The steady-state growth conditions of neoclassical growth model and Uzawa theorem revisited

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The Steady-State Growth Conditions of Neoclassical Growth Model and Uzawa Theorem Revisited*

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Abstract: Based on a neoclassical growth model including adjustment costs of investment, this paper proves that the essential condition for neoclassical model to have steady-state growth path is that the sum of change rate of the *marginal efficiency of capital accumulation* (MECA) and the rate of *capital-augmenting technical change* (CATC) be zero. We further confirm that Uzawa(1961)'s steady-state growth theorem that says the steady-state technical change of neoclassical growth model should exclusively be Harrod neutral, holds only if the marginal efficiency of capital accumulation is constant, which in turn implies that the capital supply should be infinitely elastic. Uzawa's theorem has been misleading the development of growth theorem by not explicitly specifying this prerequisite, and thus should be revisited.

Keywords: Neoclassical Growth Model; Uzawa's Theorem; Direction of Technical Change; Adjustment Cost

JEL Classifications:E13, O30, O41

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1 Introduction

The steady-state growth theorem put forward by Uzawa's (1961) (Uzawa theorem, thereafter) states that for a neoclassical growth model to exhibit steady-state equilibrium, it is required either for the production function to be Cobb-Douglas or equivalently for the technical change to be Harrod-neutral. Uzawa theorem has long been a critical guideline for growth modeling. It is so "authoritative" a judgment that most of macroeconomic models—and an even larger fraction of growth literatures—make strong assumptions about the shape of the production function or the direction of technical change (see Jones, 2005). However, are these constraints really necessary? Theoretically, besides Harrod neutrality, it is also likely for technical change to be Hicks neutral and Solow neutral. As a matter of fact, Hicksian neutrality has been applied far more frequently than Harrod neutrality in empirical studies, for example, in the measurement of total factor productivity (TFP).Further, Harrod neutrality cannot be justified with economic intuition in the real world. For example, the decreasing shares of labor in most countries lead many economists believe that technical change is capital-augmenting or Solow neutrality rather than labor-augmenting (Karabarbounis and Neiman, 2014). To us, what's more important is that, if Harrod-neutral technical change is set just to obtain steady-state path, neoclassical growth model cannot be used to analyze the determination of direction of long-run technical change. As a result, topics such as whether one economy can affect with policy the directions of technical change under different circumstances or not cannot be analyzed with the in-being framework.

Over the last decades, researchers have tries to improve Uzawa theorem by either providing more simplified proofs (see Barro and Sala-i-Martin, 2004, chapter 1; Schlicht, 2006; Acemoglu, 2009,chapter 2) or seeking for more satisfactory economic explanations (see Fellner, 1961; Kennedy, 1964; Samuelson, 1965; Drandakis and Phelps, 1966; Acemoglu, 2003; Jones, 2005; Jones and Scrimgeour, 2008). However, these endeavors have not lessened the doubts we have about this theorem.

Aghion and Howitt(1998, p16) clearly doubt the reasonableness of technical change being restricted to be Harrod neutral. Sato.etc (Sato, Ramachandran, and Lian, 1999; Sato and Ramachandran, 2000) points out that when capital accumulation was the nonlinear function of investment, steady-state technical change can be non-Harrod neutral in neoclassical growth model. Based on Schlicht's(2006) methodology, Irmen(2013) proves that technical change can be capital-augmenting in the steady-state growth by including the adjustment cost of investment.

