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Abstract:

In this paper, we propose a quick, efficient, and easy method to examine whether a time series Y_t possesses any nonlinear feature. The advantage of our proposed nonlinearity test is that it is not required to know the exact nonlinear features and the detailed nonlinear forms of Y_t. We find that our proposed test can be used to detect any nonlinearity for the variable being examined and detect GARCH models in the innovations. It can also be used to test whether the hypothesized model, including linear and nonlinear, to the variable being examined is appropriate as long as the residuals of the model being used can be estimated. Our simulation study shows that our proposed test is stable and powerful. We apply our proposed statistic to test whether there is any nonlinear feature in the sunspot data and whether the S&P 500 index follows a random walk model. The conclusion drawn from our proposed test is consistent those from other tests.

Keywords: Nonlinearity, U-statistics, Volterra expansion, sunspots, efficient market

JEL Classification: C01, C12, G10

1 Introduction

It is well-known that nonlinearity always appears in many time series like natural data and economic and financial time series, including some well-known datasets like the sunspots (Moran, 1954), Canadian lynx (Tong, 1990), and inflation rate (Engle, 1982). In practice, nonlinearity is common in both stationary or non-stationary time series. Nevertheless, detecting nonlinearity in time series is very important because very often academics and practitioners have to know this feature in the data before conducting their analysis. For example, Fourier analysis assumes the time series to be linear and stationary while, on the other hand, the wavelet analysis (Cheng, et al., 1996) is raised for linear but nonstationary. Thus, before academics and practitioners apply Fourier analysis and/or wavelet analysis in their work, they have to examine whether there is any nonlinearity in the time series.

It is a growing interest in the testing, estimation, specification, and developing properties for nonlinearity for decades. There are many nonlinear features including asymmetric cycles, nonlinear relationship among the variables being studied and their lags, time irreversibility, sensitivity to initial conditions, and others. The early development of nonlinear models include bilinear models (Granger and Andersen, 1978), threshold autoregressive models (Tong, 1978), state-dependent model (Priestley, 1980), exponential autoregressive model (Haggan and Ozaki, 1981), ARCH model (Engle, 1982), Markov switching model (Hamilton, 1989), and nonlinear state-space model (Carlin, et al., 1992). In addition, Chen and Tsay (1993a) use an arranged local regression procedure to construct functional-coefficient autoregressive models while Chen and Tsay (1993b) develop some new techniques for a class of nonlinear additive autoregressive models with exogenous variables. On the other hand, Tjϕstheim (1994) uses nonparametric regression techniques as an alternative nonlinear time series model. Tiao and Tsay (1994) discuss the advances in non-linear modelling and in Bayesian inference via the Gibbs sampler.

Nonetheless, the most general form of a nonlinear stationary process is the Volterra expansion. Using the idea of Volterra expansions, Keenan (1985) applies the one-degree-of freedom test (Tukey, 1949) for nonadditivity to derive a time-domain statistic for discriminating nonlinear from linear models. Tsay (1986) extends the work of Keenan to establish a more powerful test. Other nonlinear tests include a simple portmanteau test (Petruccelli and Davies, 1986), the quasi-likelihood ratio test (Chan and Tong, 1990), and the Wald test (Hansen, 1996). In addition, Li and Li (2011) develop a quasi-likelihood ratio test statistic for an autoregressive moving average model against its threshold extension.

Since the number of parameters of the nonlinearity part could be very large, this could affect the performance of the existing nonlinear tests. In addition, nonlinearity may occur in many and could be infinitely ways. The advantage of our proposed nonlinearity test is that it is not required to know the exact nonlinear features and the detailed nonlinear forms of a time series. Residuals of an appropriate linear model is independent under the linearity hypothesis. In this paper we use this idea to develop a new nonlinearity test to examine whether there is any nonlinearity in a time series.

The objective in this paper is to circumvent the limitation of Volterra expansion or other similar approaches that result in many parameters in the estimation by developing a new method to test the nonlinearity for a time series that does not involve many parameters. We find that our proposed test can be used to detect any nonlinearity for the variable being examined and detect GARCH models in the innovations. It can also be used to test whether the hypothesized model, including linear and nonlinear, to the variable being examined is appropriate as long as the residuals of the model being used can be estimated. We will discuss this feature more in the conclusion section.

To demonstrate the performance of our proposed nonlinear test, we conduct simulation study on two types of threshold autoregressive models and GARCH models. Our simulation reveals that for the GARCH models, our proposed test is dominantly more powerful than Tsay's test for large sample size. On the other hand, for the threshold autoregressive models, our simulation shows that Tsay's test is more powerful than our proposed test in a region while our test is more powerful in another region. We note that this finding is not surprising because there are many different forms of nonlinearity, and thus, there may not exist any test that could outperform the others in detecting nonlinearity. However, our simulation shows that our proposed test has three more desirable features when comparing with Tsay's test: our proposed test is more stable, the power of our proposed test increases while that of Tsay's test could decrease when the magnitude of parameter increases, and the power of our proposed test reaches one quickly while that of Tsay's test may not reach one when the magnitude of parameter increases. Thus, the result of our simulation supports our claim that our proposed test is a more desirable test.

At last, to demonstrate the applicability of our proposed test, we first apply both Tsay's test and the nonlinearity test we developed in this paper to test whether there exists any nonlinear feature in the sunspot data, and thereafter, apply our proposed nonlinearity test, Tsay's test, and Chow-Denning's variance ratio tests to test whether the S&P 500 index follows a random walk model. Both our proposed nonlinearity test and the Tsay's test conclude that there exists nonlinearity component in the sunspot data. On the other hand, our proposed nonlinearity test, Tsay's test, and Chow-Denning's variance ratio tests conclude that the S&P 500 index does not follow the random walk model. The conclusion drawn from our proposed test is consistent with those drawn from Tsay (1986) and others. Thus, our illustration supports our claim that our proposed statistic is useful.

The remainder of the paper is organized as follows. In Section 2, we first discuss the Volterra expansion and state the nonlinearity test developed by Tsay (1986). Thereafter, we develop our proposed new nonlinearity test to circumvent the limitation of Volterra expansion. In Section 3, we illustrate the superiority of the nonlinearity test we developed in Section 2 by conducting a simulation to examine its performance over the test developed by Tsay (1986). In Section 4, we illustrate the applicability of our proposed nonlinearity test by applying it to examine whether there is any nonlinear feature in the sunspot data and whether the S&P 500 index follows a random walk model. Section 5 wraps up the paper by providing several well-grounded observations while the proof is provided in the appendix.

2 Theory

We suppose that Y_t follows a time series model of the current and past independent and identically distributed (i.i.d.) shocks such that $Y_t = f(\varepsilon_t, \varepsilon_{t-1}, \cdots)$. If $f(\cdot)$ is a linear function of the shocks, the model is linear; otherwise, it is nonlinear. One of the most commonly used linear models is an ARMA process that could be presented as an AR and/or MA representation (Box, et al., 1994). There are many approaches, for example, parametric, semi-parametric, and nonparametric approaches, to identify the nonlinear forms of the models. There are also several nonlinearity tests available. For example, Fan and Yao (2003) establish a likelihood ratio test to test for a linear model versus a TAR model with two regimes. Cox (1981) suggests using quadratic or cubic regression to test for nonlinearity.

One of the most commonly used approaches is to apply the Volterra expansion (Wiener, 1958) to expand a nonlinear and stationary time series, say, Y_t, to be in terms of the linear, quadratic, cubic, etc. such that

$$Y_t = \mu + \sum_{-\infty}^{\infty} a_u \varepsilon_{t-u} + \sum_{u,v=-\infty}^{\infty} a_{uv} \varepsilon_{t-u} \varepsilon_{t-v} + \sum_{u,v,w=-\infty}^{\infty} a_{uvw} \varepsilon_{t-u} \varepsilon_{t-v} \varepsilon_{t-w} + \cdots \ , \qquad (1)$$

where ε_t $(-\infty < t < \infty)$ is an i.i.d. innovation with zero mean.

