The multivariate Beveridge–Nelson decomposition with I(1) and I(2) series

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Abstract

The consumption Euler equation implies that the output growth rate and the real interest rate are of the same order of integration; i.e., if the real interest rate is I(1), then so is the output growth rate and hence log output is I(2). To estimate the natural rates and gaps of macroeconomic variables jointly, this paper develops the multivariate Beveridge–Nelson decomposition with I(1) and I(2) series. The paper applies the method to Japanese data during 1980Q1–2013Q3 to estimate the natural rates and gaps of output, inflation, interest, and unemployment jointly.

JEL classification: C32, C82, E32

Keywords: Gap, Natural rate, Trend–cycle decomposition, Unit root

Highlights:

- We consider joint estimation of the natural rates and gaps of macroeconomic variables.
- By the consumption Euler equation, if the real interest rate is I(1), then log output is I(2).
- We develop the multivariate Beveridge–Nelson decomposition with I(1) and I(2) series.
- We apply the method to Japanese data during 1980Q1–2013Q3.

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1 Introduction

According to Kiley (2013, pp. 3, 10), the Beveridge–Nelson (B–N) decomposition gives the 'cleanest definition' of the natural rates and gaps of macroeconomic variables, relying only on a reduced-form time series model, e.g., a VAR model. Using a DSGE model of the US economy developed at the FRB, Kiley (2013) compares three definitions of the output gap (the B–N decomposition, the production function approach, and deviation from the flexible-price output), and finds that the resulting gap estimates are similar. Thus the B–N decomposition gives at least useful benchmark estimates of the natural rates and gaps.

The B–N decomposition is originally a method for decomposing I(1) series into random walk (permanent) and I(0) (transitory) components; cf. Beveridge and Nelson (1981). Hence, to estimate the natural rate and gap of output by the B–N decomposition, one assumes that log output is I(1). Similarly, for the multivariate B–N decomposition, one assumes that all variables are I(1).[1]

Since unit root tests often find that US log output is I(1), this requirement is not restrictive for the US data. In other countries, however, log output may be I(2). In fact, the consumption Euler equation in a simple macroeconomic model implies that the output growth rate and the real interest rate are of the same order of integration, i.e., for all t,

$$r_t^* = \delta + \rho \Delta \ln Y_{t+1}^* \tag{1}$$

where r_t^* is the natural rate of interest, Y_t^* is the natural rate of output, δ is the time preference rate, and ρ is the Arrow–Pratt measure of relative risk aversion; see Laubach and Williams (2003, p. 1063). Hence if the real interest rate is I(1), which is often the case, then log output must be I(2).[2]

To estimate the natural rates and gaps of macroeconomic variables jointly in such a case, this paper develops the multivariate B–N decomposition with I(1) and I(2) series. This is an extension of Newbold and Vougas (1996), Oh and Zivot (2006), and Oh et al. (2008), who develop the univariate B–N decomposition of I(2) series. The resulting B–N transitory components, or gaps, are simple linear transformations of the observable variables, with no need for the Kalman filter and smoother.

As an illustration, the paper applies the method to quarterly macroeconomic time series in Japan during 1980Q1–2013Q3 to estimate the natural rates and gaps of output, inflation, interest, and unemployment jointly. Unit root tests find that log output is I(2) in Japan during this period. Thus the multivariate B–N decomposition assuming I(1) log output gives an unreasonable output gap estimate. Our method gives a more sensible result.

The paper proceeds as follows. Section 2 specifies a VAR model, and gives a state space representation. Section 3 derives the multivariate B–N decompo-

[1]If log output is I(2), then it is common to use an unobserved components (UC) model and estimate the components by the Kalman filter and smoother. UC models require identification restrictions, however. A simple identification restriction is independence of the components, which often fails to hold in practice; see Morley et al. (2003). Weaker identification restrictions are complex in general, even for univariate UC models; see Sbrana (2013) and Iwata and Li (2015) for recent works.

[2]The consumption Euler equation (1) also implies cointegration between the real interest rate and the output growth rate if they are both I(1). We leave such consideration for future work, and focus on the multivariate B–N decomposition with I(1) and I(2) series in this paper. See Garratt et al. (2006) for the multivariate B–N decomposition of cointegrated series.

sition with I(1) and I(2) series. Section 4 illustrates the method using Japanese data. Section 5 concludes.

