Harris and Wilson (1978) Model Revisited: The Spatial Period-doubling Cascade in an Urban Retail Model

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19. November 2015

Harris and Wilson (1978) Model Revisited: The Spatial Period-doubling Cascade in an Urban Retail Model*

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November 19, 2015

Abstract

Harris and Wilson (1978)'s retail location model is one of the pioneering works in regional sciences on the combination of the "fast" and "slow" dynamic describing spatial pattern formation processes in the economic landscape, which is a current well-established modeling technique. Although proposed some time ago, the comparative static (bifurcation) properties of the model have not yet been sufficiently explored. We employ a simple analytical approach developed by Akamatsu et al. (2012) to reveal previously unknown bifurcation properties of the model in a space with a large number of locations. It is analytically shown that the evolutionary path of spatial structure exhibits a remarkable property, namely "spatial period-doubling cascade," which we cannot observe in the popular two-location setup. We also discuss strong linkages between the model and the models of "new economic geography" regarding the modeling strategies and their bifurcation properties.

Keywords: agglomeration, multiple agglomerations, stability, bifurcation, new economic geography model

JEL Classification: R12, R13, C62, F12, F15

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*We are grateful for the comments and suggestions made by Tomoki Ishikura, Tomoya Mori, Se-il Mun, Takatoshi Tabuchi, and especially Alan G. Wilson, as well as participants at the 28th Applied Regional Science Conference Meeting in Naha. We greatly acknowledge the funding form the Japan Society for the Promotion of Science, Grant-in-Aid for JSPS Fellows 14J02901, Grant-in-Aid for Challenging Exploratory Research 15K14044, and Grant-in-Aid for Young Scientists (B) 15K18136.

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1 Introduction

Economic activities are highly localized in space. For any spatial scale such as countries, regions, or within cities, we observe unequal spatial concentrations of population, firms, or shops. Fascinated by the regularity, numerous scholars, including location theorists, geographers, economists, and physicists have attempted to explain why and how these spatial structures emerge and evolve over time. Apart form underlying spatial inhomogeneities or locally embedded contexts (e.g., natural advantages such as rivers or harbors, institutional regulations, and other cultural contexts), one of the basic reasons for spatial inequality is the existence of various forms of agglomeration economies—spatial increasing returns to scale, whose origins are usually explained by mutually reinforcing externalities (see Duranton and Puga, 2004, for a survey).

Over the past three decades, research has emphasized the importance of the interplay between such agglomeration economies and dynamic self-organization processes in shaping spatial structure. A pioneering model in the field of geography is that of Harris and Wilson (1978) (hereafter "the HW model"). Based on a static urban retail model of Huff (1963) and Lakshmanan and Hansen (1965), the paper formulated a simple, dynamic model of agglomeration with spatial return to scale. As early as the 1970s, their research emphasized that such models inevitably encounter problems such as (i) multiple equilibria, (ii) path dependence, that is, strong dependence on the initial condition, and (iii) catastrophic phase transitions (bifurcations), all of which are popular ideas in regional sciences today. The most striking problem is the third: gradual changes of structural parameters (e.g., transport cost) may destabilize spatial configurations that were once stable, resulting in the *emergence* of other spatial structures—including lumpy, spatially unequal agglomeration.[1]

A number of explorations on the model's analytical properties have since been conducted by geographers (e.g., Clarke, 1981; Wilson, 1981; Rijk and Vorst, 1983a, 1983b). There have been, however, huge obstacles to the detailed analysis of the model. The three combined characteristics of the model already mentioned prevent us form analyzing the model's intrinsic bifurcation properties. There is, fundamentally, only one possible way to analytically study quantitative properties of equilibria in the model beyond the qualitative properties (e.g., the existence or (non-)uniqueness of equilibria): the two-location setup.

For analytical tractability, there is a long tradition in regional sciences to elucidate essential properties of dynamic spatial agglomeration models in the two-location setup. For the HW model, the existence and uniqueness of equilibrium points are rigorously addressed in a general case by Rijk and Vorst (1983a,b). The studies, however, depend upon a two-location setup to draw further concrete implications of the model, such as the critical points at which catastrophic bifurcations occur. To the authors' knowledge, there are no sufficient analytical studies that address quantitative results under a multi-locational setting beyond the two-location setup. Other studies by Wilson, such as Wilson (1981), employs a graphical trick that focuses only single location at once to draw useful insights into the bifurcation mechanism of the model. The analysis is, however, insufficient if we are interested in the behavior of the spatial system as a whole.[2]

Although the two-location setup is an effective starting point, it has several limitations. First, these models are not capable of describing or explaining rich varieties of *poly-centric* spatial

[1] Papageorgiou and Smith (1983) is a pioneering work of such an approach in economics. The study demonstrated that the emergence spatial agglomeration can be explained by *instability* of the uniform, flat-earth equilibrium.

[2] Geographers resorted to numerical approaches once the difficulties and limitations in analytical treatments of their models were realized. The next section reviews related studies.

concentrations of economic activities observed in the real world (Anas et al., 1998). Second, to what extent implications in the two-location setting can be generalized to a multi-locational version of the model is unclear. For instance, indistinguishable models in a two-location world can exhibit a significantly different bifurcation pattern in a multi-locational world.[3] Such a setting is too degenerated in the spatial dimension and does not have enough resolution for the modeler to determine the difference. Because there are undoubtedly many locations in the real world, not just two, the heavy reliance on the two-location setup requires resolution.[4]

This paper advances the discussion and identifies intrinsic bifurcation properties of the HW model in a *multi-location setting beyond two*. Allowing arbitrary numbers of zones, provided the zones are located on a symmetric circumference,[5] we analytically follow a complete path of the evolution of spatial structure in the course of changing a structural parameter. Specifically, starting form the uniform spatial distribution of retailers, we consider the process of *improvement in the global level of transportation technologies* that is captured by a global transport cost parameter. We derive the closed and semi-closed form formulae for the critical points of the transport cost parameter at which catastrophic phase transitions occur. We also identify the characteristics of these bifurcations, that is, the emergent spatial configuration after each bifurcation. This study also shows that the evolutionary path of spatial structure in the HW model exhibits a remarkable *recursive* property that is called the *spatial period-doubling cascade*. We then compare the bifurcation properties of the HW model and that of a new economic geography (NEG) model to show the strong resemblance between the two.

The analysis in this paper utilizes a method proposed by Akamatsu et al. (2012), which is tailored for the analytical treatment of general spatial agglomeration models. The study demonstrated the effectiveness of the method taking a multi-regional generalization of Pflüger (2004)'s NEG model as an example. The approach can treat essentially any model that fits the Boltzmann–Lotka–Volterra form—reviewed in the next section—as long as the spatial evolution form the uniform equilibrium is considered. Although we will explain the ideas behind this approach at the relevant points, we note that the key factors for the analysis are (a) the *spatial discounting matrix*, (b) the *racetrack economy*, and (c) *discrete Fourier transformation (DFT)*. Akamatsu et al. (2012)'s introduction provides a detailed explanation and discussion.

The remainder of this paper is organized as follows. Section 2 reviews the related literature. Section 3 introduces the HW model. In Section 4, employing Akamatsu et al. (2012)'s approach, we study the bifurcation properties of the model in the course of decreasing transportation costs. The main results are presented in this section. Section 5 discusses the relationship between the HW model and Pflüger (2004)'s model. Section 6 offers concluding remarks.

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[3]Recently, Akamatsu et al. (2015) showed that there is a fundamental difference in agglomeration patterns of Forslid and Ottaviano (2003)'s model and Helpman (1998)'s model in a multi-region economy with more than two locations. In the two-location world, however, these models exhibit quite similar bifurcation behavior.

[4]See Behrens and Thisse (2007) for a thorough discussion on the "dimensionality issue" and its empirical relevance.

[5]We assume one-dimensionality of the underlying physical space. The one-dimensionality is, however, not an essential assumption. A two-dimensional plain can be equivalently treated in a similar manner (see, for example, Ikeda et al., 2014b). We focus on the one-dimensional system for clarity in presentation.

2 Related Literature

A recent review on the HW model can be found in Wilson (2008). The author terms a generalization of the modeling strategy by Harris and Wilson (1978) as the *Boltzmann–Lotka–Volterra* (BLV) method. The BLV method is a synthesis of the *fast dynamic* (the "Boltzmann" component), and the *slow dynamic* (the "Lotka–Volterra" component). The fast dynamic describes the short-run spatial interaction patterns (i.e., the *flows* between the nodes, such as the strip distribution patterns between the origin–destination pairs), whereas the slow dynamic describes the gradual evolution of spatial distribution that governs the flow generation/attraction processes (i.e., the *stocks* at the nodes, such as population at the origins and destinations[6]). The entropy-maximizing framework, which is introduced to regional sciences by Wilson (1967) and further developed in Wilson (1970a), is one of the most unified flow-based *static* spatial interaction modeling paradigms. Accounting for the flow-dependent evolutions of stock values at the nodes, the BLV formalism adds a *dynamic* aspect to these models. In 3.1, we review the BLV method in more detail. The HW model is considered a canonical example of the BLV method.

The BLV method is sufficiently general to include a large number of modeling techniques in regional sciences as subsets. A good example is Krugman (1991)'s core–periphery (CP) model, which opened up a new branch of economics, namely NEG. NEG models are different form those of classical location theories because they are able "to combine old ingredients in a new recipe" (Ottaviano and Thisse, 2005) that employs a full-fledged general equilibrium framework: NEG models succeed in bonding up firm-level increasing return, transportation cost between regions, and factor mobility into compact, simplified general equilibrium models. From the BLV point of view, models in the NEG literature are a subclass of the BLV method whose fast dynamics are based on conventional microeconomic modeling techniques. The NEG literature also emphasizes, as geographers have done, self-organization and phase transitions in spatial structure.[7] The dependence on the two-region setup is more prevalent in the NEG literature because of economists' desire for clear exposition. Krugman (1991) also relied on two-location models to cement his ideas. There have been few analytical studies under multi-location setup more than two.

Compared to NEG models, the BLV method allows a considerably wide class of short-run spatial interaction behaviors because it does not require general equilibrium condition. Hence, it is likely that, depending on the specifications of the details of a model, the BLV method can still provide wide varieties of new insights into the nature of self-organizing processes of spatial structures—the method could identify "how the main forces acting at each spatial scale interact to generate the space-economy" (Thisse, 2010) beyond the scope of conventional modeling techniques such as those employed in NEG models. The power of the method is, however, yet to be fully demonstrated because of insufficient understanding of the analytical properties of the BLV models.

To explore more general properties of the BLV models (including the HW model) beyond the

[6]The studies by Wilson and other geographers may reflect the *synergetics* and related literatures in the 1970s and 1980s (**?**). Synergetics emphasize the role of the combination of the fast and slow dynamic and changing structural parameters in the processes of spatio-temporal pattern formation, which strongly resembles the approaches employed by regional scientists including Wilson. Haken discusses possible applications of his theory to the field of geography (Haken, 1985).

[7]There is almost no cross-references between BLV and NEG-type models. Considering the strong similarities between their methodologies, the fact is surprising and seems inappropriate. Our sub-aim is to add a cross-reference to acknowledge the contributions form both sides.

two-location setup, geographers have relied heavily on computer simulations. Clarke and Wilson (1983) and Clarke and Wilson (1985) report some of the results form their extensive numerical studies on the BLV models. Systematically changing structural parameters of the models in hand, the authors ran numerous experiments to determine the type of spatial structures that eventually emerge, at what point phase transitions occur, or how the initial conditions affect resulting spatial patterns (see Clarke et al. (1998), Wilson (2010), Wilson and Dearden (2011), Dearden and Wilson (2015), for recent explorations). Because we can easily add any "realistic" conditions to numerical simulations, they enjoy great generalities including two-dimensional space, or systems with multi-class mobile agents. An interesting finding form these studies is the self-organization of hexagonal spatial agglomeration patterns, which are quite similar to those proposed in the classical central place theory of Christaller (1933) and Lösch (1940). The great generality of computer simulations has, however, inevitably limited the clarity of the implications of these studies. For example, Weidlich and Haag (1987), Weidlich and Munz (1990), Munz and Weidlich (1990) also showed emergence of hexagonal agglomeration pattern numerically using a combination of the fast and slow dynamic. However, the model is so complicated that it is practically impossible to determine why and how such a result is obtained. In fact, there has been no accepted rigorous proof for the emergence of the Christaller–Lösch hexagon form the BLV models. Lacking a concrete understanding of the bifurcation mechanisms that govern models in hand may considerably limit the effectiveness of the conclusions and the possibilities of their empirical applications.

Numerical and analytical approaches for the BLV models—including the HW model—to date seem placed on two extremes of the trade-off of generality and clarity: the former numerical approach assumes generality, while the latter assumes clarity. This study is an attempt to bridge this gap form the "clarity" side.[8]

3 The Model

3.1 Brief Introduction to the Boltzmann–Lotka–Volterra Method

For self-containedness, we first outline the BLV method by Wilson (2008) and clarify its economic interpretation before introducing the HW model. Consider a space equipped with K discrete locations. We assume an index set of locations $\mathcal{K} \equiv \{0, 1, \ldots, K-1\}$. To introduce a spatial dimension on $\mathcal{K}$, the transport cost patterns between the locations are exogenously given by $\boldsymbol{T} = [t_{ij} \mid i, j \in \mathcal{K}]$. $\boldsymbol{T}$ is the so-called generalized transport cost: it is assumed that $\boldsymbol{T}$ includes any cost associated with the travel between locations. We want to model spatial structures, possibly spatial agglomeration patterns, at *equilibria* (in some sense) on this space. Let $\boldsymbol{h} \equiv [\ldots, h_i, \ldots]^\top \in \mathbb{R}_+^K$ be a non-negative K-dimensional real that represents the spatial structure. $h_i \geq 0$ may be interpreted as the number of residents, shops, or firms at location i depending on the modeler's interest. Of significance here is that $\boldsymbol{h}$ is endogenously determined by the model.

The "Boltzmann" component, or the *fast dynamic*, of the BLV method models the *short-run* equilibrium. In the short-run, $\boldsymbol{h}$ is assumed to be fixed. Modeled here is the *spatial interaction pattern* $\boldsymbol{S} \equiv [S_{ij} \mid i, j \in \mathcal{K}] \in \mathbb{R}_+^{K \times K}$ between locations. $\boldsymbol{S}$ is a matrix whose elements can be

[8] Ikeda et al. (2012a) and Ikeda et al. (2014a) are in the same research direction. Ikeda et al. (2012b) shows the emergence of the Christaller–Lösch hexagon form a NEG model by the group-theoretic bifurcation theory. Ikeda et al. (2014a) use computational bifurcation theory to numerically reveal the emergence of the hexagonal pattern.

interpreted as, for example, the monetary flows or the trip patterns form origins i to destinations j. $\boldsymbol{S}$ should depend on $\boldsymbol{h}$ to obtain non-trivial results. As the analysis of the HW model will show, $\boldsymbol{S}(\boldsymbol{h})$ may be subject to some constraints (e.g., the conservation of trip demands at the origins, the total transport cost spent by the spatial interaction). The BLV method assumes that $\boldsymbol{S}(\boldsymbol{h})$ arises as a result of an entropy-maximizing problem with such constraints. Let the set of all feasible spatial interaction patterns at $\boldsymbol{h}$ be $\mathcal{S}(\boldsymbol{h}) \subseteq \mathbb{R}_+^{K\times K}$. Then, the "most probable" spatial interaction pattern is obtained by solving the following problem (Wilson, 1970a, 1970b):

$$\max_{\boldsymbol{S}\in\mathcal{S}(\boldsymbol{h})} . \mathcal{H}(\boldsymbol{S}) \equiv -\sum_{i\in\mathcal{K}}\sum_{j\in\mathcal{K}} S_{ij}\log[S_{ij}] \tag{1}$$

where $\mathcal{H}(\cdot)$ is the Boltzmann–Shannon entropy.

