Strategy-proof Rules on Top-connected Single-peaked and Partially Single-peaked Domains

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Abstract

We characterize all domains on which (i) every unanimous and strategy-proof social choice function is a min-max rule, and (ii) every min-max rule is strategy-proof. As an application of this result, we obtain a characterization of the unanimous and strategy-proof social choice functions on maximal single-peaked domains (Moulin (1980), Weymark (2011)), minimally rich single-peaked domains (Peters et al. (2014)), maximal regular single-crossing domains (Saporiti (2009), Saporiti (2014)), and distance based single-peaked domains. We further consider domains that exhibit single-peakedness only over a subset of alternatives. We call such domains top-connected partially single-peaked domains and provide a characterization of the unanimous and strategy-proof social choice functions on these domains. As an application of this result, we obtain a characterization of the unanimous and strategy-proof social choice functions on multiple single-peaked domains (Reffgen (2015)) and single-peaked domains on graphs. As a by-product of our results, it follows that strategy-proofness implies tops-onlyness on these domains. Moreover, we show that strategy-proofness and group strategy-proofness are equivalent on these domains.

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1. INTRODUCTION

We consider a standard social choice problem where an alternative has to be chosen based on privately known preferences of the agents in a society. Such a procedure is known as a *social choice function* (SCF). Agents are strategic in the sense that they misreport their preferences whenever it is strictly beneficial for them. An SCF is called *strategy-proof* if no agent can benefit by misreporting her preferences, and is called *unanimous* if whenever all the agents in the society unanimously agree on their best alternative, that alternative is chosen.

Most of the subject matter of social choice theory concerns the study of the unanimous and strategy-proof SCFs for different admissible domains of preferences. In the seminal works by Gibbard (1973) and Satterthwaite (1975), it is shown that if a society has at least three alternatives and there is no particular restriction on the preferences of the agents, then every unanimous and strategy-proof SCF is *dictatorial*, that is, a particular agent in the society determines the outcome regardless of the preferences of the others. The celebrated Gibbard-Satterthwaite theorem hinges crucially on the assumption that the admissible domain of each agent is unrestricted. However, it is well established that in many economic and political applications, there are natural restrictions on such domains. For instance, in the models of locating a firm in a unidimensional spatial market (Hotelling (1929)), setting the rate of carbon dioxide emissions (Black (1948)), setting the level of public expenditure (Romer and Rosenthal (1979)), and so on, preferences admit a natural restriction widely known as *single-peakedness*. Roughly speaking, the crucial property of a single-peaked preference is that there is a prior order over the alternatives such that the preference decreases as one moves away (with respect to the prior order) from her best alternative.

The study of single-peaked domains dates back to Black (1948). Moulin (1980) and Weymark (2011) have characterized the unanimous and strategy-proof SCFs on such domains as *min-max rules*.[1,2] The characterization by Moulin (1980) and Weymark (2011) rests upon the as-

[1] Barberà et al. (1993) and Ching (1997) provide equivalent presentations of this class of SCFs.

[2] A rich literature has developed around the single-peaked restriction by considering various generalizations and extensions (see Barberà et al. (1993), Demange (1982), Schummer and Vohra (2002), Nehring and Puppe (2007a), and Nehring and Puppe (2007b)).

sumption that the underlying domain is the *maximal* single-peaked domain, i.e., it contains all single-peaked preferences with respect to a given prior order over the alternatives. However, demanding the existence of all single-peaked preferences is a strong prerequisite in many practical situations.[3] Therefore, it is important to relax the maximality assumption on a single-peaked domain. On the other hand, min-max rules are quite popular for their desirable properties like *tops-onlyness* and *Pareto efficiency*. Moreover, a subclass of min-max rules known as *median rules* satisfies another desirable property called *anonymity*.

In continuity with the above discussion, we characterize all domains on which (i) every unanimous and strategy-proof social choice function is a min-max rule, and (ii) every min-max rule is strategy-proof. We call such a domain a *min-max domain*. We show that a domain is a min-max domain if and only if it is a single-peaked domain satisfying the *top-connectedness* property. The top-connectedness property with respect to a prior order requires that for every two consecutive (in that prior order) alternatives x and y, there exists a preference that places x at the top and y at the second-ranked position.[4] It is worth noting that in a social choice problem with m alternatives, the number of preferences in a min-max domain can range from $2m-2$ to 2^{m-1}, whereas that in the maximal single-peaked domain is exactly 2^{m-1}. Thus, min-max domains include a large class of restricted single-peaked domains.

A *top-connected regular single-crossing domain* (Saporiti (2009), Saporiti (2014)) is an example of a min-max domain.[5,6] Saporiti (2014) shows that an SCF is unanimous and strategy-proof on a *maximal single-crossing domain* if and only if it is a min-max rule.[7] Our result shows that an SCF is unanimous and strategy-proof on a top-connected regular single-crossing domain if and only if it is a min-max rule. Thus, we extend Saporiti (2014)'s result by relaxing the maximality assumption on a single-crossing domain. However, we assume the domains to be regular. Note that a maximal single-crossing domain requires $m(m-1)/2$ preferences, whereas a top-connected

[3]See, for instance, the domain restriction considered in models of voting (Tullock (1967), Arrow (1969)), taxation and redistribution (Epple and Romer (1991)), determining the levels of income redistribution (Hamada (1973), Slesnick (1988)), and measuring tax reforms in the presence of horizontal inequity (Hettich (1979)). Recently, Puppe (2015) shows that under mild conditions these domains form subsets of the maximal single-peaked domain.

[4]The top-connectedness property is well studied in the literature (see Barberà and Peleg (1990), Aswal et al. (2003), Chatterji and Sen (2011), Chatterji et al. (2014), Chatterji and Zeng (2015), and Puppe (2015)).

[5]A domain is *regular* if every alternative appears as the top-ranked alternative of some preference in the domain.

[6]Single-crossing domains appear in models of taxation and redistribution (Roberts (1977), Meltzer and Richard (1981)), local public goods and stratification (Westhoff (1977), Epple and Platt (1998), Epple et al. (2001)), coalition formation (Demange (1994), Kung (2006)), selecting constitutional and voting rules (Barberà and Jackson (2004)), and designing policies in the market for higher education (Epple et al. (2006)).

[7]Saporiti (2014) provides a different but equivalent functional form of these SCFs which he calls *augmented representative voter schemes*.

regular single-crossing domain requires $2m - 2$ preferences.

Although single-peaked domains are used to model many practical situations, several empirical studies (Niemi and Wright (1987), Feld and Grofman (1988), Pappi and Eckstein (1998)) fail to support the assumption that *all* the preferences of an agent are single-peaked. In view of this, we consider domains which exhibit single-peakedness only over a subset of alternatives. We call such domains *top-connected partially single-peaked domains*. We characterize the unanimous and strategy-proof SCFs on such domains as *partly dictatorial generalized median voter schemes* (PDGMVS). Loosely put, a PDGMVS acts like a min-max rule over the subset of the domain where single-peakedness is satisfied and like a dictatorial rule everywhere else.

Reffgen (2015) introduces the notion of *multiple single-peaked domains* and characterizes the unanimous and strategy-proof SCFs on such domains. A multiple single-peaked domain is the union of several maximal single-peaked domains with respect to different prior orders over the alternatives. A plausible justification for such a domain restriction is provided by Niemi (1969) who argues that the alternatives can be ordered differently using different criteria (which he calls an *impartial culture*) and it is not publicly known which agent uses what criterion. On one extreme, such a domain becomes an unrestricted domain if there is no consensus among the agents on the prior order, and on the other extreme, it becomes a maximal single-peaked domain if all the agents agree on a single prior order.

We extend Reffgen (2015)'s result in two directions: (i) by requiring minimum knowledge about the prior orders perceived by the agents, and (ii) by requiring a minimal set of single-peaked preferences for each of these prior orders. We further show that top-connected partially single-peaked domains contain almost all domains on which (i) every unanimous and strategy-proof SCF is a PDGMVS, and (ii) every PDGMVS is strategy-proof. It is worthwhile to mention that in a social choice problem with m alternatives, the number of preferences in a top-connected partially single-peaked domain can range from $2m$ to $m!$, whereas that in a multiple single-peaked domain with respect to k prior orders is approximately (depending on the prior orders) $k \times 2^{m-1}$. Thus, the class of top-connected partially single-peaked domains is quite large including both single-peaked and unrestricted domains.

A crucial step in the proof of our characterization results is to establish the tops-onlyness property. In case of multiple single-peaked domains (Reffgen (2015)), tops-onlyness property follows from the sufficient condition provided in Chatterji and Sen (2011). However, top-connected single-peaked and top-connected partially single-peaked domains do not satisfy their condition, and the novelty of our results lies in establishing the tops-onlyness property on these domains.

Lastly, we consider group strategy-proofness. Barberà et al. (2010) provides a sufficient condition for the equivalence of strategy-proofness and group strategy-proofness on a domain. Top-connected single-peaked domains satisfy their condition, and consequently, we obtain a characterization of the unanimous and group strategy-proof SCFs on these domains as a corollary of their result. However, top-connected partially single-peaked domains do not satisfy their condition. Therefore, we independently establish the equivalence of strategy-proofness and group strategy-proofness on these domains.

To put our results in perspective, we conclude this section by comparing them with a few related articles. Owing to the desirable properties of min-max rules, Barberà et al. (1999) characterize maximal domains on which a *given* min-max rule is strategy-proof. In contrast, we characterize domains where *all* min-max rules are strategy-proof. Recently, Arribillaga and Massó (2016) provide necessary and sufficient conditions for the comparability of two min-max rules in terms of their vulnerability to manipulation. However, our results identify the min-max rules that are manipulable if single-peakedness is violated over a subset of alternatives. Chatterji et al. (2013) study a related restricted domain known as a *semi-single-peaked domain*. Such a domain violates single-peakedness around the *tails* of the prior order. They show that if a domain admits an anonymous (and hence non-dictatorial), tops-only, unanimous, and strategy-proof SCF, then it is a semi-single-peaked domain. However, we show that if single-peakedness is violated around the *middle* of the prior order, then there is *no* unanimous, strategy-proof, and anonymous SCF. Thus, our characterization result on top-connected partially single-peaked domains complements that in Chatterji et al. (2013).

The rest of the paper is organized as follows. We describe the usual social choice framework in Section 2. In Section 3, we study the unanimous and strategy-proof SCFs on top-connected single-peaked domains. Section 4 studies the unanimous and strategy-proof SCFs on top-connected partially single-peaked domains. Section 5 deals with group strategy-proofness, and the last section concludes the paper. All the omitted proofs are collected in Appendix A and Appendix B.

2. PRELIMINARIES

Let $N = \{1, ..., n\}$ be a set of at least two agents, who collectively choose an element from a finite set $X = \{a, a+1, \dots, b-1, b\}$ of at least three alternatives, where a is an integer. For $x, y \in X$ such that $x \leq y$, we define the intervals $[x, y] = \{z \in X \mid x \leq z \leq y\}$, $[x, y) = [x, y] \setminus \{y\}$, $(x, y] = [x, y] \setminus \{x\}$, and $(x, y) = [x, y] \setminus \{x, y\}$. For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, i.e., we denote sets $\{i\}$ by i.

A *preference* P over X is a complete, transitive, and antisymmetric binary relation (also called a linear order) defined on X. We denote by $\mathbb{L}(X)$ the set of all preferences over X. An alternative $x \in X$ is called the k^{th} *ranked alternative* in a preference $P \in \mathbb{L}(X)$, denoted by $r_k(P)$, if $|\{a \in X \mid aPx\}| = k - 1$. A domain of admissible preferences, denoted by $\mathcal{D}$, is a subset of $\mathbb{L}(X)$. An element $P_N = (P_1, \dots, P_n) \in \mathcal{D}^n$ is called a *preference profile*. The *top-set* of a preference profile P_N, denoted by $\tau(P_N)$, is defined as $\tau(P_N) = \{x \in X \mid r_1(P_i) = x \text{ for some } i \in N\}$. A domain $\mathcal{D}$ of preferences is *regular* if for all $x \in X$, there exists a preference $P \in \mathcal{D}$ such that $r_1(P) = x$. All the domains we consider in this paper are assumed to be regular.

Definition 2.1. A *social choice function* (SCF) f on $\mathcal{D}^n$ is a mapping $f : \mathcal{D}^n \to X$.

Definition 2.2. An SCF $f : \mathcal{D}^n \to X$ is *unanimous* if for all $P_N \in \mathcal{D}^n$ such that $r_1(P_i) = x$ for all $i \in N$ and some $x \in X$, we have $f(P_N) = x$.

Definition 2.3. An SCF $f : \mathcal{D}^n \to X$ is *manipulable* if there exists $i \in N$, $P_N \in \mathcal{D}^n$, and $P_i^{'} \in \mathcal{D}$ such that $f(P_i^{'}, P_{N\setminus i}) P_i f(P_N)$. An SCF f is *strategy-proof* if it is not manipulable.

Definition 2.4. An SCF $f : \mathcal{D}^n \to X$ is called *dictatorial* if there exists $i \in N$ such that for all $P_N \in \mathcal{D}^n$, $f(P_N) = r_1(P_i)$.

Definition 2.5. A domain $\mathcal{D}$ is called *dictatorial* if every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to X$ is dictatorial.

Definition 2.6. Two preference profiles P_N, P'_N are called *tops-equivalent* if $r_1(P_i) = r_1(P'_i)$ for all agents $i \in N$.

Definition 2.7. An SCF $f : \mathcal{D}^n \to X$ is called *tops-only* if for any two tops-equivalent $P_N, P'_N \in \mathcal{D}^n$, $f(P_N) = f(P'_N)$.

Definition 2.8. A domain $\mathcal{D}$ is called *tops-only* if every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to X$ is tops-only.

Definition 2.9. A preference $P \in \mathbb{L}(X)$ is called *single-peaked* if for all $x, y \in X$, $[x < y \leq r_1(P)$ or $r_1(P) \leq y < x]$ implies yPx. A domain $\mathcal{S}$ is called a *single-peaked domain* if each preference in it is single-peaked, and a domain $\bar{\mathcal{S}}$ is called a *maximal single-peaked* domain if it contains all single-peaked preferences.

Definition 2.10. An SCF $f : \mathcal{D}^n \to X$ is called *uncompromising* if for all $P_N \in \mathcal{D}^n$, all $i \in N$, and all $P'_i \in \mathcal{D}$:

(i) if $r_1(P_i) < f(P_N)$ and $r_1(P'_i) \leq f(P_N)$, then $f(P_N) = f(P'_i, P_{-i})$, and

(ii) if $f(P_N) < r_1(P_i)$ and $f(P_N) \leq r_1(P'_i)$, then $f(P_N) = f(P'_i, P_{-i})$.

REMARK 2.1. If an SCF satisfies uncompromisingness, then by definition, it is tops-only.