2 Model specification

2.1 VAR model

Let for $d = 1, 2$, $\{\boldsymbol{x}_{t,d}\}$ be an N_d-variate I(d) sequence. Let $N := N_1 + N_2$. Let for all t, $\boldsymbol{x}_t := (\boldsymbol{x}'_{t,1}, \boldsymbol{x}'_{t,2})'$, $\boldsymbol{y}_{t,1} := \boldsymbol{x}_{t,1}$, $\boldsymbol{y}_{t,2} := \Delta \boldsymbol{x}_{t,2}$, and $\boldsymbol{y}_t := (\boldsymbol{y}'_{t,1}, \boldsymbol{y}'_{t,2})'$, so that $\{\boldsymbol{y}_t\}$ is I(1). Let for $d = 1, 2$, $\boldsymbol{\mu}_d := \mathrm{E}(\Delta \boldsymbol{y}_{t,d})$. Let $\boldsymbol{\mu} := (\boldsymbol{\mu}'_1, \boldsymbol{\mu}'_2)'$. Assume a VAR($p$) model for $\{\Delta \boldsymbol{y}_t\}$ such that for all t,

$$\boldsymbol{\Phi}(\mathrm{L})(\Delta \boldsymbol{y}_t - \boldsymbol{\mu}) = \boldsymbol{w}_t \tag{2}$$

$$\{\boldsymbol{w}_t\} \sim \mathrm{WN}(\boldsymbol{\Sigma}) \tag{3}$$

2.2 State space representation

Define a state vector such that for all t,

$$\boldsymbol{s}_t := \begin{pmatrix} \Delta \boldsymbol{y}_t - \boldsymbol{\mu} \\ \vdots \\ \Delta \boldsymbol{y}_{t-p+1} - \boldsymbol{\mu} \end{pmatrix}$$

Then a state space representation of the VAR(p) model is for all t,

$$\boldsymbol{s}_t = \boldsymbol{A}\boldsymbol{s}_{t-1} + \boldsymbol{B}\boldsymbol{z}_t \tag{4}$$

$$\Delta \boldsymbol{y}_t = \boldsymbol{\mu} + \boldsymbol{C}\boldsymbol{s}_t \tag{5}$$

$$\{\boldsymbol{z}_t\} \sim \mathrm{WN}(\boldsymbol{I}_N) \tag{6}$$

where

$$\boldsymbol{A} := \begin{bmatrix} \boldsymbol{\Phi}_1 & \dots & \boldsymbol{\Phi}_p \\ & \boldsymbol{I}_{(p-1)N} & \mathbf{O}_{(p-1)N \times N} \end{bmatrix}$$

$$\boldsymbol{B} := \begin{bmatrix} \boldsymbol{\Sigma}^{1/2} \\ \mathbf{O}_{(p-1)N \times N} \end{bmatrix}$$

$$\boldsymbol{C} := \begin{bmatrix} \boldsymbol{I}_N & \mathbf{O}_{N \times (p-1)N} \end{bmatrix}$$

Since $\{\Delta \boldsymbol{y}_t\}$ is I(0), the roots of $\det(\boldsymbol{\Phi}(z)) = 0$ lie outside the unit circle, or the eigenvalues of $\boldsymbol{A}$ lie inside the unit circle.

We have for all t, for $s \geq 1$,

$$\mathrm{E}_t(\Delta \boldsymbol{y}_{t+s}) = \boldsymbol{\mu} + \boldsymbol{C}\boldsymbol{A}^s \boldsymbol{s}_t$$

or

$$\mathrm{E}_t(\Delta \boldsymbol{x}_{t+s,1}) = \boldsymbol{\mu}_1 + \boldsymbol{C}_1 \boldsymbol{A}^s \boldsymbol{s}_t \tag{7}$$

$$\mathrm{E}_t\left(\Delta^2 \boldsymbol{x}_{t+s,2}\right) = \boldsymbol{\mu}_2 + \boldsymbol{C}_2 \boldsymbol{A}^s \boldsymbol{s}_t \tag{8}$$

where

$$\boldsymbol{C}_1 := \begin{bmatrix} \boldsymbol{I}_{N_1} & \mathbf{O}_{N_1 \times N_2} & \mathbf{O}_{N_1 \times (p-1)N} \end{bmatrix}$$

$$\boldsymbol{C}_2 := \begin{bmatrix} \mathbf{O}_{N_2 \times N_1} & \boldsymbol{I}_{N_2} & \mathbf{O}_{N_2 \times (p-1)N} \end{bmatrix}$$

3 Multivariate B–N decomposition with I(1) and I(2) series

We introduce a lemma before stating our main result.

