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Abstract

This paper examines mild conditions under which a macroscopic fundamental diagram (MFD) emerges, relating space-averaged speed to occupancy in some area. These conditions are validated against empirical data. We allow local speed-occupancy relationships and, in particular, require no equilibrating process to be in operation. This means that merely observing the stable relationship between the space-averages of speed, flow and occupancy are not sufficient to infer a robust relationship and the emerging MFD cannot be guaranteed to be stable if traffic interventions are implemented.

Keywords: Traffic variables; Congestion; Macroscopic Fundamental diagram.

1   Introduction

Traffic flow theory is concerned with fundamental indicators such as speed, density/occupancy and flow. Speed, flow and occupancy at a point on a road are related through an identity whereby the flow, measured as the number of vehicles passing a point per time unit, is equal to the speed, in distance per time unit, multiplied by occupancy, which is the number of vehicles per unit of road length. The fundamental diagram of traffic flow, due to Greenshields (1935), provides another relationship between these three variables. It may be expressed as a relationship between flow and occupancy, according to which flow increases with the occupancy up to the capacity of the road and then decreases to zero

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due to congestion. The fundamental diagram may equivalently be expressed as a decreasing relationship between speed and occupancy or as a backward-bending two-valued relationship between speed and flow.[1]

The macroscopic fundamental diagram (MFD) is similar to the fundamental diagram, but rather than relating to a point on a road, it relates space-averages for some area of speed, flow and occupancy. The MFD was first investigated by Godfrey (1969). Recently, it has generated a lot of excitement, due to its promise that the complications of traffic networks can be ignored for many purposes. With the MFD, traffic in large urban areas can be modeled dynamically at an aggregate level. In response to the empirical MFD literature, a theoretical literature is emerging that explores the implications of the existence of an MFD for the regulation of urban congestion through metering, pricing, and provision of road and transit capacity.

The bottleneck model (Vickrey, 1969) allows the dynamics of congestion to be accounted for in a simple way in a model that accounts for the preferences of travelers regarding the timing of trips. Vickrey's model implies that the flow remains constant after a critical level of congestion is reached. Steps have been taken to incorporate capacity decrease due to congestion in Vickrey type models, see Arnott (2013), Fosgerau and Small 2013, and Fosgerau (2015).

Early work has investigated the macroscopic characteristics of traffic flow based on data from lightly congested real-world networks (Godfrey, 1969; Ardekani and Herman, 1987; Olszewski et al., 1995), while other contributions have used simulation data with artificial routing rules and static demand (Mahmassani et al., 1987; Williams et al., 1987; Mahmassani and Peeta, 1993). However, the existence of an invariant MFD for a real urban network had not been demonstrated mainly due to the difficulty in obtaining sufficient traffic data for a large network.

Later, with the help of rapid development of intelligent transportation systems, it has become possible to obtain traffic data for a urban network on a large scale. Geroliminis and Daganzo (2008) analyzed the relation between the average flow and average occupancy of Yokohama (Japan) with data collected from both loop detectors and GPS equipped taxis, and found that these data revealed a well-defined MFD. More specifically, they plotted the average flow against the average occupancy with data points representing different points in time. Data points were found to be concentrated with low scatter around a well-defined function that relates each level of the average occupancy to a single value for the average flow. To ascertain the universality of a well-defined MFD, Daganzo and Geroliminis (2008) and Helbing (2009) have developed analytical theories for the urban MFD. These studies build on the assumption of a mechanism that equilibrates traffic

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[1]In traffic engineering, congested conditions are said to occur when the occupancy is above the capacity level. In transportation economics, the term congestion refers to the phenomenon that the speed decreases below the free-flow speed due to high occupancy. The economists use the term hypercongestion for the situation where the occupancy is above the capacity level and an increase in speed is (somewhat paradoxically) associated with an increase in flow. This paper uses the terminology from engineering.

density across space.

Some empirical properties of MFD have been explored. Geroliminis and Sun (2011a) investigated the properties of the MFD under congested conditions and found that the spatial distribution of vehicle densities in the network is a key component affecting the scatter of an MFD and its shape. Geroliminis and Sun (2011b) exhibit the existence of hysteresis on freeway networks, which leads to a two-valued relationship from average density to average flow. They show that freeway network systems not only have curves with high scatter, but they also exhibit hysteresis phenomena, where higher network flows are observed for the same average network density in the onset and lower in the offset of congestion.

The previous literature has assumed either no congestion or the existence of some equilibration effect that tends to equalize congestion across space. This paper establishes conditions that ensure that an MFD emerges, even in the absence of an equilibration effect. We impose first mild conditions on the statistical dependency of traffic variables across space that are sufficient to ensure the convergence of the space-averages of traffic variables to fixed values at any point in time. Second, we show that these averages will be time-invariant conditional on the average occupancy, provided that the spatial distribution of the traffic variables is the same conditional on each value of the average occupancy. That is, if we consider a time slice and pool the occupancies from different locations into one distribution, then we require that this distribution is the same whenever the average occupancy is the same. This allows for traffic patterns that may be different at different times of day, as long as the pooled distribution is constant conditional on the number of vehicles in motion.

The fact that an MFD may emerge without equilibration also means that the existence of an equilibration process cannot be inferred from the observation that an MFD exists. It implies that merely observing a stable relationship between space-averages of speed or flow versus occupancy is not sufficient to infer the existence of a robust relationship that can be used for traffic control. The dynamics of the traffic variables across a city is equally important as the existence of an MFD, and any traffic regulation must consider the impact of the regulation on the dynamics of the traffic variables.

The rest of the paper is presented as follows: Section 2 provides the preliminaries from probability theory required for our model. In Section 3, we present our main result regarding the existence of the MFD, and explain the assumptions of our model and their validity. Section 4 validates the model assumptions using a small simulation exercise and then using empirical data from Stockholm and the Geneva region. Section 5 concludes. Proofs are given in the Appendix.

2 Preliminaries

In this section, we introduce some concepts and results from probability theory that we will use in the model that we present later in Section 3. We will present the probability theory in an abstract and general manner, connecting to the probability theory where the results come from. To fix thoughts, it may be useful to have in mind that we will be describing data that have a spatial dimension and a time dimension. We will seek minimal conditions that ensure that the space-averages converge to a unique limiting value and that the space-averages are stable in the time dimension.

Consider random variables $\{X_n\}_{n \in \mathbb{N}}$ defined on a fixed probability space $(\Omega, \mathcal{A}, \mathbf{P})$. We will use the following concept of dependence.

Definition 1. *Random variables, $X_1, X_2, \cdots$ are negatively associated (NA) if for every pair of finite disjoint subsets A_1, A_2 of $\mathbb{N}$, and for all coordinate-wise nondecreasing functions $f, g : \mathbb{R}^{|A_i|} \to \mathbb{R}$, we have $Cov\left[f\left(X_i, i \in A_1\right), g\left(X_j, j \in A_2\right)\right] \leq 0$.*

The concept of negative association was introduced by Block et al. (1982) and Joag-Dev and Proschan (1983). Negatively associated random variables are also negatively correlated, which may be seen by taking $f\left(X_i, i \in A_1\right) = X_i$ and $g\left(X_j, j \in A_2\right) = X_j$, where $i \in A_1, j \in A_2$. Negative association is thus a stronger concept than negative correlation. Based on Joag-Dev and Proschan (1983), Proposition 1 gives some examples of NA sequences of random variables that serve to illustrate the concept of NA.