Definition 2.11. Let $\beta = (\beta_S)_{S \subseteq N}$ be a list of 2^n parameters satisfying: (i) $\beta_S \in X$ for all $S \subseteq N$, (ii) $\beta_\emptyset = b$, $\beta_N = a$, and (iii) for any $S \subseteq T$, $\beta_T \leq \beta_S$. Then, an SCF $f^\beta : \mathcal{D}^n \to X$ is called a *min-max rule with respect to* β if

$$f^\beta(P_N) = \min_{S \subseteq N}\{\max_{i \in S}\{r_1(P_i), \beta_S\}\}.$$

REMARK 2.2. Every min-max rule is uncompromising.[8]

Now, we introduce a few graph theoretic notions. A *directed graph* G is defined as a pair $\langle V, E \rangle$, where V is the set of *nodes* and $E \subseteq V \times V$ is the set of *directed edges*, and an *undirected graph* G is defined as a pair $\langle V, E \rangle$, where V is the set of nodes and $E \subseteq \{\{u, v\} \mid u, v \in V \text{ and } u \neq v\}$ is the set of *undirected edges*. For a graph (directed or undirected) $G = \langle V, E \rangle$, a *subgraph* G' of G is defined as a graph $G' = \langle V, E' \rangle$, where $E' \subseteq E$. For two graphs $G_1 = \langle V_1, E_1 \rangle$ and $G_2 = \langle V_2, E_2 \rangle$, the graph $G_1 \cup G_2$ is defined as $G_1 \cup G_2 = \langle V_1 \cup V_2, E_1 \cup E_2 \rangle$.

All the graphs we consider in this paper are of the kind $G = \langle X, E \rangle$, i.e., whose node set is the set of alternatives.

Definition 2.12. A directed (undirected) graph $G = \langle X, E \rangle$ is called the *directed (undirected) line graph* on X if $(x, y) \in E$ ($\{x, y\} \in E$) if and only if $|x - y| = 1$.

[8]For details, see Weymark (2011).

Definition 2.13. Let $x, y \in X$ be such that $x < y - 1$. Then, a graph G is called a *directed (undirected) partial line graph with respect to x and y* if G can be expressed as $G_1 \cup G_2$, where $G_1 = \langle X, E_1 \rangle$ is the directed (undirected) line graph on X and $G_2 = \langle [x, y], E_2 \rangle$ is a directed (undirected) graph such that $(x, x'), (y, y') \in E_2$ ($\{x, x'\}, \{y, y'\} \in E_2$) for some $x' \in (x+1, y]$ and $y' \in [x, y-1)$.

In Figure 1, we present a directed partial line graph on $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ with respect to x_3 and x_6.

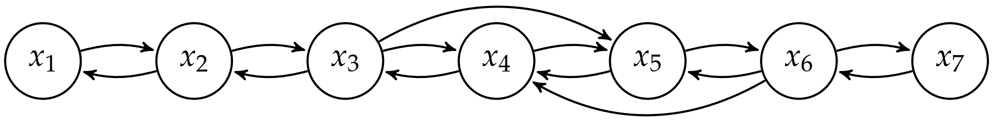


Figure 1: A directed partial line graph

Definition 2.14. The *top-graph* of a domain $\mathcal{D}$ is defined as the directed graph $\langle X, E \rangle$ such that $(x, y) \in E$ if and only if there exists a preference $P \in \mathcal{D}$ with $r_1(P) = x$ and $r_2(P) = y$.

3. Top-connected Single-peaked Domains

In this section, we introduce the notion of top-connected single-peaked domains and characterize the unanimous and strategy-proof SCFs on these domains. We begin with a few formal definitions.

Definition 3.1. A domain $\mathcal{D}$ satisfies the *top-connectedness* property if for all $x, x+1 \in X$, there are $P, P' \in \mathcal{D}$ such that $r_1(P) = r_2(P') = x$ and $r_2(P) = r_1(P') = x + 1$.

Note that a domain satisfies the top-connectedness property if and only if its top-graph is the directed line graph on X.

Definition 3.2. A domain $\hat{\mathcal{S}}$ is called a *top-connected single-peaked domain* if it is a single-peaked domain and it satisfies the top-connectedness property.

Note that the minimum cardinality of a top-connected single-peaked domain with m alternatives is $2m - 2$. Also, since the maximal single-peaked domain is also top-connected single-peaked, the maximum cardinality of such domains is 2^{m-1}. Thus, the class of top-connected single-peaked domains is quite large. In what follows, we provide an example of a top-connected single-peaked domain with five alternatives.

Example 3.1. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. Then, the domain in Table 1 is a top-connected single-peaked domain.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$
$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_4}$
x_3	x_3	x_4	x_1	x_4	x_4	x_2	x_5	x_2	x_5	x_3	x_3
x_4	x_4	x_1	x_4	x_5	x_5	x_5	x_2	x_1	x_2	x_2	x_2
x_5	x_5	x_5	x_5	x_1	x_1	x_1	x_1	x_5	x_1	x_1	x_1

Table 1: A top-connected single-peaked domain

In Figure 2, we present the top-graph of the domain in Example 3.1.

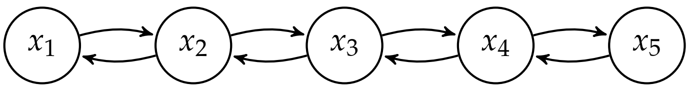


Figure 2: Top-graph of the domain in Example 3.1

3.1 Unanimous and Strategy-proof SCFs

In this subsection, we provide a characterization of the unanimous and strategy-proof SCFs on top-connected single-peaked domains.

Theorem 3.1. *Let $\hat{\mathcal{S}}$ be a top-connected single-peaked domain. Then, an SCF $f : \hat{\mathcal{S}}^n \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

The proof of the Theorem 3.1 is relegated to Appendix A.

The following corollary is immediate from Theorem 3.1.

Corollary 3.1 (Moulin (1980); Weymark (2011))**.** *Let $\bar{\mathcal{S}}$ be the maximal single-peaked domain. Then, an SCF $f : \bar{\mathcal{S}}^n \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

3.2 Min-max Domains

In this subsection, we introduce the notion of min-max domains and provide a characterization of these domains. In the following, we provide a formal definition of min-max domains.

Definition 3.3. A domain $\mathcal{D}$ is called a *min-max domain* if

(i) every unanimous and strategy-proof SCF on $\mathcal{D}^n$ is a min-max rule, and

(ii) every min-max rule on $\mathcal{D}^n$ is strategy-proof.

Our next theorem provides a characterization of the min-max domains.

Theorem 3.2. *A domain is a min-max domain if and only if it is a top-connected single-peaked domain.*

Proof. The proof of the if part follows from Theorem 3.1. We proceed to prove the only-if part. Let $\mathcal{D}$ be a min-max domain. We show that $\mathcal{D}$ is a top-connected single-peaked domain. First, we show that $\mathcal{D}$ is a single-peaked domain. Assume for contradiction that there is $Q \in \mathcal{D}$ and $x, y \in X$ such that $x < y < r_1(Q)$ and xQy. Consider the min-max rule f^β with respect to $(\beta_S)_{S\subseteq N}$ such that $\beta_S = x$ for all $\emptyset \subsetneq S \subsetneq N$. Take $P_N \in \mathcal{D}^n$ such that $P_1 = Q$ and $r_1(P_i) = y$ for all $i \in N \setminus 1$. By the definition of f^β, $f^\beta(P_N) = y$. Now, take $P'_1 \in \mathcal{D}$ with $r_1(P'_1) = x$. Again, by the definition of f^β, $f^\beta(P'_1, P_{N\setminus 1}) = x$. This means agent 1 manipulates at P_N via P'_1, which is a contradiction to the assumption that $\mathcal{D}$ is a min-max domain. Hence, $\mathcal{D}$ must be a single-peaked domain.

Now, we show that $\mathcal{D}$ satisfies the top-connectedness property. Note that since $\mathcal{D}$ is single-peaked, $r_1(P) = a$ (or b) implies $r_2(P) = a + 1$ (or $b - 1$). Consider some $x \in X \setminus \{a, b\}$. Since $\mathcal{D}$ is single-peaked, for all $P \in \mathcal{D}$, $r_1(P) = x$ implies $r_2(P) \in \{x - 1, x + 1\}$. Without loss of generality, assume for contradiction to the top-connectedness property that for all $P \in \mathcal{D}$, $r_1(P) = x$ implies $r_2(P) = x - 1$. Consider the following SCF:[9]

$$f(P_N) = \begin{cases} x \text{ if } r_1(P_1) = x \text{ and } xP_j(x-1) \text{ for all } j \in N \setminus 1, \\ x - 1 \text{ if } r_1(P_1) = x \text{ and } (x-1)P_jx \text{ for some } j \in N \setminus 1, \\ r_1(P_1) \text{ otherwise.} \end{cases}$$

It is left to the reader to verify that f is unanimous and strategy-proof. We show that f is not uncompromising, which in turn means that f is not a min-max rule. Let $P_N \in \mathcal{D}^n$ be such that $r_1(P_1) = x$ and $r_1(P_j) = x - 1$ for some $j \neq 1$, and let $P'_1 \in \mathcal{D}$ be such that $r_1(P'_1) = x + 1$. Then, by the definition of f, $f(P_N) = x - 1$ and $f(P'_1, P_{N\setminus 1}) = x + 1$. Therefore, f violates uncompromisingness. Thus, the proof of the only-if part is complete. ∎

[9]Here $\mathcal{D}$ satisfies the *unique seconds* property defined in Aswal et al. (2003) and the SCF f considered here is similar to the one used in the proof of Theorem 5.1 in Aswal et al. (2003).

3.3 Applications

3.3.1 Regular Single-crossing Domains

In this subsection, we introduce the notion of regular single-crossing domains and provide a characterization of the unanimous and strategy-proof SCFs on these domains. First, we present a formal definition of single-crossing domains.

Definition 3.4. A domain $\mathcal{S}_c$ is called a *single-crossing domain* if there is a linear order $\lhd$ on $\mathcal{S}_c$ such that for all $x, y \in X$ and all $P, \hat{P} \in \mathcal{S}_c$,

$$[x < y, P \lhd \hat{P}, \text{ and } x\hat{P}y] \Rightarrow xPy.$$

Definition 3.5. A single-crossing domain $\bar{\mathcal{S}}_c$ is called *maximal* if there is no single-crossing domain $\mathcal{S}_c$ such that $\bar{\mathcal{S}}_c \subsetneq \mathcal{S}_c$.

In what follows, we provide an example of a maximal regular single-crossing domain with five alternatives.

Example 3.2. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. Then, the domain $\mathcal{D}$ in Table 2 is a maximal regular single-crossing domain with respect to the linear order $\lhd \in \mathbb{L}(\mathcal{D})$ given by $P_1 \lhd P_2 \lhd P_3 \lhd P_4 \lhd P_5 \lhd P_6 \lhd P_7 \lhd P_8 \lhd P_9 \lhd P_{10} \lhd P_{11}$. To see this, consider two alternatives, say x_2 and x_4. Then, x_2Px_4 for all $P \in \{P_1, P_2, P_3, P_4, P_5, P_6\}$ and x_4Px_2 for all $P \in \{P_7, P_8, P_9, P_{10}, P_{11}\}$. Therefore, $x_2\hat{P}x_4$ for some $\hat{P} \in \mathcal{D}$ implies x_2Px_4 for all $P \lhd \hat{P}$.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}
$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$
$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_4}$
x_3	x_3	x_1	x_4	x_4	x_4	x_2	x_5	x_5	x_3	x_3
x_4	x_4	x_4	x_1	x_5	x_5	x_5	x_2	x_2	x_2	x_2
x_5	x_5	x_5	x_5	x_1	x_1	x_1	x_1	x_1	x_1	x_1

Table 2: A maximal regular single-crossing domain

In the following two lemmas, we establish two crucial properties of a (maximal) regular single-crossing domain.

Lemma 3.1. *Every regular single-crossing domain is a single-peaked domain.*

Proof. Let $\mathcal{S}_c$ be a regular single-crossing domain. Let $\lhd \in \mathbb{L}(\mathcal{S}_c)$ be such that for all $x, y \in X$ and all $P, \hat{P} \in \mathcal{S}_c$,

$$[x < y, P \lhd \hat{P}, \text{ and } x\hat{P}y] \Rightarrow xPy.$$

We show that each $P \in \mathcal{S}_c$ is single-peaked. Without loss of generality, assume for contradiction that there are $u, v \in X$ and $Q \in \mathcal{S}_c$ such that $u < v < r_1(Q)$ and uQv. Since $u < v$ and uQv, by the definition of a single-crossing domain, uPv for all $P \in \mathcal{S}_c$ with $P \lhd Q$. This, in particular, means $r_1(P) \neq v$ for all $P \in \mathcal{S}_c$ with $P \lhd Q$. Moreover, since $v < r_1(Q)$, by the definition of a single-crossing domain, $r_1(Q)Pv$ for all $P \in \mathcal{S}_c$ with $Q \lhd P$. This, in particular, means $r_1(P) \neq v$ for all $P \in \mathcal{S}_c$ with $Q \lhd P$. This, together with the fact that $r_1(Q) \neq v$, means $r_1(P) \neq v$ for all $P \in \mathcal{S}_c$, which is a contradiction to the regularity of $\mathcal{S}_c$. Therefore, $\mathcal{S}_c$ is single-peaked. ■

Lemma 3.2. *Every maximal regular single-crossing domain satisfies the top-connectedness property.*

Proof. Let $\bar{\mathcal{S}}_c$ be a maximal regular single-crossing domain. Then, by Lemma 3.1, $\bar{\mathcal{S}}_c$ is a regular single-peaked domain. Take $x \in X \setminus \{b\}$. We show that there exist $P, P' \in \bar{\mathcal{S}}_c$ such that $r_1(P) = r_2(P') = x$ and $r_2(P) = r_1(P') = x + 1$. Without loss of generality, assume for contradiction that for all $P \in \bar{\mathcal{S}}_c$ with $r_1(P) = x$, $r_2(P) \neq x + 1$. Because $\bar{\mathcal{S}}_c$ is single-peaked, if $x = a$, then $r_2(P) = a + 1$ for all $P \in \bar{\mathcal{S}}_c$ with $r_1(P) = a$, which is a contradiction. So, assume $x \neq a$. Because $\bar{\mathcal{S}}_c$ is single-peaked and $x \notin X \setminus \{a, b\}$, for all $P \in \bar{\mathcal{S}}_c$ with $r_1(P) = x$, $r_2(P) \neq x + 1$ implies $r_2(P) = x - 1$. Let $\lhd \in \mathbb{L}(\bar{\mathcal{S}}_c)$ be such that for all $u, v \in X$ and all $P, \hat{P} \in \bar{\mathcal{S}}_c$,

$$[u < v, P \lhd \hat{P}, \text{ and } u\hat{P}v] \Rightarrow uPv.$$

Take $\hat{P} \in \bar{\mathcal{S}}_c$ with $r_1(\hat{P}) = x$ such that for all $P \in \bar{\mathcal{S}}_c$ with $\hat{P} \lhd P$, $r_1(P) \neq x$. Consider the preference $\tilde{P}$ with $r_1(\tilde{P}) = x$ and $r_2(\tilde{P}) = x + 1$ such that for all $u, v \in X \setminus \{x, x + 1\}$, $u\tilde{P}v$ if and only if $u\hat{P}v$. Because $r_1(\tilde{P}) = x$ and $r_2(\tilde{P}) = x + 1$, by our assumption, $\tilde{P} \notin \bar{\mathcal{S}}_c$. Therefore, since $\bar{\mathcal{S}}_c$ is regular single-crossing, it follows that $\bar{\mathcal{S}}_c \cup \tilde{P}$ is also single-crossing with respect to the ordering $\lhd' \in \mathbb{L}(\bar{\mathcal{S}}_c \cup \tilde{P})$, where $\lhd'$ is obtained by placing $\tilde{P}$ just after $\hat{P}$ in the ordering $\lhd$, i.e., for all $P, P' \in \bar{\mathcal{S}}_c$, $P \lhd' P'$ if and only if $P \lhd P'$, and there is no $P \in \bar{\mathcal{S}}_c$ with $\hat{P} \lhd' P \lhd' \tilde{P}$. However, this contradicts the maximality of $\bar{\mathcal{S}}_c$, which completes the proof. ■

The following corollaries are obtained from Theorem 3.1, Lemma 3.1, and Lemma 3.2. They characterize the unanimous and strategy-proof SCFs on top-connected regular single-crossing domains and maximal regular single-crossing domains. Note that a top-connected regular single-crossing domain with m alternatives can be constructed with $2m - 2$ preferences, whereas a maximal regular single-crossing domain requires $m(m-1)/2$ preferences.