Lemma 1. *Suppose that $\boldsymbol{I}_{pN} - \boldsymbol{A}$ is invertible. Then for $T \geq 1$,*

$$T\boldsymbol{I}_{pN} + (T-1)\boldsymbol{A} + \cdots + \boldsymbol{A}^{T-1} = T(\boldsymbol{I}_{pN} - \boldsymbol{A})^{-1} - (\boldsymbol{I}_{pN} - \boldsymbol{A})^{-2}\left(\boldsymbol{I}_{pN} - \boldsymbol{A}^{T}\right)\boldsymbol{A}$$

Proof. We have for $T \geq 1$,

$$(\boldsymbol{I}_{pN} - \boldsymbol{A})\left(\boldsymbol{I}_{pN} + \boldsymbol{A} + \cdots + \boldsymbol{A}^{T-1}\right) = \boldsymbol{I}_{pN} - \boldsymbol{A}^{T}$$

or

$$\boldsymbol{I}_{pN} + \boldsymbol{A} + \cdots + \boldsymbol{A}^{T-1} = (\boldsymbol{I}_{pN} - \boldsymbol{A})^{-1}\left(\boldsymbol{I}_{pN} - \boldsymbol{A}^{T}\right)$$

Hence for $T \geq 1$,

$$\begin{aligned}
&(\boldsymbol{I}_{pN} - \boldsymbol{A})\left[T\boldsymbol{I}_{pN} + (T-1)\boldsymbol{A} + \cdots + \boldsymbol{A}^{T-1}\right] \\
&= T\boldsymbol{I}_{pN} - \boldsymbol{A} - \cdots - \boldsymbol{A}^{T} \\
&= T\boldsymbol{I}_{pN} - \left(\boldsymbol{I}_{pN} + \boldsymbol{A} + \cdots + \boldsymbol{A}^{T-1}\right)\boldsymbol{A} \\
&= T\boldsymbol{I}_{pN} - (\boldsymbol{I}_{pN} - \boldsymbol{A})^{-1}\left(\boldsymbol{I}_{pN} - \boldsymbol{A}^{T}\right)\boldsymbol{A}
\end{aligned}$$

□

Our main result is a straightforward extension of Morley (2002) and Oh and Zivot (2006). Let $\boldsymbol{x}_t^*$ and $\boldsymbol{c}_t$ be the B–N permanent and transitory components in $\boldsymbol{x}_t$, respectively.

Theorem 1. *Suppose that the eigenvalues of $\boldsymbol{A}$ lie inside the unit circle. Then for all t,*

$$\begin{aligned}
\boldsymbol{x}_{t,1}^* &= \lim_{T\to\infty}(\mathrm{E}_t(\boldsymbol{x}_{t+T,1}) - T\boldsymbol{\mu}_1) \\
\boldsymbol{x}_{t,2}^* &= \lim_{T\to\infty}\left\{\mathrm{E}_t(\boldsymbol{x}_{t+T,2}) - T^2\frac{\boldsymbol{\mu}_2}{2} - T\left[\frac{\boldsymbol{\mu}_2}{2} + \Delta\boldsymbol{x}_{t,2} + \boldsymbol{C}_2(\boldsymbol{I}_{pN} - \boldsymbol{A})^{-1}\boldsymbol{A}\boldsymbol{s}_t\right]\right\} \\
\boldsymbol{c}_{t,1} &= -\boldsymbol{C}_1(\boldsymbol{I}_{pN} - \boldsymbol{A})^{-1}\boldsymbol{A}\boldsymbol{s}_t \\
\boldsymbol{c}_{t,2} &= \boldsymbol{C}_2(\boldsymbol{I}_{pN} - \boldsymbol{A})^{-2}\boldsymbol{A}^2\boldsymbol{s}_t
\end{aligned}$$

Proof. Consider the decomposition of $\{\boldsymbol{x}_{t,1}\}$. We have for all t, for $T \geq 1$,

$$\begin{aligned}
\mathrm{E}_t(\boldsymbol{x}_{t+T,1}) &= \boldsymbol{x}_{t,1} + \sum_{s=1}^{T}\mathrm{E}_t(\Delta\boldsymbol{x}_{t+s,1}) \\
&= \boldsymbol{x}_{t,1} + \sum_{s=1}^{T}(\boldsymbol{\mu}_1 + \boldsymbol{C}_1\boldsymbol{A}^s\boldsymbol{s}_t) \\
&= \boldsymbol{x}_{t,1} + T\boldsymbol{\mu}_1 + \boldsymbol{C}_1\sum_{s=1}^{T}\boldsymbol{A}^s\boldsymbol{s}_t \\
&= \boldsymbol{x}_{t,1} + T\boldsymbol{\mu}_1 + \boldsymbol{C}_1(\boldsymbol{I}_{pN} - \boldsymbol{A})^{-1}\left(\boldsymbol{I}_{pN} - \boldsymbol{A}^{T}\right)\boldsymbol{A}\boldsymbol{s}_t
\end{aligned}$$

