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2016

Online at <https://mpra.ub.uni-muenchen.de/72392/>  
MPRA Paper No. 72392, posted 7 July 2016 19:27 UTC

# Oligarchy and soft incompleteness\*

Conal Duddy<sup>†</sup> and Ashley Piggins<sup>‡</sup>

May 11, 2016

## Abstract

The assumption that the social preference relation is complete is demanding. We distinguish between “hard” and “soft” incompleteness, and explore the social choice implications of the latter. Under soft incompleteness, social preferences can take values in the unit interval. We motivate interest in soft incompleteness by presenting a version of the strong Pareto rule that is suited to the context of a  $[0, 1]$ -valued social preference relation. Using a novel approach to the quasi-transitivity of this relation we prove a general oligarchy theorem. Our framework allows us to make a distinction between a “strong” and a “weak” oligarchy, and our theorem identifies when the oligarchy must be strong and when it can be weak. Weak oligarchy need not be undesirable.

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\*Financial support from the Irish Research Council for the Humanities and Social Sciences co-funding from the European Commission, and from the Spanish Ministry of Economy and Competitiveness through MEC/FEDER grant ECO2013-44483-P is gratefully acknowledged.

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**Keywords:** Oligarchy; Gibbard's theorem; Incompleteness; Max-star transitivity

**JEL Classification:** D71

## 1 Introduction

A social welfare function is a mapping from profiles of individual preference orderings (reflexive, transitive and complete binary relations) into a social ordering of the alternatives. Arrow's (1951, 1963) impossibility theorem demonstrates that the only social welfare functions that satisfy unrestricted domain, independence of irrelevant alternatives, and the weak Pareto principle are dictatorial.<sup>1</sup> One approach that has been taken to circumvent this negative result is to weaken the requirement that the social preference relation is an ordering. For example, research has focussed on weakening the transitivity requirement on the social preference relation to quasi-transitivity, acyclicity or Suzumura consistency.<sup>2</sup> Other work has retained Arrow's original transitivity assumption, but dropped the requirement that the social preference relation is complete. Pioneering contributions have come from Sen (1969, 1970a), Mas-Colell and Sonnenschein (1972), Plott (1973), Brown (1975), Blair and Pollak (1982), Weymark (1984), Banks (1995), Bossert and Suzumura (2008, 2010) and Gibbard (2014).<sup>3</sup>

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<sup>1</sup>A thorough survey of impossibility results in the Arrovian framework is Campbell and Kelly (2002).

<sup>2</sup>Formal definitions of these concepts can be found in Bossert and Suzumura (2010, chapter 2). Intuitively, a quasi-transitive relation is one where the strict preference relation is transitive, whereas the indifference relation need not be. An acyclic relation is one with no cycles in the strict preference relation. A Suzumura consistent relation rules out not only strict preference cycles, but all cycles that involve at least one strict preference. Acyclicity is the weakest concept of the three. Suzumura consistency and quasi-transitivity are logically independent.

<sup>3</sup>Gibbard's paper was written in the 1969-70 academic year but remained unpublished until Gibbard (2014). A historical introduction to the paper is given by Weymark (2014).

One of the earliest contributions to this literature was provided by Allan Gibbard, who proved a celebrated theorem in a paper that remained unpublished until recently. Gibbard was motivated by what Sen called the Pareto-extension rule (Sen 1969, 1970a). To explain this rule, consider the profile in Table 1.

	$x$ vs. $y$	$y$ vs. $z$	$x$ vs. $z$
Individual 1	$xP_1y$	$yP_1z$	$xP_1z$
Individual 2	$yP_2x$	$yP_2z$	$xP_2z$
Individual 3	$xP_3y$	$zP_3y$	$xP_3z$
<b>Pareto extension rule</b>	$xIy$	$yIz$	$xPz$
<b>Strong Pareto rule</b>	$xNy$	$yNz$	$xPz$
<b>Proportional rule</b>	$p(x, y) = 2/3$	$p(y, z) = 2/3$	$p(x, z) = 1$

Table 1: A profile.

The notation used is standard.<sup>4</sup> In Table 1 and throughout this paper, individual preferences are taken to be orderings, i.e. reflexive, complete and transitive binary relations (Sen 1970a, p.8).<sup>5</sup>

Under both the Pareto extension rule and the strong Pareto rule (Weymark, 1984), whenever everyone strictly prefers one social state to another then so does society (hence,  $xPz$  in the example). The two rules differ whenever there are two individuals with opposing strict preferences. Under the Pareto extension rule, these disagreements result in social indifference. A consequence of this is that the weak social preference relation  $R$  can be intransitive, i.e. in the example we have  $yRx$  and  $zRy$  but not  $zRx$ .  $R$  is, however, complete. Since individual preferences

<sup>4</sup>The primitive concept,  $R \subseteq X \times X$ , is a binary relation on a set of social states  $X$ . Writing  $xRy$  means that  $x$  is at least as good as  $y$ . Strict preference,  $P$ , is defined as  $\forall x, y \in X : xPy \leftrightarrow [xRy \wedge \neg yRx]$  while indifference,  $I$ , is defined as  $\forall x, y \in X : xIy \leftrightarrow [xRy \wedge yRx]$ . The relation of non-comparability,  $N$ , is defined as  $\forall x, y \in X : xNy \leftrightarrow [\neg xRy \wedge \neg yRx]$ . Writing  $xNy$  means that  $x$  and  $y$  are not ranked. Subscripts are used to denote individual preference relations. The absence of a subscript denotes a social preference relation.

<sup>5</sup>Given a binary relation  $R$ , reflexivity is defined as:  $\forall x \in X : xRx$ . Completeness is defined as:  $\forall x, y \in X : (x \neq y) \rightarrow (xRy \vee yRx)$ . Transitivity is defined as:  $\forall x, y, z \in X : (xRy \wedge yRz) \rightarrow xRz$ .

are assumed to be transitive, under this rule the strict social preference relation  $P$  must be transitive as well.<sup>6</sup> In contrast, under the strong Pareto rule, opposing strict preferences result in social non-comparability. This means that  $R$  need not be complete under this rule, but it is transitive nonetheless (again, because individual preferences are assumed to be).

These rules are interesting because they can be placed on a defensible normative footing. For example, assuming that the social preference relation is reflexive, quasi-transitive and complete, the Pareto extension rule can be shown to be the only collective choice rule that satisfies Arrow's conditions of unrestricted domain, independence of irrelevant alternatives, a strengthening of Arrow's Pareto principle, and a reasonable anonymity requirement.<sup>7</sup> Weymark (1984, Theorem 3) proves something similar for the strong Pareto rule under the assumption that the social preference relation is reflexive and transitive, but not necessarily complete (i.e. it is a quasi-ordering). There is an intuitive sense in which these rules represent the "closest" Arrowian rules, i.e. they come as near as possible to satisfying Arrow's requirements without resulting in a dictatorship.

Of these two complementary approaches to Arrow's theorem (Sen's and Weymark's), Weymark's is arguably more attractive. Transitivity seems a more compelling assumption about  $R$  than completeness. Aumann (1962, p.446) is often quoted in discussions of completeness: "Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms,

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<sup>6</sup>In other words,  $R$  is quasi-transitive. Quasi-transitivity is defined as:  $\forall x, y, z \in X : (xPy \wedge yPz) \rightarrow xPz$ . This formal definition plays a role in the argument that follows.

<sup>7</sup>This is Theorem 5\*3 in Sen (1970a) and Theorem 2 in Weymark (1984). A collective choice rule is a function that takes a profile of orderings as its input (one ordering for each individual) and produces a social preference relation as its output. As we have seen, this social preference relation need not be an ordering.

it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from the normative viewpoint”. Aumann is referring to individual preferences in this quote, but the argument that social preferences are incomplete is, if anything, even stronger.<sup>8</sup>

If the social preference relation is incomplete then there exists a pair of alternatives,  $x, y \in X$ , such that  $xNy$ . As noted in footnote 4,  $xNy$  is defined as  $\neg xRy \wedge \neg yRx$  which is equivalent to  $\neg xPy \wedge \neg yPx \wedge \neg xIy$ . In this paper, we call this social non-comparability, “hard” incompleteness. The strong Pareto rule, characterized by Weymark, is a collective choice rule with hard incompleteness. Our objective is to present an alternative approach to the social preference relation that lies in between completeness and hard incompleteness. We call it “soft” incompleteness and we explore the logical space for collective choice rules opened up by the possibility of soft incompleteness.<sup>9</sup>

In order to motivate the idea of soft incompleteness, let us return to the collective choice rules in Table 1. Although the Pareto extension rule and the strong Pareto rule can be given a normative justification by appealing to a certain social choice framework, there is something unsatisfactory about them. If just one person holds an opposing strict preference (i.e. there is “virtual” but not “complete” unanimity on the ranking of a particular pair), then the result is either social indifference or social non-comparability. Contrast this with what we call the pro-

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<sup>8</sup>Sen (1970b) himself developed a model involving “partial” comparability of welfare units. This gives rise to an aggregate quasi-ordering that may be incomplete. In philosophy, incompleteness is often taken to be a consequence of the “small improvement argument”. For an introduction to this literature, see Chang (1997) and Espinoza (2008).

<sup>9</sup>The words “hard” and “soft” (referring to incompleteness) are taken from a paper by Broome (1997) in which he makes the same distinction that we are making. See also Piggins and Salles (2007).

portional rule.<sup>10</sup> Let  $p(x, y)$  denote the degree to which it is true that  $x$  is socially preferred to  $y$ . Loosely speaking, under the proportional rule,  $p(x, y)$  is determined by the proportion of people who prefer  $x$  to  $y$ . In this way, we can think of the underlying social preference relation as taking numerical values weakly between 0 and 1.<sup>11</sup> Interestingly, the proportional rule is still like the strong Pareto rule; if everyone prefers  $x$  to  $y$  then  $p(x, y) = 1$ . However, unlike the standard version of this rule, social non-comparability does not follow when there is virtual unanimity. As we see in Table 1,  $p(x, y) = 2/3$  when individual 2 holds an opposing strict preference. Moving from unanimity (if individual 2 reverses her  $x$  vs.  $y$  preference then  $p(x, y) = 1$ ) to virtual unanimity still induces a transition in the social preference, but it is intuitively “softer”. In our language, the proportional rule is a collective choice rule with soft incompleteness. We say that the incompleteness is “soft” because while it is not true that  $x$  is socially preferred to  $y$  ( $p(x, y) < 1$ ), it is not false either ( $p(x, y) > 0$ ).<sup>12</sup>

Despite their differences, the Pareto extension rule, the strong Pareto rule *and* the proportional rule share one thing in common. All of them are “oligarchic” rules (a term that was first used, we believe, in Gibbard’s seminal paper). Under an oligarchic rule, (i) social preferences respect the unanimous strict preferences

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<sup>10</sup>A formal definition of the proportional rule appears in section 4.

