format and information assumptions matter if bidders use the auction outcome for signaling their valuation to an uninformed party. Although the present setting seems a natural and interesting benchmark case, a broad variety of alternative settings merit attention. Different information assumptions can be equally plausible, depending on the particular application one has in mind. If we maintain all other assumptions, but assume like Goeree (2003) that only the winner cares about the receiver's beliefs, then we still expect the second-price auction to outperform the first-price auction. If we maintain all other assumptions, but assume that the receiver observes all bids, then the conditions for the existence of a fully separating equilibrium become far more stringent.[17] Besides different information settings, different auction formats deserve further attention. Moreover, bidders may care in different ways about the inferences of receiver. Haile (2003) shows that the foresight of a resale auction can make bidders prefer to signal a low valuation. This question is also considered in Goeree (2003) and Giovannoni and Makris (2014). In this case, the bidding equilibrium is surely not a mirror image of the equilibria in the present paper, and the existence of a fully separating equilibrium seems unlikely in many settings.

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[17] To see this in the present context, assume that F is uniform on $[0,1]$, and assume a fully separating equilibrium. The problem of a v type bidder in the first-price auction is then $\max_{\tilde{v}} \tilde{v}^{n-1}\left(v-\beta\left(\tilde{v}\right)\right)+\tilde{v}$, from which we obtain $\beta\left(v\right)=\frac{n-1}{n}v+\frac{1}{v^{n-2}}$. In the second-price auction, a type v bidder's problem in a fully separating equilibrium is $\tilde{v}^{n-1}v-\int_0^{\tilde{v}}\beta\left(x\right)dx^{n-1}+\tilde{v}$, from which $\beta\left(v\right)=v+\frac{1}{(n-1)v^{n-2}}$. Neither bidding function is strictly increasing everywhere, such that the existence of a fully separating equilibrium requires considerably stronger restrictions in the present setting.

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A Appendix

A.1 Proof of Lemma 1

We proceed in three steps: 1. for any D1 PBE the bidding function β is weakly increasing, 2. In any D1 PBE, there is no pooling with the $\underline{\mathrm{v}}$ type and 3. In any D1 PBE, there is no pooling above the $\underline{\mathrm{v}}$ type.

Claim 1 (β weakly increasing) *In any D1 PBE, if type v' chooses b', then no $v'' < v'$ bids $b'' > b'$.*

Proof. To save on notation, let $p(b)$ denote the probability of winning the auction with bid b and $E_w(b)$ and $E_l(b)$ the expected values of the receiver's inference about respectively a winning and losing bidder who bids b. Assume that type v' bids b' in equilibrium and gets expected inferences $E_w(b')$ and $E_l'(b')$. Let (E_w'', E_l'') a pair of inferences such that type v' is indifferent between bidding b'' and getting inference (E_w'', E_l'') and her equilibrium payoff, i.e.

$$\begin{aligned} & p(b')(v' - b') + E_l(b') + p(b')[E_w(b') - E_l(b')] \\ & = p(b'')(v' - b'') + E_l'' + p(b'')[E_w'' - E_l''] \end{aligned}$$

or

$$\begin{aligned} [p(b'') - p(b')]\, v' = A \equiv p(b'')\, b'' - p(b')\, b' + E_l(b') + \\ p(b')[E_w(b') - E_l(b')] - (E_l'' + p(b')[E_w'' - E_l'']) \end{aligned}$$

and note that $p(b'') - p(b') \geq 0$. Then if $p(b'') - p(b') > 0$, it must be that

$$[p(b'') - p(b')]\, v'' < A,$$

such that

$$p(b')(v'' - b') + E_l(b') + p(b')[E_w(b') - E_l(b')] > p(b'')(v'' - b'') + E_l'' + p(b')[E_w'' - E_l''].$$

Hence, the lower valuation type needs a higher compensation in terms of inference for a higher bid.

Assume then that the equilibrium expected utility of type v'' is low enough to make $M^{+}(b'', v') \subseteq M^{+}(b'', v'') \cup M^{0}(b'', v'')$. Then it must be that the v'' strictly prefers bundle $(b', E_w(b'), E_l(b'))$ to her equilibrium strategy, a contradiction. Therefore $\mu(v''|b'') = 0$ in the D1 PBE, and no v'' type with $v'' < v'$ chooses a b'' bid with $b'' > b'$ if type v' bids b' in equilibrium. ■

Claim 2 (No pooling with $\underline{v}$) *No type $v >\underline{v}$ pools with type $\underline{v}$ in the D1 PBE.*

Assume an equilibrium in which a non-degenerate set of types $O = \left\{v|\beta(v) = \tilde{b}\right\}$ pool at $\tilde{b}$, such that $\underline{v}\in O$. By Claim 1, O is a convex set. If $\tilde{b} >\underline{v}$, then a type $\underline{v}$ bidder can strictly improve herself by deviating to $\underline{v}$. Such deviation is never observed, such that the receiver's inference about the $\underline{v}$ bidder is not worse, but she avoids winning the auction to pay $\tilde{b}$ in excess of her valuation $\underline{v}$.
If $\tilde{b} \leq\underline{v}$, then note that the expected inference about a bidder in O is $E_w\left(\tilde{b}\right) = \frac{1}{|O|}\int_O v dF(v)$ and $E_l\left(\tilde{b}\right)$. The probability of winning when pooling at $\tilde{b}$ is $\frac{F(\sup(O))^{n-1}}{n}$. Consider then type $\sup(O)$. If she bids a $\tilde{b}+\varepsilon$, with $\varepsilon > 0$, she wins at least with probability $F(\sup(O))^{n-1}$, in which case $E_w\left(\tilde{b}+\varepsilon\right) \geq \sup(O)$ and $E_l\left(\tilde{b}+\varepsilon\right) > E_l\left(\tilde{b}\right)$, while $\sup(O) - \tilde{b} - \varepsilon > 0$ for ε sufficiently small. But in equilibrium it must be that

$$\frac{F(\sup(O))^{n-1}}{n}\left(\sup(O) - \tilde{b} + E_w\left(\tilde{b}\right)\right) + \left(1 - \frac{F(\sup(O))^{n-1}}{n}\right) E_l\left(\tilde{b}\right)$$
$$\geq F(\sup(O))^{n-1}\left(2\sup(O) - \tilde{b} - \varepsilon\right) + \left(1 - F(\sup(O))^{n-1}\right) E_l\left(\tilde{b}+\varepsilon\right),$$

which is only true for $\varepsilon \to 0$ if $\sup(O) =\underline{v}$ and $n = 1$.

