Bertrand-Edgeworth games under triopoly: the payoffs

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Abstract

The paper extends the analysis of price competition among capacity constrained sellers beyond duopoly and symmetric oligopoly. The main focus is on the equilibrium payoffs under triopoly. The paper also includes insightful examples highlighting features of equilibrium which can arise in a triopoly but not in a duopoly. Most notably, the supports of the equilibrium strategies need not be connected, nor need be connected the union of the supports; further, an atom may exist for a firm different from the largest one.

1 Introduction

The issue of price competition among capacity-constrained sellers has attracted considerable interest since Levitan and Shubik's [16] modern reappraisal of Bertrand and Edgeworth. Assume a given number of firms producing a homogeneous good at constant and identical unit variable cost up to some fixed capacity. Further, assume that rationing takes place according to the surplus maximizing rule and that demand is a continuous, non-increasing, and non-negative function defined on the set of non-negative prices and is positive, strictly decreasing, twice differentiable and (weakly) concave on a bounded initial interval. Then there are a few well-established facts about equilibrium of the price game. First, at any pure strategy equilibrium the firms earn competitive profit. However, a pure strategy equilibrium need not exist. In this case existence of a mixed strategy equilibrium is guaranteed by Theorem 5 of [3] for discontinuous games. Under similar assumptions on demand and cost, the set of mixed strategy equilibria was characterized by Kreps and Scheinkman [15] for the duopoly within a two-stage capacity and price game. This model was subsequently extended to allow significant convexities in the demand function (by Osborne and Pitchik, [18]) or differences in unit cost among the duopolists (by Deneckere and Kovenock, [12]). This led to the discovery of new phenomena, such as the possibility of the supports of the equilibrium strategies being disconnected and non-identical for the duopolists.

Progress has also been made on the characterization of mixed strategy equilibria under oligopoly under the assumption of constant and identical unit cost and the standard restrictions upon demand. Vives [20], amongst others, characterized the (symmetric) mixed strategy equilibrium for the case of equal capacities among all firms. In a previous paper we [10] generalized Vives result to the case in which the capacities of the largest and smallest firm are sufficiently close. Within an analysis concerning horizontal merging of firms Davidson and Deneckere [4] provided the complete analysis (apart from the fact that attention is restricted to equilibria in which strategies of equally-sized firms are symmetrical) of a Bertrand-Edgeworth game with linear demand, equally-sized small firms and one large firm with a capacity that is a multiple of the small firm's capacity.[1]

An important equilibrium property was seen to hold for general oligopoly: the equilibrium payoff of (any of) the largest firm(s) is equal to the payoff of the Stackelberg follower when the rivals supply their entire capacity ([2] and [7]).[2] As under duopoly, such a property appears to be a major building block for the study of equilibria of the price game under oligopoly. As an example, in a still unpublished paper Ubeda [19] compares discriminatory and uniform auctions among capacity-constrained producers and obtains a number of novel results on discriminatory auctions. (A discriminatory auction could be designed in such a way to be equivalent to Bertrand-Edgeworth competition under the efficient rationing rule.) Based on the above mentioned property, Ubeda showed, among other things, that the maximum

[1] Davidson and Deneckere [4] assumed a given number of equally-sized firms some of which merge. To see whether merging facilitates collusion in a repeated price game, they had to characterize equilibria of the static price game for the resulting special asymmetric oligopoly and hence mixed strategy equilibria when the new capacity configuration falls in the mixed strategy region of the capacity space. Our study shows that, even in a triopoly, a continuum of equilibria may exist even if each firm's equilibrium payoff and the strategy of the largest firm are the same at any equilibrium: see Example 1 in Section 5 Davidson and Deneckere overlooked this possibility since rei restricted their attention to equilibria that treat small firms symmetrically ([4], footnote 10, p. 123).

[2] The proof in [2] is carried out along the lines in [15] for the analogous result under duopoly. After pointing out a mistake in that proof, [7] establishes correctly that result along the same lines.

and the minimum over all the supports of equilibrium strategies belong to the support of the equilibrium strategies of any firm with the largest capacity. More recently Hirata [14] has provided an extensive analysis of triopoly with concave demand and efficient rationing: having highlighted some basic features of mixed strategy equilibria under triopoly, he is able to analyze how mergers between two firms would affect profitability in the different circumstances. Most importantly, the characterization of the equilibrium payoff of any of the largest firms has proven very effective when addressing oligopolistic two-stage capacity and price game, at least under the assumptions of convex cost of capacity, the standard restrictions upon demand, and efficient rationing: based on that property it can easily be shown that, in fact, the Cournot outcome extends to oligopoly (see, for instance, [2] and [17]).

The survey above suggests that the study of price competition with capacity constraints is relevant in many respects, such as mergers (hence regulation), auctions, and price leadership.[3] Yet, in the current state of the art, a complete characterization of equilibria of the price game only exists for special cases although a number of partial results have also been provided for general oligopoly.

This paper is the first of a trilogy in which we provide a general analysis of the triopoly. This study proves to be rewarding in terms of equilibrium properties that are shown to possibly arise in the triopoly but not in the duopoly, which is interesting *per se* but also as insights for the study of general oligopoly. Our analysis differs in scope from Hirata's since we provide a complete characterization of mixed strategy equilibria: we reveal all qualitative features possibly arising in the triopoly, including the facts highlighted in [14].[4] The main focus of the present paper is the equilibrium payoffs of the firms. The payoff of the largest firm has been determined by Boccard and Wauthy and others (see [2], [7], [19], [17], and [14]). Here we determine the payoff of the middle sized firm (but see also [19]) and, by appropriately partitioning the region where pure strategy equilibria do not exist, we identify the circumstances under which the payoff/capacity ratios of the smallest

[3]The relevance of mixed strategy equilibria of price games for the analysis of mergers might also be viewed in a longer-run perspective, allowing for capacity decisions by the merged firm and outsiders (on this, see Baik [1]). Characterizing mixed strategy equilibrium of the price game in a duopoly allows Deneckere and Kovenock [11] to endogenize price leadership by the dominant firm when the capacity vector lies in the mixed strategy region.

[4]Our own research and Hirata's were conducted independently. Results were made publicly available, in [8] and [13], respectively.

firm and of the middle sized firm are the same and the circumstances under which the smallest firm enjoys a higher payoff/capacity ratio. Furthermore, in the latter circumstances, we identify the range in which the payoff of the smallest firm must lie. Moreover we provide examples showing that the supports of the optimal strategies and their union may not be connected and the maximum of the support of the equilibrium strategy of the smallest firm may be charged with positive probability by that firm.

This research has led to several other discoveries. Several properties of a duopolistic mixed strategy equilibrium prove to generalize to triopoly: the values of the minimum and the maximum of the support of the equilibrium strategy for any firm with the highest capacity (equal to p_m and p_M, respectively, as defined in Section 3); the equilibrium payoff of any firm with the second highest capacity. On the other hand, in a duopoly the supports of the equilibrium strategies completely overlap, which need not be the case in a triopoly.[5] In a duopoly the region of the capacity space where no pure strategy equilibrium exists can be partitioned in two subsets: one in which both firms get the same payoff per unit of capacity and one in which the smaller firm gets a higher payoff per unit of capacity. The latter subset is characterized by the fact that the capacity of the larger firm is higher than total demand at p_m. In the triopoly, on the contrary, there are several relevant subsets of the region where no pure strategy equilibrium exists.

- In one subset, as in the duopoly, the capacity of the largest firm is larger than or equal to demand at p_m. In this subset the other firms get the same payoff per unit of capacity, higher than that of the largest firm.[6]

- In another subset the sum of the capacities of the two largest firm is smaller than or equal to demand at p_m. In this subset all firms get the same payoff per unit of capacity.

- In another subset both the smallest firms have the same size and the capacity of the largest firm is smaller than demand at p_m. In this

[5] That minima of the supports of the equilibfium strategies may differ has also been recognized in [13] and [14].

[6] However, differently from the analogous subset in the duopoly, the equilibrium strategies of the smallest firms are constrained but not uniquely determined (there is a continuum of equilibria). This will be shown in the second paper of the trilogy where we will also show that there are other subsets in which the equilibrium strategies of the two smallest firms are similarly constrained and not uniquely determined, but not in the whole union of the supports of equilibrium strategies. In these subsets the largest firm can meet total demand at prices close to p_M and all firms get the same payoff per unit of capacity.

subset all firms get the same payoff per unit of capacity.

- The complement of the previous three subsets can be partitioned in two parts. In one part the smallest firm gets a higher payoff per unit of capacity than theothers, that in turn get the same payoff per unit of capacity, a fact also discovered by [14]. Yet we determine the interval where the payoff of the smallest firm must be and provide examples for the exact determination of that payoff (a general rule for determining that payoff will be provided in the third paper of the trilogy). In the other part all firms get the same payoff per unit of capacity and the supports of the largest and the smallest firms have a lower bound equal to the lower bound of the overall price distribution, whereas the middle sized firm set prices only at higher levels. This is an unusual result and somewhat at odds with the rest of the parameter space.

Osborne and Pitchik [18] clarified that in duopoly, under the set of assumptions on demand adopted here, the supports of equilibrium strategies are connected, otherwise supports need not be connected. Quite differently, we will prove that under triopoly the supports need not be connected and even its union may not be connected, even with a concave demand function.

The paper is organized as follows. Section 2 contains definitions and the basic assumptions of the model along with a few basic results on pure strategy equilibrium. Section 3 deals with some general results concerning mixed strategy equilibria under triopoly when pure strategy equilibria do not exist (even if we will not prove it, many of the results presented in Section 3 can be generalized to oligopoly). Section 4 provides the partition mentioned above and determines also the constraints that the payoff of the smallest firm need to fulfill. Sections 5 is devoted to some examples.

2 Preliminaries

Assumption 1. There are 3 firms producing a homogeneous good at the same constant unit cost (normalized to zero), up to capacity. Without loss of generality, we consider the subset of the capacity space (K_1, K_2, K_3) where

$$K_1 \geqslant K_2 \geqslant K_3 > 0 \tag{1}$$

and we define $K = K_1 + K_2 + K_3$.

Assumption 2. The market demand function is given by $D(p)$ (demand as a function of price p) and $P(x)$ (price as a function of quantity x). The function $D(p)$ is strictly positive on some bounded interval $(0, p^*)$, on which it is continuously differentiable, strictly decreasing and such that $pD(p)$ is strictly concave; it is continuous for $p \geqslant 0$ and equals 0 for $p \geqslant p^*$; $X = D(0) < \infty$. $P(x) = D^{-1}(x)$ on the bounded interval $(0, X)$; the function $P(x)$ is continuous for $x \geqslant 0$ and equals 0 for $x \geqslant X$; $p^* = P(0) < \infty$.

Assumption 3. It is assumed throughout that any rationing is according to the efficient rule. Consequently, let $\Omega(p)$ be the set of firms charging price p: the residual demand forthcoming to all firms in $\Omega(p)$ is $\max\left\{0, D(p) - \sum_{j:p_j<p} K_j\right\} = Y(p)$. If $\sum_{i\in\Omega(p)} K_i > Y(p)$, the residual demand forthcoming to any firm $i \in \Omega(p)$ is a fraction $\alpha_i(\Omega(p), Y(p))$ of $Y(p)$, namely, $D_i(p_1, p_2, p_n) = \alpha_i(\Omega(p), Y(p))Y(p)$.

