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Abstract: This paper proposes the spectral corrected methodology to estimate the Global Minimum Variance Portfolio (GMVP) for the high dimensional data. In this paper, we analysis the limiting properties of the spectral corrected GMVP estimator as the dimension and the number of the sample set increase to infinity proportionally. In addition, we compare the spectral corrected estimation with the linear shrinkage and nonlinear shrinkage estimations and obtain that the performance of the spectral corrected methodology is best in the simulation study.

Keyword: Global Minimum Variance Portfolio, Spectral Corrected Covariance, Sample Covariance

1 Introduction

Since Markowitz mean-variance (MV) portfolio has been invented in 1952, it has become a cornerstone of modern finance. This theory incorporates the investors' preference and their expectation of returns and risks. By this theory, investors could construct the optimal portfolio by minimizing the portfolio variance for a given level of the expected return or maximizing the portfolio return for a given level of the portfolio risk. In MV portfolio theory the global minimum variance portfolio (GMVP) is a remarkable and mostly useful portfolio. This portfolio has the smallest variance over all portfolios and do not depend on the expected return. In fact, much empirical work reports underperformance of market capitalisation-weighted portfolios relatives to the GMV portfolio (Clarke et al. [2006], Baker et al. [2011] and so on).

Suppose there are p assets and r_i be the random return of the ith one $(i = 1, ..., p)$. Denote $\mathbf{w} = (\omega_1, ..., \omega_p)'$ as an asset allocation. Then the theoretical GMVP is the unique solution of the following quadratic program:

$$\text{minimize} \quad \mathbf{w}'\Sigma_p\mathbf{w} \quad \text{subject to} \quad \mathbf{w}'\mathbf{1} = 1. \tag{1}$$

Here $\mathbf{1}$ is a vector of which the elements are 1s and Σ_p stands for the covariance matrix of the random return vector $\mathbf{r} = (r_1, ..., r_p)'$. Then the theoretical GMVP

$$\mathbf{w}_o = \frac{\Sigma_p^{-1}\mathbf{1}}{\mathbf{1}'\Sigma_p^{-1}\mathbf{1}}, \tag{2}$$

the expected GMVP return $R_o = \boldsymbol{\mu}'\Sigma_p^{-1}\mathbf{1}/\mathbf{1}'\Sigma_p^{-1}\mathbf{1}$ and the GMVP risk $\sigma_o^2 = 1/\mathbf{1}'\Sigma_p^{-1}\mathbf{1}$.

There are a great deal of papers to estimate GMVP (see Kempf and Memmel [2003], Bodnar and Schmid [2008], Bodnar and Schmid [2007], Frahm and Memmel [2010], Clarke et al. [2011], Bodnar and Gupta [2012], Wied et al. [2013], Green et al. [2013] and so on). GMVP has nice theoretical properties in many ways, but it is inevitable to estimate the population covariance matrix of the random returns in practice. The classical estimator is usually constructed by plugging the sample covariance matrix into GMVP expression (2) instead of the unknown parameter Σ_p. When the number of observations n is large enough compared with the number of assets p, the sample covariance matrix is not bad choice (see Okhrin and Schmid [2006], Memmel and Kempf, Bodnar and Schmid [2009] and so on), but it tends to be far from the population covariance matrix (Bai et al. [2004]) when the number of assets p can not be ignored with respect to the sample size n and thus it is not a appropriate estimator of Σ.

When p is large compared with the sample size n, how to estimate a covariance matrix and/or of its inverse has been a hot issue for recent ten or even more years (Bickel and Levina [2008], Cai and Zhou [2012], Rohde et al. [2011], Khare et al. [2011], Rajaratnam et al. [2008], Fan et al. [2008],Ledoit and Wolf [2004a] , Ledoit and Wolf [2004b], Golosnoy and Okhrin [2007], Frahm and Memmel [2010] and so on).

In this paper, we estimate GMVP for the high dimensional data by the spectral corrected Methodology. Here we propose the spectral corrected covariance as the population covariance estimator and plug it into (2). In this paper, we compare the spectral corrected estimation with the classic estimation, the linear shrinkage estimation and the nonlinear shrinkage estimation and find the performance of the spectral corrected estimation is best in the simulation study.

In Section 2, we will introduce the spectral corrected estimation for GMVP. In Sections 3 and 4, the Linear shrinkage estimation and the nonlinear shrinkage estimation are provided. We provide the simulation study in Section 5 and the conclusion in Section 6.

2 Spectral corrected estimation

In this part we develop the spectral corrected GMVP by plugging the spectral-corrected covariance as the estimation of Σ_p into (2). The spectral corrected covariance is constructed by correcting the spectrum of the sample covariance matrix. Suppose that $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ are identified independent distributed (i.i.d.) p dimensional random return vectors with the mean $\boldsymbol{\mu}$ and the covariance matrix Σ_p. Write the spectral decomposition of the sample covariance $\mathbf{S}_n = \sum_{i=1}^n \mathbf{x}_i\mathbf{x}' - \overline{\mathbf{x}}\,\overline{\mathbf{x}}'$ can as $\mathbf{T}_n\Lambda_n\mathbf{T}_n'$ where $\Lambda_n = diag\{\lambda_{n,1}, \lambda_{n,2}, ..., \lambda_{n,p}\}$ ($\lambda_{n,1} \geq \lambda_{n,2} \geq ... \geq \lambda_{n,p}$) and $\mathbf{T}_n$ is the matrix with the corresponding eigenvectors.

