Unified quasi-maximum likelihood estimation theory for stable and unstable Markov bilinear processes

Abdelhakim Aknouche

University of Science and Technology Houari Boumediene, Qassim university

2015

Unified quasi-maximum likelihood estimation theory for stable and unstable Markov bilinear processes

Abdelhakim Aknouche*

March 9, 2015

Abstract

A unified quasi-maximum likelihood (QML) estimation theory for stationary and nonstationary simple Markov bilinear ($SMBL$) models is proposed. Such models may be seen as generalized random coefficient autoregressions ($GRCA$) in which the innovation and the random coefficient processes are fully correlated. It is shown that the QML estimate ($QMLE$) for the $SMBL$ model is always asymptotically Gaussian without assuming strict stationarity, meaning that there is no knife edge effect. The asymptotic variance of the $QMLE$ is different in the stationary and nonstationary cases but is consistently estimated using the same estimator. A perhaps surprising result is that in the nonstationary domain, all $SMBL$ parameters are consistently estimated in contrast with unstable $GARCH$ and $GRCA$ models where the $QMLE$ of the conditional variance intercept is inconsistent. As a result, strict stationarity testing for the $SMBL$ is studied. Simulation experiments and a real application to strict stationarity testing for some financial stock returns illustrate the theory in finite samples.

Keywords: Markov bilinear process, random coefficient process, stability, instability, Quasi-maximum likelihood, knife edge effect, strict stationarity testing.

AMS Subject Classification (2000) Primary 62M10; Secondary 62M04.

Proposed running head: Inference for stable and unstable $SMBL$

*University of Science and Technology Houari Boumediene, Algeria, e-mail: aknouche_ab@yahoo.com

1. Introduction

Over the past few decades, there has been a very abundant literature on *conditional mean and volatility* (CMV) models because of their ability to describe both level and variability of a broad array of observed time series such as financial stock returns (see e.g. Engle, 1982; Nicholls and Quinn, 1982; Weiss, 1984; Bollerslev, 1986; Taylor, 1986; Tsay, 1987, 2002; Holan et *al*, 2010; Francq and Zakoïan, 2010). An essential common specification for such models is that their conditional mean and conditional variance are stochastic, generally function of the past of the observed phenomenon, from which they can be evaluated for level and volatility predictions. In particular, when the conditional variance (resp. conditional mean) is non-stochastic the CMV model is simply called purely conditional mean (resp. purely conditional volatility) model. Among the most popular specifications are: the $ARMA$ model with a $GARCH$ innovation ($ARMA$-$GARCH$), the $ARMA$ model with a stochastic volatility ($ARMA$-SV) innovation, the $ARMA$ model with a bilinear innovation ($ARMA$-BL), the subdiagonal bilinear (BL) model, the conditionally heteroskedastic $ARMA$ ($CHARMA$) model, the double autoregressions (DAR) (Ling and Li, 2008; Chen et *al*, 2014) and the random coefficient autoregression (RCA) with a special case in which the random coefficient is finite-valued like the Markov mixture autoregression (MAR) and the threshold autoregression (TAR). In fact, all aforementioned models are subclasses of the general class of weak (or nonlinear) $ARMA$ models (e.g. Amendola and Francq, 2009) which consist of $ARMA$ equations with uncorrelated, but not necessarily independent innovations. When the innovation is independent, the $ARMA$ model is simply called strong (or linear).

While $(G)ARCH$-type models seem to have dominated the literature on CMV models, a renewed interest has been paid recently to RCA models which were initially considered as purely conditional mean models. The most popular RCA model is an autoregressive equation driven by an independent and identically distributed (iid) innovation where the corresponding autoregressive coefficient is an iid process. Statistical analysis for RCA models usually assumes that the random coefficient and the innovation processes are uncorrelated (e.g. Nicholls and Quinn, 1982; Feigin and Tweedie, 1985; Schick, 1996; Aue et *al*, 2006;

Berkes et *al*, 2009; Aue and Horváth, 2011; Aknouche, 2013 etc.). The case of RCA models in which the random coefficient and the innovation are permitted to be correlated (which is called generalized RCA) has seen less interest despite its practical importance as it allows more flexible volatility representation including asymmetry in level and volatility (e.g. Hwang and Basawa, 1998; Zhao and Wang, 2012, 2013; Truquet and Yao, 2012; Aknouche, 2015a). A special case of generalized RCA models in which the random coefficient and the innovation are fully correlated is the $SMBL(1)$ given by the stochastic equation

$$y_t = (\phi + \beta\varepsilon_t)\, y_{t-1} + \varepsilon_t, \qquad t \in \mathbb{N}^*, \tag{1.1}$$

where y_0 is a given random variable and

$$\{\varepsilon_t, t \in \mathbb{N}\} \text{ is an independent and identically distributed } (iid) \text{ process} \tag{\textbf{A1}}$$

with

$$E(\varepsilon_1) = 0 \text{ and } E\left(\varepsilon_1^2\right) = \sigma^2 > 0, \tag{\textbf{A2}}$$

$\mathbb{N}^* = \mathbb{N} - \{0\}$ being the set of positive integers. The $SMBL$ equation introduced by Tong (1981) is related to many volatility models. Indeed, it can be seen as a double autoregression, a subdiagonal bilinear model or a generalized RCA in which the random coefficient is fully correlated with the innovation. Probabilistic properties of the $SMBL$ model (1.1) such as stationarity, ergodicity, geometric ergodicity and some Markov chain solidarity properties have been extensively studied (e.g. Tong, 1981; Feigin and Tweedie, 1985; Goldie and Maller, 2000; Cline and Pu, 2002; Meyn and Tweedie, 2009) where some singular properties on the stochastic unit root ($\phi = 1$) have been revealed (Cline and Pu, 2002). Some generalizations of the original formulation have been developed and their structures have been studied (e.g. Ferrante et *al*, 2003; Cline, 2007). However, statistical properties of the $SMBL$ model have received much less interest. Indeed, at the knowledge of the author, it appears that the first work concerning estimation of the $SMBL$ model (1.1) is the one of Aknouche (2013, Section 3.2) who studied asymptotic distribution of the $QMLE$ for a nonstationary $SMBL$ model (1.1) with $\beta = 1$. It turns out that the $QMLE$ coincides with the two-stage weighted least squares estimate, $2SWLSE$ (cf. Aknouche, 2012a, 2012b, 2013, 2014, 2015a).

This Chapter proposes a unified quasi-maximum likelihood (QML) estimation theory for stable and unstable $SMBL$ models (assuming β known, say $\beta = 1$), i.e.

$$y_t = (\phi + \varepsilon_t)\, y_{t-1} + \varepsilon_t, \qquad t \in \mathbb{N}^*. \tag{1.2}$$

Our aim is threefold. i) First, under stability of (1.2) with respect to strict stationarity, we show that the $QMLE$ of $(\phi, \sigma^2)'$ is asymptotically Gaussian when $\phi \neq 1$ and inconsistent in the stochastic unit root case $\phi = 1$. The result is valid regardless of any moment requirement on the observed process $\{y_t, t \in \mathbb{N}\}$. ii) Second, we shall see that when $\phi \neq 1$, the $QMLE$ of $(\phi, \sigma^2)'$ is always $\sqrt{n}$-Gaussian irrespective of the strict stationarity requirement, meaning that there is no knife edge effect (Lumsdaine, 1996; Jensen and Rahbek, 2004) for the $SMBL$ model. The corresponding asymptotic distribution is different in the stationary and nonstationary cases but is consistently estimated using the same estimator. This parallels recent results by Aue and Horváth (2011) for $RCA(1)$ models (see also Hwang and Basawa, 2005) and Francq and Zakoïan (2012, 2013a) for $GARCH(1,1)$ and asymmetric $GARCH\,(1,1)$ models, respectively. iii) Third, as an application of the proposed unified estimation theory, strict stationarity testing for the $SMBL$ equation is studied. A perhaps surprising result is that all parameters of the $SMBL$ are consistently estimated when $\phi \neq 1$. This is in contrast with $RCA(1)$ and $GARCH(1,1)$ models where the $QMLE$ of the conditional variance intercept is inconsistent in the nonstationary domain (see Aue and Horváth, 2011; Francq and Zakoïan, 2012; Aknouche, 2013, 2015a). Moreover, in the nonstationary stochastic unit root case, the $QMLE$ is still consistent when (1.2) is appropriately started.

The rest of this Chapter proceeds as follows. In Section 2, stability of the $SMBL$ equation (1.1) with arbitrary β is revisited. A necessary and sufficient condition for the $SMBL$ model with $\phi \neq 1$ to admit a unique (asymptotically) strictly stationary solution is provided. Furthermore, various modes of divergence to infinity in the nonstationary case are also presented. Assuming strict stationarity of the model and $\beta = 1$, Section 3 establishes asymptotic normality of $QMLE$ of $(\phi, \sigma^2)'$ when $\phi \neq 1$ and its inconsistency when $\phi = 1$. In Section 4, a consistent estimate for the asymptotic variance of the $QMLE$ in both strict stationarity and non strict stationarity situations is given when $\phi \neq 1$. Then, a unified asymptotic theory for the $QMLE$ in both stable and unstable situations is provided. Section 5 proposes strict stationarity and non-strict stationarity testing procedures for the $SMBL$. In particular, consistent interval estimates for the parameters are given without assuming strict stationarity. In addition, a simulation study is conducted to assess the theory in finite samples and application to strict stationarity testing for some financial stock returns is provided. Finally, Section 6 concludes.