Proposition 1. *A set of random variables $\mathcal{X} = \{X_i \mid i \in \mathcal{S} \subset \mathbb{N}\}$ is negatively associated if one of the following holds:*

1. *$\mathcal{X}$ is a sequence of independent random variables.*

2. *$\mathcal{X}$ is a sequence of negatively correlated normal random variables.*

3. *$\mathcal{X}$ is a permutation distribution.*

4. *$\mathcal{X}$ is a union of independent sets of NA random variables.*

5. *$\mathcal{X}$ is a set of independent random variables with log concave densities and $\sum\limits_{X \in \mathcal{X}} X = $ constant.*

6. *$\mathcal{X}$ is a set of increasing functions applied on disjoint subsets of a set of NA random variables.*

In the model below, the place of random variables X_i will be taken by location-specific occupancies. We may allow location-specific speed-occupancy relationships. Then Proposition 1 implies that local speeds will be NA if the occupancies are NA. NA for

local occupancies carries over to NA of local speeds through the monotonicity of the speed-occupancy relationships. This will be useful for establishing convergence of the space-averaged speed.

It is, however, fair to assume that occupancies will be positively correlated locally, simply because the occupancy on some road is likely to be high if the occupancy is high on an adjacent road. Then the occupancies cannot be strictly NA, since NA implies negative correlation. Moreover, we may expect the correlation to be higher in a central region (congested area), and that it will decrease as we go outside city center (into a non-congested area). Chandra and Ghosal (1996a,b) introduced the following weaker dependence concept, which allows for local positive correlation while still being sufficient for obtaining the convergence results that we will state below.

Definition 2. *A sequence $\{X_n\}$ of random variables is called asymptotically almost negatively associated (AANA) if there is a nonnegative sequence $q(n) \to 0$ such that for all $n, k \geq 1$,*

$$Cov(f(X_n), g(X_{n+1}, \cdots, X_{n+k})) \leq q(n)(Var(f(X_n))Var(g(X_{n+1}, \cdots, X_{n+k})))^{\frac{1}{2}} \quad (2.1)$$

for all coordinate-wise non-increasing continuous functions f and g so that the right-hand side of eq. (2.1) is finite.

So an AANA sequence $\{X_n\}$ may have positive local correlations but the correlations must become small as n increases. The sequence $q(n)$ in the definition of AANA is called the mixing coefficient. Letting $q(n) = 0$ for $n \geq 1$ shows that any NA sequence is also AANA.

AANA is in fact a strictly weaker dependence concept than NA, since there exists AANA sequences that are not NA. Consider for example the sequence: $X_n = (1 + a_n^2)^{-\frac{1}{2}}(Y_n + a_n Y_{n+1})$, where $Y_1, Y_2, \cdots$ are independent identically distributed standard normal random variables, $a_n > 0$ and $a_n \to 0$ as $n \to \infty$. Chandra and Ghosal (1996a) showed that this sequence is indeed AANA. This sequence even has local positive correlation since $Cov(X_n, X_{n+1}) > 0$, but the sequence of local correlations converges to 0. The following example is an extension of the above example which is relevant for the model that we develop in the Section 3.

Example 1. *Let $\{Y_n\}_{n=1}^{\infty}$ be independent identically distributed standard normal random variables. Define $X_n = (1 + a_n^2 + a_{n+1}^2 + \cdots + a_{n+p}^2)^{-\frac{1}{2}}(Y_n + a_n Y_{n+1} + \cdots + a_{n+p} Y_{n+p+1})$ with $a_n > 0$ and $a_n \to 0$ as $n \to \infty$. Then $\{X_n\}_{n=1}^{\infty}$ is AANA.*

In example 1, X_n can be considered as occupancy at various places in a road network. Subscript n indexes road segments, beginning from inside a congested city center. Then the example describes a case where occupancies are positively correlated with occupancies of p neighboring road segments and where the correlations decrease as we go away from the congested region of the city.

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The following Lemma, proved by Yuan and An (2009), says that the AANA property is invariant under monotonic transformations of the random variables in an AANA sequence. As we mentioned in the discussion of the NA property, this is extremely useful since it means that the AANA property carries over from local occupancies to local speeds through monotone local speed-occupancy relationships.

Lemma 1. *Let $\{X_n\}$ be a sequence of AANA random variables with mixing coefficient $q(n)$, and let $f_1, f_2, \cdots$ be all nondecreasing (or nonincreasing) functions, then $\{f_n(X_n)\}$ is also a sequence of AANA random variables with mixing coefficient $q(n)$.*

Finally, before providing the main convergence result for AANA random variables, we recall two convergence concepts for sequences of random variables.

Definition 3. *A sequence $\{X_n\}$ of random variables on a sample space Ω is said to be almost surely converging to a random variable X defined on Ω if $Pr(\omega \in \Omega \mid \lim_{n\to\infty} X_n(\omega) = X(\omega)) = 1$.*

Definition 4. *A sequence $\{X_n\}$ of random variables on a sample space Ω is said to be converging in probability to a random variable X defined on Ω if $\lim_{n\to\infty} Pr(\omega \in \Omega \mid |X_n(\omega) - X(\omega)| > \epsilon) = 0 \ \forall \ \epsilon > 0$.*

Almost sure convergence implies convergence in probability, but the converse is not true. An example of a sequence of random variables that converges in probability but not almost surely is obtained defining X_{ij} on $\Omega = [0, 1]$ for $1 \leq i \leq j$ as follows:

$$X_{ij}(\omega) = \begin{cases} 1 & \text{if } \omega \in \left[\frac{i-1}{j}, \frac{i}{j}\right] \\ 0 & \text{otherwise} \end{cases}$$

Then, the sequence $X_{11}, X_{12}, X_{22}, X_{13}, \cdots, X_{1n}, \cdots, X_{nn}, X_{1(n+1)}, \cdots$ converges to 0 in probability, but not almost surely. In fact, there is no realization of the sequence that converges: any realization is a sequence of zeros interspersed with ones.

Using AANA sequences of random variables enables us to use the following theorem, which was established by Wang et al. (2010).

Theorem 1. *Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with mixing coefficient satisfying $\sum_{k=1}^{\infty} q^2(k) < \infty$. Denote $Q_n = \max_{1 \leq k \leq n} \mathbb{E}(X_k)^2$ for $n \geq 1$ and $Q_0 = 0$. Then $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (X_i - \mathbb{E}(X_i)) = 0$ almost surely if $\sum_{n=1}^{\infty} \frac{Q_n}{n^2} < \infty$.*

Theorem 1 establishes that the average of a sequence of random variables converges to the average of the expected value of the random variables under weak conditions that allow

these variables to be dependent. In particular, the random variables may be positively correlated locally.

