A counterexample on the completion of preferences with single crossing differences

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Abstract

We provide an example of a data set where all the revealed preference relations seem to be consistent with single crossing differences and yet the revealed preference relations cannot be extended to a complete preference obeying that property.

Keywords: monotone comparative statics, single crossing differences, interval dominance, supermodular games, lattices

JEL classification numbers: C6, C7, D7

1 Introduction

To motivate the counterexample presented in this note, we first give a quick review of some concepts and results in monotone comparative statics. Let $X \subset \mathbb{R}$ be the set of all possible actions of an agent and let Ξ be a partially ordered set which we shall interpret as a set of exogenous parameters that may affect the agent's preference over the actions in X. By a *preference* we mean a binary relation $\gtrsim$ on $X \times \Xi$ such that for every $\xi \in \Xi$, it is reflexive, complete, and

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transitive on X. A preference relation $\succsim$ obeys *strict single crossing differences* (or SSCD, for short), if for every $x'' > x'$ and $\xi'' > \xi'$,

$$(x'', \xi') \succsim (x', \xi') \implies (x'', \xi'') \succ (x', \xi). \tag{1}$$

Suppose the agent has a preference $\succsim$; for every $A \subset X$ and $\xi \in \Xi$, we denote the agent's set of best responses by $\mathrm{BR}(A, \xi)$, i.e., $\mathrm{BR}(A, \xi) = \{x \in A : (x, \xi) \succsim (x', \xi) \text{ for all } x' \in A\}$. We say that the best response set is increasing in ξ if for every $\xi'' > \xi'$,

$$x'' \in \mathrm{BR}(A, \xi''), x' \in \mathrm{BR}(A, \xi') \implies x'' \geqslant x'. \tag{2}$$

Milgrom and Shannon (1994) shows that $\mathrm{BR}(\xi, A)$ is increasing in ξ for every set $A \subset X$ if and only if the agent's preference $\succsim$ obeys SSCD. It is possible to weaken the SSCD property if we consider a smaller class of sets A. For example, we may restrict an agent's feasible action sets to *intervals of* X;[1] then $\mathrm{BR}(A, \xi)$ is increasing in ξ for all intervals A if and only if $\succsim$ obeys the *strict interval dominance* (SID) order (see Quah and Strulovici (2009)). A preference relation $\succsim$ obeys SID, if for every $x'' > x'$ and $\xi'' > \xi'$,

$$(x'', \xi') \succsim (x, \xi) \text{ for all } x \in [x', x''] \implies (x'', \xi'') \succ (x', \xi''). \tag{3}$$

Lazzati, Quah, and Shirai (2016) (henceforth LQS) have recently developed a revealed preference theory of SID. Their starting point is a finite set of observations $\mathcal{O} = \{(x^t, A^t, \xi^t)\}_{t \in \mathcal{T}}$ collected from an agent, where x^t is the action chosen by the agent from a compact interval A^t (contained in a set $X \subseteq \mathbb{R}$ of all possible actions) when ξ^t is the prevailing parameter value. LQS identify a condition on $\mathcal{O}$ that is necessary and sufficient for $\mathcal{O}$ to be *rationalizable by an SID preference*; this means that there exists an SID preference relation $\succsim$ on $X \times \Xi$ such that, with this preference, x^t is a best response from the interval A^t when the parameter is ξ^t.

The condition obtained by LQS involves a restriction on the revealed preference relations. We define the *direct revealed preference* relation $\succsim^R$ on $X \times \Xi$ in the following manner: $(x'', \xi) \succsim^R (x', \xi)$ if there exists some $t \in \mathcal{T}$ with $x^t = x''$, $\xi^t = \xi$, and $x' \in A^t$. The *indirect revealed*

[1] A is an interval of X if for every $x'', x' \in A$ with $x'' > x'$, if $z \in X$ satisfies $x' < z < x''$ then $z \in A$.