For any symmetry matrix $\mathbf{A}$ with the dimension p, define the empirical spectral distribution (SD) as following

$$F^{\mathbf{A}}(x) = \frac{1}{p}\sum_{i=1}^{p} 1_{[\lambda_i,\infty)}(x), \quad x \in \mathbb{R} \tag{3}$$

in which $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p$ are the eigenvalues of $\mathbf{A}$ and $1_{[\lambda_i,\infty)}(x)$ equals to 1 for $\lambda_i \leq x$ and 0 otherwise. Then according to the large dimensional random matrix theory, under some reasonable conditions, $F^{\mathbf{S}_n}$ weakly converges to a deterministic distribution F as p and n increase to infinity proportionally, which is called limiting spectral distribution (LSD) (Marčenko and Pastur [1967], Silverstein [1995] and Silverstein and Bai [1995]). Denote the stieltjes transform of F as $m(z) = \int (x-z)^{-1}dF(x)$ ($z \in \mathbb{C}$ and $\Im z \neq 0$). Then $m(z)$ is the unique solution of

$$z(\underline{m}) = \frac{1}{\underline{m}(z)} + y\int \frac{t}{1+t\underline{m}}dH(t) \tag{4}$$

on the upper half complex plane in which H is the limiting spectral distribution of the population covariance matrix Σ_p and $\underline{m} = -(1-y)/z + ym$ ($z \in \mathbb{C}^+$). (4) provides a chance to recover the spectral information of the population covariance Σ_p. Mestre [2008] and Li and Yao [2013] provide the spectral estimation based on the contour integration under eigenvalue splitting condition and no splitting condition, respectively. Karoui [2008], Rao et al. [2008] , Chen et al. [2011] and so on introduce more methods to estimate the spectral construction of the population Σ_p.

In this paper, we do not focus on the estimation of F^{Σ_p} and suppose the spectral construction of Σ_p is known or estimated well. We correct the spectrum of the sample covariance and obtain the spectral corrected variance $\widetilde{\mathbf{S}}_n = \mathbf{T}_n\Delta_p\mathbf{T}_n'$ in which $\Delta_p = diag\{\tau_1, ..., \tau_p\}$ and $\tau_1 \geq ... \geq \tau_p$ are the spectral elements of Σ_p. Compared with the quartic form of the sample covariance matrix inverse, that of the spectral corrected covariance one performs more stably in the simulation study. We explain more details in the Theorem 2.1 and 2.2 under some reasonable assumptions:

Assumption 2.1. $\mathbf{Y}_n = (\mathbf{y}_1, \cdots, \mathbf{y}_n) = (y_{i,j})_{p,n}$ *in which* $y_{i,j}$ $(i = 1, \cdots, p, j = 1, \cdots, n)$ *are i.i.d. random variables with* $Ey_{ij} = 0$, $E|y_{ij}|^2 = 1$, $E|y_{ij}|^4 < \infty$, *and* $\mathbf{x}_k = \Sigma_p^{1/2}\mathbf{y}_k$ *for each* n *and for* $k = 1, 2, \cdots, n$;

Assumption 2.2. $\mathbf{a}_p, \mathbf{b}_p \in \mathbb{C}^p = \{\mathbf{x} \in \mathbb{C}^p\}$ *are uniformly bounded vectors.*

Assumption 2.3. $\Sigma_p = \mathbf{U}_p\Delta_p\mathbf{U}_p^*$ *is nonrandom symmetry and nonnegative definite with its spectral norm bounded in* p *where*

$$\Delta_p = diag(\tau_1\mathbf{I}_1, \tau_2\mathbf{I}_2, ..., \tau_L\mathbf{I}_L) \tag{5}$$

$\tau_1 > \tau_2 > \cdots > \tau_L$, $\mathbf{U}_p = (\mathbf{U}_{p,1}, \mathbf{U}_{p,2}, \cdots, \mathbf{U}_{p,L})$ *and* $\mathbf{I}_i$ *is a* $p_i \times p_i$ *unit matrix* $(i = 1, 2, ..., L)$.

Theorem 2.1. Under Assumptions from 2.1 to 2.3, as $p, n \to \infty$, $p/n \to y, (0 < y < 1)$, we have

$$\frac{\mathbf{a}_p'\mathbf{S}_n^{-1}\mathbf{b}_p}{\mathbf{a}_p'\Sigma_p^{-1}\mathbf{b}_p} \longrightarrow \frac{1}{(1-y)} \quad \text{a.s.} \tag{6}$$

According to Theorem 2.1, $\mathbf{a}_p'\mathbf{S}_n^{-1}\mathbf{b}_p$ is asymptotically $(1-y)^{-1}$ time of $\mathbf{a}_p'\Sigma_p^{-1}\mathbf{b}_p$ and when $y \to 1$, $\mathbf{a}_p'\mathbf{S}_n^{-1}\mathbf{b}_p$ is close to infinity. This is because the minimum eigenvalue of the sample covariance is very close to zero as $y \to 1$ and thus its inverse goes to infinity which mainly decides the value of $\mathbf{a}_p'\mathbf{S}_n^{-1}\mathbf{b}_p$. Then it is a natural idea of correcting eigenvalues of $\mathbf{S}_n$ to be the spectral corrected covariance $\widetilde{\mathbf{S}}_n$. The following theorem explains the limiting behavior about the quartic form $\mathbf{a}_p'\widetilde{\mathbf{S}}_n^{-1}\mathbf{b}_p$ under general conditions.