Similar to the spirit of Irmen(2013), this paper also considers the effects of investment adjustment costs (adjustment cost thereafter) on capital accumulation. What's different is that, we arrive at the steady-state equilibrium of neoclassical growth model in a Ramsey manner, that is, we solve the inter-temporal optimization for both consumers and firms.[3] This paper attempts to clarify that the Harrod neutrality requirement is not indispensable for the steady-state equilibrium of neoclassical growth model: neither Harrod nor non-Harrod neutral technical change can guarantee steady-state equilibrium if adjustment cost included. More precisely, in order to reach steady-state equilibrium, the sum of change rate of marginal efficiency of capital accumulation (*MECA* thereafter) and rate of capital-augmenting technical change (*CATC*) must be equal to zero. According to this new condition, steady-state technical change can be non-Harrod neutral in some cases. Therefore, steady-state equilibrium of neoclassical growth model does not exclude capital-augmenting technical change. On the contrary, it requires that technical change be capital-augmenting when marginal efficiency of capital accumulation decreases.[4]According to the new condition, it is soon obvious that widely-cited Uzawa theorem stands only under the circumstance of constant marginal efficiency of capital accumulation. All of the existing studies that prove Uzawa theorem is based on the neoclassical growth model without adjustment costs, and it is required that marginal efficiency of capital accumulation is constant. To our study, this is only a special case. In order to eliminate the misunderstandings and misuses about Uzawa theorem, we must point out clearly and accurately the additional prerequisite, i.e., that marginal efficiency of capital accumulation is constant. The next question is why steady-state equilibrium requires technical change being exclusively Harrod neutral when marginal efficiency of capital accumulation is constant? We prove that it is because of that capital accumulation has infinite price elasticity.

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[3]Different with Schlicht (2006), this paper not only proves that the steady-state equilibrium including capital-augmenting technical change is consistent with the optimization of micro agents, but also analyzes the effects of marginal efficiency of capital accumulation on Euler equation of intertemporal optimization of consumers.

[4]Though Irmen(2013) also proved that technical change can be capital-augmenting in the steady-state growth and pointed out the conditions to exhibit the steady-state growth including capital-augmenting technical change, but he didn't point out the general condition of neoclassical growth model to exhibit steady-state growth.

The rest of the paper is organized as follows. Section 2 of this paper demonstrates a neoclassical growth model with adjustment costs. Section 3 specifies the differences between steady-state growth and balanced growth based on existing literatures, and provides the conditions of their realization in the neoclassical growth model. Section 4 presents the shortcomings of Uzawa theorem and its amendments. Section 5 concludes.

2 A Neoclassical Growth Model with Adjustment Costs

2.1 Formulation of the Model

Consider a representative consumer in the economy with the usual constant relative risk aversion (CRRA) preferences. The lifetime utility of the representative consumer can be expressed as

$$\int_{t=0}^{\infty} \frac{C(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt, \quad (1)$$

where $C(t)$ is the consumption at the period, θ is the coefficient of relative risk aversion, and ρ is the rate of time preferences.

The production function satisfies the standard neoclassical properties,[5] and allows for both capital-augmenting and labor-augmenting technologies. That is,

$$Y(t) = F[B(t)K(t), A(t)L(t)], \quad (2)$$

where $Y(t), K(t), L(t)$ denotes output, capital stock and labor at the time, $B(t)$ and $A(t)$ refer to capital-augmenting and labor-augmenting technologies. Thus, $B(t)K(t)$ represents the effective capital and $A(t)L(t)$ represents effective labor at the time t. Assuming the initial endowment is no less than one, i.e. $A(0), B(0), L(0) \geq 1$. In addition, both technologies are given exogenously, that is, $\dot{A}(t)/A(t) = a \geq 0$, $\dot{B}(t)/B(t) = b \geq 0$, and growth rates of labor is also assumed to be constant: $\dot{L}(t)/L(t) = n \geq 0$.

The income of a representative consumer includes interest income (rent) and wage, and expenditure contains consumption and investment. The budget constraint of the representative consumer is thus given by

$$C(t) + I(t) = rK + wL, \quad (3)$$

Where r represents the market price of capital, w represents the market wage of labor, and $C(t) > 0, I(t) > 0$.

[5]That is, constant returns to scale (CRS), positive but diminishing marginal products, Inada conditions, and essentiality of each input (Barro and Sala-i-Martin, 2004, chapter 1).

Assuming that investment can change capital stock but requires corresponding adjustment cost. We further suppose that adjustment cost adopts linearly additive format, so the investment function can be expressed as follows:

$$I(t) = I_k(t) + h[I_k(t)], \qquad (4)$$

where $I_k(t)$is the investment that can be used to increase new capital stock, and $h[I_k(t)]$ is the corresponding adjustment cost. Assuming that the cost is positive and rises in an increasing rate with the investment, namely $h[\cdot] > 0, h[0] = 0, \partial h/\partial I_k > 0, \partial^2 h/\partial I_K^2 \geq 0$. Our capital accumulation function can be formulated as follows:

$$\dot{K}(t) = I_k(t) - \delta K(t), \qquad (5)$$

where $K(0) > 0, \delta \geq 0,\ I_k(t) > 0$. Note that one key difference between our paper and existing literature is that what appears in the accumulation function is $I_k(t)$other instead of $I(t)$.