The "Lotka–Volterra" component of the BLV method, or the *slow dynamic*, on the other hand, models the long-run equilibration of $\boldsymbol{h}$. Given the spatial distribution $\boldsymbol{h}$ and the associated spatial interaction pattern $\boldsymbol{S}(\boldsymbol{h})$ in the short-run, the modeler specifies the *payoff function* $\boldsymbol{v}(\boldsymbol{h}, \boldsymbol{S}(\boldsymbol{h})) = [\ldots, v_i(\boldsymbol{h}, \boldsymbol{S}(\boldsymbol{h})), \ldots]^\top$ that essentially maps the state $\boldsymbol{h}$ to the incentive landscape on $\mathcal{K}$. In the long-run, $\boldsymbol{h}$ is allowed to dynamically evolve depending on the payoff $\boldsymbol{v}(\boldsymbol{h}, \boldsymbol{S})$ and the state $\boldsymbol{h}$ itself. The dynamic change of $\boldsymbol{h}$ is assumed to be governed by some evolutionary dynamic $\boldsymbol{F}$ that takes $\boldsymbol{h}$ and $\boldsymbol{v}(\boldsymbol{h}, \boldsymbol{S})$ as the inputs:

$$\dot{\boldsymbol{h}} = \boldsymbol{F}(\boldsymbol{h}, \boldsymbol{v}(\boldsymbol{h}, \boldsymbol{S})) \tag{2}$$

Typically, the locations with relatively larger values of $v_i(\boldsymbol{h})$ are assumed to glow faster, and vice versa, under the dynamic $\boldsymbol{F}$. Such dynamics are very similar to—or perhaps, to some extent, inspired by—population dynamics model in mathematical and theoretical biology, hence the name "Lotka–Volterra." The *equilibrium* for a BLV model is defined as a stationary point of the above long-run dynamic: a point $\boldsymbol{h}^*$ that satisfies $\boldsymbol{F}(\boldsymbol{h}^*) = \boldsymbol{0}$. This combination of the fast and slow dynamic is the basic theoretical framework of the BLV method. Although we used the terminologies "location," "flows," or "space" to fix the ideas consistent with the focus of this paper (the retailer model), the applicability of the BLV method is not limited to spatial problems.[9]

Implicit in the BLV method is utility/profit-maximizing economic agents who select their own strategies and the standard spatial equilibrium condition based on no arbitrage. As it is noted by Wilson (2008), there is a close linkage between such formalism and the BLV method. Actually, it is just a matter of perception, or the preference of the modeler. From an economics perspective, $\boldsymbol{h}$ can be interpreted as an aggregated variable determined by the infinitesimally small *agents' choices* (e.g., the choices by consumers or firms): h_i is the number of agents selecting location i. The payoff function $v_i(\boldsymbol{h})$ is the indirect utility (for consumers) or profit (for firms) enjoyed by those who are choosing location $i \in \mathcal{K}$ at the given spatial distribution, or state of the system, $\boldsymbol{h}$. In formal economic models such as those of NEG, the spatial interaction pattern $\boldsymbol{S}(\boldsymbol{h})$ and payoff vector $\boldsymbol{v}(\boldsymbol{h}, \boldsymbol{S})$ are obtained form the result of market interaction between utility/profit-maximizing agents. The "Boltzmann" component of the BLV method models this mechanism in a reduced-form way that is characterized by the entropy-maximization problem.[10]

[9]For instance, Wilson (2008) discusses an application of the method to the analysis of scale-free networks.

[10]Note that there is concrete correspondence between the entropy-maximization problem and utility-maximization in the representative consumer model with the constant elasticity of substitution (CES) preference (Anderson et al., 1992). The former is, therefore, actually a reduced-form model of the latter. It is also equivalent to a random utility model (namely logit model).

In the BLV method, equilibria are defined as the stationary points of the slow dynamic. The standard spatial equilibrium condition in economics is different. It is formulated as the following no arbitrage condition

$$\begin{cases} V = v_i(\boldsymbol{h}) & \text{if } h_i > 0 \\ V \geq v_i(\boldsymbol{h}) & \text{if } h_i = 0 \end{cases} \qquad \forall i \in \mathcal{K} \tag{3}$$

where V is an equilibrium payoff. When the total number of agents is an exogenous constant H, that is, when the following conservation equation

$$\sum_{i\in\mathcal{K}} h_i = H, \tag{4}$$

holds, V is endogenously determined to satisfy the equilibrium condition. In the literature of Urban Economics (Fujita, 1989), such formulation is called the *closed-city* model reflecting no change in the number of agents in the system. A slightly different formulation is also possible. When V is exogenously given (e.g., when V is assumed to be zero), whereas the total number of agents H is endogenously determined. In contrast to the closed-city model, this type of formulation is called the *open-city* model because the total number of agents may vary form one equilibrium to another form the migration of agents form/to the outside of the system. For either type of model, we define the equilibrium condition and the dynamic such that the two conditions are mutually consistent.

In the following subsections, we construct the retailer model by Harris and Wilson (1978), which is the main focus of this paper. The model is a canonical example of the BLV method. The HW model can be viewed as an open-city type model described above. Although we use a slightly different interpretation on the variables for consistency with the economic literature, essential bifurcation properties are not changed.

3.2 The Fast Dynamic: Spatial Interaction Because of Consumers' Shopping

Consider a city that is discretized into K zones and associated centroids. We denote the set of these discrete zones by $\mathcal{K} \equiv \{0, 1, \ldots, K-1\}$. Generalized transport costs between zones are given exogenously by $\boldsymbol{T} \equiv [t_{ij} \mid i, j \in \mathcal{K}]$, as in the previous subsection. In each zone, there is a continuum of retailing firms. Each operates a retailing shop. The number of retailers at zone i is denoted by $h_i \geq 0$. There is a fixed portion of consumers residing at each zone. Consumers are assumed to inelastically buy retail goods. The total per capita consumer demand for shopping activity is an exogenously given constant O_i in each zone $i \in \mathcal{K}$. In the HW model, we are interested in the *equilibrium*—we provide a precise definition later in this section—spatial distribution $\boldsymbol{h}^*$ of the retailers.

We assume that, in the short-run, the consumers' shopping behavior is captured by a set of origin-constrained gravity equations. We denote the spatial distribution of retailers by a K-dimensional vector $\boldsymbol{h} \equiv [\ldots, h_i, \ldots]^\top \in \mathbb{R}_+^K$. In the short time scale, $\boldsymbol{h}$ is assumed to be *fixed constants*. The consumer demand $S_{ij}(\boldsymbol{h})$ form zone i to j, measured as a cash flow, is modeled as

$$S_{ij}(\boldsymbol{h}) = \frac{1}{\Delta_i(\boldsymbol{h})} h_j^\alpha \exp[-\beta t_{ij}] O_i \qquad \forall i, j \in \mathcal{K} \tag{5}$$

where $\Delta_i(\boldsymbol{h})$ is a normalizing function

$$\Delta_i(\boldsymbol{h}) \equiv \sum_{k\in\mathcal{K}} h_k^{\alpha} \exp[-\beta t_{ik}] \qquad \forall i \in \mathcal{K} \tag{6}$$

that ensures the conservation of demand form zone i:

$$\sum_{j\in\mathcal{K}} S_{ij}(\boldsymbol{h}) = O_i \qquad \forall i \in \mathcal{K} \tag{7}$$

In $S_{ij}(\boldsymbol{h})$, $\alpha, \beta > 0$ are assumed to be exogenous parameters. The term h_i^{α} is interpreted as the *attractiveness* of the retailers in the zone i where α determines the economy of scale. When $\alpha < 1$, it represents diminishing return with respect to scale h_i, $\alpha = 1$ constant return, and $\alpha > 1$ increasing return. As we will discuss, an interesting case is when $\alpha > 1$. On the other hand, β dictates how fast the demand decreases with travel cost t_{ij}. Therefore, $\exp[-\beta t_{ij}]$ as a whole, is interpreted as impedance of interaction form zone i to j.[11] Thus, β can be interpreted as a *global transportation cost parameter* that controls the level of spatial interaction in the city. Later in this paper, we assess the effect of gradually lowering β (i.e., improvement in the transportation technology). Note that in the HW model, the price of the retail good is absent.[12]

As summarized in the previous subsection, this specification of the spatial interaction $\boldsymbol{S}(\boldsymbol{h})$ between zones corresponds to the "Boltzmann" component of the BLV method. Simple computation shows that the spatial interaction function arises form the following entropy maximization problem

$$\max_{\boldsymbol{S}\in\mathbb{R}_+^{K\times K}} . \ \mathcal{H}(\boldsymbol{S}) \tag{8a}$$

$$\text{s.t.} \ \sum_{j} S_{ij} = O_i \tag{8b}$$

$$\sum_{i}\sum_{j} S_{ij} \log h_j = A \tag{8c}$$

$$\sum_{i}\sum_{j} S_{ij} t_{ij} = B \tag{8d}$$

with α, β are the associated Lagrange multipliers for the constraints (8c),(8d), respectively, and A, B are given constants (Wilson, 2008).

3.3 Short-run Retailer Profits

In the following three subsections, we focus on the long-run equilibrium for $\boldsymbol{h}$. We first define retailer profit and specify the incentive landscape induced by the short-run spatial interaction. First, summing up the consumers' demands form all other zones, the total revenue $S_i(\boldsymbol{h})$ of all retailers locating in zone i is given by

$$S_i(\boldsymbol{h}) \equiv \sum_{j\in\mathcal{K}} S_{ji}(\boldsymbol{h}) = \sum_{j\in\mathcal{K}} \frac{1}{\Delta_j(\boldsymbol{h})} h_i^{\alpha} \exp[-\beta t_{ji}] O_j. \tag{9}$$

[11] Note that a change of coordinate $\beta := \log \tau$ with $\tau > 1$ yields the iceberg transport technology usually assumed in NEG models, because then we can write $\exp[-\beta t_{ik}] = \tau^{-t_{ik}}$.

[12] If we allow some reduced-form, the price of the retail good (and the land rent) can be easily added to the model (see, Wilson, 2000, for a survey).

We assume that the revenue is equally distributed for all firms in the zone. In other words, after choosing zone i for the shopping destination, the visiting consumers select every shop (firm) at zone i with equal probability $1/h_i$. Then, the revenue of a single firm at zone i is given by $S_i(\boldsymbol{h})/h_i$. We also assume that the cost that a firm must pay at each zone consists of only a fixed entry cost $\kappa_i > 0$. Then, the profit of a firm at zone i is[13]

$$\Pi_i(\boldsymbol{h}) = \frac{S_i(\boldsymbol{h})}{h_i} - \kappa_i = \sum_{j\in\mathcal{K}} \frac{1}{\Delta_j(\boldsymbol{h})} h_i^{\alpha-1} \exp[-\beta t_{ji}] O_j - \kappa_i. \tag{10}$$

Using the above profit function Π_i, the "profit function" of firms in the original Harris and Wilson paper is $\hat{\Pi}_i \equiv h_i \Pi_i = S_i - \kappa_i h_i$. The original profit function is hence interpreted as the aggregated profit of zone i in our model. In effect, the original model assumes a single, large firm that is oligopolistically operating a large retailing shop at each zone. These large firms are assumed to change h_i, which is interpreted as the *capacity* of the shop zone i (e.g., the floorspace of the shop), in response to the profit $\hat{\Pi}_i$.

It is convenient to define the *spatial discounting matrix* $\boldsymbol{D}$ for further analysis. The spatial discounting matrix is a K-by-K matrix, whose (i, j)-th element is defined by $d_{ij} \equiv \exp[-\beta t_{ij}]$. Using $\boldsymbol{D}$, we have a useful vector-form expression of the profit function

$$\boldsymbol{\Pi}(\boldsymbol{h}) = \boldsymbol{M}^\top \boldsymbol{O} - \boldsymbol{\kappa} \tag{11a}$$

$$\boldsymbol{M} \equiv \operatorname{diag}[\boldsymbol{\Delta}]^{-1} \boldsymbol{D} \operatorname{diag}[\boldsymbol{h}]^{\alpha-1} \tag{11b}$$

$$\boldsymbol{\Delta} \equiv \boldsymbol{D} \operatorname{diag}[\boldsymbol{h}]^{\alpha} \mathbf{1} \tag{11c}$$

where $\boldsymbol{O} \equiv [\ldots, O_i, \ldots]^\top$, $\boldsymbol{\kappa} \equiv [\ldots, \kappa_i, \ldots]^\top$. $\mathbf{1}$ denotes a vector of appropriate dimension whose elements are all 1. Throughout this paper, $\operatorname{diag}[\boldsymbol{a}]$ denotes a diagonal matrix whose diagonal entries are given by the vector $\boldsymbol{a}$ and off-diagonals are all 0.

The vector-form expression using $\boldsymbol{D}$ has some utility other than its apparent simplicity. By its definition, the transport cost structure $\boldsymbol{T}$ and the associated impedance structure $\{d_{ij}\}$ of the spatial interaction are completely encapsulated in $\boldsymbol{D}$. In other words, $\boldsymbol{D}$ contains all the relevant information of the underlying physical space. From this, the functional form of the profit function, in relation to $\boldsymbol{D}$, reveals the way in which it depends on the physical space. In our payoff function (11), the first term arises form spatial interaction between zones (more accurately consumers' shopping behavior). The second term is a zone-specific term that does not depend on the physical space because it is a fixed cost at each zone (as we assumed). The former depends on $\boldsymbol{D}$ while the latter does not. Moreover, using the vector-form expression, we see *how* the former depends on $\boldsymbol{D}$ in a macroscopic manner compared to looking directly at the element-wise equation (10). In Section 5, we show the true value of this vector-form expression by comparing two distinct agglomeration models side by side.

3.4 Firms' Entry–Exit Behavior and the Equilibrium Condition

In the previous subsection we defined the payoff structure for the model. We now formulate the equilibrium condition. We assume that in the retailing market there are infinitely many potential

[13] A problem occurs, namely, division by zero, when $\alpha < 1$ and $h_i = 0$. However, throughout this paper, we focus our attention on the case $\alpha > 1$ that does not cause any mathematical problem.

entrants seeking profit opportunities. We also assume a tie-breaking rule: if the retailers' profit is zero at some zones, new firms enter these zones. Therefore, in the long run, retailer profits are exhausted by the entry–exit behavior of the retailing firms. In effect, for a spatial distribution of firms $\boldsymbol{h}$ to be stationary, we require the following zero-profit condition:

$$\begin{cases} \Pi_i(\boldsymbol{h}) = 0 & \text{if } \ h_i > 0 \\ \Pi_i(\boldsymbol{h}) \leq 0 & \text{if } \ h_i = 0 \end{cases} \qquad \forall i \in \mathcal{K} \tag{12}$$

We call a spatial distribution of firms $\boldsymbol{h}$ an *equilibrium* if it satisfies the above condition. Comparing the condition to the standard equilibrium condition (3), our model is an open-city model: the level of profit at any equilibrium is exogenously given by zero.

The total number of the firms at an equilibrium is thus determined form the equilibrium condition itself. The equilibrium condition is equivalent to the following complementarity condition:

$$h_i \Pi_i(\boldsymbol{h}) = 0, \ h_i \geq 0, \ \Pi_i(\boldsymbol{h}) \leq 0 \qquad \forall i \in \mathcal{K} \tag{13}$$

Adding up (13), we have the following equation that holds at any equilibrium:

$$\sum_{i \in \mathcal{K}} h_i \Pi_i(\boldsymbol{h}) = \sum_{i \in \mathcal{K}} S_i(\boldsymbol{h}) - \sum_{i \in \mathcal{K}} \kappa_i h_i = 0 \tag{14}$$

Because the first term in the middle equation is the total demand form all the consumers in the city, it reduces to the following relation:

$$\sum_{i \in \mathcal{K}} O_i - \sum_{i \in \mathcal{K}} \kappa_i h_i = 0 \tag{15}$$

This is a conservation equation that constrains the total number of firms at any equilibrium.

The conservation equation is the same as the one derived by Harris and Wilson (1978). For the equilibrium condition, the original HW model employs a "balancing condition"

$$\hat{\Pi}_i(\boldsymbol{h}) \equiv S_i(\boldsymbol{h}) - \kappa_i h_i = 0 \qquad \forall i \in \mathcal{K}, \tag{16}$$

In the original model, each of the large firms is assumed to adjust their floorspace to meet the balancing condition: if $\hat{\Pi}_i$ is positive, then increase h_i, and vice versa. From an economics perspective, the assumption is somewhat implausible because, if firms are oligopolistic, given the demand function of consumers, every firm can adjust the floorspace of its shop strategically to maximize the profit instead of forcing it to zero. The combination of continuum-of-firms setting and zero-profit condition in our formulation is designed to avoid this deficiency in economic interpretation while preserving the structure of the original HW model. We can easily show that the "balancing condition" (16) and the zero-profit condition (12) are mathematically equivalent.