Our analysis does not depend on the specific assumption being made on $\alpha_i(\Omega(p), Y(p))$: for example, it is consistent with $\alpha_i(\Omega(p), Y(p)) = K_i/\sum_{r\in\Omega(p)} K_r$ as well as with the assumption that residual demand is shared evenly, apart from capacity constraints, among firms in $\Omega(p)$.[7]

Let p^c be the competitive price, that is

$$p^c = P(K). \tag{2}$$

We now provide necessary and sufficient conditions for the existence of a pure strategy equilibrium and show that no pure-strategy equilibrium actually exists when the competitive price is not an equilibrium. These results are straightforward generalizations of similar results for the duopoly.

Proposition 1 *Let Assumptions 1, 2, and 3 hold. (i)* $(p_1, p_2, p_3) = (p^c, p^c, p^c)$ *is an equilibrium if and only if either*

$$K - K_1 \geqslant X, \text{ if } X \leqslant K, \tag{3}$$

or

$$K_1 \leqslant -p^c \left[D'(p)\right]_{p=p^c} = -\frac{P(K)}{P'(K)}, \text{ if } X > K. \tag{4}$$

In the former case the set of equilibria includes any strategy profile such that $\Omega(0) \neq \emptyset$ *and* $\sum_{s\in\Omega(0)-\{j\}} K_s \geqslant X$ *for each* $j \in \Omega(0)$. . *In the latter,* (p^c, p^c, p^c) *is the unique equilibrium.*

(ii) No pure strategy equilibrium exists if neither (3) nor (4) holds.

[7] In this case, $\alpha_i(\Omega(p), Y(p)) = \min\{K_i/Y(p), \hat{\alpha}(p)\}$ where $\hat{\alpha}(p)$ is the solution in α of equation $\sum_{i\in\Omega(p)} \min\{K_i/Y(p), \alpha\} = 1$. Let $M \in \Omega(p)$ and $K_M \geqslant K_i$ (each $i \in \Omega(p)$). Then function $\sum_{i\in\Omega(p)} \min\{K_i/Y(p), \alpha\}$ is increasing in α over the range $[0, K_M/Y(p)]$ and equal to $\sum_{i\in\Omega(p)} K_i/Y(p) > 1$ for $\alpha = K_M/Y(p)$.

Proof (i) If $K \geqslant X$, for firm i charging $p^c = 0$ is a best response to rivals charging p^c if and only if $\sum_{j \neq i} K_j \geqslant X$. This holds for each i if and only if $\sum_{j \neq 1} K_j \geqslant X$. Then any strategy profile such that $\Omega(0) \neq \emptyset$ and $\sum_{s \in \Omega(0 - \{j\})} K_s \geqslant X$ for each $j \in \Omega(0)$ is an equilibrium. If $X > K$, for firm i charging p^c is a best response to rivals charging p^c if and only if $\left[d[p(D(p) - \sum_{j \neq i} K_j)]/dp \right]_{p = p^c} \leqslant 0$. This holds for each i if and only if $K_1 \leqslant -p^c \left[D'(p) \right]_{p = p^c}$. Then there are no further equilibria, in pure or mixed strategies. Indeed, consider a pure strategy profile such that $\overline{p} = \max\{p_1, p_2, p_3\} > p^c$. If $\#\Omega(\overline{p}) = 1$, then firm $i \in \Omega(\overline{p})$ earns $\overline{p} \max\{0, D(\overline{p}) - \sum_{j: p_j < \overline{p}} K_j\} < p^c K_i$. If $\#\Omega(\overline{p}) > 1$ and $D(\overline{p}) - \sum_{j: p_j < \overline{p}} K_j > 0$ (with $D(\overline{p}) - \sum_{j: p_j < \overline{p}} K_j \leqslant 0$ the above argument obviously applies), then for at least some firm $i \in \Omega(\overline{p})$ the residual demand $[D(\overline{p}) - \sum_{j: p_j < \overline{p}} K_j] \alpha_i(\Omega(\overline{p}), Y(\overline{p}))$ is less than K_i, so that deviating to price $\overline{p} - \epsilon$, negligibly less than $\overline{p}$, results in an upward jump of i's output, up to $\min\left\{ K_i, D(\overline{p} - \epsilon) - \sum_{j: p_j < \overline{p}} K_j \right\}$. This argument can easily be adapted to rule out strategy profiles where some firm is playing a mixed strategy.

(ii) In the assumed circumstances (p^c, p^c, p^c) is not an equilibrium. Hence we just have to rule out strategy profiles such that $\overline{p} = \max\{p_1, p_2, p_3\} > p^c$. Assume first $D(\overline{p}) - \sum_{j: p_j < \overline{p}} K_j > 0$. If $\#\Omega(\overline{p}) < 3$, then any firm $j \notin \Omega(\overline{p})$ is selling its entire capacity, but it would still do so if it raised the price to any level less than $\overline{p}$. If $\#\Omega(\overline{p}) = 3$, then residual demand is less than capacity for at least some firm i, whereas its output would jump up to $\min\{K_i, D(\overline{p} - \epsilon)\}$ if undercut. Next assume $D(\overline{p}) - \sum_{j: p_j < \overline{p}} K_j \leqslant 0$. Any $i \in \Omega(\overline{p})$ has failed to make a best response unless $p^c = 0$ and $\sum_{j: p_j = 0} K_j \geqslant X$ (the latter requiring that $K \geqslant X$). But this cannot be so if $p_1 > 0$ given that $\sum_{j \neq 1} K_j < X$; if, instead, $p_1 = 0$, then firm 1 has not made a best response since $\sum_{j \neq 1} K_j < X$. ■

Remark. Condition (3) gives rise to the classic Bertrand equilibrium. Condition (4) can also be interpreted in terms of the Cournot model of quantity competition among capacity-unconstrained firms. In fact condition (4) identifies, in the (K_1, K_2, K_3)-space, the region in which each firm's capacity is not higher than its best (capacity-unconstrained) quantity response when the rivals supply their entire capacity (namely, the region that is bounded above by the lower envelope of the Cournot best-response functions).[8]

[8] It should be noted that Assumption 1 does not guarantee the uniqueness of the Cournot equilibrium. Uniqueness would be ensured if, for instance, one assumed $D'(p) + pD''(p) < 0$ on $(0, p^*)$. (On this, see [6])

Before studying equilibria in the region where pure strategy equilibria do not exist, we need to enrich our notation. A strategy by firm i is denoted by $\sigma_i : (0, \infty) \to [0, 1]$, where $\sigma_i(p) = \Pr_{\sigma_i}(p_i < p)$ is the probability of firm i charging less than p under strategy σ_i. Of course, any function $\sigma_i(p)$ is non-decreasing and everywhere continuous except at p° such that $\Pr_{\sigma_i}(p_i = p^\circ) > 0$, where it is left-continuous ($\lim_{p \to p^\circ -} \sigma_i(p) = \sigma_i(p^\circ)$), but not continuous. An equilibrium is denoted by $\phi = (\phi_1, \phi_2, \phi_3)$, where $\phi_i(p) = \Pr_{\phi_i}(p_i < p)$. We use $\Pi_i^*(\phi) = \Pi_i^*(\phi_i, \phi_{-i})$ to denote firm i's expected profit at the equilibrium strategy profile ϕ and $\Pi_i(p, \phi_{-i})$ for firm i's expected profit when it charges p with certainty and the rivals are playing the equilibrium profile of strategies ϕ_{-i}. Of course, $\Pi_i^*(\phi_i, \phi_{-i}) \geqslant \Pi_i^*(\sigma_i, \phi_{-i})$ for each i and $\Pi_i^*(\phi_i, \phi_{-i}) \geqslant \Pi_i(p, \phi_{-i})$ for each i and each p. When no doubt can arise, and for the sake of brevity, we also write Π_i^* rather than $\Pi_i^*(\phi_i, \phi_{-i})$ and $\Pi_i(p)$ rather than $\Pi_i(p, \phi_{-i})$. Further, we denote by $S_i(\phi_i)$ the support of ϕ_i and by $p_M^{(i)}(\phi_i)$ and $p_m^{(i)}(\phi_i)$ the maximum and minimum of $S_i(\phi_i)$, respectively. More specifically, we say that $p \in S_i(\phi_i)$ when $\phi_i(\cdot)$ is increasing at p, that is, when there is $\delta > 0$ such that $\phi_i(p + h) > \phi_i(p - h)$ for any $0 < h < \delta$, whereas $p \notin S_i(\phi_i)$ if $\phi_i(p + h) = \phi_i(p - h)$ for some $h > 0$. Obviously, $\Pi_i^* = \Pi_i(p)$ almost everywhere in $S_i(\phi_i)$. Once again, when no doubt can arise and for the sake of brevity, we also write S_i rather than $S_i(\phi_i)$, $p_M^{(i)}$ rather than $p_M^{(i)}(\phi_i)$, and $p_m^{(i)}$ rather than $p_m^{(i)}(\phi_i)$. If S_i is not connected, i.e. if $\phi_i(p)$ is constant in an open interval $(\widetilde{p}, \widetilde{\widetilde{p}})$ whose endpoints are in S_i ($\widetilde{p} \in S_i$ and $\widetilde{\widetilde{p}} \in S_i$), then the interval $(\widetilde{p}, \widetilde{\widetilde{p}})$ will be referred to as a *gap* in S_i. In order to shorten notation, we denote $\lim_{p \to h+} \Pi_i(p)$ and $\lim_{p \to h-} \Pi_i(p)$ as $\Pi_i(h+)$ and $\Pi_i(h-)$, respectively, and $\lim_{p \to h+} \phi_i(p)$ as $\phi_i(h+)$.

Some more notation is needed to investigate further the properties of $\Pi_i(p)$. Let $N = \{1, 2, 3\}$ be the set of firms, $N_{-i} = N - \{i\}$, and $\mathcal{P}(N_{-i}) = \{\psi\}$ be the power set of N_{-i}. Then, so long as firm i's rivals' equilibrium strategies $\phi_{-i}(p)$ are continuous in p, $\Pi_i(p) = Z_i(p; \phi_{-i}(p))$, where

$$Z_i(p; \varphi_{-i}) := p \sum_{\psi \in \mathcal{P}(N_{-i})} q_{i,\psi}(p) \prod_{r \in \psi} \varphi_r \prod_{s \in N_{-i} - \psi} (1 - \varphi_s), \tag{5}$$

φ's are taken as independent variables (with the obvious constraints that $\varphi_j \in [0, 1]$, each j), and $q_{i,\psi}(p) = \max\{0, \min\{D(p) - \sum_{r \in \psi} K_r, K_i\}\}$ is firm i's output when it charges p, any firm $r \in \psi$ charges less than p and any firm $s \in N_{-i} - \psi$ charges more than p.[9] If instead $\Pr_{\phi_j}(p_j = p^\circ) > 0$ for

[9]Note that $\prod_{r \in \psi} \varphi_r$ is the empty product, hence equal to 1, when $\psi = \emptyset$; and it is similarly $\prod_{s \in N_{-i} - \psi} (1 - \varphi_s) = 1$ when $\psi = N_{-i}$.

some $j \neq i$, then $Z_i(p^\circ; \phi_{-i}(p^\circ)) \geqslant \Pi_i(p^\circ) \geqslant \lim_{p \to p^\circ+} Z_i(p; \phi_{-i}(p))$.[10] Note that since $\sum_{\psi \in \mathcal{P}(N_{-i})} \prod_{r \in \psi} \varphi_r \prod_{s \in N_{-i} - \psi}(1 - \varphi_s) = 1$, if $\varphi_i \in [0, 1]$, then the RHS of (5) is an average of the functions $pq_{i,\psi}(p)$'s. As a consequence $pq_{i,N-i}(p) \leqslant Z_i(p; \phi_{-i}) \leqslant pq_{i,\emptyset}(p)$.