Claim 3 (No pooling above $\underline{v}$) *In the D1 PBE, no bid $\tilde{b}$ is chosen by two types $v' \neq v''$.*

Proof. Assume a D1 PBE in which $\tilde{b}$ is the lowest bid chosen by a nondegenerate set of types $O = \left\{v|\beta(v) = \tilde{b}\right\}$. Note that O is convex by Claim 1 and $\inf(O) > \underline{v}$ by Claim 2. By the continuity of f and of the utility function w.r.t. all arguments, the $\inf(O)$ must in equilibrium be indifferent between separating at $\lim_{v\to\inf(O)^-}\beta(v)$ and pooling at $\tilde{b}$. Note then that the indirect utility difference between $\sup(O)$ and $\inf(O)$ in the pooling equilibrium is

$$p\left(\tilde{b}\right)(\sup(O)-\inf(O)).$$

In the separating equilibrium this is by the envelope theorem

$$\int_{\inf(O)}^{\sup(O)} F^{n-1}(x)\,dx = \sup(O)\,F^{n-1}(\sup(O)) - \inf(O)\,F^{n-1}(\inf(O)) - \int_{\inf(O)}^{\sup(O)} x dF^{n-1}(x).$$

We now show that

$$\int_{\inf(O)}^{\sup(O)} F^{n-1}(x)\,dx > p\left(\bar{b}\right)(\sup(O)-\inf(O)) \tag{8}$$

if $f'(.) \leq 0$.
First write the probability of winning the auction while bidding $\tilde{b}$

$$p\left(\tilde{b}\right) = \sum_{i=0}^{n-1}\binom{n-1}{i}\frac{F^{n-1-i}(\inf(O))\,(F(\sup(O))-F(\inf(O)))^{i}}{i+1}.$$

Note then that $p\left(\bar{b}\right)(\sup(O)-\inf(O)) = \int_{\inf(O)}^{\sup(O)} F^{n-1}(x)\,dx = 0$ for $\sup(O) = \inf(O)$. Differentiate both sides of (8) to $\sup(O)$, to obtain

$$\frac{\partial p\left(\bar{b}\right)}{\partial \sup(O)}(\sup(O)-\inf(O)) + p\left(\bar{b}\right) < F^{n-1}(\sup(O)),$$

which can be written as

$$\alpha\left[F^{n-1}(\sup(O)) - p\left(\bar{b}\right) - F^{n-1-i}(\inf(O))\right] < F^{n-1}(\sup(O)) - p\left(\bar{b}\right), \tag{9}$$

with

$$\alpha = \frac{f(\sup(O))}{\frac{F(\sup(O))-F(\inf(O))}{\sup(O)-\inf(O)}}$$

because

$$\frac{\partial p\left(\bar{b}\right)}{\partial \sup\left(O\right)} = \sum_{i=1}^{n-1}\binom{n-1}{i} F^{n-1-i}\left(\inf\left(O\right)\right)\left(F\left(\sup\left(O\right)\right)-F\left(\inf\left(O\right)\right)\right)^{i-1}\frac{i}{i+1}f\left(\sup\left(O\right)\right)$$
$$= \left[F^{n-1}\left(\sup\left(O\right)\right)-p\left(\bar{b}\right)-F^{n-1-i}\left(\inf\left(O\right)\right)\right]\frac{f\left(\sup\left(O\right)\right)}{\left(F\left(\sup\left(O\right)\right)-F\left(\inf\left(O\right)\right)\right)},$$

in which the last equality uses

$$1-p\left(\bar{b}\right) = \left(1-F^{n-1}\left(\sup\left(O\right)\right)\right)+\sum_{i=0}^{n-1}\binom{n-1}{i} F^{n-1-i}\left(\inf\left(O\right)\right)\left(F\left(\sup\left(O\right)\right)-F\left(\inf\left(O\right)\right)\right)^{i}\frac{i}{i+1}.$$

Note then that $f'\left(.\right) \leq 0$ implies $\alpha \leq 1$, such that (9) and therefore (8) are always satisfied for $F\left(\sup\left(O\right)\right) > F\left(\inf\left(O\right)\right)$ and $f'\left(.\right) \leq 0$. Then the $\sup\left(O\right)$ type can achieve a strictly higher expected utility if she would deviate to the bid she makes in the fully separating equilibrium because the expected inference after such a deviation is at least the expected inference she gets in the fully separating equilibrium. This excludes any different types pooling in a D1 PBE. ■

A.2 Proof of Proposition 1

We proceed again in 3 steps: 1. establish shape of the bidding function; 2. show that $\beta'\left(.\right) > 0$ implies that the second order condition is satisfied and 3. show that $\beta'\left(.\right) > 0$.

Claim 4 (Bidding function) *β is as written in Proposition 1.*

Proof. Substitute $\tilde{v} = v$ in (2) to obtain

$$\frac{\partial}{\partial v}\left(\beta\left(v\right)F^{n-1}\left(v\right)\right) = 2v\left(F^{n-1}\left(v\right)\right)'+F^{n-1}\left(v\right)-\int_{\underline{v}}^{v} xdF\left(x\right)\left(F^{n-1}\left(v\right)\right)'.$$

Integrate and divide both sides by $F^{n-1}\left(\tilde{v}\right)$ to find

$$\beta\left(v\right) = \frac{2}{F^{n-1}\left(v\right)}\int_{\underline{v}}^{v} xdF^{n-1}\left(x\right)+\frac{1}{F^{n-1}\left(v\right)}\int_{\underline{v}}^{v} F^{n-1}\left(x\right)dx - \frac{1}{F^{n-1}\left(v\right)}\int_{\underline{v}}^{v}\frac{1}{F\left(y\right)}\int_{\underline{v}}^{y} xdF\left(x\right)dF^{n-1}\left(y\right). \tag{10}$$

Apply partial integration on the second and last RHS term in (10), to obtain respectively

$$\frac{1}{F^{n-1}(v)}\int_{\underline{v}}^{v} F^{n-1}(x)\,dx = v - \frac{1}{F^{n-1}(v)}\int_{\underline{v}}^{v} x dF^{n-1}(x)$$

and

$$\begin{aligned}
&\frac{1}{F^{n-1}(v)}\int_{\underline{v}}^{v}\frac{1}{F(y)}\int_{\underline{v}}^{y} x dF(x)\, d\left(F^{n-1}(y)\right) \\
&= \frac{n-1}{F^{n-1}(v)}\int_{\underline{v}}^{v}\int_{\underline{v}}^{y} x dF(x)\left(F^{n-3}(y)\right) dF(y) \\
&= \frac{n-1}{n-2}\frac{\int_{\underline{v}}^{v} x dF(x)}{F(v)} - \frac{1}{n-2}\frac{\int_{\underline{v}}^{v} x dF^{n-1}(x)}{F^{n-1}(x)},
\end{aligned}$$

and substitute these in (10) to obtain

$$\beta(v) = v + \frac{n-1}{n-2}\left(\frac{\int_{\underline{v}}^{v} x dF^{n-1}(x)}{F^{n-1}(v)} - \frac{\int_{\underline{v}}^{v} x dF(x)}{F(v)}\right).$$