Theorem 2.2. Suppose the limiting spectral distribution of $\mathbf{S}_n$ is spectral separated and $\mathbf{a}_p'\mathbf{U}_{p,i}\mathbf{U}_{p,i}^T\mathbf{b}_p = f_i$ $(i = 1, 2, \cdots, L)$. Under Assumptions 2.1 to 2.3, we have

$$\mathbf{a}_p'\widetilde{\mathbf{S}}_n^{-1}\mathbf{b}_p \longrightarrow \sum_{k=1}^{L} f_k d_k \quad \text{a.s.} \tag{7}$$

as $p, n \to \infty$ and $p/n \to y$. Here $d_k = \sum_{j=1}^{L} \frac{(u_j-\tau_j)}{\tau_j(u_j-\tau_k)}$ and u_j is the solution of $1+y\int \frac{t}{u-t}dH(t) = 0$ for any $j = 1, \cdots, L$ with $\tau_1 > u_1 > \tau_2 > \cdots > \tau_L > u_L > 0$.

Note: in (7), it is difficult to explain the relationship between the limiting behaves of $\mathbf{a}_p'\mathbf{S}_n^{-1}\mathbf{b}_p$ and $\mathbf{a}'\widetilde{\mathbf{S}}_n^{-1}\mathbf{b}_p$ theoretically. In this paper, we provide some Monte-Carlo experiments to describe the performance of $\mathbf{a}_p'\widetilde{\mathbf{S}}_n^{-1}\mathbf{b}_p$ and the conclusion of Theorem 2.1 and 2.2.

For the population covariance Σ_p, select series $\{\mathbf{a}_p\}$ and $\{\mathbf{b}_p\}$ such that $\mathbf{a}_p'\Sigma_p^{-1}\mathbf{b}_p$ is a constant for any dimension p. Generate the sample set $\mathbf{X}_1, ...\mathbf{X}_n$ from the population with a non zero mean $\boldsymbol{\mu}$ and the covariance matrix Σ_p and compute $\mathbf{a}_p'\mathbf{S}_n^{-1}\mathbf{b}_p$ and $\mathbf{a}_p'\widetilde{\mathbf{S}}_n^{-1}\mathbf{b}_p$. Here we use $\boldsymbol{\tau} = [\tau_1, \tau_2, ..., \tau_k]$ and $\mathbf{w} = [w_1, w_2, ..., w_p]$ $(w_i = p_i/p,\ i = 1, 2, ..., p)$ to denote the different eigenvalues of Σ_p and the corresponding weights on the whole p dimension. Repeat this process for N times and report their means and the standard deviations in Table 1.

From Table 1, we can find some better performances of the spectral corrected covariance than that of the sample covariance in the estimation of the quartic form $\mathbf{a}_p'\Sigma_p^{-1}\mathbf{b}_p$. First, the estimate of $\mathbf{a}_p'\widetilde{\mathbf{S}}_n^{-1}\mathbf{b}_p$ is more accurate than that of $\mathbf{a}_p'\mathbf{S}_n^{-1}\mathbf{b}_p$. In Table 1, for a given $\mathbf{a}_p'\Sigma_P^{-1}\mathbf{b}_p$, the average error of $\mathbf{a}_p'\widetilde{\mathbf{S}}_n^{-1}\mathbf{b}_p$ is very small as $y = 0.1$ and increases slowly with the increasing of y from 0.1 to 0.9 but still is bounded by 0.5 in all three Panels. For the sample covariance, the average error of $\mathbf{a}_p'\mathbf{S}_n^{-1}\mathbf{b}_p$ is smallest when $y = 0.1$ and still is about 0.2 in Panel A, B and C. It increases with increasing of y from 0.1 to 0.9 rapidly. When $y = 0.9$, $\mathbf{a}_p'\mathbf{S}_n^{-1}\mathbf{b}_p$ is asymptotically ten times of $\mathbf{a}_p'\Sigma_p^{-1}\mathbf{b}_p$. In addition, the estimate of $\mathbf{a}_p'\mathbf{S}_n^{-1}\mathbf{b}_p$ is much unstabler compared with that of $\mathbf{a}_p'\widetilde{\mathbf{S}}_n^{-1}\mathbf{b}_p$ according to their standard deviation reports. Though both of the standard deviations associated with $\widetilde{\mathbf{S}}_n$ and $\mathbf{S}_n$ increase as the increasing of y, the largest one of $\mathbf{a}_p'\widetilde{\mathbf{S}}_n^{-1}\mathbf{b}_p$ is still not more than 0.4 while that of $\mathbf{a}_p'\mathbf{S}_n\mathbf{b}_p$ already reaches 16.

From above analysis and the simulation study, it is natural to plug $\widetilde{\mathbf{S}}_n$ into (2) and have the spectral corrected global minimum variance portfolio (SCGMVP) as follows:

$$\widetilde{\mathbf{w}}_{sc} = \frac{\widetilde{\mathbf{S}}_n^{-1}\mathbf{1}}{\mathbf{1}'\widetilde{\mathbf{S}}_n^{-1}\mathbf{1}}. \tag{8}$$

Then the expected return of SCGMVP $R_{sc} = \boldsymbol{\mu}'\widetilde{\mathbf{w}}_{sc}$ and the corresponding risk $\sigma_{sc}^2 = \widetilde{\mathbf{w}}_{sc}'\Sigma_p\widetilde{\mathbf{w}}_{sc}$. In the structure of $\widetilde{\mathbf{w}}_{sc}$, the eigenvector matrix $\mathbf{T}_n$ is the main random part and determines the final performance of SCGMV.