3.5 The Slow Dynamic: Adjustment and the Stability of Equilibria

Because of the return to scale modeled in the spatial interaction function (consumers' demand), the HW model admits multiple equilibria in a wide range of parameter values, particularly when $\alpha > 1$. Therefore, we must select the set of reasonable—under some criteria—equilibria among them. This paper focuses on the set of *stable* equilibria under an adjustment dynamic of $\boldsymbol{h}$. The dynamic introduced here corresponds to the slow dynamic, or the "Lotka–Volterra" component of the BLV method.

To define the stability of a given equilibrium, we employ stability under small perturbations (i.e., local stability). Specifically, we first define the adjustment dynamic $\boldsymbol{F}$ for the state variable $\boldsymbol{h}$. We restrict our attention to the neighborhood of an equilibrium $\boldsymbol{h}^*$—in our setting a point that satisfies the zero-profit condition (12). The dynamic $\boldsymbol{F}$ should include all the equilibrium points in the set of its stationary point. The stability of $\boldsymbol{h}^*$ is defined in the sense of linear asymptotic stability under $\boldsymbol{F}$. The theory of dynamical systems posits that a stationary point $\boldsymbol{h}^*$ of $\boldsymbol{F}$ is linearly stable if all the eigenvalues of the Jacobian matrix at the point, $\nabla \boldsymbol{F}(\boldsymbol{h}^*) \equiv [\partial F_i(\boldsymbol{h}^*)/\partial h_j]$, have negative real parts; $\boldsymbol{h}^*$ is linearly unstable if at least one of the eigenvalues has a positive real part. These facts facilitate the investigation of the stability of a given equilibrium $\boldsymbol{h}^*$ by analyzing the eigenvalues of the Jacobian matrix at the point.

The adjustment dynamic $\boldsymbol{F}$ is the slow dynamic that determines the long-run spatial distribution of the firms. So far, we have not discussed *how* the equilibrium condition (12) is achieved. In this paper, consistent with Harris and Wilson (1978), we assume that the spatial pattern $\boldsymbol{h}$ gradually evolves in proportion to the profit $\boldsymbol{\Pi}(\boldsymbol{h})$ and the state $\boldsymbol{h}$ itself. Specifically, we assume that the time evolution of $\boldsymbol{h}$ is governed by the following dynamic:

$$\dot{\boldsymbol{h}} = \boldsymbol{F}(\boldsymbol{h}) \equiv \operatorname{diag}[\boldsymbol{h}]\boldsymbol{\Pi}(\boldsymbol{h}) \tag{17a}$$

or in the element-wise manner

$$\dot{h}_i = h_i \cdot \Pi_i(\boldsymbol{h}) = S_i(\boldsymbol{h}) - \kappa_i h_i. \qquad \forall k \in \mathcal{K} \tag{18}$$

Note that the dynamic is consistent with the aforementioned entry–exit behavior of firms and, hence, the set of stationary points of the dynamic coincides with the set of equilibrium points. Every stationary point satisfies the equilibrium condition (12) and vice versa.

The Jacobian matrix for the above dynamic is given as follows:

$$\nabla \boldsymbol{F}(\boldsymbol{h}) = \operatorname{diag}[\boldsymbol{\Pi}(\boldsymbol{h})] + \operatorname{diag}[\boldsymbol{h}]\nabla\boldsymbol{\Pi}(\boldsymbol{h}) \tag{19}$$

where $\nabla\boldsymbol{\Pi}(\boldsymbol{h})$ is the Jacobian matrix of firms' profit at $\boldsymbol{h}$:

$$\nabla\boldsymbol{\Pi}(\boldsymbol{h}) = \operatorname{diag}[\tilde{\boldsymbol{M}}^\top \boldsymbol{O}] - \alpha \boldsymbol{M}^\top \operatorname{diag}[\boldsymbol{O}]\boldsymbol{M} \tag{20a}$$

$$\tilde{\boldsymbol{M}} \equiv (\alpha - 1)\operatorname{diag}[\boldsymbol{\Delta}]^{-1}\boldsymbol{D}\operatorname{diag}[\boldsymbol{h}]^{\alpha-2} \tag{20b}$$

As we have summarized, given equilibrium spatial distribution of firms, we can assess the stability of the equilibrium by inspecting the eigenvalues of this Jacobian matrix.

Although we employ an explicit eigenvalue analysis of $\nabla\boldsymbol{F}$ to reveal the bifurcation properties of the HW model, we note another approach to find the set of locally stable equilibria for the HW model. Observe that the vector field defined by the profit function (11) admits a *potential function* discussed in Rijk and Vorst (1983a). Specifically, if we define a scalar valued function $W : \mathbb{R}_+^K \to \mathbb{R}$ by

$$W(\boldsymbol{h}) = \frac{1}{\alpha}\sum_{i\in\mathcal{K}} O_i \log\left(\sum_j h_j^\alpha \exp[-\beta t_{ij}]\right) - \sum_{i\in\mathcal{K}} \kappa_i h_i \tag{21}$$

then $W(\boldsymbol{h})$ satisfies a remarkable relationship between $\boldsymbol{\Pi}(\boldsymbol{h})$: $\nabla W(\boldsymbol{h}) = \boldsymbol{\Pi}(\boldsymbol{h})$ holds. In other words, $\boldsymbol{\Pi}(\boldsymbol{h})$ is a *potential field*. In the literature of evolutionary games, if the payoff vector field admits a potential function, any isolated local maximizer of the potential is locally asymptotically stable under a wide class of adjustment dynamics (see Sandholm (2010) for a recent textbook treatise). Therefore, given all the relevant parameters (i.e., $\alpha, \beta, \boldsymbol{O}, \boldsymbol{\kappa}$ and $\boldsymbol{T}$), the set of stable equilibria coincides with the set of locally maximizing isolated Karush–Kuhn–Tucker (KKT) points for the following parametric optimization problem:

$$\max_{\boldsymbol{h} \geq \boldsymbol{0}} . \, W(\boldsymbol{h} \mid \alpha, \beta, \boldsymbol{\kappa}, \boldsymbol{O}, \boldsymbol{T}) \tag{22}$$

At least in principle, we can obtain the set of all the stable rest points and assess the properties by analyzing the optimization problem. For example, if $\alpha \in (0, 1)$, we can easily prove that $W(\boldsymbol{h})$ is a strictly concave function in the interior of $\mathbb{R}_+^K$ and goes to $-\infty$ at the boundary. Hence, the optimization problem is a convex programming problem: we have a single, isolated global maximizer. In such a case, the equilibrium solution is unique and globally asymptotically stable. This result coincides with that of Rijk and Vorst (1983a).

The set of all the isolated locally maximizing KKT points for a given optimization problem, however, can not be analytically obtained in general. Particularly when $\alpha > 1$, $W(\boldsymbol{h})$ becomes a non-convex function and a multiple isolated local maximum may exist. In such a case, it is practically impossible to enumerate all the possible local maximizers. Moreover, many of these locally maximizing points may be irrelevant when looking at some specific evolutionary paths of spatial structure: some equilibria are not achievable through reasonable evolutionary paths. In the following sections, we focus on the case $\alpha > 1$ where a spatial increasing return causes multiple equilibria. In this situation, it is suitable to employ a more direct approach that follows evolutionary paths of spatial structure using the eigenvalues and eigenvectors of $\nabla \boldsymbol{F}(\boldsymbol{h}^*)$. At Lemma 4, we provide another justification for requiring $\alpha > 1$.

4 Lowering Transport Cost and the Evolution of Spatial Structure

4.1 Changing Structural Parameters and Bifurcations

In the previous section, we formulated the HW model. The definition of its equilibria and their stability has been introduced. What has been ignored so far is *the effect of changes in exogenous structural parameters* such as α, β, $\boldsymbol{O}$, $\boldsymbol{\kappa}$, and $\boldsymbol{T}$. What will happen to a stable equilibrium pattern when parameters, say, α or β gradually change? The answer is that we may encounter catastrophic phase transitions.

Such comparative static (bifurcation) analysis have been done by Harris and Wilson (1978) or Clarke (1981), and it is suggested that there will be catastrophic phase transitions in spatial patterns at some critical parameter values although not in a fully systematic way. Their analysis depend upon graphical tricks that pay attention to only a single zone at once. Although one can draw some qualitative conclusions form such analysis, it does not provide any insight into the behavior of the system of many zones. More systematic treatise can be seen in Rijk and Vorst (1983a), but the focus is on qualitative properties such as existence and uniqueness. Clarke and Wilson (1985), and more

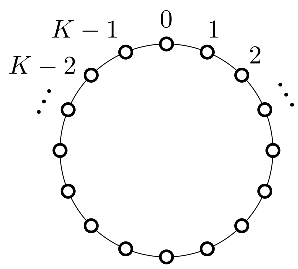


Figure 1: The racetrack economy ($K = 16$). Small white circles represent possible entry locations (i.e., zones) for a retailing firm. Lines connect neighboring zones.

recently Dearden and Wilson (2015), report the result of extensive numerical assessments on the effect of changing α, β and revealed the properties of equilibrium spatial distributions at different (α, β) pairs. Such an approach, with high-quality visualizations, is useful in providing qualitative insights but does not provide clear-cut conclusions.

This section, through explicit stability analysis, unveils previously unknown (at least in an explicit sense) bifurcation properties of the HW model in line with a decreasing transportation cost parameter β. We employ a method consistent with Akamatsu et al. (2012). The key components of the method are (i) the spatial discounting matrix $\boldsymbol{D}$, which we have already mentioned, (ii) the racetrack zone system, and (iii) discrete Fourier transformation (DFT). The combination of these three components allows us to analytically derive eigenpairs for the Jacobian matrix $\nabla \boldsymbol{F}$ of the adjustment dynamics at the equilibrium points of interest. Hence, we can explicitly study the (in)stability of equilibria and bifurcations.

4.2 Racetrack Zone System and the Uniform Equilibrium

Following the classical tradition of geography, we consider a homogeneous space over which all the underlying parameters are uniform: $O_i = O$, $\kappa_i = \kappa$. Let us assume that all zones are equivalent, and there is no zone with better access to consumers. Specifically, for the underlying physical space, we assume the *racetrack zone system*, or the *racetrack economy*, in which all K zones are placed equidistantly on a circumference (see Figure 1). For instance, in a line segment, the zones near the boundaries have fewer opportunities to access consumers; the central portion of the segment has better access. In a racetrack economy, however, every zone has the same level of accessibility to the other zones. This assumption may also be interpreted as an approximation of an infinite line.

In the racetrack zone system, the travel cost t_{ij} between two zones i and j is defined as the shortest path length on the circumference, i.e.,

$$t_{ij} = (2\pi/K) \cdot m(i, j) \tag{23}$$

$$m(i, j) \equiv \min.\{|i - j|, K - |i - j|\} \tag{24}$$

Under this setting, (i, j)-th entry of spatial discounting matrix $\boldsymbol{D}$, d_{ij}, is given by

$$d_{ij} = \exp[-\beta t_{ij}] = r^{m(i,j)} \tag{25}$$

where r is the *spatial discounting factor* that captures the accessibility between two consecutive zones on the circumference:

$$r \equiv \exp[-\beta(2\pi/K)]. \tag{26}$$

By definition, r is a strictly decreasing function of β. Corresponding to $\beta \in (0, \infty)$, the feasible range of r is $(0, 1)$: $\beta = 0 \Leftrightarrow r = 1$ and $\beta \to \infty \Leftrightarrow r \to 0$. Because our focus is decreasing β, it corresponds to increasing r. For convenience, we use r, not β, in the remainder of the paper. We interpret $r \in (0, 1)$ as a *global freeness parameter of spatial interaction between zones*.

In racetrack zone system, $\boldsymbol{D}$ is a matrix with a special structure called a *circulant* property. A K-dimensional circulant matrix is a matrix for which each row is the previous row cycled forward one step; the entries in each row are a cyclic permutation of the entries in the first row. In the racetrack economy, $\boldsymbol{D}$ is a circulant matrix generated by the row vector

$$\boldsymbol{d}_0 \equiv [1, r, r^2, \ldots, r^M, \ldots, r^2, r] \tag{27}$$

where $M \equiv K/2$ (we assume K is an even). For example, when $K = 4$, $\boldsymbol{d}_0 = [1, r, r^2, r]$. $\boldsymbol{D}$ is then

$$\boldsymbol{D} = \begin{bmatrix} 1 & r & r^2 & r \\ r & 1 & r & r^2 \\ r^2 & r & 1 & r \\ r & r^2 & r & 1 \end{bmatrix} \tag{28}$$

The k-th row $(k = 1, 2, 3)$ of the above $\boldsymbol{D}$ is given by rotating the first row right k times. The fact that $\boldsymbol{D}$ is a circulant that is parametrized by a single global parameter r plays a key role in the analysis of this paper.

We assume that the spatial distribution of retailers is initially uniform. We call this spatial structure the "flat-earth equilibrium" and denote it by $\bar{\boldsymbol{h}} = [h, h, \ldots, h]^\top$. $\bar{\boldsymbol{h}}$ is actually an equilibrium that satisfies the equilibrium condition (12). From the conservation condition (15), h is given by

$$h = \frac{O}{\kappa} \tag{29}$$

We assume $r \approx 0$ in the first place (i.e., β is quite large). Starting form $\bar{\boldsymbol{h}}$ and small r, we follow the evolution of spatial structure in the line of increasing accessibility r. In other words, we start form "corner-shop" spatial economy (Wilson and Oulton, 1983) in which the consumer's travel distance is short, and all shopping demands are met by retail shops within each zone.

4.3 Emergence of Agglomeration: Destabilization of the Uniform Equilibrium

We first investigate the stability of the flat-earth equilibrium $\bar{\boldsymbol{h}}$. To analytically examine the stability of the uniform equilibrium, we must derive the eigenvalues of the Jacobian matrix $\nabla \boldsymbol{F}(\bar{\boldsymbol{h}})$. The Jacobian matrix under the configuration, $\nabla \boldsymbol{F}(\bar{\boldsymbol{h}})$, is obtained as

$$\nabla \boldsymbol{F}(\bar{\boldsymbol{h}}) = h \nabla \boldsymbol{\Pi}(\bar{\boldsymbol{h}}) \tag{30a}$$

$$\nabla\mathbf{\Pi}(\bar{\boldsymbol{h}}) = \frac{\alpha O}{h^2}\left(-\bar{\boldsymbol{D}}^2 + \hat{\alpha}\boldsymbol{I}\right) \tag{30b}$$

where $\hat{\alpha} \equiv 1 - \alpha^{-1}$, $\bar{\boldsymbol{D}} \equiv \boldsymbol{D}/d$ with $d \equiv \boldsymbol{d}_0 \cdot \mathbf{1}$ is the row-normalized spatial discounting matrix. Because $\bar{\boldsymbol{D}}$ and $\boldsymbol{I}$ are both circulant matrices, $\nabla\mathbf{\Pi}(\bar{\boldsymbol{h}})$ and $\nabla\boldsymbol{F}(\bar{\boldsymbol{h}})$ are also circulant.[14]

Under general, inhomogeneous transport cost structures $\boldsymbol{T}$ between zones, we cannot obtain the eigenvalues and associated eigenvectors of $\nabla\boldsymbol{F}(\bar{\boldsymbol{h}})$ analytically. In such a setting, stability analysis at any equilibrium point is purely a numerical task and does not provide any clear insight into the essential bifurcation properties of the model. This is one of the most significant reasons that many previous analytical studies in geography and NEG have focused on spatially degenerated two-location models with great symmetry.