or

$$\boldsymbol{x}_{t,1} = \mathrm{E}_t(\boldsymbol{x}_{t+T,1}) - T\boldsymbol{\mu}_1 - \boldsymbol{C}_1(\boldsymbol{I}_{pN} - \boldsymbol{A})^{-1}\left(\boldsymbol{I}_{pN} - \boldsymbol{A}^T\right)\boldsymbol{A}\boldsymbol{s}_t$$

Take $T \to \infty$ and the result follows.

Consider the decomposition of $\{\boldsymbol{x}_{t,2}\}$. We have for all t, for $T \geq 1$,

$$\begin{aligned}
\boldsymbol{x}_{t+T,2} &= \boldsymbol{x}_{t,2} + \Delta\boldsymbol{x}_{t+1,2} + \Delta\boldsymbol{x}_{t+2,2} + \cdots + \Delta\boldsymbol{x}_{t+T,2} \\
&= \boldsymbol{x}_{t,2} + \Delta\boldsymbol{x}_{t,2} + \Delta^2\boldsymbol{x}_{t+1,2} \\
&\quad + \Delta\boldsymbol{x}_{t,2} + \Delta^2\boldsymbol{x}_{t+1,2} + \Delta^2\boldsymbol{x}_{t+2,2} \\
&\quad + \cdots \\
&\quad + \Delta\boldsymbol{x}_{t,2} + \Delta^2\boldsymbol{x}_{t+1,2} + \cdots + \Delta^2\boldsymbol{x}_{t+T,2} \\
&= \boldsymbol{x}_{t,2} + T\Delta\boldsymbol{x}_{t,2} + T\Delta^2\boldsymbol{x}_{t+1,2} + (T-1)\Delta^2\boldsymbol{x}_{t+2,2} + \cdots + \Delta^2\boldsymbol{x}_{t+T,2}
\end{aligned}$$

By the previous lemma, for $T \geq 1$,

$$\begin{aligned}
&\mathrm{E}_t(\boldsymbol{x}_{t+T,2}) \\
&= \boldsymbol{x}_{t,2} + T\Delta\boldsymbol{x}_{t,2} + T\,\mathrm{E}_t\left(\Delta^2\boldsymbol{x}_{t+1,2}\right) + (T-1)\,\mathrm{E}_t\left(\Delta^2\boldsymbol{x}_{t+2,2}\right) + \cdots \\
&\quad + \mathrm{E}_t\left(\Delta^2\boldsymbol{x}_{t+T,2}\right) \\
&= \boldsymbol{x}_{t,2} + T\Delta\boldsymbol{x}_{t,2} + T(\boldsymbol{\mu}_2 + \boldsymbol{C}_2\boldsymbol{A}\boldsymbol{s}_t) + (T-1)\left(\boldsymbol{\mu}_2 + \boldsymbol{C}_2\boldsymbol{A}^2\boldsymbol{s}_t\right) + \cdots \\
&\quad + \boldsymbol{\mu}_2 + \boldsymbol{C}_2\boldsymbol{A}^T\boldsymbol{s}_t \\
&= \boldsymbol{x}_{t,2} + T\Delta\boldsymbol{x}_{t,2} + \boldsymbol{\mu}_2\sum_{s=1}^{T} s + \boldsymbol{C}_2\left[T\boldsymbol{A} + (T-1)\boldsymbol{A}^2 + \cdots + \boldsymbol{A}^T\right]\boldsymbol{s}_t \\
&= \boldsymbol{x}_{t,2} + T\Delta\boldsymbol{x}_{t,2} + \boldsymbol{\mu}_2\frac{T(T+1)}{2} \\
&\quad + \boldsymbol{C}_2\left[T(\boldsymbol{I}_{pN} - \boldsymbol{A})^{-1} - (\boldsymbol{I}_{pN} - \boldsymbol{A})^{-2}\left(\boldsymbol{I}_{pN} - \boldsymbol{A}^T\right)\boldsymbol{A}\right]\boldsymbol{A}\boldsymbol{s}_t
\end{aligned}$$

or

$$\begin{aligned}
\boldsymbol{x}_{t,2} = \mathrm{E}_t(\boldsymbol{x}_{t+T,2}) - T\Delta\boldsymbol{x}_{t,2} - \boldsymbol{\mu}_2\frac{T(T+1)}{2} - \boldsymbol{C}_2 T(\boldsymbol{I}_{pN} - \boldsymbol{A})^{-1}\boldsymbol{A}\boldsymbol{s}_t \\
+ \boldsymbol{C}_2(\boldsymbol{I}_{pN} - \boldsymbol{A})^{-2}\left(\boldsymbol{I}_{pN} - \boldsymbol{A}^T\right)\boldsymbol{A}^2\boldsymbol{s}_t
\end{aligned}$$