preference relation $\succsim^{RT}$ is the transitive closure of $\succsim^R$, i.e. $(x'', \xi) \succsim^{RT} (x', \xi)$, if there exists a finite sequence of actions $z^1, z^2, ..., z^K$ such that $x'' = z^1$, $x' = z^k$, and $(z^k, \xi) \succsim^R (z^{k+1}, \xi)$ for every $k \in \{1, 2, ..., K-1\}$. It is obvious that so long as an agent is maximizing some preference (whether or not it is SID) then $\succsim^{RT} \subset \succsim$. LQS show that $\mathcal{O}$ can be rationalized by an SID preference if and only if it satisfies the following property, which they call the *axiom of revealed complementarity* (ARC):

$$\xi^t > \xi^s, x^t < x^s, \text{ and } (x^t, \xi^t) \succsim^{RT} (x^s, \xi^t) \Longrightarrow (x^s, \xi^s) \not\succsim^{RT} (x^t, \xi^s). \tag{4}$$

In fact, LQS reach a stronger conclusion: whenever $\mathcal{O}$ obeys ARC, it can be rationalized by a preference $\succsim$ obeying SSCD.

Suppose that, instead of being intervals, the observed feasible action sets A^t in the data set $\mathcal{O}$ are arbitrary nonempty sets. In this case, ARC is no longer necessary for rationalizability by an SID preference but it is obvious that the property is still necessary for $\mathcal{O}$ to be rationalized by an SSCD preference. Given this, it would be natural to conjecture that, even allowing for arbitrary feasible sets, ARC is both necessary *and sufficient* for $\mathcal{O}$ to be rationalized by an SSCD preference. However, Example 2 in LQS shows that this conjecture is false.

We may think that this conjecture fails because ARC does not capture all the implications of rationalizability by an SSCD preference and that there is a more stringent but still intuitive and easy-to-check condition on the data set that is both necessary and sufficient for this rationalizability property. An ideal scenario would be for the characterization to take the form of a no-cycling condition, similar to the ones used in Afriat's (1967) Theorem or Richter's (1966) Theorem. Indeed, there is a natural way of constructing a condition of this type, which we shall now explain. Given $\succsim^{RT}$, we define its *single crossing extension* as the binary relation $\succ^{RTS}$ such that for $x'' > (<) x'$, $(x'', \xi) \succ^{RTS} (x', \xi)$, if there exists some $\xi' < (>) \xi$ with $(x'', \xi') \succsim^{RT} (x', \xi')$. We also define $\succsim^{RTS} = \succsim^{RT} \cup \succ^{RTS}$. (Note that $\succ^{RTS}$ is not the asymmetric part of $\succsim^{RTS}$.) It is straightforward to check that if $\mathcal{O}$ is rationalizable by an SSCD preference $\succsim$, then $\succsim^{RTS} \subset \succsim$ and $\succ^{RTS} \subset \succ$ (where $\succ$ is the asymmetric part of $\succsim$). It follows that $\succsim^{RTS}$ must be *cyclically* *consistent* in the sense that

$$(z^1, \xi) \gtrsim^{RTS} (z^2, \xi) \gtrsim^{RTS} \cdots \gtrsim^{RTS} (z^K, \xi) \Longrightarrow (z^K, \xi) \not\succ^{RTS} (z^1, \xi). \quad (5)$$

A data set that violates ARC will not have $\gtrsim^{RTS}$ obeying cyclical consistency but it is possible for a data set (with non-interval feasible sets) to obey ARC and yet have $\gtrsim^{RTS}$ be cyclically inconsistent (see Example 2 in LQS). So perhaps data sets that are rationalizable by SSCD preferences are characterized by the cyclical consistency of $\gtrsim^{RTS}$, rather than ARC? But we can push this logic further.

Assuming that $\gtrsim^{RTS}$ is cyclically consistent, we can construct its transitive closure $\gtrsim^{RTST}$, which is defined as follows: $(x_i'', \xi_i) \gtrsim_i^{RTST} (x_i', \xi_i)$ if there exists a sequence $z_i^1, z_i^2, ..., z_i^k$ such that

$$(x_i'', \xi_i) \gtrsim_i^{RTS} (z_i^1, \xi_i) \gtrsim_i^{RTS} (z_i^2, \xi_i) \gtrsim_i^{RTS} ... \gtrsim_i^{RTS} (z_i^k, \xi_i) \gtrsim_i^{RTS} (x_i', \xi_i). \quad (6)$$