In the racetrack zone system, however, we can analytically obtain eigenpairs of $\nabla\boldsymbol{F}(\bar{\boldsymbol{h}})$ because $\nabla\boldsymbol{F}(\bar{\boldsymbol{h}})$ is a circulant. The eigenvalues and eigenvectors of a K-dimensional circulant matrix are obtained using the discrete Fourier transformation (DFT) matrix of the same dimension, $\boldsymbol{Z} = [z_{jk}]$ $(j, k = 0, 1, \ldots, K-1)$ with $z_{jk} \equiv \omega^{jk}$ where $\omega \equiv \exp[\mathrm{i}(2\pi/K)]$. Specifically, the following lemma holds:

Lemma 1 (e.g., Horn and Johnson (2012))**.** Let $\boldsymbol{A}$ be a K-dimensional circulant matrix with its first row vector being $\boldsymbol{a}_0 = [a_{0,i}]$. Then, the matrix $\boldsymbol{A}$ is of full rank and diagonalizable by similarity transformation by the K-dimensional discrete Fourier transformation matrix $\boldsymbol{Z}$: let $\boldsymbol{\lambda} \equiv [\lambda_0, \lambda_1, \ldots, \lambda_{K-1}]^\top$ be the eigenvalues of $\boldsymbol{A}$, then $\mathrm{diag}[\boldsymbol{\lambda}] = \boldsymbol{Z}\boldsymbol{A}\boldsymbol{Z}^{-1}$ holds. Moreover, the eigenvalues $\boldsymbol{\lambda}$ are given by the discrete Fourier transformation of $\boldsymbol{a}_0$. That is, $\boldsymbol{\lambda}$ satisfies

$$\boldsymbol{\lambda} = \boldsymbol{Z}\boldsymbol{a}_0^\top. \tag{31}$$

The eigenvectors of $\boldsymbol{A}$ are given by column vectors of $\boldsymbol{Z}$, i.e., the eigenvector associated with λ_k is

$$\boldsymbol{z}_k = [1, \omega^k, \omega^{2k}, \ldots, \omega^{k(K-1)}]^\top \qquad k = 0, 1, \ldots, K-1 \tag{32}$$

Using Lemma 1, $\nabla\boldsymbol{F}(\bar{\boldsymbol{h}})$ is diagonalizable by $\boldsymbol{Z}$, and the eigenvalues of $\nabla\boldsymbol{F}(\bar{\boldsymbol{h}})$ are obtained by the discrete Fourier transformation of its first row. By multiplying the K-dimensional DFT matrix $\boldsymbol{Z}$ form the left and $\boldsymbol{Z}^{-1}$ form the right side of (30a) and (30b), we obtain, respectively,

$$\mathrm{diag}[\boldsymbol{g}] = h \cdot \mathrm{diag}[\boldsymbol{e}] \tag{33a}$$

$$\mathrm{diag}[\boldsymbol{e}] = \frac{\alpha O}{h^2}\left(-\,\mathrm{diag}[\boldsymbol{f}]^2 + \hat{\alpha}\boldsymbol{I}\right) \tag{33b}$$

where $\boldsymbol{g}$, $\boldsymbol{e}$, $\boldsymbol{f}$ are the eigenvalues of $\nabla\boldsymbol{F}(\bar{\boldsymbol{h}})$, $\nabla\mathbf{\Pi}(\bar{\boldsymbol{h}})$, $\bar{\boldsymbol{D}}$, respectively. If we allow a convention $[\boldsymbol{v}] \cdot [\boldsymbol{w}] \equiv [v_i w_i]$ for component-wise product of two vectors $\boldsymbol{v}, \boldsymbol{w}$ and $[\boldsymbol{v}]^2 \equiv [\boldsymbol{v}] \cdot [\boldsymbol{v}]$, we obtain the expression shown below

$$\boldsymbol{g} = (\alpha\kappa)\left(-[\boldsymbol{f}]^2 + \hat{\alpha}\mathbf{1}\right) \tag{34}$$

in which we use the relation $h = O/\kappa$. Comparing equations (34) with (30), we notice that the functional relations between the vectors $\boldsymbol{g}, \boldsymbol{e}, \boldsymbol{f}$, and $\mathbf{1}$ are exactly the same as those between $\nabla\boldsymbol{F}$, $\nabla\mathbf{\Pi}$, $\bar{\boldsymbol{D}}$, and $\boldsymbol{I}$.

Summarizing this discussion, we have the following lemma concerning the eigenvalues $\boldsymbol{g}$ of the Jacobian matrix of the dynamic at the flat-earth equilibrium $\bar{\boldsymbol{h}}$:

[14] It is straightforward to show that the set of all circulant matrices is closed under addition and multiplication.

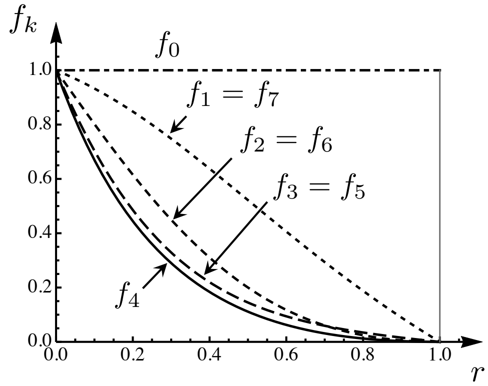


Figure 2: The eigenvalues $\boldsymbol{f}$ of $\bar{\boldsymbol{D}}$ for $K = 8$ as functions of r. Each f_k $(k = 1, 2, \ldots, 7)$ is strictly decreasing in r. Among them the minimal eigenvalue is f_4 for all r.

Lemma 2. Let $\boldsymbol{g}$ be the eigenvalues of the Jacobian matrix of the adjustment dynamic, $\nabla \boldsymbol{F}(\bar{\boldsymbol{h}})$, at the flat-earth equilibrium $\bar{\boldsymbol{h}}$ in the economy. Then, $\boldsymbol{g}$ satisfies

$$g_k = (\alpha\kappa) \cdot G(f_k) \tag{35a}$$

$$G(x) = -x^2 + \hat{\alpha} \tag{35b}$$

where $\boldsymbol{f} \equiv [f_0, f_1, \ldots, f_{K-1}]^\top$ are the eigenvalues of $\bar{\boldsymbol{D}}$. The k-th eigenvector associated with the k-th eigenvalue g_k is given by the k-th column vector $\boldsymbol{z}_k$ of the DFT matrix.

Proof. Straightforward application of Lemma 1, as we have discussed in the above. □

For the eigenvalues $\boldsymbol{f}$ of $\bar{\boldsymbol{D}}$ in a racetrack system, we have the following characterization form Akamatsu et al. (2012), which is illustrated in Figure 2 for the case $K = 8$.

Lemma 3 (Akamatsu et al. (2012), Lemma 4.2)**.** Let $\boldsymbol{f} \equiv [f_0, f_1, \ldots, f_{K-1}]^\top$ be the eigenvalues of the row-wise normalized spatial discounting matrix $\bar{\boldsymbol{D}} \equiv \boldsymbol{D}/d$ in the racetrack economy with K zones. Assume that K is a multiple of 4. Then, the eigenvalues $\boldsymbol{f}$ are obtained analytically by the discrete Fourier transformation of the first row $\boldsymbol{d}_0$. Moreover, the following holds:

(a) $f_0 = 1$. For $k = 1, 2, \ldots, K-1$, each f_k is a monotonically decreasing function of spatial discounting factor r, which takes a value on $(0, 1)$.

(b) The minimal eigenvalue $\min_k\{f_k\}$ is always f_M where $M \equiv K/2$. f_M and the associated eigenvector $\boldsymbol{z}_M$ are explicitly given as

$$f_M(r) = \left(\frac{1-r}{1+r}\right)^2 \tag{36a}$$

$$\boldsymbol{z}_M = [1, -1, 1, -1, \ldots, 1, -1]^\top \tag{36b}$$

Combining Lemma 2 and Lemma 3, we have the analytical expression of $\boldsymbol{g}$ in relation to relevant exogenous parameters α, κ, and, particularly, the spatial discounting factor r. The eigenvalues $\boldsymbol{g}$ of the Jacobian matrix of the dynamic can be interpreted as *net agglomeration forces* in the direction

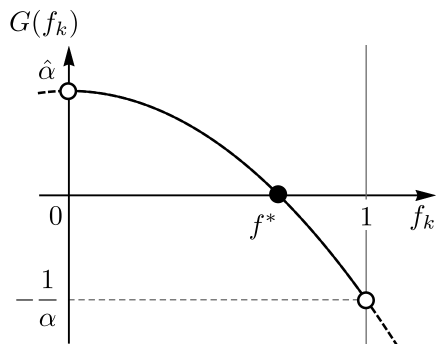


Figure 3: $G(f_k)$ as a function of f_k. $G(f_k)$ is a negative quadratic that takes the maximum value $\hat{\alpha}$ when $f_k = 0$ (or when $r = 1$). $G(f_k)$ is strictly decreasing in f_k and crosses the f_k-axis once at $f^* \equiv \sqrt{\hat{\alpha}}$ if and only if $\alpha > 1$.

of the associated eigenvectors. In $G(x)$, the first term is always negative. It is a dispersion force that arises form competition between retailers at different locations modeled in the gravity equations of consumer demand. The second term, $\hat{\alpha} \equiv 1 - \alpha^{-1}$, captures the effect of α. It is positive if $\alpha > 1$, zero if $\alpha = 1$, and negative if $\alpha < 1$. It is immediate that if $\alpha \leq 1$, $G(x)$ is always negative for $x \in (0, 1)$, implying that there are no agglomeration forces in the model. The flat-earth equilibrium $\bar{\boldsymbol{h}}$ is, therefore, always asymptotically stable at any r. On the other hand, if $\alpha > 1$, the second term $\hat{\alpha}$ is positive. In this case, it is interpreted as an agglomeration force. Therefore, we can interpret $G(x)$ as net agglomeration force in the sense that it is the agglomeration force (the second term) minus the dispersion force (the first term).

This paper is interested in the process of symmetry breaking and self-organization of the spatial structure. Depending on the value of α, there is a possibility that the destabilization of the flat-earth equilibrium never occurs. For instance, if for some k we have $g_k > 0\ \forall r$, flat-earth equilibrium $\bar{\boldsymbol{h}}$ cannot be stable at any level of transport cost r. On the other hand, $g_k < 0\ \forall r, k$ implies that $\bar{\boldsymbol{h}}$ is locally asymptotically stable for all r. Thus, we require that α lies in some specific range to ensure $G(f) = 0$ has at least one solution in $f \in (0, 1)$, so that some $\boldsymbol{g}$ changes its sign at some $f \in (0, 1)$. Elementary algebra shows the following lemma:

Lemma 4. The parameter α should satisfy

$$\alpha > 1 \tag{37}$$

to ensure that (a) the flat-earth equilibrium $\bar{\boldsymbol{h}}$ is stable for some small r (large f_k) and (b) a destabilization of $\bar{\boldsymbol{h}}$ occurs according to the increase of r (decrease of f_k).

Proof. Because $G(x)$ is strictly decreasing for $x \in [0, 1]$, and $f_k(r)$ is strictly decreasing for $r \in (0, 1)$, $G(f_k(r))$ is strictly increasing for $r \in (0, 1)$. Therefore, $G(f_k(r)) = 0$ has a unique solution r_k^* in $(0, 1)$ for each k if and only if $G(0) > 0$ and $G(1) < 0$. This yields the above condition. □

Figure 3 illustrates $G(f_k)$ as a function of f_k when $\alpha > 1$ holds. The condition states that if there is no increasing return effect in the spatial interaction function, the flat-earth equilibrium cannot destabilize even when the transport cost is low. We have seen another proof for this claim using the potential function $W(\boldsymbol{h})$ in the previous section.

We assume (37) holds in the remainder of the paper. In this situation, we have complete characterization of the first bifurcation form the flat-earth equilibrium. First, the solution for $G(f) = 0,\ f \in (0,1)$ is $f^* \equiv \sqrt{\hat{\alpha}}$ (see Figure 3). Second, we conclude that the critical value of r is given by the equation $f_M(r) = f^*$ because, form Lemma 3, we know f_k is strictly decreasing in r and $\min_k f_k = f_M$. Moreover, the bifurcation direction is $\boldsymbol{z}_M$ for the eigenvector associated with f_M. Thus, the next proposition holds:

Proposition 5. Suppose that α satisfies (37). We start form a high level of transportation cost (small r) in which the flat-earth equilibrium $\bar{\boldsymbol{h}}$ is stable and consider the steady decrease in transportation cost, i.e., steady increase of r.

(a) $\bar{\boldsymbol{h}}$ becomes unstable at the critical value $r^{(0)}$, which is given by

$$r^{(0)} \equiv \frac{1 - \sqrt[4]{\hat{\alpha}}}{1 + \sqrt[4]{\hat{\alpha}}},\ \hat{\alpha} \equiv 1 - \frac{1}{\alpha} \tag{38}$$

(b) The bifurcation at $r^{(0)}$ leads to a spatial structure

$$\boldsymbol{h} = \bar{\boldsymbol{h}} + \delta \boldsymbol{z}_M = [h+\delta, h-\delta, h+\delta, h-\delta, \ldots, h+\delta, h-\delta]^\top \quad (0 \leq \delta \leq h) \tag{39}$$

in which retailers in alternate regions increase.

(c) $r^{(0)}$ decreases as the increasing return parameter α increases.

Proof. (a) Solving $G(f_M(r)) = 0,\ r \in (0,1)$ we obtain $r^{(0)}$ in (38). (b) The destabilization of $\bar{\boldsymbol{h}}$ causes a phase transition in the direction of $\boldsymbol{z}_M$, which is the associated eigenvector for $g_M = (\alpha\kappa)G(f_M)$. (c) It is immediate that $\mathrm{d}r^{(0)}/\mathrm{d}\alpha < 0$. □

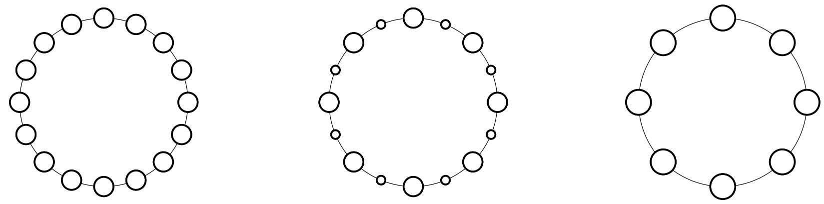
See Figure 4 for an illustration of the flat-earth equilibrium and the resulting pattern after the first bifurcation. The maximality of g_M—and the minimality of f_M—has an intuitive interpretation. Because $g_k = (\alpha\kappa) \cdot G(f_k)$ is the net agglomeration force *in the direction of* $\boldsymbol{z}_k$, the maximality of g_M implies that the most profitable direction of the aggregate motion of firms, compared to the other directions, is $\boldsymbol{z}_M$.

4.4 Successive Emergence of Spatial Structure

4.4.1 The Stability of $K/2$-Centric Pattern and the Second Bifurcation

After the first bifurcation form the flat-earth equilibrium, δ increases rapidly according to the increase in r. We expect that spacial configuration converges to another symmetric equilibrium pattern in which only alternate zones in $\mathcal{K}$ have retailers. We denote the $K/2$-centric spatial configuration by $\boldsymbol{h}^{(1)}$ (see Figure 4 (c)). The bracketed superscript indicates how much symmetry breaking has been experienced by increasing r. $\boldsymbol{h}^{(1)}$ is given by

$$\boldsymbol{h}^{(1)} \equiv \left[2h, 0, 2h, 0, \ldots, 2h, 0\right]^\top \tag{40}$$



(a) $\bar{\boldsymbol{h}}$ (b) $\bar{\boldsymbol{h}} + \delta \boldsymbol{z}_M$ (c) $\boldsymbol{h}^{(1)}$

Figure 4: The spatial patterns relevant to the first bifurcation ($K = 16$): (a) the flat-earth equilibrium $\bar{\boldsymbol{h}} = [h, h, \dots, h]^\top$; (b) the transient spatial structure $\bar{\boldsymbol{h}} + \delta \boldsymbol{z}_M = [h + \delta, h - \delta, h + \delta, \dots, h - \delta]^\top$ immediately after the first bifurcation at $r^{(0)}$; (c) $K/2$-centric equilibrium $\boldsymbol{h}^{(1)} = [2h, 0, 2h, 0, \dots, 2h, 0]^\top$.

in which the even-numbered zones $\{0, 2, \dots, K-2\}$ have twice the firms of the flat-earth equilibrium $\bar{\boldsymbol{h}}$ while the other zones have none because of the preceding bifurcation form $\bar{\boldsymbol{h}}$. It is immediate that the spacial configuration satisfies the equilibrium condition (12) and hence the conservation equation (15). Compared to the "corner-shop" interpretation of $\bar{\boldsymbol{h}}$, the spatial pattern $\boldsymbol{h}^{(1)}$ may be interpreted as a city with slight inhomogeneity because there are zones with no retailers.

There is an apparent symmetry and a resemblance to $\bar{\boldsymbol{h}}$ because, if we ignore zones without retailers, we obtain a "flat-earth" equilibrium although the number of zones is reduced to $K/2$. The similarity implies a hypothesis that the bifurcation form this equilibrium pattern, analogous to the bifurcation form $\bar{\boldsymbol{h}}$, further halves the number of market centers and leads to a $K/2^2$-centric pattern in which alternate zones in $\{0, 2, 4, \dots, K-2\}$ increase their retailing firms while the other zones decrease their retailing firms. We prove this intuition in the following.