2. Stability analysis for the $SMBL$ model

Existence of a nonanticipative strictly stationary solution of (1.1) is now considered. It is clear that studying stationarity of the one-sided equation (1.1) translates immediately into studying stationarity of the two-sided version of (1.1)

$$y_t = (\phi + \beta\varepsilon_t)\, y_{t-1} + \varepsilon_t, \qquad t \in \mathbb{Z}, \tag{2.1}$$

($\mathbb{Z}$ being the set of integers). This of course implies that y_0 in (1.1) should have the same distribution as the unique strictly stationary solution of (2.1) when exists. Otherwise, we rather speak about the unique "*asymptotically*" strictly stationary solution $\{y_t, t \in \mathbb{N}\}$ in the sense that the limiting distribution of y_t (as $t \to \infty$) exists and is unchanged whatever the distribution of y_0. For both situations we are then interested in the stability of (1.1) with respect to strict stationarity. Notice that the finite second moment assumption **A2** on the innovation sequence $\{\varepsilon_t, t \in \mathbb{Z}\}$ is unnecessary for that purpose and is replaced by the weaker condition of finiteness of absolute log-moments:

$$E\left(\left|\log\left|\varepsilon_1\right|\right|\right) < \infty \text{ and } E\left(\left|\log\left|\phi + \beta\varepsilon_1\right|\right|\right) < \infty. \tag{\textbf{A3}}$$

For model (2.1), assumption **A1** corresponds to

$$\{\varepsilon_t, t \in \mathbb{Z}\} \text{ is an independent and identically distributed } (iid) \text{ process.} \tag{\textbf{A1}'}$$

The following result, by now classical, provides a necessary and sufficient condition for strict stationarity of model (2.1) and hence stability of (1.1) with respect to strict stationarity.

Theorem 2.1 *Consider equation* (2.1) *subject to* ($\mathbf{A1'}$) *and* ($\mathbf{A3}$).

i) (2.1) *admits a unique nonanticipative strictly stationary and ergodic solution given by*

$$y_t = \sum_{j=0}^{\infty} \prod_{i=0}^{j-1} (\phi + \beta \varepsilon_{t-i})\, \varepsilon_{t-j}, \qquad t \in \mathbb{Z}, \tag{2.2}$$

where the latter series converges absolutely almost surely if

$$\gamma := E\left(\log |\phi + \beta \varepsilon_1|\right) < 0. \tag{2.3}$$

ii) Conversely, if (2.1) *admits a nonanticipative strictly stationary solution,* $\phi \neq 1$ *and*

$$P\left(\varepsilon_1 = c\right) < 1, \tag{2.4}$$

for all $c \in \mathbb{R}$*, then* (2.3) *holds.*

iii) If $\phi = 1$ *then model* (2.1) *is not irreducible in the sense of Bougerol and Picard* (1992) *and the Markov chain* $\{y_t, t \in \mathbb{N}\}$ *defined by* (1.1)*, starting from* y_0*, is not ergodic. Moreover, under* (2.3) *and assuming that* $E\left(\log\left|y_0 + \frac{1}{\beta}\right|\right) < \infty$*,*

$$y_t \underset{t\to\infty}{\overset{a.s.}{\to}} -\tfrac{1}{\beta}. \tag{2.5}$$

Proof i) The first part of the theorem follows from Brandt (1986).

ii) It is clear that when $\phi \neq 1$ and ε_1 is nondegenerate (i.e. (2.4) holds), model (2.1) is irreducible in the sense of Bougerol and Picard (1992), so ii) follows from their Theorem 2.5.

iii) If $\phi = 1$ then (2.2) reduces to $y_t = -\frac{1}{\beta}$ for all $t \in \mathbb{Z}$ (cf. Cline and Pu, 2002, p. 287) which is a strictly stationary solution whatever $\gamma \in [-\infty, +\infty)$. Considering the one-sided equation (1.1), if $y_0 = -\frac{1}{\beta}$ $a.s.$ then $y_1 = (1 + \beta\varepsilon_1)\, y_0 + \varepsilon_1 = -\frac{1}{\beta}$ $a.s.$, so any subspace of $\mathbb{R}$ containing $\left\{-\frac{1}{\beta}\right\}$ is invariant under (2.1). This shows that model (2.1) is not irreducible in the sense of Bougerol and Picard (1992). Moreover, non ergodicity of the Markov chain $\{y_t, t \in \mathbb{N}\}$ starting from y_0 has been proved by Cline and Pu (2002, Theorem 2.1). Finally, (2.5) trivially follows when $y_0 = -\frac{1}{\beta}$ $a.s.$ since as seen above $y_t = -\frac{1}{\beta}$ $a.s.$ for all $t \in \mathbb{N}$. If, however, $P\left(y_0 \neq -\frac{1}{\beta}\right) < 1$, then iterating (1.1) with $\phi = 1$, we have

$$\begin{aligned} y_t + \tfrac{1}{\beta} &= (1+\beta\varepsilon_t)\, y_{t-1} + \varepsilon_t + \tfrac{1}{\beta} \\ &= (1+\beta\varepsilon_t)\left(y_{t-1} + \tfrac{1}{\beta}\right) \\ &= \ldots = \prod_{k=1}^{t} (1+\beta\varepsilon_k)\left(y_0 + \tfrac{1}{\beta}\right), \qquad t \in \mathbb{N}^*. \end{aligned}$$

From the strong law of large numbers and under (2.3) and $E\left(\log\left|y_0 + \frac{1}{\beta}\right|\right) < \infty$, it follows that

$$\begin{aligned} \frac{1}{t}\log\left|y_t + \tfrac{1}{\beta}\right| &= \frac{1}{t}\sum_{k=1}^{t}\log|1+\beta\varepsilon_k| + \frac{1}{t}\log\left|y_0 + \tfrac{1}{\beta}\right| \\ \underset{t\to\infty}{\overset{a.s.}{\to}} \gamma &< 0. \end{aligned}$$

This shows that $\log\left|y_t + \frac{1}{\beta}\right| \underset{t\to\infty}{\overset{a.s.}{\to}} -\infty$, so $\left|y_t + \frac{1}{\beta}\right| \underset{t\to\infty}{\overset{a.s.}{\to}} 0$ proving (2.5). ■

So in all, assuming (**A1**), (**A3**), $\phi \neq 1$ and (2.4), condition (2.3) is the necessary and sufficient condition for model (2.1) to have a unique (nonanticipative) strictly stationary and ergodic solution. For $\phi = 1$ the $SMBL$ model (1.1) is (tied-down line) degenerate in the sense of Goldie and Maller (2000, p. 1199) and Babillot et *al* (1997, p. 480) since when $c = -\frac{1}{\beta}$, then $c = (1+\beta\varepsilon_t)\, c + \varepsilon_t$ for all $t \in \mathbb{N}$. As a consequence, if $y_0 = -\frac{1}{\beta}$ *a.s.* then $y_t = -\frac{1}{\beta}$ *a.s.* for all $t \in \mathbb{N}$. However, when $\gamma < 0$, even though the Markov chain $\{y_t, t \in \mathbb{N}\}$ is not ergodic, it has a unique stationary distribution given by $\delta_{-\frac{1}{\beta}}$ (Cline and Pu, 2002), where δ_x denotes the degenerate distribution concentrated at x.

Existence condition of a unique strictly stationary solution to (2.1) with a finite second moment is given by the following result.

Theorem 2.2 *Under (**A1′**), (**A3**) and* (2.4)*, equation* (2.1) *admits a unique nonanticipative strictly stationary solution given by* (2.2) *with $E\left(y_1^2\right) < \infty$, where the corresponding series converges a.s. and in mean square, if and only if*

$$\phi^2 + \beta^2\sigma^2 < 1. \tag{2.6}$$

Proof See e.g. Nicholls and Quinn (1982) and Feigin and Tweedie (1985) for the sufficiency part. For the necessity part, assume that $\{y_t, t \in \mathbb{Z}\}$ is a stationary solution to (2.1) with $E\left(y_1^2\right) < \infty$. Then, from (2.1) we have

$$y_t^2 = \phi^2 y_{t-1}^2 + \varepsilon_t^2 \left(1 + \beta y_{t-1}\right)^2 + 2\phi y_{t-1} \left(1 + \beta y_{t-1}\right) \varepsilon_t,$$

so

$$E\left(y_t^2\right) = \phi^2 E\left(y_{t-1}^2\right) + \sigma^2 E\left(1 + \beta y_{t-1}\right)^2,$$

and

$$\left(1 - \left(\phi^2 + \beta^2 \sigma^2\right)\right) E\left(y_t^2\right) = \sigma^2,$$