<sup>11</sup>The idea that preferences can be numerically valued (or “fuzzy”) has, of course, been considered before. A sample of the literature is Orlovsky (1978), Ovchinnikov (1981), Basu (1984), Billot (1995), Dutta, Panda, Pattanaik (1986), Dutta (1987), Ponsard (1990), Dasgupta and Deb (1991, 1996, 2001), Ovchinnikov and Roubens (1991, 1992) and Banerjee (1993, 1994). Salles (1998) surveys the field. Throughout this paper, we use the words “valued” and “fuzzy” interchangeably.

<sup>12</sup>Hard incompleteness is inescapable if we accept the principle of bivalence, that for every proposition “ $P$ ”, either “ $P$ ” is true or “ $P$ ” is false. However, many philosophers do not believe in the principle of bivalence. These philosophers appeal to the existence of vagueness. If Jim is borderline thin then it seems that the proposition “Jim is thin” is neither true nor false, violating the principle of bivalence. Keefe and Smith (1997, pp.1-57) is an excellent introduction to the philosophical literature on vagueness. Williamson (1994) and Keefe (2000) are also recommended.

of the oligarchs, and (ii) if any two oligarchs hold opposing strict preferences over a pair of alternatives then neither can be overruled. Part (i) means that when the oligarchs act in concert with one another (in terms of their strict preferences) they can determine the social ranking irrespective of the wishes of everyone else. In other words, they act as a collective like an Arrovian dictator. Part (ii) means that each oligarch possesses a “veto” in that  $xPy$  is not true when an oligarch prefers  $y$  to  $x$ . Under the Pareto extension rule, the strong Pareto rule and the proportional rule, the entire society is the oligarchy.

Gibbard writes (2014, p.5), “A liberum veto oligarchy is a terrible way to govern a large group. If the set of oligarchs is small, the system is as undemocratic as my name for it suggests, for a small group can get its way whatever anyone outside the group wants. If the set of oligarchs is large, the system is more democratic but likely to be paralyzed. Numerous oligarchs will rarely be unanimous, and when they are not, society will have no preference. In the egalitarian case when everyone is an “oligarch”, no one’s preference can ever be over-ridden in forming the social preference, and social decision is paralyzed by the slightest controversy. Thus, however large or small the set of oligarchs, a liberum veto oligarchy is a poor system”.

Despite the fact that the proportional rule is oligarchic, each person exercises only a weak form of veto power, i.e. anyone who prefers  $x$  to  $y$  can ensure that  $p(y, x) < 1$ . However, no one can unilaterally ensure that  $p(y, x) = 0$ . The latter can be thought of as reflecting a stronger form of veto power (and this kind of power is in the hands of all individuals under both the strong Pareto rule and the Pareto extension rule). The proportional rule is a weakly oligarchic rule, with each individual possessing weak veto power. This distinction between a weak and



a strong oligarchy is central to this paper. The example of the proportional rule suggests that a large, weak oligarchy is not necessarily undesirable. Intuitively, it is a less troubling kind of oligarchy than Gibbard's quote would appear to suggest.

Gibbard's original oligarchy theorem says that any collective choice rule that satisfies Arrow's conditions together with the weaker requirement of a quasi-transitive social preference relation (in addition to it being reflexive and complete), must be an oligarchic rule in which the opposing strict preferences of any two oligarchs over a pair of alternatives results in social indifference between that pair.<sup>13</sup> Weymark (1984) calls this an  $\alpha$ -oligarchy. Sen's Pareto extension rule is then characterized by strengthening Arrow's Pareto condition and introducing a natural anonymity requirement.<sup>14</sup> Complementing Gibbard's result, Weymark proves that any collective choice rule that satisfies Arrow's conditions together with the weaker requirement that the social preference relation is a quasi-ordering must be an oligarchic rule in which the opposing strict preferences of any two oligarchs over a pair of alternatives results in social non-comparability between that pair.<sup>15</sup> He calls this a  $\beta$ -oligarchy. Weymark's characterization of the strong Pareto rule then follows from a strengthening of Arrow's Pareto condition and the introduction of a natural anonymity requirement.<sup>16</sup>

As a foundation for these results, Weymark first proves a general oligarchy theorem.<sup>17</sup> For any collective choice rule that satisfies Arrow's conditions along with the requirement that the social preference relation is reflexive and quasi-transitive,

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<sup>13</sup>The Pareto extension rule is one oligarchic rule that satisfies Arrow's axioms when transitivity is weakened to quasi-transitivity. Gibbard's theorem tells us that all such rules must be oligarchic.

<sup>14</sup>This is Theorem 2 in Weymark (1984).

<sup>15</sup>This is Corollary 2 in Weymark (1984).

<sup>16</sup>This is Theorem 3 in Weymark (1984).

<sup>17</sup>This is Theorem 1 in Weymark (1984).

there exists a unique oligarchy. The veto power exercised by each oligarch takes the following form: when an oligarch prefers  $x$  to  $y$  then it is false that  $y$  is socially preferred to  $x$  (i.e.  $p(y, x) = 0$ ). Strengthening the social rationality requirement in this theorem leads either to  $\alpha$ -oligarchy (when we add completeness),  $\beta$ -oligarchy (when we add full transitivity), or dictatorship (when we add both).

The objective of this paper is to prove a counterpart to Weymark's general oligarchy theorem in the framework of a  $[0, 1]$ -valued social preference relation. In fact, Weymark's original result is a special case, obtained by requiring the social preference relation to take values in  $\{0, 1\}$  only. At a technical level, this paper can be viewed as a contribution to the literature on social choice with fuzzy preferences.<sup>18</sup> However, as we will see, there are important differences (both formal and philosophical) between the approach taken in this paper and other approaches considered in this literature. We explain some of these differences later in this section.

Our central result says the following. For any collective choice rule that satisfies Arrow's conditions along with the requirement that the  $[0, 1]$ -valued social preference relation is reflexive and quasi-transitive, there exists an oligarchy. For  $[0, 1]$ -valued relations, our definition of reflexivity is standard but our definition of quasi-transitivity is new, and we argue in favor of it below. In our theorem, just like in the classical case, whenever the oligarchs all strictly prefer  $x$  to  $y$  then  $p(x, y) > 0$ . Further, the veto power exercised by each oligarch in our theorem takes the following form: when an oligarch prefers  $x$  to  $y$  then it is not true that  $y$

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<sup>18</sup>Barrett and Salles (2011) survey the field. A sample of the literature is Barrett, Pattanaik and Salles (1986), Dutta (1987), Ovchinnikov (1991), Banerjee (1994), Billot (1995), Richardson (1998), Dasgupta and Deb (1999), García-Lapresta and Llamazares (2000), Fono and Andjiga (2005), Perote-Peña and Piggins (2007), Duddy, Perote-Peña and Piggins (2010, 2011) and Gibilisco et.al (2014). See also Leclerc (1984, 1991) and Leclerc and Monjardet (1995).

is socially preferred to  $x$  (i.e.  $p(y, x) < 1$ ). On the surface, our theorem looks very similar to Weymark’s. However, our framework permits subtle differences. First, we show that if a certain technical property (called “no zero divisor”) is satisfied by our quasi-transitivity condition, then each oligarch in fact exercises strong veto power (i.e.  $p(y, x) = 0$ ). This is identical to the veto power exercised by oligarchs in Weymark’s theorem. This part of our theorem has a negative interpretation; the oligarchy must be strong. A consequence of this is that the proportional rule (which is weakly oligarchic) must fail to generate quasi-transitive social preferences when the no zero divisor condition is in place.<sup>19</sup>

This oligarchy result is essentially what Barrett, Pattanaik and Salles (1986) prove in their landmark paper on fuzzy social choice.<sup>20</sup> Although they do not mention the concept of a zero divisor in their paper, something very similar to the no zero divisor condition holds under their formulation of quasi-transitivity.<sup>21</sup> This seminal result by Barrett, Pattanaik and Salles has typically been given a negative interpretation; that admitting a  $[0, 1]$ -valued social preference relation still leads to an “undesirable distribution of ‘veto’ power” (page 2 of their paper) in the Arrovian framework. Our initial result reinforces this message for Weymark’s framework; fuzziness does not help us escape from strong oligarchy.

Despite this, our central theorem also shows that if the no zero divisor condition does *not* hold (in which case there exists a zero divisor), then an oligarchy can exist that is not strong. In this case, each oligarch can exercise weak veto power

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<sup>19</sup>The proportional rule satisfies unrestricted domain, independence of irrelevant alternatives and the weak Pareto principle.

<sup>20</sup>This is Theorem 3.5 in that paper.

<sup>21</sup>Note that Barrett, Pattanaik and Salles (1986) use a fuzzy strict social preference relation as their primitive concept, not a weak one. The distinction between transitivity and quasi-transitivity does not, therefore, apply. The transitivity condition they use is Condition 2.3 in their paper.

(like under the proportional rule). This part of our theorem has a more positive interpretation; the oligarchy can be weak and weak oligarchy is less objectionable than strong oligarchy. The proportional rule demonstrates that weak veto power need not be troubling, and the fact that everyone can exercise it can be viewed (arguably) as a requirement of fairness.

Our theorem also allows us to establish two corollaries, which are generalizations of Weymark’s Corollary 1 and Corollary 2.<sup>22</sup> First, if we add the requirement that the  $[0, 1]$ -valued social preference relation is connected, then Arrow’s conditions jointly imply a counterpart of Weymark’s  $\alpha$ -oligarchy.<sup>23</sup> The opposing strict preferences of any two oligarchs over a pair of alternatives results in a social preference over the pair that cannot be “hard” incomplete, nor can there be a strict preference for one alternative over the other (Gibbard’s original “indifference” over the pair is a special case). Second, if we assume that the  $[0, 1]$ -valued social preference relation is fully transitive (in addition to reflexive), then Arrow’s conditions jointly imply a counterpart of Weymark’s  $\beta$ -oligarchy. The opposing strict preferences of any two oligarchs over a pair of alternatives results in a social preference over the pair that cannot correspond to indifference, nor can it correspond to a strict preference for one alternative over the other. Hard incompleteness is possible in this case (and so Weymark’s original “non-comparability” is a special case). Our final result says that if the  $[0, 1]$ -valued social preference relation is reflexive, fully transitive and connected (and the no zero divisor condition is satisfied), then Arrow’s conditions jointly imply a dictator in the sense of Arrow. Therefore, all of our results are consistent with (and generalize) the classical theory of Arrow,

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<sup>22</sup>Weymark (1984, pp.240-241).