To find the lower bound in the first-price auction, note that by the intermediate value theorem two values v_1 and v_2 exist such that

$$\beta(v) = v - \frac{n-1}{n-2}\left(v_1\frac{\int_{\underline{v}}^{v} dF(x)}{F(v)} - v_2\frac{\int_{\underline{v}}^{v} dF^{n-1}(x)}{F^{n-1}(v)}\right)$$

Moreover,

$$\lim_{v\to\underline{v}^+} v_1 = \lim_{v\to\underline{v}^+} v_2 = \underline{v},$$

such that $\lim_{v\to\underline{v}^+}\beta(v) = \underline{v}$. ■

Claim 5 (Second order condition) *The second order conditions are satisfied iff* $\beta'(.) > 0$

Proof. We first show that a strictly increasing bidding function implies local strict concavity of the bidder's problem, and then that the equilibrium bid is a global expected utility maximizing choice for each bidder.

First, use the first order condition (2) to define

$$G(\tilde{v},v) \equiv \left(F^{n-1}(\tilde{v})\right)'(v-\beta(\tilde{v})+\tilde{v})+(1-\beta'(\tilde{v}))\left(F^{n-1}(\tilde{v})\right)-\frac{1}{F(\tilde{v})}\int_{\underline{v}}^{\tilde{v}} xdF(x)\left(F^{n-1}(\tilde{v})\right)'=0,$$

which defines $\beta(v)$ for $\tilde{v}=v$. By the implicit function theorem $\beta'(v)>0$ if and only if strictly higher v prefer to imitate a strictly higher $\tilde{v}$, i.e. if

$$-\frac{G_2(\tilde{v},v)}{G_1(\tilde{v},v)}=-\frac{\left(F^{n-1}(\tilde{v})\right)'}{G_1(\tilde{v},v)}>0,$$

which is only satisfied if $G_1(\tilde{v},v)<0$ for all v at $\tilde{v}=v$.
By construction, $G(\tilde{v},v)=0$ is satisfied at $\tilde{v}=v$, while $G_2(\tilde{v},v)>0$ for all $\tilde{v}>v$, such that type v's utility reaches a unique maximum at $\tilde{v}=v$. ■

Claim 6 (Strictly increasing β) *β is strictly increasing.*

Proof. Write

$$\begin{aligned}\beta'(v) &= 1+\frac{n-1}{n-2}\frac{f(v)}{F(v)}\left((n-1)\left(v-\frac{\int_{\underline{v}}^{v} xdF^{n-1}(x)}{F^{n-1}(v)}\right)-\left(v-\frac{\int_{\underline{v}}^{v} xdF(x)}{F(v)}\right)\right)\\ &= 1+\frac{n-1}{n-2}\frac{f(v)}{F(v)}\left((n-1)\frac{\int_{\underline{v}}^{v} F^{n-1}(x)\,dx}{F^{n-1}(v)}-\frac{\int_{\underline{v}}^{v} F(x)\,dx}{F(v)}\right),\end{aligned}$$

with the last equation by partial integration. To see that

$$(n-1)\frac{\int_{\underline{v}}^{v} F^{n-1}(x)\,dx}{F^{n-1}(v)}-\frac{\int_{\underline{v}}^{v} F(x)\,dx}{F(v)}\geq 0,$$

note that this term is 0 for $n=2$, and that

$$\begin{aligned}&\frac{\partial}{\partial n}\left((n-1)\frac{\int_{\underline{v}}^{v} F^{n-1}(x)\,dx}{F^{n-1}(v)}\right)\\ &=\frac{\int_{\underline{v}}^{v} F^{n-1}(x)\,dx}{F^{n-1}(v)}+\frac{(n-1)^2}{F(v)}\left(\frac{\int_{\underline{v}}^{v} F^{n-2}(x)\,dx}{F^{n-2}(v)}-\frac{\int_{\underline{v}}^{v} F^{n-1}(x)\,dx}{F^{n-1}(v)}\right)>0,\end{aligned}$$

such that $\beta'(v)>1$ for $n\geq 3$. ■

A.3 Proof of Proposition 2

The proof that all D1 PBE bidding functions of the all-pay auction satisfy $\beta'(.) > 0$ is almost identical to the proof of Lemma 1, and therefore omitted. Let denote $EP^k(\tilde{v})$ denote the expected payment of a bidder choosing type $\tilde{v}$'s equilibrium bid, with $k = I, A$ indicating resp. the first-price and all-pay auction. Let

$$E(\tilde{v}) = F^{n-1}(\tilde{v})\,\tilde{v} + \left(1 - \left(F^{n-1}(\tilde{v})\right)\right) \frac{\int_{\tilde{v}}^{\bar{v}} \frac{1}{F(y)} \int_{\underline{\mathrm{v}}}^{y} x dF(x)\, dF^{n-1}(y)}{\left(1 - \left(F^{n-1}(\tilde{v})\right)\right)}$$

represent the receiver's expected inference about a bidder choosing type $\tilde{v}$'s equilibrium bid. Then the expected payoff of a valuation v bidder choosing a $\tilde{v}$ type's equilibrium bid is

$$F(\tilde{v})^{n-1} v - EP^k(\tilde{v}) + E(\tilde{v}).$$

The first order condition for expected payoff maximization is

$$(F(\tilde{v})^{n-1})' v - \left(EP^k(\tilde{v})\right)' + \left(E(\tilde{v})\right)' = 0.$$

Substituting $\tilde{v} = v$ solving for EP^k, we obtain

$$EP^k(v) = EP^k(v) + \int_v^v x dF(x)^{n-1} + \int_v^v \left(E(x)\right)' dx$$

As $EP^I(v) = F(v)^{n-1}\beta^I(v)$, $EP^A(v) = \beta^A(v)$ and $EP^I(\underline{\mathrm{v}}) = EP^A(\underline{\mathrm{v}}) = 0$, it follows that $\beta^A(v) = F(v)^{n-1}\beta^I(v)$.

