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Abstract

This note corrects Blanchard and Kahn's (1980) solution for a linear dynamic rational expectations model with one state variable and one control variable.

1. Introduction

Blanchard and Kahn (1980) [BK] derived the solution for an important class of dynamic linear rational expectations models. The BK algorithm has become a standard tool for economic modelers.[2] In general, the model solution is analytically intractable. However, as pointed out by BK, models with one predetermined and one non-predetermined endogenous variable can be handled analytically (which may facilitate an intuitive understanding of the model solution). That special case is important as it includes, e.g., the basic Real Business Cycle model with fixed labor (King and Rebelo (1999)). In this note, we show that the formula provided by BK, for this key special case, is incorrect; we also provide the correct formula.

2. A linear rational expectations model with one state and one control

Consider the following model (the notation follows BK):

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[2]The BK algorithm is e.g. often used to solve linearized dynamic general equilibrium models, the workhorses of modern macroeconomics (King and Rebelo (1999)). Google Scholar records 2342 cites (03/2016) for the BK paper.

$$\begin{bmatrix} x_{t+1} \\ E_t p_t \end{bmatrix} = A \begin{bmatrix} x_t \\ p_t \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} Z_t , \tag{1}$$

where x_t is a predetermined variable ('state'), and p_t is a non-predetermined variable ('control'). Z_t is a (kx1) vector of exogenous variables. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a (2x2) matrix, and γ_1, γ_2 are (1xk) vectors. Let λ_1, λ_2 be the eigenvalues of *A*, and let $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ be the matrix of eigenvectors of *A*, i.e. $AB = BJ$, with $J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Finally, let $C \equiv B^{-1}$, $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$. Note that $A = BJC$.

Proposition 1 of BK (p.1308) shows that model (1) has a unique (non-exploding) solution if and only if one eigenvalue of *A* is outside the unit circle, while the other eigenvalue is inside (or on) the unit circle. Assume that this condition holds, and let $|\lambda_1| \le 1, |\lambda_2| > 1.$ BK (p.1309) state that then the solution of (1) is:

$$x_t = \lambda_1 x_{t-1} + \gamma_1 Z_{t-1} + \mu \sum_{i=0}^{\infty} \lambda_2^{-i-1} E_{t-1} Z_{t+i-1} , \tag{2}$$

$$p_t = a_{12}^{-1} [(\lambda_1 - a_{11}) x_t + \mu \sum_{i=0}^{\infty} \lambda_2^{-i-1} E_t Z_{t+i}], \tag{3}$$

$$\text{with} \quad \mu \equiv (\lambda_1 - a_{11}) \lambda_1 - a_{12} \lambda_2. \tag{4}$$

Comment: When μ is defined by (4), then $\mu \sum_{i=0}^{\infty} \lambda_2^{-i-1} E_t Z_{t+i-1}$ is a (kx1) vector. This implies that (2) and (3) cannot hold for k>1 when quantity μ is given by (4) (as x_t and p_t are scalars). This suggests that the formula for μ is incorrect.

We now derive the correct formula for μ.

Equations (2) and (3) are special cases of the solution for general linear difference models (with arbitrary numbers of states and controls) given in Proposition 1 of BK (p.1308). For convenience, the general case is shown in the Appendix. The general solution for predetermined variable x_t indicates that the correct expression for the vector μ in equation (2) above is

$$\mu = -(b_{11} \lambda_1 c_{12} + b_{12} \lambda_2 c_{22}) c_{22}^{-1} (c_{21} \gamma_1 + c_{22} \gamma_2).$$