3 Equilibria under triopoly when no pure strategy equilibrium exists: some general results

The analysis developed in this section refers to the region of the capacity space where no pure strategy equilibrium exists, i.e. the region where

$$K_1 > \max\left\{K - X, -\frac{P(K)}{P'(K)}\right\} \tag{6}$$

and inequalities (1) hold.[11]

Since [15] it has been known that, in a duopoly,[12]

D.1 $\Pi_1^* = \max_p pq_{1,N_{-1}}(p)$;

D.2 $p_M^{(1)} = p_M^{(2)} = p_M$, where

$$p_M = \arg\max_p pq_{1,N_{-1}}(p); \tag{7}$$

D.3 $p_m^{(1)} = p_m^{(2)} = p_m$, where p_m is the price such that if firm 1 charges this price and any other firm charges a higher price, then firm 1 gets exactly Π_1^*, as defined in D.1, i.e.

$$p_m = \min\left\{p : pq_{1,\emptyset}(p) = \Pi_1^*\right\}; \tag{8}$$

D.4 $\Pi_2^* = p_m K_2$;

D.5 if $K_1 = K_2$, then $\phi_1(p_M) = \phi_2(p_M) = 1$ whereas if $K_1 > K_2$, $\phi_1(p_M) < \phi_2(p_M) = 1$.

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[10] The exact value of $\Pi_i(p^\circ)$ when $\Pr_{\phi_j}(p_j = p^\circ) > 0$ for some $j \neq i$ depends on function $\alpha_i(\Omega(p), Y(p))$.

[11] Statements **D.1**, **D.2** (with a small modification when $K - K_1 \geqslant X$), **D.3**, and **D.4** also applies to the region of the capacity space where pure strategy equilibrium exists.

[12] In the list each item is referred to with a capital D to indicate duopoly. However, definitions are given in such a way that they are valid also in the triopoly. In a duopoly $N_{-1} = \{K_2\}$, obviously.

Some of these results also hold in a triopoly, as will be shown in this section.

The definitions of p_M and p_m also make it possible to characterize the region where inequalities (6) and (1) hold by substituting inequality (6) with inequality

$$P(K) < p_m. \tag{6'}$$

Indeed, if $K_1 \leqslant K - X$, then $p_m = P(K) = 0$ whereas if $K_1 \leqslant -\frac{P(K)}{P'(K)}$, then $p_m = p_M = P(K) \geqslant 0$. Conversely, if inequality (6) holds, then inequality (6′) holds too. Finally, note that in the region where inequalities (6) and (1) hold we have:

$$p_M = \arg\max_p p\left[D(p) - \sum_{j\neq 1} K_j\right] \tag{9}$$

$$p_m = \max\{\widehat{p}, \widehat{\widehat{p}}\}, \tag{10}$$

where

$$\widehat{p} = \frac{\max_p p[D(p) - \sum_{j\neq 1} K_j]}{K_1} \tag{11}$$

$$\widehat{\widehat{p}} = \min\left\{p : pD(p) = \max_p p\left[D(p) - \sum_{j\neq 1} K_j\right]\right\}. \tag{12}$$

Note that $\widehat{\widehat{p}} \geqslant \widehat{p}$ if and only if $D(\widehat{p}) \leqslant D(\widehat{\widehat{p}}) \leqslant K_1$. This is so since $\widehat{\widehat{p}}D(\widehat{\widehat{p}}) = \widehat{p}K_1$ and the demand function is decreasing. In the remainder of this section we see how statements D.1-D.5 generalize to triopoly. The following proposition states in our formalism a proposition concerning oligopoly available in the literature. It generalizes to triopoly statements **D.1** and **D.2**. For a complete proof see [2] and [7]. See also [19], [17], and [14].

Proposition 2 *Let Assumptions 1, 2, and 3 and inequality (6′) hold. In any equilibrium $\phi_j(p_M) = 1$ for any j such that $K_j < K_1$; $p_M^{(i)} = p_M$ for some i such that $K_i = K_1$, and*

$$\Pi_i^* = \max_p p\left[D(p) - \sum_{j\neq 1} K_j\right] \tag{13}$$

for any i such that $K_i = K_1$.

Corollary 1 $\max_i p_M^{(i)} = p_M$.

Corollary 2 If $\widehat{p} > \widehat{\widehat{p}}$, then for any i such that $K_i = K_1$, the equilibrium payoff can also be written $\Pi_i^* = p_m K_1$.

Corollary 3 If $\widehat{\widehat{p}} \geqslant \widehat{p}$, then the equilibrium payoff of firm 1 can also be written $\Pi_1^* = p_m D(p_m)$ and $K_1 \geqslant D(p_m) > D(p_M) > K_2 + K_3$.

The following proposition generalizes statement **D.3** to triopoly. Similar generalizations were also provided by Ubeda [19] in a different context.

Proposition 3 *Let Assumptions 1, 2, and 3 and inequality* $(6')$ *hold. In any equilibrium* (ϕ_1, ϕ_2, ϕ_3)*:*

(i) $p_m^{(j)} \geqslant p_m^{(1)}$ *for any firm* j*.*

(ii) $p_m^{(i)} = p_m$ *for any* i *such that* $K_i = K_1$*.*

Proof (i) Let $p_m^{(j)} < p_m^{(1)}$ for some $j \in N_{-1}$. Since $D(p_M) > \sum_{j \neq 1} K_j$, then it would be $\Pi_j(p) = pK_j > p_m^{(j)} K_j = \Pi_j^*$ for $p \in (p_m^{(j)}, p_m^{(1)})$, an obvious contradiction.

(ii) If $p_m^{(1)} > p_m$, then $\Pi_1(p) = pq_{1,\emptyset}(p) > \Pi_1^*$ in the interval $(p_m, p_m^{(1)})$ because of part (i). Hence $p_m^{(1)} \leqslant p_m$. If $p_m^{(1)} < p_m$, then $\Pi_1(p) \leqslant pq_{1,\emptyset}(p) < p_m q_{1,\emptyset}(p_m) = \Pi_1^*$ in the interval $[p_m^{(1)}, p_m)$. Hence $p_m^{(1)} = p_m$. ■

Corollary 4 $\min_i p_m^{(i)} = p_m$.

Let $M = \{i \in N : p_M^{(i)} = p_M\}$ and $L = \{i \in N : p_m^{(i)} = p_m\}$. The following proposition establishes quite expected properties of equilibria in the region defined by inequalities (6) and (1).

Proposition 4 *Let Assumptions 1, 2, and 3 and inequality* $(6')$ *hold. In any equilibrium* (ϕ_1, ϕ_2, ϕ_3)*:*

(i) for any $i \in N$, $\Pi_i^* = \Pi_i(p)$ *for* p *in the interior of* S_i *and for* $p = p_m^{(i)}$*;*

(ii) for any $p^\circ \in (p_m, p_M)$, $D(p^\circ) < \sum_{i: p_m^{(i)} < p^\circ} K_i$*;*

(iii) $\#L \geqslant 2$*;*

(iv) If $(p^\circ, p^{\circ\circ}) \subset S_i$*, then* $(p^\circ, p^{\circ\circ}) \subset \cup_{j \neq i} S_j$*;*

(v) For any $i \in L - \{1\}$, $\Pi_i^* = p_m K_i$*;*

(vi) $D(p_m) < \sum_{j \in L} K_j$*;*

(vii) For any $i \neq 1$ *such that* $p_M^{(i)} \geqslant P(K_1)$, $\Pi_i^* = p_m K_i$*;*

(viii) if $K_2 > K_3$ *and* $\Pi_i^* = p_m K_i$ *(each* i*), then either* $D(p_m) \geq K_1 + K_2$ *or* $D(p_m) \leq K_1 + K_3$*;*

(ix) $\#M \geqslant 2$*.*

Proof

(i) Suppose contrariwise that $\Pi_i^* > \Pi_i(p^\circ)$ for some p° in the interior of S_i. Then, since $\Pi_i(p^\circ) \geq \Pi_i(p^\circ+)$[13] it would be $\Pi_i(p) < \Pi_i^*$ on a right neighbourhood of p°, contrary to the fact that p° is internal to S_i. Nor can it be $\Pi_i^* > \Pi_i(p_m^{(i)})$. This derives from the argument above if $p \in S_i$ for $p > p_m^{(i)}$ and sufficiently close to $p_m^{(i)}$. If instead $p_m^{(i)}$ is an isolated point of S_i, the contradiction is that $\Pr_{\phi_i}(p_i = p_m^{(i)}) > 0$ even though $\Pi_i(p_m^{(i)}) < \Pi_i^*$.

(ii) Otherwise $\Pi_i(p) = pK_i > \Pi_i(p_m^{(i)}) = \Pi_i^*$ for $p \in (p_m^{(i)}, p^\circ]$, an obvious contradiction.

(iii) Assume contrariwise that $L = \{i\}$. Then, on a right neighborhood of p_m, $\Pi_i(p) = pq_{i,\emptyset}(p) > p_m q_{i,\emptyset}(p_m) = \Pi_i(p_m) = \Pi_i^*$: an obvious contradiction.

(iv) See Appendix A.

(v) $p_m K_i = \Pi_i(p_m-) \leqslant \Pi_i(p_m) \leqslant p_m q_{i,\emptyset}(p_m) \leqslant p_m K_i$: inequalities are obvious; the equality holds since $D(p_m) > D(p_M) > \sum_{j \neq 1} K_j > K_i$.

(vi) If $D(p_m) > \sum_{j \in L} K_j$, then $\Pi_i(p)$ (each $i \in L$) would be increasing on a right neighborhood of p_m. If $D(p_m) = \sum_{j \in L} K_j$, then $L = \{1, i\}$, by Proposition 3(i) and inequality (6'). On a neighborhood of p_m either $p \in S_1 \cap S_i$ or $p \notin S_1 \cup S_2 \cup S_3$ because of part (iv). In the latter case $\phi_2(p_m+) > 0$ and $\Pi_1(p) = p\phi_i(p_m+)[D(p) - K_i] + p\,[1 - \phi_i(p_m+)]\,K_1$, which is increasing in p, on a neighborhood of p_m. In the former case

$$\Pi_1^* = \Pi_1(p) = p\phi_i(p)[D(p) - K_i] + p\,[1 - \phi_i(p)]\,K_1$$

on a neighborhood of p_m. By Corollary 2, $\Pi_1^* = p_m K_1$ and hence $\phi_i(p) = \frac{(p-p_m)K_1}{p[D(p_m)-D(p)]}$. But then $\phi_i(p_m+) = \frac{K_1}{-p_m D'(p_m)} > 1$ since $K_1 > D(p_m) - K_2 - K_3 > -p_m D'(p_m)$.

(vii) If $i \in L$ the claim follows from part (v). Let $i \notin L$ and therefore, because of parts (iii) and (v), $j \in L$ and $\Pi_j^* = p_m K_j$. If, for some $p \geqslant P(K_1)$, $\Pi_i(p) = pK_i(1 - \phi_1(p)) > p_m K_i$, then also $\Pi_j(p) = pK_j(1 - \phi_1(p)) > p_m K_j$ and firm j has not made a best response.

(viii) If $K_1 + K_3 < D(p_m) < K_1 + K_2$, then

- $L \neq \{1, 2, 3\}$ otherwise $\phi_2(p) = \sqrt{\frac{K_1}{K_2}\frac{p-p_m}{p}}$, $\phi_3(p) = \frac{D(p)-K_1-K_2}{K_3} + \frac{K_2}{K_3}\phi_2(p)$, and $\phi_3(p) < 0$ on a right neighbourhood of p_m since $\phi_3(p_m+) = \frac{D(p_m)-K_1-K_2}{K_3} < 0$;

[13] $\Pi_i(p^\circ) > \lim_{p \to p^\circ+} \Pi_i(p)$ only if firm $j \neq i$ charges p° with positive probability.