Take $T \to \infty$ and the result follows. □

Let

$$\boldsymbol{W} := \begin{bmatrix} -\boldsymbol{C}_1(\boldsymbol{I}_{pN} - \boldsymbol{A})^{-1}\boldsymbol{A} \\ \boldsymbol{C}_2(\boldsymbol{I}_{pN} - \boldsymbol{A})^{-2}\boldsymbol{A}^2 \end{bmatrix}$$

Then for all t,

$$\boldsymbol{c}_t = \boldsymbol{W}\boldsymbol{s}_t \tag{9}$$

where $\boldsymbol{W}$ depends only on $\boldsymbol{\Phi}_1, \ldots, \boldsymbol{\Phi}_p$ and $\{\boldsymbol{s}_t\}$ is observable. Thus given the VAR coefficients, computation of $\boldsymbol{c}_t$ is straightforward. Moreover, since $\boldsymbol{c}_t$ is a linear transformation of $\Delta\boldsymbol{y}_t, \ldots, \Delta\boldsymbol{y}_{t-p+1}$, it is essential that $\{\Delta\boldsymbol{y}_t\}$ is indeed I(0) in order to obtain a reasonable estimate of $\{\boldsymbol{c}_t\}$.

Table 1: Data

Variable	Description	Source
Y_t	Real GDP (billions of chained 2005 yen, SA, AR)	Cabinet Office
P_t	GDP deflater (2005=100, SA)	Cabinet Office
I_t	Average interest rates on certificates of deposit by maturity (new issues) / 90 days to 179 days (%, AR)	Japanese Bankers Association (–1995), Bank of Japan (1996–)
L_t	Labor force (ten thousand persons, SA)	Statistics Bureau
E_t	Employed person (ten thousand persons, SA)	Statistics Bureau

Note: SA means 'seasonally-adjusted' and AR means 'annual rate.'

4 Application

4.1 Data

We consider joint estimation of the natural rates and gaps of the following four macroeconomic variables:

output Let Y_t be output. Assume that $\{\Delta \ln Y_t\}$ is I(1) without drift; i.e., $\{\ln Y_t\}$ is I(2) and $\mathrm{E}\left(\Delta^2 \ln Y_t\right) = 0$.

inflation rate Let P_t be the price level and $\pi_t := \ln(P_t/P_{t-1})$ be the inflation rate. Assume that $\{\pi_t\}$ is I(1).

interest rate Let I_t be the 3-month nominal interest rate (annual rate in per cent), $i_t := \ln(1 + I_t/400)$, $r_t := i_t - \mathrm{E}_t(\pi_{t+1})$ be the ex ante real interest rate, and $\hat{r}_t := i_t - \pi_{t+1}$ be the ex post real interest rate. Assume that $\{r_t\}$ is I(1).[3]

unemployment rate Let L_t be the labor force, E_t be employment, and $U_t := -\ln(E_t/L_t)$ be the unemployment rate. Assume that $\{U_t\}$ is I(1).

Table 1 lists the data used. Consistent long time series on real GDP and the GDP deflater with the most recent benchmark year 2005 are available only for 1980Q1–2013Q3.

Except for real GDP and the GDP deflater, we transform monthly series into quarterly series. For the nominal interest rate, we take the 3-month arithmetic means of the monthly series in each quarter. For the unemployment rate, we take the 3-month arithmetic means of monthly labor force and employment to obtain the quarterly series, from which we derive the quarterly unemployment rate. Let for all t,

$$\boldsymbol{x}_t := \begin{pmatrix} \pi_t \\ \hat{r}_t \\ U_t \\ \ln Y_t \end{pmatrix}, \quad \boldsymbol{y}_t := \begin{pmatrix} \pi_t \\ \hat{r}_t \\ U_t \\ \Delta \ln Y_t \end{pmatrix} \tag{10}$$

[3]We can estimate the interest rate gap even if we observe $\{\hat{r}_t\}$ instead of $\{r_t\}$. See Murasawa (2014, pp. 499–500).