To allow for both a temporal and a spatial dimension, we need to use two-dimensional arrays of random variables. We conclude the preliminaries by providing some terminology for talking about these. An array of random variables $X_{in}, i, n \in \mathbb{N}$ is row-wise AANA if the sequence $\{X_{in}, n \in \mathbb{N}\}$ is AANA for every $i \in \mathbb{N}$. We will also need to introduce concepts of asymptotic boundedness and distant convergence relevant for a two-dimensional array a_{ij} of real numbers. The concept of distant convergence is a contribution of this paper.

Definition 5. *An array a_{ij} is said to asymptotically bounded by 0 if there exists a non-negative sequence $q(n) \to 0$ such that for all $n, k \geq 1$: $a_{n,n+k} < q(n)$.*

Definition 6. *An array a_{ij} is said to distant convergent to a if for every $\epsilon > 0$ there exist N so that if $|i - j| > N$, then $|a_{ij} - a| < \epsilon$.*

It is important to notice that asymptotically bounded by 0 is not same as distant convergent to 0. Definition 5 says that $a_i j$ is asymptotically bounded by 0 if $\lim\limits_{i \to \infty} a_{ij} = 0$, whereas $a_i j$ is distant convergent to 0 if $\lim\limits_{|i-j| \to \infty} a_{ij} = 0$ by definition 6. The asymptotically bounded property implies convergence with respect to first index only, and the distant convergent implies convergence with the difference of the two indices. Now, We establish convergence in probability of average speed, average flow and average occupancy to a constant at any given time using the following lemmas:

Lemma 2. *Let $\{X_n\}$ be a sequence of random variables with uniformly bounded means μ_n and covariances σ_{ij} that are asymptotically bounded by 0. Then, $\frac{1}{n} \sum\limits_{i=1}^{n} X_i$ converges in probability to $\lim\limits_{n \to \infty} \frac{1}{n} \sum\limits_{i=1}^{n} \mu_i$.*

Lemma 3. *Let $\{X_n\}$ be a sequence of random variables with uniformly bounded means μ_n and covariances σ_{ij} that are distant convergent to 0. Then, $\frac{1}{n} \sum\limits_{i=1}^{n} X_i$ converges in probability to $\lim\limits_{n \to \infty} \frac{1}{n} \sum\limits_{i=1}^{n} \mu_i$.*

3 The model and the main result

We are now ready to introduce the model that we use to relate the concepts and results from probability theory just introduced to the existence of an MFD in a road network. We consider a road network partitioned into an infinite number of segments $i \in \mathbb{N}$. The assumption that there are infinitely many segments is a mathematical idealization that

allows us to use the asymptotical results obtained in Section 2. We observe traffic variables at each segment and at each time t in a set T.

Indexing of the road segments should begin in the center, and the index should increase as we go away from the center. It does not matter that we use a one-dimensional index over two-dimensional space: in practice it is possible to spiral out form the center. The sequence of indexing is important since our assumptions require almost no positive local correlation for traffic variables away from the center (with higher index). A city with multiple congested areas can be included as long as there are sub-urban regions (higher index) with no significant positive local correlation.

Let o_i^t be a positive random variable denoting the occupancy on segment $i \in \mathbb{N}$ at time $t \in T$. For every segment i, there is a local speed-occupancy function $v_i : \mathbb{R}^+ \to \mathbb{R}^+$ and a local flow-occupancy function $q_i : \mathbb{R}^+ \to \mathbb{R}^+$. The local speed-density function is non-increasing, i.e., speed decreases or remains constant as the occupancy increases.

We assume that local occupancies, speeds and flows are uniformly bounded. This is clearly a realistic assumption as actual occupancies and speeds are bounded physically. This ensures that the average expected occupancy $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(o_i^t)$ exists and similarly for speed and flow.

We will use the following assumptions.

Assumption 1. *One of the following holds:*

1. *For every $t \in T$, $Cov(o_i^t, o_j^t)$, $Cov(v_i(o_i^t), v_j(o_j^t))$, and $Cov(q_i(o_i^t), q_j(o_j^t))$ are asymptotically bounded by 0.*

2. *For every $t \in T$, $Cov(o_i^t, o_j^t)$, $Cov(v_i(o_i^t), v_j(o_j^t))$, and $Cov(q_i(o_i^t), q_j(o_j^t))$ are distant convergent to 0.*

Assumption 1.1 below requires that the covariance between adjacent streets decreases with an increase in index. We note that asymptotic boundedness of the covariance is implied by AANA. It is thus a strictly weaker assumption. Assumption 1.2 holds when the covariance between the occupancy of segments decreases with increasing distance between two streets, i.e., the occupancy on adjacent segments may have higher covariance whereas occupancy on distant segments will have smaller covariance. The same assumption is also introduced for speed and flow.

Using Lemma 2 and 3, Assumption 1 guarantees the convergence in probability of the space-averaged occupancy, speed and flow. This means that spatial averages have a limiting value and the probability that spatial averages deviate by some fixed amount from their limiting values becomes arbitrarily small as the number of observations increases in the spatial dimension.

Assuming almost asymptotically negative association between the occupancies on different segments enables us to derive stronger results for the space-averaged speed and occupancy.

Assumption 2. *For every t, the sequence $\{o_i^t\}_{i\in\mathbb{N}}$ is AANA with mixing coefficient satisfying $\sum_{k=1}^{\infty} q^2(k) < \infty$.*

Assumption 2 allows us to establish almost sure convergence of space-averaged occupancy and speed as stated in the following lemma:

Lemma 4. *Under Assumption 2, at any time $t \in T$, space-averaged occupancy $\bar{o}_{nt} = \frac{1}{n}\sum_{i=1}^{n} o_i^t$ and space-averaged speed $\bar{v}_{nt} = \frac{1}{n}\sum_{i=1}^{n} v_i(o_i^t)$ converge almost surely.*

Lemma 2, 3 and 4 show that empirical averages of speed or flow against occupancy will be close to their expected values, where the meaning of closeness depends on whether we have convergence in probability or convergence almost surely. It may still happen, however, that the average speed or average flow converge on different values for the same value of the average occupancy at different times of day. A further assumption is required to guarantee that the limiting values are the same across times in the set T where the limiting value for the average occupancy is constant. The assumption that we will make is the weakest we can find that achieves this purpose. Importantly, it allows the distribution across space to vary over time. We do thus *not* require that the distribution of traffic variables at a location i is independent of time: it is sufficient to put a restriction on the distribution of speeds and flows, pooling them across space.

Denote the limiting values

$$\hat{o}_t = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left(o_i^t\right), \hat{v}_t = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left(v_i^t\right), \hat{q}_t = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left(q_i^t\right)$$

and let $T_{\hat{o}}$ be the set of times t where $\hat{o}_t = \hat{o}$.

Assumption 3. *For all $s \geq 0$, $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} 1_{\{v_i^t \leq s\}}$ and $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} 1_{\{q_i^t \leq s\}}$ are constant a.s. independent of $t \in T_{\hat{o}}$.*

Using data from Yokohama, Geroliminis and Sun (2011a) compare the distribution of occupancies across space at different times, conditional on the average occupancy. They find distributions that are highly dispersed. Nevertheless, they look quite stable and this evidence seems to be in good agreement with the present Assumption 3.

Then we can state our result.