As explained, we can assess the stability of an equilibrium using the eigenvalues of the Jacobian matrix of the dynamic. Repeating a similar discussion, we have the following lemma:

Lemma 6. Let $\boldsymbol{g}^{(1)} \equiv [g_0^{(1)}, g_1^{(1)}, \dots, g_{K-1}^{(1)}]^\top$ be the eigenvalues of the Jacobian of the adjustment dynamic at $K/2$-centric equilibrium $\boldsymbol{h}^{(1)}$. The maximal eigenvalue $g_{\text{max}}^{(1)} \equiv \max_k g_k^{(1)}$ is given by:

$$g_{\text{max}}^{(1)} = (\alpha\kappa) \cdot G^{(1)}(r) \tag{41a}$$

$$G^{(1)}(r) \equiv -\frac{1}{2}\left(\frac{1-r^2}{1+r^2}\right)^2 + \hat{\alpha} \tag{41b}$$

and the associated eigenvector is

$$\boldsymbol{z}_M^{(1)} \equiv [\underbrace{1, 0, -1, 0}, \underbrace{1, 0, -1, 0}, \dots, \underbrace{1, 0, -1, 0}]^\top \tag{42}$$

Proof. See Appendix A.1. □

As in $\bar{\boldsymbol{h}}$, depending on the value of α, there is a possibility that destabilization of $\boldsymbol{h}^{(1)}$ does not occur. Following the same line of logic as the uniform equilibrium $\bar{\boldsymbol{h}}$, we conclude the following:

Lemma 7. For the $K/2$-centric equilibrium $\boldsymbol{h}^{(1)}$ to be stable at some r and bifurcation to occur according to the increase of r, the parameter α should satisfy

$$1 < \alpha < 2 \tag{43}$$

Proof. We show that $\mathrm{d}G^{(1)}(r)/\mathrm{d}r > 0$ for $r \in (0,1)$ and, hence, $G^{(1)}(r)$ is a monotonically increasing function of r. Therefore, destabilization of $\boldsymbol{h}^{(1)}$ occurs if and only if $G^{(1)}(0) < 0$ and $G^{(1)}(1) > 0$, which is equivalent to (43). □

The condition $\alpha < 2$ requires that the increasing return effect should not be too strong so that $\boldsymbol{h}^{(1)}$ can be a stable equilibrium. Otherwise, $\boldsymbol{h}^{(1)}$ cannot be stable at any transport cost level r. $\alpha > 1$ is a condition which, as we have discussed for the bifurcation form $\bar{\boldsymbol{h}}$, requires that the increasing return effect should *exist* so that $\boldsymbol{h}^{(1)}$ becomes unstable according to an increase in r.

Starting form a small value of r at which $\boldsymbol{h}^{(1)}$ is stable, we gradually increase r. Under the condition (43), $\boldsymbol{h}^{(1)}$ destabilizes at some r. The racetrack zone system and symmetry of $\boldsymbol{h}^{(1)}$ reveals a complete characterization of this bifurcation. Specifically, we prove that the second bifurcation form $\boldsymbol{h}^{(1)}$ leads to the emergence of a $K/2^2$-centric pattern $\boldsymbol{h}^{(2)}$, as expected.

Proposition 8. Suppose that α satisfies (43). Let us start form a sufficiently high level of transportation cost (sufficiently small r) in which $\boldsymbol{h}^{(1)}$ is stable and consider the steady decrease in transportation cost. Then

(a) $\boldsymbol{h}^{(1)}$ becomes unstable at the critical value of r, which is given by

$$r^{(1)} = \sqrt{\frac{1 - \sqrt[4]{2\hat{\alpha}}}{1 + \sqrt[4]{2\hat{\alpha}}}}, \tag{44}$$

(b) The bifurcation leads to a pattern

$$\boldsymbol{h} = \boldsymbol{h}^{(1)} + \delta \boldsymbol{z}_M^{(1)} \qquad (0 \leq \delta \leq 2h) \tag{45}$$

in which retailers in alternate, even-numbered zones $\{0, 2, 4, \ldots, K-2\}$ increase.

(c) $r^{(1)}$ decreases as the increasing return parameter α increases.

Proof. (a) Solving $G^{(1)}(r) = 0$, we obtain $r^{(1)}$. (b) See Appendix A.1. (c) It is immediate to verify $\mathrm{d}r^{(1)}/\mathrm{d}\alpha < 0$. □

4.4.2 Recursive Emergence of Spatial Structure: The Spatial-period Doubling Cascade

Our analysis so far has shown that under suitable values of α, the first and second bifurcation lead to a spatially alternate pattern, each time reducing the number of the market centers (zones with retailers) by half: $K \to K/2 \to K/4$. This property implies the hypothesis that this recursive bifurcation process continues. If K is a power of 2, then this process may continue until the monocentric pattern is attained (see Figure 5). We call such an evolutionary path of spatial structure the *spatial period-doubling cascade* because each bifurcation doubles the distance between consecutive market centers. For clarity in presentation, we assume that $K = 2^J$ with $J \geq 2$ in the following.[15] Then, on the spatial period-doubling cascade path, there would be, at most, J such bifurcations.

[15] As discussed in Akamatsu et al. (2012), the applicability of our method is not limited exclusively to such a situation.

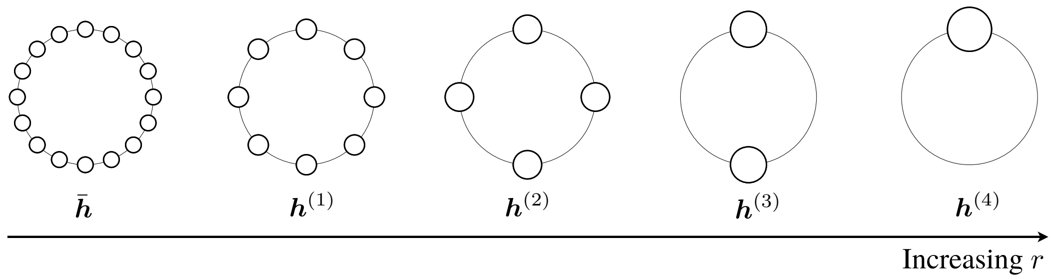


Figure 5: The evolutionary path of stable spatial structure in the spatial period-doubling cascade ($K = 2^4 = 16$). Each phase transition reduces the symmetry of the spatial structure.

To verify the occurrence of the spatial period-doubling cascade, we first examine the stability of $K/2^n$-centric equilibria $\boldsymbol{h}^{(n)}$ ($n = 1, 2, 3, \ldots, J-1$), which is defined as

$$\boldsymbol{h}^{(n)} \equiv \Big[\underbrace{2^n h, 0, 0, \ldots, 0}_{2^n}, \underbrace{2^n h, 0, 0, \ldots, 0}_{2^n}, \ldots, \underbrace{2^n h, 0, 0, \ldots, 0}_{2^n}\Big]^\top \tag{46}$$

and examine the bifurcation form these equilibria. We verify that these spatial configurations satisfy the equilibrium condition. Using essentially the same approach that we employed at the equilibria $\bar{\boldsymbol{h}}$ and $\boldsymbol{h}^{(1)}$, we have the following lemmas:

Lemma 9. Let $\boldsymbol{g}^{(n)} \equiv [g_0^{(n)}, g_1^{(n)}, \ldots, g_{K-1}^{(n)}]^\top$ be the eigenvalues of the Jacobian matrix of the adjustment dynamic at $K/2^n$-centric equilibrium $\boldsymbol{h}^{(n)}$ ($n = 1, 2, 3, \ldots, J-1$). The maximal eigenvalue $g_{\text{max}}^{(n)} \equiv \max_k g_k^{(n)}$ is given by

$$g_{\text{max}}^{(n)} = (\alpha\kappa) \cdot G^{(n)}(r) \tag{47a}$$
$$G^{(n)}(r) \equiv -\tilde{G}^{(n)}(r) + \hat{\alpha} \tag{47b}$$

where $\tilde{G}^{(n)}$ is a monotonically decreasing function of r, and its associated eigenvector is

$$\boldsymbol{z}_M^{(n)} \equiv \Big[\underbrace{1, 0, 0, \ldots, 0}_{2^n}, \underbrace{-1, 0, \ldots, 0}_{2^n}, \ldots, \underbrace{-1, 0, \ldots, 0}_{2^n}\Big]^\top \tag{48}$$

Proof. See Appendix A.2 for the proof and the exact definition of $\tilde{G}^{(n)}(r)$. □

Lemma 10. For the $K/2^n$-centric equilibrium $\boldsymbol{h}^{(n)}$ ($n = 1, 2, \ldots, J-1$) to be stable at some r, and bifurcation to occur according to the decrease in r, the parameter α should satisfy

$$1 < \alpha < 2^n \tag{49}$$

Proof. See Appendix A.3. Note that the first part of the inequality yields $G^{(n)}(1) > 0$ so that the destabilization occurs at a high r and in the pattern $G^{(n)}(0)$ so that there is a range of r where $\boldsymbol{h}^{(n)}$ is stable. □

Note that condition (49) relaxes as n increases, that is, if it is satisfied at some n, then all $n' > n$ satisfies (49). For instance, when $\alpha < 2$, $\alpha < 2^n$ for all $n \geq 2$.

Combining the above two lemmas, we have the following proposition that characterizes the bifurcation form $\boldsymbol{h}^{(n)}$:

Proposition 11. Suppose that α satisfies (49). We start form a sufficiently high level of transportation cost (sufficiently small r) in which $\boldsymbol{h}^{(n)}$ $(n = 1, 2, \ldots, J-1)$ is stable, and we consider the steady decrease in transportation cost.

(a) $\boldsymbol{h}^{(n)}$ becomes unstable at the critical value of transportation cost, $r^{(n)}$, which is given as the solution for the equation

$$G^{(n)}(r) = 0 \qquad r \in (0, 1) \tag{50}$$

(b) The bifurcation leads to a spatial pattern

$$\boldsymbol{h} = \boldsymbol{h}^{(n)} + \delta \boldsymbol{z}_M^{(n)} \qquad (0 < \delta < 2^n h) \tag{51}$$

in which retailers of the alternate market centers in $\boldsymbol{h}^{(n)}$ increase.

Proof. (a) Appendix A.3 shows that $G^{(n)}(r)$ is a monotonically increasing function for $r \in (0, 1)$. Thus, $G^{(n)}(r) = 0$ has a unique solution if (49) is satisfied, which gives the critical value $r^{(n)}$. (b) The direction of the bifurcation is $g_{\max}^{(n)}$'s associated eigenvector $\boldsymbol{z}_M^{(n)}$ given by (48). □

We show that Proposition 11 is a generalization of Proposition 8, although in this case (50) is not analytically tractable for $n \geq 2$.

We now have the (semi-)closed-form formulae of the critical points $r^{(0)}, r^{(1)}$, and $r^{(n)} (n = 0, 1, 2, \ldots, J-1)$. We have yet to show, however, under what condition $\boldsymbol{h}^{(n+1)}$ emerge after the bifurcation form $\boldsymbol{h}^{(n)}$. Specifically, $\boldsymbol{h}^{(n+1)}$ can be already unstable before $r^{(n)}$ is attained. If the critical values $\{r^{(n)}\}$ are increasing in n, each bifurcation reduces the number of market centers (zones with positive numbers of retailing firms) by half—the successive emergence of $\boldsymbol{h}^{(n)}$ is ensured. The required condition for the spatial period-doubling cascade is

$$r^{(n)} < r^{(n+1)} \qquad (n = 0, 1, 2, \ldots, J-2) \tag{52}$$

Akamatsu et al. (2012) has shown that a multi-regional extension of the Pflüger (2004)'s NEG model satisfies this property. Yet, because there are different types of agglomeration mechanisms in the HW model compared to Pflüger's model, there is no reason to expect that the HW model will have this property in general. We can prove that this property does hold for suitable values of α. The next section discusses the source of this similarity.

To verify the property (52), we at least require $r^{(0)} < r^{(1)}$ provided that these bifurcations occur (i.e., (37) and (43) holds). That is, α should be in the range $(1, 2)$, and the following inequality should hold:

$$\frac{1 - \sqrt[4]{\hat{\alpha}}}{1 + \sqrt[4]{\hat{\alpha}}} < \sqrt{\frac{1 - \sqrt[4]{2\hat{\alpha}}}{1 + \sqrt[4]{2\hat{\alpha}}}} \tag{53}$$

The inequality is analytically solvable, and we obtain the exact condition for α as shown below:

$$1 < \alpha < \bar{\alpha} \equiv \frac{\sqrt{2}}{4(\sqrt[4]{2} - 1)} \quad (\approx 1.87) \tag{54}$$

Given the above $\bar{\alpha}$, we have the following result:

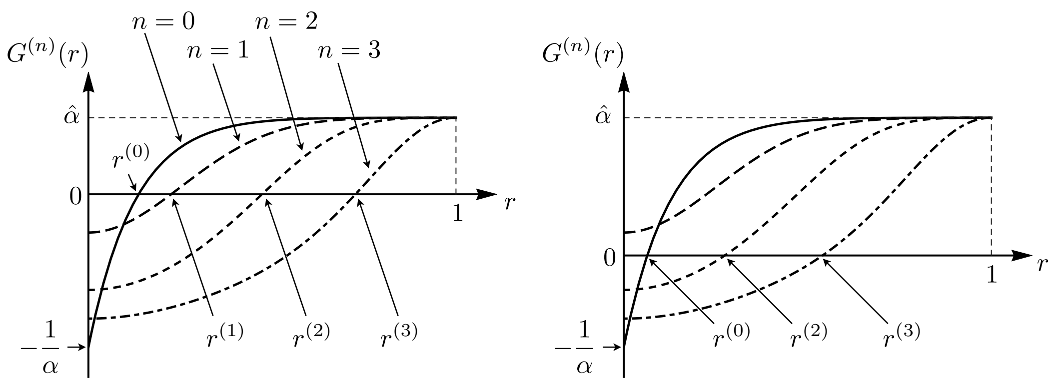


Figure 6: $G^{(n)}(r)$ $(n = 0, 1, 2, 3)$ and the critical points $r^{(n)}$ for $K = 2^4 = 16$. Left: the spatial period-doubling cascade $(1 < \alpha < \bar{\alpha})$; right: the spatial period-doubling cascade with skipping $(\bar{\alpha} < 2 < \alpha < 2^2)$.

Proposition 12 (The spatial period-doubling cascade)**.** Let the number of zones $K \geq 4$ be a power of 2. Starting form the flat-earth equilibrium $\bar{\boldsymbol{h}}$, we consider the evolutionary path of spatial structure in the course of increasing r. The HW model exhibits a series of spatial period-doubling bifurcations if and only if (54) holds. That is, the number of market centers is reduced by half, the spacings between neighboring market centers doubles after each bifurcation, and the recursive process continues until the mono-centric pattern is attained if and only if (54) holds.

Proof. See Appendix A.4. □

The proof for Proposition 12 (Appendix A.4) is tedious, but the intuition is straightforward. When (54) holds, (49) holds for all $n = 0, 1, \ldots, J - 1$ and, thus, all the relevant bifurcations will occur. The remaining task is to show that the relations (52) hold. Because $r^{(n)}$ are the solutions for $G^{(n)}(r) = 0$, we compare $G^{(n)}(r)$ for different n. Figure 6 illustrates this. In the left pane of Figure 6, we graphically show that $r^{(n)} < r^{(n+1)}$ for all $n = 0, 1, \ldots, J - 2$ provided that $r^{(0)} < r^{(1)}$.

A component of NEG models is seen in a special case of the HW model. Approximately, spatial interaction terms that appear in these models are the case $\alpha = 1$.[16] From this viewpoint, it is natural that the HW model exhibits similar bifurcation properties as NEG models in the range of α that approximates 1.

4.4.3 The Spatial Period-doubling Cascade with “Skipping"

In the previous subsection, we proved that when $1 < \alpha < \bar{\alpha}$, the HW model exhibits the spatial period-doubling cascade, which has previously been addressed in Pflüger’s NEG model. In this section, we show that when $\alpha > \bar{\alpha}$, the HW model shows somewhat different bifurcation properties compared to the simple spatial period-doubling cascade. We have the following proposition.

[16] When $\alpha = 1$, we do not have any agglomeration force at work in the HW model. Because NEG models have different agglomeration mechanisms compared to the HW model, agglomeration occurs in these models even when we set $\alpha = 1$.