<sup>23</sup>We use the term connected here, as we reserve “complete” for relations where, for all  $x, y \in X$ ,  $xPy \vee yPx \vee xIy$ . A  $[0, 1]$ -valued relation can be connected without being complete.

Gibbard and Weymark. Interestingly, for this last result, if we dispense with the assumption of no zero divisor, then anonymous rules are possible that satisfy all of the remaining assumptions.

## 1.1 Literature

In terms of interpretation, we contend that fuzziness gives us a way of thinking about the logic of “softly incomplete” relations. Although this interpretation is new, it is compatible with other justifications for the fuzzy approach that have appeared in the literature. The typical justification refers to the presence of unresolved conflict, i.e. that  $x$  can be preferred to  $y$  to some extent, with  $y$  being preferred to  $x$  to some extent.<sup>24</sup> We contend that this reflects a kind of incompleteness; it is not true that one of these alternatives is preferred to the other, nor is it true that they are equally good. However, this incompleteness is not necessarily “hard”.

Next, a formal difference. It is common in the literature to assume that both individual and social preferences are fuzzy. However, we take individual preferences to be “crisp” orderings of the set of alternatives, and allow only the social preference relation to be fuzzy. Although an argument can be made for treating individual preferences as fuzzy, we wish to remain as close as possible to the literature on revising Arrow’s collective rationality requirement. This literature makes the same assumption about individual preferences as we do. Moreover, we would intuitively expect a fuzzy social preference relation to “smooth” the aggregation of preferences somewhat. One objective of this paper is to explore the extent to which this is possible.

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<sup>24</sup>Barrett, Pattanaik and Salles (1986, p.1).

We consider next definitional differences. Let the social weak preference relation be a function  $r : X \times X \rightarrow [0, 1]$ , where  $X$  is a set of three or more social states (we will retain this assumption about the cardinality of  $X$  throughout). The value  $r(x, y)$  can be interpreted as the degree to which it is true that  $x$  is (socially) at least as good as  $y$ . The function  $r$  is called a fuzzy weak social preference relation (FWSPR). Considerable debate has taken place as to the appropriate properties to impose on this function if it is to be a coherent preference relation. One reasonable requirement is that  $r(x, x) = 1$  for all  $x \in X$ . This is the natural counterpart to the standard reflexivity condition and we adopt it in this paper. Well-known difficulties arise when we try to determine a counterpart to the transitivity condition. These difficulties are compounded by the various ways in which it is possible to factor out of an FWSPR an asymmetric component (corresponding to strict preference) and a symmetric component (corresponding to indifference). Each of these approaches is valid in the sense that the definitions adopted will collapse into the classical ones when the range of the function  $r$  is  $\{0, 1\}$ .

We highlight some of these differences in Table 3 by focusing on Dutta (1987), Banerjee (1994), Richardson (1998) and Dasgupta and Deb (1999). We exclude Barrett, Pattanaik and Salles (1986) from this list since, as noted earlier, they work with an asymmetric relation as a primitive and so the issue of factorization does not arise. That said, we will discuss their transitivity condition in the next section. In Table 3,  $p$  is the asymmetric component of  $r$ .

Assumption	Dutta	Banerjee	Richardson	Dasgupta & Deb
Transitivity	Max-min	Max- $\delta$	Minimal	Weak max-min
$p(x, y) =$	$r(x, y)$ if $r(x, y) > r(y, x)$ , 0 otherwise.	$1 - r(y, x)$	$\text{Max}(r(x, y) - r(y, x), 0)$	Same as Richardson
Connectedness	$r(x, y) + r(y, x) \geq 1$	Same as Dutta	$\text{Max}(r(x, y), r(y, x)) = 1$	Same as Dutta

Table 2: Approaches adopted in the literature.

The names of various transitivity conditions appear in Table 3 and they are defined as follows.

**Max-min transitivity:** for all  $x, y, z \in X$ ,  $r(x, z) \geq \min(r(x, y), r(y, z))$ .

**Max- $\delta$  transitivity:** for all  $x, y, z \in X$ ,  $r(x, z) \geq r(x, y) + r(y, z) - 1$ .

**Minimal transitivity:** for all  $x, y, z \in X$ , if  $r(x, y) = 1$  and  $r(y, z) = 1$  then  $r(x, z) = 1$ .

**Weak max-min transitivity:** for all  $x, y, z \in X$ , if  $r(x, y) \geq r(y, x)$  and  $r(y, z) \geq r(z, y)$ , then  $r(x, z) \geq \min(r(x, y), r(y, z))$ .

Much has been written about the appropriateness of these various formulations. In addition to the papers themselves, Barrett and Salles (2011) and Dasgupta and Deb (1996, 2001) contain important comments.

For our central theorem, in addition to reflexivity, we assume that  $r$  is quasi-transitive and so the asymmetric component  $p$  is transitive. The form of transitivity we adopt is discussed thoroughly in the next section, where we provide an argument in support of it. We should emphasize as well that our central theorem does not require us to commit to a particular method of factoring  $p$  from  $r$ , nor does it require that  $r$  is connected. Instead, in section 2, we propose a set of criteria that we assume any reasonable method of factorization will satisfy. It is

straightforward to verify that all of the methods of factorization described in Table 2 satisfy these criteria. This approach makes our results more general than they otherwise would have been.

## 1.2 Quasi-transitivity

Our central assumption about  $r$  is that it is quasi-transitive. Just as in the crisp case, this means that  $p$  is transitive (not necessarily  $r$ ). However, this raises the question of how we should model the transitivity of  $p$ . One possibility is to adopt a form of max-min transitivity: for all  $x, y, z \in X, p(x, z) \geq \min(p(x, y), p(y, z))$ . This formulation has the advantage of insisting that  $p(x, z) = 1$  whenever  $p(x, y) = 1$  and  $p(y, z) = 1$ . This accords with intuition.

Notice that we can generalize max-min transitivity by requiring  $p$  to be max-star transitive. That is, for all  $x, y, z \in X$ ,

$$p(x, z) \geq p(x, y) \star p(y, z) \tag{1}$$

where the  $\star$  operator is a triangular norm,<sup>25</sup> i.e. a function  $T$  from  $[0, 1]^2$  to  $[0, 1]$  such that for all  $a, b, c \in [0, 1]$  the following conditions are satisfied,

- (i)  $T(a, b) = T(b, a)$ ,
- (ii)  $T(a, T(b, c)) = T(T(a, b), c)$ ,
- (iii)  $T(a, b) \leq T(a, c)$  if  $b \leq c$ ,
- (iv)  $T(a, 1) = a$ .

It is easy to see that max-min transitivity is a special case of a max-star transitive relation, as the function  $T_M(a, b) = \min(a, b)$  for  $a, b \in [0, 1]$  satis-

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<sup>25</sup>Klement, Mesiar and Pap (2000) is a comprehensive account of triangular norms.



fies criteria (i) to (iv). Another special case is Lukasiewicz transitivity: for all  $x, y, z \in X$ ,  $p(x, z) \geq \max(p(x, y) + p(y, z) - 1, 0)$ . It is easy to verify that the function  $T_L(a, b) = \max(a + b - 1, 0)$  for  $a, b \in [0, 1]$  also satisfies criteria (i) to (iv).

In order to justify our approach, it is worth recalling the definition of quasi-transitivity in the crisp case:

$$\text{for all } x, y, z \in X, (xPy \wedge yPz) \rightarrow xPz. \quad (2)$$

We argue that (1) is the natural “fuzzification” of (2). Under (1) it must be the case via property (iv) of a triangular norm that we have  $p(x, z) = 1$  whenever  $p(x, y) = 1$  and  $p(y, z) = 1$ . From an intuitive point of view, the fuzzy strict social preference relation ought to satisfy this; if it is true that  $x$  is socially preferred to  $y$ , and true that  $y$  is socially preferred to  $z$ , then it should also be true that  $x$  is socially preferred to  $z$ .

Other approaches to the transitivity of  $p$  do not share this property. For example, Barrett, Pattanaik and Salles (1986) take the fuzzy strict relation  $p^{BPS}$  as a primitive, and do not derive it from a weak relation. They require, for all  $x, y, z \in X$ ,

$$p^{BPS}(x, y) > 0 \text{ and } p^{BPS}(y, z) > 0 \text{ implies } p^{BPS}(x, z) > 0. \quad (3)$$

However,  $p^{BPS}(x, y) = 1, p^{BPS}(y, z) = 1$  and  $p^{BPS}(x, z) = 0.01$  is transitive under this definition, and this is arguably problematic. It also demonstrates that (3) does not imply (1). To see that (1) does not imply (3) consider Lukasiewicz

transitivity. We have  $p(x, y) = 0.5, p(y, z) = 0.5$  and  $p(x, z) = 0$ . However, this violates (3). The same is true under Tang's (1994) formulation of the transitivity of  $p$ . Tang requires, for all  $x, y, z \in X$ ,

$$\text{if } p^{TA}(x, y) > p^{TA}(y, x) \text{ and } p^{TA}(y, z) > p^{TA}(z, y) \text{ then } p^{TA}(x, z) > p^{TA}(z, x). \quad (4)$$

Formulation (4) suffers from exactly the same intuitive difficulty as (3) and, like (3), is logically independent of (1). Similarly, (4) and (3) are independent.

Given that these three conditions are logically independent, the choice of an appropriate condition must be determined by *a priori* reasoning. We have already noted an intuitive difficulty with (3) and (4), not shared by (1). We now give our main argument in favor of (1). It is based on the logical form of (2).

First, note that we can take  $p(x, y)$  to be the degree of truth of the proposition “ $x$  is socially preferred to  $y$ ”. The set of truth degrees is, therefore,  $[0, 1]$ . Triangular norms (t-norms) are regarded as the natural way of modeling conjunction in infinitely-many valued (i.e. fuzzy) logic (Hájek (1998)). In this logic, if a proposition  $P$  is true to degree 0.8 and proposition  $Q$  is true to degree 0.2, then the conjunction  $P \wedge Q$  is true to degree  $T(0.8, 0.2)$ .

Property (i) of a t-norm is a natural commutativity property, expressing the idea that the order of propositions is immaterial in conjunction. Property (ii) is a natural associativity property, expressing the idea that the order of performing conjunction is immaterial. Property (iii) is a monotonicity property, expressing the idea that increasing the truth degree of a conjunct should not decrease the truth degree of the conjunction. Finally, property (iv) corresponds to regarding the truth degree 1 as full truth, conjunction with which should equal the truth

value of the other conjunct. The conditions imply that  $T(0, a) = 0$  for all  $a \in [0, 1]$ , which corresponds to regarding the truth degree 0 as full falsity, conjunction with which is always fully false.