If we can find at least one strict relation $\succ_i^{RTS}$ in the sequence (6), then, we let $(x_i'', \xi_i) \succ_i^{RTST} (x_i', \xi_i)$. (Once again, note that $\succ_i^{RTST}$ is *not* the asymmetric part of $\gtrsim_i^{RTST}$.) Given $\gtrsim^{RTST}$, we can construct its single crossing extension $\gtrsim^{RTSTS}$; this relation must also be cyclically consistent if $\mathcal{O}$ is rationalizable by an SSCD preference. If so, we can take its transitive closure, the single crossing extension of that transitive closure, check for cyclical consistency and, if it passes, repeat the process. Since the data set has only finitely many observations, this process will terminate after a finite number of steps. Furthermore, if $\mathcal{O}$ is rationalizable by an SSCD preference then the process will terminate without a violation of cyclical consistency. It is natural to speculate that this property (i.e., the absence of violations of cyclical consistency) is also *sufficient* for $\mathcal{O}$ to be rationalized by an SSCD preference. We provide a counter example to show that this conjecture is also false.

2 A counter-example

Consider the following direct revealed preference relations:

(a) $\underline{\xi = 0}$: $(5,0) \gtrsim^R (1,0)$, $(1,0) \gtrsim^R (-5,0)$,$(3,0) \gtrsim^R (1,0)$, $(1,0) \gtrsim^R (-3,0)$. $(-1,0) \gtrsim^R (2,0)$, $(-1,0) \gtrsim^R (4,0)$, $(9,0) \gtrsim^R (6,0)$, $(-8,0) \gtrsim^R (-7,0)$, $(-6,0) \gtrsim^R (-9,0)$, $(-2,0) \gtrsim^R$

$(-1,0)$, $(-4,0) \succsim^R (-1,0)$, and $(7,0) \succsim^R (8,0)$.

(b) $\underline{\xi = 1}$: $(6,1) \succsim^R (3,1)$, $(2,1) \succsim^R (7,1)$, $(8,1) \succsim^R (5,1)$, $(4,1) \succsim^R (9,1)$, $(-7,1) \succsim^R (-2,1)$, $(-3,1) \succsim^R (-6,1)$, $(-9,1) \succsim^R (-4,1)$, and $(-5,1) \succsim^R (-8,1)$.

These relations can be formed from observing an agent choosing from pairs of alternatives; for example, $(5,0) \succsim^R (1,0)$ because 5 was chosen from the feasible action set $\{1,5\}$ when the prevailing parameter value is 0. Figures 1 and 2 show the graph of $\succsim^R$ at each value of ξ_i. We claim that this data set has the following features:

(i) it is not rationalizable by an SSCD preference,

(ii) the binary relation $\succsim^{RTS}$ obeys cyclical consistency, and

(iii) the relation $\succsim^{RTST}$ does not have a nontrivial single crossing extension.

To establish claim (i), suppose that there exists an SSCD preference relation $\succsim^*$ that rationalizes the data. We show that is not possible for either $(1,0) \succsim^* (-1,0)$ or $(-1,0) \succsim^* (1,0)$ to hold. First, we show that $(1,0) \succsim^* (-1,0)$ is impossible. If $(1,0) \succsim^* (-1,0)$, then it implies that $(3,0) \succsim^* (2,0)$ and $(5,0) \succsim^* (4,0)$ since $(3,0) \succsim^R (1,0)$, $(-1,0) \succsim^R (2,0)$, and $\succsim^*$ is transitive. Since $\succsim^*$ obeys SSCD, we obtain $(3,1) \succ^* (2,1)$ and $(5,1) \succ^* (4,1)$. The relations $(6,1) \succsim^R (3,1)$ and $(2,1) \succsim^R (7,1)$ and the transitivity of $\succsim^*$ in turn imply that $(6,1) \succ^* (7,1)$ and $(8,1) \succ^* (9,1)$. Again by SSCD, it holds that $(6,0) \succ^* (7,0)$ and $(8,0) \succ^* (9,0)$. These relations, together with $(9,0) \succsim^R (6,0)$ and $(7,0) \succsim^R (8,0)$ together with transitivity imply that $(6,0) \succ^* (6,0)$, which is impossible.