Because of $\partial I(t)/\partial I_k(t) = 1 + \partial h/\partial I_k(t) \geq 1$, the investment $I(t)$ is surely a monotonically increasing function of $I_K(t)$. By equation (4) we can solve implicitly the relationship between new capital and investment, that is, the efficiency function of capital accumulation as follows,

$$I_k(t) = G[I(t)] \leq I(t), \qquad (6)$$

where $G[I(t)]$ is the efficiency function of capital accumulation, which reflects the degree to which investment is converted to new capital goods. By simply inserting formula (6) into (5), we obtain the capital accumulation equation with adjustment costs:

$$\dot{K}(t) = G[I(t)] - \delta K(t). \qquad (7)$$

It is evident from equations (6) and (7) that $\dot{K}(t) = G[I(t)] - \delta K(t) \leq I(t) - \delta K(t)$, which shows that the speed of capital accumulation depends not only on the level of investment$I(t)$, but also on the conversion efficiency from investment to capital $G(\cdot)$. By the property of the inverse function, we obtain the following relations:

$$\begin{cases} G_I \equiv \dfrac{\partial G}{\partial I(t)} = \dfrac{1}{\partial I(t)/\partial I_k(t)} = \dfrac{1}{1 + \partial h/\partial I_k(t)} > 0 \\ G_{II} \equiv \dfrac{\partial^2 G}{\partial I(t)^2} = \dfrac{\partial\{[1 + \partial h/\partial I_k(t)]^{-1}\}}{\partial I(t)} = -\dfrac{\partial^2 h/\partial I_k(t)^2}{[1 + \partial h/\partial I_k(t)]^3} \leq 0 \end{cases} \qquad (8)$$

where G_I and G_{II} refer to the *marginal efficiency of capital accumulation* and its first-order derivative, respectively. Equation group (8) shows that the *marginal efficiency of capital accumulation* diminishes with additional investment that incurs adjustment costs. Intuitively, adjustment costs of investment increases with investment increases, thus the conversion efficiency from investment to capital decreases correspondingly.

2.2 Market Equilibrium

We can analyze this optimization problem by setting up the following Hamiltonian

$$H(C,K,\lambda)=\frac{C(t)^{1-\theta}}{1-\theta}e^{-\rho t}+\lambda(t)\{G[rK(t)+wL(t)-C(t)]-\delta K(t)\}. \quad (9)$$

Where $\lambda(t)$ is covarite. The usual transversality condition is expressed as:

$$\lim_{t\to\infty}\lambda(t)K(t)=0. \quad (10)$$

The first-order conditions of equation (9) thus are:

$$\begin{cases}\frac{\partial H}{\partial C}=C^{-\theta}e^{-\rho t}-\lambda G_I=0\\ \dot{\lambda}=-\frac{\partial H}{\partial K}=-\lambda(G_I r-\delta)\end{cases}. \quad (11)$$

After some mathematical manipulations on the first-order conditions, we obtain the Euler equation:

$$\frac{\dot{C}}{C}=[G_I r-\frac{\dot{G}_I}{G_I}-\rho-\delta]/\theta. \quad (12)$$

According to equation (12), it is noteworthy that the case that the growth rate of consumption is a constant, namely $\dot{C}/C$ is being a constant, cannot lead to the result that capital price r must also be a constant. Mathematically, whether r is a constant or not depends also on whether marginal efficiency of capital accumulation G_I is a constant or not. For example, we can assume that the capital accumulation formula takes the form similar to what has been used by Irmen (2013),

$$\dot{K}(t)=I(t)^{\beta}-\delta K(t), \quad (13)$$

where $0\le\beta\le 1$.