If the null hypothesis of linearity is true, residuals of the hypothesized linear model are independent. This is the basic idea used in the development of various nonlinearity tests.

2.1 Tsay's F Test

Tsay (1986) develops a nonlinearity test based on the idea of using the Volterra expansion. His test is popular and is well-known to have decent power on detecting nonlinearity in a sequence, say, $\{Y_t\}$. In his test, the following null hypothesis is used:

$$H_0 \ : \ \text{there is no nonlinearity in the time series being examined.} \qquad (2)$$

The test mainly consists three steps:

Step 1: Applying the linear regression model $Y_t = \mathbf{W}_t \mathbf{\Phi} + e_t$ to fit Y_t on $\{1, Y_{t-1}, \cdots, Y_{t-M}\}$ and obtain the estimate of its innovation $\{\hat{e}_t\}$, for $t = M+1, \cdots, T$, where $\mathbf{W}_t = (1, Y_{t-1}, \cdots, Y_{t-M})$, M is a pre-specified positive integer, and T is the length of sequence $\{Y_t\}$.

Step 2: Adopting the multivariate regression model $\mathbf{Z}_t = \mathbf{W}_t\mathbf{H} + \mathbf{X}_t$ to fit $\mathbf{Z}_t$ on $\{1, Y_{t-1}, \cdots, Y_{t-M}\}$ and obtain the error term $\{\hat{\mathbf{X}}_t\}$, for $t = M+1, \cdots, T$, where $\mathbf{Z}_t^T = \text{vech}(\mathbf{V}_t^T\mathbf{V}_t)$ with $\mathbf{V}_t = (Y_{t-1}, \cdots, Y_{t-M})$ and vech denotes the half stacking vector.

Step 3: Thereafter, fit $\hat{e}_t = \hat{\mathbf{X}}_t\boldsymbol{\beta} + \varepsilon_t, (t = M+1, \cdots, T)$ to obtain the Tsay's test:

$$\hat{F} = \frac{(\sum_{t=M+1}^{T} \hat{\mathbf{X}}_t\hat{e}_t)(\sum_{t=M+1}^{T} \hat{\mathbf{X}}_t^T\hat{\mathbf{X}}_t)^{-1}(\sum_{t=M+1}^{T} \hat{\mathbf{X}}_t^T\hat{e}_t)/M^*}{\sum_{t=M+1}^{T} \hat{\varepsilon}_t^2/(T - M - M^* - 1)} \ . \tag{3}$$

Under the null hypothesis of linearity and for large T, the statistic $\hat{F}$ follows approximately a F-distribution with $\frac{1}{2}M(M+1)$ and $T - \frac{1}{2}M(M+3) - 1$ degrees of freedom. Thus, for the test level α, one could reject the null hypothesis of linearity if

$$\hat{F} > F_{\left(\frac{1}{2}M(M+1), T-\frac{1}{2}M(M+3)-1\right)}(\alpha) \ . \tag{4}$$

Readers may refer to Tsay (1986) for more details for his test.

2.2 New Non-Linearity Test

The major drawback of applying the Volterra expansion is that the number of parameters is too large. To circumvent the limitation, one could assume a_u, a_{uv}, and a_{uvw} in equation (1) to be functions of small numbers of parameters. However, the problem of this approach is that we do not know the forms of "functions" and, in fact, such "functions" may not exist. Thus, in this paper we introduce another approach to circumvent the limitation of the Volterra expansion of getting too many parameters. To identify any nonlinearity of the time series $\{Y_t\}$, we first follow the idea from Tsay (1986) to use the following AR model to remove any autocorrelation in the data:

$$Y_t = \sum_{i=1}^{p} \phi_i Y_{t-i} + e_t \ , \tag{5}$$

where $\varepsilon_t \sim \text{WN}(0, \sigma^2)$ and WN stands for 'white noise.' After removing the linear components in $\{Y_t\}$ by introducing the linear model in (5), we proceed to examine whether there is any remaining incremental power from time t to the later time $t+h$ in the residuals sequence. If such power is identified, there exists nonlinear feature in the corresponding residuals, $\{\hat{e}_t\}$. We use this concept to develop a nonlinearity test to the residual series $\{\hat{e}_t\}$ of the variables being studied to examine whether there is any remaining nonlinearity in the residuals. For simplicity, we denote Y_t to be the corresponding residuals of the variable being examined. We first state the following definition:

Definition 2.1 *Let $\{Y_t\}$ be a strictly stationary and weakly dependent series, the m-length lead vector of Y_t is*

$$Y_t^m \equiv \big(Y_t, Y_{t+1}, \cdots, Y_{t+m-1}\big),\ m = 1, 2, \cdots,\ t = 1, 2, \cdots$$

and L_y-length lag vector of Y_t is

$$Y_{t-L_y}^{L_y} \equiv \big(Y_{t-L_y}, Y_{t-L_y+1}, \cdots, Y_{t-1}\big), L_y = 1, 2, \cdots, t = L_y + 1, L_y + 2, \cdots .$$

In addition,

$$Y_{t-L_y}^{m+L_y} \equiv \big(Y_{t-L_y}, \cdots, Y_{t-1}, Y_t, Y_{t+1}, \cdots, Y_{t+m-1}\big), L_y = 1, 2, \cdots, t = L_y + 1, L_y + 2, \cdots .$$

Series $\{Y_t\}$ ***does not possess any nonlinearity*** *if and only if*

$$Pr\left(\|Y_t^m - Y_s^m\| < e \,\middle|\, \| Y_{t-L_y}^{L_y} - Y_{s-L_y}^{L_y} \| < e\right) = Pr\left(\|Y_t^m - Y_s^m\| < e\right), \qquad (6)$$

at any time t and s, for any length m and lag length L_y, and for any $e > 0$,where $Pr(\cdot\,|\,\cdot)$ denotes conditional probability and $\|\cdot\|$ denotes the maximum norm which is defined as

$$\|X - Y\| = max\big(|x_1 - y_1|, |x_2 - y_2|, \cdots, |x_n - y_n|\big),$$

for any two vectors $X = \big(x_1, \cdots, x_n\big)$ and $Y = \big(y_1, \cdots, y_n\big)$.

In addition, we define

$$C_1\big(m + L_y, e\big) \equiv Pr\left(\| Y_{t-L_y}^{m+L_y} - Y_{s-L_y}^{m+L_y} \| < e\right),\ C_2\big(L_y, e\big) \equiv Pr\left(\| Y_{t-L_y}^{L_y} - Y_{s-L_y}^{L_y} \| < e\right),$$
$$C_3\big(m, e\big) \equiv Pr\left(\|Y_t^m - Y_s^m\| < e\right). \qquad (7)$$

Because

$$Pr\left(\| Y_t^m - Y_s^m \| < e \mid \| Y_{t-L_y}^{L_y} - Y_{s-L_y}^{L_y} \| < e\right) = \frac{Pr\left(\| Y_t^m - Y_s^m \| < e, \| Y_{t-L_y}^{L_y} - Y_{s-L_y}^{L_y} \| < e\right)}{Pr\left(\| Y_{t-L_y}^{L_y} - Y_{s-L_y}^{L_y} \| < e\right)} = \frac{C_1\big(m + L_y, e\big)}{C_2\big(L_y, e\big)},$$

when one tests the existence of the nonlinearity of a sequence $\{Y_t\}$, instead of testing the linearity hypothesis stated in (2), one could test the following hypothesis:

$$H_0 : \frac{C_1\big(m + L_y, e\big)}{C_2\big(L_y, e\big)} - C_3\big(m, e\big) = 0 \tag{8}$$

where c_i is defined in (7). The series $\{Y_t\}$ is said to possess nonlinearity if the hypothesis H_0 in (8) is rejected.