implying that (2.6) should be satisfied. ■

It is clear that (2.6) implies (2.3), so the second-order stationarity domain is strictly included in the strict stationarity one. Therefore, there is non-invariance of the stability domains. When the strict stationarity condition (2.3) is dropped, the two-sided equation (2.1) has no interest, but asymptotic behavior of the solutions of the one-sided equation (1.1) could be studied. The following result (cf. Aknouche, 2013 when $\beta = 1$) gives the limit of y_t as $t \to \infty$ under each one of the following instability conditions

$$\gamma = 0. \tag{2.7a}$$

$$\gamma > 0. \tag{2.7b}$$

Theorem 2.3 *Consider model* (1.1) *subject to* (**A1**) *and* (**A3**).
i) Under $\phi \neq 1$ *and* (2.7*a*),

$$|y_t| \underset{t\to\infty}{\overset{p}{\to}} \infty. \tag{2.8a}$$

ii) Under $\phi \neq 1$ *and* (2.7*b*), *there exists* $0 < \lambda < 1$ *such that*

$$\lambda^t |y_t| \underset{t\to\infty}{\overset{a.s.}{\to}} \infty. \tag{2.8b}$$

iii) Under $\phi = 1$, (2.7*b*) *and* $P\left(y_0 \neq -\frac{1}{\beta}\right) = 1$, *there exists* $0 < \lambda < 1$ *such that*

$$\lambda^t |y_t| \underset{t\to\infty}{\overset{a.s.}{\to}} \infty. \tag{2.9}$$

Proof See Aknouche (2013, Lemma 1) when $\beta = 1$. ■

Thus, the asymptotic behavior of y_t can be summarized for the two cases $\phi \neq 1$ and $\phi = 1$ as follows:

i) When $\phi \neq 1$:

- Under stability ($\gamma < 0$) (Vervaat, 1979),

$$y_t \underset{t\to\infty}{\overset{\mathcal{L}}{\to}} \sum_{j=1}^{\infty} \prod_{i=1}^{j-1} (\phi + \beta\varepsilon_i)\, \varepsilon_j.$$

- Under instability ($\gamma = 0$),

$$|y_t| \underset{t\to\infty}{\overset{p}{\to}} \infty.$$

- Under strict instability ($\gamma > 0$),

$$\lambda^t |y_t| \underset{t\to\infty}{\overset{a.s.}{\to}} \infty \quad \text{for some } 0 < \lambda < 1.$$

ii) When $\phi = 1$:

- Under stability ($\gamma < 0$) and $E\left(\log\left|y_0 + \frac{1}{\beta}\right|\right) < \infty$,

$$y_t \underset{t\to\infty}{\overset{a.s.}{\to}} -\tfrac{1}{\beta}.$$

- Under strict instability ($\gamma > 0$) and $P\left(y_0 \neq -\frac{1}{\beta}\right) = 1$,

$$\lambda^t |y_t| \underset{t\to\infty}{\overset{a.s.}{\to}} \infty, \quad \text{for some } \ 0 < \lambda < 1.$$

- If $P\left(y_0 = -\frac{1}{\beta}\right) = 1$ then whatever $\gamma \in [-\infty, +\infty)$,

$$y_t = -\tfrac{1}{\beta}, \ \ a.s.\ \forall t \in \mathbb{N}.$$

iii) The case $\phi = 1$, $\gamma = 0$ and $P\left(y_0 \neq -\frac{1}{\beta}\right) < 1$ remains open.

3. QML estimation for stable $SMBL$ models

In the sequel, we consider model (1.2) (i.e. with $\beta = 1$) started with an arbitrary random variable y_0 and subject to (**A1**), (**A2**), the fourth moment assumption

$$E\left(\varepsilon_1^4\right) < \infty, \qquad \textbf{(A4)}$$

and the non-degeneracy condition

$$P\left(\varepsilon_1 = 0\right) = 0. \tag{\textbf{A5}}$$

The parameter of the model about which we will make inference is denoted by $\theta = (\phi, \sigma^2)'$. Notice that the conditional mean and conditional variance of the $SMBL$ process given the past information are respectively given by $E\left(y_t/\mathcal{F}_{t-1}\right) = \phi y_{t-1}$ and $Var\left(y_t/\mathcal{F}_{t-1}\right) = \sigma^2\left(1+y_{t-1}\right)^2$, where $\mathcal{F}_t$ denotes the σ-algebra generated by $\{\varepsilon_s, s \leq t\}$. Observe that the $SMBL$ model is with an endogenous volatility since $Var\left(y_t/\mathcal{F}_{t-1}\right)$ depends on $\{y_t, t \in \mathbb{N}\}$.

Therefore, given a series $y_1, y_2, ..., y_n$ generated from (1.2) the logarithmed (Gaussian) quasi-likelihood function of θ conditional on y_0 is written as follows

$$\log l = -\frac{1}{2}\sum_{t=1}^{n}\log\left(\sqrt{2\pi}\sigma\left|1+y_{t-1}\right|\right) - \frac{1}{2\sigma^2}\sum_{t=1}^{n}\frac{\left(y_t - \phi y_{t-1}\right)^2}{\left(1+y_{t-1}\right)^2}. \tag{3.1}$$

Thanks to the form of the log-likelihood in (3.1), the $QMLE$, $\widehat{\theta}'_{QML} = \left(\widehat{\phi}_{QML}, \widehat{\sigma}^2_{QML}\right)$, which is the maximizer of (3.1), is given in a closed form

$$\widehat{\phi}_{QML} = \left(\sum_{t=1}^{n}\frac{y_{t-1}^2}{\left(1+y_{t-1}\right)^2}\right)^{-1}\sum_{t=1}^{n}\frac{y_{t-1}y_t}{\left(1+y_{t-1}\right)^2}. \tag{3.2}$$

$$\widehat{\sigma}^2_{QML} = \frac{1}{n}\sum_{t=1}^{n}\frac{\left(y_t - \widehat{\phi}_{QML}y_{t-1}\right)^2}{\left(1+y_{t-1}\right)^2}. \tag{3.3}$$

It turns out that the $QMLE$ defined by (3.2)-(3.3) is also the two-stage weighted least squares estimate ($2SWLSE$) in which the weight is the inverse of the conditional variance (see Aknouche, 2013). Consistency and asymptotic normality of the $QMLE$ given by (3.2)-(3.3) are now established under in particular the stability condition (2.3).

Theorem 3.1 *Let* $\{y_t, t \in \mathbb{N}\}$ *be the unique (asymptotically) strictly stationary solution of model* (1.2) *which is subject to* (**A1**), (**A2**), (2.3) *and* (**A5**) *and let* $\widehat{\phi}_{QML}$ *and* $\widehat{\sigma}^2_{QML}$ *given by* (3.2)-(3.3). *Then:*

i) When $\phi \neq 1$,

$$\widehat{\phi}_{QML} \underset{n\to\infty}{\overset{a.s.}{\to}} \phi. \tag{3.4a}$$

$$\widehat{\sigma}^2_{QML} \underset{n\to\infty}{\overset{a.s.}{\to}} \sigma^2. \tag{3.4b}$$

ii) When $\phi = 1$ and $E(\log|y_0+1|) < \infty$, $\widehat{\theta}_{QML}$ is inconsistent.

Proof i) From (3.2) and (1.2) we have

$$\widehat{\phi}_{QML} - \phi = \left(\frac{1}{n}\sum_{t=1}^{n}\frac{y_{t-1}^2}{(1+y_{t-1})^2}\right)^{-1}\frac{1}{n}\sum_{t=1}^{n}\frac{y_{t-1}}{1+y_{t-1}}\varepsilon_t. \tag{3.5}$$

So (3.4a) follows from the ergodic theorem, **(A5)** and the fact that $E(\varepsilon_1) = 0$. To show (3.4b), we rewrite (3.3) as follows:

$$\begin{aligned}
\widehat{\sigma}_{QML}^2 &= \frac{1}{n}\sum_{t=1}^{n}\frac{\left(y_t - \phi y_{t-1} - \left(\widehat{\phi}_{QML} - \phi\right)y_{t-1}\right)^2}{(1+y_{t-1})^2} \\
&= \frac{1}{n}\sum_{t=1}^{n}\frac{(y_t - \phi y_{t-1})^2}{(1+y_{t-1})^2} + \frac{\left(\widehat{\phi}_{QML} - \phi\right)^2 y_{t-1}^2}{(1+y_{t-1})^2} - \frac{2(y_t - \phi y_{t-1})\left(\widehat{\phi}_{QML} - \phi\right)y_{t-1}}{(1+y_{t-1})^2} \\
&= \frac{1}{n}\sum_{t=1}^{n}\varepsilon_t^2 + \frac{1}{n}\sum_{t=1}^{n}\frac{\left(\widehat{\phi}_{QML} - \phi\right)^2 y_{t-1}^2}{(1+y_{t-1})^2} - \frac{2}{n}\sum_{t=1}^{n}\frac{\left(\widehat{\phi}_{QML} - \phi\right)y_{t-1}\varepsilon_t}{(1+y_{t-1})^2}.
\end{aligned} \tag{3.6}$$

Using (3.4a) and the Césaro lemma, the last two terms of the right hand side of (3.6) converge $a.s.$ to zero. Thus, (3.4b) follows from the strong law of large numbers and (**A2**).