A.4 Proof of Lemma 2

This proof proceeds in the same 3 steps as the proof of Lemma 1, and the first and third step are similar to those in the proof of Lemma 1. Let $\Pr(1|b)$, $\Pr(2|b)$ and $\Pr(3|b) = 1 - \Pr(1|b) - \Pr(2|b)$ resp. denote the probabilities of winning, having the second highest bid and having a lower bid with bid b, and let $E^1(b)$, $E^2(b)$ and $E^l(b)$ be the expected inferences of the receiver if a bidder with bid b resp. wins, has the second highest bid and loses, and let $E^p(b)$ be the expected payment of a winner with bid b.

Claim 7 (Weakly increasing) *If a v' type bids b' in equilibrium, then no $v'' < v'$ bids $b'' > b'$ in equilibrium.*

Proof. Assume the opposite. Because both b' and b'' are sent in equilibrium, it must be that $\Pr(1|b'') > \Pr(1|b')$. Then if v'' bids b'' in equilibrium, it must be that

$$\begin{aligned}
&\Pr(1|b'')\left(v'' - E^p(b'') + E^1(b'')\right) + \Pr(2|b'')\, E^2(b'') \\
&+ (1 - \Pr(1|b'') - \Pr(2|b''))\, E^l(b'') \\
&\geq \Pr(1|b')\left(v'' - E^p(b') + E^1(b')\right) \\
&+ \Pr(2|b')\, E^2(b') + (1 - \Pr(1|b') - \Pr(2|b'))\, E^l(b').
\end{aligned}$$

But given that $\Pr(1|b'') > \Pr(1|b')$, this implies that the v' type strictly prefers a b'' bid above b', which contradicts the equilibrium. Assume then that in equilibrium the v'' type's equilibrium expected utility is so low that $M^+(b'', v') \subseteq M^+(b'', v'') \cup M^0(b'', v'')$, then it must be that the v'' strictly prefers the bundle $\left(b', E^1(b'), E^2(b'), E^l(b'), E^p(b')\right)$ to her equilibrium strategy, a contradiction. Hence, if type v' bids b' in equilibrium, then $\mu(v''|b'') = 0$, and no v'' type with $v'' < v'$ chooses a b'' bid, with $b'' > b'$. ■

Claim 8 (No pooling with $\underline{v}$) *In the D1 PBE, no other type pools with $\underline{v}$.*

Proof. Suppose a nondegenerate set of types $O = \left\{v|\beta(v) = \tilde{b}\right\}$, with $\underline{v} \in O$, pool in equilibrium at bid $\tilde{b}$. If $n \geq 3$, then if type $\underline{v}$ (or a type just above her) deviates to a bid $\tilde{b} - \varepsilon$, for $\varepsilon > 0$, she has zero probability of having the highest or second highest bid, while the expected inference if she loses remains unchanged at $E^l\left(\tilde{b}\right)$. In equilibrium, such a deviation cannot be profitable such that:

$$\Pr\left(1|\tilde{b}\right)\left(\underline{v} - \tilde{b} + E^1\left(\tilde{b}\right)\right) + \Pr\left(2|\tilde{b}\right) E^2\left(\tilde{b}\right) + \left(1 - \Pr\left(1|\tilde{b}\right) - \Pr\left(2|\tilde{b}\right)\right) E^l\left(\tilde{b}\right) \geq E^l\left(\tilde{b}\right)$$

or that

$$\frac{\Pr\left(1|\tilde{b}\right)}{\Pr\left(1|\tilde{b}\right)+\Pr\left(2|\tilde{b}\right)}\left(\underline{\mathrm{v}}-\tilde{b}+E^{1}\left(\tilde{b}\right)\right)+\frac{\Pr\left(2|\tilde{b}\right)}{\Pr\left(1|\tilde{b}\right)+\Pr\left(2|\tilde{b}\right)}E^{2}\left(\tilde{b}\right)\geq E^{l}\left(\tilde{b}\right).$$

Note that because $E^{2}\left(\tilde{b}\right)\leq E^{l}\left(\tilde{b}\right)$, it must be that

$$\underline{\mathrm{v}}-\tilde{b}+E^{1}\left(\tilde{b}\right)\geq E^{2}\left(\tilde{b}\right). \tag{11}$$

If the $\sup(O)$ type would deviate to a bid $\tilde{b}+\varepsilon$, for $\varepsilon>0$ small enough such that $\tilde{b}+\varepsilon$ is out-of-equilibrium and no equilibrium bids are in $\left(\tilde{b},\tilde{b}+\varepsilon\right)$, she still pays $\tilde{b}$ and gets expected inference $E^{1}\left(\tilde{b}\right)$ if winning, is inferred as $E^{2}\left(\tilde{b}+\varepsilon\right)>E^{2}\left(\tilde{b}\right)$ if having the second highest bid and has expected inference $E^{l}\left(\tilde{b}+\varepsilon\right)$ if losing. For $\sup(O)$ to bid $\tilde{b}$ in equilibrium, it must be that

$$\begin{gathered}\Pr\left(1|\tilde{b}+\varepsilon\right)\left(\sup(O)-\tilde{b}+E^{1}\left(\tilde{b}\right)\right)+ \\ \Pr\left(2|\tilde{b}+\varepsilon\right)E^{2}\left(\tilde{b}+\varepsilon\right)+\left(1-\Pr\left(1|\tilde{b}+\varepsilon\right)-\Pr\left(2|\tilde{b}+\varepsilon\right)\right)E^{l}\left(\tilde{b}+\varepsilon\right)\leq \\ \Pr\left(1|\tilde{b}\right)\left(\sup(O)-\tilde{b}+E^{1}\left(\tilde{b}\right)\right)+\Pr\left(2|\tilde{b}\right)E^{2}\left(\tilde{b}\right)+\Pr\left(3|\tilde{b}\right)E^{l}\left(\tilde{b}\right).\end{gathered} \tag{12}$$

Note then that

$$E^{l}\left(\tilde{b}\right)=\frac{\Pr\left(3|\tilde{b}+\varepsilon\right)}{\Pr\left(3|\tilde{b}\right)}E^{l}\left(\tilde{b}+\varepsilon\right)+\frac{\Pr\left(3|\tilde{b}\right)-\Pr\left(3|\tilde{b}+\varepsilon\right)}{\Pr\left(3|\tilde{b}\right)}E^{2}\left(\tilde{b}\right), \tag{13}$$

i.e. if the $\sup(O)$ type is neither winning nor second when pooling at $\tilde{b}$, then the second highest bidder either has a higher valuation or she is in O. In the former case, the receiver's expected inference is $E^{l}\left(\tilde{b}+\varepsilon\right)$. In the latter case it must be $E^{2}\left(\tilde{b}\right)$. Substituting (13) this in (12), we obtain

$$\left(\Pr\left(1|\tilde{b}+\varepsilon\right)-\Pr\left(1|\tilde{b}\right)\right)\left(\sup(O)-\tilde{b}+E^1\left(\tilde{b}\right)-E^2\left(\tilde{b}\right)\right)+$$
$$\Pr\left(2|\tilde{b}+\varepsilon\right)\left(E^2\left(\tilde{b}+\varepsilon\right)-E^2\left(\tilde{b}\right)\right)\leq 0,$$

which can only be satisfied is $\sup(O) = \underline{v}$. ■

Claim 9 (No Pooling) *In the D1 PBE there is no pooling at bids strictly above* $\underline{v}$.