Corollary 2.1. Under the conditions of Theorem 2.2,

$$\frac{R_{sc}}{R_o} \to \frac{\sum_{i,j=1}^{L} f_{r,i} f_{1,j} d_i \tau_j}{\sum_{i,j=1}^{L} f_{r,i} f_{1,j} d_j \tau_i} \quad \text{a.s.}$$

as $p, n \to \infty$ and $p/n \to y$. Here d_i $(i = 1, ..., L)$ are defined in Theorem 2.2.

The proof of Corollary 2.1 can be deduced by Theorem 2.1 and 2.2 directly.

By the spectral corrected methodology, we can not obtain a portfolio estimator with a consistent expected return as R_o, but SCGMVP has higher expected return and lower risk than the classical GMVP estimator with the sample covariance does. And compared with the Linear shrinkage and nonlinear shrinkage methodologies, the performance of our SCGMVP is still better. Now we review some contents about these two methodologies.

3 Linear shrinkage Estimation

Ledoit and Wolf [2004b] propose a well conditioned estimator for large dimensional covariance matrices. In this paper, they focus on the optimal linear combination $\Sigma^* = \rho_1 \mathbf{I} + \rho_2 \mathbf{S}_n$ of the identity matrix $\mathbf{I}$ and the sample covariance matrix who minimizes the expected quadratic loss $E\|\Sigma^* - \Sigma\|^2$ for all $\rho_1, \rho_2 \in \mathbb{R}$. That is

$$\min_{\rho_1,\rho_2} E\|\Sigma^* - \Sigma\|^2 \quad \text{s.t.} \quad \Sigma^* = \rho_1 \mathbf{I} + \rho_2 \mathbf{S}_n. \tag{9}$$

Here $\|\cdot\|$ is the Frobenius norm: $\|\mathbf{M}\| = \sqrt{\mathrm{Tr}(\mathbf{M}\mathbf{M}')/r}$ for any $r \times m$ matrix $\mathbf{M}$. By the computation, the solution of the optimization problem is

$$\Sigma^* = \frac{\beta^2}{\delta^2}\mu\mathbf{I} + \frac{\alpha^2}{\delta^2}\mathbf{S}_n \tag{10}$$

Here $\mu = \mathrm{tr}(\Sigma)/p$, $\alpha^2 = \|\Sigma - \mu\mathbf{I}\|^2$, $\beta^2 = E\|\mathbf{S}_n - \Sigma\|^2$ and $\delta^2 = E\|\mathbf{S}_n - \mu\mathbf{I}\|^2$. By the large dimensional random theory, the consistent estimations for these parameters is $\hat{\mu} = \mathrm{tr}(\mathbf{S}_n)/p$, $\hat{\delta}^2 = \|\mathbf{S}_n - \hat{\mu}\mathbf{I}\|^2$, $\hat{\beta}^2 = \min(\overline{b}_n^2, \hat{\delta}_n^2)$ and $\hat{\alpha}_n^2 = \hat{\delta}_n^2 - \hat{\beta}_n^2$ where $\overline{b}_n^2 = \frac{1}{n^2}\sum_{k=1}^n \|\mathbf{y}_{\cdot k}\mathbf{y}'_{\cdot k} - \mathbf{S}_n\|^2$ and $\mathbf{y}_{\cdot k}$ is the k-th column of the data matrix $\mathbf{Y}_n$. Then the corresponding linear shrinkage estimator is given as

$$\overline{\Sigma}_n = \frac{\hat{\beta}^2}{\hat{\delta}^2}\hat{\mu}\mathbf{I} + \frac{\hat{\alpha}^2}{\hat{\delta}^2}\mathbf{S}_n.$$

4 Nonlinear shrinkage Estimation

Nonlinear shrinkage estimation of the covariance matrix was introduced by Ledoit et al. [2012]. Now I will introduce this methodology shortly. Suppose $\widehat{\Sigma} \equiv \widehat{\Sigma}(\mathbf{Y}_n)$ be an estimator of Σ under the data matrix $\mathbf{Y}_n$. Then for any arbitrary orthogonal matrix $\mathbf{A}$, if $\widehat{\Sigma}(\mathbf{A}\mathbf{Y}_n) = \mathbf{A}\widehat{\Sigma}(\mathbf{Y}_n)\mathbf{A}$, the estimation $\widehat{\Sigma}$ is said to be rotation-equivariant. With the rotation equivariant property, the estimator of Σ can be written as the form $\mathbf{V}_n\mathbf{D}_n\mathbf{V}'_n$ where $\mathbf{D}_n$ is a diagonal matrix with elements $d_1, ..., d_p$ and $\mathbf{V}_n$ is the sample eigenvectors matrix of the data set. In the rotation-equivariant estimator set, Ledoit et al. [2012] consider the optimal estimator under the following loss function

$$\left\|\mathbf{V_n}\mathbf{D}_n\mathbf{V}'_n - \Sigma\right\| \tag{11}$$

where $\|\cdot\|$ is the Frobenius norm defined as $\|\mathbf{M}\| = \sqrt{\mathrm{Tr}(\mathbf{M}\mathbf{M}')/r}$ for any $r \times m$ matrix $\mathbf{M}$. Minimize the loss function (11) and get the solution is $\mathbf{D}_n^* \equiv \mathrm{Diag}(d_1^*, ..., d_p^*)$ where $d_i^* = v_i'\Sigma v_i$ and v_i is i-th column of $\mathbf{V}$ for $i = 1, ..., p$. Then the optimal rotation-equivariant estimator of Σ is $\mathbf{S}_n^* = \mathbf{V}_n\mathbf{D}_n^*\mathbf{V}'_n$.