ii) When $y_0 = -1$ $a.s.$, we have seen that $y_t = -1$ $a.s.$ for all $t \in \mathbb{N}$. So $\widehat{\theta}_{QML}$ given by (3.2)-(3.3) is undefined and hence inconsistent. If, however, $P(y_0 = -1) < 1$ then under (2.3) and $E(\log|y_0+1|) < \infty$, result (2.5) clearly holds, so $\widehat{\theta}_{QML}$ is still inconsistent. ■

Now we establish asymptotic normality of $\widehat{\theta}_{QML}$ under in particular the stability condition (2.3). For an asymptotically stationary process $\{z_t, t \in \mathbb{N}\}$ denote by $E_\infty(z_t) = \lim_{t\to\infty} E(z_t)$. Let

$$\Sigma = \begin{pmatrix} \sigma^2\left(E_\infty\left(\frac{y_t^2}{(1+y_t)^2}\right)\right)^{-1} & E(\varepsilon_1^3)E_\infty\left(\frac{y_t}{1+y_t}\right)\left(E_\infty\left(\frac{y_t^2}{(1+y_t)^2}\right)\right)^{-1} \\ E(\varepsilon_1^3)E_\infty\left(\frac{y_t}{1+y_t}\right)\left(E_\infty\left(\frac{y_t^2}{(1+y_t)^2}\right)\right)^{-1} & Var(\varepsilon_1^2) \end{pmatrix}. \tag{3.7}$$

In order that Σ exists, y_t^2 should be non-null almost surely as $t \to \infty$. This holds if we assume that $\{\varepsilon_t, t \in \mathbb{N}\}$ is non-degenerate in the sense of **(A5)**. Thus, we have the following result.

Theorem 3.2 *Let $\{y_t, t \in \mathbb{N}\}$ be the unique (asymptotically) strictly stationary solution to equation* (1.2) *which is subject to* (**A1**), (**A2**), (**A4**), (2.3), **(A5)** *and* $\phi \neq 1$. *Then,*

$$\sqrt{n}\left(\widehat{\theta}_{QML} - \theta\right) \underset{n\to\infty}{\overset{\mathcal{L}}{\to}} N\left(0, \Sigma\right), \tag{3.8}$$

where Σ *is given by* (3.7).

Proof First, we rewrite (3.5) and (3.6) as follows

$$\sqrt{n}\left(\widehat{\phi}_{QML} - \phi\right) = \left(\frac{1}{n}\sum_{t=1}^{n}\frac{y_{t-1}^2}{\left(1+y_{t-1}\right)^2}\right)^{-1}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{y_{t-1}\varepsilon_t}{1+y_{t-1}}. \tag{3.9}$$

$$\begin{aligned}\sqrt{n}\left(\widehat{\sigma}_{QML}^2 - \sigma^2\right) &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left(\varepsilon_t^2 - \sigma^2\right) + \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{\left(\widehat{\phi}_{QML} - \phi\right)^2 y_{t-1}^2}{\left(1+y_{t-1}\right)^2} \\ &\quad - \frac{2}{\sqrt{n}}\sum_{t=1}^{n}\frac{\left(\widehat{\phi}_{QML} - \phi\right) y_{t-1}\varepsilon_t}{\left(1+y_{t-1}\right)^2}.\end{aligned} \tag{3.10}$$

Using strong consistency of $\widehat{\phi}_{QML}$ (see (3.4a)) we have (see e.g. Nicholls and Quinn, 1982; Aknouche, 2015a)

$$\widehat{\phi}_{QML} - \phi = n^{-\frac{1}{2}}O_p\left(1\right),$$

so from Césaro lemma and the ergodic theorem (3.10) becomes

$$\sqrt{n}\left(\widehat{\sigma}_{QML}^2 - \sigma^2\right) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left(\varepsilon_t^2 - \sigma^2\right) + o_p\left(1\right). \tag{3.11}$$

In vector form, (3.9) and (3.11) may be expressed as follows

$$\sqrt{n}\left(\widehat{\theta}_{QML} - \theta\right) = \begin{pmatrix} \frac{1}{n}\sum_{t=1}^{n}\frac{y_{t-1}^2}{\left(1+y_{t-1}\right)^2} & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{y_{t-1}\varepsilon_t}{1+y_{t-1}} \\ \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left(\varepsilon_t^2 - \sigma^2\right) \end{pmatrix} + o_p\left(1\right). \tag{3.12}$$

Using the ergodic theorem we have

$$\begin{pmatrix} \frac{1}{n}\sum_{t=1}^{n}\frac{y_{t-1}^2}{\left(1+y_{t-1}\right)^2} & 0 \\ 0 & 1 \end{pmatrix} \underset{n\to\infty}{\overset{a.s.}{\to}} \begin{pmatrix} E_{\infty}\left(\frac{y_t^2}{\left(1+y_t\right)^2}\right) & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.13}$$

On the other hand, the sequence $\{\mathbf{W}_t, t \in \mathbb{N}\}$ defined by $\mathbf{W}_t = \left(\frac{y_{t-1}\varepsilon_t}{1+y_{t-1}}, \varepsilon_t^2 - \sigma^2\right)'$ is clearly a bounded Martingale difference with respect to $\{\mathcal{F}_t, t \in \mathbb{N}\}$. Moreover, using again the ergodic theorem it follows that

$$\begin{aligned}
\frac{1}{n}\sum_{t=1}^{n} E\left(\mathbf{W}_t\mathbf{W}_t'/\mathcal{F}_{t-1}\right) &= \frac{1}{n}\sum_{t=1}^{n}\begin{pmatrix} \frac{\sigma^2 y_{t-1}^2}{(1+y_t)^2} & \frac{y_{t-1}E\left(\varepsilon_1\left(\varepsilon_1^2-\sigma^2\right)\right)}{1+y_{t-1}} \\ \frac{y_{t-1}E\left(\varepsilon_1\left(\varepsilon_1^2-\sigma^2\right)\right)}{1+y_{t-1}} & E\left(\varepsilon_1^2-\sigma^2\right)^2 \end{pmatrix} \\
&\underset{n\to\infty}{\overset{a.s.}{\to}} \begin{pmatrix} \sigma^2 E_\infty\left(\frac{y_t^2}{(1+y_t)^2}\right) & E\left(\varepsilon_1^3\right)E_\infty\left(\frac{y_t}{1+y_t}\right) \\ E\left(\varepsilon_1^3\right)E_\infty\left(\frac{y_t}{1+y_t}\right) & E\left(\varepsilon_1^2-\sigma^2\right)^2 \end{pmatrix} := \Omega.
\end{aligned}$$

Therefore, the Martingale central limit theorem yields

$$\frac{1}{\sqrt{n}}\left(\sum_{t=1}^{n}\frac{y_{t-1}\varepsilon_t}{1+y_{t-1}}, \sum_{t=1}^{n}\left(\varepsilon_t^2-\sigma^2\right)\right)' \underset{n\to\infty}{\overset{\mathcal{L}}{\to}} N(0,\Omega). \tag{3.14}$$

So result (3.8) follows while combining (3.12)-(3.14). ■

4. Unified QML estimation theory for stable and unstable $SMBL$ models

Having established asymptotics for the $QMLE$ in the stable case, we now use asymptotic results by Aknouche (2013, Section 3.2) for the $QMLE$ in the unstable $SMBL$ case, giving unified theory for the $QMLE$ irrespective of stability issues.

Theorem 4.1 *Let* $\{y_t, t \in \mathbb{N}\}$ *be a solution to equation* (1.2) *which is subject to* (**A1**), (**A2**), (**A4**) *and* (**A5**).

i) If $\phi \neq 1$,

$$\widehat{\theta}_{QML} \underset{n\to\infty}{\overset{a.s.}{\to}} \theta \qquad \text{if} \quad E\left(\log|\phi+\varepsilon_1|\right) \neq 0. \tag{4.1a}$$

$$\widehat{\theta}_{QML} \underset{n\to\infty}{\overset{p}{\to}} \theta \qquad \text{if} \quad E\left(\log|\phi+\varepsilon_1|\right) = 0. \tag{4.1b}$$

ii) In addition,

$$\sqrt{n}\left(\widehat{\theta}_{QML}-\theta\right) \underset{n\to\infty}{\overset{\mathcal{L}}{\to}} N(0,\Delta), \tag{4.1c}$$

where

$$\Delta = \begin{cases} \Sigma & \text{if } E\left(\log|\phi+\varepsilon_1|\right) < 0, \\ \begin{pmatrix} \sigma^2 & E\left(\varepsilon_1^3\right) \\ E\left(\varepsilon_1^3\right) & Var\left(\varepsilon_1^2\right) \end{pmatrix} & \text{if } E\left(\log|\phi+\varepsilon_1|\right) \geq 0, \end{cases} \tag{4.2}$$

and Σ *is given by* (3.7).

iii) *If, however,* $\phi = 1$, $E\left(\log|\phi+\varepsilon_1|\right) \geq 0$ *and* $P\left(y=-1\right)=0$ *then* (4.1c) *still holds.*

Proof i) (4.1a) follows from (2.9) and (3.5) when $E\left(\log|\phi+\varepsilon_1|\right) > 0$ (see Aknouche, 2013), and from (3.4) when $E\left(\log|\phi+\varepsilon_1|\right) < 0$. Result (4.1$b$) easily follows from (2.8a) and (3.5) (see Aknouche, 2013).

ii) See Aknouche (2013, Theorem 4, (i)) for the proof of (4.2) in the case where (2.3) is not satisfied. If, however, (2.3) holds then (4.2) reduces to (3.8) which has been already proved.

iii) See Aknouche (2013, Theorem 4, (ii)) for the proof. ■

Assuming $\phi \neq 1$, we now propose for the asymptotic variance Δ given by (4.2), an estimate that is consistent in the strict stationary and nonstationary cases. Set

$$\widehat{\varepsilon}_t = \frac{y_t - \widehat{\phi}_{QML} y_{t-1}}{1+y_{t-1}}, \tag{4.3a}$$

$$\widehat{\mu}_r = \frac{1}{n}\sum_{t=1}^{n} \widehat{\varepsilon}_t^r, \tag{4.3b}$$

for some $r \in \{1,...,4\}$. Clearly, $\widehat{\mu}_2$ reduces to $\widehat{\sigma}^2_{QML}$.