Under Definition 2.1, the nonlinearity test statistic is given by

$$T_n = \sqrt{n}\left(\frac{C_1\big(m + L_y, e, n\big)}{C_2\big(L_y, e, n\big)} - C_3\big(m, e, n\big)\right), \tag{9}$$

where

$$\begin{aligned}
C_1\big(m + L_y, e, n\big) &\equiv \frac{2}{n(n-1)} \sum\sum_{t<s} I\big(y_{t-L_y}^{m+L_y}, y_{s-L_y}^{m+L_y}, e\big)\,; \\
C_2\big(L_y, e, n\big) &\equiv \frac{2}{n(n-1)} \sum\sum_{t<s} I\left(y_{t-L_y}^{L_y}, y_{s-L_y}^{L_y}, e\right)\,; \\
C_3\big(m, e, n\big) &\equiv \frac{2}{n(n-1)} \sum\sum_{t<s} I\left(y_t^m, y_s^m, e\right)\,; \\
I(x, y, e) &= \begin{cases} 0, & \text{if } \|x - y\| > e \\ 1, & \text{if } \|x - y\| \le e \end{cases};
\end{aligned}$$

$t, s = L_y + 1, \cdots, T - m + 1$; $n = T + 1 - m - L_y$; and T is the length of sequence Y_t.

We note that the idea of nonlinearity used in Definition 2.1 is that if A and B are independent, then $Pr\,(A|B) = Pr\,(A)$. If equation (6) holds, we will have $\big\{ \parallel Y_{t-L_y}^{L_y} - Y_{s-L_y}^{L_y} \parallel < e \big\}$ is independent of $\big\{\|Y_t^m - Y_s^m\| < e\big\}$, and thus, the past of Y_t could not be used to explain the present and the future of Y_t and, in this situation, we claim that Y_t does not contain any nonlinearity. We establish the following property for our proposed test statistic T_n defined in (9):

Theorem 2.1 *Assuming that $\{Y_t\}$ is strictly stationary, weakly dependent, and satisfies the conditions*[1] *stated in Denker and Keller (1983) and for any given values of m,*

[1] *See Conditions (a), (b), and (c) in Theorem A1 in the Appendix.*

L_y, and $e > 0$ defined in Definition 2.1, if $\{Y_t\}$ does not possess any nonlinear feature, then the test statistic defined in (9) is distributed as $N\big(0, \sigma^2(m, L_y, e)\big)$ asymptotically. A consistent estimator of the variance $\sigma^2(m, L_y, e)$ follows:

$$\hat{\sigma}^2\big(m, L_y, e\big) = \widehat{\nabla f(\theta)}^T \cdot \hat{\Sigma} \cdot \widehat{\nabla f(\theta)}^T ,$$

in which

$$\widehat{\nabla f(\theta)} = \left[\frac{1}{\hat{\theta}_2}, \frac{-\hat{\theta}_1}{\hat{\theta}_2^2}, -1\right]^T = \left[\frac{1}{C_2\big(L_y, e, n\big)}, -\frac{C_1\big(m + L_y, e, n\big)}{C_2^2\big(L_y, e, n\big)}, -1\right]^T$$

and each component $\Sigma_{i,j}$ $(i, j = 1, 2, 3)$ of the covariance matrix Σ is given by

$$\Sigma_{i,j} = 4 \cdot \textstyle\sum_{k \geq 1} \omega_k E(A_{i,t} \cdot A_{j,t+k-1}),$$

where

$$\omega_k = \begin{cases} 1 & \text{if } k = 1 \\ 2, & \text{otherwise} \end{cases} \quad , \quad A_{1,t} = h_{11}\left(y_{t-L_y}^{m+L_y}, e\right) - C_1(m + L_y, e)\,,$$
$$A_{2,t} = h_{12}\left(y_{t-L_y}^{L_y}, e\right) - C_2(L_y, e) \quad , \quad A_{3,t} = h_{13}\left(y_t^m, e\right) - C_3(m, e)\,,$$

$\mathbf{z}_t = Y_{t-L_y}^{m+L_y}$, and $h_{1i}(\mathbf{z}_t)$, $i = 1, \cdots, 3$, is the conditional expectation of $h_i(\mathbf{z}_t, \mathbf{z}_s)$ given the value of $\mathbf{z}_t$ as follows:

$$h_{11}\left(y_{t-L_y}^{m+L_y}, e\right) = E\big(h_1 \,\big|\, y_{t-L_y}^{m+L_y}\big) \quad , \quad h_{12}\left(y_{t-L_y}^{L_y}, e\right) = E\big(h_2 \,\big|\, y_{t-L_y}^{L_y}\big) \quad , \quad h_{13}\left(y_t^m, e\right) = E\big(h_3 \,\big|\, y_t^m\big)\,.$$

Moreover, a consistent estimator of $\Sigma_{i,j}$ element is given by:

$$\hat{\Sigma}_{i,j} = 4 \cdot \sum_{k=1}^{K(n)} \omega_k(n) \left[\frac{1}{2(n-k+1)} \sum_t \left(\hat{A}_{i,t}(n) \cdot \hat{A}_{j,t-k+1}(n) + \hat{A}_{i,t-k+1}(n) \cdot \hat{A}_{j,t}(n)\right)\right] ,$$

in which $t = L_y + k, \cdots, T - m + 1$, $K(n) = [n^{1/4}]$, $[x]$ *is the integer part of* x,

$$
\begin{aligned}
\omega_k(n) &= \begin{cases} 1, & \text{if } k = 1 \\ 2\big(1 - [(k-1)/K(n)]\big), & \text{otherwise} \end{cases}, \\
\hat{A}_{1,t}(n) &= \frac{1}{n-1}\left(\sum_{s \neq t} I\big(Y_{t-L_y}^{m+L_y}, Y_{s-L_y}^{m+L_y}, e\big)\right) - C_1\big(m + L_y, e, n\big), \\
\hat{A}_{2,t}(n) &= \frac{1}{n-1}\left(\sum_{s \neq t} I\big(Y_{t-L_y}^{L_y}, Y_{s-L_y}^{L_y}, e\big)\right) - C_2\big(L_y, e, n\big), \\
\hat{A}_{3,t}(n) &= \frac{1}{n-1}\left(\sum_{s \neq t} I\big(Y_t^m, Y_s^m, e\big)\right) - C_3\big(L_y, e, n\big), \\
& \quad t, s = L_y + 1, \cdots, T - m + 1 .
\end{aligned}
$$

The hypothesis H_0 defined in (2) is rejected at level α if

$$
\left|T_n\right| > N\Big(\frac{\alpha}{2}; 0, \hat{\sigma}^2(m, L_y, e)\Big),
$$

where T_n is defined in (9). In this situation, Y_t is concluded to possess nonlinearity.

We suggest academics and practitioners could consider to standardize the variable, say, for example, Y_t, before conducting the test. The reason for standardization is that the value of e to be chosen depends on the standard deviation σ of Y_t. The larger the standard deviation, the larger e should be chosen. Thus, standardizing the variable under examination allows us to choose a similar value of e for different magnitudes of Y_t in practice.

3 Simulation

In this section, we illustrate the superiority of the nonlinearity test we developed in Section 2 by conducting simulation to compare the performance of our proposed test and that of the test developed by Tsay (1986). For simplicity, we call the test developed by Tsay (1986) "Tsay test" and the test developed in this paper "HWBZ test."

As Volterra expansion in (1) is one of the most commonly used forms for a nonlinear and stationary time series while threshold autoregressive model is another popular method in nonlinear analysis, in this paper we first use the following two models in our simulation study:

$$\begin{aligned}&\text{Model A}: \quad Y_t = \varepsilon_t + \beta\varepsilon_{t-1}\varepsilon_{t-2} \;\;, \quad \text{and} \\ &\text{Model B}: \quad Y_t = \begin{cases} -\beta Y_{t-1} + \varepsilon_t & Y_{t-1} \geq 0 \\ \beta Y_{t-1} + \varepsilon_t & Y_{t-1} < 0 \end{cases} ,\end{aligned} \tag{10}$$

where $\{\varepsilon_t\}$ is assumed to be i.i.d. $N(0,1)$ for both Models A and B and $|\beta| < 1$ for Model B. Readers may refer to Tsay (1986) for more information about Model A and we modify a simple threshold autoregressive model in Fan and Yao (2003) to get Model B. We use 10,000 replications to generate different samples in our simulation to compare the performance of our test with Tsay's test.