Table 2: Unit root tests

Variable	Const.	Trend	ADF			ADF-GLS	
			Lags	τ	p-value	Lags	τ
π_t	yes	yes	1	-6.80	.00	5	-1.24
$\hat{r}_t$	yes	yes	1	-5.79	.00	5	-1.58
U_t	yes	yes	11	-2.23	.47	8	-1.53
$\Delta \ln Y_t$	yes	no	0	-9.34	.00	12	$-.43$
$\Delta\pi_t$	yes	no	4	-9.25	.00	0	-5.00^{***}
$\Delta\hat{r}_t$	yes	no	4	-9.47	.00	0	-13.99^{***}
ΔU_t	yes	no	12	-2.06	.26	12	-1.09
$\Delta^2 \ln Y_t$	no	no	7	-7.24	.00		

Notes: For the ADF-GLS test, *** denotes significance at the 1% level. For the number of lags included in the ADF regression, we use the default choice in gretl 1.10.1 with maximum 12, where the lag order selection criteria are AIC for the ADF test, and a modified AIC using the Perron and Qu (2007) method for the ADF-GLS test. With no constant nor trend in the ADF regression, the ADF test is asymptotically point optimal; hence the ADF-GLS test is unnecessary for $\left\{\Delta^2 \ln Y_t\right\}$.

The sample period of $\{\boldsymbol{y}_t\}$ is 1980Q2–2013Q2 (133 observations).

Table 2 shows the results of the ADF and ADF-GLS tests for unit root. The results depend on the number of lags included in the ADF regression. The ADF test suffers from size distortion with short lags and low power with long lags. The ADF-GLS test remedies the problems except when there is no constant nor trend in the ADF regression, in which case the ADF test is asymptotically point optimal. The level .05 ADF-GLS test fails to reject $H_0 : \{y_{t,i}\} \sim \mathrm{I}(1)$ against $H_1 : \{y_{t,i}\} \sim \mathrm{I}(0)$ for each variable, and rejects $H_0 : \{\Delta y_{t,i}\} \sim \mathrm{I}(1)$ in favor of $H_1 : \{\Delta y_{t,i}\} \sim \mathrm{I}(0)$ except for $\{U_t\}$ (use the ADF test for $\left\{\Delta^2 \ln Y_t\right\}$). Thus the unit root tests suggest that $\{\boldsymbol{y}_t\}$ is I(1) except for $\{U_t\}$, which may be I(2).

Table 3 shows the results of the KPSS stationarity tests. The results depend on the lag truncation parameter for the Newey–West estimator of the long-run error variance. The level .05 KPSS test rejects $H_0 : \{y_{t,i}\} \sim \mathrm{I}(0)$ in favor of $H_1 : \{y_{t,i}\} \sim \mathrm{I}(1)$ except for $\{\pi_t\}$, and fails to reject $H_0 : \{\Delta y_{t,i}\} \sim \mathrm{I}(0)$ against $H_1 : \{\Delta y_{t,i}\} \sim \mathrm{I}(1)$ for each variable. Thus the stationarity tests suggest that $\{\boldsymbol{y}_t\}$ is I(1) except for $\{\pi_t\}$, which may be I(0).

Given the testing results, we proceed with our assumption that $\{\boldsymbol{y}_t\}$ is I(1).

4.2 Model specification

For convenience, we center $\{\Delta\boldsymbol{y}_t\}$ except for $\Delta^2 \ln Y_t$, for which the mean is 0 by assumption, and delete the constant term from the model.

To select p, we compute model selection criteria for $p = 1, \dots, 8$. The common estimation period is 1982Q4–2013Q2. Table 4 summarizes the results of lag order selection. The level .05 LR test rejects $H_0 : \{\Delta\boldsymbol{y}_t\} \sim \mathrm{VAR}(p-1)$ in favor of $H_1 : \{\Delta\boldsymbol{y}_t\} \sim \mathrm{VAR}(p)$ even for $p = 8$, whereas AIC, BIC, and HQC select smaller models. Since a high-order VAR model covers low-order VAR models as special cases, we choose $p = 8$ to be on the safe side.

We estimate the VAR(8) model by OLS, and compute the B–N transitory

Table 3: KPSS stationarity tests

Variable	Trend	LM
π_t	yes	$.13^{*}$
$\hat{r}_t$	yes	$.35^{***}$
U_t	yes	$.24^{***}$
$\Delta \ln Y_t$	no	1.01^{***}
$\Delta\pi_t$	no	.17
$\Delta\hat{r}_t$	no	.06
ΔU_t	no	.15
$\Delta^2 \ln Y_t$	no	.04

Notes: * and *** denote significance at the 10% and 1% levels respectively. The lag truncation parameter for the Newey–West estimator of the long-run error variance is 4 (the default value for our sample length in gretl 1.10.1).