Proof. Assume that $\tilde{b}$ is the lowest bid at which a nondegenerate set of types $O = \left\{v|\beta(v) = \tilde{b}\right\}$ pool. The same envelope theorem argument as for the first-price auction also works for the second. Note then again that the expected utility difference between $\sup(O)$ and $\inf(O)$ while pooling at $\tilde{b}$ is $p\left(1|\bar{b}\right)(\sup(O) - \inf(O))$, while in separation this is by the envelope theorem

$$\int_{\inf(O)}^{\sup(O)} F^{N-1}(x)\,dx = \sup(O)\,F^{N-1}(\sup(O)) - \inf(O)\,F^{N-1}(\inf(O)) - \int_{\inf(O)}^{\sup(O)} x dF^{N-1}(x)\,.$$

If $\inf(O) = \sup(O)$, these are both equal to zero, but by the same differential argument as for Claim 3,

$$p\left(\bar{b}\right)(\sup(O) - \inf(O)) < \int_{\inf(O)}^{\sup(O)} F^{N-1}(x)\,dx.$$

The condition $f'(.) \leq 0$, imposed to guarantee the existence of a separating equilibrium, always guarantees this inequality. ■

A.5 Proof of Proposition 3

The proof proceeds in three steps: deriving the bidding function, showing that the second order condition is satisfied if the bidding functions is strictly increasing and showing that the proposed bidding function is strictly increasing. The second step is almost identical to Claim 5, and is omitted.

Claim 10 (Bidding function) *β is as written in Proposition 3*

Proof. From (4), collect terms to obtain

$$\beta(v) = \frac{n-2}{n-1}\frac{1}{F(v)}\left(v - \frac{\int_{\underline{v}}^{v} x dF(x)}{F(v)}\right) + \frac{\int_{v}^{\bar{v}} x dF(x)}{1-F(v)} + \frac{1}{(n-1)}\frac{(1-F(v))}{f(v)} \tag{14}$$

$$= \frac{n-2}{n-1}\frac{\int_{\underline{v}}^{v} F(x)\,dx}{F^2(v)} + \frac{1}{(n-1)}\frac{\int_{v}^{\bar{v}} f(x)\,dx}{f(v)} + \frac{\int_{v}^{\bar{v}} x dF(x)}{1-F(v)}, \tag{15}$$

where (15) is obtained from (14) by partially integrating the first term. Then by L'Hôpital's rule, $\lim_{v\to\underline{v}} \frac{\int_{\underline{v}}^{v} F(x)dx}{F^2(v)} = \frac{F(\underline{v})}{2F(\underline{v})f(\underline{v})}$, while $\lim_{v\to\bar{v}} \frac{\int_{v}^{\bar{v}} xdF(x)}{1-F(v)} = \frac{-\bar{v}f(\bar{v})}{-f(\bar{v})} = \bar{v}$, such that

$$\lim_{v\to\underline{v}} \beta(v) = E(V) + \frac{n}{2(n-1)f(\underline{v})}$$
$$\lim_{v\to\bar{v}} \beta(v) = \bar{v} + \frac{n-2}{n-1}(\bar{v} - E(V)).$$

■

Claim 11 (Second order condition) *The second order condition is satisfied iff* $\beta'(.) > 0$.

Proof. The proof that $\beta'(.) > 0$ implies that the second order condition is satisfied is identical to that of Claim 5. ■

Claim 12 (Strictly increasing β) β *is strictly increasing if* $n \geq 4$ *and* $f'(.) \leq 0$ *or if* $n = 3$ *and* $f'(.) < 0$.

Proof. Write

$$\beta'(v) = \frac{n-2}{n-1}\frac{1}{F(v)}\left(1 - 2f(v)\frac{\int_{\underline{v}}^{v} F(x)\,dx}{F^2(v)}\right) + \frac{f(v)}{1-F(v)}\left(\frac{\int_{v}^{\bar{v}} xdF(x)}{1-F(v)} - v\right)$$
$$- \frac{1}{(n-1)}\left(1 + \frac{f'(v)\int_{v}^{\bar{v}} f(x)\,dx}{(f(v))^2}\right), \tag{16}$$

and apply partial integration on the second RHS term in (16) to find

$$\beta'(v) = \frac{n-2}{n-1}\frac{1}{F(v)}\left(1 - 2f(v)\frac{\int_{\underline{v}}^{v} F(x)\,dx}{F^2(v)}\right) + \frac{f(v)\int_{v}^{\bar{v}}(1-F(x))\,dx}{(1-F(v))^2} \tag{17}$$

$$-\frac{1}{(n-1)} - \frac{1}{(n-1)}\frac{f'(v)\int_{v}^{\bar{v}} f(x)\,dx}{(f(v))^2}.$$

Note then that all RHS terms in (17) are nonnegative if $f'(.) \leq 0$, except $-\frac{1}{(n-1)}$. If $f'(.) < 0$ and $v > \underline{v}$, then $2f(v)\frac{\int_{\underline{v}}^{v} F(x)dx}{F^2(v)} < \frac{\int_{\underline{v}}^{v} dF^2(x)}{F^2(v)} = 1$, such that $1 - 2f(v)\frac{\int_{\underline{v}}^{v} F(x)dx}{F^2(v)} > 0$. At the other hand, the last term $-\frac{1}{(n-1)}\frac{f'(v)\int_{v}^{\bar{v}} f(x)dx}{(f(v))^2}$ is strictly positive for $v < \bar{v}$. Both terms are zero for $f'(.) = 0$. The main step is now to prove that $f'(.) \leq 0$ implies