In addition, since GARCH models are found in many financial data, in this paper we also conduct simulation for the GARCH model. We will conduct simulation for the GARCH(1,1) model such that

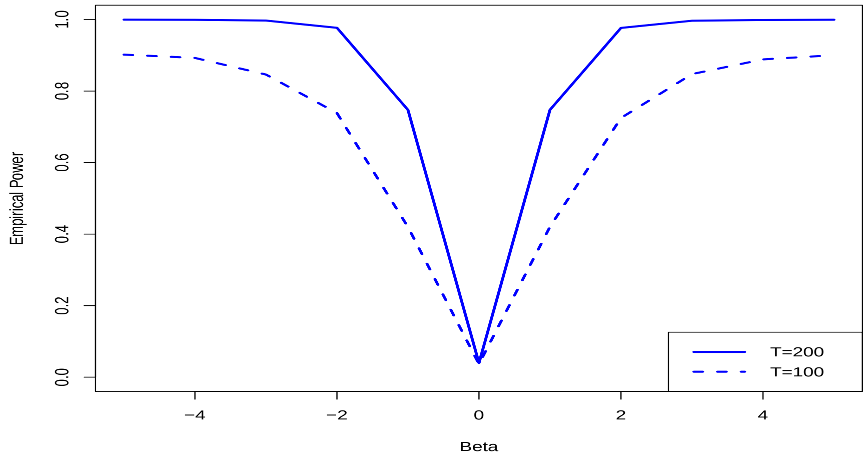
$$\text{Model C}: \quad Y_t = \phi_0 + \phi_1 Y_{t-1} + \varepsilon_t, \varepsilon_t = h_t \cdot e_t \tag{11}$$

in which $h_t = \sqrt{\alpha + \beta_1\varepsilon_{t-1}^2 + \beta_2 h_{t-1}^2}$, e_t is assumed to be i.i.d. $N(0,1)$. For simplicity, we consider $\phi_0 = \phi_1 = \alpha = 0.5$, $0 \leq \beta_1, \beta_2 < 1$ and $\beta = \beta_1 + \beta_2 < 1$, and $\beta_1 = \beta_2 = \beta/2$, $\beta = 0.1, 0.2, \cdots, 0.9$ in our simulation.

Let R be the times of rejecting the null hypothesis that Y_t does not possess any nonlinearity in the 10,000 replications at level 5%, and thus, the empirical power is then $R/10,000$. To conduct our simulation, we let $L_y = m = 1$ and $e = 1.5$ for the HWBZ test and let $M = 4$ for the Tsay's F test, this is the same M used in Tsay (1986) in his simulation.

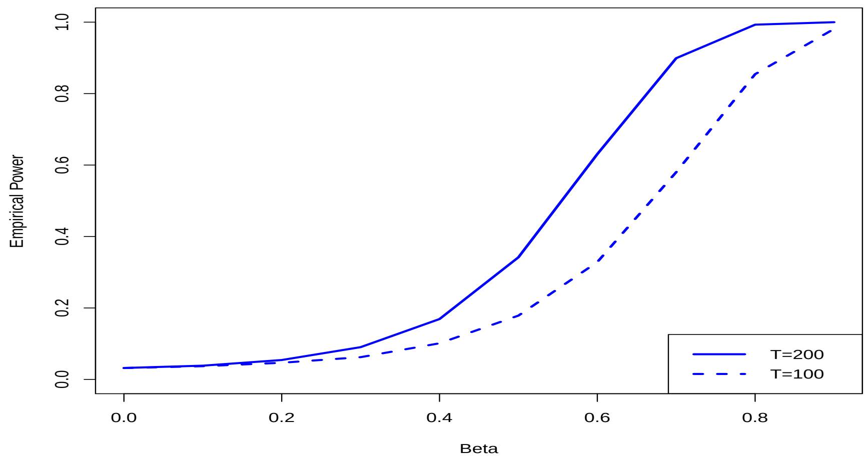
We first conduct simulation for the HWBZ test for the sample size $T = 100$ and 200 for both Models A and B. The results are plotted in Figures 1 and 2, respectively. For Model B, we only conduct simulation for $\beta \geq 0$ due to the symmetry property of the model. From both Figures 1 and 2, our findings show that (1) for both $T = 100$ and 200, our test gets higher power when nonlinear feature weights more in absolute values, (2) for

Figure 1: Empirical Power of the HWBZ test for different values of β in Model A.



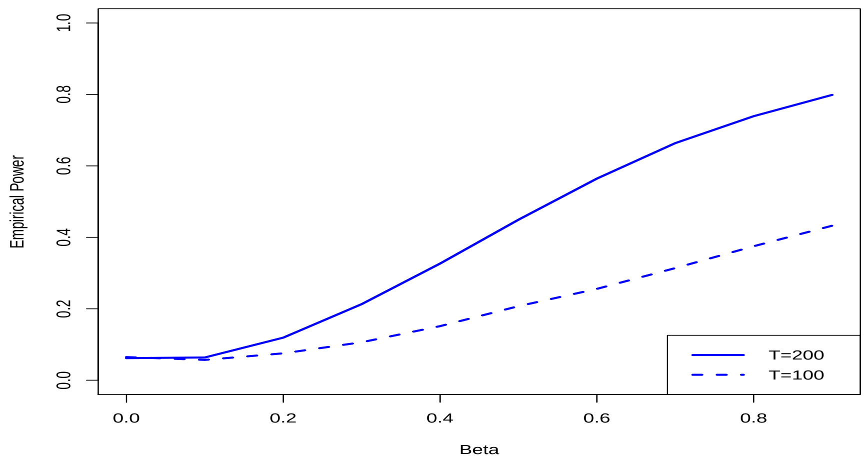
Note: The solid and dotted lines show the powers of the HWBZ test for different values of β in Model A for the sample size $T = 100$ and 200, respectively. Simulation is conducted with the test level $\alpha = 5\%$ and 10,000 replications.

Figure 2: Empirical Power of the HWBZ test for different values of β in Model B.



Note: The solid and dotted lines show the power of the HWBZ Test for different values of β in Model B for the sample size $T = 100$ and 200, respectively. Simulation is conducted with the test level $\alpha = 5\%$ and 10,000 replications.

Figure 3: Empirical Power of the HWBZ test for different values of β in Model C.



Note: The solid and dotted lines show the power of the HWBZ test for different values of β in Model D for the sample size $T = 100$ and 200, respectively. Simulation is conducted with the test level $\alpha = 5\%$ and 10,000 replications.

any β, the empirical power increases as the length T increases, and (3) when $T = 200$, our test's power quickly reach 1, inferring that our test is powerful and stable.

We turn to compare the power and size of our test with those of Tsay's test for different values of β in Models A and B. To do so, we conduct simulation and report the simulation results in Figures 4 and 5. For Model A, we observe from Figure 4 that Tsay's test is more powerful than our proposed test for $0 < |\beta| < 1$ whereas our proposed test is much more powerful than Tsay's test when $|\beta| > 1$. However, our simulation shows that (1) the empirical power of Tsay's test decreases sharply when $|\beta| > 1$ and (2) it decreases further when the magnitude of $|\beta|$ increases further after 1 and becomes stabilized at power below 0.4 when $|\beta| > 2$. On the contrary, the empirical power of our proposed test increases steadily as nonlinear weight $|\beta|$ increases, and quickly increases to 1 when the length $T = 200$. This shows that our proposed test is more stable than Tsay's test.