In fact, the structures of the optimal rotation-equivariant estimator $\mathbf{S}_n^* = \mathbf{V}_n\mathbf{D}_n^*\mathbf{V}'_n$ and the spectral-corrected estimator $\widetilde{\mathbf{S}}_n = \mathbf{T}_n\Delta_p\mathbf{T}'_n$ are same. They all keep the eigenvector matrix of sample covariance and change it's spectral elements and $\mathbf{S}_n^*$ has smaller loss than $\widetilde{\mathbf{S}}_n$. But the problem is d_i^* $(i = 1, ..., p)$ are unknown even as the spectral element of Σ_p is given. It not only increases the difficulty of estimations but also reduces the accuracy.

For the estimation of d_i^* $(i = 1, ..., p)$, Ledoit and Péché [2011] show that they can be approximated by

$$d_i^{or} \equiv \frac{\lambda_i}{\left|1 - c - c\lambda_i \breve{m}_F(\lambda_i)\right|^2}. \tag{12}$$

Here λ_i is the ith eigenvectvalue of sample covariance, $\breve{m}_F(\lambda_i) = \frac{1-c}{c\lambda_i} - \frac{1}{cz_{\lambda_i}}$ and z_{λ_i} is the solution of the following equation

$$z - cz\int_{-\infty}^{+\infty} \frac{\tau}{\tau - z} dH(\tau) = \lambda_i \quad \text{for} \quad i = 1, ...p \quad \text{and} \quad z \in \mathbb{C}^+.$$

According to (12), the oracle estimator is given as

$$\mathbf{S}_n^{or} = \mathbf{V}_n\mathbf{D}_n^{or}\mathbf{V}'_n \quad \text{where} \quad \mathbf{D}_n^{or} \equiv \mathrm{Diag}(d_1^{or}, ..., d_p^{or}).$$

By the same way, among the class of rotation-equivant estimators, the optimal estimator of Σ_p^{-1} is given by $\mathbf{P}_n^* \equiv \mathbf{V}_n\mathbf{A}_n^*\mathbf{V}'_n$ where $\mathbf{A}_n^* \equiv \mathrm{Diag}(a_1^*, ..., a_p^*)$. Here $a_i^* \equiv v_i'\Sigma^{-1}v_i$ and can be approximated by $a_i^{or} \equiv \lambda_i^{-1}(1 - c - 2c\lambda_i\mathrm{Re}[\breve{m}_F(\lambda_i)])$ for $i = 1, ..., p$. Then the corresponding oracle estimator of Σ^{-1} is given as $\mathbf{P}_n^{or} \equiv \mathbf{V}_n\mathbf{A}_n^{or}\mathbf{V}_n$ in which $\mathbf{A}_n^{or} = \mathrm{Diag}(a_1^{or}, ..., a_p^{or})$.

5 Simulation study

In this part, we compare the behaviors of four estimators of Σ_p—the spectral-corrected estimator $\widetilde{\mathbf{S}}_n$, the sample covariance $\mathbf{S}_n$, the linear shrinkage estimator $\overline{\Sigma}_n$ and the nonlinear shrinkage estimators $\mathbf{S}_n^{or}$ and $\mathbf{P}_n^{or}$ in GMV model. In particular, note that $\mathbf{P}_n^{or}$ is the estimator of Σ^{-1} and not equal to $(\mathbf{S}_n^{or})^{-1}$.

Plug the estimators of Σ_p or Σ_p^{-1} into R_o and σ_o^2 and compare their behaviors in the portfolio expected return and risk.

Suppose there are p assets with a random return vector $\mathbf{r} = (r_1, r_2, ..., r_p)'$ with nonzero mean $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_p)'$ and the covariance matrix Σ_p. Here we use $\boldsymbol{\tau} = [\tau_1, \tau_2, ..., \tau_k]$ and $\mathbf{w} = [w_1, w_2, ..., w_p]$ ($w_i = p_i/p$) to denote the different eigenvalues of Σ_p and the corresponding weights on the whole p dimension. The simulation is designed in the following steps:

(i) Generate n i.i.d. p dimensional sample vectors $\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_n$ with the mean $\boldsymbol{\mu}$ and the covariance matrix Σ_p.

(ii) Compute the covariance matrix estimators—the sample covariance $\mathbf{S}_n$, the spectral-corrected sample covariance $\widetilde{\mathbf{S}}_n$, the linear shrinkage covariance $\overline{\Sigma}_n$, the nonlinear shrinkage covariance $\mathbf{S}_n^{or}$ and the nonlinear shrinkage inverse covariance $\mathbf{P}_n^{or}$.