Table 4: Lag order selection

p	Log-lik	LR	p-value	AIC	BIC	HQC
1	2208.40			-35.649	-35.283^{*}	-35.500
2	2245.72	37.32	.00	-35.996	-35.264	-35.698^{*}
3	2253.71	7.98	.46	-35.865	-34.768	-35.419
4	2284.81	31.10	.00	-36.111^{*}	-34.647	-35.516
5	2300.58	15.77	.01	-36.107	-34.278	-35.364
6	2313.95	13.37	.04	-36.064	-33.869	-35.173
7	2327.77	13.82	.03	-36.029	-33.468	-34.989
8	2346.87	19.10	.00	-36.079	-33.153	-34.891

Notes: For AIC, BIC, and HQC, * denotes the selected model. The LR test statistic for testing $H_0 : \{\Delta \boldsymbol{y}_t\} \sim \text{VAR}(p-1)$ vs $H_1 : \{\Delta \boldsymbol{y}_t\} \sim \text{VAR}(p)$ follows $\chi^2(16)$ under H_0.

components according to (9). We use R 3.2.2 by R Core Team (2015) for the computation.

4.3 Empirical results

We compare the results of the multivariate B–N decomposition of $\{\boldsymbol{x}_t\}$ under alternative assumptions about log output, i.e., I(1) or I(2), using the Japanese data. For the former, we estimate a VAR(8) model for $\{\Delta\boldsymbol{x}_t\}$, and apply the usual multivariate B–N decomposition to $\{\boldsymbol{x}_t\}$. Our gaps are the B–N transitory components and hence should be I(0) with mean 0.

Figure 1 plots the gap estimates assuming I(1) log output. Since the largest magnitude eigenvalue of the estimated $\boldsymbol{A}$ is .946, the multivariate B–N decomposition is applicable. In the bottom panel, the output gap does not seem I(0) with mean 0, but has an upward trend. The output gap is mostly negative until mid-1990s, and positive afterwards. Moreover, the output gap seems too large, sometimes exceeding $\pm 10\%$. The unemployment rate gap has an upward trend as well, and the interest rate gap may have a downward trend. Except for the inflation rate gap, these gap estimates do not seem I(0) with mean 0 and hence unreasonable.

Figure 2 plots the gap estimates assuming I(2) log output. Since the largest magnitude eigenvalue of the estimated $\boldsymbol{A}$ is .909, the multivariate B–N decomposition is applicable. In the top panel, the inflation rate gap hardly changes from the previous estimate. The interest rate and unemployment rate gaps are also similar to the previous estimates except for trend. The output gap differs substantially from the previous estimate, however, with no trend and smaller size. Overall, these gap estimates seem I(0) with mean 0 and hence make more sense.

The gap estimates depend on p, the order of the VAR model. However, for any p, assuming I(2) log output gives gap estimates that seem I(0) with mean 0, whereas assuming I(1) log output does not, at least for our Japanese data. In such a case, the multivariate B–N decomposition with I(1) and I(2) series is a simple and useful alternative to the decomposition based on a UC model.

5 Conclusion

The consumption Euler equation implies that if the real interest rate is I(1), then log output is I(2). To estimate the natural rates and gaps jointly in such a case, this paper develops the multivariate B–N decomposition with I(1) and I(2) series. The proposed method seems useful at least for the Japanese data.

The consumption Euler equation also implies cointegration between the real interest rate and the output growth rate if they are both I(1). To introduce cointegration into the multivariate B–N decomposition, one can use a VECM instead of a VAR model; see Garratt et al. (2006).

We never know for sure if log output is I(1) or I(2), but only have posterior probabilities. Perhaps another interesting and important next step is to apply Bayesian model averaging in this context.