For Model B, as displayed in Figure 5, the conclusion drawn from the results of our simulation are similar to those for Model A: (1) Tsay's test is more powerful than our proposed test when $|\beta| < 0.65$ while our proposed test is more powerful afterward for both $T = 100$ and 200, and (2) the empirical power of Tsay's test decreases sharply when $|\beta| > 0.65$ and decreases further when the magnitude of β increases further whereas the empirical power of our proposed test increases steadily as β increases, and quickly increases to 1 for both $T = 100$ and 200. Thus, our proposed test is more stable than Tsay's test and is more powerful for large magnitude of β.

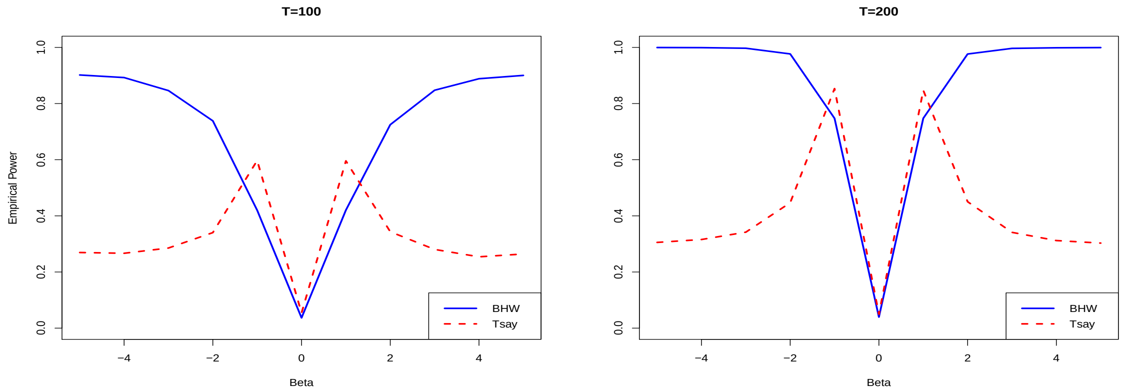
For Model C, we display our simulation results in Figure 6. From the figure, we first find that our proposed test is more powerful than Tsay's test in one region while Tsay's test is more powerful in another region for $T = 100$. However, when $T = 200$, our proposed test is more powerful than Tsay's test in nearly the entire range. Since the quality of a sequence is more reliable when the sequence is longer, we could say that our propose test is more powerful than Tsay's test for larger T.

In short, our simulation shows that for Model C our proposed test is more powerful than Tsay's test for larger T. However, for Models A and B, our proposed test is more powerful than Tsay's test in one region while Tsay's test is more powerful in another region. We note that since there are too many nonlinearity forms, it is not surprised that no single test will dominate the others in testing nonlinearity feature. Thus, we are not surprised that Tsay's test is more powerful than ours in a region while our test is more powerful in another region. Nonetheless, to be stable is one of the most important features for a test statistic and since our proposed test more stable than Tsay's. In addition, the power of our proposed test reaches one quickly when the magnitude of β increases is a desirable property while the power of Tsay's test is decreasing for Models A and B when the magnitude of β increases is not a desirable feature. Thus, we claim that our test is a more desirable test.

4 Illustration

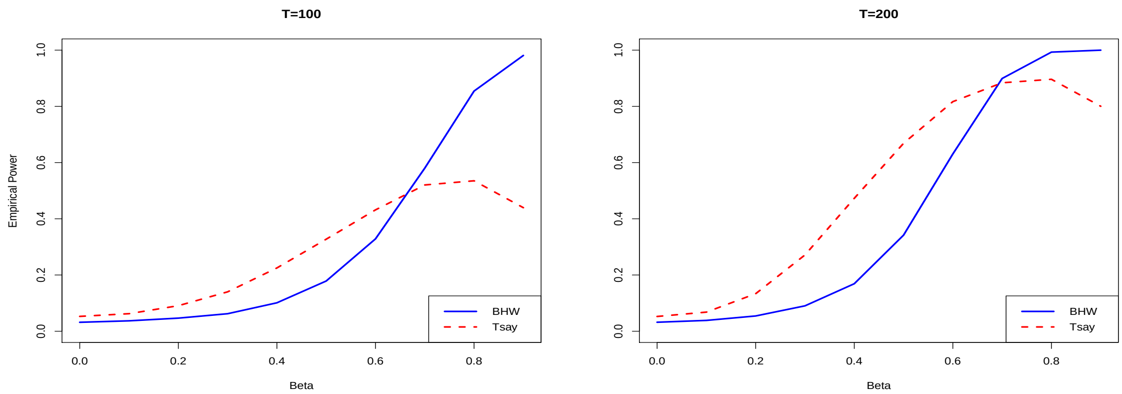
In this section, we illustrate the applicability of the nonlinearity test we have developed in Section 2 by applying our proposed nonlinearity test, Tsay's test and some other related statistics to test whether there exists any nonlinear feature in the sunspot data and stock

Figure 4: Empirical Power of Tsay's and HWBZ's Tests for different values of β in Model A.



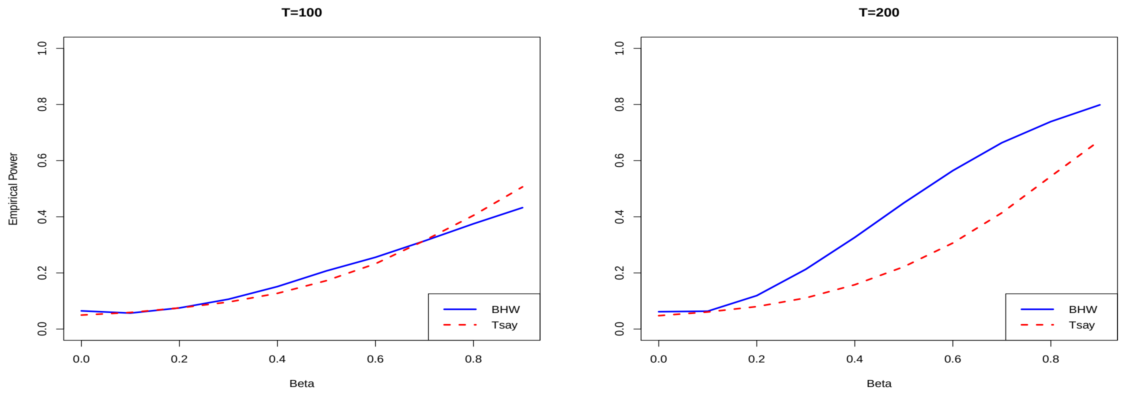
Note: The left panel shows the plot for the sample size $T = 100$ and the right panel displays the plot for $T = 200$. The solid line exhibits the HWBZ's test while the dashed line shows the power of Tsay's Test for different values of β in Model A. Simulation is conducted with the test level $\alpha = 5\%$ and 10,000 replications.

Figure 5: Empirical Power of Tsay's and HWBZ's Tests for different values of β in Model B.



Note: The left panel shows the plot for $T = 100$ and the right panel displays the plot for $T = 200$. The solid line exhibits the HWBZ's test while the dashed line shows the power of Tsay's Test for different values of β in Model B. Simulation is conducted with the test level $\alpha = 5\%$ and 10,000 replications.

Figure 6: Empirical Power of Tsay's and HWBZ's Tests for different values of β in Model C.



Note: The left panel shows the plot for $T = 100$ and the right panel displays the plot for $T = 200$. The solid line exhibits the HWBZ's test while the dashed line shows the power of Tsay's Test for different values of β in Model C. Simulation is conducted with the test level $\alpha = 5\%$ and 10,000 replications.

market returns in this section.