- $L \neq \{1,3\}$ otherwise $\Pi_i(p) = pK_i > p_m K_i = \Pi_i^*$ (each $i \in L$) on a right neighbourhood of p_m;
- $L \neq \{1,2\}$ otherwise $\Pi_3(p) = (1 - \phi_1(p)\phi_2(p))K_3 p > p_m K_3$ on a right neighbourhood of p_m: the inequality is equivalent to $(p-p_m)\left[K_1 + K_2(1 - \phi_2(p)) - D(p)\right] > 0$ since $\phi_1(p) = \frac{(p-p_m)K_2}{(K_1+K_2-D(p))p}$.

(ix) See Appendix A. ■

Corollary 5 $\Pi_2^* \geqslant p_m K_2$, $\Pi_3^* \geqslant p_m K_3$.

The following proposition generalizes to triopoly statement **D.5**; furthermore, it shows that if several firms have the largest capacity, then their equilibrium strategies are necessarily the same (this symmetry need not arise for equally-sized firms that are smaller than the largest one, as will be clarified by Proposition 6).

Proposition 5 *Let Assumptions 1, 2, and 3 and inequality (6′) hold. In any equilibrium* (ϕ_1, ϕ_2, ϕ_3)*:*

(i) if $K_1 > K_2$*, then* $\phi_1(p_M) < 1$*;*

(ii) if $K_1 = K_2 > K_3$*, then: (ii.a)* $\phi_1(p_M) = \phi_2(p_M) = 1$*, (ii.b)* $p_M^{(3)} < p_M$*, (ii.c)* $p_M^{(1)} = p_M^{(2)} = p_M$*, (ii.d)* $\phi_2(p) = \phi_1(p)$ *throughout* $[p_m, p_M]$*.*

(iii) if $K_1 = K_2 = K_3$*, then:* $p_M^{(1)} = p_M^{(2)} = p_M^{(3)} = p_M$*,* $\phi_3(p) = \phi_2(p) = \phi_1(p)$ *throughout* $[p_m, p_M]$*,* $\phi_1(p_M) = \phi_2(p_M) = \phi_3(p_M) = 1$*.*

Proof (i) If $\phi_1(p_M) = 1$, then $\phi_j(p_M) = 1$ (each j) because of Proposition 2. Let $i \in M - \{1\}$, then $\Pi_i^* = \Pi_i(p_M-) = Z_i(p_M; 1, 1) = p_M q_{i,N_{-i}}(p_M)$. This implies an obvious contradiction if $p_M q_{i,N_{-i}}(p_M) \leqslant 0$. A similar contradiction holds if $p_M q_{i,N_{-i}}(p_M) > 0$ too since $\left[\frac{d}{dp} p q_{i,N_{-i}}(p)\right]_{p=p_M} < 0$ and therefore $\Pi_i(p) \geqslant p q_{i,N_{-i}}(p) > p_M q_{i,N_{-i}}(p_M) = \Pi_i^*$ for p in a left neighbourhood of p_M.

(ii.a) If, say, $\phi_1(p_M) < \phi_2(p_M) = 1$, then $\Pi_2(p_M-) > \Pi_1(p_M) = \Pi_1^*$, contrary to Proposition 2. Nor can $\phi_2(p_M)$ and $\phi_1(p_M)$ be both less than 1, since then $\Pi_2(p_M-) > \Pi_2(p_M)$.

(ii.b) Because of part (ii.a), if $p_M^{(3)} = p_M$, then the contradiction pointed out in the proof of part (i) holds.

(ii.c) Because of part (ii.b) and Proposition 4(ix).

(ii.d) and (iii) See Appendix A. ■

Finally the following proposition, whose proof is in Appendix A, generalizes to triopoly statement **D.4**.

Proposition 6 *Let Assumptions 1, 2, and 3 and inequality* $(6')$ *hold. In any equilibrium* (ϕ_1, ϕ_2, ϕ_3)*:*

(i) $\Pi_2^* = p_m K_2$*;*

(ii) if $K_2 = K_3$*, then* $\Pi_2^* = \Pi_3^*$*;*

(iii) if $K_2 = K_3$ *and* $\phi_i(p^\circ) < \phi_j(p^\circ)$ $(i, j \neq 1)$ *for some* $p^\circ \in S_2 \cup S_3$*, then (iii.a)* $K_2 < K_1$*; further, (iii.b) if* $p^\circ \in S_2 \cap S_3$*, then* $p^\circ \geqslant P(K_1)$*, whereas (iii.c) if* $p^\circ < P(K_1)$*, then* $S_i \cap [p^\circ, P(K_1)) = \emptyset$*.*

4 On the equilibrium payoff of firm 3

Our next major task is to determine Π_3^* when $K_3 < K_2$. We know from Proposition 4(v)&(vii) that if $3 \in L$ or $p_M^{(3)} \geq P(K_1)$, then $\Pi_3^* = p_m K_3$; but we do not know yet when this is the case. We know also, from Proposition 4(viii) and Corollary 5, that Π_3^* may be larger than $p_m K_3$, but we have to determine when this holds and the level of Π_3^*. We introduce the following partition of the region defined by inequalities (6) and (1).

$A = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 \geqslant K_3, D(\widehat{p}) \leqslant K_1\}$.

$B = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, K > D(\widehat{p}) \geqslant K_1 + K_2\}$.

$C_1 = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, K_1 + K_2 > D(\widehat{p}) > K_1 + K_3\}$.

$C_2 = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, K_1 + K_3 \geqslant D(\widehat{p}), D(p_M) \geqslant K_1\}$

$C_3 = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, K_1 + K_3 \geqslant D(\widehat{p}), D(p_M) < K_1 < D\left(\frac{\widehat{p}K_1}{K_1 - K_3}\right)\}$

$D = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 > K_3, K_1 + K_3 \geqslant D(\widehat{p}), D(p_M) < D\left(\frac{\widehat{p}K_1}{K_1 - K_3}\right) \leqslant K_1 < D(\widehat{p})\}$[14]

$E = \{(K_1, K_2, K_3) : K_1 \geqslant K_2 = K_3, K_1 < D(\widehat{p})\}$.

We will prove that $\Pi_3^* = p_m K_3$ in sets A, B, D, and E: in sets A and D because $p_M^{(3)} \geq P(K_1)$, in sets B and D because $3 \in L$ (but $L = \{1, 2, 3\}$ in B and $L = \{1, 3\}$ in D), in set E because $K_3 = K_2$. We will prove also that $p_m K_3 < \Pi_3^* \leqslant \pi_m$ in set $C_1 \cup C_2 \cup C_3$, where π_m will be defined in this section. The exact value of Π_3^* in this set cannot be determined without determining at the same time also the profile of equilibrium strategies and therefore the determination of Π_3^* in this set will be postponed to another paper.

[14] If $K_1 + K_3 \geqslant D(\widehat{p})$, then $D(p_M) < D\left(\frac{\widehat{p}K_1}{K_1 - K_3}\right)$ since the latter inequality is equivalent to $p_M(K_1 - K_3) > \widehat{p}K_1 = p_M(D(p_M) - K_2 - K_3)$.

In order to recognize that the intersection of any two of the above sets is empty whereas their union is set $\{(K_1, K_2, K_3) : K_1 \geqslant K_2 \geqslant K_3, K > D(p_m)\}$, i.e. the region defined by inequalities (1) and (6′), we construct the partition through a chain of increasingly finer partitions. First of all, we distinguish three sub-regions, that in which $\widehat{\widehat{p}} \geqslant \widehat{p}$, that in which $\widehat{p} > \widehat{\widehat{p}}$ and $K_2 > K_3$, and that in which $\widehat{p} > \widehat{\widehat{p}}$ and $K_2 = K_3$. The first sub-region is actually set A and the third sub-region is actually set E. The second sub-region is partitioned into three parts, defined by conditions $K > D(\widehat{p}) \geqslant K_1 + K_2$, $K_1 + K_2 > D(\widehat{p}) > K_1 + K_3$, and $K_1 + K_3 \geqslant D(\widehat{p}) > K_1$, respectively. The first part consists of set B. The second part consists of set C_1. The third part is partitioned into the sets C_2 $(D(p_M) \geqslant K_1)$, C_3 $\left(D(p_M) < K_1 < D\left(\frac{\widehat{p}K_1}{K_1-K_3}\right)\right)$, and D $\left(D(p_M) < D\left(\frac{\widehat{p}K_1}{K_1-K_3}\right) \leqslant K_1 < D(\widehat{p})\right)$.

It is checked that actually

- $K_1 > K_2$ whenever $D(p_M) \leqslant K_1 + K_3$, hence in $A \cup C_2 \cup C_3 \cup D$, and in part of $B \cup C_1 \cup E$,
- $K_1 > K_2 + K_3$ whenever $D(p_M) \leqslant K_1$, hence in $A \cup C_3 \cup D$ and in part of $B \cup C_1 \cup C_2 \cup E$, and
- $K_1 + K_3 > D(\widehat{p})$ whenever $D\left(\frac{\widehat{p}K_1}{K_1-K_3}\right) \leqslant K_1 \leqslant D(\widehat{p})$, hence in D and in part of E.[15]

The aim of this section is to state the following Theorem 1. From the previous section we know the values of Π_1^*, Π_2^* and, of course, Π_3^* when $K_3 = K_2$ (see Propositions 2 and 6). Among other things, the theorem states that $\Pi_3^* = p_m K_3$ everywhere except in $C_1 \cup C_2 \cup C_3$ and determines the maximum value that Π_3^* can assume in this set. The following Proposition 7 (proof in Appendix A) introduces functions $\phi_{1j}^\star(p)$, $\phi_j^\star(p)$, $\phi_{1j}^{\star\star}(p)$, and $\phi_j^{\star\star}(p)$ $(j = 2, 3)$ to be used in Theorem 1 and in next section.

Proposition 7 *Let $K_1 + K_j > D(p_m) > K_1$ (some $j \neq 1$). (i) Denote by $\phi_{1j}^\star(p) = \frac{(p-p_m)K_j}{p[K_1+K_j-D(p)]}$ and $\phi_j^\star(p) = \frac{(p-p_m)K_1}{p[K_1+K_j-D(p)]}$ the solutions of equations $p_m K_j = Z_j(p; \varphi_1, 0)$ and $p_m K_1 = Z_1(p; \varphi_j, 0)$, respectively, over the range $\{p_m, \min\{P(K_1), p_M\})$. Then $\phi_{1j}^\star(p)$ and $\phi_j^\star(p)$ are increasing over the range*

[15] The first two remarks are obvious consequences of the fact that $D(p_M) > K_2+K_3$. The third remark is a consequence of inequalities $\widehat{p}K_1 \geq P(K_1)[D(P(K_1)) - K_3] \geq \widehat{p}[D(\widehat{p}) - K_3]$. These two inequalities hold since the former is equivalent to $\frac{\widehat{p}K_1}{K_1-K_3} \geq P(K_1)$ and the latter is a consequence of the facts that function $p(D(p) - K_3)$ is increasing over the range $[p_m, p_M]$.

$[p_m, \min\{\tilde{p}_M^{(j)}, P(K_1)\}]$, *where* $\tilde{p}_M^{(j)}$ *is the unique solution in* $[p_m, p_M]$ *of the equation* $K_1 p_m = [D(p) - K_j]p$.