Theorem 2. *Under assumptions 1 and 3, the limiting value for the space-averaged flow and speed is unique given the limiting value for space-averaged occupancy. In other words, $\hat{o}_{t_1} = \hat{o}_{t_2} = \hat{o}$ implies that $\hat{v}_{t_1} = \hat{v}_{t_2}$ and $\hat{q}_{t_1} = \hat{q}_{t_2}$.*

We have noted that AANA implies asymptotic boundedness by 0. Then the stronger Assumption 2 for occupancies together with Assumption 3 ensures Theorem 2 as well as the stronger almost sure convergence result of Lemma 4.

Theorem 2 ensures unique limiting values for $\hat{v}_t$ and $\hat{q}_t$ for all $t \in T_{\hat{o}}$. Extending this to all values of $\hat{o}$ implies that a scatter plot of space-averages of speed or flow against occupancy will converge to a single-valued function. This establishes then the existence of a well-defined MFD.

4 Model validation

We have established mathematical conditions that are sufficient to ensure the emergence of an MFD. We have also already noted that the empirical evidence in Geroliminis and Sun (2011a) supports the present Assumption 3 that the distribution of occupancies across space is independent of time conditional on the average occupancy. In this section, we will provide further checks on the validity of those conditions by confronting some testable implications of these conditions with simulated and empirical evidence.

4.1 Simulated data

We consider 300 road segments and choose randomly a speed-occupancy relationship for each segment in the following way: Speed limits are randomly chosen between 1 and 2. For each segment we choose a random occupancy level between 1 and 2 at which the speed becomes 0. Then we define a decreasing local speed-occupancy function for each road segment. The local speed-ocupancy function is piece wise linear and concave. To define the local speed-occupancy function, we choose a random occupancy level between 1 and the occupancy level, where speed becomes zero, and we assign a random speed between 1 and the speed limit of the street. Then, we join three points using linear curve. Flow is the product of speed and occupancy multiplied by the length of each segment which is a random number between 1 and 2.

The occupancies are taken to be independent random variables with identical distributions, which makes the occupancies AANA. Furthermore, functions of independent random variables are also independent, and therefore both speed and flow are AANA random variables. In this set up, we expect almost sure convergence for all traffic variables: occupancy, speed and flow.

We replicate this 1000 times with different means for occupancies to obtain simulations over a range of values for the average occupancy. This leads to the following occupancy-flow, speed-flow and speed-occupancy curves shown in Figure 1. We notice that a well-defined MFD emerges. This illustrates the basic insight underlying this paper, namely the

average of random speeds, occupancies and flows converges to well-defined values under weak conditions even when there is no process to equilibrate speeds across space and even when local speed-occupancy relationships vary completely at random.
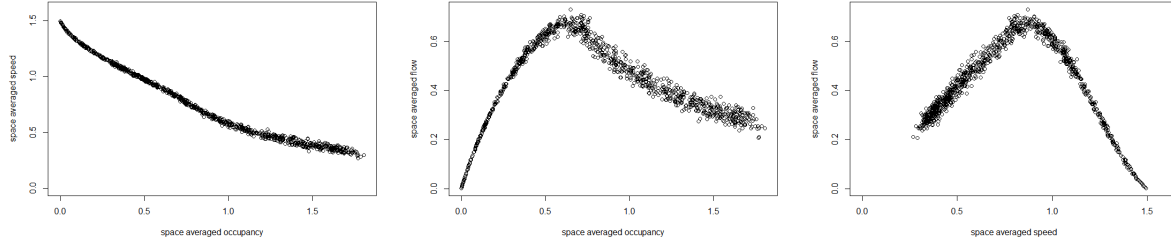

Figure 1: MFD for Simulated data. Left most picture represent the relation between speed and occupancy, middle picture represents flow and occupancy and the right most picture represents flow and speed relationship

4.2   Stockholm taxi data

Our first empirical test of the model assumptions utilizes GPS traces from taxis in Stockholm. The data are used to estimate the distribution of speed across space and time. We can then test our assumptions related to speed. We do not attempt to estimate occupancy or flow since the distribution of taxis in space and time may be quite different from that of the overall traffic.

We focus on the network on Södermalm island, which covers about 2×5 km^2 and is shown in Figure 2. The data consist of location reports with coordinates and timestamps from a fleet of about 1500 taxis. When active, each taxi reports its location once every 1-2 minutes. The data source is described in more detail in Rahmani et al. (2010). We use observations from weekdays between September 29 and October 10, 2014 and all hours of the day are used.

For each pair of consecutive reports from the same taxi, the speed is computed based on the Euclidean distance between the locations and the time difference between the timestamps. The speed observations are filtered to discard very low or high values that represent stopped vehicles or are atypical for normal traffic. After filtering the data set contains 113,823 speed observations. Each speed observation is then spatially associated with the mid-point between the start and end locations of the pair.

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Figure 2: The network and region of Södermalm, Stockholm used in the model validation.

4.2.1 Test of Assumption 1

To study the relation between distance and the covariance of speed, the region is spatially discretized in a square grid with 0.25 km distance between grid points. The speed observations are binned spatially according to the nearest grid point, and temporally according to 15-minute clock-time intervals and days. There are 119 grid points covering the region, 96 clock-time intervals and 10 days, generating in total 114,240 time-space bins. The local average speed is computed for each bin. For each grid point and clock-time interval the mean value across days is subtracted from the average speed to remove the effect of varying speeds during the day. Since the taxis do not cover the whole network at all times, values are missing for 61% of the bins.

For each pair of grid points the covariance of local speeds across all clock-time intervals and days is estimated, taking missing values into account. Kernel regression is then used to estimate the mean covariance of local speeds as a function of the Euclidean distance between the grid points. A Gaussian kernel with bandwidth 0.25 km is used, and a 95% confidence interval is estimated with bootstrapping.

The speed covariance shows large variability between grid points, which is partly an effect of the noisy nature of the taxi FCD (floating car data) (Figure 3, left). Still, the average covariance decays with distance, and the confidence interval shows that it is not statistically significantly positive for distances beyond 0.75 km (Figure 3, right). This supports the assumption that the covariance of speed is distant convergent to 0 (Assumption 1b).
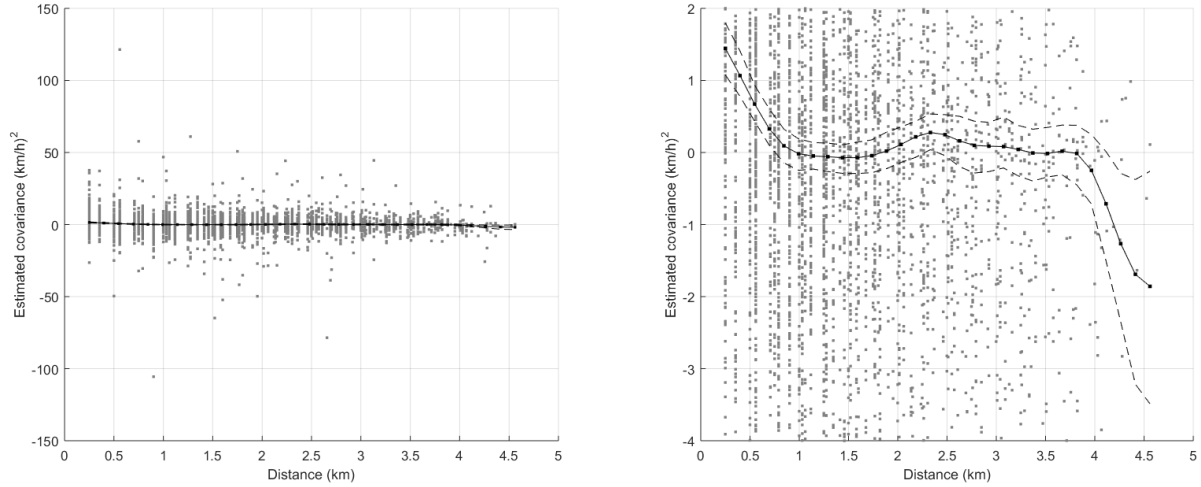
Figure 3: Covariance of local speed vs. distance for Södermalm, Stockholm based on binning of taxi FCD. Left figure shows all data points, right figure truncates the y-axis. Grey dots: Covariance between pairs of grid points. Solid line: Covariance function estimated with kernel regression. Dashed line: Bootstrap 95% confidence interval.