(iii) Plug the covariance estimators into

$$\widehat{\mathbf{w}_{GMV}} = \frac{\widehat{\Sigma}_p^{-1}\mathbf{1}}{\mathbf{1}'\widehat{\Sigma}_p^{-1}\mathbf{1}}, \quad R_{\widehat{\mathbf{w}_{GMV}}} = \frac{\boldsymbol{\mu}'\widehat{\Sigma}_p^{-1}\mathbf{1}}{\mathbf{1}'\widehat{\Sigma}_p^{-1}\mathbf{1}}$$

in which $\widehat{\Sigma}_p = \mathbf{S}_n$, $\widetilde{\mathbf{S}}_n$, $\overline{\Sigma}_n$ and $\mathbf{S}_n^{or}$. Plug the inverse estimation $\mathbf{P}_n^{or}$ into

$$\widehat{\mathbf{w}_{GMV}}* = \frac{\mathbf{P}_n^{or}\mathbf{1}}{\mathbf{1}'\mathbf{P}_n^{or}\mathbf{1}}, \quad R_{\widehat{\mathbf{w}_{GMV}}}* = \frac{\boldsymbol{\mu}'\mathbf{P}_n^{or}\mathbf{1}}{\mathbf{1}'\mathbf{P}_n^{or}\mathbf{1}}.$$

(iv) Repeat steps from (i) to (ii) for N times.

Figures 1 and 2 report the expected returns and risks of the GMV portfolio estimates associated with different covariance estimators—the spectral corrected covariance, the sample covariance, the linear shrinkage covariance, the nonlinear shrinkage covariance. In these two figures, we provide 8 pairs of the expected return and risk box plots for two population covariance structures as $y = 0.1, 0.2, ..., 0.8$, respectively. The numbers from 1 to 5 on the x-axes record the expected return and risk of the theoretical GMVP and that of the GMVP estimator associated with the covariance estimators—$\widetilde{\mathbf{S}}_n$, $\mathbf{S}_n$, $\overline{\Sigma}_n$, $\mathbf{S}_n^{or}$ and $\mathbf{P}_n^{or}$. In these two figures, the repeating time is 10000 and thus the computation scientific precision is asymptotically down to the second decimal point.

For the expected returns of the GMVP estimates, the performance of SCGMVP is not significantly better than the others. The medias of the expected returns of the every estimation almost are located between $R_o \pm 0.01$ and the interquartile ranges are smaller than 0.005 which means these portfolio estimations perform well without significant differences in the expected return.

For the risks of the GMVP estimates, the performance of the spectral corrected GMV portfolio is not significant different from the other as $y = 0.1$ and is significantly better than the others. For all ys from 0.2 to 0.8, the media and the corresponding interquartile range of the risks associated with the spectral corrected portfolio is smallest compared with that of the other estimations. The medias of the expected returns of SCGMVP are larger than the theoretical risk of GMVP at least 0.02 and smaller than the others at least 0.01. According to the interquartile ranges of the risks associated with the GMVP estimators, SCGMVP's risks are stablest since the difference between the range of SCGMVP's risks and the others is at least 0.02 as $y \geq 0.2$. Thus, from Figures 1 and 2, it is reasonable to use the spectral corrected methodology to estimate the global minimum variance portfolio.

6 Conclusion

In this paper, we introduce the spectral corrected methodology to correct the eigenvalues of the sample covariance matrix and construct the spectral corrected global minimum variance portfolio. This methodology overcomes the serious disturbance deduced by the departure of the spectrum of the sample covariance from that of the population covariance matrix in the estimation of the global minimum variance portfolio. In addition, compared with the linear shrinkage and nonlinear shrinkage estimations, SCGMVP has more expected return and Lower average risk. Therefore, we have enough reason to consider the spectral corrected methodology to construct GMVP estimator.

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Table 1: Comparison of $\mathbf{a}_p'\Sigma_p^{-1}\mathbf{b}_p$, $\lim \mathbf{a}_p\widetilde{\mathbf{S}}_n^{-1}\mathbf{b}_p$, $\mathbf{a}_p'\widetilde{\mathbf{S}}_n^{-1}\mathbf{b}_p$, $\mathbf{a}_p'\mathbf{S}_n^{-1}\mathbf{b}_p$ and $\frac{\mathbf{a}_p'\Sigma^{-1}\mathbf{b}_p}{1-y}$.