Theorem 4.2 *i) Under* ***(A1)****,* ***(A2)****,* ***(A5)*** *and* $\phi \neq 1$,

$$\widehat{\varepsilon}_t - \varepsilon_t \underset{t\to\infty}{\overset{a.s.}{\to}} 0 \quad \text{if } E\left(\log|\phi+\varepsilon_1|\right) \neq 0. \tag{4.4a}$$

$$\widehat{\varepsilon}_t - \varepsilon_t \underset{t\to\infty}{\overset{p}{\to}} 0 \quad \text{if } E\left(\log|\phi+\varepsilon_1|\right) = 0. \tag{4.4b}$$

ii) If, in addition, $E\left(\varepsilon_1^r\right) < \infty$, *then*

$$\widehat{\mu}_r \underset{n\to\infty}{\overset{a.s.}{\to}} E\left(\varepsilon_1^r\right) \quad \text{if } E\left(\log|\phi+\varepsilon_1|\right) \neq 0. \tag{4.5a}$$

$$\widehat{\mu}_r \underset{n\to\infty}{\overset{p}{\to}} E\left(\varepsilon_1^r\right) \quad \text{if } E\left(\log|\phi+\varepsilon_1|\right) = 0. \tag{4.5b}$$

Proof i) From (4.3a) and (1.2) we have

$$\widehat{\varepsilon}_t - \varepsilon_t = \left(\phi - \widehat{\phi}_{QML}\right) \frac{y_{t-1}}{1 + y_{t-1}}. \tag{4.6}$$

Hence, (4.4a) follows from (4.1a) and the $a.s.$ boundedness of $\frac{y_{t-1}}{1+y_{t-1}}$. Result (4.5$b$) follows from (4.6), (4.1b) and the boundedness in probability of $\frac{y_{t-1}}{1+y_{t-1}}$.

ii) (4.6) and elementary algebras yield

$$\begin{aligned}
\widehat{\mu}_r &= \frac{1}{n}\sum_{t=1}^{n} \left(\varepsilon_t + (\widehat{\varepsilon}_t - \varepsilon_t)\right)^r \\
&= \frac{1}{n}\sum_{t=1}^{n} \varepsilon_t^r + \frac{1}{n}\sum_{t=1}^{n}\sum_{i=0}^{r-1} \binom{r}{i} \varepsilon_t^i (\widehat{\varepsilon}_t - \varepsilon_t)^{r-i} \\
&= \frac{1}{n}\sum_{t=1}^{n} \varepsilon_t^r + \frac{1}{n}\sum_{t=1}^{n}\sum_{i=0}^{r-1} \binom{r}{i} \varepsilon_t^i \left(\left(\phi - \widehat{\phi}_{QML}\right) \frac{y_{t-1}}{1 + y_{t-1}}\right)^{r-i}. \qquad (4.7)
\end{aligned}$$

From (4.1a) and the Césaro lemma, (4.7) becomes

$$\widehat{\mu}_r = \frac{1}{n}\sum_{t=1}^{n} \varepsilon_t^r + o_{a.s.}(1),$$

so (4.5a) follows from the ergodic theorem. If, however, $E(\log|\phi + \varepsilon_1|) = 0$, then we can use (4.1$b$) to easily show that the last term in the right hand side of (4.7) is $o_p(1)$. So (4.5b) is established from the ergodic theorem. ■

Using Theorem 4.2, a consistent estimate for the asymptotic covariance matrix Δ is now given. Define $\widehat{\Delta}$ by

$$\widehat{\Delta}_{11} = \widehat{\sigma}^2_{QML} \left(\frac{1}{n}\sum_{t=1}^{n} \frac{y_t^2}{(1+y_t)^2}\right)^{-1}. \tag{4.8a}$$

$$\widehat{\Delta}_{12} = \widehat{\Delta}_{21} = \widehat{\mu}_3 \frac{1}{n}\sum_{t=1}^{n} \frac{y_t}{1+y_t} \left(\frac{1}{n}\sum_{t=1}^{n} \frac{y_t^2}{(1+y_t)^2}\right)^{-1}. \tag{4.8b}$$

$$\widehat{\Delta}_{22} = \frac{1}{n}\sum_{t=1}^{n} \left(\varepsilon_t^2 - \widehat{\mu}_2\right)^2. \tag{4.8c}$$

Then, we state the main result of this Section.

Corollary 4.1 *Under* ***(A1)***, ***(A2)***, ***(A4)***, $\phi \neq 1$ *and* ***(A5)***,

$$\widehat{\Delta} \underset{n\to\infty}{\overset{a.s.}{\to}} \Delta \quad \textit{if } E\left(\log\left|\phi+\varepsilon_1\right|\right) \neq 0. \tag{4.9a}$$

$$\widehat{\Delta} \underset{n\to\infty}{\overset{p}{\to}} \Delta \quad \textit{if } E\left(\log\left|\phi+\varepsilon_1\right|\right) = 0. \tag{4.9b}$$

In addition,

$$\sqrt{n}\widehat{\Delta}^{-1}\left(\widehat{\theta}_{QML}-\theta\right) \underset{n\to\infty}{\overset{\mathcal{L}}{\to}} N\left(0,I\right), \tag{4.10}$$

where I denotes the identity matrix of dimension 2.

Proof i) (4.9) follows from (4.8), (4.4), (4.5) and the ergodic theorem.

ii) (4.10) is a consequence of (4.1c) and (4.9). ■

In practice, result (4.10) is useful in getting confidence interval estimates and significancy tests for the $SMBL$ parameters (see Section 5). It is the analog of results by Aue and Horváth (2011) for RCA models and Francq and Zakoïan (2012, 2013a) for $GARCH$ and asymmetric $GARCH$ models (see also Aknouche, 2012a, 2012b, 2014, 2015a; Aknouche and Al-Eid, 2012; Aknouche et *al*, 2011).

5. Strict stationarity testing and illustrations

5.1. Strict stationarity testing

For CMV models with endogenous volatility, $EnCMV$ (e.g. $GARCH$, RCA, DAR, $SMBL$), second-order stationarity and unit root testing seem to have a little interest compared to CMV models with exogenous volatility (e.g. strong $ARMA$, $ARMA$-$GARCH$) because outside the second-order stationarity domain, the observed process may still remain strictly stationary. This is in contrast with CMV models (e.g. strong $ARMA$, $ARMA$-$GARCH$) with exogenous volatility in which both regions of strict and second-order stationarities (with respect to the conditional mean parameter) coincide. An important consequence is that the asymptotic distribution of the $QMLE$ for such endogenous volatility models is invariant inside or outside the second-order stationary domain and only depends on strict stationarity (see e.g. Francq and Zakoïan 2012, 2013a; Aue and Horváth, 2011; Aknouche, 2013 and the references therein). Thus, for $SMBL$ modeling, strict stationarity and non-strict stationarity testing are appealing.

For the strict stationarity testing problems

$$H_0 : \gamma < 0 \quad \text{against} \quad H_1 : \gamma \geq 0, \tag{5.1}$$

and

$$H_0 : \gamma \geq 0 \quad \text{against} \quad H_1 : \gamma < 0, \tag{5.2}$$

$(\gamma = E \log |\phi + \varepsilon_1|)$ consider the estimate $\widehat{\gamma}_n$ of γ given by

$$\widehat{\gamma}_n = \frac{1}{n} \sum_{t=1}^{n} \log \left| \widehat{\phi}_{QML} + \widehat{\varepsilon}_t \right|,$$

where $\widehat{\varepsilon}_t$ is obtained from $(4.3a)$. If we set

$$\gamma_n (\varphi) = \frac{1}{n} \sum_{t=1}^{n} \log \left| \varphi + \frac{y_t - \varphi y_{t-1}}{1 + y_{t-1}} \right|,$$

for some φ, then clearly $\widehat{\gamma}_n = \gamma_n \left(\widehat{\phi}_{QML} \right)$.