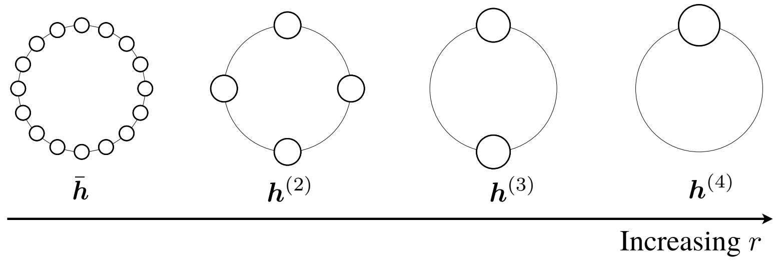


Figure 7: The spatial period-doubling cascade with "skipping" ($K = 16$). 8-centric pattern $\boldsymbol{h}^{(1)}$ is skipped; immediately after the first bifurcation, 4-centric pattern $\boldsymbol{h}^{(2)}$ emerges.

Proposition 13 (Spatial period-doubling cascade with skipping)**.** Let the number of zones $K \geq 4$ be a power of 2. Suppose that (54) does not hold: $\alpha \geq \bar{\alpha}$. We start form a sufficiently high level of transportation cost (sufficiently small r) in which $\bar{\boldsymbol{h}}$ is stable and consider the steady decrease in transportation cost (steady increase in r). Destabilization of the flat-earth equilibrium $\bar{\boldsymbol{h}}$ results in the emergence of $\boldsymbol{h}^{(\hat{n})}$, where $\hat{n}$ is the first n that satisfies $r^{(n)} > r^{(0)}$. A further decrease will trigger a spatial period-doubling cascade originating form $\boldsymbol{h}^{(\hat{n})}$.

Proof. We require $1 < \alpha < 2^n$ for the bifurcation form a stable $K/2^n$-centric pattern $\boldsymbol{h}^{(n)}$ to occur (Lemma 10). We also require that the latter part of this inequality ensures stabilization of $\boldsymbol{h}^{(n)}$ for some low r: $G^{(n)}(0) < 0$; the first part of the inequality ensures destabilization of $\boldsymbol{h}^{(n)}$ by increasing r. Because $\alpha > \bar{\alpha}$ ensures $\alpha > 1$, we only assess the consequence of violation of the latter part, $\alpha < \bar{\alpha}$. Let us first assume that α satisfies $2^{\bar{n}-1} \leq \alpha < 2^{\bar{n}}$ for some $\bar{n} \geq 2$. $\boldsymbol{h}^{(1)}, \ldots, \boldsymbol{h}^{(\bar{n}-1)}$ are unstable for all r. Hence, possible emergent patterns after the first bifurcation are $\boldsymbol{h}^{(\bar{n})}, \boldsymbol{h}^{(\bar{n}+1)}, \ldots$. The emergent pattern is $\boldsymbol{h}^{(\hat{n})}$, where $\hat{n}$ is the first $n \geq \bar{n}$ that satisfies $r^{(\hat{n})} > r^{(0)}$. We see form Appendix A.4 that $r^{(n)} < r^{(n+1)}$ for $n \geq 1$. Thus, after the emergence of $\boldsymbol{h}^{(\hat{n})}$, a spatial period-doubling cascade starts. □

The right pane in Figure 6 illustrates the patterns of $G^{(n)}(x)$ in such a case. Figure 7 illustrates the evolution of spatial structure under the same setting as Figure 6 right. In this figure, $\boldsymbol{h}^{(1)}$ is already unstable at $r^{(0)}$, and destabilization of $\bar{\boldsymbol{h}}$ results in the emergence of a perturbed spatial configuration $\boldsymbol{h}^{\times(1)} \equiv \boldsymbol{h}^{(1)} + \boldsymbol{z}_M^{(1)}$. As r increases, $\boldsymbol{h}^{\times(1)}$ converges to $\boldsymbol{h}^{(2)}$ and a spacial period-doubling cascade starts form $\boldsymbol{h}^{(2)}$. Therefore, a symmetric $K/2$-centric pattern $\boldsymbol{h}^{(1)}$ is "skipped" in this evolutionary path of spatial structure.

Under the two-zonal setup (when $K = 2$), we do not observe "skipping" of a possible equilibrium pattern in the line of decreasing transport cost because we have only two alternative equilibrium patterns: the flat-earth equilibrium $\bar{\boldsymbol{h}} \equiv [h, h]^\top$ and the so-called *core–periphery* equilibrium $\boldsymbol{h}^{(1)} \equiv [2h, 0]^\top$. Recalling the "spatial resolution" issue of two-location models we discussed in the introduction, the "skip" observed in the HW model should be a concrete example that indicates insufficiency of the analytical approaches that resort to the two-location setup. Although qualitative bifurcation properties of models would be, perhaps, similar in the multi-locational world, there would probably be hidden rich implications that cannot be figured out in the two-location world.

4.5 Sustainability of Agglomeration in the HW model

We have so far focused on the bifurcation properties in the direction of monotonically increasing r (decreasing transport cost). We are also interested in the effect of *increasing* transportation cost after an agglomeration form the flat-earth equilibrium is established— or *sustainability* of a given equilibrium. Typical results in the NEG literature show that when the transport cost level gradually increases again, there is a critical point for the transport cost level, often termed a *sustain point*, at which once established agglomeration is no longer stable. This property is illustrated by a tomahawk diagram in Krugman (1991).

In the HW model, however, we do not have such sustain points. More precisely, an established equilibrium, for example, $\boldsymbol{h}^{(n)}$, is always asymptotically stable in the whole range $r \in (0, r^{(n)})$. Hence, when starting form a value of r at which $\boldsymbol{h}^{(n)}$ is stable with decreasing r (increasing the transport cost), there is no destabilization of the equilibrium—$\boldsymbol{h}^{(n)}$ is fully sustained. Once an agglomeration is formed, there is no path escaping form the agglomeration. This extreme hysteresis property is because of the characteristic of the consumer demand function for the retailing activity (i.e., equation (5)). The consumers' demands for a firm completely vanishes if the number of firms in the location equals zero. This prevents potential entrants form locating the zone because the profit at the zone is negative because of the fixed entry cost κ. For this reason, once a zone is left behind by retailers, the zone will never obtain a new retailer regardless of the extent of the transport cost (provided that it is finite).

Some readers may think this is an implausible or even unnatural property for an agglomeration model. If undesirable, an extra spatial interaction term can be added such that there is always positive demand for every location, which is the case in many NEG models.

4.6 Numerical Examples

We provide some numerical examples of spatial structural evolution in the HW model to illustrate and highlight the theoretical results obtained in the previous section. Figure 8 shows typical examples of bifurcation diagrams for different values of α. The number of zones, K, is 16. The values of α are selected to cover all the essential cases that emerge: relatively small α where the spatial period-doubling cascade occurs, relatively large α where the spatial period-doubling cascade with skipping occurs, and an intermediate case.

In each sub-figure in Figure 8, the black thick lines represent the number of firms located at the largest market center at stable equilibria for different r. Placed above are the spatial agglomeration patterns that will emerge with a steady and gradually increasing r (i.e., decreasing β) form $r = 0$ ($\beta = +\infty$) to $r = 1$ ($\beta = 0$). Each critical value of $r^{(n)}$ for the bifurcation form $\boldsymbol{h}^{(n)}$, if any, is obtained by solving (50). As discussed, every symmetric agglomeration pattern $\boldsymbol{h}^{(n)}$ is asymptotically stable—or "sustained"—in the whole range $(0, r^{(n)})$.

Figure 8 (a) is the case $\alpha = 1.5$. $\alpha = 1.5$ satisfies the condition $1 < \alpha < \bar{\alpha}$. Therefore, form Proposition 12, we expect that spatial period-doubling cascade occurs. Actually, we have the path: the evolutionary path of spatial structure follows the pattern

$$\bar{\boldsymbol{h}} \to \boldsymbol{h}^{(1)} \to \boldsymbol{h}^{(2)} \to \boldsymbol{h}^{(3)} \to \boldsymbol{h}^{(4)}$$

and each bifurcation at the critical values, or r, $r^{(n)}$ $(n = 0, 1, 2, 3)$ doubles the number of firms at each market center while halving the number of market centers.

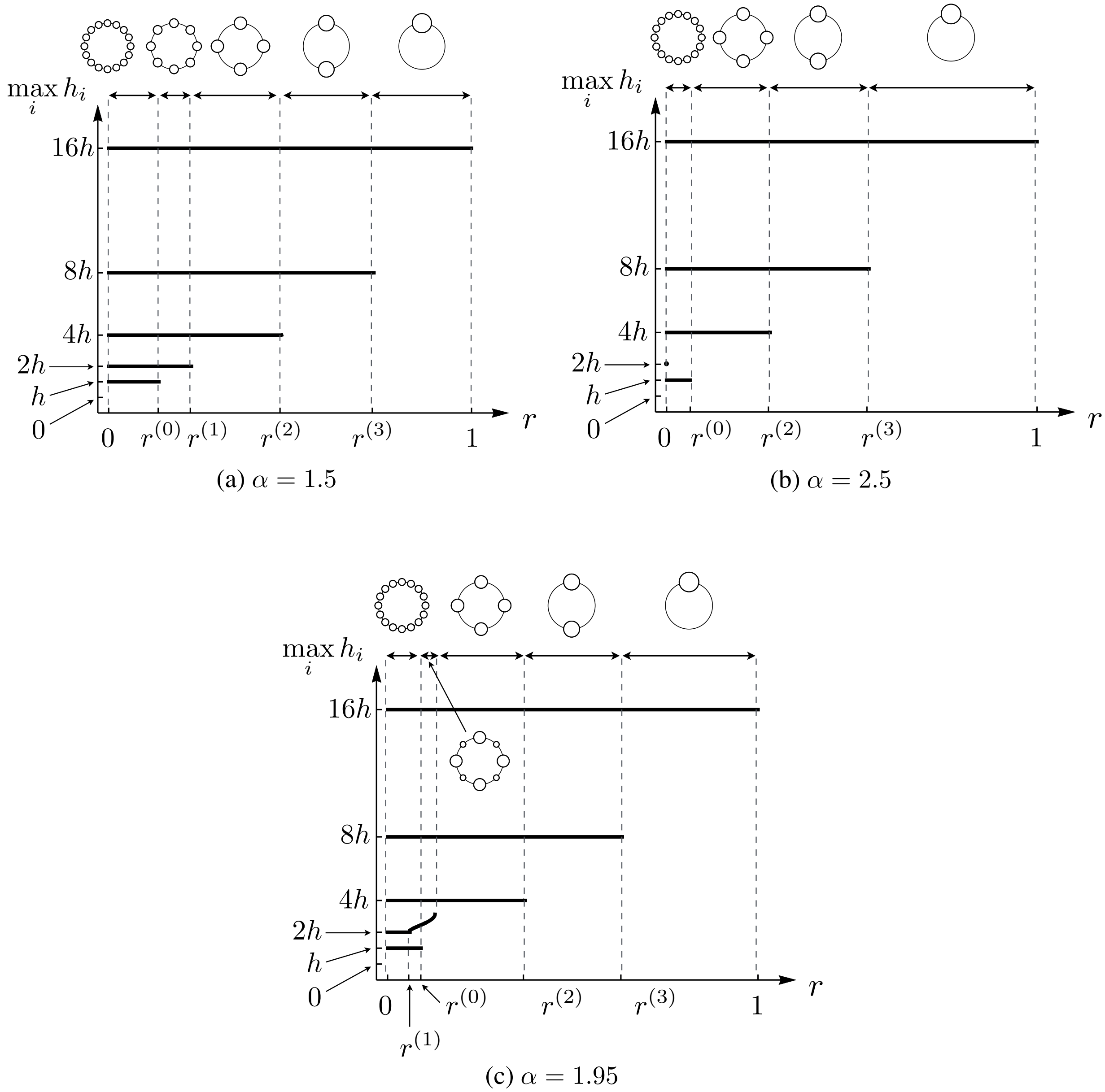


Figure 8: Typical examples of the bifurcation diagram for different α. $K = 16$. Thick black lines depict the number of firms located at the largest market center at stable equilibria. (a) An example of the spatial period-doubling bifurcation path ($\alpha = 1.5$). (b) An instance of the spatial period-doubling bifurcation with skipping path ($\alpha = 2.5$). (c) A borderline case ($\alpha = 1.95$) in which a deformed eight-centric pattern emerges after the first bifurcation.

Figure 8 (b) is the case $\alpha = 2.5$. This case is an example of the spatial period-doubling cascade *with skipping*. $\alpha = 2.5$ does not satisfy the condition $1 < \alpha < \bar{\alpha}$. Moreover, $\alpha > 2$ holds. So, $\boldsymbol{h}^{(1)}$ cannot be a stable equilibrium for essentially all values of r, except for the extremal value

$r = 0$. The first n that satisfies $r^{(0)} < r^{(n)}$ is $n = 2$. The emergent spatial structure after the first bifurcation is, hence, $\boldsymbol{h}^{(2)}$. In the course of increasing r, the path is

$$\bar{\boldsymbol{h}} \to \boldsymbol{h}^{(2)} \to \boldsymbol{h}^{(3)} \to \boldsymbol{h}^{(4)}$$

in which $\boldsymbol{h}^{(1)}$ is completely skipped compared to the first case.

Figure 8 (c) is the case $\alpha = 1.95$. This case is an intermediate case between (a) and (c). In this case, $\alpha > \bar{\alpha}$ holds, but α is not large enough to destabilize $\boldsymbol{h}^{(1)}$ for all r (i.e., $\alpha < 2$). After the first bifurcation, a deformed eight-centric pattern $\boldsymbol{h}^{\times(1)} \equiv \boldsymbol{h}^{(1)} + \delta \boldsymbol{z}_M^{(1)}$ emerges. Thus the evolutionary behavior is quite similar to the spatial period-doubling cascade, except that $\boldsymbol{h}^{(1)}$ itself is skipped in the line of increasing r.

5 Connection to the New Economic Geography Models

Krugman (1991)'s core–periphery (CP) model proposed one of the most successful micro-founded modeling paradigms of economic agglomeration. The "Dixit–Stiglitz, icebergs, evolution, and the computer" modeling technique (Fujita et al., 1999) introduced by the paper have opened up a fruitful field called NEG today. We have demonstrated that there is a strong similarity between the bifurcation properties of the HW model and those of an NEG model (Pflüger's model), and both models exhibit a characteristic evolutionary path called the spatial period-doubling cascade. This section makes a concrete comparison between the HW model and Pflüger's model to reveal their close connection. At the same time, we address the strong connection between the BLV method and the NEG methodology.

5.1 Footloose-entrepreneur Class of New Economic Geography Models

Pflüger's model is an example of a particular class, namely, the "footloose-entrepreneur" type, of NEG models. This subsection minimally but sufficiently—at least for the purposes of comparing it with HW models in the next subsection—introduces "footloose-entrepreneur" NEG models. In the "footloose-entrepreneur" class of NEG models, the key economic agents are mobile consumers that can choose their residing locations. Although there are other agents as immobile consumers and firms, in a reduced form, this class of NEG models focuses on the equilibrium spatial distribution of mobile consumers.

Again, consider a space that consists of K discrete regions. Let us denote the spatial distribution of mobile consumers by $\boldsymbol{h}$. $\boldsymbol{h}$ is a K-dimensional vector whose i-th element represents the number of mobile consumers at location i. The total number of mobile consumers, H, is an exogenously given constant. In the *short-run*, the spatial distribution mobile consumer, $\boldsymbol{h}$, is fixed. Mobile consumers cannot change their location. Given $\boldsymbol{h}$, the general equilibrium condition determines the indirect utility level $\boldsymbol{v}$ of consumers at each location. If fortunate, we have a closed-form expression of the indirect utility $\boldsymbol{v}(\boldsymbol{h})$ as a function of $\boldsymbol{h}$. In some cases, (e.g., Krugman's original case) we are not so fortunate, but at least we can calculate it numerically. In the *long-run*, every mobile consumer is free to choose a residing location. At the long-run equilibrium, there is no incentive for a mobile consumer to change their choice of location. This condition is the following no arbitrage condition

$$\begin{cases} V = v_i(\boldsymbol{h}) & \text{if } h_i > 0 \\ V \geq v_i(\boldsymbol{h}) & \text{if } h_i = 0 \end{cases} \qquad \forall i \tag{55a}$$

combined with conservation of mobile consumers

$$\sum_{i \in \mathcal{K}} h_i = H \tag{55b}$$

Therefore, the footloose-entrepreneur class of NEG models corresponds to the closed-city class of models in urban economics (recall here that the HW model is of open-city class).