4.2.2 Test of Assumption 3

Assumption 3 states that the pooled distribution of speed across space should be invariant across time conditional on the average network occupancy. Since reliable occupancy information is not available from the taxi FCD, we instead consider the distribution of speed conditional on clock-time intervals where we expect the average occupancy to be the same. In particular, since a large part of traffic consists of trips that are first made in one direction and then repeated in the opposite direction (e.g., commute trips), it can be assumed that the average network occupancy should be similar during the first half of the day (0:00-11:45) and the second half of the day (12:00-23:45). Our model would then assume that the pooled speed distribution is the same for these two intervals.

To study the invariance of the speed distribution, the empirical cumulative distribution function (cdf) of the local speeds across grid points, clock-time intervals and days is computed. When observations are split between before and after mid-day, the cdf:s for the two periods are close to each other (Figure 4). The results are thus in agreement with the assumption that the speed distribution is invariant for a given average occupancy; when the average occupancy varies, the speed distribution can also vary.
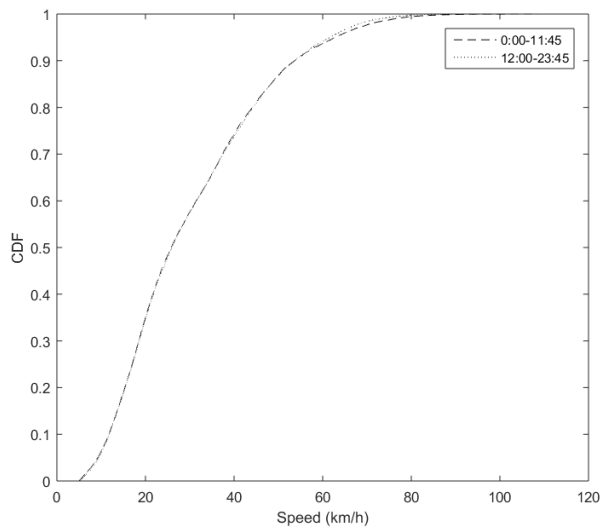
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Figure 4: Cumulative distribution function of local speeds across grid points, clock-time intervals and days for Södermalm, Stockholm based on binning of taxi FCD. It splits clock-time into before mid-day (0:00–11:45) and after mid-day (12:00-23:45).

4.3 Geneva loop-detector data

To further validate our assumptions, we study loop-detector data for the city of Geneva.[2] The Geneva network is shown in Figure 5. The data consists of occupancy, flow and speed measured by 254 detectors during September, 2014. These detectors are not physical detectors, but aggregated measurements used by the DGT (the transport authority La Direction générale des transports). The DGT typically aggregates different loops at the same level on different lanes into "one counting point".

The measurements are recorded every 3 minute and 20 seconds, and some of the detectors work only during day time. Occupancy is measured as the number of car on a street, whereas the speed and the flow are measured in kilometer per hour and number of cars passing per hour, respectively. The macroscopic fundamental diagram observed from the data is shown in Figure 6. We can notice two curves, and possible explanation is due to the spread in the empirical cumulative distribution function when flow is less. We discuss it in greater details when we test Assumption 3.

4.3.1 Test of Assumption 1

To study the relation between distance and the covariance of speed, we calculate the spatial distance between detectors. For each detector and clock-time interval the mean

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Figure 5: The network and region of Geneva city used in the model validation.

value across days is subtracted from the average speed to remove the effect of varying speeds during the day. For each pair of detectors, the covariance of speed, flow and occupancy is calculated, taking missing values in account. For better visualization, we have approximated distance to every 0.5 km. This is shown in Figure 7.

We notice that covariance decays with distance for all variables. Similar results are observed for flow and occupancy. Covariance for Speed and Covariance for occupancy converges much faster, whereas covariance for flow converges slower respect to occupancy and flow. Figure 7 help us conclude that covariances are distant convergent to 0.

4.3.2 Test of Assumption 3

We divide the average occupancy in different intervals of length 1 (car) and see the empirical cumulative distribution function for speed and flow when the average occupancy lies in that interval. The range of average occupancy is $(0, 26)$ and we round these values
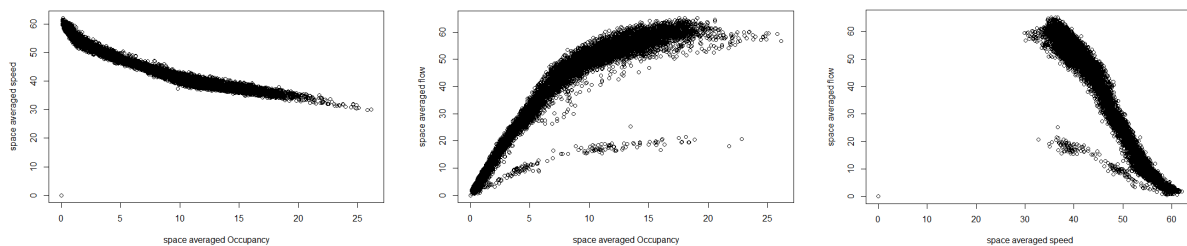
Figure 6: MFD for Geneva city. Left most picture represent the relation between speed and occupancy, middle picture represents flow and occupancy and the right most picture represents speed and flow relationship for the month of September 2014
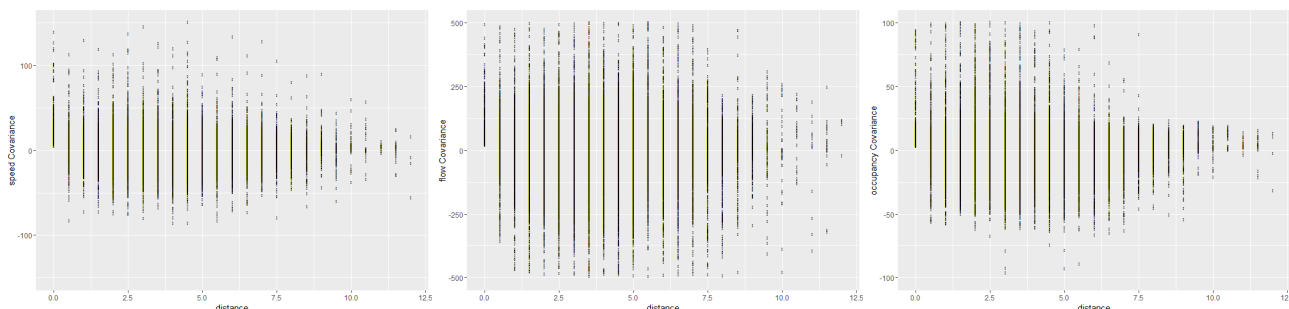

Figure 7: Covariances against distance for traffic variables: speed(left), flow(center) and occupancy(right).

to integers. We present the results at average occupancy level $2, 9, 15, 19$, though we have checked for every rounded occupancy level. We choose these intervals based on the frequency table of the average occupancy, and display the results where frequency of the average occupancy has significant change.