(ii) Denote by $\phi_{1j}^{\star\star}(p)$ *and* $\phi_j^{\star\star}(p)$ *the solutions of equations* $p_m K_j = Z_j(p; \varphi_1, 1)$ *and* $p_m K_1 = Z_1(p; \varphi_j, 1)$*, respectively. Then:*

(ii.a) Over the range $[p_m, \min\{P(K_1 + K_3), p_M\}]$, $\phi_{12}^{\star\star}(p) = \frac{(p-p_m)K_2}{p[K-D(p)]}$ *and* $\phi_2^{\star\star}(p) = \frac{K_1}{K_2}\phi_1^{\star\star}(p)$*, which are both increasing.*

(ii.b) Over the range $[p_m, \min\{P(K_1 + K_3), p_M\}]$, $\phi_{13}^{\star\star}(p) = \frac{p-p_m}{p}$ *and* $\phi_3^{\star\star}(p) = \frac{p[D(p)-K_2]-K_1 p_m}{K_3 p}$*, which are both increasing.*

(ii.c) Over the range $[\max\{P(K_1 + K_3), p_m\}, p_M]$, $\phi_{1j}^{\star\star}(p) = \frac{p-p_m}{p}$ *and* $\phi_j^{\star\star}(p) = \frac{p[D(p)-K_i]-K_1 p_m}{K_j p}$*, $(i \neq 1, j)$ which are both increasing.*

Theorem 1.[16] *Let the region defined by inequalities (6) and (1) be partitioned as above.* (a) *If* $(K_1, K_2, K_3) \in A$*, then in any equilibrium* $p_m = \widehat{\widehat{p}}$ *and* $\Pi_1^* = p_m D(p_m)$*;* $\Pi_j^* = p_m K_j$ *(each* $j \neq 1$*).*

(b) *If* $(K_1, K_2, K_3) \in B$*, then in any equilibrium* $p_m = \widehat{p}$*,* $\Pi_i^* = p_m K_i$ *for all i,* $L = \{1, 2, 3\}$*.*

(c) *If* $(K_1, K_2, K_3) \in C_1 \cup C_2 \cup C_3$*, then* (c.i) *in any equilibrium* $p_m = \widehat{p}$*,* $\Pi_i^* = p_m K_i$ *for* $i \neq 3$*,* $L = \{1, 2\}$*, and* $p_m K_3 < \Pi_3^* \leqslant \pi_m$*, where* $\pi_m = \max_{p \in [p_m, \min\{P(K_1), \tilde{p}_M^{(2)}\}]} F(p) > F(\min\{P(K_1), \tilde{p}_M^{(2)}\}) = p_m K_3$*,* $F(p) = Z_3(p; \phi_{12}^\star(p), \phi_2^\star(p))$*. Furthermore,* $p_M^{(3)} < P(K_1)$ *and (c.ii)* $M = \{1, 2\}$*.*

(d) *If* $(K_1, K_2, K_3) \in D$*, then in any equilibrium* $p_m = \widehat{p}$*,* $\Pi_i^* = p_m K_i$ *for all i,* $L = \{1, 3\}$ *and* $p_m^{(2)} \geqslant P(K_1)$*.*

(e) *If* $(K_1, K_2, K_3) \in E$*, then in any equilibrium* $p_m = \widehat{p}$ *and* $\Pi_i^* = p_m K_i$ *for all i.*

Proof The assertions about p_m, Π_1^* and Π_2^* in the various parts follow straightforwardly from Propositions 2, 3, and 6(i) and Corollaries 2 and 3.

(a) Since $D(\widehat{p})\widehat{p} \leqslant K_1\widehat{p} = D(\widehat{\widehat{p}})\widehat{\widehat{p}}$, $p_m = \widehat{\widehat{p}}$. Then Proposition 4(vii) completes the proof.

(b) $L = \{1, 2, 3\}$ because of Proposition 4(vi); $\Pi_3^* = p_m K_3$ because of Proposition 4(v).

(c.i) See Appendix A.

[16] Hirata discovered to a large extent that $L = \{1, 2, 3\}$ in sets $A \cup B \cup E$ ([14], Claims 3 and 6), but he was not concerned with $\Pr_{\phi_i}(p_i = p_m) = 0$. He recognized the fact that $p_m^{(3)} > p_m$ and $\Pi_3^* > p_m K_3$ in what is here called C_1, C_2, and C_3 ([14], Claims 4 and 5), but he was not concerned with how $p_m^{(3)}$ and Π_3^* are then determined. Hirata also recognized that $p_m^{(2)} > p_m$ and $\Pi_3^* = p_m K_3$ in our set D ([14], Claim 5),

(c.ii) Let us partition set C_1 into subsets C_{11} ($D(p_M) > K_1 + K_3$), C_{12} ($K_1 + K_3 \geqslant D(p_M) > K_1$), C_{13} ($D(p_M) \leqslant K_1$). The claim is already proved in C_{13} and C_3 (see part (c.i)). As to the other subsets, we begin by ruling out the event $M = \{1, 2, 3\}$. Under this event, $\Pi_2(p_M-) = Z_2(p_M; \phi_{-2}(p_M)) = \Pi_2^*$ and $\Pi_3(p_M-) = Z_3(p_M; \phi_{-3}(p_M)) = \Pi_3^*$. These two equations contradict each other in these subsets when, according to Proposition 2, $\phi_2(p_M) = \phi_3(p_M) = 1$: if the former holds, then $\Pi_3(p_M-) < \Pi_3^*$ and the latter cannot hold. Let us see how this works in each case. Note that in $C_2 \cup C_{12}$, $D(p_M) \leqslant K_1 + K_3$. Hence, under our working assumption, $\Pi_2^* = p_m K_2 = p_M[1 - \phi_1(p_M)]K_2$: thus, $\phi_1(p_M) = 1 - p_m/p_M$, in turn implying $Z_3(p_M) = p_M[1 - \phi_1(p_M)]K_3 = p_m K_3$, contrary to part (c.i). In C_{11}, $\Pi_2^* = p_m K_2 = Z_2(p_M) = p_M[\phi_1(p_M)(D(p_M) - K_1 - K_3) + (1 - \phi_1(p_M))K_2]$, that is, $\phi_1(p_M) = \frac{p_M - p_m}{p_M} \frac{K_2}{K - D(p_M)}$. By substituting this into $Z_3(p_M) = p_M[1 - \phi_1(p_M)]K_3$ obtain $Z_3(p_M) = \frac{p_M[K_1 + K_3 - D(p_M)] + p_m K_2}{K - D(p_M)} K_3 < p_m K_3$ since $K_1 + K_3 < D(p_M)$; hence again part (c.i) is contradicted. It remains to dismiss the event of $M = \{1, 3\}$ in $C_{11} \cup C_{12} \cup C_2$. This is done by showing that otherwise $\Pi_2(p) > \Pi_2^*$ in a left neighborhood of p_M. If $p_M^{(2)} < p_M$ in $C_{11} \cup C_{12} \cup C_2$, then $\Pi_3(p_M-) = p_M[1 - \phi_1(p_M)]K_3 = \Pi_3^* > p_m K_3$, implying $\phi_1(p_M) = 1 - \frac{\Pi_3^*}{p_M K_3} < 1 - \frac{p_m}{p_M}$ and hence $\Pi_2(p_M-) = p_M \phi_1(p_M) \max\{0, D(p_M) - K_1 - K_3\} + p_M[1 - \phi_1(p_M)]K_2 > p_m K_2 = \Pi_2^*$.

(d) See Appendix A.

(e) Follows from Proposition 6(ii). ■

5 Examples

The first example is devoted to illustrate Proposition 6(iii).

Example 1. Let $D(p) = 1 - p$, and $(K_1, K_2, K_3) = (\frac{3}{4}, \frac{1}{8}, \frac{1}{8})$. Then, $p_M = \frac{3}{8}$, $\Pi_1^* = \frac{9}{64}$, $p_m = \frac{3}{16}$, and $\Pi_2^* = \Pi_3^* = \frac{3}{128}$. Note that $P(K_1) = \frac{3}{4} < p_M$. Let real number $h \in [\frac{1}{2}, 1]$, then it is easily checked that the following Nash equilibrium exists.

- For $p \in [p_m, p_h]$, where p_h is such that $\phi_2(p_h) = h$,

$$\phi_1(p) = \sqrt{\left\{ \left[\frac{p\,(K_1 + K_2 - D\,(p))}{(p - p_m)\,K_2} \right]^2 + \frac{K_1}{K_2^2} \frac{p\,(D\,(p) - K_1)}{(p - p_m)} \right\}^{-1}},$$

$$\phi_2(p) = \phi_3(p) = 1 - \frac{K_2}{D(p) - K_1} \left[1 - \frac{p - p_m}{p \phi_1(p)} \right];$$

- for $p \in [p_h, P(K_1)]$

$$\phi_1(p) = \frac{(p-p_m)K_2}{p\left[(1-h)(K_1+K_2-D(p))+hK_2\right]}$$

$$\phi_2(p) = \frac{(p-p_m)K_1+ph(D(p)-K_1-K_2)}{p\left[(1-h)(K_1+K_2-D(p))+hK_2\right]}$$

$$\phi_3(p) = h;$$

- for $p \in [P(K_1), p_M] = [\frac{1}{4}, \frac{3}{8}]$, $\phi_1(p) = \frac{16p-3}{16p}$, $\phi_2(p) = \frac{64p(1-p)-9-8p\phi_3(p)}{8p}$ where $\phi_3(p)$ is any non-decreasing function whose derivative is not higher than $\frac{9-64p^2}{8p^2}$, $\phi_3(P(K_1)) = h$, and $\phi_3(p_M) = 1$.

Note that $\phi_2(p) = \phi_3(p)$ for each p if and only if $h = \frac{3}{4}$ and $\phi_3(p) = \frac{64p(1-p)-9}{16p}$ for $p \geqslant P(K_1)$. Further, if $h < \frac{1}{2}$, then $\phi_2(P(K_1)) \leqslant 1$ does not hold.

The other examples are devoted to illustrate Theorem 1(c). In Example 2 $\Pi_3^* = \pi_m$, whereas in Examples 3 and 4 $\Pi_3^* < \pi_m$. In Examples 2 and 3 $S_1 = S_2 \cup S_3 = [p_m, p_M]$ whereas in Example 4 $S_1 \cup S_2 \cup S_3 \neq [p_m, p_M]$. Moreover in Example 4 firm 3 charges price $p_M^{(3)}$ with positive probability.

Example 2. Let $D(p) = 20 - p$ and $(K_1, K_2, K_3) = (15, 4, 0.5)$. Then, $p_M = 7.75$, $\Pi_1^* = 60.0625$, $p_m = 4.0041\overline{6}$, and $\Pi_2^* = 16.01\overline{6}$. Note that $(15, 4, 0.5) \in C_1$ since $P(K_1+K_2) = 1 < p_m = 4.0041\overline{6} < P(K_1+K_3) = 4.5$. We partition $[p_m, p_M]$ into $\alpha = [p_m, p_m^{(3)})$, $\beta = [p_m^{(3)}, p_M^{(3)})$, and $\gamma = [p_M^{(3)}, p_M]$. In α, $\phi_1(p) = \phi_{12}^*(p) = \frac{4(4.0041\overline{6}-p)}{p(1-p)}$ and $\phi_2(p) = \phi_2^*(p) = \frac{15(4.0041\overline{6}-p)}{p(1-p)}$. One can easily check that $\arg\max_{p \in [p_m, P(K_1)]} Z_3(p, \phi_{12}^*(p), \phi_2^*(p)) = P(K_1+K_3)$. Let $p_m^{(3)} = P(K_1+K_3) = 4.5$ and $\Pi_3^* = Z_3(p_m^{(3)}, \phi_{12}^*(p_m^{(3)}), \phi_2^*(p_m^{(3)})) \approx 2.11620$. To find $p_M^{(3)}$, note that, in γ, $\phi_{12}^{**}(p) = 1-(p_m/p) = 1-(4.0041\overline{6}/p)$ and $\phi_2^{**}(p) = \frac{p(D(p)-K_3)-\Pi_1^*}{pK_2} = \frac{p(19.5-p)-60.0625}{4p}$. Then by solving equation $Z_3(p, \phi_{12}^{**}(p), \phi_{12}^{**}(p)) = \Pi_3^*$ over the range $[p_m^{(3)}, P(K_1)]$, we obtain $p_M^{(3)} \approx 4.66038$. Turning to range β, denote the solutions to system

$$\Pi_i^* = Z_i(p; \varphi_{-i}), \qquad i \in \{1,2,3\} \tag{14}$$

by $\phi_1^\circ(p), \phi_2^\circ(p)$, and $\phi_3^\circ(p)$.[17] One can check that $\left[\phi_2^{\circ\prime}(p)\right]_{p=p_M^{(3)}} < 0$. However a gap $(\widetilde{p}, p_M^{(3)})$ in S_2 gives the solution; $\widetilde{p}$ is found by solving $\phi_2^\circ(p) =$

[17] System (14) leads to a second-degree algebraic equation, only one of the solutions for $\phi_2^\circ(p)$ being nonnegative.