Notice that  $T$  provides a generalization of classical (2-valued) conjunction, i.e. when  $P$  and  $Q$  take values in  $\{0, 1\}$  then  $T(P, Q)$  will always be equal to the classical valuation. All of these attractive properties justify modeling the antecedent in (2) as a triangular norm in the fuzzy context, and this seems to be widely accepted in the literature on fuzzy logic.

An equally attractive way of modeling the consequent in (2) is provided by the following principle: if  $(P \wedge Q) \rightarrow R$  then  $R$  is at least as true as  $(P \wedge Q)$ . Again, if we require  $P, Q$  and  $R$  to be 2-valued then this principle holds in classical logic. Combining this and the argument above about conjunction leads us to (1). Therefore, we hold that (1) is the natural fuzzification of (2).

Note that it is possible to partition the set of max-star transitive relations into two parts, and this partitioning is critical to the results that we present in this paper. This partitioning is possible because it is also the case that the set of t-norms can be partitioned into two parts. To identify this partitioning property of t-norms, note that the Lukasiewicz t-norm possesses a technical property that is not shared by the min t-norm. It contains a zero divisor.

A triangular norm  $T$  contains a *zero divisor* if there exists an element  $a \in ]0, 1[$  such that for some  $b \in ]0, 1[$  we have  $T(a, b) = 0$ . The element  $a$  is called a zero divisor of  $T$ . For example, 0.4 is a zero divisor of the Lukasiewicz t-norm,  $T_L$ . In fact, for  $T_L$ , the set of zero divisors is  $]0, 1[$ . If there is no  $a \in ]0, 1[$  and no  $b \in ]0, 1[$  such that  $T(a, b) = 0$  then we say that the t-norm  $T$  contains *no zero divisor*.

Unlike the Lukasiewicz t-norm, the min t-norm,  $T_M$ , contains no zero divisor

because  $T_M(a, b) = 0$  only if  $a = 0$  or  $b = 0$ . Two further examples of t-norms will reinforce this partitioning property. The product t-norm is defined by  $T_P(a, b) = a \times b$  and so has no zero divisor. The “drastic” t-norm is defined by  $T_D(a, b) = b$  (if  $a = 1$ ),  $a$  (if  $b = 1$ ), and 0 otherwise.  $T_D$  has a zero divisor. Like  $T_L$ , the set of zero divisors for  $T_D$  is  $]0, 1[$ .<sup>26</sup> Note that all t-norms are equal, i.e.  $T(a, b) = T'(a, b)$ , when  $a, b \in \{0, 1\}$ . This paper can be viewed as an examination of the social choice consequences of the zero divisor/no zero divisor distinction.<sup>27</sup> As a consequence of this partitioning of t-norms, we can also partition the set of max-star transitive relations into two parts; those whose t-norm contains a zero divisor and those whose t-norm does not.

Note that the formulation of transitivity adopted by Barrett, Pattanaik and Salles (condition (3) above) can be interpreted (in the framework of this paper) as a “no zero divisor” requirement. We cannot say that their formulation of transitivity is a special case of ours, as they do not assume that  $p^{BPS}$  is max-star transitive. However, it would be natural for them to do so on foot of the arguments given above. An appropriate reformulation would yield  $p^{BPS}(x, z) \geq p^{BPS}(x, y) \star p^{BPS}(y, z)$  with the requirement that  $\star$  is a t-norm with no zero divisor. Their condition (3) then follows from this.

In a series of important papers, Dasgupta and Deb (1996, 1999, 2001) raise an objection to max-min transitivity, which is a special case of the transitivity

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<sup>26</sup>The drastic t-norm is the smallest t-norm in that, for all  $a, b \in [0, 1]$ ,  $T_D(a, b) \leq T_O(a, b)$  where  $T_O$  is any other t-norm.

<sup>27</sup>The role of these concepts in judgment aggregation is explored in Duddy and Piggins (2013). In Duddy, Perote-Peña and Piggins (2011) we considered these concepts with respect to a fuzzy version of Arrow’s theorem. However, in this paper, we are concerned with Weymark’s theorem, not Arrow’s. Additionally, in our earlier paper we used a strong version of the independence condition. The version we use here is weaker than that one, and is closer to the standard version employed in the social choice literature.

condition presented here. First, we should emphasize that *none* of our results require us to assume max-min transitivity. Second, even if we did assume max-min transitivity (as opposed to the more general max-star transitivity) Dasgupta and Deb's objection would still not apply to the setting of this paper. We briefly elaborate on this point.

Dasgupta and Deb raise the following concern.<sup>28</sup> Suppose that  $r$  is max-min transitive and connected. Assume that  $r(y, z) = 1$  and  $r(z, y) = 0$  and so, intuitively, it is true that society prefers  $y$  to  $z$ . If  $r(x, y) > 0$  then  $r(z, y) \geq \min(r(z, x), r(x, y))$  and so  $r(z, x) = 0$ . Connectedness implies that  $r(x, z) = 1$ . As Dasgupta and Deb observe, if it is true that  $y$  is preferred to  $z$  then we should not be able to deduce from this that  $x$  is preferred to  $z$  simply because  $r(x, y) > 0$ . The value of  $r(x, y)$  may be close to zero, for instance.

Note that Dasgupta and Deb state their objection to max-min transitivity when that condition is applied to  $r$ , and not to  $p$ . For most of our results we make no assumption about the transitivity of  $r$  itself (only that the  $p$  derived from  $r$  is max-star transitive).<sup>29</sup> Moreover, when stating their objection, they assume that  $r$  is connected. If we construct a counterpart to their example for  $p$ , then we can show that  $p(x, y) > 0$ ,  $p(y, z) = 1$  and  $p(z, y) = 0$  implies  $p(z, x) = 0$ . However, it does not follow from this that  $p(x, z) = 1$  as we do not assume that  $p$  is connected (nor do we assume that  $r$  is connected). This objection to max-min transitivity does not hold in our framework.

To outline the paper, section 2 contains the model and the description of the

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<sup>28</sup>This is Proposition 3 in Dasgupta and Deb (1996).

<sup>29</sup>Further, note that the max-star transitivity of  $r$  does not follow from the max-star transitivity of  $p$ . Consider the FWSPR defined as  $r(x, y) = 1, r(y, x) = 0.5, r(y, z) = 1, r(z, y) = 0.5, r(x, z) = 0.7, r(z, x) = 0.2$ . The max-star transitivity of  $r$  is violated in this example (since  $r(x, z) < 1$ ). However,  $p$  is max-star transitive if we use Richardson's factorization and the minimum t-norm.

central axioms. Section 3 contains our results. Section 4 discusses other aggregation rules (including the proportional rule). Section 5 concludes.

## 2 Model

Let  $X$  denote a finite set of at least three social alternatives. Let  $N = \{1, \dots, n\}$  denote the finite set of individuals with  $n \geq 2$ . The individuals in  $N$  are ordered (and “labeled”) from 1 to  $n$ . Let  $\mathbf{R}$  denote the set of all binary relations over  $X$  that are reflexive, transitive and complete. A profile is an  $n$ -tuple  $(R_1, \dots, R_n)$  in  $\mathbf{R}^n$ . We write  $P_i$  for the asymmetric part of individual  $i$ ’s weak preference relation  $R_i$ .

A collective choice rule (a rule)  $\phi$  is a function from the set of admissible profiles  $D$ , a subset of  $\mathbf{R}^n$ , to a set  $C_{T,F}$ .  $C_{T,F}$  is the set of all FWSPRs on  $X$  which are reflexive and quasi-transitive with respect to some specified t-norm ( $T$ ) and some specified method of factorization ( $F$ ). The quasi-transitivity requirement means that the  $p$  relation derived from the method of factorization  $F$  is max-star transitive with respect to the t-norm  $T$ .

A function from  $D$  to  $C_{T_M, \text{Richardson}}$  is an example of a collective choice rule. Another example is a function from  $D$  to  $C_{T_L, \text{Banerjee}}$ . In what follows, we denote  $\phi((R_1, \dots, R_n))$  by  $r$  and  $\phi((R'_1, \dots, R'_n))$  by  $r'$  and so on.

For some of our results, properties of collective choice rules are derived that are independent of the particular co-domain (e.g. they would hold for both  $C_{T_M, \text{Richardson}}$  and  $C_{T_L, \text{Banerjee}}$ ). Other results hold for particular co-domains (e.g. when the t-norm used contains no zero divisor).

The analysis is simplified by the fact that we do not need to commit to any

particular method of factoring  $p$  from  $r$  for our results. However, we assume that whichever method is used satisfies the following requirements:

- (A)  $r(x, y) = 1$  and  $r(y, x) = 0$  implies  $p(x, y) = 1$  and  $p(y, x) = 0$ ,
- (B)  $r(x, y) > 0$  and  $r(y, x) = 0$  implies  $p(x, y) > 0$ ,
- (C)  $p(x, y) = 0$  implies  $r(x, y) \leq r(y, x)$ ,
- (D)  $r(x, y) \geq p(x, y)$  and
- (E)  $r(x, y) \geq r(z, w)$  and  $r(y, x) \leq r(w, z)$  implies  $p(x, y) \geq p(z, w)$ .

It is straightforward to verify that all of the methods of factorization listed in Table 3 satisfy these properties.<sup>30</sup> This means that, when stating our results, the co-domain of  $\phi$  can be simply written  $C_T$  where only the t-norm is specified. It should be understood, however, that some method of factorization is being used to derive  $p$  from  $r$ . That method satisfies conditions (A) to (E).

The assumptions we impose on  $\phi$  are:

**Unrestricted domain.**  $D = \mathbf{R}^n$ .

This condition says that all logically possible orderings of the elements in  $X$  are permissible as inputs into the aggregation exercise.

**Weak Pareto principle.** For all  $x, y \in X$  and all  $(R_1, \dots, R_n) \in D$ ,  $xP_iy$  for all  $i \in N$  implies  $r(x, y) = 1$  and  $r(y, x) = 0$ .

This condition says that if everyone strictly prefers  $x$  to  $y$ , then it is definitely true that society does as well (by virtue of factorization requirement (A)).

**Independence of irrelevant alternatives.** For all  $x, y \in X$  and all  $(R_1, \dots, R_n), (R'_1, \dots, R'_n) \in D$ ,  $[xR_iy \leftrightarrow xR'_iy]$  and  $[yR_ix \leftrightarrow yR'_ix]$  for all  $i \in N$  implies  $r(x, y) = r'(x, y)$  and  $r(y, x) = r'(y, x)$ .