Figure 8 shows that the distribution of speeds across the city of Geneva overlap at given average occupancy level. Furthermore, Figure 6 shows a very small variability in the speed-occupancy diagram. We also observe that at higher average occupancy, the overlapping of the speed distribution is clearly represented by the diagram, and therefore we have almost no variability in the occupancy-speed curve.

Figure 9 shows the distribution of flows across city overlapping at a given average occupancy with few exceptions. We notice that flows have high variability in the distribution at lower occupancy level ($\bar{o} = 2, 5$ in the figure), but it overlaps at higher occupancy level. The higher variability in the empirical CDF at lower frequency explains why we have different curves at lower flow in the Figure 6 for flow-occupancy and flow-speed diagram.
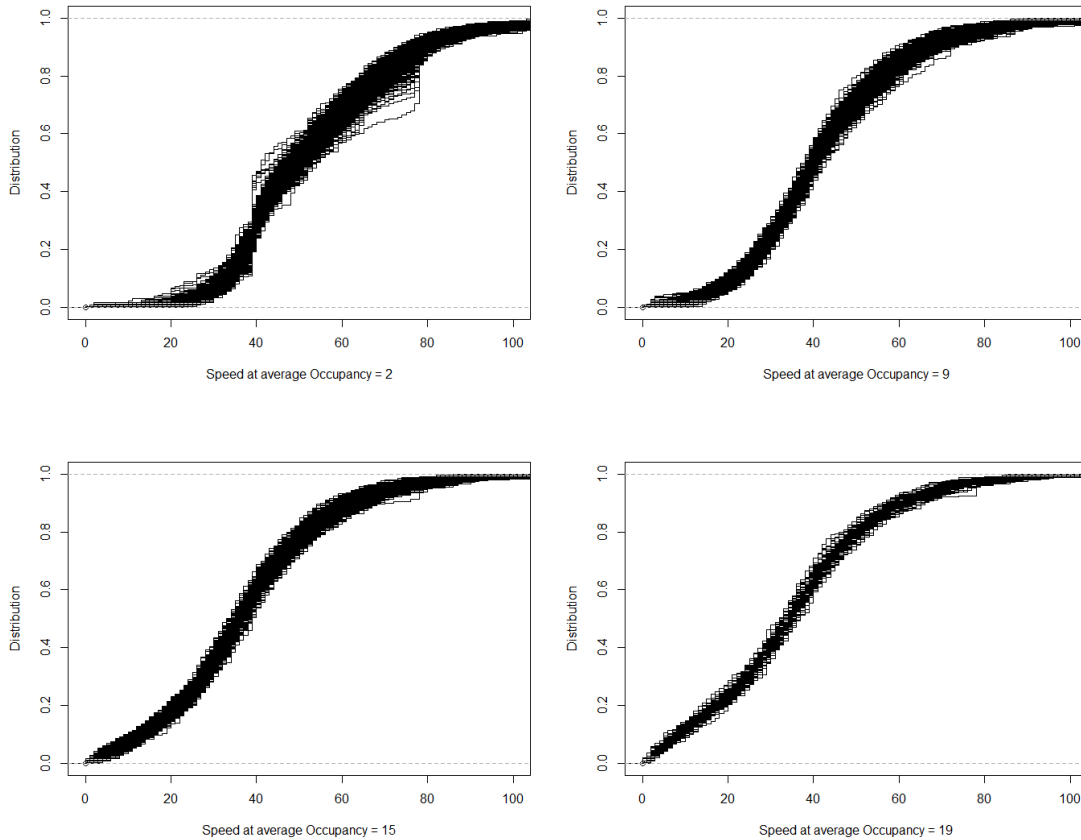
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Figure 8: Distribution of speeds at average occupancy $2, 9, 15, 19$ respectively.

5   Concluding remarks

We have established conditions that ensure that an MFD emerges in the limit as we increase the number of measurement locations. We have confirmed using empirical data that the correlation of speed across space decreases with distance, as required by the theory.

In practice, this means that we would expect an MFD to emerge when the number of measurement locations is sufficiently large. This occurs without any assumption of any equilibrating process. In particular, it is *not* required that speeds tend to be equal across space.

What does this imply for the use of an MFD for traffic control? To illustrate, we can think of a city consisting of two independent parts with no connecting roads between. Let us say that the conditions for the emergence of an MFD are satisfied in both parts
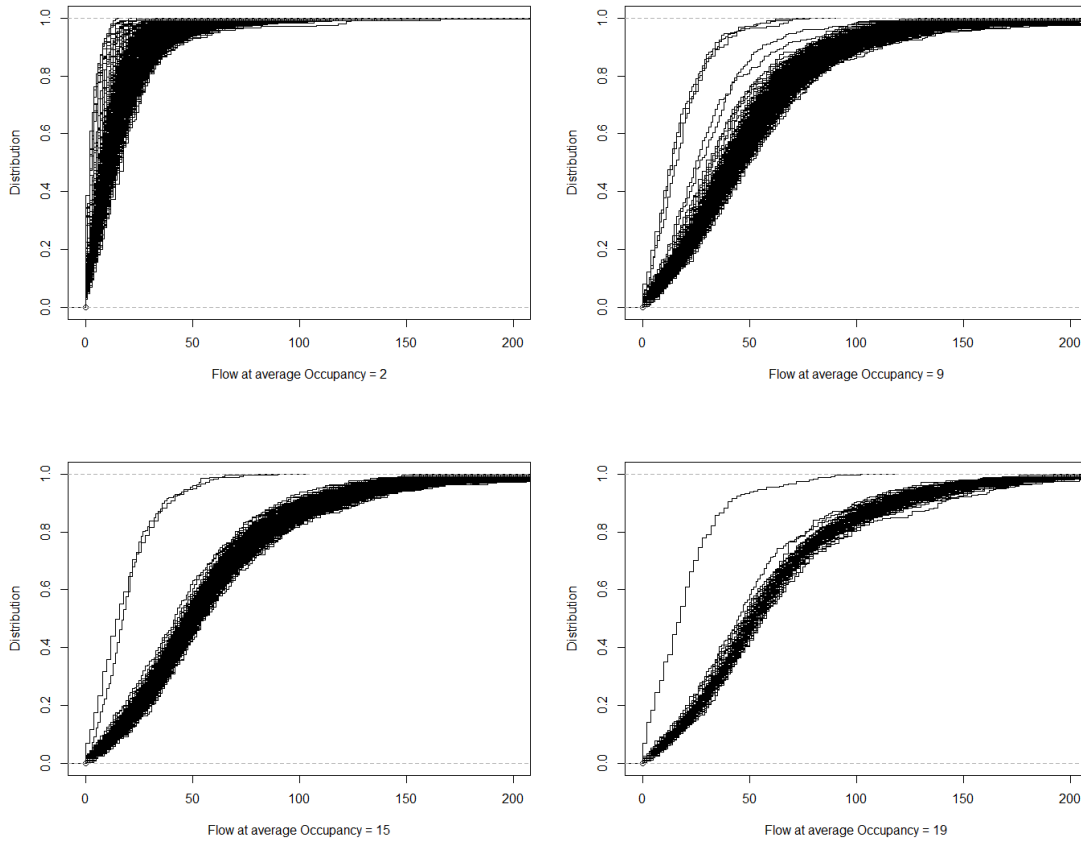
Figure 9: Distribution of flows at average occupancy $2, 9, 15, 19$ respectively.