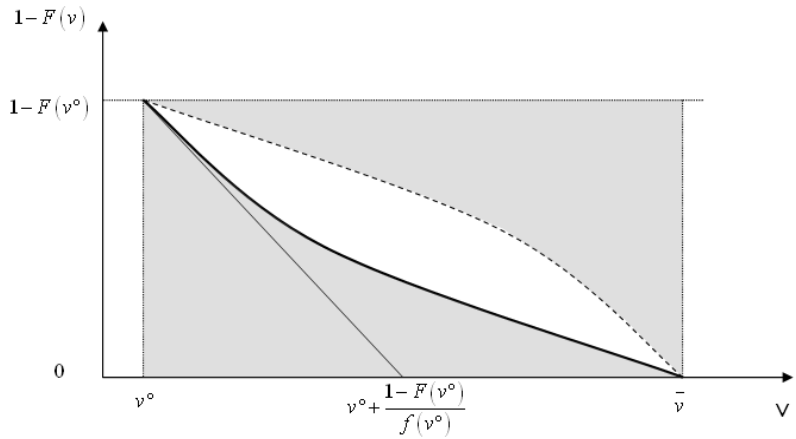
$$\frac{f(v)\int_{v}^{\bar{v}}(1-F(x))\,dx}{(1-F(v))^2} \geq \frac{1}{2}. \tag{18}$$

First note that F is the uniform distribution, inequality (18) is satisfied with equality. Note that $f'(.) \leq 0$ implies that $1 - F(.)$ is convex and write the inequality as

$$2\frac{\int_{v}^{\bar{v}}(1-F(x))\,dx}{1-F(v)} \geq \frac{1-F(v)}{f(v)}. \tag{19}$$

In figure A.5, that the LHS of (19), for $v = v^\circ$, is the grey area divided by the distance $1-F(v^\circ)$. Moreover, $\frac{\partial(1-F(v^\circ))}{\partial v^\circ} = -f(v^\circ)$, such that this tangent line through $(v^\circ, 1-F(v^\circ))$ crosses the X-axis at $v^\circ + \frac{1-F(v^\circ)}{f(v^\circ)}$. For $f'(.) = 0$, it must be that $v^\circ + \frac{1-F(v^\circ)}{f(v^\circ)} = \bar{v}$, such that the inequality in (19) is always satisfied with equality. If however $f'(v) < 0$ at some $v > v^\circ$, this strictly increases the LHS but not the RHS of (19), such that the inequality is strictly satisfied. Thus, $f'(.) \leq 0$ implies that

$$\frac{f(v)\int_{v}^{\bar{v}}(1-F(x))\,dx}{(1-F(v))^2} \geq \frac{1}{2} > \frac{1}{(n-1)}.$$



Hence, $\beta'(.) > 0$ for $n > 3$, while for $n = 3$, we need $f'(.) < 0$ to guarantee $\beta'(.) > 0$. ■

A.6 Proof of Lemma 3

Claim 13 (β weakly increasing) *If in a D1 PBE v' exits at b', then no $v'' < v'$ exits at $b'' > b'$.*

Proof. Assume that v'' stays until b''. If type v' exits at b', then what she can win by staying is not better than what can be expected by exiting. The expected payoff of exiting at b' is identical for the v' and v'' types, while v' benefits strictly more from winning than v'', such that v'' should strictly prefer to exit at b'. ■

Assume then a PBE with v' exiting at b', $v'' < v'$ and $b'' > b'$ an out-of-equilibrium exit strategy. Then if type v'' equilibrium strategy is so low that $M^+(b'', v') \subseteq M^+(b'', v'') \cup M^0(b'', v'')$, then type v'' would strictly prefer to exit at b' above her equilibrium strategy, a contradiction. Hence, $M^+(b'', v'') \cup M^0(b'', v'') \subset M^+(b'', v')$, such that in any D1 PBE we have $\beta'(.) \geq 0$.

Claim 14 (No pooling) *In any D1 PBE, no two types $v' \neq v''$ exit at the same price $\tilde{b}$.*

Let $\tilde{b}$ be the lowest price at which a nondegenerate set of types $O = \left\{v|\beta(v) = \tilde{b}\right\}$ exit. By Claim 13, O is convex. For a non-degenerate set O, a sufficiently small $\varepsilon > 0$ can be found for which the winning equilibrium payoff at price $\tilde{b}+\varepsilon$ is strictly greater than at $\tilde{b}$. If $\tilde{b}+\varepsilon$ is out-of-equilibrium, then for ε sufficiently small and O nondegenerate

$$\sup(O)-\tilde{b}-\varepsilon+\frac{\int_{\sup(O)}^{\bar{v}} xdF(x)}{1-F(\sup(O))} > \sup(O)-\tilde{b}+\frac{\int_{\inf(O)}^{\bar{v}} xdF(x)}{1-F(\inf(O))}, \quad (20)$$

while the expected payoff of a loser exiting at $\tilde{b}+\varepsilon$ is at least as large as that of a loser exiting at $\tilde{b}$. If $\tilde{b}$ is chosen in PBE by higher types, this increases the RHS of inequality 20. Hence, $\beta'(.) > 0$.

A.7 Proof of Proposition 4

Equation (7) is obtained by setting $\tilde{v} = v$ in (5) and solving for β. To see that $f'(.) \leq 0$ implies $\beta'(.) > 0$, write

$$\beta'(v) = \frac{n-2}{n-1}\left[1-\frac{f(v)}{F(v)}\left(v-\frac{\int_{\underline{v}}^{v} xdF(x)}{F(v)}\right)\right]+\frac{f(v)}{1-F(v)}\left(\frac{\int_{v}^{\bar{v}} xdF(x)}{1-F(v)}-v\right).$$

The second RHS term is always strictly positive for $v \in [\underline{v}, \bar{v})$. To see that the first RHS term is always positive, note that the term between square brackets is strictly positive if

$$F(v) > \left(v-\frac{\int_{\underline{v}}^{v} xdF(x)}{F(v)}\right)F'(v),$$

which is always satisfied. Indeed for $f'(.) \leq 0$, F is concave such that $F(v) \geq F'(v)(v-\underline{v}) \geq \left(v-\frac{\int_{\underline{v}}^{v} xdF(x)}{F(v)}\right)F'(v)$, with the last inequality strict for $v \in (\underline{v}, \bar{v}]$.