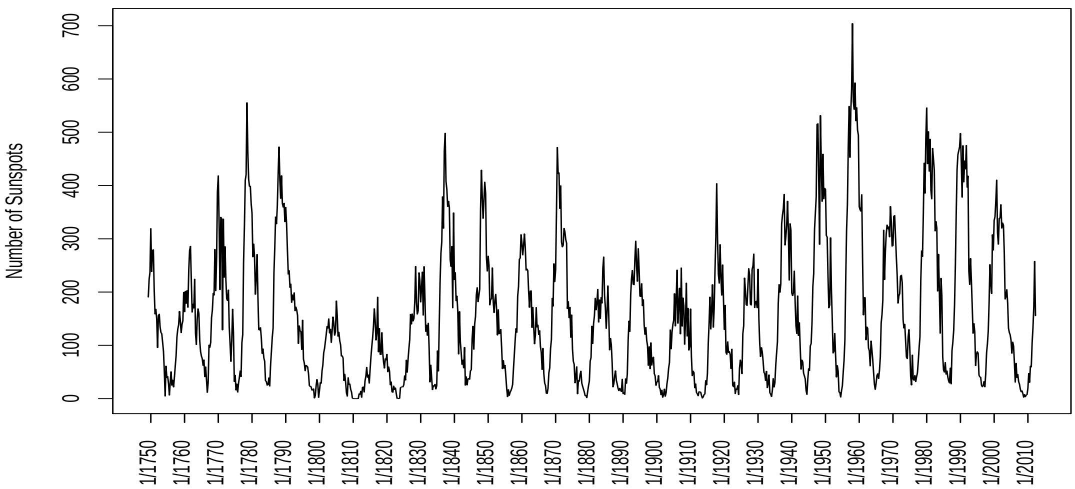
4.1 Sunspots

Sunspots refer to dark spots on the surface of the sun related to the motion of the solar dynamo. Johann Rudolf Wolf (1816-1893) introduces a formula for calculating the sunspot numbers: $R = k(10g + f)$, where g is the number of groups of sunspots, f is the total number of individual spots, and k is a constant for the observations. To honor the contribution by Johann Rudolf Wolf, it is common to call sunspot number "Wolf's sunspot number" (Izenman, 1983)

The earliest linear model built for the sunspot data is probably done by Yule (1927) who introduces the class of linear autoregressive models to analyze the data. Since then, the literature, see, for example, Moran (1954), of linear time series analysis of the sunspot data has been growing exponentially. However, some works, see, for example, Tong and Lim (1980) points out that linear model is not adequate for fitting the data and forecasting.

In this paper we illustrate the applicability of our proposed test and Tray's test to examine the nonlinearity in the quarterly Wolf's sunspot numbers from the first quarter

Figure 7: Wolf's Sunspots Numbers



Note: Quarterly Wolf's sunspot numbers from first quarter of 1749 to first quarter of 2012.

of 1749 to the first quarter of 2012. Let Y_t be Wolf's quarterly sunspot numbers from the first quarter of 1749 to the first quarter of 2012, we exhibit the time series plot of the sunspot data in Figure 7. We first discuss how to use our test statistic to examine whether there is any nonlinearity in $\{Y_t\}$. To do so, as discussed in Section 2, we first fit the data by using the following AR(p) model:

$$Y_t = \sum_{i=1}^{p} \phi_i Y_{t-i} + e_t, \;\; e_t \sim \mathrm{WN}(0, \sigma^2) \tag{12}$$

to the sunspot data. We find that the "best" linear model for the sunspot data is

$$\begin{aligned} Y_t &= 19.8849 - 0.7051Y_{t-1} - 0.1549Y_{t-2} - 0.1873Y_{t-3} - 0.0834Y_{t-4} \; . \\ &\quad +0.1055Y_{t-6} + 0.0712Y_{t-7} + 0.0810Y_{t-9} + e_t \; . \end{aligned} \tag{13}$$

We exhibit the results in Table 1. Thereafter, we apply the Ljung-Box test to test the hypothesis of no autocorrelations up to lag k for the residuals and display the results in Table 2. In addition, we display the autocorrelations of the residuals in Figure 8. The results from Table 2 and Figure 8 show that the autocorrelations of the residuals are not significantly different from zero for any lag up to 42,[2] and thus, one may conclude that the AR model in (13) is adequate and there is no other linear relationship remained in the residuals.

Table 1: The Results of the Linear AR Model

Parameter	Estimate	Standard Error	t Value
intercept	19.8849	2.2872	8.694***
Y_{t-1}	0.7029	0.0305	23.004***
Y_{t-2}	0.1545	0.0375	4.114***
Y_{t-3}	0.1872	0.0378	4.948***
Y_{t-4}	0.0883	0.0353	2.497**
Y_{t-6}	-0.1049	0.0353	-2.965***
Y_{t-7}	-0.0722	0.0346	-2.083**
Y_{t-9}	-0.0830	0.0247	-3.355***

Note: This table exhibits the results of the linear AR model as shown in (13).
*, **, and *** mean significant at levels 10%, 5%, and 1%, respectively.

Table 2: Autocorrelation Check: The Result of Ljung-Box Test

Check for Sunspots Numbers			Check for Residuals		
Lag (k)	df	$\chi^2(k)$	Lag (k)	df	$\chi^2(k)$
6	6	4075.119***	12	5	6.632
12	12	4708.268***	18	11	13.377
18	18	5146.997***	24	17	18.366
24	24	6232.194***	30	23	25.434
30	30	6540.412***	36	29	33.231
36	36	7060.406***	42	35	46.582

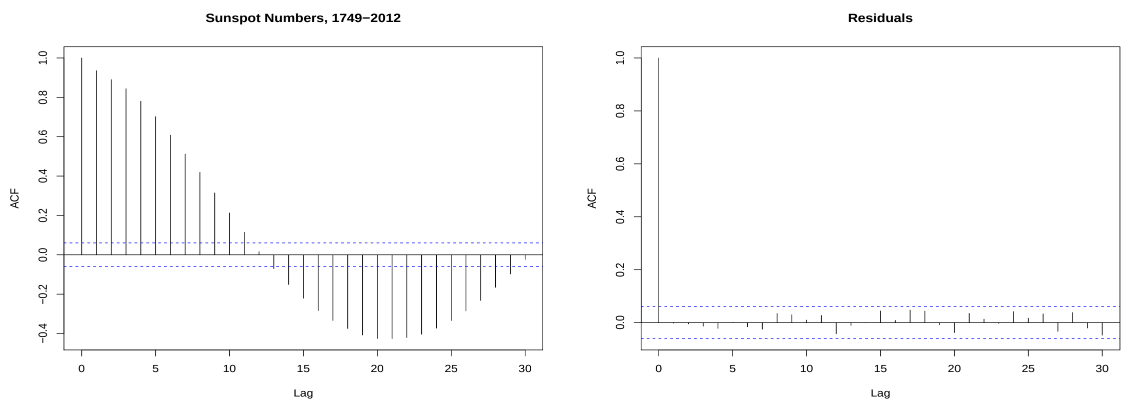
Note: The null hypothesis of Ljung-Box test is that the autocorrelations up to lag k in the population from which the sample is taken are 0. $\chi^2(k)$ is the test statistic with k degrees of freedom. Readers may refer to Ljung and Box (1978) for more details of the test. The left panel displays the values of $\chi^2(k)$ for the Sunspots numbers while the right panel shows the values for the residuals after fitting the linear AR model as shown in (13).
*, **, and *** mean significant at levels 10%, 5%, and 1%, respectively.

One may believe that the linear model in (13) fits the sunspot data well. To check whether this is true, we apply the test we developed in Section 2 to examine whether there

[2] Readers may consider to apply the approach developed by Li (1992) to correct the residual autocorrelations for nonlinear time series models.

is sequential dependence within the standardized residuals, $\left(\hat{e}_t - \text{mean}(\hat{e}_t)\right)/\sqrt{\text{var}(\hat{e}_t)}$, obtained from fitting the linear model in (12). To do so, we use $L_y = m = 1$ and $e = 1.5$ in our proposed test, as the same values being used our simulation. The p value of the HWBZ test is $6.9837e^{-7}$, which strongly reveals some kind of dependence within the residuals. Thus, applying our test, one could realize that there still exists nonlinearity component in the sunspot data. This result is consistent with the findings by Tong and Lim (1980), Tong (1983), and many others. In addition, we use Tsay's test to detect the nonlinearity in the Wolf's Sunspots numbers. Its p value is $3.5416e^{-14}$, inferring that both our proposed test and Tsay's test draw the same conclusion that there exists nonlinearity in the Wolf's Sunspots numbers.