According to our formulation, the marginal efficiency of capital accumulation G_I can be expressed as $G_I=\beta I^{\beta-1}$.When $\beta<1$,Euler equation is $\frac{\dot{C}}{C}=[\beta I^{\beta-1}r-(\beta-1)\frac{\dot{I}}{I}-\rho-\delta]/\theta$. If growth rate of I is above zero, $\beta I^{\beta-1}r$ must be a positive constant in order to ensure consumption growth rate is a constant. $I^{\beta-1}$ will drop as I increases continuously. This means that price of capital r must rise steadily as well. The growth rate can be written as $\frac{\dot{r}}{r}=(1-\beta)\frac{\dot{I}}{I}$. Intuitively, because of the existence of adjustment costs of investment, market price of capital must increase continuously so that the net returns of capital stock is guaranteed to be higher than time discount rate of consumers.

If and only if $\beta=1$, and $G_I=1$, investment can be totally converted to the increased capital stock, which means that adjustment costs is zero. In this case, the

Euler equation is $\dot{C}/C = [r - \rho - \delta]/\theta$. So r must also be a constant if consumption growth rate is a constant. Thus, Euler equation of consumption is closely related to the format of capital accumulation function, and the equation of this format $\dot{C}/C = [r - \rho - \delta]/\theta$ can not be obtained from all kinds of capital accumulation functions.[6]

From production function (2) we can get that the first-order condition of profit maximization of representative firm requires market price of capital being equal to marginal efficiency of capital:

$$r = \partial Y/\partial K = B[\partial Y/\partial(BK)] \qquad (14)$$

Putting equation (14) into (12), we can obtain the Euler equation of Market Equilibrium:

$$\theta\frac{\dot{C}}{C} = G_I B\frac{\partial Y}{\partial(BK)} - \frac{\dot{G}_I}{G_I} - \rho - \delta. \qquad (15)$$

Thus equation (15) is the Euler equation achieved by solving the optimization problems of both consumption and production.

3. Conditions of Steady State Growth and Balanced Growth

3.1 Steady State Growth and Balanced Growth

In existing literature, steady state growth and balanced growth are closely related, and used interchangeably sometimes. However, if we consider the case where the marginal efficiency of capital accumulation is variable, these two concepts and their requirements are quite different. Barro and Sala-i-Martin (2004) defined steady state growth as the case that each endogenous variables within the economy have a constant exponential growth rate, and Jones and Scrimgeour (2008) defined balanced growth as all variables within the economy having a constant exponential growth rate. In Temple (2008) and Acemoglu (2009), however, balanced growth requires not only that each endogenous variable within the economy has a constant exponential growth rate, but also that the capital-output ratio K/Y and interest rate r keep unchanged. Temple (2008) and Acemoglu (2009) have not provided the definitions of steady state growth and balanced growth simultaneously, so we are not clear whether they have recognized the differences or not. Schlicht (2006) considers that Balanced Growth is a special case of exponential growth in the setting that time is continuous, and it not only requires a constant growth rate of each variable, but also that part of the variables keep a special proportional relationship. In other words, the requirement of *Balanced Growth* is more stringent than that of Steady State Growth. Thus, we combine the

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[6] Acemoglu(2009, chapter 15, p520) assumed capital accumulation formula was $\dot{K}(t) = sK(t)$, where s was the parameter given exogenously. As a result, $\dot{C}/C = [r - \rho - \delta]/\theta$ was not valid at that time. So that, Acemoglu still use the equation to prove technical change at that time can still only be Harrod neutral is not valid (proposition15.12).

definition of Schlicht (2006) and adjustment cost in Irmen (2003), and define steady state growth and balanced growth as follows:[7]

Steady State Growth: In addition to adjustment cost h in the foregoing neoclassical growth model, each endogenous variable grows in constant exponential rate.

Balanced Growth: In the foregoing neoclassical growth model, when interest rate r and capital-output ratio K / Y keep unchanged. Each endogenous variable grows in constant exponential rate.

Next, we derive and obtain the conditions of Steady State Growth and Balanced Growth in turn.