y	$f(\Sigma_p)$	$\lim f(\widetilde{\mathbf{S}}_n)$	$f\left(\widetilde{\mathbf{S}}_n\right)$	$f(\mathbf{S}_n)$	$\frac{f(\Sigma_p)}{1-y}$
A: $\boldsymbol{\tau} = (25, 10, 5, 1)$, $\mathbf{w} = \frac{1}{4}(1, 1, 1, 1)$.					
0.1	1.86	1.8857	1.8832(0.0938)	2.0667(0.1308)	2.066
0.2	1.86	1.9153	1.9175(0.1330)	2.3315(0.2095)	2.325
0.3	1.86	1.9497	1.9482(0.1644)	2.6678(0.3085)	2.657
0.4	1.86	1.9896	1.9840(0.2065)	3.1142(0.4673)	3.1
0.5	1.86	2.0370	2.0253(0.2459)	3.7495(0.7119)	3.72
0.6	1.86	2.0953	2.0822(0.2783)	4.7594(1.0897)	4.65
0.7	1.86	2.1661	2.1402(0.3138)	6.4346(1.8411)	6.2
0.8	1.86	2.2479	2.2027(0.3458)	9.6998(3.7428)	9.3
0.9	1.86	2.3540	2.2479(0.4005)	20.638(14.465)	18.6
B: $\boldsymbol{\tau} = (10, 5, 1)$, $\mathbf{w} = \frac{1}{10}(4, 3, 3)$.					
0.1	1.7	1.7161	1.7159(0.0783)	1.8914(0.1124)	1.888
0.2	1.7	1.7348	1.7348(0.1149)	2.1294(0.1921)	2.125
0.3	1.7	1.7567	1.7574(0.1527)	2.4432(0.3064)	2.428
0.4	1.7	1.7823	1.7829(0.1719)	2.8605(0.4222)	2.833
0.5	1.7	1.8126	1.8105(0.1938)	3.4308(0.5982)	3.4
0.6	1.7	1.8498	1.8452(0.2431)	4.3315(1.0416)	4.25
0.7	1.7	1.8943	1.8846(0.2519)	5.9039(1.6676)	5.666
0.8	1.7	1.9444	1.9236(0.2736)	8.9074(3.4104)	8.5
0.9	1.7	2.0066	1.9514(0.2913)	19.060(11.968)	17
C: $\boldsymbol{\tau} = (5, 3, 1)$, $\mathbf{w} = \frac{1}{10}(4, 3, 3)$.					
0.1	2.2666	2.3016	2.3017(0.1102)	2.5216(0.1528)	2.5185
0.2	2.2666	2.3421	2.3396(0.1563)	2.8384(0.2550)	2.8333
0.3	2.2666	2.3892	2.3862(0.2061)	3.2562(0.4079)	3.2380
0.4	2.2666	2.4435	2.4343(0.2265)	3.8107(0.5633)	3.7777
0.5	2.2666	2.5066	2.4757(0.2483)	4.5773(0.8110)	4.5333
0.6	2.2666	2.5809	2.5069(0.2810)	5.7787(1.3933)	5.6666
0.7	2.2666	2.6643	2.5382(0.2793)	7.8695(2.2318)	7.5555
0.8	2.2666	2.7502	2.5699(0.2882)	11.881(4.5272)	11.333
0.9	2.2666	2.8458	2.5890(0.2989)	25.446(16.054)	22.666

Note: here $f(\mathbf{A}) = \mathbf{a}_p'\mathbf{A}^{-1}\mathbf{b}_p$, $n = 100$, $y = p/n$ and $N = 10000$