For the out-of-equilibrium adjustment dynamic of $\boldsymbol{h}$, NEG models often assume the *replicator dynamic*

$$\dot{\boldsymbol{h}} = \mathrm{diag}[\boldsymbol{h}](\boldsymbol{v}(\boldsymbol{h}) - \bar{v}(\boldsymbol{h})\mathbf{1}) \tag{56}$$

where $\bar{v}(\boldsymbol{h}) \equiv (1/H) \sum_{i \in \mathcal{K}} v_i(\boldsymbol{h}) h_i$ is the economy's average indirect utility level. The rest points of the dynamic have a close relation with the equilibrium points in the model (i.e., the spatial distribution $\boldsymbol{h}$ of consumers that satisfies (55a)). The set of locally stable rest points coincides with the rest points of long-run equilibrium points of interest.

The above construction shows that this class of model belongs to the BLV formalism category. The short-run equilibrium condition in NEG models corresponds to the "Boltzmann" component, whereas the long-run evolutionary dynamic corresponds to the "Lotka–Volterra" component. The most characteristic modeling philosophy of NEG models, which is emphasized by many scholars including Krugman (e.g., Krugman (2011)), is that the "Boltzmann" component of these models is equipped with a full-fledged general equilibrium model according to the combination of the Dixit–Stiglitz model and the iceberg transportation cost. As a reduced-form model, however, NEG models comfortably fall into the general framework of the BLV method.

To make a concrete comparison, we introduce Pflüger (2004)'s "solvable" variant of the footloose-entrepreneur model. Pflüger's model (hereafter, the Pf model) enjoys analytical tractability compared to Krugman's original CP model, yet preserves basic properties such as the *forward linkage* (or price index effect) and *backward linkage* (or demand effect). In the Pf model, we have the closed-form formula for the indirect utility function $\boldsymbol{v}(\boldsymbol{h})$, that is, the short-run equilibrium is "solvable." The indirect utility function in the reduced-form model is given by

$$\boldsymbol{v}(\boldsymbol{h}) = \frac{\mu}{\sigma - 1} \log[\boldsymbol{D}^\top \boldsymbol{h}] + \frac{\mu}{\sigma} \boldsymbol{M}^\top (\boldsymbol{h} + l\mathbf{1}) \tag{57a}$$

$$\boldsymbol{M} \equiv \{\mathrm{diag}[\boldsymbol{D}^\top \boldsymbol{h}]\}^{-1} \boldsymbol{D} \tag{57b}$$

or, in the element-wise manner,

$$v_i(\boldsymbol{h}) = \frac{\mu}{\sigma - 1} \log \sum_{k \in \mathcal{K}} d_{ik} h_k + \frac{\mu}{\sigma} \sum_{j \in \mathcal{K}} \frac{d_{ij}}{\sum_{k \in \mathcal{K}} d_{kj} h_k} (h_j + l) \qquad \forall i \in \mathcal{K} \tag{58}$$

in which $\mu > 0$, $\sigma \geq 1$, $l > 0$ are constants each interpreted as the expenditure ratio on manufacturing goods, the elasticity of substitution of manufacturing goods, and the number of immobile consumer at each location, respectively[17]. $\log[\boldsymbol{a}]$ for a vector $\boldsymbol{a}$ with strictly positive elements is defined as a component-wise $\log[\cdot]$ operation: $\log[\boldsymbol{a}] \equiv [\log[a_i]]$. Observe that the second term in (57a) is similar to the first term of the profit function (11a) of the HW model. This is because these terms both capture total inflow at each location that is induced by gravity-type equations capturing trade flows between locations.

5.2 Harris and Wilson's Model v.s. Pflüger's Model

We assume again that the underlying physical space is a circumference with K discrete locations. We compare the HW and Pf model on this space. As with the analysis of the HW model, we start form a high level of transport cost (i.e., $r \approx 0$) at which the flat-earth equilibrium is stable and gradually increases r. For simplicity, we focus only on the first bifurcation: the destabilization of the flat-earth equilibrium, $\bar{\boldsymbol{h}} \equiv [h, h, \ldots, h]^\top$. For the Pf model, $h = H/K$.

Although there are apparent differences in the equilibrium conditions and dynamics, it is sufficient to compare the Jacobian matrices $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ and $\nabla \boldsymbol{\Pi}(\bar{\boldsymbol{h}})$ to compare the character of the first bifurcation form $\bar{\boldsymbol{h}}$. This is because the equilibrium conditions are basically equivalent if we change variables, and the dynamics also share a similar character[18]: both dynamics share a rationality property that ensures that the direction of adjustment is in accordance with the payoff vector (i.e., $\boldsymbol{v}(\boldsymbol{h})$ or $\boldsymbol{\Pi}(\boldsymbol{h})$) at every non-stationary point. As an immediate consequence, the task reduces to a comparison of the characteristics of the payoff function of each model without explicit consideration of the dynamics: comparing $\boldsymbol{v}(\boldsymbol{h})$ and $\boldsymbol{\Pi}(\boldsymbol{h})$ suffices. This claim is verified by comparing the Jacobian matrix of the dynamic $\nabla \boldsymbol{F}(\bar{\boldsymbol{h}})$ and that of $\nabla \boldsymbol{\Pi}(\bar{\boldsymbol{h}})$ for the HW model: these equations coincide up to a constant multiple.

As with the analysis of the HW model, we derive eigenvalues and eigenvectors of the Jacobian matrix $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ to assess the characteristics of the first bifurcation in the Pf model. $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ is given by

$$\nabla \boldsymbol{v}(\bar{\boldsymbol{h}}) = \frac{1}{h}\left(a\bar{\boldsymbol{D}} - b\bar{\boldsymbol{D}}^2\right) \tag{59}$$

where $a \equiv \mu(\sigma-1)^{-1} + \sigma^{-1}$ and $b \equiv \mu\sigma^{-1}(1 + lh^{-1})$. The eigenvalues $\boldsymbol{e}^{\text{Pf}}$ of the Jacobian matrix $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ take the same functional form with respect to $\bar{\boldsymbol{D}}$ as (59). Applying Lemma 1 to $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$, we can immediately show that its eigenvalues $\boldsymbol{e}^{\text{Pf}}$ and $\boldsymbol{f}$ and the eigenvalues of $\bar{\boldsymbol{D}}$, satisfy

$$e_k^{\text{Pf}} = \frac{1}{h} \cdot G^{\text{Pf}}(f_k) \qquad k = 0, 1, \ldots, K-1 \tag{60a}$$

$$G^{\text{Pf}}(x) \equiv ax - bx^2 \tag{60b}$$

and the associated eigenvector for the e_k^{Pf} is k-th column vector of K-dimensional DFT matrix, $\boldsymbol{z}_k$.

For the HW model, as shown in the previous section, $\nabla \boldsymbol{\Pi}(\bar{\boldsymbol{h}})$ is given by

$$\nabla \boldsymbol{\Pi}(\bar{\boldsymbol{h}}) = \frac{O\alpha}{h^2}\left(\hat{\alpha}\boldsymbol{I} - \bar{\boldsymbol{D}}^2\right). \tag{61}$$

[17] For the details and derivation of the vector-matrix representation of the multi-regional Pf model, consult Akamatsu et al. (2012).

[18] Both of the dynamics belong to the *imitative dynamics* category and satisfy a property called *positive correlation*. Imitative dynamics is a class of evolutionary dynamics under which the growth rate of a strategy is proportional to the number of agents using that strategy. For an elaborate discussion, see Sandholm (2010).

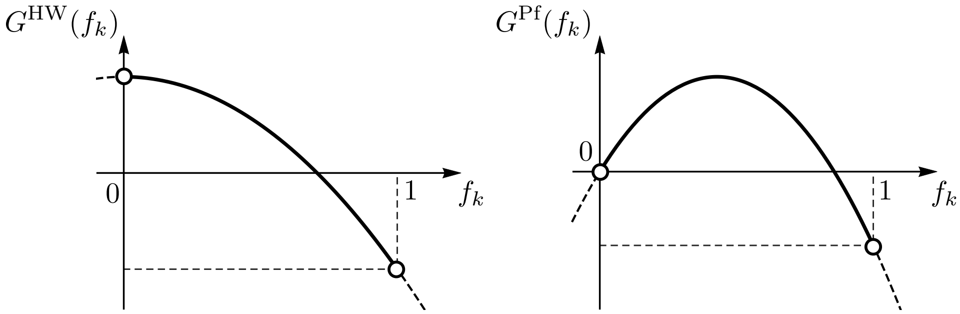


Figure 9: $G^{\mathrm{HW}}(f_k)$ and $G^{\mathrm{Pf}}(f_k)$ as a function of f_k. Both are concave quadratics that cross f_k axis only once in $(0, 1)$. At the origin $f_k = 0$, the former takes the maximum; the latter equals zero.

where $\hat{\alpha} = 1 - \alpha^{-1}$. For this model, the eigenvalues $\boldsymbol{e}^{\mathrm{HW}}$ are concave quadratic of $\boldsymbol{f}$ as[19]

$$e_k^{\mathrm{HW}} = \frac{O\alpha}{h^2} \cdot G^{\mathrm{HW}}(f_k) \qquad k = 0, 1, \ldots, K-1 \tag{62a}$$

$$G^{\mathrm{HW}}(x) \equiv \hat{\alpha} - x^2 \tag{62b}$$

and the associated eigenvector for e_k^{HW} is, again, $\boldsymbol{z}_k$. See Figure 9 for an illustration of $G^{\mathrm{HW}}(\cdot)$ and $G^{\mathrm{Pf}}(\cdot)$.

In either of (60) and (62), the positive term (the first term) represents the agglomeration force of the model while the negative term (the second) represents the dispersion force. Each of (60) and (62) can be interpreted as *net agglomeration force* as a whole, by which we mean (agglomeration force) − (dispersion force). Each $e_k^{\cdot}$ can thus be seen as the net strength of the agglomeration force in the deviation direction of $\boldsymbol{z}_k$. If $e_k^{\cdot} < 0\ \forall k$, for mobile consumers, there is no incentive to disturb the flat-earth equilibrium $\bar{\boldsymbol{h}}$; if some $e_k^{\cdot}$ is positive for some direction, agglomeration force in this direction overcomes the dispersion force and $\bar{\boldsymbol{h}}$ becomes unstable.

Following a similar line of logic, we show that the first bifurcation, if any, in the Pf model, results in a $K/2$-centric pattern: *maximal* the eigenvalue of $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ corresponds to the *minimal* eigenvalue of $\bar{\boldsymbol{D}}$. The eigenvalue of $\bar{\boldsymbol{D}}$ being $f_M \equiv f_{K/2}$. Its associated eigenvector $\boldsymbol{z}_M$ is the direction by which mobile consumers in alternate locations disappear, and remaining locations double the number of their inhabitants. We further prove that the spatial period-doubling bifurcation occurs in the Pf model (Akamatsu et al., 2012). It is a striking similarity.

For a comparison of (60) and (62), first, both are a concave quadratic of f_k because of the second term in each equation. Second, the other term differs between models: in the Pf model, the first term is linear in f_k while, in the HW model, it is a constant with respect to f_k. These similarities and differences have clear economic interpretations based on the model assumptions. The similarity stems form the fact that both models have a spatial competition effect (or, equivalently, a market crowding effect) captured by the consumer demand function (gravity equation) and appear as the second order term of $\bar{\boldsymbol{D}}$ in $\nabla \boldsymbol{v}(\bar{\boldsymbol{h}})$ or $\nabla \boldsymbol{\Pi}(\bar{\boldsymbol{h}})$. There is a difference in the agglomeration force in each model. In the Pf model, the agglomeration force is a space-dependent agglomeration one—price index effect and demand effect, which is strongly affected by the spatial distribution of

[19](62) is the same equation as (33b). We redefine the equation here for comparison.

mobile consumers and, hence, the distance between locations. On the other hand, in the HW model, the space-independent agglomeration force is at work: the attractiveness of a zone i is determined exclusively by $(h_i)^\alpha$, which does not require any spatial dimension but only the number of firms at the zone. The agglomeration force in the Pf model (i.e., the first term: af_k) gradually changes in the line of changing transportation cost while that of the HW model (i.e., $\hat{\alpha}$), remains constant.

We may hypothesize that the existence of spatial competition is a necessary condition for the emergence of the spatial period-doubling cascade in the line of decreasing transportation cost. The common property between the HW model and the Pf model is that the Jacobian matrices are negative quadratic of $\bar{\boldsymbol{D}}$, which is because of the spatial competition effect modeled in the gravity equation.

Additionally, in the two-location setting ($K = 2$), we do not have a way to qualitatively distinguish between the HW model and the Pf model when we gradually increase r. Both of the models exhibit a transition form the flat-earth equilibrium $\bar{\boldsymbol{h}} \equiv [h, h]^\top$ to the core–periphery equilibrium $\boldsymbol{h}^{(1)} \equiv [2h, 0]^\top$. On the other hand, if the parameters of the Pf model are such that the destabilization of the flat-earth equilibrium occurs, then the model *always* exhibits the spatial period-doubling cascade—in contrast to the HW model, there is no possibility of any skipping. This, again, demonstrates the limitation of the two-location assumption.

6 Concluding Remarks

This paper analytically unveiled previously unknown bifurcation properties of Harris and Wilson (1978)'s dynamic retail location model under a multi-locational setting other than two. We demonstrated that the approach of Akamatsu et al. (2012) enables us to reveal the characteristics of agglomeration/dispersion forces in the model and to analytically trace the process of recursive emergence of spatial structures. Specifically, we analytically derived the critical points of a structural parameter, namely, the transport cost parameter r, at which the first and second bifurcation occur. Moreover, we showed that the evolutionary path of spatial structure exhibits a striking property that is called the *spatial period-doubling cascade*, previously found only for the NEG model of Pflüger (2004). We clarified the similarities and differences between Harris and Wilson (1978)'s model and Pflüger's NEG model: (i) both models exhibit the spatial period-doubling cascade, and (ii) only the HW model exhibits extreme sustainability of equilibria and a skip in equilibrium patterns. These results demonstrate the power of the method utilized in this paper and suggest the possibility of a unified and systematic understanding of agglomeration models in both geography and spatial economics.

Clarke and Wilson (1985) showed that in a two-dimensional plain, the HW model exhibits remarkable spatial patterns that resemble hexagonal agglomeration patterns proposed in the classical central place theory of Christaller (1933) and Lösch (1940). The underlying mechanism for this striking result, however, had not been fully explained previously because of the lack of analytical approaches. Recently, Ikeda et al. (2014a) gave group-theoretic proof that two-dimensional NEG models admit hexagonal spatial configuration as a stable equilibrium that arises form destabilization of the flat-earth equilibrium. Judging form the strong linkage between NEG models and the HW model, there is a possibility that we can rigorously explain Clarke and Wilson (1985)'s numerical result. Moreover, there should be a much wider class of spatial interaction-type models that result in the self-organization of hexagonal spatial patterns of central place theory. Identifying the necessary conditions, if any, for the emergence of stable hexagonal spatial patterns should be an interesting and challenging research task. This is, however, beyond the scope of this paper.

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A Proofs

A.1 Proof for Lemma 6

To analytically derive the eigenvalues of the Jacobian matrix $\nabla \boldsymbol{F}(\boldsymbol{h}^{(1)})$, we must change the coordinate system because $\nabla \boldsymbol{F}(\boldsymbol{h}^{(1)})$ is not a circulant matrix. Specifically, we must consider a permutation of row/column indices. We define permutation matrix $\boldsymbol{P}$ by

$$\boldsymbol{P} \equiv \begin{bmatrix} \boldsymbol{P}_{\text{even}} \\ \hdashline \boldsymbol{P}_{\text{odd}} \end{bmatrix} \tag{63}$$

in which $\boldsymbol{P}_{\text{even}}$ and $\boldsymbol{P}_{\text{even}}$ are both a $K/2$-by-K matrix defined by $\boldsymbol{P}_{\text{even}} = [\delta_{i,j/2}]$ and $\boldsymbol{P}_{\text{odd}} = [\delta_{i,(j-1)/2}]$, where $\delta_{\cdot,\cdot}$ is Kronecker's delta. $\boldsymbol{P}_{\text{even}}$ extracts even-numbered zones; $\boldsymbol{P}_{\text{odd}}$ extracts odd-numbered zones.