Write this as $\mu = \phi_1\gamma_1 + \phi_2\gamma_2$, with $\phi_1 \equiv -(b_{11}\lambda_1 c_{12}c_{22}^{-1}c_{21} + b_{12}\lambda_2 c_{21})$ and $\phi_2 = -(b_{11}\lambda_1 c_{12} + b_{12}\lambda_2 c_{22})$. $A = BJC$ implies that $a_{11} = b_{11}\lambda_1 c_{11} + b_{12}\lambda_2 c_{21}$ and $a_{12} = b_{11}\lambda_1 c_{12} + b_{12}\lambda_2 c_{22}$. We thus see that $\phi_2 = -a_{12}$ holds. Substituting $b_{12}\lambda_2 c_{21} = a_{11} - b_{11}\lambda_1 c_{11}$ into the definition of ϕ_1 gives $\phi_1 = -(b_{11}\lambda_1 c_{12}c_{22}^{-1}c_{21} + a_{11} - b_{11}\lambda_1 c_{11})$ $\Leftrightarrow$ $\phi_1 \equiv -(a_{11} + b_{11}\lambda_1[c_{12}c_{22}^{-1}c_{21} - c_{11}])$. $B = C^{-1}$ implies $b_{11} = c_{22}/(c_{11}c_{22} - c_{12}c_{21})$ and $c_{12}c_{22}^{-1}c_{21} - c_{11} = -b_{11}^{-1}$. Thus $\phi_1 = \lambda_1 - a_{11}$. In summary, the <u>correct</u> formula for μ is:

$$\mu \equiv (\lambda_1 - a_{11})\gamma_1 - a_{12}\gamma_2. \tag{5}$$

It can readily be verified from the general solution for the non-predetermined variable p_t (see Appendix) that equation (3) above holds when the quantity μ is defined by (5).

References

Blanchard, O. and C. Kahn, 1980. The Solution of Linear Difference Models Under Rational Expectations. Econometrica 48, 1305-1311.

King, R. and S. Rebelo, S., 1999. Resuscitating Real Business Cycles, in: Handbook of Macroeconomics (J. Taylor and M. Woodford, eds.), Elsevier, Vol. 1B, pp. 927-1007.

Appendix

Blanchard and Kahn (1980): the general model

Consider the model

$$\begin{bmatrix} X_{t+1} \\ E_t P_t \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_t , \tag{A1}$$

where X_t is an nx1 vector of predetermined variable, and p_t is an mx1 vector of non-predetermined variables; Z_t is a (kx1) vector of exogenous variables. A is an (n+m)x(n+m) matrix, and γ is an (n+m)xk matrix. Consider the Jordan canonical form $A = C^{-1} J C$, where C and J are (n+m)x(n+m) matrices. Let the diagonal elements of J (i.e. the eigenvalues of A) be ordered by increasing absolute value. Let $\bar{n}$ ($\bar{m}$) denote the number of eigenvalues of A that are on or inside the unit circle (outside the unit circle). Partition J as $J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$, where J_1 and J_2 are matrices of dimensions $(\bar{n} \text{x} \bar{n})$ and $(\bar{m} \text{x} \bar{m})$, respectively. Decompose C, $B \equiv C^{-1}$ and γ as $C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ and $\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$, where $C_{11}, C_{12}, C_{21}, C_{22}$ are matrices of dimensions $(\bar{n} \text{x} n)$, $(\bar{n} \text{x} m)$, $(\bar{m} \text{x} n)$ and $(\bar{m} \text{x} m)$, respectively; $B_{11}, B_{12}, B_{21}, B_{22}$ have dimensions $(n \text{x} \bar{n})$, $(n \text{x} \bar{m})$, $(m \text{x} \bar{n})$ and $(m \text{x} \bar{m})$, respectively, while γ_1 and γ_2 have dimensions $(\bar{n} \text{x} k)$ and $(\bar{m} \text{x} k)$, respectively. Proposition 1 in Blanchard and Kahn (1980) states that the model (A1) has a unique (non-explosive) solution if and only if the number of non-predetermined variables equals the number of eigenvalues of A outside the unit circle: $m = \bar{m}$. If that condition is met, then the solution is:

$$X_t = B_{11} J_1 B_{11}^{-1} X_{t-1} + \gamma_1 Z_{t-1} - (B_{11} J_1 C_{12} + B_{12} J_2 C_{22}) C_{22}^{-1} \sum_{i=0}^{\infty} J_2^{-i-1} (C_{21}\gamma_1 + C_{22}\gamma_2) E_{t-1} Z_{t+i-1},$$

$$P_t = -C_{22}^{-1} C_{21} X_t + C_{22}^{-1} \sum_{i=0}^{\infty} J_2^{-i-1} (C_{21}\gamma_1 + C_{22}\gamma_2) E_t Z_{t+i}.$$