Figure 8: Plots of the Autocorrelation Functions



Note: The left panel exhibits the ACF for Sunspots numbers whereas the right panel displays the ACF for the residuals after fitting the linear AR model as shown in (13).

4.2 Random Walk Hypothesis and Nonlinearity in the Efficient Market

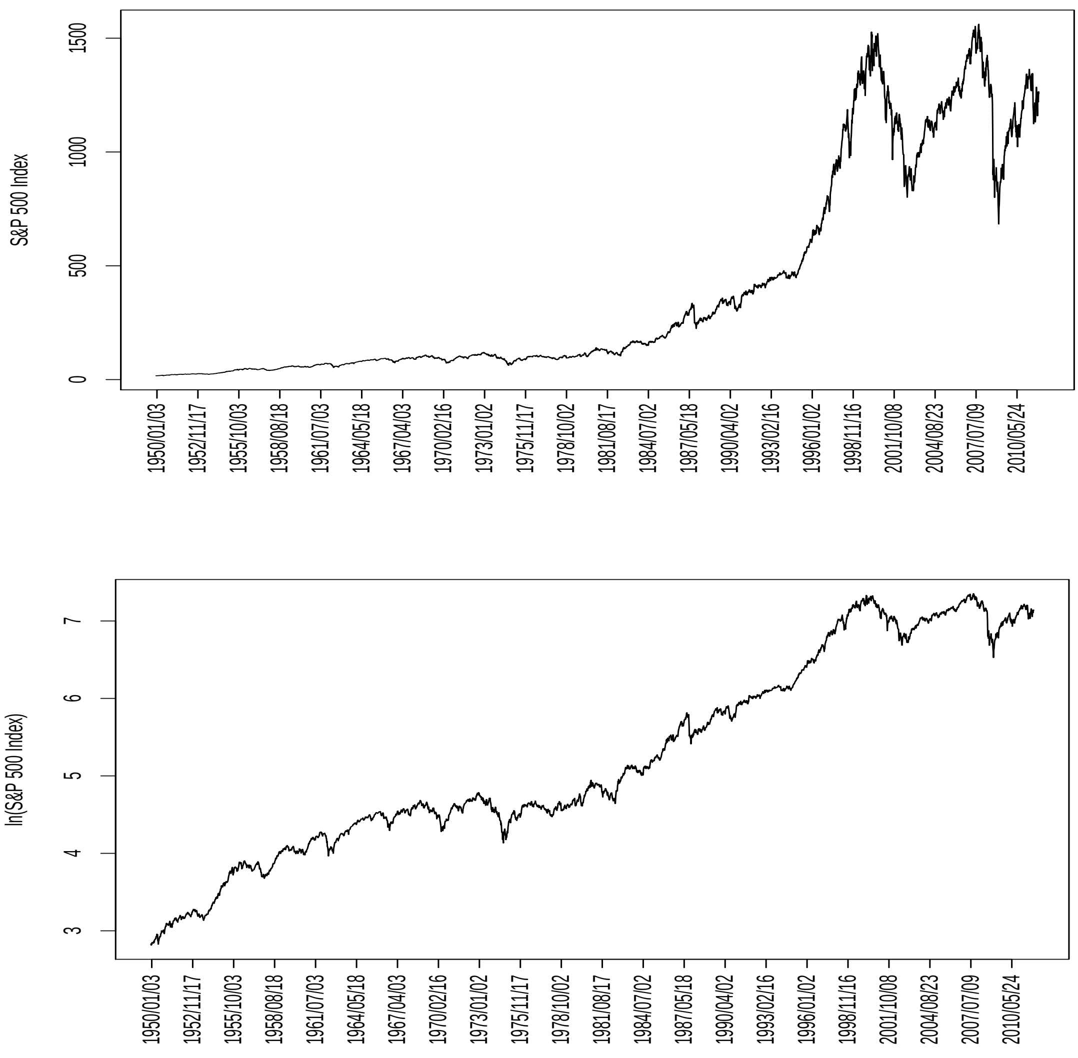
We turn to apply our proposed statistics to test for the random walk hypothesis (RWH). There are several approaches to test for RWH. For example, one could apply a unit root test, see, for example, Tiku and Wong (1998) and the references therein to test whether the data are stationary. If the stationarity is rejected, the RWH is supported; otherwise, the RWH is rejected. Lo and MacKinlay (LM, 1988) develop the variance ratio (VR) tests to test for the RWH. In addition, Chow and Denning (CD, 1993) use several VRs for different holding periods to test whether the multiple VRs are jointly equal to one. Wright (2000) modifies the VR tests by considering "rank and sign" whereas Whang and Kim (2003) propose to use "subsampling" technical to improve the VR tests. Kim (2006) adopts a resampling method to estimate the sampling distribution of the VR statistic that can be used to data with conditional and unconditional heteroscedasticity. Most of literature of the RWH examine the RWH on stock prices. However, there are still some papers examine the RWH on other variables, e.g. exchange rates, currency futures prices, gross national product. In this paper, we illustrate the applicability of the nonlinearity test we developed to test for the RWH for the Standard & Poor's 500 index (S&P 500), to test whether it follows a random walk model.

The S&P 500 is a capitalization-weighted index of the prices of 500 USA stocks listed on the NYSE or NASDAQ with largest capitalization. The data are obtained from Datastream. In this paper we analyze the weekly index of S&P500 from January 1, 1950 to December 31, 2011, totally 62 years. We denote the S&P500 index at week t as P_t and follow Lo and MacKinlay (1988) to analyze its logarithm so that we define $X_t = \ln P_t$, as the logarithm of the index process. In this paper we use the following null hypothesis H_0:

$$X_t = \mu + X_{t-1} + \varepsilon_t \tag{14}$$

to test whether the log S&P 500 follows a random walk (RW) model stated in (14) in which μ is a drift parameter and ε_t is the random disturbance term with zero mean and does not possess any nonlinearity. We exhibit the time series plots of the S&P 500 and its logarithm in Figure 9. If $\hat{\varepsilon}_t$ obtained from equation (14) possesses any nonlinear feature, we will reject the RWH and conclude that X_t does not follow the RW model in (14). We note that the advantage of using our statistic is that if we reject H_0, we not only conclude that X_t does not follow the RW model, we also know that there is nonlinear feature for X_t, and thus, academics and practitioners could think of any nonlinear component to be included in (14) to improve the model. We also note that our statistic can test not only the RW model stated in (14), but also any RW model with and without drift, any RW model with break(s) in intercept(s) and/or trend(s), and actually any linear and nonlinear model as long as their residuals could be estimated. Once the residuals can be estimated, one could use our proposed statistic to test whether there is any (additional) nonlinear feature that should be included in the model.

Figure 9: Time Series Plots of S&P 500 Index and Its Logarithm



Note: The data are from January 1950 to December 2011, totally 62 years.

We first fit X_t by the model in (14) and obtain its residuals $\hat{\varepsilon}_t$. We then standardize the residuals and follow the idea from Hiemstra and Jones (1994) and Bai, et al. (2010, 2011) from causality to choose $L_y = m = 1$ and $e = 1.5$. In order to get more information of the underlying series, we follow Lo and MacKinlay (1988) to examine the full period as well as different sub-periods. To form the sub-periods, we first cut the entire period into two equally-distanced sub-periods, we then further cut them into 4 equally-distanced sub-periods and report the test results in Table 3. For comparison, we also employ the LM, CD, and Tsay tests in our analysis. The p-values of the HWBZ's nonlinearity test reported in the second column of Table 3 strongly reject H_0, leading us to conclude that $\hat{\varepsilon}_t$ is nonlinear, and thus, we claim that ln S&P500 index X_t does not follow a random walk model for the whole period as well as for any of the sub-periods. This result is consistent with the results of Tsay's test (reported in the third column of Table 3), LM tests (reported in the fourth and fifth columns of Table 3) and CD tests (reported in the sixth and seventh columns of Table 3) for the whole period as well as for any of the sub-periods.