3.2 Steady State Growth Conditions

Define $k \equiv BK/AL$ be the radio of effective capital and effective labor, the intensive form of the production function can be rewritten as $f(k) = F(BK/AL, 1)$. This implies that the marginal product of effective capital is $f'(k) = \partial Y/\partial(BK)$. Define $\hat{c} \equiv C/AL$ as the consumption per effective labor. Combined with equations (7) and (15), we get:

$$\begin{cases} \dfrac{\dot{k}}{k} = b + \dfrac{G[I]}{K} - \delta - a - n \\ \dfrac{\dot{\hat{c}}}{\hat{c}} = \dfrac{1}{\theta}\left[G_I B f'(k) - \dfrac{\dot{G}_I}{G_I} - \rho - \delta\right] - a - n \end{cases} . \tag{16}$$

Assume that, after some time t_0, the economy is on its steady-state equilibrium path. According to the definitions above, effective labor, capital and consumption have the same exponential growth rate. We have $\dot{\hat{c}}(t)/\hat{c}(t) = 0$ and $\dot{k}(t)/k(t) = 0$ from the definitions of variables. Combined with equations (16), we get:

$$\begin{cases} \dfrac{G[I]}{K} = a + n + \delta - b \\ G_I B f'(k) - \dfrac{\dot{G}_I}{G_I} = \rho + \delta + \theta(a + n) \end{cases} . \tag{17}$$

From the second equation (17), we get:

$$f'(k^*) = \frac{\rho + \delta + \theta(a + n) + \dot{G}_I/G_I}{G_I B}. \tag{18}$$

Since $\dot{k}(t)/k(t) = 0$ after time t_0, both the left-hand side and right-hand side of equation (18) are positively constant. Thus, after time t_0, $G_I B$ must be a constant. That is, when $t > t_0$, we have:

$$\dot{G}_I/G_I + \dot{B}/B = 0 \tag{19}$$

[7] Of course, as Barro and Sala-i-Martin (2004) pointed out, the state that the growth rates of the variables are constant is represented by Balanced Growth by someone. And Steady State Growth refers specifically to the special case that growth rate is 0.

Because that k* being a constant is a necessary condition for steady-state growth, equation (19) is a necessary condition for neoclassical growth model to exhibit steady-state growth. Equation (19) shows that on the steady-state equilibrium path, the change rate of the marginal efficiency of capital accumulation plus the rate of capital-augmenting technological progress equals 0. Intuitively, consumers can accumulate physical capital through savings. However, due to the increasing adjustment cost, effective capital has a constant exponential growth rate only when the increasing speed of adjustment cost can be offset by capital-augmenting technological progress.

Let $\dot{G}_I/G_I + \dot{B}/B = 0$, we obtain:

$$f'(k^*) = [\rho + \delta + \theta(a+n) - b]/G_I B \tag{20}$$

By equation (7), we obtain the steady-state growth rate of capital:

$$\dot{K}^*/K^* = \dot{I}_K^*/I_K^* = G(I)/K - \delta = a + n - b \tag{21}$$

Combining equations (2) and (3) and $\hat{c} = C/AL$, we obtain the steady-state growth rate of the other endogenous variables as follows:

$$\dot{Y}^*/Y^* = \dot{I}^*/I^* = \dot{C}^*/C^* = a + n \tag{22}$$

Let r and w denote the price of capital and labor, and equal their marginal product in the steady-state equilibrium. Then we have $r = \frac{\partial Y}{\partial K} = Bf'(k^*)$, $w = \frac{\partial Y}{\partial L} = A[f(k^*) - k^* f'(k^*)]$. The radio of factor income is:

$$\frac{rK}{wL} = \frac{k^* f'(k^*)}{f(k^*) - k^* f'(k^*)} \tag{23}$$

Since that k* is a constant, factor income shares keeps unchanged. And when k* is a constant, growth rates of two factors price is:

$$\begin{cases} \dot{r}/r = \dot{B}/B = b \\ \dot{w}/w = \dot{A}/A = a \end{cases} \tag{24}$$

By investment function including adjustment costs, $I(t) = I_k(t) + h[I_k(t)]$ including adjustment costs, we obtain that the growth rate of adjustment costs h will not be a constant when $\dot{I}^*/I^* > \dot{I}_K^*/I_K^*$, just as Irmen (2013) pointes out.