Suppose instead that $(-1,0) \succsim^* (1,0)$. By $(-2,0) \succsim^R (-1,0)$, $(-4,0) \succsim^R (-1,0)$, and the transitivity of $\succsim^*$, we obtain $(-2,0) \succsim^* (-3,0)$ and $(-4,0) \succsim^* (-5,0)$. SSCD then guarantees that $(-2,1) \succ^* (-3,1)$ and $(-4,1) \succ^* (-5,1)$. The transitivity of $\succsim^*$, $(-7,1) \succsim^R (-2,1)$ and $(-3,1) \succsim^R (-6,1)$ together imply that $(-7,1) \succ^* (-6,1)$. Similarly, $(-9,1) \succsim^R (-4,1)$ and $(-5,1) \succsim^R (-8,1)$ imply that $(-9,1) \succ^* (-8,1)$. Again by SSCD, it holds that $(-7,0) \succ^* (-6,0)$ and $(-9,0) \succ^* (-8,0)$. Finally, by $(-6,0) \succsim^R (-9,0)$ and $(7,0) \succsim^R (8,0)$, we have $(-7,0) \succ^* (-7,0)$, which is impossible.

To show that claims (ii) and (iii) are also true, we first take the transitive closure of $\succsim^R$. This gives the following relations besides the ones already related by $\succsim^R$ at $\xi = 0$: $(5,0) \succsim^{RT} (-5,0), (5,0) \succsim^{RT} (-3,0), (3,0) \succsim^{RT} (-3,0), (3,0) \succsim^{RT} (-5,0), (-2,0) \succsim^{RT} (4,0), (-2,0) \succsim^{RT}$

$(2,0), (-4,0) \succsim^{RT} (4,0), (-4,0) \succsim^{RT} (2,0), (9,0) \succsim^{RT} (6,0)$, and $(-6,0) \succsim^{RT} (-9,0)$. Note that, at $\xi = 1$, no new relation is added by taking the transitive closure.

By taking the single crossing extension, we obtain the binary relation $\succsim^{RTS}$. In the list below, for each value of ξ, Type 1 relations consist of pairs that are also related by $\succsim^{RT}$ and where the relation could be "transimitted" to the other value of ξ via single crossing extension; Type 2 consists of pairs related by $\succsim^{RT}$ which cannot be transimitted to the other value of ξ via single crossing extension, and Type 3 consists of pairs related by $\succ^{RTS}$, which are formed via extension from the other value of ξ.

(a) $\xi = 0$:

- Type 1: $(5,0) \succsim^{RTS} (1,0)$, $(1,0) \succsim^{RTS} (-5,0)$, $(3,0) \succsim^{RTS} (1,0)$, $(1,0) \succsim^{RTS} (-3,0)$, $(5,0) \succsim^{RTS} (-5,0)$, $(5,0) \succsim^{RTS} (-3,0)$, $(3,0) \succsim^{RTS} (-3,0)$, $(3,0) \succsim^{RTS} (-5,0)$, $(9,0) \succsim^{RTS} (6,0)$, and $(-6,0) \succsim^{RTS} (-9,0)$.
- Type 2: $(-1,0) \succsim^{RTS} (2,0)$, $(-8,0) \succsim^{RTS} (-7,0)$, $(-2,0) \succsim^{RTS} (-1,0)$, $(-4,0) \succsim^{RTS} (-1,0)$, $(7,0) \succsim^{RTS} (8,0)$, $(-2,0) \succsim^{RTS} (4,0)$, $(-2,0) \succsim^{RTS} (2,0)$, $(-4,0) \succsim^{RTS} (4,0)$, and $(-4,0) \succsim^{RTS} (2,0)$.
- Type 3: $(2,0) \succ^{RTS} (7,0)$, $(-7,0) \succ^{RTS} (-2,0)$, $(4,0) \succ^{RTS} (9,0)$, and $(-9,0) \succ^{RTS} (-4,0)$.

(b) $\xi = 1$

- Type 1: $(2,1) \succsim^{RTS} (7,1)$, $(-7,1) \succsim^{RTS} (-2,1)$, $(4,1) \succsim^{RTS} (9,1)$, and $(-9,1) \succsim^{RTS} (-4,1)$,
- Type 2: $(6,1) \succsim^{RTS} (3,1)$, $(8,1) \succsim^{RTS} (5,1)$, $(-3,1) \succsim^{RTS} (-6,1)$, and $(-5,1) \succsim^{RTS} (-8,1)$,
- Type 3: $(5,1) \succ^{RTS} (1,1)$, $(1,1) \succ^{RTS} (-5,1)$, $(3,1) \succ^{RTS} (1,1)$, $(1,1) \succ^{RTS} (-3,1)$, $(5,1) \succ^{RTS} (-5,1)$, $(5,1) \succ^{RTS} (-3,1)$, $(3,1) \succ^{RTS} (-3,1)$, $(3,1) \succ^{RTS} (-5,1)$, $(9,1) \succ^{RTS} (6,1)$, and $(-6,1) \succ^{RTS} (-9,1)$