and that these MFDs are different. Then, as this paper shows, an MFD will also emerge for the averages covering both parts of the city due to the assumed independence. If we now reduce traffic in one part of the city, the MFD for the whole city that would then emerge would be dominated by the MFD of the other part. Then the averaged MFD would change as a result of the reduction in traffic. This shows that metering would affect the observed shape of the MFD and that the MFD therefore does not predict the effect of metering.

So it should now be clear that merely observing a stable relationship between space-averages of speed or flow versus occupancy is not sufficient to infer the existence of a robust relationship that can be used for traffic control. It seems that some kind of equilibrating mechanism is required, as assumed by the papers mentioned in Introduction. This raises the question of how the existence and strength of such an equilibrating mechanism can be validated with empirical data. This seems to be an important topic for future research.

A Proofs

We refer to Yuan and An (2009) and Wang et al. (2010) for proofs of Lemma 1 and Theorem 1, respectively.

Proof of Lemma 2. Define $\bar{X}_n = \frac{1}{n} \sum\limits_{i=1}^{n} X_i$ and denote also $\sigma = \max_i Var(X_i)$, $\bar{\mu}_n = \frac{1}{n} \sum\limits_{i=1}^{n} \mu_i$, and $\bar{\mu} = \lim_{n\to\infty} \frac{1}{n} \sum\limits_{i=1}^{n} \mu_i$. Then $\mathbb{E}(\bar{X}_n) = \bar{\mu}_n$.

Given some $\epsilon > 0$, use the asymptotical bound on the covariances to choose k such that $q(i) < \epsilon$ for all $i > k$. Then

$$Var(\bar{X}_n) = \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i) + \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sigma_{ij}$$

$$\leq \frac{\sigma}{n} + \frac{2}{n} \sum_{i=1}^{n} q(i)$$

$$\leq \frac{\sigma}{n} + \frac{2}{n}(k \max_{i \leq k} q(i) + n\epsilon)$$

$$\leq \frac{\sigma}{n} + \frac{2k \max\limits_{i \leq k} q(i)}{n} + 2\epsilon,$$

which means that $Var(\bar{X}_n) \leq 3\epsilon$ for sufficiently large n. By Chebychev's inequality we then have

$$Prob(|\bar{X}_n - \mathbb{E}(\bar{X}_n)| \geq a) \leq \frac{Var(\bar{X}_n)}{a^2} \leq \frac{3\epsilon}{a^2}$$

for sufficiently large n. Then also

$$Prob(|\bar{X}_n - \bar{\mu}| \geq a) \leq Prob(|\bar{X}_n - \bar{\mu}_n| \geq a - |\bar{\mu}_n - \bar{\mu}|) \leq \frac{3\epsilon}{(a - |\bar{\mu}_n - \bar{\mu}|)^2}$$

for sufficiently large n. Since this is true for all $\epsilon > 0$, we conclude that $Prob(|\bar{X}_n - \bar{\mu}| \geq a)$ tends to zero as n tends to infinity. This is the required result that $\bar{X}_n$ converges in probability to $\bar{\mu}$. $\qquad\square$

Proof of Lemma 3. Define $\bar{X}_n = \frac{1}{n} \sum\limits_{i=1}^{n} X_i$. Then $\mathbb{E}(\bar{X}_n) = \frac{1}{n} \sum\limits_{i=1}^{n} \mathbb{E}(X_i)$. Denote $\sigma = \max\limits_{i} Var(X_i)$, and $\delta = \max\limits_{i,j} Cov(X_i, X_j)$, $\bar{\mu}_n = \frac{1}{n} \sum\limits_{i=1}^{n} \mu_i$, and $\bar{\mu} = \frac{1}{n} \sum\limits_{i=1}^{\infty} \mu_i$.

$$Var(\bar{X}_n) = \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i) + \frac{1}{n^2} \sum_{1 \leq i \neq j \leq n} Cov(X_i, X_j)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{0 < |i-j| \leq k} Cov(X_i, X_j) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{|i-j| > k} Cov(X_i, X_j).$$

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From the definition of distant convergence, for all $\epsilon > 0$, there exist k such that $|Cov(X_i, X_j)| < \epsilon$ for all i, j such that $|i - j| > k$. Note that k does not depend on n. Therefore,

$$Var(\bar{X}_n) \leq \frac{\sigma}{n} + \frac{k}{n^2} \sum_{i=1}^{n} \max_{0 < |i-j| \leq k} Cov(X_i, X_j) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{|i-j| > k)} \epsilon$$

$$\leq \frac{\sigma}{n} + \frac{k}{n}\delta + \epsilon$$

$$\leq 3\epsilon.$$

By Chebychev's inequality we then have

$$Prob(|\bar{X}_n - \mathbb{E}(\bar{X}_n)| \geq a) \leq \frac{Var(\bar{X}_n)}{a^2} \leq \frac{3\epsilon}{a^2}$$

for sufficiently large n. Then also

$$Prob(|\bar{X}_n - \bar{\mu}| \geq a) \leq Prob(|\bar{X}_n - \bar{\mu}_n| \geq a - |\bar{\mu}_n - \bar{\mu}|) \leq \frac{3\epsilon}{(a - |\bar{\mu}_n - \bar{\mu}|)^2}$$

for sufficiently large n. Since this is true for all $\epsilon > 0$, we conclude that $Prob(|\bar{X}_n - \bar{\mu}| \geq a)$ tends to zero as n tends to infinity. This is the required result that $\bar{X}_n$ converges in probability to $\bar{\mu}$. $\qquad \square$

Proof of Lemma 4. The conditions of Theorem 1 are satisfied since occupancies and speeds are uniformly bounded. Then we have $\lim\limits_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n}(o_{it} - \mathbb{E}(o_{it})) = 0$ almost surely. Since

$$\left| \frac{1}{n} \sum_{i=1}^{n} o_{it} - \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(o_{it}) \right| < \left| \frac{1}{n} \sum_{i=1}^{n} o_{it} - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(o_{it}) \right| + \left| \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(o_{it}) - \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(o_{it}) \right|$$

and both right-hand side terms tend to zero almost surely, we find that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} o_{it} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(o_{it})$$

almost surely.