Let

$$\begin{aligned} e_t &= \log |\phi + \varepsilon_t| - E \log |\phi + \varepsilon_1| , \quad t \in \mathbb{N}) \\ \sigma_e^2 &= E \left(e_1^2 \right), \end{aligned}$$

and assume that

$$E \left((\log |\phi + \varepsilon_1|)^2 \right) < \infty. \tag{\textbf{A6}}$$

Therefore, the following result provides the asymptotic distribution of $\widehat{\gamma}_n$ under $\gamma \in [-\infty, +\infty)$.

Theorem 5.1 *Consider model* (1.2) *subject to* **A1**, **A3**, **A4**, **A5**, **A6** *and* $\phi \neq 1$. *Then,*

$$\sqrt{n} \left(\widehat{\gamma}_n - \gamma \right) \underset{n \to \infty}{\overset{\mathcal{L}}{\to}} N \left(0, \sigma_\gamma^2 \right), \tag{5.3a}$$

where

$$\sigma_\gamma^2 = \begin{cases} \sigma_e^2 + \sigma^2 \left(E_\infty \left(\frac{y_t^2}{(1 + y_t)^2} \right) \right)^{-1} \left(E_\infty \left(\frac{1}{\phi + y_t} \right) \right)^2 & \text{if } \gamma < 0, \\ \sigma_e^2 & \text{if } \gamma \geq 0. \end{cases} \tag{5.3b}$$

Proof The Taylor formula gives

$$\begin{aligned}
\widehat{\gamma}_n &= \gamma_n\left(\widehat{\phi}_{QML}\right) \\
&= \gamma_n(\phi) + \left(\widehat{\phi}_{QML} - \phi\right)\frac{\partial \gamma_n(\phi)}{\partial \varphi} + o_p\left(n^{-\frac{1}{2}}\right) \\
&= \gamma_n(\phi) + \frac{1}{n}\left(\widehat{\phi}_{QML} - \phi\right)\sum_{t=1}^{n}\frac{1}{\phi + y_t} + o_p\left(n^{-\frac{1}{2}}\right).
\end{aligned}$$

So

$$\begin{aligned}
\sqrt{n}\left(\widehat{\gamma}_n - \gamma\right) &= \sqrt{n}\left(\gamma_n(\phi) - \gamma\right) + \sqrt{n}\left(\gamma_n\left(\widehat{\phi}_{QML}\right) - \gamma_n(\phi)\right) \\
&= \sqrt{n}\left(\gamma_n(\phi) - \gamma\right) + \sqrt{n}\left(\widehat{\phi}_{QML} - \phi\right)\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\phi + y_t} + o_p(1). \qquad (5.4)
\end{aligned}$$

If $\gamma < 0$ the ergodic theorem yields

$$\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\phi + y_t} \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} E_\infty\left(\frac{1}{\phi + y_t}\right). \qquad (5.5)$$

If, however, $\gamma \geq 0$ then from (2.8) we have

$$\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\phi + y_t} \underset{n\to\infty}{\overset{p}{\longrightarrow}} 0. \qquad (5.6)$$

Thus (5.3) follows from (5.4), (5.5), (5.6) and (4.1c). ■

Like the $GARCH$ model (cf. Francq and Zakoïan, 2012, Theorem 3.1), the asymptotic variance of $\widehat{\gamma}_n$ is larger in the strict stationarity domain than in the non strict stationarity one.

To make inference about $\widehat{\gamma}_n$ we need to estimate its asymptotic variance σ_γ^2. Let

$$\widehat{\sigma}_\gamma^2 = \widehat{\sigma}_e^2 + \widehat{\sigma}_{QML}^2\left(\frac{1}{n}\sum_{t=1}^{n}\frac{y_t^2}{(1+y_t)^2}\right)^{-1}\left(\frac{1}{n}\sum_{t=1}^{n}\frac{1}{\widehat{\phi}_{QML} + y_t}\right)^2,$$

where

$$\widehat{\sigma}_e^2 = \frac{1}{n}\sum_{t=1}^{n}\left(\log\left|\widehat{\phi}_{QML} + \widehat{\varepsilon}_t\right| - \widehat{\gamma}_n\right)^2.$$

The following result establishes consistency of $\widehat{\sigma}_\gamma^2$.

Corollary 5.1 *Under the same assumptions of Theorem 5.1 we have*

$$\begin{aligned} \widehat{\sigma}_\gamma^2 &\underset{n\to\infty}{\overset{a.s.}{\to}} \sigma_\gamma^2 \quad \textit{if } E\left(\log\left|\phi+\varepsilon_1\right|\right) &\neq& \ 0. \\ \widehat{\sigma}_\gamma^2 &\underset{n\to\infty}{\overset{p}{\to}} \sigma_\gamma^2 \quad \textit{if } E\left(\log\left|\phi+\varepsilon_1\right|\right) &=& \ 0. \end{aligned}$$

An important consequence of Theorem 5.1 and Corollary 5.1 is that we can get a consistent interval estimate for γ.

Corollary 5.2 *Under the same assumptions of Theorem 5.1, a confidence interval for* γ *at the asymptotic nominal level* $\alpha \in (0,1)$ *is*

$$\left[\widehat{\gamma}_n - \frac{\widehat{\sigma}_\gamma}{\sqrt{n}}\Phi^{-1}\left(1-\frac{\alpha}{2}\right), \widehat{\gamma}_n - \frac{\widehat{\sigma}_\gamma}{\sqrt{n}}\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right],$$

where Φ *denotes the standard normal (*$N(0,1)$*) cumulative distribution.*

Let $T_n = \dfrac{\sqrt{n}\widehat{\gamma}_n}{\widehat{\sigma}_e}$ be the test statistic for the problems (5.1) and (5.2). Thanks to the form of σ_γ^2 in Theorem 5.1, we have taken T_n to be a function of $\widehat{\sigma}_e$ not of $\widehat{\sigma}_\gamma$, allowing to simplify the procedure. The same has been considered earlier by Francq and Zakoïan (2012, 2013a) in the context of $GARCH$ and asymmetric power $GARCH$ models. The following result gives the asymptotic critical regions for the testing problems (5.1) and (5.2).

Corollary 5.3 *Under the same assumptions of Theorem 5.1:*

i) The asymptotic level of the test STS *defined for the problem* (5.1) *by the critical region*

$$C^{STS} = \left\{T_n > \Phi^{-1}(1-\alpha)\right\},$$

is bounded by α *and is equal to* α *under* $\gamma = 0$*. Moreover, the test* STS *is consistent for all* $\gamma > 0$.

ii) The asymptotic level of the test NSS *defined for the problem* (5.2) *by the critical region*

$$C^{NSS} = \left\{T_n < \Phi^{-1}(\alpha)\right\},$$

is bounded by α *and is equal to* α *under* $\gamma = 0$*. Moreover, the test* NSS *is consistent for all* $\gamma < 0$.

The proofs of Corollary 5.1-5.3 are based on arguments already used in the proofs of Theorem 4.2 and Theorem 5.1 and hence they are omitted.

It is worth noting that as in the $GARCH\,(1,1)$ case (see Francq and Zakoïan, 2012), the test statistic $T_n = \sqrt{n}\dfrac{\widehat{\gamma}_n - \gamma}{\widehat{\sigma}_e} + \sqrt{n}\dfrac{\gamma}{\widehat{\sigma}_e}$ is such that

$$\begin{aligned} T_n &\overset{a.s.}{\underset{n\to\infty}{\rightarrow}} -\infty \qquad \text{if } \gamma < 0. \\ T_n &\overset{a.s.}{\underset{n\to\infty}{\rightarrow}} +\infty \qquad \text{if } \gamma > 0. \end{aligned} \tag{5.7}$$

5.2. Finite sample properties of the proposed inference procedures

This subsection studies the behavior of the $QMLE$ and the strict stationarity tests STS and NSS in finite sample through some simulation experiments and real stock return series.

5.2.1. Finite sample properties of the $QMLE$

The $QMLE$ has been run on 1000 simulated series generated from Gaussian $SMBL$ models with sample sizes 100 and 1000. Three set of parameters have been considered. The first one corresponds to $(\phi, \sigma^2) = (0.5, 0.7)$ for which the model is strictly stationary ($\gamma = -0.6451 < 0$, StS) with finite variance ($\phi^2+\sigma^2 = 0.95 < 1$, $2nS$). For the second one, $(\phi, \sigma^2) = (0.8, 0.7)$, the model is strictly ($\gamma = -0.6451 < 0$) but not second-order stationary ($N2S$), having an infinite variance ($\phi^2+\sigma^2 = 1.34 > 1$). For the third one, $(\phi, \sigma^2) = (2, 1)$, the model is neither strictly stationary ($\gamma = 0.5203 > 0$, NSS) nor second-order stationary ($\phi^2 + \sigma^2 = 5 > 1$, $N2S$). For all instances, we have obtained bias and standard deviations (Std) for the $QMLE$

over the 1000 replications (cf. Table 5.1).