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<sup>30</sup>Note that Banerjee's method of factorization satisfies (D) only if connectedness is assumed.

Given that  $p(x, y)$  is factored out of  $r(x, y)$  and  $r(y, x)$ , this condition says that the degree to which it is true that  $x$  is (socially) preferred to  $y$  depends on (and only on) the orderings of the individuals over  $x$  and  $y$ .<sup>31</sup> Unrestricted domain, the weak Pareto principle and independence of irrelevant alternatives are our basic axioms.

The following definitions play a role in what follows. These are definitions, not axioms.

**Veto power.** An individual  $i$  has veto power if, for all  $x, y \in X$  and all  $(R_1, \dots, R_n) \in D$ ,  $xP_iy$  implies  $p(y, x) < 1$ .

**Strong veto power.** An individual  $i$  has strong veto power if, for all  $x, y \in X$  and all  $(R_1, \dots, R_n) \in D$ ,  $xP_iy$  implies  $p(y, x) = 0$ .

**Decisive.** A group of individuals  $V$  is decisive if, for all  $x, y \in X$  and all  $(R_1, \dots, R_n) \in D$ ,  $xP_iy$  for all  $i \in V$  implies  $p(x, y) > 0$ .

**Oligarchy.** A set of individuals  $V$  is an oligarchy if  $V$  is decisive and every individual in  $V$  has veto power.

**Strong oligarchy.** A set of individuals  $V$  is a strong oligarchy if  $V$  is decisive and every individual in  $V$  has strong veto power.

**$\alpha$ -oligarchy.** A set of individuals  $V$  is an  $\alpha$ -oligarchy if  $V$  is an oligarchy and, for all  $i \in V$ , all  $x, y \in X$  and all  $(R_1, \dots, R_n) \in D$ ,  $xP_iy$  implies  $r(x, y) > \alpha$ .

**Strong  $\alpha$ -oligarchy.** A set of individuals  $V$  is a strong  $\alpha$ -oligarchy if  $V$  is an oligarchy and, for all  $i \in V$ , all  $x, y \in X$  and all  $(R_1, \dots, R_n) \in D$ ,  $xP_iy$  implies  $r(x, y) \geq \alpha$ .

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<sup>31</sup>Translating the independence condition of Duddy, Perote-Peña and Piggins (2011) into the setting of this paper yields: For all  $x, y \in X$  and all  $(R_1, \dots, R_n), (R'_1, \dots, R'_n) \in D$ ,  $[xR_iy \leftrightarrow xR'_iy]$  for all  $i \in N$  implies  $r(x, y) = r'(x, y)$ .



**$\beta$ -oligarchy.** A set of individuals  $V$  is a  $\beta$ -oligarchy if  $V$  is an oligarchy and, for all  $i \in V$ , all  $x, y \in X$  and all  $(R_1, \dots, R_n) \in D$ ,  $xP_i y$  implies  $r(y, x) < 1$ .

**Strong  $\beta$ -oligarchy.** A set of individuals  $V$  is a strong  $\beta$ -oligarchy if  $V$  is an oligarchy and, for all  $i \in V$ , all  $x, y \in X$  and all  $(R_1, \dots, R_n) \in D$ ,  $xP_i y$  implies  $r(y, x) = 0$ .

### 3 Theorems

We note the following condition that a rule  $\phi$  may satisfy.

**Strict-ranking neutrality.** For all  $w, x, y, z \in X$ , and all  $(R_1, \dots, R_n), (R'_1, \dots, R'_n) \in D$ , if  $[wP_i x \text{ or } xP_i w]$  and  $[wP_i x \leftrightarrow yP'_i z]$  for all  $i \in N$  then  $p(w, x) = p'(y, z)$ .

This condition says that if everyone ranks  $w$  and  $x$  at profile  $(R_1, \dots, R_n)$  in the same way as they rank  $y$  and  $z$ , respectively, at profile  $(R'_1, \dots, R'_n)$ , and no one is indifferent between  $w$  and  $x$  at  $(R_1, \dots, R_n)$ , then  $p(w, x)$  equals  $p'(y, z)$ . We can now state the following proposition.

**Lemma 1.** *Irrespective of the  $t$ -norm, any rule  $\phi$  that satisfies unrestricted domain, the weak Pareto principle and independence of irrelevant alternatives satisfies strict-ranking neutrality.*

*Proof.* Let  $\phi$  be a collective choice rule. Case 1: If  $w = y$  and  $x = z$  then the result follows immediately from the fact that  $\phi$  satisfies independence of irrelevant alternatives.

Case 2: Assume that  $w = y$  and  $x \neq z$ . Consider  $(R_1^*, \dots, R_n^*) \in D$ . At this profile, each individual's ranking of  $w$  and  $x$  is the same as in  $(R_1, \dots, R_n)$ , each

individual's ranking of  $w$  and  $z$  is the same as in  $(R'_1, \dots, R'_n)$ , and all individuals strictly prefer  $x$  to  $z$ . By weak Pareto and factorization requirement (A),  $p^*(x, z) = 1$ . Since  $r$  is quasi-transitive, we have  $p^*(w, z) \geq p^*(w, x)$ . Independence of irrelevant alternatives implies that  $p'(w, z) \geq p(w, x)$ . Repeating this argument with everyone strictly preferring  $z$  to  $x$  leads to  $p(w, x) \geq p'(w, z)$  and so  $p(w, x) = p'(w, z)$ .

Case 3: The case where  $w \neq y$  and  $x = z$  is similar to case 2.

Case 4: Assume that  $w, x, y$  and  $z$  are distinct. Consider  $(R_1^*, \dots, R_n^*) \in D$ . At this profile, each individual's ranking of  $w$  and  $x$  is the same as in  $(R_1, \dots, R_n)$  and each individual's ranking of  $y$  and  $z$  is the same as in  $(R'_1, \dots, R'_n)$ . Further, each individual's ranking of  $y$  and  $x$  is identical to their ranking of  $w$  and  $x$  at  $(R_1, \dots, R_n)$ . Finally, all individuals strictly prefer  $w$  to  $y$ , and  $x$  to  $z$ . A similar argument to that in case 1 shows that  $p^*(w, x) \geq p^*(y, x)$  and  $p^*(y, z) \geq p^*(y, x)$ . Independence of irrelevant alternatives implies that  $p(w, x) \geq p^*(y, x)$  and  $p'(y, z) \geq p^*(y, x)$ . Simply repeating this argument with everyone strictly preferring  $y$  to  $w$ , and  $z$  to  $x$  (instead of  $w$  to  $y$ , and  $x$  to  $z$ ) leads to  $p(w, x) \leq p^*(y, x)$  and  $p'(y, z) \leq p^*(y, x)$ , and so  $p(w, x) = p'(y, z)$ .

Case 5: Assume that  $w = z$  and  $x \neq y$ . Consider  $(R_1^*, \dots, R_n^*) \in D$ . At this profile, each individual's ranking of  $w$  and  $x$  is the same as in  $(R_1, \dots, R_n)$ , and each individual's ranking of  $y$  and  $x$  is the same as their ranking of  $w$  and  $x$ . From case 3 we know that  $p^*(w, x) = p^*(y, x)$ . Similarly, we know from case 2 that a profile  $(R''_1, \dots, R''_n)$  in which each individual's ranking of  $y$  and  $x$  is the same as their ranking of  $y$  and  $w$  leads to  $p''(y, x) = p''(y, w)$ . Independence of irrelevant alternatives leads to  $p(w, x) = p'(y, w)$ .

Case 6: The case where  $w \neq z$  and  $x = y$  is similar to case 5.

Case 7: Assume that  $w = z$  and  $x = y$ . Let  $e$  be an alternative distinct from  $w$  and  $x$ . We know that such an alternative exists since  $\#X \geq 3$ . Repeatedly combining cases 2 and 3 yields the desired result that  $p(w, x) = p'(x, w)$ .  $\square$

**Lemma 2.** *Irrespective of the t-norm, if a rule  $\phi$  satisfies unrestricted domain, the weak Pareto principle and independence of irrelevant alternatives then there is an oligarchy. Moreover, if the t-norm has no zero divisor then it is a strong oligarchy.*

*Proof.* Take any two alternatives  $x$  and  $y$  and let  $(R_1, \dots, R_n)$  be a profile where everyone strictly prefers  $x$  to  $y$  and let  $(R'_1, \dots, R'_n)$  be a profile where everyone strictly prefers  $y$  to  $x$ . The weak Pareto principle and factorization requirement (A) implies that  $p(y, x) = 0$  and  $p'(y, x) = 1$ . If we move from  $(R_1, \dots, R_n)$  to  $(R'_1, \dots, R'_n)$  in  $n$  steps, changing the preference of one individual  $i$  from  $R_i$  to  $R'_i$  at each step, there must be a step at which the value of the strict social preference relation over  $(y, x)$  rises above zero. We say that the individual whose switch of preference causes that value to rise above zero is “pivotal”.<sup>32</sup> There are  $n!$  possible ways to sequence the set of  $n$  individuals and so there are  $n!$  paths from  $(R_1, \dots, R_n)$  to  $(R'_1, \dots, R'_n)$ . For each path there must be a pivotal individual. We will show that if an individual is pivotal for any path then that individual has veto power. If the t-norm has no zero divisor then that individual has strong veto power.

Without loss of generality, suppose that individual 2 is pivotal in *one* of the paths from  $(R_1, \dots, R_n)$  to  $(R'_1, \dots, R'_n)$ . There are two adjacent profiles in that

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<sup>32</sup>The idea of a pivotal voter originates with Barberá (1980). This technique has been used in various proofs of Arrow’s theorem, e.g. Geanakoplos (2005). We use the same technique here to prove the existence of an oligarchy.

sequence where individual 2 is the only one whose preference has changed. Let  $\mathbf{R}^{(1)}$  denote the earlier of these two profiles, where individual 2 still strictly prefers  $x$  to  $y$ , and let  $\mathbf{R}^{(2)}$  denote the other, where individual 2 has switched to strictly preferring  $y$  to  $x$ . Let  $N_{xy}$  denote those individuals who strictly prefer  $x$  to  $y$  at both  $\mathbf{R}^{(1)}$  and  $\mathbf{R}^{(2)}$  (they have yet to switch), and let  $N_{yx}$  denote the set of individuals who strictly prefer  $y$  to  $x$  at both of those profiles (they have already switched).