These relations are depicted in Figures 3 and 4, where thick arrows represent $\succ^{RTS}$, while dotted arrows represent $\succsim^{RTS} \setminus \succ^{RTS}$ (which equals $\succsim^{RT}$). We can check from these graphs that $\succsim^{RTS}$ is cyclically consistent and that $\succsim^{RTST}$ obeys single crossing differences in the following sense: if $x'' > x'$ and $\xi'' > \xi'$, then (a) $(x'', \xi') \succsim^{RTST} (x', \xi')$ implies that $(x'', \xi'') \succ^{RTST} (x', \xi'')$ and

(b) $(x', \xi'') \succsim^{RTST} (x'', \xi'')$ implies that $(x', \xi') \succ^{RTST} (x'', \xi')$. It follows that $\succsim^{RTST}$ admits no further single crossing extension.

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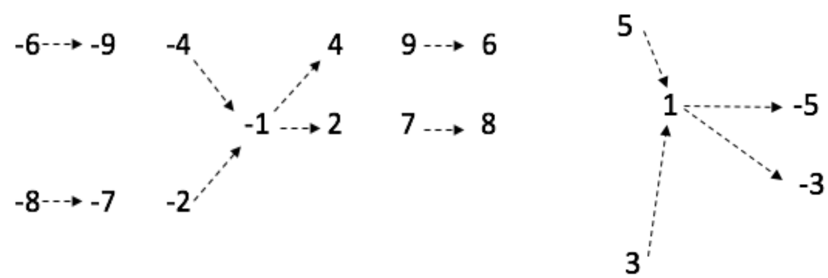

Figure 1: $\succsim^R$ at $\xi = 0$

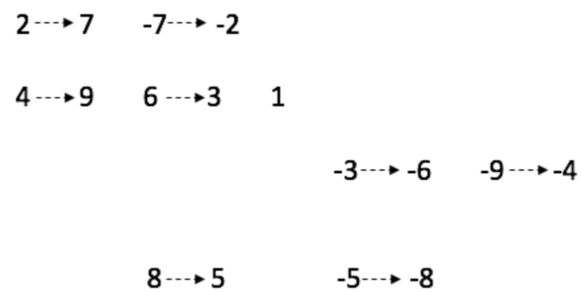

Figure 2: $\succsim^R$ at $\xi = 1$

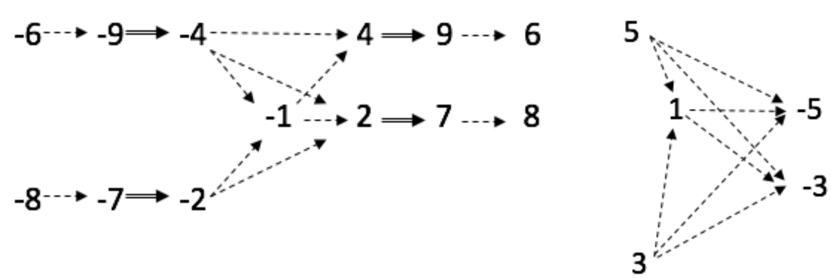

Figure 3: $\succsim^{RTS}$ at $\xi = 0$

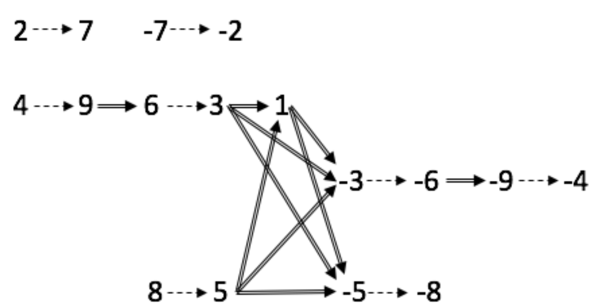

Figure 4: $\succsim^{RTS}$ at $\xi = 1$