Corollary 3.2. *Let $\mathcal{S}_c$ be a top-connected regular single-crossing domain. Then, an SCF $f : \mathcal{S}_c^n \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

Corollary 3.3 (Saporiti (2014))**.** *Let $\bar{\mathcal{S}}_c$ be a maximal regular single-crossing domain. Then, an SCF $f : \bar{\mathcal{S}}_c^n \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

3.3.2 Minimally Rich Single-peaked Domains

In this subsection, we present a characterization of the unanimous and strategy-proof SCFs on minimally rich single-peaked domains. The notion of minimally rich single-peaked domains is introduced in Peters et al. (2014). For the sake of completeness, we present below a formal definition of such domains.

Definition 3.6. A single-peaked preference P is called *left single-peaked* (*right single-peaked*) if for all $u < r_1(P) < v$, we have uPv (vPu). Moreover, a single-peaked domain $\mathcal{S}_m$ is called *minimally rich* if it contains all left and all right single-peaked preferences.

Clearly, a minimally rich single-peaked domain is a top-connected single-peaked domain. So, we have the following corollary from Theorem 3.1.

Corollary 3.4. *Let $\mathcal{S}_m$ be a minimally rich single-peaked domain. Then, an SCF $f : \mathcal{S}_m^n \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

3.3.3 Distance based Single-peaked Domains

In this subsection, we introduce the notion of single-peaked domains that are based on distances. Consider the situation where a public facility has to be developed at one of the locations $x_1, \ldots, x_m$. Suppose that there is a street connecting those locations, and for every two locations x_i and x_{i+1}, there are two types of distances, a forward distance from x_i to x_{i+1} and a backward distance from x_{i+1} to x_i. An agent bases her preferences on such distances, i.e., whenever a location is strictly closer than another to her most preferred location, she prefers the former to the latter. Moreover, ties are broken on both sides. We show that such a domain is a top-connected single-peaked domain under some condition on the distances. Below, we present a formal definition of such domains.

Consider the directed line graph $G = \langle X, E \rangle$ on X. A function $d : E \to (0, \infty)$ is called a *distance function* on G. Given a distance function d, define the *distance between two nodes* $x, y \in X$ as $d(x,y) = d(x, x+1) + \ldots + d(y-1, y)$ if $y > x$ and as $d(x,y) = d(x, x-1) + \ldots + d(y+1, y)$ if $y < x$. A distance function satisfies *adjacent symmetry* if $d(x, x+1) = d(x, x-1)$ for all $x \in X \setminus \{a, b\}$. A preference P *respects a distance function* d if for all $x, y \in X$, $d(r_1(P), x) < d(r_1(P), y)$ implies xPy. A domain $\mathcal{S}_d$ is called a *single-peaked domain based on a distance function* d if $\mathcal{S}_d = \{P \in \mathbb{L}(X) \mid P \text{ respects } d\}$.

Below, we provide an example of a single-peaked domain based on a distance function.

Example 3.3. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. The directed line graph $G = \langle X, E \rangle$ on X and the adjacent symmetric distance function d on E are as given below.

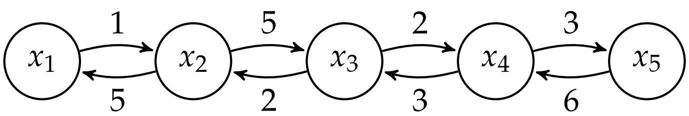


Figure 3: The directed line graph G on X and an adjacent symmetric distance function d on G

Then, the domain in Table 3 is a single-peaked domain based on the distance function d.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$
$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_1}$	$\mathbf{x_4}$	$\mathbf{x_2}$	$\mathbf{x_5}$	$\mathbf{x_3}$	$\mathbf{x_4}$
x_3	x_1	x_3	x_2	x_4	x_3	x_5	x_3
x_4	x_4	x_4	x_5	x_5	x_2	x_2	x_2
x_5	x_5	x_5	x_1	x_1	x_1	x_1	x_1

Table 3: A single-peaked domain based on the distance function d

Let $G = \langle X, E \rangle$ be the directed line graph on X and let $d : E \to (0, \infty)$ be an adjacent symmetric distance function. Then, it is easy to verify that a single-peaked domain based on the distance function d is a top-connected single-peaked domain. Therefore, we have the following corollary from Theorem 3.1.

Corollary 3.5. *Let $G = \langle X, E \rangle$ be the directed line graph on X and let $d : E \to (0, \infty)$ be an adjacent symmetric distance function. Let $\mathcal{S}_d$ be a single-peaked domain based on the distance function d. Then, $f : \mathcal{S}_d^n \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

4. Partially Single-peaked Domains

In this section, we consider a class of non-single-peaked domains. These domains exhibit single-peakedness only over a strict subset of alternatives. We call such domains partially single-peaked domains which are formally defined below.

Definition 4.1. Let $x, y \in X$ such that $x < y - 1$. Then, a domain $\tilde{\mathcal{S}}$ is called *partially single-peaked* with respect to x and y if

(i) for all $P \in \tilde{\mathcal{S}}$ with $r_1(P) \in [x, y]$ and all $u, v \notin (x, y)$, $\big[v < u \leq r_1(P)$ or $r_1(P) \leq u < v\big]$ implies uPv,

(ii) for all $P \in \tilde{\mathcal{S}}$ with $r_1(P) \notin [x, y]$ and all $u, v \in X$ such that $u \notin (x, y)$, $\big[v < u \leq r_1(P)$ or $r_1(P) \leq u < v\big]$ implies uPv, and

(iii) there exist $Q, Q' \in \tilde{\mathcal{S}}$ with $r_1(Q) = x$ and $r_1(Q') = y$ such that either $\big[r_2(Q) \in (x+1, y)$ and $r_2(Q') \in (x, y-1)\big]$ or $\big[r_2(Q) = y$ and $r_2(Q') = x\big]$.

Condition (i) in Definition 4.1 says that if the top-ranked alternative of a preference in a partially single-peaked domain lies in the interval $[x, y]$, then it maintains single-peakedness over the intervals $[a, x]$ and $[y, b]$. Note that this condition does not impose any restriction on the relative ordering of an alternative in $[x, y]$ and an alternative outside $[x, y]$. The interpretation of Condition (ii) is as follows. Consider a preference P in a partially single-peaked domain such that $r_1(P) \notin [x, y]$. Suppose, for instance, $r_1(P) \in [a, x)$. Then, P maintains single-peakedness over the interval $[a, r_1(P)]$. Moreover, if an alternative u lies in the interval $(r_1(P), x]$ or in the interval $[y, b]$, then it is preferred to any alternative v in the interval $(u, b]$. Therefore, in contrast to Condition (i), Condition (ii) imposes a mild restriction on the relative ordering of an alternative in $[x, y]$ and an alternative outside $[x, y]$. Further, both these conditions do not impose any restriction on the relative ordering of two alternatives in the interval $[x, y]$. Finally, Condition (iii) ensures that the intervals $[a, x]$ and $[y, b]$ are the maximal intervals over which every preference in a partially single-peaked domain maintains single-peakedness. To see this, first note that all the preferences in a partially single-peaked domain with respect to x and y maintain single-peakedness over the intervals $[a, x]$ and $[y, b]$. This, together with the facts that $r_1(Q) = x$, $r_2(Q) > x + 1$ and $r_1(Q') = y$, $r_2(Q') < y - 1$, ensures that the intervals $[a, x]$ and $[y, b]$ are the maximal intervals with the said property. In Section 4.2, we show that the particular restrictions on the second-ranked alternatives of Q and Q' given in Condition (iii) are necessary for our results.

We illustrate the notion of partially single-peaked domains in Figure 4. Figure 4(a) and Figure 4(b) present partially single-peaked preferences P with $r_1(P) \in [x, y]$ and $r_1(P) \in [a, x)$, respectively. Figure 4(c) presents the partially single-peaked preferences Q and Q' with $r_1(Q) = x$, $r_2(Q) \in (x + 1, y)$, $r_1(Q') = y$, and $r_2(Q') \in (x, y - 1)$, and Figure 4(d) presents the partially single-peaked preferences Q and Q' with $r_1(Q) = x$, $r_2(Q) = y$, $r_1(Q') = y$, and $r_2(Q') = x$. Note that all these preferences are single-peaked over the intervals $[a, x]$ and $[y, b]$. Furthermore, for the preference depicted in Figure 4(a), there is no restriction on the ranking of the alternatives in the interval (x, y), and for that shown in Figure 4(b), there is no restriction on the ranking of the alternatives in the interval (x, y) except that x is preferred to all the alternatives in $(x, b]$. Also, for the preferences in Figures 4(c) and 4(d), there is no restriction on the ranking of the alternatives in (x, y) other than the restriction on the second-ranked alternatives.

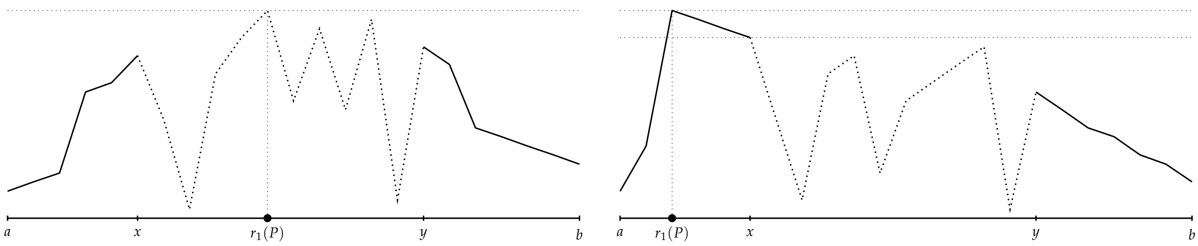
In the following, we define a top-connected partially single-peaked domain.

Definition 4.2. A domain $\tilde{S}$ is called a *top-connected partially single-peaked domain* with respect to alternatives x and y with $x < y - 1$ if

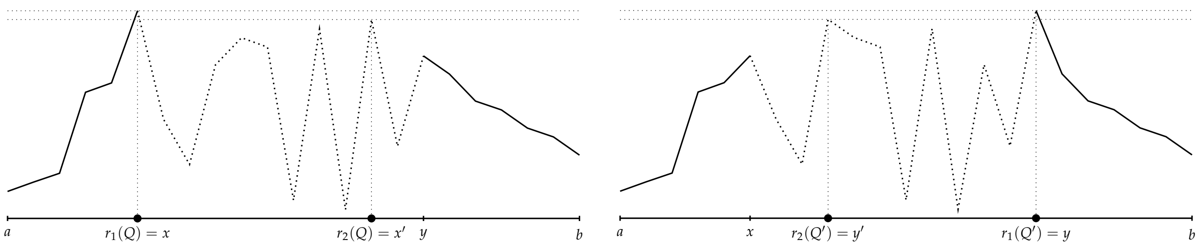
(i) $\tilde{S}$ is a partially single-peaked domain with respect to x and y, and

(ii) $\tilde{S}$ contains a top-connected single-peaked domain.

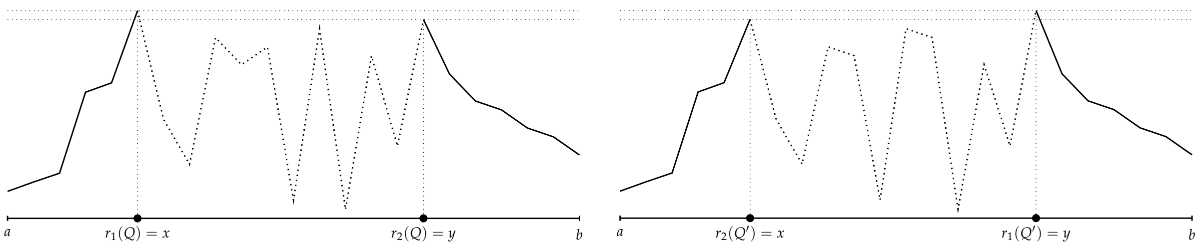
We interpret Definition 4.2 in terms of its top-graph. Let G be the top-graph of a top-connected partially single-peaked domain with respect to alternatives x and y. Then, G is a directed partial line graph with respect to x and y. To see this, note that G can be written as $G_1 \cup G_2$, where $G_1 = \langle X, E_1 \rangle$ is the directed line graph on X and $G_2 = \langle [x, y], E_2 \rangle$ is a directed graph such that $(x, r_2(Q)), (y, r_2(Q')) \in E_2$ where $r_2(Q) \in (x + 1, y]$ and $r_2(Q') \in [x, y - 1)$. In Example 4.1, we present a top-connected partially single-peaked domain with seven alternatives, and in Figure 5, we present the top-graph of that domain.



(a) Partially single-peaked preference P with $r_1(P) \in [x, y]$ (b) Partially single-peaked preference P with $r_1(P) \in [a, x)$



(c) Partially single-peaked preferences Q, Q' with $x + 1 < r_2(Q) < y$ and $x < r_2(Q') < y - 1$



(d) Partially single-peaked preferences Q, Q' with $r_2(Q) = y$ and $r_2(Q') = x$

Figure 4: Partially single-peaked preferences

Example 4.1. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, where $x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < x_7$. Then, the domain in Table 4 is a top-connected partially single-peaked domain with respect to x_3 and x_6. To see this, first consider a preference with its top-ranked alternative in the interval $[x_3, x_6]$, say P_7. Note that $x_3 P_7 x_2 P_7 x_1$ and $x_6 P_7 x_7$, which means P_7 is single-peaked over the intervals $[x_1, x_3]$ and $[x_6, x_7]$. Moreover, the position of x_5 is completely unrestricted (here at the bottom) in P_7. Next, consider a preference with its top-ranked alternative in the interval $[x_1, x_3]$, say P_2. Once again, note that P_2 is single-peaked over the intervals $[x_1, x_3]$ and $[x_6, x_7]$. Further, x_3 is preferred to the alternatives x_4, x_5, x_6, x_7, and there is no restriction on the relative ordering of the alternatives x_4 and x_5 (here $x_5 P_2 x_4$). Finally, consider the preferences Q and Q'. Since $r_1(Q) = x_3$, $r_2(Q) = x_5$, $r_1(Q') = x_6$, and $r_2(Q') = x_4$, they satisfy Condition (iii) in Definition 4.1.