	$\gamma = -0.6451$ (STS)		$\gamma = -0.4183$ (STS)		$\gamma = 0.5203$ (NSS)	
	$\phi^2+\sigma^2= 0.95$ $(2nS)$		$\phi^2+\sigma^2= 1.34$ $(N2S)$		$\phi^2+\sigma^2= 5$ $(N2S)$	
	$\phi = 0.5$	$\sigma^2= 0.7$	$\phi = 0.8$	$\sigma^2= 0.7$	$\phi = 2$	$\sigma^2= 1$
$n = 100$						
Bias	-0.0007	-0.0124	-0.0070	-0.0153	0.0018	-0.0149
Std	0.0170	0.0997	0.0110	0.0988	0.0898	0.1402
$n = 1000$						
Bias	0.0001	-0.0015	0.0000	-0.0013	0.0006	-0.0023
Std	0.0019	0.0344	0.0011	0.0310	0.0308	0.0464

Table 5.1 $Bias$ and Std of the $QMLE$ for the Gaussian $SMBL$ under second-order stationarity $(2nS)$, strict stationarity (STS) with infinite variance $(N2S)$ and non-strict stationnarity (NSS).

It may be observed from Table 5.1 that the $QMLE$ results are totally consistent with asymptotic theory. Indeed, for all instances, the $QMLE$ has very small bias and Std irrespective of the stationarity conditions. Moreover, in the unstable case the $QMLE$ of all parameters is consistent contrary to the unstable $GARCH$ (Francq and Zakoïan, 2012) and the unstable RCA (Aue and Horváth, 2011) where the $QMLE$ of the conditional variance intercept is inconsistent.

5.2.2. Finite sample properties of the tests

We have applied the tests STS and NSS on 1000 replications of Gaussian $SMBL$ series with sample sizes 100, 500 and 3000. Various sets of parameters, inside $(\gamma < 0)$, (approximately) on the boundary $(\gamma \simeq 0)$ and outside the strict stationarity domain $(\gamma > 0)$ have been taken (cf. Table 5.2 and Table 5.3). For all instances, we have obtained relative frequency of rejection of the tests STS (cf. Table 5.2) and NSS (cf. Table 5.3) at the nominal level

$\alpha = 5\%$.

	(ϕ, σ^2)						
	$(0.5, 0.7)$	$(0.9, 0.7)$	$(0.8, 2)$	$(0.8, 2.87)$	$(0.8, 2.88)$	$(1.1, 3)$	$(2, 2)$
	γ						
	-0.4625	-0.3312	-0.1368	-0.0005	0.0008	0.1029	0.4508
n							
100	0.0	0.0	0.3	7.1	7.5	27.1	99.3
500	0.0	0.0	0.0	5.8	6.6	68.2	100.0
3000	0.0	0.0	0.0	4.6	4.8	99.8	100.0

Table 5.2 Percentage of rejection of the strict stationarity test STS, $H_0 : \gamma < 0$, at the nominal level $\alpha = 5\%$ for the Gaussian $SMBL$ model.

It may be observed from Table 5.2 that the relative frequency of rejection of the test STS:

i) tends to be close to 0% as γ decreases negatively ($\gamma < 0$),

ii) tends to be close to 100% as γ increases positively ($\gamma > 0$) and,

iii) is close to the nominal level $\alpha = 5\%$ around $\gamma = 0$.

These conclusions tend to be true as n increases confirming consistency of the STS.

	(ϕ, σ^2)						
	$(0.5, 0.7)$	$(0.9, 0.7)$	$(0.8, 2)$	$(0.8, 2.87)$	$(0.8, 2.88)$	$(1.1, 3)$	$(2, 2)$
	γ						
	-0.4625	-0.3312	-0.1368	-0.0005	0.0008	0.1029	0.4508
n							
100	100.0	97.0	33.8	3.9	4.5	0.4	0.0
500	100.0	100.0	88.9	4.2	3.2	0.0	0.0
3000	100.0	100.0	100.0	4.9	3.8	0.0	0.0

Table 5.3 Percentage of rejection of the non strict stationarity test NTS, $H_0 : \gamma \geq 0$, at the nominal level $\alpha = 5\%$ for the Gaussian $SMBL$ model.

From Table 5.3 the same conclusion may be done as above: the relative frequency of rejection of the non-strict stationarity test NSS:

i) tends to be close to 100% as γ decreases negatively ($\gamma < 0$),

ii) is close to 0% whenever γ increases positively ($\gamma > 0$) and

iii) is close to the nominal level $\alpha = 5\%$ when $\gamma \simeq 0$ and n increases.

5.2.3. Application: strict stationarity testing for some financial stock returns

We have applied the proposed strict stationarity tests to daily returns of three stock market indices and two oil prices. We have considered the $SP500$ from 01/02/1997 to 06/06/2000, the $CAC40$ from 06/11/2010 to 06/10/2013, the KV Pharmaceutical ($NYSE$: KV-A) from 09/18/ 2008 to 02/07/2011,the $BRENT$ oil price from 01/02/2008 to 03/14/2013 and the WTI oil price from 01/11/2010 to 03/14/2013 (see also Aknouche and Touche, 2015). The KV-A series has been taken from Francq and Zakoïan (2012). For the WTI oil price series, missing data have been removed. Table 5.4 displays the strict stationarity test statistic T_n computed on each return series. In view of the asymptotic property of T_n in (5.7), the strict stationarity hypothesis of the $SMBL$ model cannot be rejected at any reasonable level for the return series of $SP500$, $CAC40$, $BRENT$ and WTI. In contrast, a strict stationary $SMBL$ is not plausible for the KV-A return series. The same conclusion with a $GARCH(1,1)$ model has been made by Francq and Zakoïan (2012) for the KV-A return series.

	$SP500$	$CAC40$	$BRENT$	WTI	KV-A
T_n	-150.8579	-137.4617	-164.4189	-127.8241	0.7933

Table 5.4 The test statistic T_n of the strict stationarity tests STS and NSS for returns of $SP500$, $CAC40$, $BRENT$, WTI and KV-A.

6. Conclusion

In this Chapter statistical properties of the $SMBL$ model (a random coefficient autoregression in which the random coefficient coincides with the innovation) have been explored irrespective of its probabilistic structure. In addition to its parsimony and simplicity, the $SMBL$ model allows describing the level and volatility contrary to the pur $GARCH$ process which only models volatility. Testing purely conditional variance effect may then be done while considering the null hypothesis H_0: $\phi = 0$ against the alternative H_1: $\phi \neq 0$. The test may be obtained irrespective of the stationarity assumption from the distribution of $\widehat{\phi}_{QML}$ given by Corollary 4.1. An interesting statistical property of the $SMBL$ model is that its $QMLE$ has a closed form and surprisingly is consistent for all parameters in the unstable case. This is in contrast with standard RCA and $GARCH$ models where the conditional variance intercept cannot be consistently estimated in the unstable domain (cf. Aue and Horváth, 2011; Aknouche, 2013; Francq and Zakoïan, 2012). Notice that the proposed unified QML theory for the $SMBL$ model was based on the fourth moment assumption **A4** on the innovation, which may be too restrictive when modeling heavy tailed stock returns. So adapting such a theory to some robust methods which do not require **A4**, such as the least absolute deviation estimate ($LADE$) and the generalized $QMLE$ ($GQMLE$), would be of interest (see e.g. Peng and Yao 2003; Berkes and Horváth, 2004; Francq and Zakoïan, 2013b, Fan et *al*, 2014 for the $GARCH$ model and Zhu and Ling, 2013 for the DAR model).

7. Appendix: Glossary

$\underset{n\to\infty}{\overset{a.s.}{\to}}$	Almost sure convergence as $n \to \infty$.
$\underset{n\to\infty}{\overset{\mathcal{L}}{\to}}$	Convergence in distribution (law) as $n \to \infty$.
$\underset{n\to\infty}{\overset{p}{\to}}$	Convergence in probability as $n \to \infty$.
$o_p(1)$	A term converging in probability to zero as $n \to \infty$.
$o_{a.s.}(1)$	A term converging almost surely to zero as $n \to \infty$.
$O_p(1)$	A term bounded in probability as $n \to \infty$.
$\mathbb{N}$	Set of nonnegative integer numbers.
$\mathbb{N}^*$	Set of positive integer numbers.
$\mathbb{Z}$	Set of integer numbers.

$\mathbb{R}$	Set of real numbers.
$2nS$	Second-order stationary, second-order stationarity.
$2S(WLSE)$	Two-Stage (Weighted Least Squares Estimate).
$ARCH$	Autoregressive Conditionally Hetereskedastic.
$ARMA$	Autoregressive Moving Average.
$ARMA$-BL	$ARMA$ with BiLinear innovation.
$ARMA$-$GARCH$	$ARMA$ with $GARCH$ innovation.
$ARMA$-SV	$ARMA$ with Stochastic Volatility innovation.
$a.s.$	almost surely.
BL	BiLinear.
$CHARMA$	Conditionally Heteroskedastic $ARMA$.
CMV	Conditional Mean and Volatility.
DAR	Double AutoRegression.
$GARCH$	Generalized $ARCH$.
$GRCA$	Generalized RCA.
$GQMLE$	Generalized $QMLE$.
iid	independent and identically distributed.
$LADE$	Least Absolute Deviation Estimate.
MAR	Mixture Autoregression.
$N2S$	Non Second-order Stationary, Non Second-order Stationarity.
NSS	Non Strict Stationary, Non Strict Stationarity.
$QML(E)$	Quasi Maximum Likelihood (Estimate).
RCA	Random Coefficient Autoregression, Random Coefficient Autoregressive.
$SMBL$	Simple Markov BiLinear.
Std	Standard deviation.
STS	Strict Stationary, Strict Stationarity.
SV	Stochastic Volatility.
TAR	Treshold autoregression.