Take any two alternatives  $a$  and  $b$  and construct a profile  $(R_1^*, \dots, R_n^*)$  where every individual in  $N_{xy}$  strictly prefers  $z$  to both  $a$  and  $b$  while every individual in  $N_{yx}$  strictly prefers both  $a$  and  $b$  to  $z$ . Individual 2 strictly prefers  $a$  to  $z$  and  $z$  to  $b$ . The individuals in  $N - \{2\}$  are free to hold any preference over  $a$  and  $b$  (i.e. they can prefer one of these alternatives to the other, or be indifferent between them). Each individual's preference over  $z$  and  $b$  at this profile is the same as his or her preference over  $x$  and  $y$ , respectively, at profile  $\mathbf{R}^{(1)}$ . Hence, by strict-ranking neutrality, we have  $p^*(b, z) = 0$ . Each individual's preference over  $z$  and  $a$  at  $(R_1^*, \dots, R_n^*)$  is the same as his or her preference over  $x$  and  $y$ , respectively, at profile  $\mathbf{R}^{(2)}$ . By strict-ranking neutrality, we have  $p^*(a, z) > 0$ .

Since  $r$  is quasi-transitive, we have  $p^*(b, z) \geq p^*(b, a) \star p^*(a, z)$ . This means that  $0 \geq p^*(b, a) \star p^*(a, z)$  with  $p^*(a, z) > 0$ . Recall that for any t-norm, we have  $\alpha \star 1 = \alpha$  for any  $\alpha \in [0, 1]$ . So if it were the case that  $p^*(b, a) = 1$  then we would have  $0 \geq p^*(a, z)$ . Since this is not true it follows that  $p^*(b, a) < 1$ . Here we see individual 2 exerting veto power; we have  $p^*(b, a) < 1$  regardless of the preferences all other individuals have over  $a$  and  $b$ . Independence of irrelevant alternatives implies that individual 2 can ensure  $p(b, a) < 1$  at any profile at which he or she strictly prefers  $a$  to  $b$ . Since  $a$  and  $b$  are just arbitrary alternatives, individual 2

has veto power in general.

What if the t-norm has no zero divisor? This means that for any numbers  $\alpha$  and  $\beta$  in  $]0, 1]$  we have  $\alpha \star \beta > 0$ . Hence, it follows from  $0 \geq p^*(b, a) \star p^*(a, z)$  and  $p^*(a, z) > 0$  that we have  $p^*(b, a) = 0$ . Individual 2 is exerting strong veto power. Simply repeating the argument above shows that individual 2 has strong veto power in general.

We have seen that if an individual is pivotal in a path from  $(R_1, \dots, R_n)$  to  $(R'_1, \dots, R'_n)$  then that individual has veto power, with that veto power being strong if the t-norm has no zero divisor. Let  $V$  be the set of all such individuals. We will prove that  $V$  is an oligarchy. To do this, we need to show that these individuals are jointly decisive. Consider the profile “between”  $(R_1, \dots, R_n)$  and  $(R'_1, \dots, R'_n)$  with  $R'_i$  for each  $i$  in  $V$  and  $R_i$  for each  $i$  in  $N - \{V\}$ . That is, the profile where just the individuals in  $V$  have switched to preferring  $y$  to  $x$ . Let us denote this profile by  $\mathbf{R}^{(3)}$ . We will prove that the value of the strict social preference relation over  $(y, x)$  at this profile is greater than zero.

By way of contradiction, assume that  $p^{(3)}(y, x) = 0$ . Just as in the argument above, assume that the individuals in  $N - \{V\}$  switch one at a time, from strictly preferring  $x$  to  $y$ , to strictly preferring  $y$  to  $x$ . At some point the value of the strict social preference relation over  $(y, x)$  must rise above zero. From our earlier argument, the person whose switch causes this value to rise must be pivotal. However, this contradicts the assumption that all such individuals are in  $V$ . Therefore,  $p^{(3)}(y, x) > 0$ .

To see that  $V$  is decisive in general, construct a profile  $(R''_1, \dots, R''_n)$  where everyone in  $V$  strictly prefers  $a$  to  $z$  and  $z$  to  $b$ , and the individuals in  $N - \{V\}$  strictly prefer  $z$  to both  $a$  and  $b$ . The individuals in  $N - \{V\}$  are free to have any

preference over  $a$  and  $b$  (i.e. they can strictly prefer one of these alternatives to the other, or be indifferent between them). Each individual's preference over  $z$  and  $a$  at this profile is the same as his or her preference over  $x$  and  $y$ , respectively, at profile  $\mathbf{R}^{(3)}$ . Hence, strict-ranking neutrality implies that  $p''(a, z) > 0$ . By the weak Pareto principle and factorization requirement (A) we have  $p''(z, b) = 1$ . Since  $r$  is quasi-transitive, we have  $p''(a, b) > 0$ . Independence of irrelevant alternatives implies that at any profile where every individual in  $V$  strictly prefers  $a$  to  $b$  the value of the strict social preference relation over  $(a, b)$  is greater than zero. Since  $a$  and  $b$  are just arbitrary alternatives, this argument holds for all pairs of alternatives.  $\square$

The following result is useful in establishing an important property of strong oligarchies.

**Lemma 3.** *If an individual exists with strong veto power then every oligarchy must contain this individual.*

*Proof.* Assume  $i$  has strong veto power. Therefore,  $xP_iy$  implies  $p(y, x) = 0$ . Assume that there exists some oligarchy  $V$  not containing  $i$ . Assume that  $yP_jx$  for all  $j \in V$ . By definition,  $p(y, x) > 0$ . This is a contradiction.  $\square$

An implication of this result is that if a strong oligarchy exists then it must be the unique oligarchy. As we will see, a weak oligarchy (i.e. one that is not strong) need not be unique.

An interpretation of our results thus far: all rules satisfying unrestricted domain, independence of irrelevant alternatives, and the weak Pareto principle are oligarchical. Moreover, if the rule maps to  $C_T$  where the t-norm  $T$  contains no

zero divisor (for example,  $C_{T_M}$  or  $C_{T_P}$ ) then it must be strongly oligarchical, with a unique strong oligarchy.<sup>33</sup> Our next result establishes that if the collective choice rule maps to  $C_{T'}$  where  $T'$  is a t-norm containing a zero divisor, then strong oligarchy can be avoided (i.e. it does not logically follow from the other assumptions). Note that in Lemma 4 we assume that there are at least three individuals.

**Lemma 4.** *Assume that there are at least three individuals. For any t-norm with a zero divisor  $T'$  there exists a rule  $\phi$  that maps into  $C_{T'}$  that is without a strong oligarchy, and which also satisfies unrestricted domain, the weak Pareto principle and independence of irrelevant alternatives.*

*Proof.* We construct a particular rule as follows. For every pair  $(x, y)$ , let  $\pi(x, y)$  denote the number of individuals at the profile who strictly prefer  $x$  to  $y$  divided by the number of individuals who are *not* indifferent between  $x$  and  $y$ . If every individual is indifferent between  $x$  and  $y$  then let  $\pi(x, y)$  be one. Also, let  $\delta$  be a number in the open interval  $]0, 1[$  such that  $\delta \star \delta = 0$ . Such a number exists since the t-norm  $T'$  has a zero divisor. Define

$$r(x, y) = \begin{cases} 1 & \text{if } \pi(x, y) = 1 \\ \delta & \text{if } \frac{1}{2} \leq \pi(x, y) < 1 \\ 0 & \text{if } \pi(x, y) < \frac{1}{2}. \end{cases}$$

It is obvious that this rule satisfies unrestricted domain, the weak Pareto principle and independence of irrelevant alternatives. Moreover, no individual has strong veto power and so there is no strong oligarchy. To see this, take any individual

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<sup>33</sup>Recall that  $T_M$  refers to the minimum t-norm and  $T_P$  refers to the product t-norm.

$i$  and suppose that  $i$  strictly prefers  $x$  to  $y$  while every other individual strictly prefers  $y$  to  $x$ . It follows that  $r(x, y) = 0$  and  $r(y, x) = \delta$  (assuming that there are at least three individuals). By property (B) of our factorization requirements it follows that  $p(y, x) > 0$ . It remains for us to prove that  $r$  is quasi-transitive.

Assume, by way of contradiction, that the rule generates a relation  $r$  with asymmetric part  $p$  such that  $p(x, z) < p(x, y) \star p(y, z)$ . In order for this to hold, it must be the case that  $p(x, y) \star p(y, z) > 0$ . Given that  $\star$  is monotone (condition (iii) of a t-norm) and  $\delta \star \delta = 0$ , it follows that  $p(x, y) > \delta$  or  $p(y, z) > \delta$  (or both). Let us assume, without loss of generality, that  $p(x, y) > \delta$ . Since  $r(x, y) \geq p(x, y)$  (property (D) of our factorization requirements), it follows that  $r(x, y) > \delta$ . From the definition of the rule we can see that this implies  $r(x, y) = 1$ . It follows, again from that definition, that every individual *weakly* prefers  $x$  to  $y$  (i.e.  $xR_iy$  for all  $i \in N$ ). Since individual preferences are transitive, this implies that every individual who weakly prefers  $y$  to  $z$  must also weakly prefer  $x$  to  $z$ . Similarly, everyone who weakly prefers  $z$  to  $x$  must also weakly prefer  $z$  to  $y$ . So we have  $\pi(x, z) \geq \pi(y, z)$  and  $\pi(z, x) \leq \pi(z, y)$ . It follows from the definition of the rule that  $r(x, z) \geq r(y, z)$  and  $r(z, x) \leq r(z, y)$ . This implies that  $p(x, z) \geq p(y, z)$  (by property (E) of our factorization requirements). But we know, by the properties of a t-norm, that  $p(x, y) \star p(y, z)$  cannot be greater than  $p(y, z)$ .<sup>34</sup> Hence  $p(x, z) \geq p(x, y) \star p(y, z)$ . This is a contradiction.  $\square$

Of course, the rule described in Lemma 4 is actually a class of rules, as  $\delta$  can vary from t-norm to t-norm (and even within a t-norm). For example, under  $C_{T_D}$   $\delta$  could be equal to 0.8, whereas this value would not be suitable for  $C_{T_L}$ .<sup>35</sup> Each

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<sup>34</sup>The proof of this is straightforward.

<sup>35</sup>Recall that  $T_D$  refers to the drastic t-norm and  $T_L$  refers to the Lukasiewicz t-norm.



rule in this class is a weakly oligarchic rule. Every subset of  $N$  containing at least half of the individuals is a weak oligarchy. Clearly, if all members of such a subset strictly prefer  $x$  to  $y$  then  $r(x, y) > 0$  and  $r(y, x) = 0$ . Factorization requirement (B) implies  $p(x, y) > 0$ . If anyone in the subset reverses their  $x$  vs.  $y$  preference (and so now strictly prefers  $y$  to  $x$ ) then  $r(x, y) < 1$ . It follows that  $p(x, y) < 1$  by requirement (D).