$\phi_2^\circ(p_M^{(3)}) = .487931$ over $(p_m^{(3)}, p_M^{(3)})$, which yields $\widetilde{p} \approx 4.57316$. Further, one can check that $\phi_1^\circ(p)$, $\phi_2^\circ(p)$, and $\phi_3^\circ(p)$ are all increasing throughout $[p_m^{(3)}, \widetilde{p}]$. To sum up: $S_1 = [4.0041\overline{6}, 7.75]$, $S_2 = [4.0041\overline{6}, 4.57316] \cup [4.66038, 7.75]$, and $S_3 = [4.5, 4.66038]$.

Example 3. Let $D(p) = 1 - p$ and $(K_1, K_2, K_3) = (51/64, 3/16, 1/16)$. Then, $p_M = 3/8$, $\Pi_1^* = 9/64$, $p_m = 3/17$, and $\Pi_2^* = 9/272$. Note that $(51/64, 3/16, 1/16) \in C_3$ since $P(K_1 + K_3) = 9/64 < p_m = 3/17$ and $p_M = 3/8 > P(K_1) = 13/64 > \frac{K_1 p_m}{K_1 - K_3} = 153/799 > p_m = 3/17$. It is calculated that $\pi_m \approx 0.0111064372457363291$. But if we followed the procedure used in Example 2 we would obtain that $p_m^{(3)} \approx 0.185285078457503860$, and $p_M^{(3)} \approx 0.1990076200$; $\phi_2^\circ(p)$ is concave in p and $\phi_2^\circ(p_m^{(3)}) \approx 0.2234432780 > \phi_2^\circ(p_M^{(3)}) \approx 0.1699261147$. Hence this cannot be a solution. However a reduction of the payoff of firm 3 to $\Pi_3^* \approx 0.01110349997$, with $S_3^* = [p_m^{(3)}, p_M^{(3)})] \approx [0.1832877785, 0.1991977634]$ coincident with a gap in S_2^*, solves the problem.

Example 4. Let $D(p) = 20 - p$, $(K_1, K_2, K_3) = (16; 5; 0.2)$. Then $p_M = 7.40$, $\Pi_1^* = 54.76$, $p_m = 3.4225$, and $\Pi_2^* = 17.1125$. Note that $(16; 5; 0.2) \in C_1$ since $p_m = 3.4225 < P(K_1 + K_3) = 3.8$. It is found that $\arg\max Z_3(p, \phi_{12}^*(p), \phi_2^*(p)) = P(K_1 + K_3) = 3.8$ and $\pi_m = 0.7339571913$. But

$$[\partial Z_3(p; \varphi_1, \varphi_2)/\partial p]_{p=P(K_1+K_3)^+; \varphi_1=\phi_{12}^*(p); \varphi_2=\phi_2^*(p)} < 0,$$

and therefore $p_m^{(3)} < P(K_1+K_3)$. In fact, if $p_m^{(3)} = P(K_1+K_3)$ then $\Pi_3(p) < \Pi_3^* = \pi_m$ for p larger than $P(K_1+K_3)$, implying that $\Pr(p_3 = P(K_1+K_3)) = 1$. But then $Z_2(p; \phi_1^*(P(K_1+K_3), 1) < p_m K_2$ for $p \in (P(K_1+K_3); 3.817544)$ and $Z_2(p; \phi_1^*(P(K_1+K_3), 1) > p_m K_2$ on a right neighbourhood of 3.817544) whereas $Z_1(p; \phi_2^*(P(K_1+K_3), 1) < p_m K_1$ for $p \in (P(K_1+K_3); 3.823921831)$: as a consequence $Z_2(p; \phi_1(P(K_1 + K_3), 1) > \Pi_2^*$ (an obvious contradiction) for $p \in (3.817544; 3.823921831)$. However an equilibrium exists in which $(p_m^{(3)}, P(K_1 + K_3)] \cap S_2 = \emptyset$, $[p_m^{(3)}, P(K_1 + K_3)] = S_3 \subset S_1$, $\Pr(p_3 = P(K_1 + K_3)) = 1 - \phi_3(P(K_1 + K_3)) > 0$, and $p \notin S_1 \cup S_2 \cup S_3$ for p larger than and close enough to $P(K_1 + K_3)$. This equilibrium is given by the following functions

$$\phi_1(p) = \begin{cases} \frac{(p-p_m)K_2}{p[K_1+K_2-D(p)]} & p \in [p_m, p_m^{(3)}] \\ \frac{pK_3-\Pi_3^*}{pK_3}\frac{p_m^{(3)}[K_1+K_2-D(p_m^{(3)})]}{(p_m^{(3)}-p_m)K_1} & p \in [p_m^{(3)}, P(K_1+K_3)] \\ \frac{P(K_1+K_3)K_3-\Pi_3^*}{P(K_1+K_3)K_3}\frac{p_m^{(3)}[K_1+K_2-D(p_m^{(3)})]}{(p_m^{(3)}-p_m)K_1} & p \in [P(K_1+K_3), \tilde{p}] \\ \frac{p-p_m}{p} & p \in [\tilde{p}; p_M] \end{cases}$$

$$\phi_2(p) = \begin{cases} \frac{(p-p_m)K_1}{p[K_1+K_2-D(p)]} & p \in [p_m, p_m^{(3)}] \\ \frac{(p_m^{(3)}-p_m)K_1}{p_m^{(3)}[K_1+K_2-D(p_m^{(3)})]} & p \in [p_m^{(3)}, \tilde{p}] \\ \frac{p[D(p)-K_3]-p_mK_1}{pK_2} & p \in [\tilde{p}, p_M] \end{cases}$$

$$\phi_3(p) = \begin{cases} 0 & p \in [p_m, p_m^{(3)}] \\ \frac{p-p_m}{pK_3}\frac{p_m^{(3)}[K_1+K_2-D(p_m^{(3)})]}{p_m^{(3)}-p_m} + \frac{D(p)-K_1-K_2]}{K_3} & p \in [p_m^{(3)}, P(K_1+K_3)) \\ 1 & p \in [P(K_1+K_3), p_M] \end{cases}$$

where $p_m^{(3)}$, Π_3^* and $\widetilde{p}$ are the solutions of the following system:

$$\Pi_3^* = p_m^{(3)}\left[1-\left(\frac{(p_m^{(3)}-p_m)}{p_m^{(3)}(K_1+K_2-D(p_m^{(3)}))}\right)^2 K_1K_2\right]K_3 \tag{15}$$

$$[P(K_1+K_3)K_3-\Pi_3^*]\,\frac{p_m^{(3)}[K_1+K_2-D(p_m^{(3)})]}{P(K_1+K_3)(p_m^{(3)}-p_m)K_1K_3} = \frac{\widetilde{p}-p_m}{\widetilde{p}} \tag{16}$$

$$\frac{(p_m^{(3)}-p_m)K_1}{p_m^{(3)}[K_1+K_2-D(p_m^{(3)})]} = \frac{\widetilde{p}[D(\widetilde{p})-K_3]-p_mK_1}{\widetilde{p}K_2} \tag{17}$$

It is found that $p_m^{(3)} = 3.7982466455$, $\widetilde{p} = 3.821618795$, $\Pi_3^* = 0.7338170986$. This solution to system (15)-(17) is unique since equation (15) determines Π_3^* as an increasing function of $p_m^{(3)}$; equation (17) determines $\widetilde{p}$ as an increasing functions of $p_m^{(3)}$; then the RHS of equation (16) is an increasing function of $p_m^{(3)}$ whereas the LHS is decreasing. Hence, $S_1 = [p_m; 3.8] \cup [3.821618795; 7.40]$, $S_2 = [p_m, 3.7982466455] \cup [3.821618795; 7.40]$, $S_3 = [3.7982466455; 3.80]$, and $\Pr(p_3 = P(K_1+K_3) = 0.9079374281$.

6 Appendix A

Many proofs in this appendix are obtained by exploiting the properties of functions $Z_i(p;\varphi_{-i})$ (see equations (5)) in which the φ_j are independent variables. These properties are summarized in the following Lemma 1. It addresses concavity of Z_i in terms of p and clarifies how the impact of the φ_j's upon Z_i depends upon p and the firms' capacities.

Sometimes we factorize φ_j and $(1-\varphi_j)$ in equation (5)) to obtain

$$Z_i(p;\varphi_r,\varphi_j) = \varphi_j Z_i(p;\varphi_r,1) + (1-\varphi_j) Z_i(p;\varphi_r,0). \tag{18}$$

$Z_i(p;\varphi_r,1)$ and $Z_i(p;\varphi_r,0))$ have a clear interpretation: they are firm i's expected payoffs when charging p, conditional on $p_j < p$ and $p_j > p$, respectively, when $\Pr(p_r < p) = \varphi_r$ and $\Pr(p_r > p) = 1-\varphi_r$.

Lemma 1

(i) $Z_i(p;\varphi_{-i})$ (each i) is continuous and almost everywhere twice differentiable in p throughout $[p_m,p_M]$. Exceptions may arise at $p = P(K_1+K_r)$ $(r \neq 1)$ and at $p = P(K_1)$. Let $p = P(K_1+K_r)$; if $\varphi_1\varphi_r > 0$, then $[\partial Z_i(p;\varphi_{-i})/\partial p]_{p=P(K_1+K_r)^-} < [\partial Z_i(p;\varphi_{-i})/\partial p]_{p=P(K_1+K_r)^+}$ $(i \neq 1, r)$; if $\varphi_1(1-\varphi_j) > 0$ $(j \neq 1, r)$, then $[\partial Z_r(p;\varphi_{-r})/\partial p]_{p=P(K_1+K_r)^-} > [\partial Z_r(p;\varphi_{-r})/\partial p]_{p=P(K_1+K_r)^+}$ (each $r \neq 1$); if $\varphi_r(1-\varphi_j) > 0$ $(j \neq 1, r)$, then $[\partial Z_1(p;\varphi_{-1})/\partial p]_{p=P(K_1+K_r)^-} > [\partial Z_1(p;\varphi_{-1})/\partial p]_{p=P(K_1+K_r)^+}$. Let $p = P(K_1)$; if $\varphi_1(1-\varphi_r) > 0$ $(i \neq 1, r)$, then $[\partial Z_i(p;\varphi_{-i})/\partial p]_{p=P(K_1)^-} < [\partial Z_i(p;\varphi_{-i})/\partial p]_{p=P(K_1)^+}$ (each $i \neq 1$); if $(1-\varphi_2)(1-\varphi_3) > 0$, then $[\partial Z_1(p;\varphi_{-1})/\partial p]_{p=P(K_1)^-} > [\partial Z_1(p;\varphi_{-1})/\partial p]_{p=P(K_1)^+}$.