References

[1] Aknouche, A. (2012*a*). Multi-stage weighted least squares estimation of *ARCH* processes in the stable and unstable cases. *Statistical Inference for Stochastic Processes*, **15**, 241-256.

[2] Aknouche, A. (2012*b*). Implication of instability on econometric and financial time series modeling. *In Econometrics: New research*, editors: Mendez, S.A. and Vega, A.M., Nova Publishers, New York, pp. 149-186.

[3] Aknouche, A. (2013). Two-stage weighted least squares estimation of nonstationary random coefficient autoregressions. *Journal of Time Series Econometrics*, **5**, 25-47.

[4] Aknouche, A. (2014). Estimation and strict stationarity testing of *ARCH* processes based on weighted least squares. *Mathematical Methods of Statistics*, **23**, 81-102.

[5] Aknouche, A. (2015*a*). Quadratic random coefficient autoregression with linear-in-parameters volatility. *Statistical Inference for Stochastic Processes*, forthcoming, DOI 10.1007/s11203-014-9108-3.

[6] Aknouche, A. (2015*b*). Explosive strong periodic autoregression with multiplicity one. *Journal of Statistical Planning and Inference*, **161**, 50-72.

[7] Aknouche, A. and Touche, N. (2015). Weighted least squares-based inference for stable and unstable threshold power *ARCH* processes. *Statistics & Probability Letters*, **97**, 108-115.

[8] Aknouche, A., Al-Eid, E.M. and Hmeid, A.M. (2011). Offline and online weighted least squares estimation of nonstationary power *ARCH* processes. *Statistics & Probability Letters*, **81**, 1535-1540.

[9] Amendola, A. and Francq, C. (2009). Concepts and tools for nonlinear time series modelling. *Handbook of Computational Econometrics, Edts: D. Belsley and E. Kontoghiorghes*, Wiley.

[10] Aue, A. and Horváth, L. (2011). Quasi-likelihood estimation in stationary and nonstationary autoregressive models with random coefficients. *Statistica Sinica*, **21**, 973-999.

[11] Aue, A., Horváth, L. and Steinebach, J. (2006). Estimation in random coefficient autoregressive models. *Journal of Time Series Analysis*, **27**, 61-76.

[12] Babillot, M., Bougerol, P. and Elie, L. (1997). The random difference equation $X_n = A_n X_{n-1} + B_n$ in the critical case. *Annals of Probability*, **25**, 478-493.

[13] Berkes, I. and Horváth, L. (2004). The efficiency of the estimators of the parameters in $GARCH$ processes. *Annals of Statistics*, **32**, 633-655.

[14] Berkes, I., Horváth, L. and Ling, S. (2009). Estimation in nonstationary random coefficient autoregressive models. *Journal of Time Series Analysis*, **30**, 395-416.

[15] Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, **31**, 307-327.

[16] Bougerol, P. and Picard, N. (1992). Strict stationarity of generalized autoregressive processes. *Annals of Probability*, **20**, 1714-1730.

[17] Brandt, A. (1986). The stochastic equation $Y_{n+1} = A_n Y_n + B_n$ with stationary coefficients. *Advances in Applied Probability*, **18**, 211-220.

[18] Chen, M., Li, D. and Ling, S. (2014). Nonstationarity and quasi-maximum likelihood estimation on a double autoregressive model. *Journal of Time Series Analysis*, **35**, 189-202.

[19] Cline, D.B.H. (2007). Stability of nonlinear stochastic recursions with application to nonlinear *AR-GARCH* models. *Advances in Applied Probability*, **39**, 462-491.

[20] Cline, D.B.H. and Pu, H.M.H. (2002). A note on a simple Markov bilinear stochastic process. *Statistics & Probability Letters*, **56**, 283-288.

[21] Engle, R.F. (1982). Autoregressive Conditional Heteroskedasticity with estimates of variance of U.K. inflation. *Econometrica*, **50**, 987-1008.

[22] Fan, J., Qi, L. and Xiu, D. (2014). Quasi maximum likelihood estimation of $GARCH$ models with heavy-tailed likelihoods. *Journal of Business and Economic Statistics*, **32**, 178-191.

[23] Feigin, P.D. and Tweedie, R.L. (1985). Random coefficient autoregressive processes: a Markov chain analysis of stationarity and finiteness of moments. *Journal of Time Series Analysis*, **6**, 1-14.

[24] Ferrante, M., Fonseca, G. and Vidoni, P. (2003). Geometric ergodicity, regularity of the invariant distribution and inference for a threshold bilinear Markov process. *Statistica Sinica*, **13**, 367-384.

[25] Francq, C. and Zakoïan, J.M. (2010). *$GARCH$ Models: Structure, statistical inference and financial applications*. Wiley.

[26] Francq, C. and Zakoïan, J.M. (2012). Strict stationarity testing and estimation of stationary and explosive $GARCH$ models. *Econometrica,* **80**, 821-861.

[27] Francq, C. and Zakoïan, J.M. (2013a). Inference in nonstationary asymmetric $GARCH$ models. *Annals of Statistics*, **41**, 1693-2262.

[28] Francq, C. and Zakoïan, J.M. (2013b). Optimal predictions of powers of conditionally heteroskedastic processes. *Journal of the Royal Statistical Society*, **B75**, 345-367.

[29] Goldie, C. and Maller, R. (2000). Stability of perpetuities. *Annals of Probability*, **28**, 1195-1218.

[30] Holan, S.H., Lund, R. and Davis, G. (2010). The $ARMA$ alphabet soup: A tour of $ARMA$ model variants. *Statistics Survey*, **4**, 232-274.

[31] Hwang, S.Y. and Basawa, I.V. (1998). Parameter estimation for generalized random coefficient autoregressive processes. *Journal of Statistical Planning and Inference*, **68**, 323-337.

[32] Hwang, S.Y. and Basawa, I.V. (2005). Explosive random-coefficient $AR(1)$ processes and related asymptotics for least squares estimation. *Journal of Time Series Analysis*, **26**, 807-824.

[33] Jensen, S.T. and Rahbek, A. (2004). Asymptotic normality of the QML estimator of $ARCH$ in the nonstationary case. *Econometrica*, **72**, 641-646.

[34] Ling, S. and Li, D. (2008). Asymptotic inference for a nonstationary double $AR(1)$ model. *Biometrika*, **95**, 257-263.

[35] Lumsdaine, R.L. (1996). Consistency and asymptotic normality of the quasi-maximum likelihood estimator in $IGARCH(1,1)$ and covariance stationary $GARCH(1,1)$ models. *Econometrica*, **64**, 575-596.

[36] Meyn, S. and Tweedie, R. (2009). *Markov chains and stochastic stability.* 2nd edition, Springer Verlag, New York.

[37] Nicholls, D.F. and Quinn, B.G. (1982). *Random coefficient autoregressive model: An introduction.* Springer Verlag, New York.

[38] Peng, L. and Yao, Q. (2003). Least absolute deviations estimation for $ARCH$ and $GARCH$ models. *Biometrika*, **90**, 967-975.

[39] Schick, A. (1996). $\sqrt{n}$-consistent estimation in a random coefficient autoregressive model. *Australian Journal of Statistics*, **38**, 155-60.

[40] Taylor, S. (1986). *Financial time series analysis.* Wiley.

[41] Tong, H. (1981). A note on a Markov bilinear stochastic process in discrete time. *Journal of Time Series Analysis*, **2**, 279-284.

[42] Truquet, L. and Yao, J. (2012). On the quasi-likelihood estimation for random coefficient autoregressions. *Statistics*, **46**, 505-521.

[43] Tsay R.S. (1987). Conditional heteroskedastic time series models. *Journal of the American Statistical Association*, **7**, 590-604.

[44] Tsay, R.S. (2002). *Analysis of financial time series: Financial econometrics*. Wiley.

[45] Vervaat, W. (1979). On a stochastic difference equation and a representation of non negative infinitely divisible random variables. *Advances in Applied Probability*, **11**, 750-783.

[46] Weiss, A.A. (1984). $ARMA$ models with $ARCH$ errors. *Journal of Time Series Analysis*, **5**, 129-43.

[47] Zhao, Z.W. and Wang, D.H. (2012). Statistical inference for generalized random coefficient autoregressive model. *Mathematical and Computer Modelling*, **56**, 152-166.

[48] Zhao, Z.W., Wang, D.H. and Peng, C.X. (2013). Coefficient constancy test in generalized random coefficient autoregressive model. *Applied Mathematics and Computation*, **219**, 10283-10292.

[49] Zhu, K. and Ling, S. (2013). Quasi-maximum exponential likelihood estimators for a double $AR(p)$ model. *Statistica Sinica*, **23**, 251-270.