For β as in (7), the exit rule in (5) fixes for every v a unique $\tilde{v}$, as

$$\frac{\partial}{\partial \tilde{v}}\left(-\beta(\tilde{v})+\frac{1}{1-F(\tilde{v})}\int_{\tilde{v}}^{\bar{v}} xdF(x)-\frac{\tilde{v}}{n-1}-\frac{n-2}{n-1}\frac{\int_{\underline{v}}^{\tilde{v}} xdF(x)}{F(\tilde{v})}\right) = -1.$$

Note also that no type v wishes to mimic a different type $\tilde{v}$. By construction $\beta(\tilde{v})$ is such that (5) is satisfied with equality for $v = \tilde{v}$ and such that for $v > \tilde{v}$ the benefits of winning (LHS) are strictly greater than the RHS when mimicking $\tilde{v}$'s strategy. The latter is the opposite if $v < \tilde{v}$.

A.8 Proof of Proposition 5

Let $\beta^k(.)$ denote the equilibrium biding function and ER^k be the expected revenue for $k = I, II, E$ respectively the first-price auction, the second-price auction, the English auction. We first write the expected revenue of the 3 auctions in a more convenient form.

$$\begin{aligned}
ER^I &= \int_{\underline{v}}^{\bar{v}} \beta^I(x)\, dF^n(x) \\
&= \int_{\underline{v}}^{\bar{v}} x dF^n(y) - \frac{n-1}{n-2}\int_{\underline{v}}^{\bar{v}} \frac{\int_{\underline{v}}^{y} x dF(x)}{F(y)} dF^n(y) \\
&+ \frac{n-1}{n-2}\int_{\underline{v}}^{\bar{v}} \frac{\int_{\underline{v}}^{y} x dF^{n-1}(x)}{F^{n-1}(y)} dF^n(y)
\end{aligned}$$

$$\begin{aligned}
&= E\left(V_1^{(n)}\right) + \frac{n-1}{n-2}E\left(V_2^{(n)}\right) - \frac{n-1}{n-2}\int_{\underline{v}}^{\bar{v}} \frac{1}{F(y)} \int_{\underline{v}}^{\bar{v}} 1_{x\le v} x dF(x) \left(F^n(y)\right)' dy \\
&= E\left(V_1^{(n)}\right) + \frac{n-1}{n-2}E\left(V_2^{(n)}\right) - \frac{(n-1)\,n}{n-2}\int_{\underline{v}}^{\bar{v}}\int_{\underline{v}}^{\bar{v}} 1_{x\le v} f(y)\left(F^{n-2}(y)\right) dy x f(x)\, dx \\
&= \frac{n-1}{n-2}E\left(V_1^{(n)}\right) + \frac{n-1}{n-2}E\left(V_2^{(n)}\right) - \frac{n}{n-2}E(V) \qquad (21)
\end{aligned}$$

The expected revenue of the second-price auction is

$$\begin{aligned}
ER^{II} &= \int_{\underline{v}}^{\bar{v}} \beta^{II}(x)\, d\left(nF^{n-1}(x) - (n-1)F^n(x)\right) \\
&= n(n-2)\int_{v}^{\bar{v}} x(1-F(x))F^{n-3}(x)f(x)dx \\
&- n(n-2)\int_{\underline{v}}^{\bar{v}} \frac{\int_{\underline{v}}^{y} F(x)\, dx}{F^2(y)} (1-F(y))\, F^{n-2}(y)\, f(y)\, dy \\
&+ n(n-1)\int_{\underline{v}}^{\bar{v}} \frac{\int_{y}^{\bar{v}} x dF(x)}{1-F(y)} \left((1-F(y))\, F^{n-2}(y)\, f(y)\right) dy \\
&+ n\int_{\underline{v}}^{\bar{v}} \left((1-F(y))^2 F^{n-2}(y)\right) dy
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n}{n-1} E\left(V_2^{(n-1)}\right) - \frac{n}{n-3} E(V) + \frac{n}{n-3} E\left(V_1^{(n-2)}\right) \\
&- \frac{n}{n-1} E\left(V_1^{(n-1)}\right) + E\left(V_1^{(n)}\right) + n \int_{\underline{v}}^{\bar{v}} \left((1-F(y))^2 F^{n-2}(y)\right) dy
\end{aligned}
$$

The expected revenue of the English auction is:

$$
\begin{aligned}
ER^E &= (n-1) n \int_{\underline{v}}^{\bar{v}} \left(\frac{n-2}{n-1} \left(v - \frac{\int_{\underline{v}}^{v} x dF(x)}{F(v)} \right) + \frac{\int_{v}^{\bar{v}} x dF(x)}{1-F(v)} \right) F^{n-2}(v) (1-F(v)) f(v) dv \\
&= n(n-2) \int_{\underline{v}}^{\bar{v}} v F^{n-2}(v) f(v) dv - n(n-2) \int_{\underline{v}}^{\bar{v}} v F^{n-1}(v) f(v) dv \\
&- (n-2) n \int_{\underline{v}}^{\bar{v}} \frac{\int_{\underline{v}}^{v} x dF(x)}{F(v)} F^{n-2}(v) (1-F(v)) f(v) dv \\
&+ (n-1) n \int_{\underline{v}}^{\bar{v}} \frac{\int_{v}^{\bar{v}} x dF(x)}{1-F(v)} F^{n-2}(v) (1-F(v)) f(v) dv
\end{aligned}
$$

$$
\begin{aligned}
&= n \frac{n-2}{n-1} E\left(V_1^{(n-1)}\right) - (n-2) E\left(V_1^{(n)}\right) \\
&- (n-2) n \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}}^{v} x dF(x) F^{n-3}(v) (1-F(v)) f(v) dv \\
&+ (n-1) n \int_{\underline{v}}^{\bar{v}} \int_{v}^{\bar{v}} x dF(x) F^{n-2}(v) f(v) dv
\end{aligned}
$$

$$
\begin{aligned}
&= n \frac{n-2}{n-1} E\left(V_1^{(n-1)}\right) - (n-2) E\left(V_1^{(n)}\right) \\
&- (n-2) n \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}}^{\bar{v}} 1_{x<v} x dF(x) F^{n-3}(v) (1-F(v)) f(v) dv \\
&+ (n-1) n \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}}^{\bar{v}} 1_{x>v} x dF(x) F^{n-2}(v) f(v) dv
\end{aligned}
$$

$$
\begin{aligned}
&= n\frac{n-2}{n-1}E\left(V_1^{(n-1)}\right) - (n-2)E\left(V_1^{(n)}\right) \\
&- (n-2)\,n\int_{\underline{v}}^{\bar{v}}\int_{x}^{\bar{v}} F^{n-3}(v)\,(1-F(v))\,f(v)dvxdF(x) \\
&+ n\int_{\underline{v}}^{\bar{v}}\int_{\underline{v}}^{x} dF^{n-1}(v)\,xdF(x)\,,
\end{aligned}
$$

such that

$$
ER^E = -\frac{n}{n-1}E(V) + nE\left(V_1^{(n-1)}\right) - \frac{n^2-3n+1}{n-1}E\left(V_1^{(n)}\right).
$$

Claim 15 (English and first-price auction revenue) *In the D1 PBE* $ER^I > ER^E$.