By Lemma 1, speeds are also AANA with mixing coefficient $q(n)$. Then the conclusion for the space-averaged speed follows in the same way as for the space-averaged occupancy. $\square$

Proof of Theorem 2 . From Assumption 3, $lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1_{\{v_i^t \leq s\}}$ is constant a.s. for all s, and independent of $t \in T_{\hat{o}}$. We denote the limiting function by $V(s)$. Then

$$V(s) = \mathbb{E}\left(\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1_{\{v_i^t \leq s\}} \right) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} P\left(v_i^t \leq s \right)$$

by dominated convergence, noting that $\frac{1}{n}\sum_{i=1}^{n} 1_{\{v_i^t \leq s\}} \leq 1$.

That $\hat{v}_t$ is independent of t follows, again using dominated convergence, since

$$\hat{v}_t = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(v_i^t) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \int_0^\infty P(v_i^t > s))ds$$

$$= \int_0^\infty \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} P(v_i > s)ds$$

$$= \int_0^\infty (1 - V(s))ds,$$

which is independent of t by assumption.

The proof for $\hat{q}_t$ is similar. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Example 1. We shall show that the correlation coefficient between $U = f(X_m)$ and $V = g(X_{m+1}, \cdots, X_{m+k})$ is dominated in absolute value by a sequence b_m converging to 0. It is sufficient to prove this under the additional hypothesis $\mathbb{E}(U) = 0 = \mathbb{E}(V), \mathbb{E}(U^2) = 1 = \mathbb{E}(V^2)$. Then,

$$\begin{aligned}
|(Cov(U,V))| &\leq Cov(U, \mathbb{E}(U \mid X_{m+1}, \cdots, X_{m+k}) \\
&= \mathbb{E}(\mathbb{E}(U \mid X_{m+1}, \cdots, X_{m+k}))^2 \\
&\leq \mathbb{E}(\mathbb{E}(U \mid Y_{m+1}, \cdots, Y_{m+k+p+1}))^2 \\
&= \mathbb{E}(\mathbb{E}(U \mid Y_{m+1}, \cdots, Y_{m+p+1}))^2 \\
&= \mathbb{E}\left(\mathbb{E}\left(U \mid Z_{m+1} = \frac{(a_m Y_{m+1} + \cdots + a_{m+p} Y_{m+p+1})}{(a_m^2 + a_{m+1}^2 + \cdots + a_{m+p}^2)^{\frac{1}{2}}}\right)\right)^2
\end{aligned}$$
(A.1)

Clearly, Z_{m+1} is a standard normal random variable. Let $\psi_m(x, z)$ be the conditional density of X_m given Z_{m+1} and $\phi(x)$ be the density of standard normal random variable. Using the fact $\mathbb{E}(U) = 0$, we have

$$\mathbb{E}(\mathbb{E}(U \mid Z_{m+1}))^2 = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)\left(\frac{\psi_m(x,z)}{\phi(x)} - 1\right)\phi(x)dx\right)^2 \phi(z)dz$$

By using Cauchy-Schwartz inequality, the integral is at most

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\psi_m(x,z)}{\phi(x)} - 1\right)^2 \phi(x)dx\phi(z)dz = a_m^2 + a_{m+1}^2 + \cdots + a_{m+p}^2 = b_m^2$$

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Moreover, if $a_m \to 0$ then $b_m \to 0$. And $\sum\limits_{i=1}^{\infty} a_m^2 < \infty$ implies $\sum\limits_{i=1}^{\infty} b_m^2 < \infty$. $\qquad\square$

References

S. Ardekani and R. Herman. Urban network-wide traffic variables and their relations. *Transportation Science*, 21:1–16, 1987.

R. Arnott. A bathtub model of downtown traffic congestion. *Journal of Urban Economics*, 76:110–121, 2013.

H. W. Block, T. H. Savits, and M. T. Shaked. Some concepts of negative dependence. *Ann. Probab.*, 10(3):765–772, 1982.

T. K. Chandra and S. Ghosal. Extensions of the strong law of large numbers of marcinkiewicz and zygmund for dependent variables. *Acta Mathematica Hungarica*, 71(4):327–336, 1996a.

T. K. Chandra and S. Ghosal. The strong law of large numbers for weighted averages under dependence assumptions. *Journal of Theoretical Probability*, 9(3):797–809, 1996b.

C. F. Daganzo and N. Geroliminis. An analytical approximation for the macroscopic fundamental daigram of urban traffic. *Transportation Research Part B*, 42(9):771–781, 2008.

M. Fosgerau. Congestion in the bathtub. *Economics of Transportation*, 4:241–255, 2015.

N. Geroliminis and C. F. Daganzo. Existence of urban scale macroscopic fundamental diagram: some experimental findings. *Transportation Research Part B*, 42(9):759–770, 2008.

N. Geroliminis and J. Sun. Properties of a well-defined macroscopic fundamental diagram for urban traffic. *Transportation Research Part B*, 45(3):605–617, 2011a.

N. Geroliminis and J. Sun. Hysteresis phenomena of a macroscopic fundamental diagram in freeway networks. *Transportation Research Part A*, 45:966–979, 2011b.

J. W. Godfrey. The mechanism of a road network. *Traffic Engineering and Control*, 7: 323–327, 1969.

B. D. Greenshields. A study in highway capacity. *Highway research board proceedings*, 14: 448–477, 1935.

D. Helbing. Derivation of a fundamental diagram for urban traffic flow. *The European Physical Journal*, 70(2):229–241, 2009.

K. Joag-Dev and F. Proschan. Negative association of random variables, with applications. *Ann. Stat*, 11(1):286–295, 1983.

H. S. Mahmassani and S. Peeta. Network performance under system optimal and user equilibrium dynamic assignments: implication for atis. *Transportation Research record*, 1408:83–93, 1993.

H. S. Mahmassani, J. C. Williams, and R. Herman. Performance of urban traffic networks. *In: Gartner, N.H., Wilson, N:H:M. (Eds.), Proceedings of the 10th International Symposium on Transportation and Traffic Theory. Elsevier, Amsterdam, The Netherlands*, 1987.

P. Olszewski, H. S. Fan, and Y. W. Tan. Area-wide traffic speed-flow model for the singapore cbd. *Transportation Research Part A*, 29(4):273–281, 1995.

M. Rahmani, H. Koutsopoulos, and A. Ranganathan. Requirements and potential of GPS-based floating car data for traffic management: Stockholm case study. In *Intelligent Transportation Systems (ITSC), 2010 13th International IEEE Conference on*, pages 730–735, 2010.

W. S. Vickrey. Congestion theory and transport investment. *American Economic Review*, 59:251–260, 1969.

X. Wang, S. Hu, and W. Yang. Convergence properties for asymptotically almost negatively associated sequence. *Discrete Dynamics in Nature and Society, Article ID 218380, 15 pages, 2010. doi:10.1155/2010/218380*, 2010.

J. C. Williams, H. S. Mahmassani, and R. Herman. Urban traffic network flow models. *Transportation Research Record*, 1112:78–88, 1987.

D. M. Yuan and J. An. Rosenthal type inequalities for asymptotically almost negatively associated random variables and applications. *Sci. China Ser. A*, 52(9):1887–1904, 2009.