In summary, when $\dot{G}_I/G_I + \dot{B}/B = 0$ is established, growth rates of all endogenous variables in addition to adjustment costs h can be constant, and the neoclassical growth model including the adjustment costs exists steady-state equilibrium. So the formula $\dot{G}_I/G_I + \dot{B}/B = 0$ is both necessary and sufficient conditions for the neoclassical growth model including the adjustment costs to exist steady-state equilibrium.

It is to be noted that the steady-state growth above neither requires Harrod-neutrality nor requires that the form of production function be Cobb-Douglas. It only requires that $\dot{G}_I/G_I + \dot{B}/B = 0$ be established. [8]In contrast, when $\dot{G}_I/G_I < 0$,

[8]We can verify that, even for Cobb-Douglas production function, $\dot{G}_I/G_I + \dot{B}/B = 0$ must also be established in neoclassical growth model to realize steady-state growth.

$\dot{B}/B > 0$ must hold to guarantee steady-state growth path. That is, technological progress will not be under Harrod-neutrality, or that neoclassical growth model cannot exhibit steady-state equilibrium if technological progress is under Harrod-neutrality.

Further, by the Taylor expansion of equation (16) on the steady-state equilibrium $\hat{c}^*, k^*$, we get:

$$\begin{pmatrix} \frac{\dot{k}(t)}{k(t)} \\ \frac{\dot{\hat{c}}(t)}{\hat{c}(t)} \end{pmatrix} \approx \begin{pmatrix} \frac{\partial[\dot{k}(t)/k(t)]}{\partial k}\Big|_{\substack{k=k*\\ \hat{c}=\hat{c}*}}, & -\frac{G_I}{k^*} \\ \frac{1}{\theta} G_I(t_0)B(t_0)f''(k^*), & 0 \end{pmatrix} \begin{pmatrix} k \\ \hat{c} \end{pmatrix} \tag{25}$$

Coefficient determinant of equation (25) is:

$$\det\begin{bmatrix} \frac{\partial[\dot{k}(t)/k(t)]}{\partial k}\Big|_{\substack{k=k*\\ \hat{c}=\hat{c}*}}, & -\frac{G_I}{k^*} \\ \frac{1}{\theta} G_I(t_0)B(t_0)f''(k^*), & 0 \end{bmatrix} = \frac{1}{\theta} G_I(t_0)B(t_0)f''(k^*)\frac{G_I}{k^*} < 0 \tag{26}$$

Therefore, equation (26) shows that if $\dot{G}_I/G_I + \dot{B}/B = 0$ is established, the steady-state equilibrium path of neoclassical growth model is stable at saddle point.

3.3 Balanced Growth Conditions

Balanced growth requires that the growth rate of each variable be a constant, and that interest rate r and capital-output ratio K/Y keep unchanged, that is $\dot{r}/r = 0$. Thus there are more restrictions for Balanced Growth than for Steady State Growth. Since Steady-state growth requires that $\dot{r}/r = \dot{B}/B = b$ (see the equation in (24)), and the definition of Balanced Growth requires $\dot{r}/r = 0$. Thus, Balanced Growth requires that the rate of capital-augmenting technological progress equals 0, that is, $\dot{B}/B = b = 0$. Combining this requirement into our new steady-state conditions, Balanced Growth indeed requires $\dot{G}_I/G_I = 0$. Therefore, in order to get Balanced Growth, it not only requires that the change rate of the marginal efficiency of capital accumulation plus the rate of capital-augmenting technological progress equals 0, but also that both are equal to 0. That is, $\dot{G}_I/G_I = 0$ and $\dot{B}/B = 0$ hold in the Balanced growth.

When $\dot{G}_I/G_I < 0$, the economy is capable of attaining Steady State Growth, but Balanced Growth is impossible. For example, according to the assumption in Irmen (2013), capital accumulation equation (13) requires $\dot{G}_I/G_I = -(1-\beta)(a+n)$ in steady state. In this case, the steady-state equilibrium growth requires $\dot{B}/B = b = (1-\beta)(a+n)$. So, interest rate r would continue to grow in the speed of $(1-\beta)(a+n)$, which is greater than 0, and K/Y continues to decline in the speed of $-(1-\beta)(a+n)$ at this time. Obviously, the economy attains the Steady State Growth rather than Balanced Growth.[9]

[9]Since that Balanced Growth is a special case of Steady State Growth, it is impossible that there only exists Balanced Growth, but not Steady State Growth.