The top-graph G of the domain in Example 4.1 is given in Figure 5. Note that G is a partial line

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	Q	Q'
$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_6}$	$\mathbf{x_7}$	$\mathbf{x_3}$	$\mathbf{x_6}$
$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_6}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_4}$	$\mathbf{x_6}$	$\mathbf{x_5}$	$\mathbf{x_7}$	$\mathbf{x_6}$	$\mathbf{x_5}$	$\mathbf{x_4}$
x_3	x_3	x_3	x_1	x_4	x_2	x_3	x_5	x_3	x_3	x_4	x_4	x_5	x_5	x_2	x_3
x_4	x_6	x_4	x_4	x_5	x_5	x_2	x_2	x_2	x_6	x_3	x_3	x_4	x_4	x_6	x_7
x_5	x_5	x_5	x_5	x_6	x_6	x_1	x_6	x_1	x_7	x_2	x_2	x_3	x_3	x_1	x_2
x_6	x_7	x_6	x_6	x_7	x_1	x_7	x_1	x_6	x_2	x_7	x_7	x_2	x_2	x_7	x_1
x_7	x_4	x_7	x_7	x_1	x_7	x_5	x_7	x_7	x_1	x_1	x_1	x_1	x_1	x_4	x_5

Table 4: A top-connected partially single-peaked domain

graph since it can be written as $G_1 \cup G_2$, where G_1 is the directed line graph on $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and G_2 is a directed graph on $\{x_3, x_4, x_5, x_6\}$ having edges (x_3, x_5) and (x_6, x_4).

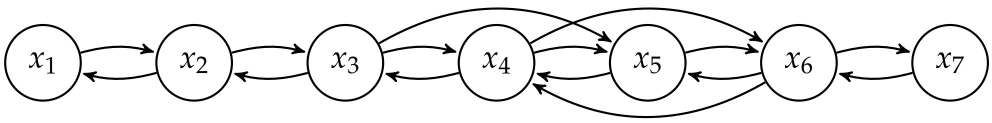


Figure 5: Top-graph of the domain in Example 4.1

4.1 Unanimous and Strategy-proof SCFs

In this subsection, we characterize the unanimous and strategy-proof SCFs on top-connected partially single-peaked domains as partly dictatorial generalized median voter schemes. A formal definition of such SCFs is presented below:

Definition 4.3. Let $x, y \in X$ be such that $x < y - 1$. Then, a min-max rule $f^\beta : \mathcal{D}^n \to X$ with parameters $\beta = (\beta_S)_{S \subseteq N}$ is a *partly dictatorial generalized median voter scheme* (PDGMVS) with respect to x and y if there exists an agent $d \in N$, called the *partial dictator* of f^β, such that $\beta_d \in [a, x]$ and $\beta_{N \setminus d} \in [y, b]$.

Remark 4.1. Reffgen (2015) defines PDGMVS in a different fashion but it can be shown that their definition is equivalent to Definition 4.3.[10]

The following lemma justifies why the agent d in Definition 4.3 is called the partial dictator.

Lemma 4.1. *Let $x, y \in X$ be such that $x < y - 1$ and let $f^\beta : \mathcal{D}^n \to X$ be a PDGMVS with respect to x and y. Suppose agent d is the partial dictator of f^β. Then,*

(i) $f^\beta(P_N) \in [a, x]$ *if* $r_1(P_d) \in [a, x)$,

[10]For details see the proof of Theorem 3.1 in Reffgen (2015).

(ii) $f^\beta(P_N) \in [y,b]$ *if* $r_1(P_d) \in (y,b]$*, and*

(iii) $f^\beta(P_N) = r_1(P_d)$ *if* $r_1(P_d) \in [x,y]$*.*

Proof. First, we prove (i). The proof of (ii) can be established using symmetric arguments. Assume for contradiction that $r_1(P_d) \in [a,x)$ and $f^\beta(P_N) > x$. Since f^β is a min-max rule, f^β is uncompromising. Therefore, $f^\beta(P'_d, P_{N\setminus d}) = f^\beta(P_N)$, where $r_1(P'_d) = a$. Again by uncompromisingness, we have $f^\beta(P'_N) \geq f^\beta(P_N)$, where $r_1(P'_i) = b$ for all $i \neq d$. Because $f^\beta(P_N) > x$, this means $f^\beta(P'_N) > x$. However, by the definition of f^β, $f^\beta(P'_N) = \beta_d$. Since $\beta_d \in [a,x]$, this is a contradiction. This completes the proof of (i).

Now, we prove (iii). Without loss of generality, assume for contradiction that $r_1(P_d) \in [x,y]$ and $f^\beta(P_N) > r_1(P_d)$. Using a similar argument as for the proof of (i), we have $f^\beta(P'_N) \geq f^\beta(P_N)$, where $r_1(P'_d) = a$ and $r_1(P'_i) = b$ for all $i \neq d$. This, in particular, means $f^\beta(P'_N) > x$. Since by the definition of f^β, $f^\beta(P'_N) = \beta_d$ and $\beta_d \in [a,x]$, this is a contradiction. This completes the proof of (iii). ■

The following theorem characterizes the unanimous and strategy-proof SCFs on top-connected partially single-peaked domains.

Theorem 4.1. *Let* $x,y \in X$ *be such that* $x < y-1$ *and let* $\tilde{\mathcal{S}}$ *be a top-connected partially single-peaked domain with respect to* x *and* y*. Then, an SCF* $f : \tilde{\mathcal{S}}^n \to X$ *is unanimous and strategy-proof if and only if it is a PDGMVS with respect to* x *and* y*.*

The proof of the Theorem 4.1 is relegated to Appendix B.

Our next corollary is a consequence of Lemma 4.1 and Theorem 4.1. It characterizes a class of dictatorial domains, and thereby it generalizes the celebrated Gibbard-Satterthwaite (Gibbard (1973), Satterthwaite (1975)) results. Note that our dictatorial result is independent of those in Aswal et al. (2003), Sato (2010), Pramanik (2015), and so on.

Corollary 4.1. *Let* $\mathcal{D}$ *be a top-connected partially single-peaked domain with respect to* a *and* b*. Then,* $\mathcal{D}$ *is a dictatorial domain.*

4.2 A Result on Partial Necessity

In this subsection, we introduce the notion of PDGMVS domains. A formal definition is given below.

Definition 4.4. A domain $\mathcal{D}$ is called a *PDGMVS domain* if there are $x, y \in X$ with $x < y - 1$ such that

(i) every unanimous and strategy-proof SCF on $\mathcal{D}^n$ is a PDGMVS with respect to x and y, and

(ii) every PDGMVS with respect to x and y on $\mathcal{D}^n$ is strategy-proof.

Conditions (i), (ii), and (iii) in Definition 4.1 are obviously strong conditions. Are they necessary for PDGMVS domains? The question appears to be extremely difficult to resolve completely. However, Lemma 4.2 shows that Conditions (i) and (ii) are necessary, and the subsequent discussion shows that Condition (iii) is also close to being necessary in an appropriate sense.

Lemma 4.2. *Let $\mathcal{D}$ be a PDGMVS domain. Then, $\mathcal{D}$ satisfies Conditions (i) and (ii) in Definition 4.1.*

Proof. First, we show that $\mathcal{D}$ satisfies Condition (i) in Definition 4.1. Without loss of generality, assume for contradiction that there exists $\tilde{P} \in \mathcal{D}$ with $r_1(\tilde{P}) \in [x, y]$ such that $u\tilde{P}v$ for some $u < v \leq x$. Consider the PDGMVS $f^\beta : \mathcal{D}^n \to X$, where

$$\beta_S = \begin{cases} v \text{ if } S = \{1\}, \\ a \text{ if } \{1\} \subsetneq S, \\ b \text{ if } 1 \notin S. \end{cases}$$

We show that f^β is not strategy-proof. Note that agent 1 is the partial dictator of f^β. Consider the preference profile $P_N \in \mathcal{D}^n$ such that $r_1(P_1) = a$, $P_2 = \tilde{P}$, and $r_1(P_j) = b$ for all $j \neq 1, 2$. Then, by the definition of f^β, $f^\beta(P_N) = v$. Let $P_2' \in \mathcal{D}$ be such that $r_1(P_2') = u$. Again, by the definition of f^β, $f^\beta(P_2', P_{N\setminus 2}) = u$. Since $u\tilde{P}v$, this means agent 2 manipulates at P_N via P_2'.

Now, we show that $\mathcal{D}$ satisfies Condition (ii) in Definition 4.1. Without loss of generality, assume for contradiction that there exist $\tilde{P} \in \mathcal{D}$ with $r_1(\tilde{P}) \in [a, x)$ and $u, v \in X$ with $u \notin (x, y)$ such that $\left[v < u \leq r_1(P) \text{ or } r_1(P) \leq u < v\right]$ and $v\tilde{P}u$. If $\left[v < u \leq r_1(\tilde{P})\right]$ and $v\tilde{P}u$, then using a similar argument as for the proof of the necessity of Condition (i), it follows that there is a PDGMVS on $\mathcal{D}^n$ that is manipulable. So, suppose $r_1(\tilde{P}) \leq u < v$ and $v\tilde{P}u$. We distinguish two cases.

CASE 1. Suppose $u \leq x$.

Consider the PDGMVS $f^\beta : \mathcal{D}^n \to X$, where

$$\beta_S = \begin{cases} u \text{ if } 1 \in S \text{ and } S \neq N, \\ b \text{ if } 1 \notin S. \end{cases}$$

We show that f^β is not strategy-proof. Let $P_N \in \mathcal{D}^n$ be such that $P_1 = \tilde{P}$ and $r_1(P_j) = b$ for all $j \neq 1$. Then, by the definition of f^β, $f^\beta(P_N) = u$. Let $P'_1 \in \mathcal{D}$ be such that $r_1(P'_1) = v$. Again, by the definition of f^β, $f^\beta(P'_1, P_{N \setminus 1}) = v$. Since $v\tilde{P}u$, agent 1 manipulates at P_N via P'_1.

CASE 2. Suppose $x < u$.
Since $u \notin (x, y)$, this means $y \leq u$. Consider the PDGMVS $f^\beta : \mathcal{D}^n \to X$, where

$$\beta_S = \begin{cases} a \text{ if } 1 \in S, \\ u \text{ if } 1 \notin S \text{ and } S \neq \emptyset. \end{cases}$$

We show that f^β is not strategy-proof. Let $P_N \in \mathcal{D}^n$ be such that $P_2 = \tilde{P}$ and $r_1(P_j) = b$ for all $j \neq 2$. Then, by the definition of f^β, $f^\beta(P_N) = u$. Let $P'_2 \in \mathcal{D}$ be such that $r_1(P'_2) = v$. Again, by the definition of f^β, $f^\beta(P'_2, P_{N \setminus 2}) = v$. Since $v\tilde{P}u$, agent 2 manipulates at P_N via P'_2. ■

Coming to Condition (iii) in Definition 4.1, it is to be noted that it can be violated in many ways. We consider those domains obtained through mild violations of the same and show that there do exist unanimous and strategy-proof SCFs on such domains that are not PDGMVS.

Recall that Condition (iii) requires two non-single-peaked preferences Q and Q' in $\mathcal{D}$ such that $r_1(Q) = x$, $r_2(Q) = x'$, $r_1(Q') = y$, and $r_2(Q') = y'$, where either $[x' \in (x+1, y)$ and $y' \in (x, y-1)]$ or $[x' = x$ and $y' = x]$. Suppose a domain $\mathcal{D}$ satisfies Conditions (i) and (ii) in Definition 4.1. Suppose further that $\mathcal{D}$ contains a non-single-peaked preference of the form Q, but no preference of the form Q'. In the following example, we construct a two-agent unanimous and strategy-proof SCF on such a domain $\mathcal{D}$ that is not a PDGMVS.

Example 4.2. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. By $P = x_1x_2x_3x_4x_5$, we mean a preference P such that $x_1Px_2Px_3Px_4Px_5$. Consider the domain as follows:

$$\mathcal{D} = \{x_1x_2x_3x_4x_5, x_1x_3x_4x_5x_2, x_2x_1x_3x_4x_5, x_2x_3x_4x_5x_1, x_3x_2x_1x_4x_5, x_3x_4x_5x_2x_1, x_4x_3x_2x_1x_5, x_4x_5x_3x_2x_1, x_5x_4x_3x_2x_1\}.$$

Note that $\mathcal{D} \setminus \{x_1x_3x_4x_5x_2\}$ is a top-connected single-peaked domain and the preference $x_1x_3x_4x_5x_2$ is of the form Q with $x = x_1$ and $x' = x_3$. However, there is no preference in $\mathcal{D}$ of the form Q', that is, for instance, no preference with top-ranked alternative x_5 and second-ranked alternative in $\{x_2, x_3\}$. In Table 5, we present a two-agent SCF that is unanimous and strategy-proof but not a PDGMVS.

P_1 \ P_2	$x_1x_2x_3x_4x_5$	$x_1x_3x_4x_5x_2$	$x_2x_1x_3x_4x_5$	$x_2x_3x_4x_5x_1$	$x_3x_2x_1x_4x_5$	$x_3x_4x_5x_2x_1$	$x_4x_3x_2x_1x_5$	$x_4x_5x_3x_2x_1$	$x_5x_4x_3x_2x_1$
$x_1x_2x_3x_4x_5$	x_1	x_1	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_1x_3x_4x_5x_2$	x_1	x_1	x_2	x_2	x_3	x_3	x_3	x_3	x_3
$x_2x_1x_3x_4x_5$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_2x_3x_4x_5x_1$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_3x_2x_1x_4x_5$	x_2	x_3	x_2	x_2	x_3	x_3	x_3	x_3	x_3
$x_3x_4x_5x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_3	x_3	x_3
$x_4x_3x_2x_1x_5$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_4
$x_4x_5x_3x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_4
$x_5x_4x_3x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_5

Table 5: A unanimous and strategy-proof SCF which is not a PDGMVS

It is left to the reader to verify that the SCF presented in Table 5 is unanimous and strategy-proof. Note that it violates tops-onlyness at the preference profiles $(x_3x_4x_5x_2x_1, x_1x_2x_3x_4x_5)$ and $(x_3x_4x_5x_2x_1, x_1x_3x_4x_5x_2)$, and hence it is not a PDGMVS.

Now, suppose that $\mathcal{D}$ contains two non-single-peaked preferences Q and Q', however, they do *not* satisfy Condition (iii) in Definition 4.1 for their second-ranked alternatives. In the following example, we construct a two-agent unanimous and strategy-proof SCF on such a domain $\mathcal{D}$ that is not a PDGMVS.

Example 4.3. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. Let $\mathcal{D}$ be the domain given in Example 4.2. Consider the domain $\mathcal{D} \cup \{x_5x_1x_4x_3x_2\}$. As pointed out in Example 4.2, $\mathcal{D} \setminus \{x_1x_3x_4x_5x_2\}$ is a top-connected single-peaked domain. Although, the preferences $x_1x_3x_4x_5x_2$ and $x_5x_1x_4x_3x_2$ are non-single-peaked, they do not satisfy the Condition (iii) in Definition 4.1 since their second-ranked alternatives are x_3 and x_1, respectively. In Table 6, we present a two-agent SCF that is unanimous and strategy-proof but not a PDGMVS.