(ii) $Z_1(p;\varphi_2,\varphi_3)$ is concave and increasing in p throughout $[p_m,p_M]$.

(iii) $Z_i(p;\varphi_{-i})$ (each $i \neq 1$) is concave in p over any range enclosed in (p_m,p_M) in which it is differentiable; local convexity only arises at $P(K_1+K_r) \in (p_m,p_M)$ $(r \neq 1, i)$, if $\varphi_1\varphi_r > 0$, and at $P(K_1) \in (p_m,p_M)$, if $\varphi_1(1-\varphi_r) > 0$ $(r \neq 1, i)$.

(iv) In any range enclosed in $[p_m,p_M)$ where $Z_i(p;\varphi_1,\varphi_j)$ $(i,j = 2,3)$ is concave in p, but not strictly concave, it is increasing in p, provided that $\varphi_1 < 1$.

(v) $Z_i(p;\varphi_{-i})$ is continuous and differentiable in φ_j and $\partial Z_i/\partial\varphi_j \leq 0$, each i and $j \neq i$. More precisely: if $p \in (p_m, P(K_1))$, then $\partial Z_i/\partial\varphi_j < 0$, each i and $j \neq i$; if $p \geq P(K_1)$, then $\partial Z_1/\partial\varphi_j < 0$, $\partial Z_i/\partial\varphi_1 < 0$, and $\partial Z_i/\partial\varphi_j = 0$ (each $i \neq 1$ and $j \neq 1, i$).

(vi) If $K_i = K_j$ and $\varphi_i \leqslant \varphi_j$, then $Z_i(p;\varphi_{-i}) \leqslant Z_j(p;\varphi_{-j})$ and $Z_i(p;\varphi_{-i}) < Z_j(p;\varphi_{-j})$ whenever $\varphi_i < \varphi_j$ and $\partial Z_i/\partial\varphi_j < 0$.

(vii) If $K_i \leqslant K_j$ and $\varphi_i > \varphi_j = 0$, then $(K_j/K_i)Z_i(p;\varphi_{-i}) \geqslant Z_j(p;\varphi_{-j})$.

Proof

(i) For given φ_{-i}, $Z_i(p;\varphi_{-i})$ is a weighted arithmetic average of functions $pq_{i,\psi}(p)$, each of which is almost everywhere twice differentiable: the only exception arises when $P(K_1+K_r)\in(p_m,p_M)$ $(r\neq 1,i)$ and when $P(K_1)\in(p_m,p_M)$. Indeed

$$\begin{aligned}
&\left[\partial pq_{i,\{1,r\}}(p)/\partial p\right]_{p=P(K_1+K_r)^-}=\left[pD'(p)\right]_{p=P(K_1+K_r)}<\\
&<\left[\partial pq_{i,\{1,r\}}(p)/\partial p\right]_{p=P(K_1+K_r)^+}=0\quad(i\neq 1,r),\\
&\left[\partial pq_{r,\{1\}}(p)/\partial p\right]_{p=P(K_1+K_r)^-}=K_r>\\
&>\left[\partial pq_{r,\{1\}}(p)/\partial p\right]_{p=P(K_1+K_r)^+}=K_r+\left[pD'(p)\right]_{p=P(K_1+K_r)},\\
&\left[\partial pq_{1,\{r\}}(p)/\partial p\right]_{p=P(K_1+K_r)^-}=K_1>\\
&>\left[\partial pq_{1,\{r\}}(p)/\partial p\right]_{p=P(K_1+K_r)^+}=K_1+\left[pD'(p)\right]_{p=P(K_1+K_r)},\\
&\left[\partial pq_{i,\{1\}}(p)/\partial p\right]_{p=P(K_1)^-}=\left[pD'(p)\right]_{p=P(K_1)}<\\
&<\left[\partial pq_{i,\{1\}}(p)/\partial p\right]_{p=P(K_1)^+}=0\quad(i\neq 1),\\
&\left[\partial pq_{1,\emptyset}(p)/\partial p\right]_{p=P(K_1)^-}=K_1>\\
&>\left[\partial pq_{1,\emptyset}(p)/\partial p\right]_{p=P(K_1)^+}=K_1+\left[pD'(p)\right]_{p=P(K_1)}.
\end{aligned}$$

(ii) For each φ_2 and φ_3, function $Z_1(p;\varphi_2,\varphi_3)$ is a weighted arithmetic average of functions of p each of which is concave and increasing over the range $[p_m,p_M]$. This derives from the concavity of $pD(p)$ and the fact that $p[D(p)-K_2-K_3]$ is increasing.

(iii) Functions $pq_{i,\psi}(p)$ are concave everywhere they are twice differentiable. Part (i) completes the proof.

(iv) By definition $\partial^2 Z_i(p;\varphi_1,\varphi_j)/\partial p^2=0$ in a range if and only if $Z_i(p;\varphi_1,\varphi_j)$ is either proportional to pK_i or equal to 0. The latter cannot hold for $p\in[p_m,p_M)$ and $\varphi_1<1$.

(v) For a given p, $Z_i(p;\varphi_j,\varphi_r)$ is a polynomial of degree 2 (or lower) in φ_j and φ_r. Hence it is everywhere continuously differentiable with respect to φ_j and φ_r. Partial differentiation of (18) yields

$$\frac{\partial Z_i}{\partial\varphi_j}=Z_i(p;\varphi_r,1)-Z_i(p;\varphi_r,0).$$

Then it is easily checked that $Z_i(p;\varphi_1,1)=Z_i(p;\varphi_1,0)=p(1-\varphi_1)K_i$ for $p\geqslant P(K_1)$, each $i\neq 1$ and each $j\neq i,1$ whereas $Z_i(p;\varphi_r,1)<Z_i(p;\varphi_r,0)$ in all the other cases.

(vi) Since $K_i = K_j$, $Z_i(p; \varphi_r, \beta) = Z_j(p; \varphi_r, \beta)$. Hence, taking account of equation (18), $Z_i(p; \varphi_{-i}) - Z_j(p; \varphi_{-j}) = \varphi_j Z_i(p; \varphi_r, 1) + (1-\varphi_j) Z_i(p; \varphi_r, 0) - \varphi_i Z_i(p; \varphi_r, 1) - (1-\varphi_i) Z_i(p; \varphi_r, 0) = (\varphi_j - \varphi_i)\left[Z_i(p; \varphi_r, 1) - Z_i(p; \varphi_r, 0)\right] = (\varphi_j - \varphi_i)\partial Z_i / \partial \varphi_j$.

(vii) From equation (18) and since $\varphi_i > \varphi_j = 0$, $Z_i(p; \varphi_{-i}) = Z_i(p; \varphi_r, 0)$ whereas $Z_j(p; \varphi_{-j}) \leq Z_j(p; \varphi_r, 0)$ because of part (vi). Thus it suffices to prove that $(K_j/K_i) Z_i(p; \varphi_r, 0) \geq Z_j(p; \varphi_r, 0)$. This is done by noting that for any $q_{i,\psi}(p)$ with a positive coefficient in $Z_i(p; \varphi_r, 0)$ there is a corresponding $q_{j,\psi}(p)$ with a positive coefficient in $Z_j(p; \varphi_r, 0)$, based on the same $\psi \in \mathcal{P}(N_{-i-j}) = \{\{r\}, \emptyset\}$, and vice versa. The claim follows since $(K_j/K_i) q_{i,\psi}(p) \geqslant q_{j,\psi}(p)$ for each $\psi \in \mathcal{P}(N_{-i-j})$. ■

Proof of part (iv) of Proposition 4

(iv) If $\phi_{-i}(p)$ is constant over $(p^\circ, p^{\circ\circ})$, then $d\Pi_i(p)/dp = \partial Z_i(p; \phi_{-i}(p^\circ))/\partial p \neq 0$ over part of S_i. The inequality is an obvious consequence of Lemma 1(ii)-(iv).

■

The following Lemma 2 deals with atoms in the support of some equilibrium strategy that are internal to the range $[p_m, p_M]$. This will be helpful when computing the equilibrium whenever such an atom exists, as is the case with the data of Example 4.

Lemma 2 Let Assumptions 1, 2, and 3 and inequality ($6'$) hold. In any equilibrium (ϕ_1, ϕ_2, ϕ_3) in which $\phi_j(p^\circ) < \phi_j(p^\circ+)$ for some j and some $p^\circ \in (p_m, p_M)$

(i) $\Pi_j^* = \Pi_j(p^\circ) = Z_j(p^\circ; \phi_{-j}(p^\circ))$;

(ii) there is $p^{\circ\circ} > p^\circ$ such that $(S_1 \cup S_2 \cup S_3) \cap (p^\circ, p^{\circ\circ}) = \emptyset$ and $p^{\circ\circ} \in S_1 \cup S_2 \cup S_3$;

(iii) $\lim_{p \to p^{\circ}+} \partial Z_j(p, \phi_{-j}(p^\circ))/\partial p \leq 0$;

(iv) $K_j < K_1$;

(v) $p^\circ < P(K_1)$.

Proof

(i) If $\Pi_j(p^\circ) < Z_j(p^\circ; \phi_{-j}(p^\circ))$, then firm j has not made a best response by charging p° with positive probability.

(ii) If there is $p^{\circ\circ\circ} > p^\circ$ such that $(p^\circ, p^{\circ\circ\circ}) \subseteq S_h$, then $(p^\circ, p^{\circ\circ\circ}) \subseteq S_h \cap S_i$ because of Proposition 4(iv). Then $\{h, i, j\} = \{2, 3\}$ otherwise, by Lemma 1(v), $\Pi_k^* = \Pi_k(p^\circ+) < \Pi_k(p^\circ-)$, $k \in \{h, i\}$ and $k \neq j$: an obvious contradiction. But then $\Pi_k^* = \Pi_k(p) = p[1 - \phi_1(p^\circ)]K_k$ on a right neighbourhood of p°: another obvious contradiction.

(iii) Otherwise $\Pi_j(p) > \Pi_j(p^\circ)$ on a right neighbourhood of p° because of part (ii).

(iv) Because of part (iii) and Lemma 1(ii).