We note that one may also be interested in testing the martingale hypothesis (which is a weak form about the efficient market) that $X_t = \ln(\text{S\&P500})$ is a martingale with respect to some filtration $(\mathcal{F}_n)$; that is, to test whether the return sequence $r_t = X_t - X_{t-1}$ forms a martingale difference $(E(r_t|\mathcal{F}_n) = \mu)$. To do so, one could use the wild bootstrap Cramer von Mises test statistic (denoted as Cp) and wild bootstrap Kolmogorov-Smirnov test statistic (denoted as Kp) (Domínguez and Lobato, 2003) to test whether the return sequence $\{r_t\}$ is a martingale difference sequence. We denote the two martingale difference test Cp and Kp as MTD and display the tests results in the fourth and fifth columns of Table 3. The results lead us not to reject X_t to be a martingale at 5% for the whole period as well as for any sub-period except for sub-period January of 1981 to December of 2011. We can find that the returns r_t are serially uncorrelated, but dependent. Because in this paper we are only interested in testing nonlinear feature in the random walk model, not testing the martingale hypothesis. Thus, we skip detailed discussion on testing the martingale hypothesis. Readers may refer to Shiryaev (1999) for more details and discussions on testing the martingale hypothesis and the conjecture of the 'martingale property' that generalizes 'random walk' conjecture in the concept of efficient market.

Table 3: Test for the martingale and random walk hypothesis of the ln S&P 500 index.

Time period	HWBZ	Tsay	MTD		Chow-Denning	
	(p)	(p)	Cp(p)	Kp(p)	CD1	CD2
Jan, 1950 - Dec, 2011	5.2e-08	1.7e-05	0.2666	0.1833	190.90	141.35
Jan, 1950 - Dec, 1980	5.5e-05	8.1e-14	0.0900	0.0666	130.94	80.23
Jan, 1981 - Dec, 2011	4.4e-05	1.6e-05	0.0200	0.0266	133.85	99.18
Jan, 1950 - Jun, 1965	0.0044	0.0015	0.1233	0.0866	90.47	66.95
Jul, 1965 - Dec, 1980	0.0023	1.1e-11	0.4566	0.6033	80.41	43.57
Jan, 1981 - Jun, 1996	0.0001	0.0953	0.0833	0.1766	91.20	67.00
Jul, 1996 - Dec, 2011	0.0027	7.4e-05	0.0433	0.0600	84.24	50.63

Note:
1) H_0 of HWBZ and Tsay test is: there is no nonlinearity in the return $r_t = X_t - X_{t-1}$; H_0 of MTD (Cp and Kp) test is: the return $r_t = X_t - X_{t-1}$ is a martingale difference sequence; H_0 of CD test is: X_t in equation (14) is a random walk.
2) We report p-values for HWBZ, Tsay and MTD (Cp and Kp) tests and critical values for CD tests.
3) For HWBZ test, we use $L_y = m = 1$ and $e = 1.5$ in the estimation. For Tsay test, we choose $M = 4$ (readers may refer to Section 2.1 for the definition of M).
4) For MTD (Cp and Kp) tests, the number of bootstrap replications is 300, and the lag value is 1.
5) For CD tests, CD1 tests for i.i.d. ε_t whereas CD2 tests for uncorrelated with possible heteroskedasticity ε_t. The 10%, 5%, and 1% critical value of CD test are 2.2262, 2.4909, and 3.0222, respectively. The vector of holding periods p is $(2, 4, 8, 16)$ for this multiple test.

5 Conclusion

There are many works in the development of nonlinearity tests. A nonlinearity test could be portmanteau. It could also be parametric, semi-parametric, or nonparametric. In general, the number of parameters in a nonlinearity test could be very large, this could affect the performance of the existing nonlinear tests. In addition, nonlinearity may occur in many and could be infinitely ways. Thus, it is not our intention to develop a single test that outperforms all others in examining nonlinearity. In this paper we focus on nonlinearity within a stationary time series, which is often ignored by academics and practitioners, especially in applied science such as finance and economics. We add a reliable, user-friendly, desirable, and powerful test to the nonparametric nonlinearity test category in the literature. As a nonlinear feature is in general more complex and more difficult to model than a linear one, it is not reasonable to restrict the form of the nonlinearities at the stage of detecting them within a sequence. Our test satisfies this criterion and circumvents the limitation of using too many parameters like those using the Volterra expansion.

To demonstrate the performance of our proposed nonlinear test, we conduct simulation study on two types of threshold autoregressive models and GARCH models. Our simulation reveals that for the GARCH models, our proposed test is more powerful than Tsay's test for large sample size. On the other hand, for the threshold autoregressive models, our simulation shows that Tsay's test is more powerful than ours in a region while our test is more powerful in another region. We note that this finding is not surprised because there are many nonlinearity features, and thus, there may not exist any single test that could outperform the others in examining nonlinearity. However, our simulation shows that our proposed test has three desirable features than Tsay's test: (1) our proposed test is more stable, (2) the power of our proposed test increases while that of Tsay's test could decrease when the magnitude of parameter increases, and (3) the power of our proposed test reaches one quickly but the power of Tsay's test is decreasing when the magnitude of β increases. Thus, the results of our simulation support our claim that our test is a more desirable test.

To demonstrate the applicability of our proposed test, we first apply both Tsay's test and the nonlinearity test we developed in this paper to test whether there exists any nonlinear feature in the sunspot data, one of the most typical nonlinear cases. Thereafter, we apply our proposed nonlinearity test, Tsay's test, and Chow-Denning's variance ratio statistics to test whether the S&P 500 index follows a random walk model. Our findings show that both our proposed test and Tsay's test draw the same conclusion that there exists nonlinearity in the Wolf's Sunspots numbers. Our findings also reveal that our proposed test, Tsay's test, and Chow-Denning's variance ratio statistics draw the same conclusion that the S&P 500 index does not follow random walk models. The test result from our proposed statistic is consistent with those of the tests developed by Tsay (1986) and others.

At last, we note that our test could not only be used to detect any nonlinearity for the variable being examined. If one believes a predetermined model could be fitted to the variable and its residuals could be estimated. Then, the test developed in this paper could also be used to examine whether there is an nonlinearity in the residuals and, in turn, test whether the model being used to fit to the variable is appropriate. For example, if one believes that a model, say Model D, which could be linear or nonlinear, is the right model for the data, and thus, she could fit Model D to the variable and obtain its residuals. Thereafter, she could apply our proposed statistic to test whether the null hypothesis of linearity is rejected for the residuals. If it does not, this infers that Model D is appropriate to be used for the variable being studied. On the other hand, if our test rejects the linearity of the residuals, this infers that the model is not appropriate and one may then think of any nonlinear component to be included in Model D to make it more appropriate to the data. However, if one could not find any model to be appropriate for the data but one could find two models, say, Model D and Model E, that could be the best choices for the data and one could estimate the residuals for both Models D and E. Then, one could still apply our proposed statistic to test for their residuals and the one with smaller p-value will be the more desirable model for the data.

There are many nonlinear time series models. One may not be able to estimate the residuals for some nonlinear time series models. However, it is still possible for academics and practitioners to estimate the residuals for some nonlinear time series models, for example, one could choose a few terms such as the linear, quadratic and cubic terms in the Volterra expansion to be the one's desired nonlinear time series models. As long as the residuals of the nonlinear time series models can be estimated, one could apply the test developed in this paper to test whether there is still any nonlinearity in the residuals. If the null hypothesis of linearity is not rejected, then one could conclude that the chosen nonlinear time series model is appropriate.

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