Note that the restriction of the SCF presented in Table 6 to $\mathcal{D}^2$ is same as the SCF presented in Table 5. It is left to the reader to verify that this SCF is unanimous and strategy-proof. However, as pointed out in Example 4.2, it violates tops-onlyness, and hence it is not a PDGMVS.

P_1 \ P_2	$x_1x_2x_3x_4x_5$	$x_1x_3x_4x_5x_2$	$x_2x_1x_3x_4x_5$	$x_2x_3x_4x_5x_1$	$x_3x_2x_1x_4x_5$	$x_3x_4x_5x_2x_1$	$x_4x_3x_2x_1x_5$	$x_4x_5x_3x_2x_1$	$x_5x_4x_3x_2x_1$	$x_5x_1x_4x_3x_2$
$x_1x_2x_3x_4x_5$	x_1	x_1	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_1
$x_1x_3x_4x_5x_2$	x_1	x_1	x_2	x_2	x_3	x_3	x_3	x_3	x_3	x_1
$x_2x_1x_3x_4x_5$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_2x_3x_4x_5x_1$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_3x_2x_1x_4x_5$	x_2	x_3	x_2	x_2	x_3	x_3	x_3	x_3	x_3	x_3
$x_3x_4x_5x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_3	x_3	x_3	x_3
$x_4x_3x_2x_1x_5$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_4	x_4
$x_4x_5x_3x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_4	x_4
$x_5x_4x_3x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_5	x_5
$x_5x_1x_4x_3x_2$	x_1	x_1	x_2	x_2	x_3	x_3	x_4	x_4	x_5	x_5

Table 6: A unanimous and strategy-proof SCF which is not a PDGMVS

4.3 Applications

4.3.1 Multiple Single-peaked Domain

In this subsection, we consider a well-known class of domains called multiple single-peaked domains and present a characterization of the unanimous and strategy-proof SCFs on such domains. However, before formally defining such domains, we introduce the notion of a single-peaked domain with respect to an arbitrary order over X.

Definition 4.5. Let $\prec \in \mathbb{L}(X)$ be a prior order over X. Then, a preference $P \in \mathbb{L}(X)$ is *single-peaked with respect to* $\prec$ if for all $x, y \in X$, $[x \prec y \preceq r_1(P)$ or $r_1(P) \preceq y \prec x]$ implies yPx. A domain $\mathcal{S}_\prec$ is called a *single-peaked domain with respect to* $\prec$ if each preference in it is single-peaked with respect to $\prec$, and a domain $\bar{\mathcal{S}}_\prec$ is called *maximal single-peaked with respect to* $\prec$ if it contains all single-peaked preferences with respect to $\prec$.

Definition 4.6. Let $\mathcal{L} = \{\prec_1, \ldots, \prec_q\}$, where $\prec_k \in \mathbb{L}(X)$ for all $1 \leq k \leq q$, be a set of q prior orders over X. Then, a domain is called a *multiple single-peaked domain with respect to* $\mathcal{L}$, denoted by $\mathcal{S}_\mathcal{L}$, if $\mathcal{S}_\mathcal{L} = \bigcup\limits_{k \in \{1,\ldots,q\}} \bar{\mathcal{S}}_{\prec_k}$, where $\bar{\mathcal{S}}_{\prec_k}$ is the maximal single-peaked domain with respect to the prior order $\prec_k$. A multiple single-peaked domain with respect to $\mathcal{L}$ is called *trivial* if $\bar{\mathcal{S}}_\prec = \bar{\mathcal{S}}_{\prec'}$ for all $\prec, \prec' \in \mathcal{L}$.

For ease of presentation, for any multiple single-peaked domain with respect to $\mathcal{L}$, we assume without loss of generality that the integer ordering $<$ is in the set $\mathcal{L}$.

Definition 4.7. Let $\mathcal{S}_\mathcal{L}$ be a non-trivial multiple single-peaked domain with respect to a set of prior orders $\mathcal{L}$. Then, alternatives $u, v \in X$ with $u < v - 1$ are called *break-points* of $\mathcal{S}_\mathcal{L}$ if

(i) for all preferences $P \in \mathcal{S}_\mathcal{L}$ and all $c, d \in X \setminus (u, v)$, $\big[d < c \leq r_1(P)$ or $r_1(P) \leq c < d\big]$ implies cPd, and

(ii) there exist $P, P' \in \mathcal{S}_{\mathcal{L}}$ such that $r_1(P) = u$, $r_2(P) \in (u+1, v]$, $r_1(P') = v$, and $r_2(P') \in [u, v-1)$.

REMARK 4.2. Let u and v be the break-points of a non-trivial multiple single-peaked domain $\mathcal{S}_{\mathcal{L}}$. Then, u, v induce the partition $\{X_L, X_M, X_R\}$ of X, where $X_L = [a, u)$, $X_M = [u, v]$, and $X_R = (v, b]$. Reffgen (2015) calls such a partition the *maximal common decomposition* of X and the sets X_L, X_M, and X_R as the *left component*, the *middle component*, and the *right component* of alternatives, respectively.

In the following, we illustrate the notion of break-points of a non-trivial multiple single-peaked domain by means of an example.

Example 4.4. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ be the set of alternatives. Consider the set of prior orders $\mathcal{L} = \{<, \prec_1, \prec_2, \prec_3\}$, where $<= x_1x_2x_3x_4x_5x_6x_7$, $\prec_1= x_1x_2x_3x_5x_4x_6x_7$, $\prec_2= x_1x_2x_5x_4x_3x_6x_7$, and $\prec_3= x_1x_2x_4x_3x_5x_6x_7$. Let $\mathcal{S}_{\mathcal{L}}$ be the multiple single-peaked domain with respect to $\mathcal{L}$. Clearly, $\mathcal{S}_{\mathcal{L}}$ is non-trivial since $\bar{\mathcal{S}}_{\prec_1} \neq \bar{\mathcal{S}}_{\prec_2}$. We claim $u = x_2$ and $v = x_6$ are the break points of $\mathcal{S}_{\mathcal{L}}$. It is easy to verify that $\mathcal{S}_{\mathcal{L}}$ satisfies Condition (i) in Definition 4.7. For Condition (ii), consider the preferences $P, P' \in \bar{\mathcal{S}}_{\prec_2} \subseteq \mathcal{S}_{\mathcal{L}}$ such that $r_1(P) = x_2$, $r_2(P) = x_5$, $r_1(P') = x_6$, and $r_2(P') = x_3$. Finally, the maximal common decomposition of X is given by $X_L = \{x_1\}$, $X_M = \{x_2, x_3, x_4, x_5, x_6\}$, and $X_R = \{x_7\}$.

Note that a non-trivial multiple single-peaked domain with break-points u and v is a top-connected partially single-peaked domain with respect to u and v. Thus, we have the following corollary.

Corollary 4.2 (Reffgen (2015))**.** *Let $\mathcal{S}_{\mathcal{L}}$ be a non-trivial multiple single-peaked domain with break-points u and v. Then, an SCF $f : \mathcal{S}^n_{\mathcal{L}} \to X$ is unanimous and strategy-proof if and only if it is a PDGMVS with respect to u and v.*

4.3.2 Single-peaked Domains on Graphs

In this subsection, we introduce the notion of single-peaked domains on graphs. All the graphs we consider in this subsection are undirected.

Definition 4.8. A *path* in an undirected graph $G = \langle X, E \rangle$ from a node x to a node y, denoted by $\pi_G(x, y)$, is defined as a sequence of nodes $(x_1, \ldots, x_k)$ such that $\{x_i, x_{i+1}\} \in E$ for all $i =$

$1, \ldots, k-1$. An undirected graph $G = \langle X, E \rangle$ is called *connected* if for all $x, y \in X$, there is a path from x to y.

Definition 4.9. An undirected graph $G = \langle X, E \rangle$ is called a *tree* if for every two distinct nodes $x, y \in X$, there is a unique path from x to y. A *spanning tree* of an undirected connected graph G is defined as a connected subgraph of G that is a tree. For an undirected connected graph G, we denote by $\mathcal{T}_G$ the set of all spanning trees of G.

Definition 4.10. Let $T = \langle X, E \rangle$ be a tree. Then, a domain is called *single-peaked with respect to T*, denoted by $\mathcal{S}_T$, if for all $P \in \mathcal{S}_T$ and all distinct $x, y \in X$,

$$[x \in \pi_T(r_1(P), y)] \implies [xPy].$$

Definition 4.11. Let $G = \langle X, E \rangle$ be an undirected connected graph. Then, a domain is called *single-peaked with respect to G*, denoted by $\mathcal{S}_G$, if $\mathcal{S}_G = \cup_{T \in \mathcal{T}_G} \mathcal{S}_T$.

Note that if T is the undirected line graph on X, then $\mathcal{S}_T$ is the maximal single-peaked domain. In Lemma 4.3, we show that if a domain is single-peaked with respect to an undirected partial line graph, then it is a top-connected partially single-peaked domain.

Lemma 4.3. *Let $x < y-1$ and let G be an undirected partial line graph with respect to x and y. Then, $\mathcal{S}_G$ is a top-connected partially single-peaked domain with respect to x and y.*

Proof. Let G be an undirected partial line graph with respect to x and y with $x < y-1$. We show that $\mathcal{S}_G$ is a top-connected partially single-peaked domain. Let $G = G_1 \cup G_2$, where $G_1 = \langle X, E_1 \rangle$ is the undirected line graph on X and $G_2 = \langle [x, y], E_2 \rangle$ is an undirected graph such that $\{x, x'\}, \{y, y'\} \in E_2$ for some $x' \in (x+1, y]$ and $y' \in [x, y-1)$.

First, we show that $\mathcal{S}_G$ is partially single-peaked, that is, $\mathcal{S}_G$ satisfies Conditions (i), (ii), and (iii) in Definition 4.1. Take $P \in \mathcal{S}_G$ with $r_1(P) \in [x, y]$ and take $u, v \in X \setminus (x, y)$. Suppose $[v < u \leq r_1(P)$ or $r_1(P) \leq u < v]$. Consider an arbitrary spanning tree T of G. Then, by the definition of G, $u \in \pi_T(r_1(P), v)$, and hence uPv. Therefore, $\mathcal{S}_G$ satisfies Condition (i). Using a similar argument, it can be shown that $\mathcal{S}_G$ satisfies Condition (ii). Now, we show that there are $Q, Q' \in \mathcal{S}_G$ satisfying Condition (iii). Consider the tree $T = \langle X, E \rangle$ such that $E = (E_1 \setminus \{x, x+1\}) \cup \{x, x'\}$. Since $G_1 = \langle X, E_1 \rangle$ is the undirected line graph on X, T is a spanning tree of G. Because $\{x, x'\} \in E$, there is a preference $Q \in \mathcal{S}_T \subseteq \mathcal{S}_G$ with $r_1(Q) = x$ and $r_2(Q) = x'$.

Similarly, there is a preference $Q' \in \mathcal{S}_G$ with $r_1(Q') = y$ and $r_2(Q') = y'$. If $x' \neq y$ and $y' \neq x$, then clearly Q and Q' satisfy Condition (iii). On the other hand, if, for instance, $x' = y$, then that means there is an edge $\{x, y\}$ in G, and consequently, y' can be chosen as x. Thus, Condition (iii) is satisfied.

Now, we show that $\mathcal{S}_G$ contains a top-connected single-peaked domain. Since G_1 is the undirected line graph on X, $\mathcal{S}_{G_1}$ is the maximal single-peaked domain. Moreover, since G_1 is a spanning tree of G, $\mathcal{S}_{G_1} \subseteq \mathcal{S}_G$. This completes the proof of the lemma. ■

Combining Theorem 4.1 with Lemma 4.3 leads immediately to the following characterization of the unanimous and strategy-proof SCFs on a single-peaked domain with respect to an undirected partial line graph.

Corollary 4.3. *Let $x, y \in X$ be such that $x < y - 1$ and let $G = \langle X, E \rangle$ be an undirected partial line graph with respect to x and y. Suppose $\mathcal{S}_G$ is the single-peaked domain with respect to G. Then, an SCF $f : \mathcal{S}_G^n \to X$ is unanimous and strategy-proof if and only if it is a PDGMVS.*

5. Group Strategy-proofness

In this section, we consider group strategy-proofness and obtain a characterization of the unanimous and group strategy-proof SCFs on top-connected single-peaked domains and top-connected partially single-peaked domains. We begin with the definition of group strategy-proofness.

Definition 5.1. An SCF $f : \mathcal{D}^n \to X$ is called *group manipulable* if there is a preference profile P_N, a non-empty coalition $C \subseteq N$, and a preference profile $P'_C \in \mathcal{D}^{|C|}$ of the agents in C such that $f(P'_C, P_{N \setminus C}) P_i f(P_N)$ for all $i \in C$. An SCF $f : \mathcal{D}^n \to X$ is called *group strategy-proof* if it is not group manipulable.

Barberà et al. (2010) establishes a sufficient condition on a domain that ensures the equivalence of strategy-proofness and group strategy-proofness on that domain. It can be easily verified that top-connected single-peaked domains satisfy their sufficient condition. Thus, we have the following corollary.

Corollary 5.1. *Let $\hat{\mathcal{S}}$ be a top-connected single-peaked domain. Then, an SCF $f : \hat{\mathcal{S}}^n \to X$ is unanimous and group strategy-proof if and only if it is a min-max rule.*

In the following theorem, we present a characterization of the unanimous and group strategy-proof SCFs on top-connected partially single-peaked domains. It is worth mentioning that these domains do not satisfy the sufficient condition for the equivalence of strategy-proofness and group strategy-proofness provided in Barberà et al. (2010).

Theorem 5.1. *Let $x, y \in X$ be such that $x < y - 1$ and let $\tilde{\mathcal{S}}$ be a top-connected partially single-peaked domain with respect to x and y. Then, an SCF $f : \tilde{\mathcal{S}}^n \to X$ is unanimous and group strategy-proof if and only if it is a PDGMVS with respect to x and y.*

Proof. Let $x, y \in X$ be such that $x < y - 1$ and let $\tilde{\mathcal{S}}$ be a top-connected partially single-peaked domain with respect to x and y. Suppose $f : \tilde{\mathcal{S}}^n \to X$ is a PDGMVS with respect to x and y where agent d is the partial dictator. It is enough to show that f is group strategy-proof. Clearly, no group can manipulate f at a preference profile $P_N \in \tilde{\mathcal{S}}^n$ where $r_1(P_d) \in [x, y]$. Consider a preference profile $P_N \in \tilde{\mathcal{S}}^n$ such that $r_1(P_d) \in [a, x)$. We show that f is group strategy-proof at P_N. Since $r_1(P_d) \in [a, x)$, by the definition of PDGMVS, $f(P_N) \in [a, x]$. Let $C' = \{i \in N \mid r_1(P_i) \leq f(P_N)\}$ and $C'' = \{i \in N \mid r_1(P_i) > f(P_N)\}$. Suppose a coalition C manipulates f at P_N. Then, there is $P'_C \in \tilde{\mathcal{S}}^{|C|}$ such that $f(P'_C, P_{N \setminus C}) P_i f(P_N)$ for all $i \in C$. If $f(P'_C, P_{N \setminus C}) < f(P_N)$, then by the definition of $\tilde{\mathcal{S}}$, we have $C \cap C'' = \emptyset$. However, by the definition of PDGMVS, $f(P'_C, P_{N \setminus C}) \geq f(P_N)$ for all $C \subseteq C'$ and all $P'_C \in \tilde{\mathcal{S}}^{|C|}$, a contradiction. Again, if $f(P'_C, P_{N \setminus C}) > f(P_N)$, then by the definition of $\tilde{\mathcal{S}}$, we have $C \cap C' = \emptyset$. However, by the definition of PDGMVS, $f(P'_C, P_{N \setminus C}) \leq f(P_N)$ for all $C \subseteq C''$ and all $P'_C \in \tilde{\mathcal{S}}^{|C|}$, a contradiction. The proof for the case where $r_1(P_d) \in (y, b]$ follows from a symmetric argument. ■

6. Conclusion

In this paper, we have introduced a class of restricted domains which we call top-connected single-peaked domains and have characterized the unanimous and strategy-proof SCFs on such domains as min-max rules. Outstanding examples of top-connected single-peaked domains are maximal single-peaked domains, minimally rich single-peaked domains, distance based single-peaked domains, and top-connected regular single-crossing domains. Further, we have introduced the notion of min-max domains for which the set of unanimous and strategy-proof SCFs coincides with that of min-max rules. We have shown that a domain is a min-max domain if and only if it is a top-connected single-peaked domain.