(v) It is an obvious consequence of parts (ii), (iii) and (iv). ■

Proof of part (ix) of Proposition 4

(ix) If $M = \{1\}$, then $p_M > p_M^{(i)} \geqslant p_M^{(j)}$ ($i \neq 1, j \neq 1, i$; if $p_M^{(2)} = p_M^{(3)}$, then with no loss of generality, $\Pr(p_j = p_M^{(j)}) = 0$). Moreover, $S_1 \cap (p_M^{(i)}, p_M) = \emptyset$ since $\Pi_1(p) = p[D(p) - K_2 - K_3] < \Pi_1^*$ for $p \in (p_M^{(i)}, p_M)$. Because of Lemma 2, $\Pr(p_1 = p_M^{(i)}) = 0$ and $Z_i(p_M^{(i)}, \phi_{-i}(p_M^{(i)})) = \Pi_i^*$, whether $\Pr(p_i = p_M^{(i)}) = 0$ or not. Then the obvious contradiction that $\Pi_i(p) > \Pi_i(p_M^{(i)})$ on the right of $p_M^{(i)}$ is obtained. Indeed if $D(p_M^{(i)}) \leq K_1 + K_j$, then $\partial Z_i(p, \phi_{-i}(p))/\partial p = [1 - \phi_1(p_M^{(i)})]K_i > 0$ on a right neighbourhood of $p_M^{(i)}$. If, on the contrary, $D(p_M^{(i)}) > K_1 + K_j$, then $D(p_m) > K_1 + K_j$ and therefore $\Pi_i^* = p_m K_i$ because of parts (v) and (vi). As a consequence $\phi_1(p_M^{(i)}) = \frac{(p_M^{(i)} - p_m)K_i}{p_M^{(i)}(K - D(p_M^{(i)}))}$ since $Z_i(p_M^{(i)}; \phi_1(p_M^{(i)}), 1) = p_m K_i$. Moreover,

$$\Pi_i(p) = Z_i(p; \phi_1(p_M^{(i)}), 1) = p[\phi_1(p_M^{(i)})(D(p) - K_1 - K_j) + \left(1 - \phi_1(p_M^{(i)})\right) K_i]$$

for $p \in [p_M^{(i)}, \min\{P(K_1 + K_j), p_M\}]$. Hence in this interval

$$\frac{d\Pi_i(p)}{dp} = \frac{\partial Z_i(p; \phi_1(p_M^{(i)}), 1)}{\partial p} = \frac{K_i}{p_M^{(i)}\left[K - D(p_M^{(i)})\right]} \times$$

$$\left\{(p_M^{(i)} - p_m)\left[D(p) + pD'(p) - K_2 - K_3\right] + p_m K_1 - p_M^{(i)}\left[D(p_M^{(i)}) - K_2 - K_3\right]\right\} > 0.$$

■

Proof of parts (ii.d) and (iii) of Proposition 5

(ii.d) If $\phi_2(p^\circ) > \phi_1(p^\circ)$ at $p^\circ \in S_1$, then, by Lemma 1(v), $\Pi_2(p^\circ) > \Pi_1(p^\circ) = \Pi_1^*$, contrary to Proposition 2. If $\phi_2(p^\circ) > \phi_1(p^\circ)$ at $p^\circ \in S_2 - S_1$, then, because of parts (ii.a) and (ii.b), $\phi_1(p)$ should jump up at some $p < p_M$ in order to avoid the previous contradictions, contrary to Lemma 2(iv).

(iii) Along the same lines of the proof of part (ii). ■

Proof of Proposition 6

(i) If $K_2 = K_1$, then $\Pi_2^* = p_m K_2$ because of Proposition 3(ii) and Proposition 4(v). Next, let $K_2 < K_1$. If $p_m^{(2)} = p_m$, then $\Pi_2^* = p_m K_2$ because of Proposition 4(v). If $p_m^{(2)} > p_m$, then $\Pi_2^* = Z_2(p_m^{(2)}; \phi_{-2}(p_m^{(2)}))$ and, by Proposition 4(iii), $p_m^{(3)} = p_m$ and $\Pi_3^* = p_m K_3$. Hence $p_m K_2 = (K_2/K_3)\Pi_3^* \geqslant$

$(K_2/K_3)Z_3(p_m^{(2)}; \phi_{-3}(p_m^{(2)})) \geqslant Z_2(p_m^{(2)}; \phi_{-2}(p_m^{(2)})) = \Pi_2^* \geqslant \Pi_2(p_m-) = p_m K_2$, the second inequality being a consequence of Lemma 1(vii).

(ii) Along the same lines of the proof of part (i).

(iii.a) An obvious consequence of Proposition 5(iii).

(iii.b) $[\partial Z_i(p; \varphi_{-i})/\partial \varphi_j]_{\varphi_{-i}=\phi_{-i}(p^\circ)} = 0$, otherwise part (ii) would be violated: hence, by Lemma 1(v), $p^\circ \geqslant P(K_1)$.

(iii.c) Because of part (iii.b) $p^\circ \notin S_2 \cap S_3$; more specifically $p^\circ \notin S_i$ and $p^\circ \in S_j$, since, by Lemma 1(v)-(vi), $Z_i(p^\circ, \phi_{-i}(p^\circ)) < Z_j(p^\circ, \phi_{-j}(p^\circ))$. Let $p^{\circ\circ} = \min_{p>p^\circ} S_i$: clearly, $\phi_i(p^{\circ\circ}) < \phi_j(p^{\circ\circ})$. Then, unless $p^{\circ\circ} \geqslant P(K_1)$, we get the following contradiction: $\Pi_j^* \geqslant Z_j(p^{\circ\circ}, \phi_1(p^{\circ\circ}), \phi_i(p^{\circ\circ})) > Z_i(p^{\circ\circ}, \phi_1(p^{\circ\circ}), \phi_j(p^{\circ\circ})) = \Pi_i^* = \Pi_j^*$, because of Lemma 1(v)-(vi). ■

Proof of Proposition 7

(i) It is an immediate consequence of Lemma 1(ii)&(v) since $\phi_{1j}^\star(p) < \phi_j^\star(p) \leqslant 1$.

(ii.a) It is immediately recognized that $\phi_2^{\star\star}(p) \leqslant 1$ whenever $p \leqslant p_M$ (the equality holds if and only if $p = p_M$). Thus, by Lemma 1(ii), $\phi_{12}^{\star\star}(p)$ is increasing over the range under concern, obviously implying the same as for $\phi_2^{\star\star}(p)$.

(ii,b)-(ii,c) It is immediately recognized that $\phi_j^{\star\star}(p_M) = 1$ and that $\phi_{1j}^{\star\star}(p)$ is increasing. To prove that $\phi_j^{\star\star}(p)$ is increasing too it is sufficient to observe that $K_1 p_m \geqslant (D(p) - K_2 - K_3)p \geqslant -D'(p)p^2$. Both inequalities are satisfied as strict inequalities for $p \in [p_m, p_M)$ and as equalities for $p = p_M$.

■

Proof of parts (c.i) and (d) of Theorem 1

(c.i) We will first establish some properties of any equilibrium in which $L = \{1, 2\}$; then we will prove that necessarily $L \neq \{1, 2, 3\}$ and $L \neq \{1, 3\}$. It is immediately recognized that if $L = \{1, 2\}$, then $\Pi_3(p)/K_3 > \Pi_2(p)/K_2$ for p larger than and sufficiently close to p_m. Hence $\Pi_3^* > p_m K_3$. Consequently, $p_M^{(3)} < P(K_1)$, by Proposition 4(vii). Hence $\phi_1(p) = \phi_{12}^\star(p)$ and $\phi_2(p) = \phi_2^\star(p)$ (see Proposition 7(i)) in a neighborhood of p_m and $p_m K_3 < \Pi_3^* \leqslant \pi_m$ since, clearly, $\Pi_3^* = \Pi_3(p_m^{(3)}) = F(p_m^{(3)})$. Now we will prove that in no equilibrium $L = \{1, 2, 3\}$. This is obvious in C_1 because of Proposition 4(viii)&(v). In $C_2 \cup C_3$, by way of contradiction, denote by $(\widehat{\phi}_1(p), \widehat{\phi}_2(p), \widehat{\phi}_3(p))$ the equilibrium strategy profile on a neighborhood of p_m. The equations $\Pi_1^* = Z_1(p; \widehat{\phi}_2(p), \widehat{\phi}_3(p))$ and $\Pi_2^* = Z_2(p; \widehat{\phi}_1(p), \widehat{\phi}_3(p))$ imply $\widehat{\phi}_1(p) < \frac{(p-p_m)K_2}{p[K_1+K_2-D(p)]}$ and $\widehat{\phi}_2(p) < \frac{(p-p_m)K_1}{p[K_1+K_2-D(p)]}$ since $\widehat{\phi}_3(p) > 0$ and Lemma 1(v) holds. Consequently, *a fortiori* $Z_3(p; \widehat{\phi}_1(p), \widehat{\phi}_2(p)) > p_m K_3$, contrary to Proposition 4(v). Finally we will prove that in no equilibrium $L = \{1,3\}$. This claim obviously holds in C_1, because of Proposition 4(viii). By simple calculation we obtain that $Z_2(p;\phi^\star_{13}(p),\phi^\star_3(p)) < p_m K_2$ (see Proposition 7(i)) over $(p_m, P(K_1))$ and therefore it should be $p^{(2)}_m \geqslant P(K_1)$ because of Proposition 6(i)). An immediate contradiction is obtained in C_2 since $p_M \leqslant P(K_1)$ (note that $p^{(2)}_m = p^{(2)}_M = p_M$ would contradict Proposition 2). As for C_3, it cannot be $[p_m, P(K_1)] \subseteq S_1 \cap S_3$ since then $\phi_3\left(\frac{\widehat{p}K_1}{K_1-K_3}\right) > 1$ (an obvious contradiction), given that $\frac{\widehat{p}K_1}{K_1-K_3} < P(K_1) \leq p_M$. Nor there exist $p^\circ \leqslant \tilde{p}^{(3)}_M$ (see Proposition 7(i)) such that $[p_m, p^\circ] \subset S_1 \cap S_3$ and $p \notin S_1 \cup S_2 \cup S_3$ on a sufficiently small right neighbourhood of p° (note that no atom may exist in p_m because of Proposition 4(v)-(vi)). This is so because $[\partial Z_1/\partial p]_{p=p^\circ,\varphi_3=\phi^\star_3(p^\circ)} > 0$ so that $Z_1(p;\phi_{-1}(p)) > p_m K_1$ on a right neighbourhood of p°.

(d) We will first establish some properties of any equilibrium in which $L = \{1,3\}$; then we will prove that necessarily $L \neq \{1,2,3\}$ and $L \neq \{1,2\}$. If $L = \{1,3\}$, following an argument in the proof of part (c.i), $\Pi_2(p)$ is lower than $p_m K_2$ at any $p < P(K_1)$ and therefore $p^{(2)}_m \geqslant P(K_1)$. No contradiction arise since $p_M > P(K_1)$ and $\frac{\widehat{p}K_1}{K_1-K_3} > P(K_1)$. Then the event of $L = \{1,2,3\}$ is ruled out as in the proof of part (c.i). Under the event $L = \{1,2\}$, following the proof of part (c.i) we obtain $\Pi^*_3 > p_m K_3$ and $p^{(3)}_M < P(K_1)$. Then $Z_1(p,\phi_{-1}(p)) = p[D(p)-K_3]p - \phi_2(p)K_2 < p_m K_1$ in the interval $(p^{(3)}_M, P(K_1)]$. The inequality holds since the function $p[D(p)-K_3]$ is increasing for any $p \leq p_M$ and $\widehat{p}K_1 \geq P(K_1)[K_1-K_3]$. Hence $(p^{(3)}_M, P(K_1)) \cap S_1 = \emptyset$ and, because of Proposition 4(iv), $(p^{(3)}_M, P(K_1)) \cap S_2 = \emptyset$ too. As a consequence $\phi_1(p) = \phi_1(p^{(3)}_M)$ over the range $(p^{(3)}_M, P(K_1)]$. Moreover $\phi_1(P(K_1)) = \phi_1(p^{(3)}_M) < \phi^\star_{13}(p^{(3)}_M) < \phi^\star_{13}(P(K_1)) = \frac{P(K_1)-p_m}{P(K_1)}$ (see Proposition 7(i)); the first inequality holds because of Lemma 1(v) and equation $\Pi^*_3 = Z_3(p,\phi_1(p^{(3)}_M),\phi_2(p^{(3)}_M))$, since $\Pi^*_3 > p_m K_3$ and $\phi_2(p^{(3)}_M) > 0$; the second inequality holds since $D\left(\frac{\widehat{p}K_1}{K_1-K_3}\right) \leq K_1$ and therefore $\phi^\star_3(P(K_1)) \leqslant 1$. Hence $Z_2(P(K_1),\phi_{-2}(P(K_1))) > p_m K_2$, which contradicts Proposition 6(i). ■

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