This class of rules is “universal” in the sense that whichever zero divisor t-norm we use, there exists a value of  $\delta$  that allows the collective choice rule described in the lemma to work. As we will see in the next section, some rules, like the proportional rule, work with some zero divisor t-norms, but not with others. That rule is not, therefore, universal.

To explore the rule in Lemma 4 some more, consider the Condorcet triplet. Suppose that there are three individuals,  $i, j$  and  $k$ , with preferences  $xP_iyP_iz, yP_jzP_jx$  and  $zP_kxP_ky$ . Applying the rule yields  $r(x, z) = 0$  and so  $p(x, z) = 0$  by factorization requirement (D). Similarly,  $\delta \geq p(x, y) > 0$  and  $\delta \geq p(y, z) > 0$  by requirements (B) and (D). Note, however, that the inequality  $0 \geq p(x, y) \star p(y, z)$  still holds and so the derived  $p$  is max-star transitive. Suppose instead that the t-norm does not contain a zero divisor. In this case the triplet does generate a  $p$  that violates max-star transitivity. What lesson are we to draw from this? If there is no zero divisor then  $p(x, y) > 0$  and  $p(y, z) > 0$  imply that  $p(x, z) > 0$ . However, under the requirement of independence of irrelevant alternatives, the profile condition required to raise  $p(x, z)$  above zero may not be satisfied. When there is a zero divisor, then  $p(x, z) > 0$  does not logically follow from  $p(x, y) > 0$  and  $p(y, z) > 0$  and so it is not necessary for that profile condition to be satisfied. Of course,  $p(x, y)$  and  $p(y, z)$  could take values that *require*  $p(x, z) > 0$  even when there is

zero divisor (if  $p(x, y) = 1$  and  $\delta \geq p(y, z) > 0$ , for example). However, in this case, the profile condition required for  $p(x, z) = p(y, z) > 0$  *must* be satisfied. For example,  $xP_iyP_iz, xP_jyP_jz$  and  $zP_kxP_ky$  is a profile that supports this outcome. Note that at this profile strict-ranking neutrality implies that  $p(x, z) = p(y, z)$ .

Lemmas 1 to 4 allow us to state our central theorem.

**Theorem 5.** *Let  $\phi$  be a collective choice rule that satisfies unrestricted domain, the weak Pareto principle and independence of irrelevant alternatives. If  $\phi$  maps into  $C_T$  where  $T$  is any  $t$ -norm with no zero divisor, then there exists a unique, strong oligarchy. However, for any  $t$ -norm  $T'$  with a zero divisor, there exists a rule  $\phi$  that maps into  $C_{T'}$ , and this  $\phi$  is weakly oligarchical, provided that there are at least three individuals.*

It is worth emphasizing that we regard Theorem 5 as a characterization theorem. As explained earlier, the “zero divisor” property partitions the set of max-star transitive relations into two parts. Therefore, we have characterized the set of reflexive and quasi-transitive FWSPRs under which strong oligarchies are logically inevitable (under the assumptions of unrestricted domain, the weak Pareto principle and independence of irrelevant alternatives). We have also characterized the set of reflexive and quasi-transitive FWSPRs under which weak oligarchy is possible (given those same assumptions).

We prove three additional results in this section. For Corollary 6 we require the FWSPR  $r$  to be connected. An FWSPR  $r$  is connected if  $r(x, y) + r(y, x) \geq 1$  for all  $x, y \in X$ . Corollary 6 generalizes Weymark’s Corollary 1 (his  $\alpha$ -oligarchy result).

**Corollary 6.** *Assume that  $\phi$  maps into a set  $C_T$  of FWSPRs that are connected in*

addition to being reflexive and quasi-transitive. If  $\phi$  satisfies unrestricted domain, the weak Pareto principle and independence of irrelevant alternatives then there exists an  $\alpha$ -oligarchy. Moreover, if the  $t$ -norm  $T$  has no zero divisor then it must be a strong  $\alpha$ -oligarchy.

*Proof.* Take any rule  $\phi$  that maps into a set  $C_T$  of FWSPRs that are reflexive, quasi-transitive and connected. If  $\phi$  satisfies unrestricted domain, the weak Pareto principle and independence of irrelevant alternatives then we know by Lemma 2 that there exists an oligarchy. Take any two alternatives  $x$  and  $y$  and an oligarch  $i$ . We know that  $xP_iy$  implies  $p(y, x) < 1$ . Note that  $p(y, x) < 1$  implies that  $r(x, y) > 0$  or  $r(y, x) < 1$ . This follows from factorization requirement (A). Since  $r$  is connected,  $r(y, x) < 1$  implies  $r(x, y) > 0$ . Hence  $xP_iy$  implies  $r(x, y) > 0$ . Therefore, the oligarchy is an  $\alpha$ -oligarchy.

Now let us assume that the  $t$ -norm  $T$  has no zero divisor. By Lemma 2 we know that there is a strong oligarchy. That is,  $xP_iy$  implies that  $p(y, x) = 0$ . This implies  $r(x, y) \geq r(y, x)$  from factorization requirement (C). Connectedness requires that  $r(y, x) \geq 1 - r(x, y)$ , and substituting this into the previous inequality yields  $r(x, y) \geq \frac{1}{2}$ . Therefore, the strong oligarchy is a strong  $\alpha$ -oligarchy.  $\square$

For Corollary 7 we require the FWSPR to be a quasi-ordering. In the crisp case, this means that both the weak social preference relation and the strict are transitive (i.e. we have full transitivity), in addition to the weak relation being reflexive. Therefore, we say that an FWSPR  $r$  is a quasi-ordering if both  $r$  and  $p$  are max-star transitive, and  $r$  is reflexive. Corollary 7 generalizes Weymark's Corollary 2 (his  $\beta$ -oligarchy result).

**Corollary 7.** *Assume that  $\phi$  maps into a set  $C_T$  of FWSPRs that are quasi-*

orderings. If  $\phi$  satisfies unrestricted domain, the weak Pareto principle and independence of irrelevant alternatives then there exists a  $\beta$ -oligarchy. Moreover, if the  $t$ -norm  $T$  has no zero divisor then it must be a strong  $\beta$ -oligarchy.

*Proof.* Take any rule  $\phi$  that maps into a set  $C_T$  of FWSPRs that are quasi-orderings. If  $\phi$  satisfies unrestricted domain, the weak Pareto principle and independence of irrelevant alternatives then we know by Lemma 2 that there exists an oligarchy. Let  $i$  be an oligarch. Assume, by way of contradiction, that there is a profile  $(R_1, \dots, R_n)$  such that  $r(y, x) = 1$  even though  $xP_i y$ . Let us construct another profile  $(R'_1, \dots, R'_n)$ , leaving every individual's preference over  $x$  and  $y$  unchanged so that independence of irrelevant alternatives implies  $r'(y, x) = 1$ . At  $(R'_1, \dots, R'_n)$  every individual strictly prefers  $x$  to  $z$ , and  $i$  strictly prefers  $z$  to  $y$ . The weak Pareto principle implies  $r'(z, x) = 0$  and  $r'(x, z) = 1$ .

Since  $r'(y, x) = 1$  and  $r'(x, z) = 1$ , max-star transitivity of  $r$  implies  $r'(y, z) = 1$ . Also, since  $r'(y, x) = 1$  and  $r'(z, x) = 0$ , max-star transitivity of  $r$  implies  $r'(z, y) = 0$ . Hence  $p'(y, z) = 1$  from factorization requirement (A). However,  $i$  is an oligarch and we have  $zP'_i y$  so we must have  $p'(y, z) < 1$ . This is a contradiction. So there cannot exist a profile  $(R_1, \dots, R_n)$  such that  $r(y, x) = 1$  when  $xP_i y$ .

Now let us assume that the  $t$ -norm  $T$  has no zero divisor. We know by Lemma 2 that there is a strong oligarchy. Let  $i$  be a strong oligarch. Assume, by way of contradiction, that there is a profile  $(R_1^*, \dots, R_n^*)$  such that  $r^*(y, x) > 0$  even though  $xP_i^* y$ . Let us construct another profile  $(R''_1, \dots, R''_n)$ , leaving every individual's preference over  $x$  and  $y$  unchanged so that independence of irrelevant alternatives implies  $r''(y, x) > 0$ . At  $(R''_1, \dots, R''_n)$  every individual strictly prefers  $x$  to  $z$  so that the weak Pareto principle implies  $r''(z, x) = 0$  and  $r''(x, z) = 1$ . Let

us also have  $zP_i''y$ .

Since  $r''(y, x) > 0$  and  $r''(x, z) = 1$ , max-star transitivity of  $r$  implies  $r''(y, z) > 0$ . Also, since  $r''(y, x) > 0$  and  $r''(z, x) = 0$ , max-star transitivity of  $r$  (with no zero divisor) implies  $r''(z, y) = 0$ . Given that  $r''(y, z) > 0$  and  $r''(z, y) = 0$  we have  $p''(y, z) > 0$ . This follows from factorization requirement (B). However,  $i$  has strong veto power and we have  $zP_i''y$  so we must have  $p''(y, z) = 0$ . This is a contradiction. So there cannot exist a profile  $(R_1^*, \dots, R_n^*)$  such that  $r^*(y, x) > 0$  with  $xP_i^*y$ .  $\square$

We complete our analysis in this section by showing how strengthening our social rationality requirements further leads to dictatorship in the sense of Arrow.

**Arrow dictator.** An individual  $i$  is an Arrow dictator if, for all  $x, y \in X$  and all

$$(R_1, \dots, R_n) \in D, xP_iy \text{ implies } p(x, y) = 1 \text{ and } p(y, x) = 0.$$

We say that an FWSPR  $r$  is an ordering if it is a connected quasi-ordering. Again, this is just like in the crisp case. We can now state the following result.