Define $\boldsymbol{D}^{\times}$ by $\boldsymbol{D}^{\times} \equiv \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{\top}$. We have

$$\boldsymbol{D}^{\times} = \left[\begin{array}{c:c} \boldsymbol{D}^{(0)} & \boldsymbol{D}^{(1)} \\ \hdashline \boldsymbol{D}^{(1)\top} & \boldsymbol{D}^{(0)} \end{array}\right] \tag{64}$$

where $\boldsymbol{D}^{(0)}$ and $\boldsymbol{D}^{(1)}$ are both $K/2$-by-$K/2$ circulant with the first row being

$$\boldsymbol{d}_0^{(0)} = [1, r^2, r^4, \ldots, r^M, \ldots, r^2], \tag{65}$$

$$\boldsymbol{d}_0^{(1)} = [r, r^3, \ldots, r^{M-1}, r^{M-1}, \ldots, r], \tag{66}$$

respectively. $\boldsymbol{D}^{(0)}$ captures the distance structure between market centers (or *core zones*) while $\boldsymbol{D}^{(1)}$ captures that of core zones and *periphery zones* without retailers. Because both $\boldsymbol{D}^{(i)}$ are circulant matrices, we diagonalize $\boldsymbol{D}^{\times}$ by a block diagonal matrix $\boldsymbol{Z}^{\times} \equiv \operatorname{diag}[\boldsymbol{Z}_{[K/2]}, \boldsymbol{Z}_{[K/2]}]$ with $\boldsymbol{Z}_{[K/2]}$ being a $K/2$-dimensional DFT matrix.

In a similar spirit, permutation by $\boldsymbol{P}$ enables us to block diagonalize $\nabla \boldsymbol{F}(\boldsymbol{h}^{(1)})$. Specifically, a little computation will show that if we define the permuted Jacobian matrix by $\nabla \boldsymbol{F}^{\times}(\boldsymbol{h}^{(1)}) \equiv \boldsymbol{P}\nabla \boldsymbol{F}(\boldsymbol{h}^{(1)})\boldsymbol{P}^{\top}$, then

$$\nabla \boldsymbol{F}^{\times}(\boldsymbol{h}^{(1)}) = \left[\begin{array}{c:c} \boldsymbol{V}^{(00)} & \boldsymbol{0} \\ \hdashline \boldsymbol{0} & -\kappa \boldsymbol{I} \end{array}\right] \tag{67a}$$

$$\boldsymbol{V}^{(00)} = (\alpha\kappa)\left\{-\frac{1}{2}\left(\{\bar{\boldsymbol{D}}^{(0)}\}^2 + \bar{\boldsymbol{D}}^{(1)}\bar{\boldsymbol{D}}^{(1)\top}\right) + \left(1 - \frac{1}{\alpha}\right)\boldsymbol{I}\right\} \tag{67b}$$

Again we can diagonalize $\nabla \boldsymbol{F}^{\times}(\boldsymbol{h}^{(1)})$ by $\boldsymbol{Z}^{\times}$, because every sub-matrix of $\nabla \boldsymbol{F}^{\times}(\boldsymbol{h}^{(1)})$ is circulant. Let $\boldsymbol{e}$ be the eigenvalues of $\boldsymbol{V}^{(00)}$. After straightforward computation using Lemma 1, we have

$$e_k = (\alpha\kappa) \cdot \hat{G}(f_k^{(0)}, f_k^{(1)} f_k^{(1\top)}) \qquad k = 0, 1, \ldots, K/2 - 1 \tag{68a}$$

$$\hat{G}(x, y) \equiv -\frac{1}{2}(x^2 + y) + \hat{\alpha} \tag{68b}$$

where $\boldsymbol{f}^{(0)}, \boldsymbol{f}^{(1)}, \boldsymbol{f}^{(1\top)}$ are the eigenvalues of $\bar{\boldsymbol{D}}^{(0)}, \bar{\boldsymbol{D}}^{(1)}, \bar{\boldsymbol{D}}^{(1)\top}$, respectively. Placing this back into the original coordinating system, the eigenvalues $\boldsymbol{g}^{(1)}$ of $\nabla \boldsymbol{F}(\boldsymbol{h}^{(1)})$ are obtained as

$$g_k = \begin{cases} e_{k/2} & \text{if } k\text{: even} \\ -\kappa & \text{if } k\text{: odd} \end{cases}. \tag{69}$$

Moreover, we show that

$$\arg\max_{k} . \, e_k = \arg\min_{k} . \, \{f_k^{(0)}\}^2 + f_k^{(1)} f_k^{(1\top)} = K/2^2 \tag{70}$$

and

$$f_{K/2^2}^{(0)} = \left(\frac{1-r^2}{1+r^2}\right)^2, \; f_{K/2^2}^{(1)} f_{K/2^2}^{(1\top)} = 0. \tag{71}$$

Substituting (71) into (67), we obtain $g_{\text{max}}^{(1)}$ in Lemma 6. The corresponding eigenvector is M-th column vector of $\boldsymbol{Z}^{(1)} \equiv \boldsymbol{P}^{\top}\boldsymbol{Z}^{\times}\boldsymbol{P}$, which is given by

$$\boldsymbol{z}_M^{(1)} \equiv [1, 0, -1, 0, \dots, -1, 0]^{\top} \tag{72}$$

A.2 Proof for Lemma 9

In this appendix, we derive the analytical expression of $\boldsymbol{g}^{(n)}$ and the eigenvalues of the Jacobian matrix $\nabla\boldsymbol{F}(\boldsymbol{h})$ of the adjustment dynamics at K_n-centric equilibrium $\boldsymbol{h}^{(n)}$ $(n = 1, 2, \dots, J-1)$ where $K_n \equiv K/2^n$ is the number of market centers (the number of zones with $h_i > 0$) in this equilibrium. As in Appendix A.1, we should permute in advance $\nabla\boldsymbol{F}(\boldsymbol{h}^{(n)})$ to ensure block circulant property and, thus, the eigenvalues are obtained by DFT.

We define the permutation matrix $\boldsymbol{P}^{(n)}$ $(n = 1, 2, \dots, J-1)$ by

$$\boldsymbol{P}^{(n)} \equiv \text{diag}\left[\underbrace{\hat{\boldsymbol{P}}^{(n)}, \hat{\boldsymbol{P}}^{(n)}, \dots, \hat{\boldsymbol{P}}^{(n)}}_{2^n}\right], \; \hat{\boldsymbol{P}}^{(n)} \equiv \begin{bmatrix} \boldsymbol{P}_{\text{even}}^{(n)} \\ \hdashline \boldsymbol{P}_{\text{odd}}^{(n)} \end{bmatrix} \tag{73}$$

For an illustrative example, see Appendix A.1, which is the case $n = 1$. In (73), $\boldsymbol{P}_{\text{even}}^{(n)}$ and $\boldsymbol{P}_{\text{odd}}^{(n)}$ are a K_n-by-K_{n-1} matrix that extracts even-numbered zones and odd-numbered zones in the *core zones*, respectively. Hence, $\boldsymbol{P}^{(n)}$ is a $2^n/2$-by-$2^n/2$ block diagonal matrix whose diagonal entries are a K_{n-1}-by-K_{n-1} matrix.

Using $\boldsymbol{P}^{(n)}$, define permutation matrix $\boldsymbol{P}$ by

$$\boldsymbol{P} \equiv \boldsymbol{P}^{(n)}\boldsymbol{P}^{(n-1)} \cdots \boldsymbol{P}^{(1)}. \tag{74a}$$

The recursive definition of the permutation matrix $\boldsymbol{P}$ can be viewed to capture that the recursive spatial period-doubling bifurcation $\boldsymbol{h}^{(n)}$ has occurred. Applying a similarity transformation using $\boldsymbol{P}$ to $\nabla\boldsymbol{F}(\boldsymbol{h}^{(n)})$, we obtain the 2^n-by-2^n block diagonal matrix whose blocks are all the K_n-by-K_n matrix:

$$\nabla\boldsymbol{F}^{\times}(\boldsymbol{h}^{(n)}) \equiv \boldsymbol{P}\nabla\boldsymbol{F}(\boldsymbol{h}^{(n)})\boldsymbol{P}^{\top} = \left[\begin{array}{c:c:c:c} \boldsymbol{V}^{(00)} & & & \\ \hdashline & -\kappa\boldsymbol{I} & & \\ \hdashline & & \ddots & \\ \hdashline & & & -\kappa\boldsymbol{I} \end{array}\right] \tag{75}$$

where

$$\boldsymbol{V}^{(00)} \equiv (\alpha\kappa)\left(-\frac{1}{2^n}\sum_{i=0}^{2^n-1}\bar{\boldsymbol{D}}^{(i)}\bar{\boldsymbol{D}}^{(i)\top} + \left(1 - \frac{1}{\alpha}\right)\boldsymbol{I}\right) \tag{76}$$

From the representation, $K - 2^n$ out of K eigenvalues are all $-\kappa < 0$. Hence, $\boldsymbol{V}^{(00)}$ is the only matrix that affects the stability of $\boldsymbol{h}^{(n)}$. Let the eigenvalues of $\boldsymbol{V}^{(00)}$ be $\boldsymbol{e}$. The eigenvalues other than $\boldsymbol{e}$ being negative constants, only the maximal eigenvalue in $\boldsymbol{e}$ should be required to assess the stability of $\boldsymbol{h}^{(n)}$.

In $\boldsymbol{V}^{(00)}$, $\bar{\boldsymbol{D}}^{(i)}$ are K_n-by-K_n matrices obtained by row-wise normalization of $\boldsymbol{D}^{(i)}$. Here, $\boldsymbol{D}^{(i)}$ are first block-row matrices of $\boldsymbol{D}^{\times} \equiv \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{\top}$, which are each circulant matrix with a first row vector $\boldsymbol{d}_0^{(i)}$ explicitly given by

$$d_{0,k}^{(i)} = \begin{cases} r^{i+k2^n} & \text{if } 0 \le k \le K_n/2 \\ r^{(K_n-k)2^n - i} & \text{if } K_n/2 \le k \le K_n - 1 \end{cases} \qquad i = 0, 1, \ldots, 2^n \tag{77}$$

All the matrices in the right hand side being circulant matrix, $\boldsymbol{V}^{(00)}$ in (76) is also a circulant matrix. Thus, its eigenvalues are obtained by applying DFT by $\boldsymbol{Z}_{[K_n]}$. After some tedious computation using explicit formula for $\boldsymbol{d}_0^{(i)}$, (77), we obtain the analytical expression of the eigenvalues $\boldsymbol{e}$ of $\boldsymbol{V}^{(00)}$. Moreover, we prove that the maximal eigenvalue among $\boldsymbol{e}$ is $e_{K_n/2}$. The associated eigenvector for $e_{K_n/2}$ is the M-th column of the matrix $\boldsymbol{P}^{\top}\boldsymbol{Z}^{\times}\boldsymbol{P}$ where $\boldsymbol{Z}^{\times} \equiv \boldsymbol{I} \otimes \boldsymbol{Z}_{[K_n]}$. $\otimes$ denotes the Kronecker product.

Explicitly writing down $e_{K_n/2}$, we have the following result, proving Lemma 9.

$$G^{(n)}(r) = (\alpha\kappa)\hat{G}^{(n)}(r) \tag{78a}$$

where

$$\hat{G}^{(n)}(r) \equiv -\tilde{G}^{(n)}(r) + \hat{\alpha} \tag{78b}$$

$$\tilde{G}^{(n)}(r) \equiv \begin{cases} \dfrac{1}{2^n}\{\psi_0^{(n)}(r)\}^2 + \dfrac{1}{2^n}\psi_0^{(n)}(r)\displaystyle\sum_{j=1}^{2^n-1}\psi_j^{(n)}(r) & \text{if } n \le J-2 \\ \dfrac{1}{2^n}\{\psi_0^{(n)}(r)\}^2 + \dfrac{1}{2^n}\displaystyle\sum_{j=1}^{2^n-1}\psi_j^{(n)}(r) & \text{if } n = J-1 \end{cases} \tag{78c}$$

$$\psi_j^{(n)}(r) \equiv \left(\frac{1 - r^{2^n - 2j}}{1 + r^{2^n - 2j}}\right)^2 \qquad \forall j \ne 0 \tag{78d}$$

$$\psi_0^{(n)}(r) \equiv \left(\frac{1 - r^{2^n}}{1 + r^{2^n}}\right)^{p(n)}, \qquad p(n) = \begin{cases} 1 & \text{if } n = J-1 \\ 2 & \text{otherwise} \end{cases} \tag{78e}$$

For a detailed discussion and proof for maximality of $e_{K_n/2}$, consult Appendix G and J of Akamatsu et al. (2012). Note that there is a mistake in Akamatsu et al. (2012)'s Lemma 6.5 for the case $n = J - 1$. Combining (J.6) and (J.7) in Appendix J of the paper, we obtain an equation similar to the above (78c).

A.3 Proof for Lemma 10

Note that $G^{(n)}(r)$ $(n = 1, 2, \ldots, J-2)$ are equivalently written as

$$G^{(n)}(r) = -\frac{1}{2^n}\psi_0^{(n)}(r)\left(\psi_0^{(n)}(r) + \epsilon_n(r)\right) + \hat{\alpha} \tag{79a}$$

$$\epsilon_n(r) \equiv 2 \sum_{k=1}^{(2^n/2)-1} C_k(r), \quad C_k(r) \equiv \left(\frac{1-r^{2k}}{1+r^{2k}}\right)^2 \tag{79b}$$

Because it is evident that $\mathrm{d}C_k/\mathrm{d}r \leq 0, \mathrm{d}\psi_0^{(n)}(r)/\mathrm{d}r \leq 0$ with equality holding iff $r=1$, $G^{(n)}(r)$ as a whole is strictly increasing for $r \in (0,1)$. Therefore, to prove that the equation $G^{(n)}(r)=0$ has a unique solution in $r \in (0,1)$, it is sufficient to show that $G^{(n)}(0) < 0$ and $G^{(n)}(1) > 0$. We see $G^{(n)}(0) = \hat{\alpha} + 2^{-n} - 1 = -\alpha^{-1} + 2^{-n}$ and $G^{(n)}(1) = \hat{\alpha}$. The case $n = J-1$ is also immediate. Combining these relations, we have Lemma 10.

A.4 Proof for Proposition 12

Under the condition (49), every $\boldsymbol{h}^{(n)}$ destabilizes. Observe that critical points for these bifurcations $r^{(n)}$ $(n = 1, 2, \ldots, J-1)$ are solutions to $G^{(n)}(r) = 0$. Because $G^{(n)}(r)$ are strictly increasing in $r \in (0,1)$, if we can show $G^{(n)}(r) > G^{(n+1)}(r)\ \forall r \in (0,1)$, this implies $r^{(n)} < r^{(n+1)}$. Using the same notation as Appendix A.3 and using $\psi_0^{(n+1)} \geq \psi_0^{(n)}$ (with equality iff $r=1$), we conclude for $r \in (0,1)$

$$G^{(n)}(r) - G^{(n+1)}(r) > \frac{1}{2^{n+1}}\psi_0^{(n)}(r)\Big(\Delta\psi_0(r) + \Delta\epsilon(r)\Big) \tag{80a}$$

$$\Delta\psi_0(r) \equiv \psi_0^{(n+1)}(r) - 2\psi_0^{(n)}(r), \tag{80b}$$

$$\Delta\epsilon(r) \equiv \epsilon_{n+1}(r) - 2\epsilon_n(r) \tag{80c}$$

We see

$$\Delta\epsilon(r) = (\epsilon_{n+1} - \epsilon_n) - \epsilon_n \tag{81}$$

$$= 2 \sum_{k=2^n/2}^{(2^{n+1}/2)-1} C_k + 2 \sum_{k=1}^{(2^n/2)-1} C_k \tag{82}$$

$$= 2C_{2^n/2} + 2 \sum_{k=1}^{(2^n/2)-1} \Big(C_{k+(2^n/2)} - C_k\Big) \tag{83}$$

and because $C_k \geq C_l$ for $k > l$ (with equality iff $r=1$), we see that the second term in (83) is non-negative. Moreover, because $\psi_0^{(n)} = C_{2^n/2}$ for $n \neq J-1$, we have for such n

$$\Delta\psi_0(r) + \Delta\epsilon(r) = \epsilon_{n+1}(r) + 2 \sum_{k=1}^{(2^n/2)-1} \Big(C_{k+(2^n/2)} - C_k\Big) \geq 0 \tag{84}$$

with equality, again, iff $r=1$. This inequality and (80a) together imply that $G^{(n)} - G^{(n+1)} > 0$ holds for all $r \in (0,1)$, thus leading to Proposition 12.