Proof. We use that

$$
E\left(V_2^{(n)}\right) = nE\left(V_1^{(n-1)}\right) - (n-1)\,E\left(V_1^{(n)}\right) \tag{22}
$$

to write

$$
ER^E = -\frac{n}{n-1}E(V) + E\left(V_2^{(n)}\right) + \frac{n}{n-1}E\left(V_1^{(n)}\right)
$$

such that

$$
\begin{aligned}
ER^I - ER^E &= -\frac{n}{(n-1)(n-2)}E(V) + \frac{1}{(n-1)(n-2)}E\left(V_1^{(n)}\right) + \frac{1}{n-2}E\left(V_2^{(n)}\right) \\
&= \frac{1}{(n-1)(n-2)}\left(-nE(V) + E\left(V_1^{(n)}\right) + (n-1)\,E\left(V_2^{(n)}\right)\right).
\end{aligned}
$$

Note then that because

$$
nE(V) = \sum_{k=1}^{n} E\left(V_k^{(n)}\right),
$$

we have for $n \geq 3$ (which is required for a D1 PBE)

$$
ER^I - ER^E = \frac{1}{(n-1)(n-2)}\left(-\sum_{k=2}^{n} E\left(V_k^{(n)}\right) + (n-1)\,E\left(V_2^{(n)}\right)\right) > 0.
$$

■

Claim 16 (First- and second-price auction revenue) *In the D1 PBE* $ER^{II} > ER^{I}$.

Proof. We use (22) to write

$$\begin{aligned} ER^{II} &= n\frac{n-2}{n-3}E\left(V_1^{(n-2)}\right) - nE\left(V_1^{(n-1)}\right) - \frac{n}{n-3}E\left(V\right) \\ &+ E\left(V_1^{(n)}\right) + n\int_{\underline{v}}^{\bar{v}} \left((1-F(y))^2 F^{n-2}(y)\right) dy \end{aligned}$$

Then

$$\begin{aligned} ER^{II} - ER^{I} &= n\frac{n-2}{n-3}E\left(V_1^{(n-2)}\right) - nE\left(V_1^{(n-1)}\right) \\ &- \frac{n}{n-3}E\left(V\right) + E\left(V_1^{(n)}\right) + n\int_{\underline{v}}^{\bar{v}} \left((1-F(y))^2 F^{n-2}(y)\right) dy \\ &- \left(-(n-1)E\left(V_1^{(n)}\right) + n\frac{n-1}{n-2}E\left(V_1^{(n-1)}\right) - \frac{n}{n-2}E\left(Y\right)\right) \\ &= n\left(\begin{array}{c} E\left(V_1^{(n)}\right) - \frac{(n-2)+(n-1)}{n-2}E\left(V_1^{(n-1)}\right) + \frac{n-2}{n-3}E\left(V_1^{(n-2)}\right) \\ -\frac{1}{(n-2)(n-3)}E\left(V\right) + \int_{\underline{v}}^{\bar{v}} \left((1-F(y))^2 F^{n-2}(y)\right) dy \end{array}\right). \end{aligned}$$

Note then that

$$\begin{aligned} \int_{\underline{v}}^{\bar{v}} \left((1-F(y))^2 F^{n-2}(y)\right) dy &= \int_{\underline{v}}^{\bar{v}} F^{n-2}(y)\,dy - 2\int_{\underline{v}}^{\bar{v}} F^{n-1}(y)\,dy + \int_{\underline{v}}^{\bar{v}} F^{n}(y)\,dy \\ &= 2E\left(V_1^{(n-1)}\right) - E\left(V_1^{(n)}\right) - E\left(V_1^{(n-2)}\right), \end{aligned}$$

because by partial integration

$$\bar{v} = \int_{\underline{v}}^{\bar{v}} \left(yF^{n-2}(y)\right)' dy = \int_{\underline{v}}^{\bar{v}} F^{n-2}(y)\,dy + \int_{\underline{v}}^{\bar{v}} y\,dF^{n-2}(y)$$

and the same for the other terms. Then

$$ER^{II} - ER^{I} = n\left(-\frac{1}{n-2}E\left(V_1^{(n-1)}\right) + \frac{1}{n-3}E\left(V_1^{(n-2)}\right) - \frac{1}{(n-2)(n-3)}E\left(Y\right)\right)$$

Then write

$$ER^{II} - ER^{I} = n\int y\left(-\frac{n-1}{n-2}F^{n-2}(y) + \frac{n-2}{n-3}F^{n-3}(y) - \frac{1}{(n-2)(n-3)}\right)dF(y)$$
$$= n\int y\left(-\frac{n-1}{n-2}u^{n-2} + \frac{n-2}{n-3}u^{n-3} - \frac{1}{(n-2)(n-3)}\right)dF(y)$$

with $u \equiv F(y)$ and define

$$G(u) = -\frac{n-1}{n-2}u^{n-2} + \frac{n-2}{n-3}u^{n-3} - \frac{1}{(n-2)(n-3)}.$$

Note then that

$$\int_0^1 G(u)\,du = -\frac{1}{n-2} + \frac{1}{n-3} - \frac{1}{(n-2)(n-3)} = 0$$

while $G(0) = -\frac{1}{(n-2)(n-3)}$ and $G(1) = 0$. Moreover,

$$G'(u) = (-(n-1)u + (n-2))\,u^{n-4} = 0$$

at $u = \frac{n-2}{n-1}$, a maximum since $G''\left(\frac{n-2}{n-1}\right) = -(n-2)\left(\frac{n-2}{n-1}\right)^{n-4} < 0$. Thus, $G(u)$ must be strictly positive on an interval $[0, u^*)$ and strictly negative on $(u^*, 1)$, while $\int_{[0,u^*)} G(u)\,du = -\int_{(u^*,1)} G(u)\,du$.

Note then that $y(u) = F^{-1}(u)$ is a strictly increasing function. Then by the intermediate value theorem we can find two values $0 < y_1 < y_2$ such that

$$ER^{I} - ER^{II} = y_1\int_{[0,u^*)} G(u)\,du + y_2\int_{(u^*,1)} G(u)\,du$$

while by the above

$$ER^{I} - ER^{II} = (y_2 - y_1)\int_{(u^*,1)} G(u)\,du < 0.$$

■