**Corollary 8.** *Assume that  $\phi$  maps into a set  $C_T$  of FWSPRs that are orderings. Further, assume that the t-norm  $T$  has no zero divisor. If  $\phi$  satisfies unrestricted domain, the weak Pareto principle and independence of irrelevant alternatives then there exists a unique Arrow dictator.*

*Proof.* Take any rule  $\phi$  that maps into a set  $C_T$  of FWSPRs that are orderings, and where  $T$  is a t-norm with no zero divisor. If  $\phi$  satisfies unrestricted domain, the weak Pareto principle and independence of irrelevant alternatives, then we know by Lemma 2 that there exists a unique, strong oligarchy. Corollary 6 and Corollary 7 imply that it is both a strong  $\alpha$  and a strong  $\beta$ -oligarchy. Assume that

the oligarchy contains more than one individual,  $i$  and  $j$ , for example. Assume  $xP_iy$  and  $yP_jx$ . Given that  $i$  has strong veto power, this implies that  $p(y, x) = 0$ . Given that  $i$  is both a strong  $\alpha$  and a strong  $\beta$ -oligarch implies that  $r(x, y) \geq 0.5$  and  $r(y, x) = 0$ . However,  $r(y, x) = 0$  contradicts the fact that  $j$  is an  $\alpha$ -oligarch. Therefore, there cannot be more than one oligarch. Call this oligarch  $i$  and so  $xP_iy$  implies  $r(x, y) \geq 0.5$  and  $r(y, x) = 0$ . Connectedness implies  $r(x, y) = 1$ . Therefore, by factorization requirement (A) we have  $p(x, y) = 1$  and  $p(y, x) = 0$ . This proves that  $i$  is an Arrow dictator.  $\square$

As we will see in the next section, when the co-domain of  $\phi$  is  $C_{T_L}$ , then the proportional rule generates an FWSPR that is an ordering and yet is also anonymous (i.e. the names of the individuals do not matter). This rule satisfies unrestricted domain, the weak Pareto principle and independence of irrelevant alternatives. Therefore, the t-norm used in the specification of the transitivity condition can have a dramatic effect on the character of the aggregation rule.

## 4 Other rules

In this section we describe some particular aggregation rules of interest. The class of rules identified in Lemma 4 are fully transitive, in that both  $r$  and  $p$  are max-star transitive. Under the first rule we describe, which we call the “square-root” rule,  $r$  fails to be transitive, but  $p$  is transitive. However,  $p$  would be intransitive if we were to use a t-norm with no zero divisor (the rule is weakly oligarchical and satisfies unrestricted domain, the weak Pareto principle and independence of irrelevant alternatives).

Suppose we use the Lukasiewicz t-norm and Banerjee’s method of factorization

$(p(x, y) = 1 - r(y, x))$ . Let  $w(x, y)$  denote the proportion of individuals who *weakly* prefer  $x$  to  $y$ . The square-root rule says that, for all  $x$  and  $y$ ,  $r(x, y) = \sqrt{w(x, y)}$ . This rule is not fully transitive. Suppose that we have only two people,  $i$  and  $j$ , and that  $xP_iyP_iz$  and  $zP_jxP_jy$ . We have  $r(y, z) = \sqrt{1/2} = 1/\sqrt{2}$ . Similarly, we have  $r(z, x) = 1/\sqrt{2}$  and  $r(y, x) = 0$ . Max-star transitivity implies that  $r(y, x) \geq \max(0, r(y, z) + r(z, x) - 1)$ . Therefore,  $r(y, x) > 0$  which is a contradiction.

However,  $p$  is transitive under this rule. By contradiction, assume that there is a profile such that  $p(x, y) + p(y, z) - 1 > p(x, z)$ . Since we are using Banerjee's method of factorisation, this implies that  $1 - r(y, x) + 1 - r(z, y) - 1 > 1 - r(z, x)$ . Rearranging this, we have  $r(z, x) > r(z, y) + r(y, x)$ . In other words,  $\sqrt{w(z, x)} > \sqrt{w(z, y)} + \sqrt{w(y, x)}$ . However, since individual preferences are transitive, we know that  $w(z, x) \leq w(z, y) + w(y, x)$ . Therefore,  $\sqrt{w(z, x)} \leq \sqrt{w(z, y) + w(y, x)}$ . Since  $\sqrt{w(z, y) + w(y, x)} \leq \sqrt{w(z, y)} + \sqrt{w(y, x)}$  we have  $\sqrt{w(z, x)} \leq \sqrt{w(z, y)} + \sqrt{w(y, x)}$ . This is a contradiction. This proves that the square-root rule is quasi-transitive. Note that this rule satisfies all of the conditions of Corollary 6, and so is an  $\alpha$ -oligarchical rule with each individual an oligarch.

We now define the proportional rule. The proportional rule is a natural rule, well-suited to the framework of this paper. We highlight the relationship that exists between this rule and the Lukasiewicz t-norm. First, we give a definition of the rule. As in Lemma 4, let  $\pi(x, y)$  denote the number of individuals at the profile who strictly prefer  $x$  to  $y$  divided by the number of individuals who are *not* indifferent between  $x$  and  $y$ . If every individual is indifferent between  $x$  and  $y$  then let  $\pi(x, y)$  equal one.

**Proportional rule.** Define  $r^{\text{pro}}(x, y) = \pi(x, y)$  and  $p^{\text{pro}}(x, y) = 1 - r^{\text{pro}}(y, x)$ .

Under this definition,  $r^{\text{pro}}$  is connected (in fact, for all distinct  $x, y \in X$ ,  $r^{\text{pro}}(x, y) + r^{\text{pro}}(y, x) = 1$ ) and  $r^{\text{pro}}(x, y) = p^{\text{pro}}(x, y)$ . It is easy to verify that the values in Tables 1 are consistent with this definition. The proportional rule satisfies unrestricted domain, the weak Pareto principle and independence of irrelevant alternatives. Given that it is a weakly oligarchical rule, Theorem 5 implies that it must violate the max-star transitivity of  $p$  when we use a t-norm with no zero divisor (such as the min t-norm). In fact, this is easily seen from the Condorcet triplet.

We prove that  $p^{\text{pro}} = r^{\text{pro}}$  is transitive under the Lukasiewicz t-norm. By way of contradiction, assume that there is a profile  $(R_1, \dots, R_n) \in D$  such that  $r^{\text{pro}}(x, z) < \max(r^{\text{pro}}(x, y) + r^{\text{pro}}(y, z) - 1, 0)$ . Since  $1 \geq r^{\text{pro}}(x, z) \geq 0$  it must be the case that for the profile to exist we have  $r^{\text{pro}}(x, z) < r^{\text{pro}}(x, y) + r^{\text{pro}}(y, z) - 1$ . Given that  $r^{\text{pro}}(x, y) + r^{\text{pro}}(y, x) = 1$  for all distinct  $x, y \in X$ , we have  $1 - r^{\text{pro}}(z, x) < 1 - r^{\text{pro}}(y, x) + 1 - r^{\text{pro}}(z, y) - 1$ . Rearranging yields  $r^{\text{pro}}(z, y) + r^{\text{pro}}(y, x) < r^{\text{pro}}(z, x)$ . However, since individual preferences are orderings,  $zP_i x$  implies  $zP_i y$  or  $yP_i x$  (or both). Therefore, it is impossible for the right hand side of the inequality to be greater than the left. This is the contradiction.

To show that the proportional rule is not universal (in the sense of section 3), consider the following zero divisor t-norm. It is known as the nilpotent minimum. Let  $T_{NM}(a, b) = \min(a, b)$  if  $a + b > 1$ , and  $T_{NM}(a, b) = 0$  otherwise. The proportional rule fails to be transitive under this norm, as is easily seen from the Condorcet triplet.

Further, the proportional rule satisfies the strong Pareto principle, not just weak Pareto. The strong Pareto principle says (i) that if everyone weakly prefers  $x$  to  $y$  then  $r(x, y) = 1$ , and (ii) that if at least one person strictly prefers  $x$



to  $y$  and everyone weakly prefers  $x$  to  $y$  then  $r(x, y) = 1$  and  $r(y, x) = 0$ . A rule proposed by Ovchinnikov (1991) which, on the surface, looks similar to the proportional rule does not satisfy strong Pareto. As before,  $w(x, y)$  denotes the proportion of individuals who weakly prefer  $x$  to  $y$ . Ovchinnikov's rule is defined by  $r^{\text{Ov}}(x, y) = w(x, y)$ .<sup>36</sup> If there are just two people, and one prefers  $x$  to  $y$  while the other is indifferent, we have  $r^{\text{Ov}}(x, y) = 1$  and  $r^{\text{Ov}}(y, x) = 1/2$  which violates strong Pareto.

## 5 Conclusion

In this paper we have established some general results on the aggregation of individual preferences into a  $[0, 1]$ -valued social preference relation. We have operated with a standard domain of individual preferences, and focussed on the properties of the social relation. It is possible to conduct our analysis with fuzzy individual as well as fuzzy social preferences, but little of extra value would be obtained. Moreover, our approach keeps us closer to the literature on revising Arrow's collective rationality requirement.

Essential to our argument is the max-star formulation of transitivity. We argued in section 1.2 that this is the natural fuzzification of the standard transitivity condition. If this argument is accepted, then it is possible to partition the set of max-star transitive relations into two parts; those whose t-norm contains a zero divisor and those whose t-norm does not. Our central theorem (Theorem 5) shows that in the latter case, the Arrow conditions imply strong oligarchy under the as-

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<sup>36</sup>Ovchinnikov's actual rule is more general than this in that he allows  $r^{\text{Ov}}(x, y) = \psi(w(x, y))$  where  $\psi$  is an automorphism of the unit interval. We are treating this as the identity automorphism.

sumption that the social preference relation is reflexive and quasi-transitive (where these conditions are adapted to the  $[0, 1]$  context). However, in the former case, strong oligarchy need not follow as a matter of logic and weak oligarchy is possible. This opens up the possibility, for example, of using the proportional rule as an attractive collective choice rule.

The first half of Theorem 5 has, therefore, the flavor of an impossibility result, whereas the second half has the flavor of a possibility result. We hope that this second half tempers somewhat the negative interpretation given to results that have been derived when applying fuzzy set theory to social choice. If our argument for max-star transitivity is accepted then everything else is relatively uncontroversial. For example, our results do not require us to commit to any particular method of factorization. Note, as well, that in the non-fuzzy case, Theorem 5 would collapse into Weymark's general oligarchy theorem (his Theorem 1). Similarly, Corollaries 6 and 8 collapse into Gibbard's and Arrow's theorems respectively.

In concluding his classic paper on Arrow's theorem with social quasi-orderings, Weymark observes (1984, p. 245) that "very few of the implications of letting social preferences be incomplete have yet been determined". By re-interpreting a fuzzy social preference relation as reflecting a kind of incompleteness (which we term soft incompleteness), we hope to have made a contribution in this regard. We have focussed on quasi-transitivity as our basic coherence condition. Future work would explore the social choice implications of fuzzy counterparts of acyclicity and Suzumura consistency.

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