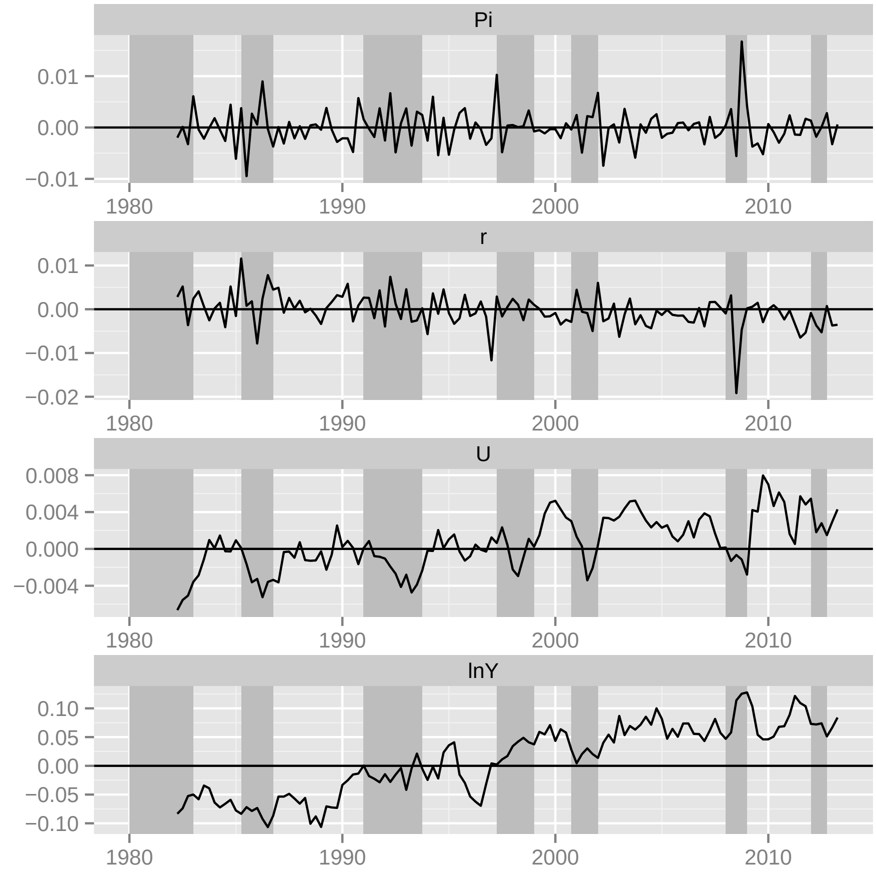

Figure 1: Gap estimates assuming I(1) log output. The shaded areas indicate the recessions determined by the Cabinet Office.

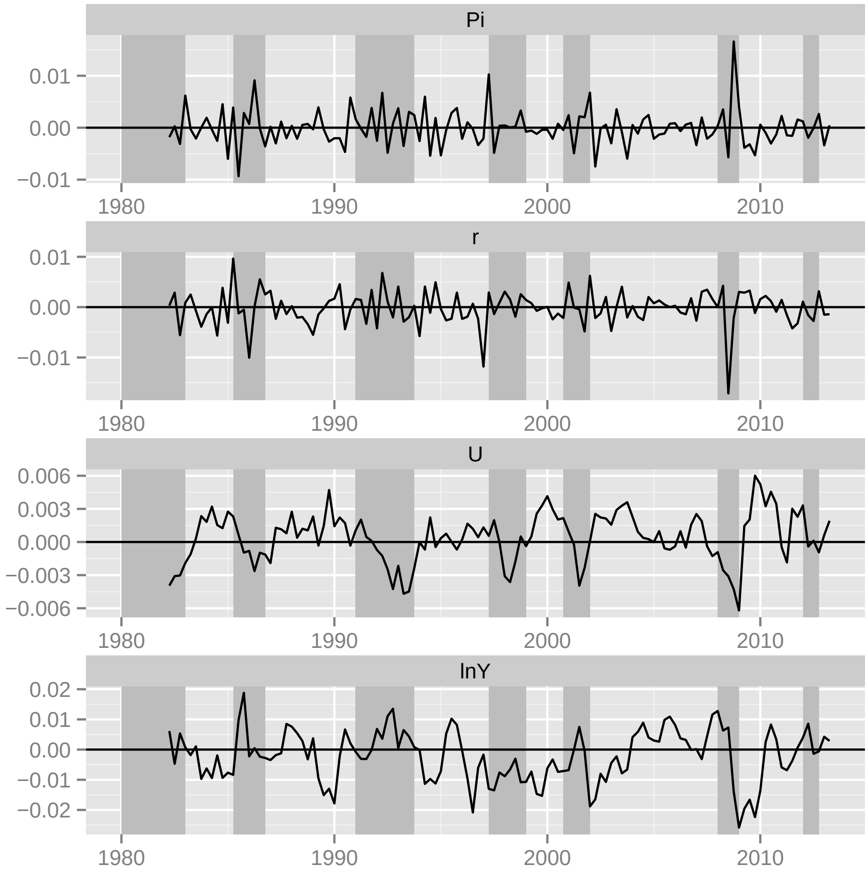

Figure 2: Gap estimates assuming I(2) log output. The shaded areas indicate the recessions determined by the Cabinet Office.

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