Next, we have considered domains that violate single-peakedness over a subset of alternatives. We call such domains top-connected partially single-peaked domains. We have shown that an SCF is unanimous and strategy-proof on such a domain if and only if it is a PDGMVS. Outstanding examples of top-connected partially single-peaked domains are multiple single-peaked domains and single-peaked domains on graphs.

Finally, we have considered group strategy-proofness and have shown that strategy-proofness and group strategy-proofness are equivalent on top-connected single-peaked and top-connected partially single-peaked domains.

Appendix A. Proof of Theorem 3.1

Proof. (If part) Let $\hat{\mathcal{S}}$ be a top-connected single-peaked domain and suppose $f^\beta : \hat{\mathcal{S}}^n \to X$ is a min-max rule. Then, f^β is unanimous by definition. Let $\bar{\mathcal{S}}$ be the maximal single-peaked domain. By (Weymark (2011)), $f^\beta : \bar{\mathcal{S}}^n \to X$ is strategy-proof. Since $\hat{\mathcal{S}} \subseteq \bar{\mathcal{S}}$, f^β is strategy-proof on $\hat{\mathcal{S}}^n$. This completes the proof of the if part.

(Only-if part) Let $f : \hat{\mathcal{S}}^n \to X$ be a unanimous and strategy-proof SCF. We show that f is a min-max rule. First, we establish a few properties of f in the following sequence of lemmas.

Lemma A.1 shows that the outcome of f at every preference profile $P_N \in \hat{\mathcal{S}}^n$ must lie in-between $\min(\tau(P_N))$ and $\max(\tau(P_N))$.

Lemma A.1. *It must be that $f(P_N) \in [\min(\tau(P_N)), \max(\tau(P_N))]$ for all $P_N \in \hat{\mathcal{S}}^n$.*

Proof. Assume to the contrary that $f(P_N) \notin [\min(\tau(P_N)), \max(\tau(P_N))]$ for some $P_N \in \hat{\mathcal{S}}^n$. Without loss of generality, assume that $f(P_N) = x < \min(\tau(P_N))$. Then, $f(P_N) = x < x+1 \leq \min(\tau(P_N)) \leq r_1(P_i)$ for all $i \in N$. Since P_i is single-peaked, this means $(x+1)P_i x$ for all $i \in N$. For each $i \in N$, consider $P'_i \in \hat{\mathcal{S}}$ such that $r_1(P'_i) = x+1$ and $r_2(P'_i) = x$. Then, by strategy-proofness, $f(P'_i, P_{N\setminus i}) = x$. By moving the agents $i \in N$ from the preference P_i to the preference P'_i one-by-one and applying strategy-proofness at every step, we have $f(P_N) = f(P'_1, P_{N\setminus 1}) = f(P'_1, P'_2, P_{N\setminus\{1,2\}}) = \ldots = f(P'_1, \ldots, P'_{n-1}, P_n) = x$. However, by unanimity, $f(P'_1, \ldots, P'_n) = x+1$. This means agent n manipulates at $(P'_1, \ldots, P'_{n-1}, P_n)$ via P'_n, a contradiction. This completes the proof of the lemma. ■

Lemma A.2 and Corollary A.1 establish a restricted version of uncompromisingness. The implication of the lemma is as follows. Consider a preference profile P_N. Fix an alternative

$y \in X$. Construct another preference profile P'_N where each agent with top-ranked alternative at P_N on the left (right) of y move to a preference with top-ranked alternative y, while all other agents keep their preferences unchanged. Then, (i) if $f(P_N)$ was on the right (left) of y, then $f(P'_N) = f(P_N)$, and (ii) if $f(P_N)$ was on the left (right) of y, then $f(P'_N) = y$.

Lemma A.2. *Let $P_N, P'_N \in \hat{\mathcal{S}}^n$ and $y \in X$ be such that for all $i \in N$, if $r_1(P_i) < y$ then $r_1(P'_i) = y$, otherwise $P_i = P'_i$. Then, $f(P'_N) = \max\{f(P_N), y\}$.*

Proof. Suppose $f(P_N) = x$. We distinguish two cases based on the relative positions of x and y.

CASE 1. Suppose that $y \leq x$.

Note that if $y \leq \min(\tau(P_N))$, then $P'_N = P_N$, and hence by Lemma A.1, $y \leq x$. Therefore, there is nothing to prove. Suppose $\min(\tau(P_N)) < y$. Let $i \in N$ be such that $r_1(P_i) = \min(\tau(P_N))$. Take $P'_i \in \hat{\mathcal{S}}$ such that $r_1(P'_i) = y$. We show that $f(P'_i, P_{N\setminus i}) = x$. Suppose $f(P'_i, P_{N\setminus i}) > x$. Since P'_i is single-peaked and $r_1(P'_i) \leq x < f(P'_i, P_{N\setminus i})$, it must be that $xP'_i f(P'_i, P_{N\setminus i})$. This means agent i manipulates at $(P'_i, P_{N\setminus i})$ via P_i, a contradiction. Now suppose $f(P'_i, P_{N\setminus i}) < x$. Since $r_1(P_i) < r_1(P'_i)$, it must be that $\min(\tau(P_N)) \leq \min(\tau(P'_i, P_{N\setminus i}))$. Because $r_1(P_i) = \min(\tau(P_N))$ and $\min(\tau(P_N)) \leq \min(\tau(P'_i, P_{N\setminus i}))$, by Lemma A.1, it follows that $r_1(P_i) \leq f(P'_i, P_{N\setminus i})$. Since P_i is single-peaked and $r_1(P_i) \leq f(P'_i, P_{N\setminus i}) < x$, we have $f(P'_i, P_{N\setminus i}) P_i x$. This means agent i manipulates at P_N via P'_i, a contradiction. Therefore, $f(P'_i, P_{N\setminus i}) = x$. Now, if $y \leq \min(\tau(P'_i, P_{N\setminus i}))$, then by the assumption of the lemma, $P'_N = (P'_i, P_{N\setminus i})$, and hence the proof is complete. Suppose $\min(\tau(P'_i, P_{N\setminus i})) < y$. Consider $j \in N$ such that $r_1(P_j) = \min(\tau(P'_i, P_{N\setminus i}))$. Let P'_j be such that $r_1(P'_j) = y$. Using a similar argument as before, it follows that $f(P'_i, P'_j, P_{N\setminus \{i,j\}}) = f(P'_i, P_{N\setminus i}) = x$. Continuing in this manner, we have $f(P'_N) = x$. This completes the proof of the lemma for Case 1.

CASE 2. Suppose that $x < y$.

Let $y = x + k$ for some positive integer k. Suppose $N_x = \{i \in N \mid r_1(P_i) \leq x\}$. Let $\hat{P}_N \in \hat{\mathcal{S}}^n$ be such that $r_1(\hat{P}_i) = x$ for all $i \in N_x$ and $\hat{P}_i = P_i$ for all $i \in N \setminus N_x$. By strategy-proofness, $f(\hat{P}_N) = x$. Suppose $\bar{P} \in \hat{\mathcal{S}}$ is such that $r_1(\bar{P}) = x + 1$ and $r_2(\bar{P}) = x$. Take $i \in N_x$ and let $\bar{P}_i = \bar{P}$. Then, by strategy-proofness, $f(\bar{P}_i, \hat{P}_{N\setminus i}) \in \{x, x+1\}$ as otherwise agent i manipulates at $(\bar{P}_i, \hat{P}_{N\setminus i})$ via $\hat{P}_i$. Using a similar argument, $f(\bar{P}_i, \bar{P}_j, \hat{P}_{N\setminus \{i,j\}}) \in \{x, x+1\}$, where $i, j \in N_x$ and $\bar{P}_j = \bar{P}$. Continuing in this manner, we have $f(\bar{P}_N) \in \{x, x+1\}$, where $\bar{P}_i = \bar{P}$ for all $i \in N_x$ and $\bar{P}_i = \hat{P}_i$ for all $i \in N \setminus N_x$. However, $\min(\tau(\bar{P}_N)) = x + 1$. Hence, by Lemma A.1,

$f(\bar{P}_N) = x+1$. Suppose $N_{x+1} = \{i \in N \mid r_1(P_i) \leq x+1\}$. Let $\tilde{P} \in \hat{\mathcal{S}}$ be such that $r_1(\tilde{P}) = x+2$ and $r_2(\tilde{P}) = x+1$. Further, let $\tilde{P}_N \in \hat{\mathcal{S}}^n$ be such that $\tilde{P}_i = \tilde{P}$ if $i \in N_{x+1}$, and $\tilde{P}_i = \bar{P}_i$ for all other agents. Then, by using a similar argument as before, we have $f(\tilde{P}_N) = x+2$. Continuing in this manner, we have $f(P'_N) = x+k$, which completes the proof of the lemma for Case 2.

Since Cases 1 and 2 are exhaustive, this completes the proof of the lemma. ∎

Corollary A.1. *Let $P_N, P'_N \in \hat{\mathcal{S}}^n$ and $y \in X$ be such that for all $i \in N$, if $r_1(P_i) > y$ then $r_1(P'_i) = y$, otherwise $P_i = P'_i$. Then, $f(P'_N) = \min\{f(P_N), y\}$.*

Our next lemma shows that f is uncompromising.[11]

Lemma A.3. *The SCF f is uncompromising.*

Proof. Let $P_N \in \hat{\mathcal{S}}^n$, $i \in N$, and $P'_i \in \hat{\mathcal{S}}$ be such that $r_1(P_i) < f(P_N)$ and $r_1(P'_i) \leq f(P_N)$. It is sufficient to show $f(P'_i, P_{N\setminus i}) = f(P_N)$. Suppose $f(P_N) = x$, $r_1(P_i) = y$, and $r_1(P'_i) = y'$. Assume for contradiction that $f(P'_i, P_{N\setminus i}) = x' \neq x$. By strategy-proofness, it must be that $x' < y$ as otherwise agent i manipulates either at P_N via P'_i or at $(P'_i, P_{N\setminus i})$ via P_i. Consider $\bar{P}_N \in \hat{\mathcal{S}}^n$ such that $r_1(\bar{P}_j) = y$ for all $j \in N$ with $r_1(P_j) \leq y$, and $\bar{P}_j = P_j$ for all other agents. Since $f(P_N) = x$, by Lemma A.2, we have $f(\bar{P}_N) = \max\{x, y\} = x$. On the other hand, since $f(P'_i, P_{N\setminus i}) = x'$, by Lemma A.2, we have $f(\bar{P}_N) = \max\{x', y\} = y$, a contradiction. This completes the proof of the lemma. ∎

The following lemma establishes that f is a min-max rule.

Lemma A.4. *The SCF f is a min-max rule.*

Proof. For all $S \subseteq N$, let $(P^a_S, P^b_{N\setminus S}) \in \hat{\mathcal{S}}^n$ be such that $r_1(P^a_i) = a$ for all $i \in S$ and $r_1(P^b_i) = b$ for all $i \in N \setminus S$. Define $\beta_S = f(P^a_S, P^b_{N\setminus S})$ for all $S \subseteq N$. Clearly, $\beta_S \in X$ for all $S \subseteq N$. By unanimity, $\beta_\varnothing = b$ and $\beta_N = a$. Also, by uncompromisingness, $\beta_S \leq \beta_T$ for all $T \subseteq S$.

Take $P_N \in \hat{\mathcal{S}}^n$. We show $f(P_N) = \min\limits_{S \subseteq N}\{\max\limits_{i \in S}\{r_1(P_i), \beta_S\}\}$. Suppose $S_1 = \{i \in N \mid r_1(P_i) < f(P_N)\}$, $S_2 = \{i \in N \mid f(P_N) < r_1(P_i)\}$, and $S_3 = \{i \in N \mid r_1(P_i) = f(P_N)\}$. By uncompromisingness, $\beta_{S_1 \cup S_3} \leq f(P_N) \leq \beta_{S_1}$. Consider the expression $\min\limits_{S \subseteq N}\{\max\limits_{i \in S}\{r_1(P_i), \beta_S\}\}$. Take $S \subseteq S_1$. Then, by Condition (iii) in Definition 2.11, $\beta_{S_1} \leq \beta_S$. Since $r_1(P_i) < f(P_N)$ for all $i \in S$ and

[11] Since every SCF satisfying uncompromisingness is tops-only, Lemma A.3 shows that a top-connected single-peaked domain is a tops-only domain. It can be easily verified that top-connected single-peaked domains fail to satisfy the sufficient conditions for a domain to be tops-only identified in Chatterji and Sen (2011) and Chatterji and Zeng (2015).