4. Problems and Revisions of Uzawa Steady-State Growth Theorem

Based on our reasoning above, this paper summarizes the following two reasons why Uzawa's theorem is surprising and confusing (Jones and Scrimgeour, 2008; Acemoglu, 2009, Chapter 2).

First, the existing Uzawa's theorem has the defect of hasty generalization, and needs to be amended. Uzawa's theorem states that technological progress must be Harrod-neutrality for a neoclassical growth model to exist steady-state equilibrium. Our reasoning above proves that this requirement is not accurate. Only when the marginal efficiency of capital accumulation keeps constant, that is, $\dot{G}_I/G_I = 0$, the steady-state equilibrium of neoclassical growth model require that the rate of capital-augmenting technological progress equal 0, that is, $\dot{B}/B = 0$. In fact, almost all of existing studies aiming to prove the existing Uzawa steady-state growth theorem assume that the capital accumulation equation is $\dot{K}(t) = I(t) - \delta K(t)$, which is equivalent to the assumption that the marginal efficiency of capital accumulation keeps constant (Uzawa, 1961; Acemoglu, 2003; Barro and Sala-i-Martin, 2004, Chapter 1; Schlicht, 2006; Jones and Scrimgeour, 2008; Acemoglu, 2009, Chapter 2). So, if we relax this restrict and assume that the marginal efficiency of capital accumulation is variable, we can easily find that this restrict is no need for us to achieve Steady-State Growth. That is, the assumption about Harrod-neutrality is unnecessary (Sato, Ramachandran, and Lian, 1999; Sato and Ramachandran, 2000; Irmen, 2013). Since that capital accumulation in the neoclassical growth model can incur adjustment costs and keep marginal efficiency diminishing, [10]that the marginal efficiency of capital accumulation being unchanged is indeed a very special case for neoclassical growth model. The existing Uzawa's theorem does not point out this prerequisite explicitly, and mistakenly treats the particular requirement under special assumption as a common requirement under general assumption. By doing so, the theorem imposes a redundant restriction on the neoclassical growth model, and lead to an unreasonable conclusion.[11] In order to make sure that the existing Uzawa's theorem holds, we must list its prerequisites explicit. The existing Uzawa's theorem should be amended as follows: *if the marginal efficiency of capital accumulation is*

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[10]For example, the capital accumulation function Irmen (2013) used ($\dot{K}(t) = I(t)^{\beta} - \delta K(t)$) is more concise and general.

[11]Balanced Growth of the neoclassical growth model requires technological progress must be Harrod neutral, but Harrod neutral technological progress does not guarantee Balanced Growth. If adjustment costs of investment increases, the marginal efficiency of capital accumulation will be less than 0. Then no matter what direction of technological progress is, it is impossible to achieve Balanced Growth. Therefore, in order to achieve Balanced Growth, neoclassical model must meet two prerequisites: First, the marginal efficiency of capital accumulation keeps unchanged, namely $\dot{G}_I/G_I = 0$; Second, technological progress must be Harrod neutral, i.e. $\dot{B}/B = 0$. Therefore, it cannot be considered that the existing Uzawa steady-state growth theorem puts forward the conditions that the neoclassical growth model achieves Balanced Growth.

constant, the steady-state equilibrium of neoclassical growth model requires technical change must be Harrod neutral.

Secondly, the existing Uzawa's theorem has not provided the intuition why the marginal efficiency of capital accumulation being constant requires the steady-state technical change to be Harrod neutral. By pointing out the prerequisites for the establishment, the revised Uzawa's theorem is already a correct proposition logically. But, why does Steady State Growth require that technical change must be Harrod neutral if the marginal efficiency of capital accumulation keeps unchanged（$\dot{G}_I/G_I = 0$）? Why does capital accumulation function $\dot{K}(t) = I(t) - \delta K(t)$ require that technical change must be Harrod neutral when economy has Steady State Growth? In the following, we will prove that: if $\dot{G}_I/G_I = 0$, the price elasticity of capital accumulation tends to infinity, that is, $\varepsilon_K = \frac{\dot{K}/K}{\dot{r}/r} \to \infty$.