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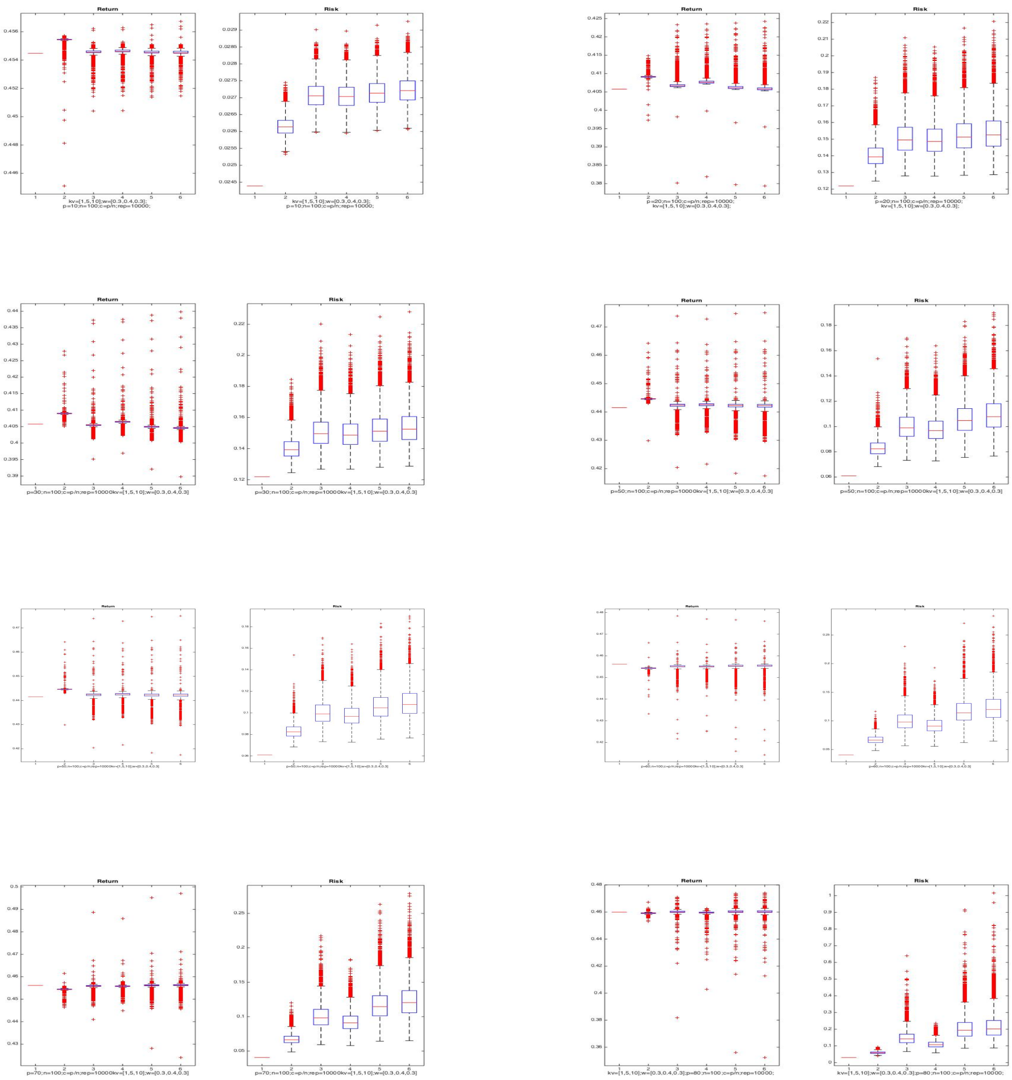
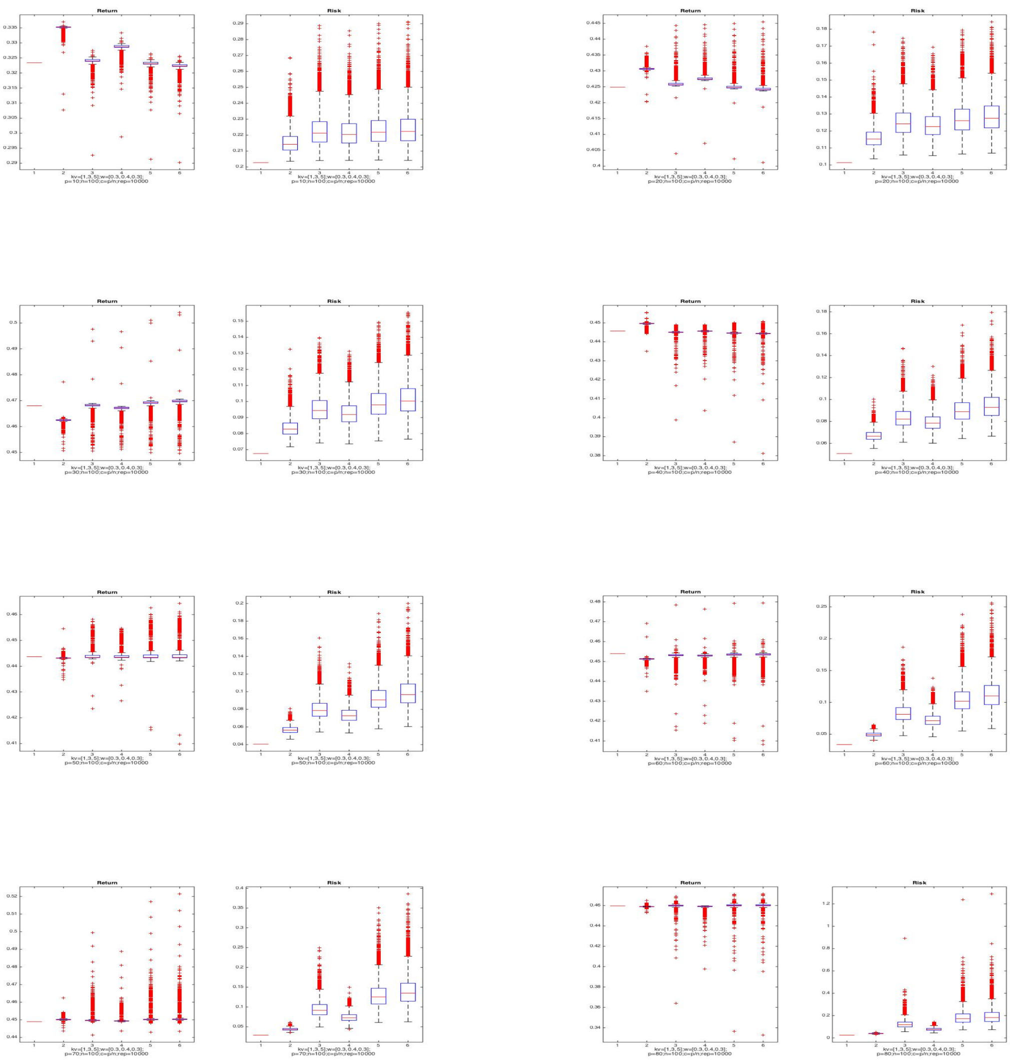
Figure 1: Comparetion of the GMV portfolio estimators associated with the spectral-corrected covariance, sample covariance, linear shrinkage covariance and non-linear shrinkage covariance.



Note: we provide 8 pairs of the expected return and risk box plots for $y = 0.1, 0.2, ..., 0.8$. Here the different eigenvalue and the corresponding weight vectors are $(1, 5, 10)$ and $(0.3, 0.4, 0.3)$, respectively. We use the number in the x-axes to denote the population result and the different estimations in which 1 is the population result and the numbers from 2 to 6 represent the estimations associated with $\widetilde{\mathbf{S}}_n$, $\mathbf{S}$, $\overline{\Sigma}_n$, $\mathbf{S}_n^{or}$ and $\mathbf{P}_n^{or}$, respectively.

Figure 2: Comparetion of the GMV portfolio estimators associated with the spectral-corrected covariance, sample covariance, linear shrinkage covariance and non-linear shrinkage covariance.



Note: we provide 8 pairs of the expected return and risk box plots for $y = 0.1, 0.2, ..., 0.8$. Here the different eigenvalue and the corresponding weight vectors are $(1, 3, 5)$ and $(0.3, 0.4, 0.3)$, respectively. We use the number in the x-axes to denote the population result and the different estimations in which 1 is the population result and the numbers from 2 to 6 represent the estimations associated with $\widetilde{\mathbf{S}}_n$, $\mathbf{S}$, $\overline{\Sigma}_n$, $\mathbf{S}_n^{or}$ and $\mathbf{P}_n^{or}$, respectively.