$f(P_N) \leq \beta_{S_1} \leq \beta_S$, we have $\max_{i \in S}\{r_1(P_i), \beta_S\} = \beta_S$. Clearly, for all $S \subseteq N$ such that $S \cap S_2 \neq \varnothing$, we have $\max_{i \in S}\{r_1(P_i), \beta_S\} > f(P_N)$. Consider $S \subseteq N$ such that $S \cap S_2 = \varnothing$ and $S \cap S_3 \neq \varnothing$. Then, $S \subseteq S_1 \cup S_3$, and hence $\beta_{S_1 \cup S_3} \leq \beta_S$. Therefore, $\max_{i \in S}\{r_1(P_i), \beta_S\} = \max\{f(P_N), \beta_S\} \geq \max\{f(P_N), \beta_{S_1 \cup S_3}\}$. Since $\beta_{S_1 \cup S_3} \leq f(P_N)$, we have $\max\{f(P_N), \beta_{S_1 \cup S_3}\} = f(P_N)$. Combining all these, we have $\min_{S \subseteq N}\{\max_{i \in S}\{r_1(P_i), \beta_S\}\} = \min\{f(P_N), \beta_{S_1}\}$. Because $f(P_N) \leq \beta_{S_1}$, we have $\min\{f(P_N), \beta_{S_1}\} = f(P_N)$. This completes the proof of the lemma. ■

The proof of the only-if part of Theorem 3.1 follows from Lemmas A.1 - A.4. ■

Appendix B. Proof of Theorem 4.1

Proof. (If part) Let $x, y \in X$ be such that $x < y - 1$ and let f^β be a PDGMVS on $\tilde{\mathcal{S}}^n$ with respect to x and y. Then, f^β is unanimous by definition. We show that f^β is strategy-proof. Let d be the partial dictator of f^β. If $r_1(P_d) \in [x, y]$, then $f^\beta(P_N) = r_1(P_d)$, and hence f^β cannot be manipulated at a preference profile $P_N \in \tilde{\mathcal{S}}^n$. Take $P_N \in \tilde{\mathcal{S}}^n$ such that $r_1(P_d) \in [a, x)$. Then, by Lemma 4.1, $f^\beta(P_N) \in [a, x]$. Take $i \in N$ such that $r_1(P_i) \leq f^\beta(P_N)$. By the definition of f^β, $f^\beta(P'_i, P_{N \setminus i}) \geq f^\beta(P_N)$ for all $P'_i \in \tilde{\mathcal{S}}$. Since $f^\beta(P_N) \leq x$, by the definition of a top-connected partially single-peaked domain, $r_1(P_i) \leq f^\beta(P_N)$ means $f^\beta(P_N) P_i z$ for all $z > f^\beta(P_N)$. Therefore, agent i cannot manipulate f^β at P_N. By a symmetric argument, agent i cannot manipulate f^β at a preference profile where $r_1(P_i) \geq f^\beta(P_N)$. Using a similar argument, it follows that f^β cannot be manipulated at a preference profile P_N with $r_1(P_d) \in (y, b]$. This completes the proof of the if part.

(Only-if part) Let $x, y \in X$ be such that $x < y - 1$ and let $\tilde{\mathcal{S}}$ be a top-connected partially single-peaked domain with respect to x and y. Suppose $f : \tilde{\mathcal{S}}^n \to X$ is a unanimous and strategy-proof SCF. We show that f is a PDGMVS with respect to x and y. Let $\hat{\mathcal{S}}$ be a top-connected single-peaked domain contained in $\tilde{\mathcal{S}}$. Such a domain must exist by Definition 4.2. By Theorem 3.1, f restricted to $\hat{\mathcal{S}}^n$ must be a min-max rule. We establish a few properties of f in the following sequence of lemmas.

Our next lemma shows that f satisfies tops-onlyness for a particular type of preference profiles. It says the following. Let c be an arbitrary alternative. Consider a preference profile P_N such that for all $i \in N$, P_i is single-peaked and $r_1(P_i) \in \{x, c\}$. Suppose the outcome of f at P_N is c. Consider a top-equivalent preference profile P'_N where the agents with top-ranked alternative

c do not change their preferences. Then, the outcome of f at P'_N must be c.

Lemma B.1. *Let $\emptyset \subsetneq S \subsetneq N$ and let $c \in X$. Suppose $(P_S, P_{N\setminus S}) \in \hat{\mathcal{S}}^n$ and $(P'_S, P_{N\setminus S}) \in \tilde{\mathcal{S}}^n$ are two tops-equivalent preference profiles such that $r_1(P_i) = x$ for all $i \in S$, and $r_1(P_j) = c$ for all $j \in N \setminus S$. Then, $f(P_S, P_{N\setminus S}) = c$ implies $f(P'_S, P_{N\setminus S}) = c$.*

Proof. Take S such that $\emptyset \subsetneq S \subsetneq N$. We prove the lemma using induction on $|c - x|$. By unanimity, the lemma holds for $c = x$. Suppose the lemma holds for all c such that $|c - x| \leq k$. We prove the lemma for all c such that $|c - x| = k + 1$. Take c such that $|c - x| = k + 1$. Let $(P_S, P_{N\setminus S}) \in \hat{\mathcal{S}}^n$ and $(P'_S, P_{N\setminus S}) \in \tilde{\mathcal{S}}^n$ be two tops-equivalent preference profiles such that $r_1(P_i) = x$ for all $i \in S$, and $r_1(P_j) = c$ for all $j \in N \setminus S$. Suppose $f(P_S, P_{N\setminus S}) = c$. We show $f(P'_S, P_{N\setminus S}) = c$. We show this for $x < c$, the proof for the case $x > c$ is similar. Since $x < c$ and $|c - x| = k + 1$, we have $c = x + k + 1$. Let $(P_S, \hat{P}_{N\setminus S}) \in \hat{\mathcal{S}}^n$ be such that $r_1(\hat{P}_j) = x + k$ and $r_2(\hat{P}_j) = x + k + 1$ for all $j \in N \setminus S$. Because f is a min-max rule on $\hat{S}^n$ and $f(P_S, P_{N\setminus S}) = x + k + 1$, we have $f(P_S, \hat{P}_{N\setminus S}) = x + k$. Since $(P_S, \hat{P}_{N\setminus S})$ and $(P'_S, \hat{P}_{N\setminus S})$ are tops-equivalent and $r_1(\hat{P}_j) = x + k$ for all $j \in N \setminus S$, we have by the induction hypothesis, $f(P'_S, \hat{P}_{N\setminus S}) = x + k$. For all $j \in N \setminus S$, let $\bar{P}_j \in \hat{\mathcal{S}}$ be such that $r_1(\bar{P}_j) = x + k + 1$ and $r_2(\bar{P}_j) = x + k$. Since $f(P'_S, \hat{P}_{N\setminus S}) = x + k$, by moving the agents $j \in N \setminus S$ from $\hat{P}_j$ to $\bar{P}_j$ one-by-one and applying strategy-proofness at every step, we have $f(P'_S, \bar{P}_{N\setminus S}) \in \{x + k, x + k + 1\}$. We claim $f(P'_S, \bar{P}_{N\setminus S}) = x + k + 1$. Assume for contradiction that $f(P'_S, \bar{P}_{N\setminus S}) = x + k$. Recall that $P_i \in \hat{\mathcal{S}}$ for all $i \in S$. Since $(x + k)P_i(x + k + 1)$ for all $i \in S$, by moving the agents $i \in S$ from P'_i to P_i one-by-one and applying strategy-proofness at every step, we have $f(P_S, \bar{P}_{N\setminus S}) \leq x + k$. Since $r_1(P_j) = r_1(\bar{P}_j) = x + k + 1$ for all $j \in N \setminus S$, by strategy-proofness, $f(P_S, P_{N\setminus S}) \neq x + k + 1$. This contradicts our assumption that $f(P_S, P_{N\setminus S}) = x + k + 1$. Therefore, $f(P'_S, \bar{P}_{N\setminus S}) = x + k + 1$. Since $r_1(P_j) = r_1(\bar{P}_j) = x + k + 1$ for all $j \in N \setminus S$, we have by strategy-proofness, $f(P'_S, P_{N\setminus S}) = x + k + 1$. This completes the proof of the lemma. ■

Corollary B.1. *Let $\emptyset \subsetneq S \subsetneq N$ and let $c \in X$. Suppose $(P_S, P_{N\setminus S}) \in \hat{\mathcal{S}}^n$ and $(P'_S, P_{N\setminus S}) \in \tilde{\mathcal{S}}^n$ are two tops-equivalent preference profiles such that $r_1(P_i) = y$ for all $i \in S$, and $r_1(P_j) = c$ for all $j \in N \setminus S$. Then, $f(P_S, P_{N\setminus S}) = c$ implies $f(P'_S, P_{N\setminus S}) = c$.*

Our next lemma shows that the outcome of f at a boundary preference profile cannot be strictly in-between x and y.[12]

Lemma B.2. *Let $P_N \in \tilde{\mathcal{S}}^n$ be such that $r_1(P_i) \in \{a, b\}$ for all $i \in N$. Then, $f(P_N) \notin (x, y)$.*

[12] A *boundary preference profile* is one where the top-ranked alternative of each agent is either a or b.

Proof. Assume for contradiction that $f(P_N) = z \in (x, y)$ for some $P_N \in \tilde{\mathcal{S}}^n$ such that $r_1(P_i) \in \{a, b\}$ for all $i \in N$. Let $S = \{i \in N \mid r_1(P_i) = a\}$. Then, it must be that $\emptyset \subsetneq S \subsetneq N$ as otherwise we are done by unanimity. Let $r_2(Q) = x'$ and $r_2(Q') = y'$, where $Q, Q' \in \tilde{\mathcal{S}}$ are as given in Condition (iii) of Definition 4.1. We distinguish three cases based on the relative positions of x', y', and z.

CASE 1. Suppose $x' \in (x+1, y-1)$, $y' \in (x+1, y-1)$, and $z \in (x, y'] \cup [x', y)$.

We consider the case where $z \in (x, y']$, the proof for the case where $z \in [x', y)$ follows from a symmetric argument. Let $P'_N \in \hat{\mathcal{S}}^n$ be such that $r_1(P'_i) = y'$ for all $i \in S$, and $r_1(P'_j) = y - 1$, $r_2(P'_j) = y$ for all $j \in N \setminus S$. Further, let $\hat{P}_N \in \hat{\mathcal{S}}^n$ be such that $r_1(\hat{P}_i) = x$ for all $i \in S$ and $r_1(\hat{P}_j) = x + 1$ for all $j \in N \setminus S$. Because f is a min-max rule on $\hat{\mathcal{S}}^n$ and $f(P_S, P_{N\setminus S}) = z$, we have $f(P'_S, P'_{N\setminus S}) = y'$ and $f(\hat{P}_S, \hat{P}_{N\setminus S}) = x + 1$. As $f(\hat{P}_S, \hat{P}_{N\setminus S}) = x + 1$, by Lemma B.1, we have $f(Q_S, \hat{P}_{N\setminus S}) = x + 1$, where $Q_i = Q$ for all $i \in S$. Consider the preference profile $(Q'_S, P'_{N\setminus S})$, where $Q'_i = Q'$ for all $i \in S$. Note that $f(P'_S, P'_{N\setminus S}) = y'$, $r_1(Q') = y$, and $r_2(Q') = y'$. Therefore, by moving the agents $i \in S$ from P'_i to Q' one-by-one and using strategy-proofness at every step, we have $f(Q'_S, P'_{N\setminus S}) \in \{y, y'\}$. We claim $f(Q'_S, P'_{N\setminus S}) = y$. Assume for contradiction that $f(Q'_S, P'_{N\setminus S}) = y'$. Since $y P'_j y'$ for all $j \in N \setminus S$, by moving the agents $j \in N \setminus S$ from P'_j to Q' one-by-one and applying strategy-proofness at every step, we have $f(Q'_S, Q'_{N\setminus S}) \neq y$. However, this contradicts unanimity. So, $f(Q'_S, P'_{N\setminus S}) = y$. For all $i \in S$, let $\tilde{P}_i \in \hat{\mathcal{S}}$ be such that $r_1(\tilde{P}_i) = y$. By strategy-proofness, $f(\tilde{P}_S, P'_{N\setminus S}) = y$. Since f is a min-max rule on $\hat{\mathcal{S}}^n$, this means $f(\tilde{P}_S, \hat{P}_{N\setminus S}) = y$. For all $i \in S$, let $\tilde{P}'_i \in \hat{\mathcal{S}}$ be such that $r_1(\tilde{P}'_i) = x'$. Because $(\tilde{P}_S, \hat{P}_{N\setminus S}), (\tilde{P}'_S, \hat{P}_{N\setminus S}) \in \hat{\mathcal{S}}^n$ and f is a min-max rule on $\hat{\mathcal{S}}^n$, $f(\tilde{P}_S, \hat{P}_{N\setminus S}) = y$ implies $f(\tilde{P}'_S, \hat{P}_{N\setminus S}) = x'$. Because $f(\tilde{P}'_S, \hat{P}_{N\setminus S}) = x'$, $r_1(Q) = x$, and $r_2(Q) = x'$, by moving the agents $i \in S$ from $\tilde{P}'_i$ to Q one-by-one and applying strategy-proofness at every step, we have $f(Q_S, \hat{P}_{N\setminus S}) \in \{x, x'\}$. Since $\{x + 1\} \cap \{x, x'\} = \emptyset$ by our assumption, this is a contradiction to our earlier finding $f(Q_S, \hat{P}_{N\setminus S}) = x + 1$. This completes the proof of the lemma for Case 1.

CASE 2. Suppose $x' \in (x+1, y-1)$, $y' \in (x+1, y-1)$, $y' < x' - 1$, and $z \in (y', x')$.

Let $P'_N, \hat{P}_N \in \hat{\mathcal{S}}^n$ be such that $r_1(P'_i) = x'$ and $r_1(\hat{P}_i) = x$ for all $i \in S$, and $r_1(P'_j) = y$ and $r_1(\hat{P}_j) = y'$ for all $j \in N \setminus S$. Because f is a min-max rule on $\hat{\mathcal{S}}^n$ and $f(P_S, P_{N\setminus S}) = z$, we have $f(P'_S, P'_{N\setminus S}) = x'$ and $f(\hat{P}_S, \hat{P}_{N\setminus S}) = y'$. As $f(\hat{P}_S, \hat{P}_{N\setminus S}) = y'$, by Lemma B.1, we have $f(Q_S, \hat{P}_{N\setminus S}) = y'$, where $Q_i = Q$ for all $i \in S$. Again, as $f(P'_S, P'_{N\setminus S}) = x'$, by Corollary B.1, we have $f(P'_S, Q'_{N\setminus S}) = x'$, where $Q'_j = Q'$ for all $j \in N \setminus S$. Because $f(Q_S, \hat{P}_{N\setminus S}) = y'$, $r_1(Q') = y$, and $r_2(Q') = y'$, by moving the agents $j \in N \setminus S$ from $\hat{P}_j$ to Q' one-by-one and using strategy-proofness at every step, we have $f(Q_S, Q'_{N\setminus S}) \in \{y, y'\}$. Again, because $f(P'_S, Q'_{N\setminus S}) = x'$, $r_1(Q) = x$, and $r_2(Q) = x'$, by moving the agents $i \in S$ from P'_i to Q one-by-one and using strategy-proofness at every step, we have $f(Q_S, Q'_{N\setminus S}) \in \{x, x'\}$. Since $\{x, x'\} \cap \{y, y'\} = \emptyset$ by our assumption, this is a contradiction. This completes the proof of the lemma for Case 2.

CASE 3. Suppose $x' = y$, $y' = x$, and $z \in (y', x')$.