Proof: when $\dot{G}_I/G_I = 0$, $G(\cdot)$ is a linear function of I. Let's assume $G(I) = \varphi I + I_0$, where $\varphi > 0$ and is a constant. Insert G(I) into equation (7) and define $s \equiv I/Y > 0$, we get:

$$\dot{K}/K = \varphi sY/K + I_0/K - \delta \quad (27)$$

Let α denote the output elasticity of capital, then by the neoclassical production function we get that the relation between the average output and market price of capital in a competitive market: $Y/K = r/\alpha$.[12] Insert it into equation (27), we get:

$$\dot{K}/K = \varphi sr/\alpha + I_0/K - \delta \quad (28)$$

Insert equation (28) into the price elasticity of capital accumulation formula, we get:

$$\varepsilon_K = [(\varphi s/\alpha)r + I_0/K - \delta]/(\dot{r}/r) \quad (29)$$

Since $\varphi s/\alpha > 0$, as long as $\dot{r}/r > 0$, r tends to infinity over time, so is ε_K. That is, capital accumulation has infinite price elasticity.

QED .

Therefore, Uzawa's theorem should be revised as follows: *If the price elasticity of capital accumulation is infinite, the steady-state equilibrium of neoclassical growth model requires technical change to be Harrod neutral.*

When capital accumulation has infinite price elasticity, the speed of capital accumulation will response infinitely to any rise of capital price. As a consequence, any rise of capital price is impossible in the long run. In other words, technological progress cannot lead to the rise of marginal productivity and price of capital in the long run. That is, either there is no technological progress, or at least technological

[12] In the previous growth and development literatures, it was always assumed that revenue of capital was used to invest, and income of labor was used to consume. At this time I=rK, capital accumulation function is $\dot{K}/K = r - \delta$. It's very clear that capital accumulation has infinite price elasticity.

change must be Harrod neutral. Because the constant marginal efficiency of capital accumulation means infinite price elasticity of capital accumulation, Steady State Growth requires non-capital-augmenting technological progress. However, it is obvious that Uzawa's theorem is just a special circumstance when capital accumulation has infinite price elasticity. If the marginal efficiency of capital accumulation declines, the price elasticity of capital is limited. This may be more realistic when physical capital accumulation is constrained by non-renewable resources, which means capital accumulation will become more and more difficult. In this case, Steady State Growth does not require technical change must be Harrod neutral, but instead capital-augmenting technological progress to continuously improve the efficiency of capital. Therefore, once the prerequisites are set clearly, the conclusion of Uzawa's theorem not only more reasonable, but provides inspiration for further study on the determinants of technological progress direction. More precisely, the price elasticity of input accumulation will be one of the key factors that affect the direction of technological progress in the steady state.

5 Conclusions

By including adjustment costs into the firm's investment function, we show that, for a neoclassical growth model to exhibit steady-state growth, it is just required that the sum of the growth rate of marginal efficiency of capital accumulation and the rate of capital-augmenting technical change equals zero. According to this condition, when marginal efficiency of capital accumulation decreases, the steady-state equilibrium of neoclassical growth model requires technical change be non-Harrod neutral, or more precisely, should include capital-augmenting technical change. The proposition that steady-state technical change of neoclassical growth model should exclusively be Harrod neutral holds only if the marginal efficiency of capital accumulation is constant. Thus, the existing version of Uzawa's theorem must be revised.

Why is steady-state technical change Harrod neutral only when the marginal efficiency of capital accumulation keeps constant? This paper proves that constant marginal efficiency of capital accumulation implies infinite price elasticity, that is, capital accumulation will respond infinitely to the increased capital price. Thus in a long term, technical change cannot lead to the increasing of marginal efficiency and price of capital. That is, technical change can only be labor-augmenting, namely Harrod neutral, rather than capital-augmenting.

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