Let $P'_N \in \hat{\mathcal{S}}^n$ be such that $r_1(P'_i) = x$ for all $i \in S$ and $r_1(P'_j) = y$ for all $j \in N \setminus S$. Because f is a min-max rule on $\hat{\mathcal{S}}^n$ and $f(P_S, P_{N\setminus S}) = z$, we have $f(P'_S, P'_{N\setminus S}) = z$. Take $i \in N$ and consider the preference profile $(Q_i, P'_{S\setminus i}, P'_{N\setminus S})$, where $Q_i = Q$. Since $r_1(P'_i) = r_1(Q_i) = x$ and $f(P'_S, P'_{N\setminus S}) \neq x$, by strategy-proofness, $f(Q_i, P'_{S\setminus i}, P'_{N\setminus S}) \neq x$. Continuing in this manner, it follows that $f(Q_S, P'_{N\setminus S}) \neq x$, where $Q_i = Q$ for all $i \in S$. Moreover, since $r_2(Q_i) = y$ for all $i \in S$ and $r_1(P'_j) = y$ for all $j \in N \setminus S$, by unanimity and strategy-proofness, $f(Q_S, P'_{N\setminus S}) \in \{x, y\}$. Since $f(Q_S, P'_{N\setminus S}) \neq x$, this means $f(Q_S, P'_{N\setminus S}) = y$. Let $Q'_j = Q'$ for all $j \in N \setminus S$. As $f(Q_S, P'_{N\setminus S}) = y$ and $r_1(Q') = y$, by strategy-proofness, $f(Q_S, Q'_{N\setminus S}) = y$. Now, if we first move the agents $j \in N \setminus S$ from P'_j to Q' and then move the agents $i \in S$ from P'_i to Q, then it follows from a similar argument that $f(Q_S, Q'_{N\setminus S}) = x$. Since $x \neq y$, this is a contradiction to our earlier finding that $f(Q_S, Q'_{N\setminus S}) = y$. This completes the proof of the lemma for Case 3.

Since Cases 1, 2 and 3 are exhaustive, this completes the proof of the lemma. ■

Let $(\beta_S)_{S\subseteq N}$ be the parameters of f restricted to $\hat{\mathcal{S}}^n$. In Lemma B.3 and Lemma B.4, we establish a few properties of these parameters.

Lemma B.3. *For all $S \subseteq N$, $\beta_S \in [a, x]$ if and only if $\beta_{N\setminus S} \in [y, b]$.*

Proof. Take $S \subseteq N$. It is enough to show that $\beta_S \in [a, x]$ implies $\beta_{N\setminus S} \in [y, b]$. Assume for contradiction that $\beta_S, \beta_{N\setminus S} \in [a, x]$. Let $Q' \in \tilde{\mathcal{S}}$ with $r_1(Q') = y$ be as given in Condition (iii) of Definition 4.1. Suppose $r_2(Q') = y'$. Take $z \in (y', y)$. Let $(P_S, P_{N\setminus S}) \in \hat{\mathcal{S}}^n$ be such that $r_1(P_i) = a$ for all $i \in S$ and $r_1(P_j) = b$ for all $j \in N \setminus S$. Since f restricted to $\hat{\mathcal{S}}^n$ is a min-max rule, $f(P_S, P_{N\setminus S}) = \beta_S \in [a, x]$. Let $(P'_S, P'_{N\setminus S}) \in \hat{\mathcal{S}}^n$ be such that $r_1(P'_i) = y'$ for all $i \in S$ and $r_1(P'_j) = z$ for all $j \in N \setminus S$. Since $f(P_S, P_{N\setminus S}) \in [a, x]$, by uncompromisingness of f restricted to $\hat{\mathcal{S}}^n$, we have $f(P'_S, P'_{N\setminus S}) = y'$. Because $r_1(Q') = y$ and $r_2(Q') = y'$, by moving the agents $i \in S$ one-by-one from P'_i to Q' and applying strategy-proofness at every step, we have $f(Q'_S, P'_{N\setminus S}) \in \{y, y'\}$, where $Q'_i = Q'$ for all $i \in S$.

Now, let $(\bar{P}_S, \bar{P}_{N\setminus S}) \in \hat{\mathcal{S}}^n$ be such that $r_1(\bar{P}_i) = b$ for all $i \in S$ and $r_1(\bar{P}_j) = a$ for all $j \in N \setminus S$. Again, since f restricted to $\hat{\mathcal{S}}^n$ is a min-max rule, $f(\bar{P}_S, \bar{P}_{N\setminus S}) = \beta_{N\setminus S} \in [a, x]$. Recall that for $j \in N \setminus S$, $P'_j \in \hat{\mathcal{S}}$ with $r_1(P'_j) = z$. Consider $(P''_S, P'_{N\setminus S}) \in \hat{\mathcal{S}}^n$ such that $r_1(P''_i) = y$ for all $i \in S$. Since $f(\bar{P}_S, \bar{P}_{N\setminus S}) \in [a, x]$, by uncompromisingness of f restricted to $\hat{\mathcal{S}}^n$, we have $f(P''_S, P'_{N\setminus S}) = z$. Because $r_1(P''_i) = y = r_1(Q')$ for all $i \in S$, by Corollary B.1, it follows that $f(Q'_S, P'_{N\setminus S}) = z$. However, as $z \notin \{y, y'\}$, this is a contradiction to our earlier finding that $f(Q'_S, P'_{N\setminus S}) \in \{y, y'\}$. This completes the proof of the lemma. ■

The following lemma says that there is exactly one agent i such that $\beta_i \in [a, x]$.

Lemma B.4. *It must be that* $|\{i \in N \mid \beta_i \in [a, x]\}| = 1$.

Proof. Suppose there are $i \neq j \in N$ such that $\beta_i, \beta_j \in [a, x]$. By Lemma B.3, $\beta_i \in [a, x]$ implies $\beta_{N\setminus i} \in [y, b]$. Since $j \in N \setminus i$ and $\beta_T \leq \beta_S$ for all $S \subseteq T$, $\beta_{N\setminus i} \in [y, b]$ implies $\beta_j \in [y, b]$, a contradiction. Hence, there can be at most one agent $i \in N$ such that $\beta_i \in [a, x]$.

Suppose $\beta_i \in [y, b]$ for all $i \in N$. By Lemma B.3, this means $\beta_{N\setminus i} \in [a, x]$ for all $i \in N$. Therefore, there must be $S \subseteq N$ such that $\beta_S \in [a, x]$ and for all $S' \subsetneq S$, $\beta_{S'} \in [y, b]$. By unanimity, $S \neq \emptyset$. If S is singleton, say $\{i\}$ for some $i \in N$, then $\beta_i \in [a, x]$ and we are done. So assume that there are $j \neq k \in S$.

Consider the preference profile $P_N \in \hat{\mathcal{S}}^n$ such that $r_1(P_j) = x + 1$, $r_2(P_j) = x$, $r_1(P_i) = x'$ for all $i \notin S$, and $r_1(P_i) = x$ for all $i \in S \setminus j$. Since $\beta_S \in [a, x]$ and $\beta_{S'} \in [y, b]$ for all $S' \subsetneq S$, it follows from the definition of a min-max rule that $f(P_N) = x + 1$. Let $P'_k \in \hat{\mathcal{S}}$ be such that $r_1(P'_k) = x'$. Since $\beta_{S\setminus k} \in [y, b]$ and f restricted to $\hat{\mathcal{S}}^n$ is a min-max rule, it follows that $f(P'_k, P_{N\setminus k}) = x'$. Consider the preference profile $(Q_k, P_{N\setminus k})$, where $Q_k = Q$. Because $f(P'_k, P_{N\setminus k}) = x'$, $r_1(Q_k) = x$, and $r_2(Q_k) = x'$, by strategy-proofness, $f(Q_k, P_{N\setminus k}) \in \{x, x'\}$. Suppose $f(Q_k, P_{N\setminus k}) = x$. Because $f(P_N) = x + 1$ and $r_1(P_k) = x$, this means agent k manipulates at P_N via Q_k. So, $f(Q_k, P_{N\setminus k}) = x'$. Let $P'_j \in \hat{\mathcal{S}}$ be such that $r_1(P'_j) = x$. Since $\beta_S \in [a, x]$ and x is the top-ranked alternative of the agents in S at preference profile $(P'_j, P_{N\setminus j})$, we have $f(P'_j, P_{N\setminus j}) = x$. As $r_1(P_k) = r_1(Q_k) = x$, this means $f(P'_j, Q_k, P_{N\setminus\{j,k\}}) = x$. Because $f(Q_k, P_{N\setminus k}) = x'$, $r_1(P_j) = x + 1$, and $r_2(P_j) = x$, agent j manipulates at $(Q_k, P_{N\setminus\{k\}})$ via P'_j. This completes the proof of the lemma. ■

REMARK B.1. By Lemma B.3 and Lemma B.4, it follows that f restricted to $\hat{\mathcal{S}}^n$ is a PDGMVS.

Our next lemma establishes that f is uncompromising.[13] First, we introduce few notations that we use in the proof of the lemma. For $P_N \in \tilde{\mathcal{S}}^n$, let $\tilde{N}(P_N) = \{i \in N \mid P_i \notin \hat{\mathcal{S}}\}$ be the set of agents who do not have single-peaked preferences at P_N. Moreover, for $0 \leq l \leq n$, let $\tilde{\mathcal{S}}^n_l = \{P_N \in \tilde{\mathcal{S}}^n \mid |\tilde{N}(P_N)| \leq l\}$ be the set of preference profiles where at most l agents have non-single-peaked preferences. Note that $\tilde{\mathcal{S}}^n_0 = \hat{\mathcal{S}}^n$ and $\tilde{\mathcal{S}}^n_n = \tilde{\mathcal{S}}^n$.

Lemma B.5. *The SCF f is uncompromising.*

Proof. Since $\tilde{\mathcal{S}}^n_0 = \hat{\mathcal{S}}^n$, f restricted to $\tilde{\mathcal{S}}^n_0$ is uncompromising. Suppose f restricted to $\tilde{\mathcal{S}}^n_k$ is uncompromising for some $k < n$. We show that f restricted to $\tilde{\mathcal{S}}^n_{k+1}$ is uncompromising. It is enough to show that f restricted to $\tilde{\mathcal{S}}^n_{k+1}$ is tops-only. To see this, note that if f restricted to $\tilde{\mathcal{S}}^n_{k+1}$ is tops-only, then f is uniquely determined on $\tilde{\mathcal{S}}^n_{k+1}$ by its outcomes on $\hat{\mathcal{S}}^n$. Therefore, since f restricted to $\hat{\mathcal{S}}^n$ is uncompromising, f is uncompromising on $\tilde{\mathcal{S}}^n_{k+1}$.

Take $P_N \in \tilde{\mathcal{S}}^n_{k+1}$ and $j \in \tilde{N}(P_N)$. Let $\hat{P}_j \in \hat{\mathcal{S}}$ be such that $r_1(\hat{P}_j) = r_1(P_j)$. Then, P_N and $(\hat{P}_j, P_{N\setminus j})$ are tops-equivalent and $(\hat{P}_j, P_{N\setminus j}) \in \tilde{\mathcal{S}}^n_k$. It is sufficient to show that $f(P_N) = f(\hat{P}_j, P_{N\setminus j})$. Assume for contradiction that $f(P_N) \neq f(\hat{P}_j, P_{N\setminus j})$. Assume, without loss of generality, that the partial dictator of f restricted to $\hat{\mathcal{S}}^n$ is agent 1. Then, by the induction hypothesis, agent 1 is the partial dictator of f restricted to $\tilde{\mathcal{S}}^n_k$, i.e., for all $P_N \in \tilde{\mathcal{S}}^n_k$, if $r_1(P_1) \in [a, x)$ then $f(P_N) \in [a, x]$, if $r_1(P_1) \in (y, b]$ then $f(P_N) \in [y, b]$, and if $r_1(P_1) \in [x, y]$ then $f(P_N) = r_1(P_1)$. We distinguish two cases based on the position of the top-ranked alternative of agent 1.

CASE 1. Suppose $r_1(P_1) \in [a, x) \cup (y, b]$.

We consider the case where $r_1(P_1) \in [a, x)$, the proof for the case where $r_1(P_1) \in (y, b]$ follows from symmetric arguments. Since $r_1(P_1) \in [a, x)$, we have $f(\hat{P}_j, P_{N\setminus j}) \in [a, x]$. Because $\hat{P}_j$ is single-peaked, if $f(\hat{P}_j, P_{N\setminus j}) < f(P_N) \leq r_1(\hat{P}_j)$ or $r_1(\hat{P}_j) \leq f(P_N) < f(\hat{P}_j, P_{N\setminus j})$, then agent j manipulates at $(\hat{P}_j, P_{N\setminus j})$ via P_j. Moreover, since $f(\hat{P}_j, P_{N\setminus j}) \in [a, x]$, if $f(P_N) < f(\hat{P}_j, P_{N\setminus j}) \leq r_1(\hat{P}_j)$ or $r_1(P_j) \leq f(\hat{P}_j, P_{N\setminus j}) < f(P_N)$, then by the definition of a top-connected partially single-peaked domain, agent j manipulates at $(P_j, P_{N\setminus j})$ via $\hat{P}_j$. Now, suppose $f(\hat{P}_j, P_{N\setminus j}) < r_1(\hat{P}_j) < f(P_N)$. Let $\bar{P}_j \in \hat{\mathcal{S}}$ be such that $r_1(\bar{P}_j) = f(P_N)$. Since f restricted to $\tilde{\mathcal{S}}^n_k$ is uncompromising and $f(\hat{P}_j, P_{N\setminus j}) < r_1(\hat{P}_j) < r_1(\bar{P}_j)$, we have $f(\bar{P}_j, P_{N\setminus j}) = f(\hat{P}_j, P_{N\setminus j})$. Because $r_1(\bar{P}_j) = f(P_N)$, it follows that agent j manipulates at $(\bar{P}_j, P_{N\setminus j})$ via P_j. Using a similar argument, it can be shown

[13]Since every SCF satisfying uncompromisingness is tops-only, Lemma B.5 shows that a top-connected partially single-peaked domain is a tops-only domain. As in the case of Lemma A.3, it can be easily verified that top-connected partially single-peaked domains fail to satisfy the sufficient conditions for a domain to be tops-only identified in Chatterji and Sen (2011) and Chatterji and Zeng (2015).

that $f(P_N) < r_1(\hat{P}_j) < f(\hat{P}_j, P_{N\setminus j})$ leads to a manipulation by agent j. Therefore, $f(P_N) = f(\hat{P}_j, P_{N\setminus j})$ when $r_1(P_1) \in [a, x)$. This completes the proof of the lemma for Case 1.

CASE 2. Suppose $r_1(P_1) \in [x, y]$.

Since agent 1 is the partial dictator, $f(\hat{P}_j, P_{N\setminus j}) = r_1(P_1)$. Consider $\bar{P}_j \in \hat{\mathcal{S}}$ such that $r_1(\bar{P}_j) = f(P_N)$. Since $(\bar{P}_j, P_{N\setminus j}) \in \tilde{\mathcal{S}}^n_k$, by the induction hypothesis, we have $f(\bar{P}_j, P_{N\setminus j}) = r_1(P_1)$. Because $r_1(\bar{P}_j) = f(P_N)$ and $f(\bar{P}_j, P_{N\setminus j}) = r_1(P_1) \neq f(P_N)$, agent j manipulates at $(\bar{P}_j, P_{N\setminus j})$ via P_j. Therefore, $f(P_N) = f(\hat{P}_j, P_{N\setminus j})$ when $r_1(P_1) \in [x, y]$. This completes the proof of the lemma for Case 2.

Since Cases 1 and 2 are exhaustive, this completes the proof of the lemma by induction. ■

Now, we complete the proof of the only-if part of Theorem 4.1. Since f is uncompromising on $\tilde{\mathcal{S}}^n$ and f restricted to $\hat{\mathcal{S}}^n$ is a min-max rule with parameters $(\beta_S)_{S\subseteq N}$ satisfying the properties as stated in Lemma B.3 and Lemma B.4, it follows that f is a PDGMVS. ■

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