Pricing Derivatives in the New Framework: OIS Discounting, CVA, DVA & FVA

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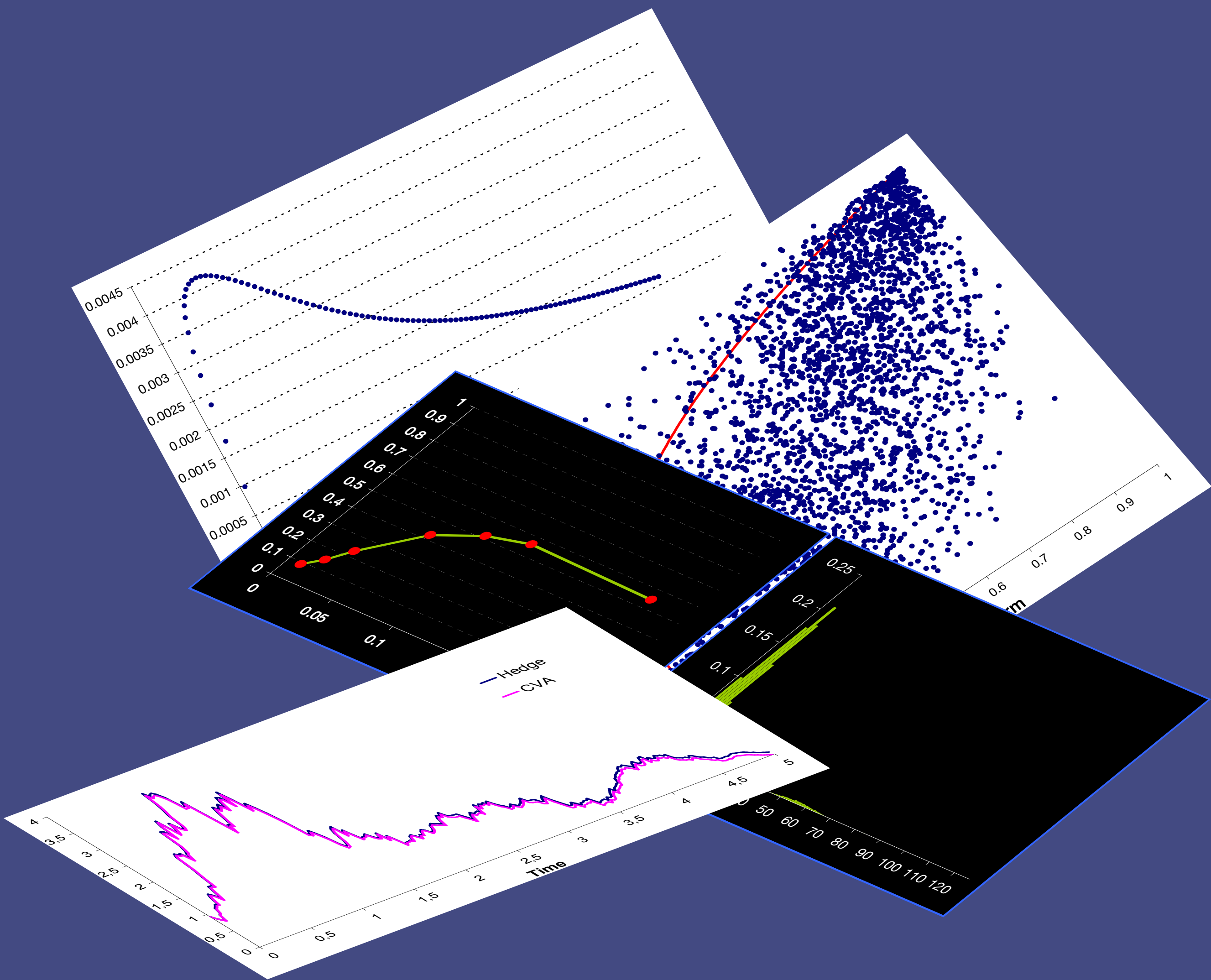

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Introduction

Introduction

What you have put your hands on is a free online quantitative finance book. In this book we have tried to summarize some methodological issues that we have faced in our work as quants since 2007 and that are not usually tackled in classical quantitative finance books. All of this issues have to do with the implications of collateral, counterparty credit risk and funding risk in the valuation and risk management of financial derivatives, therefore the book's title.

As a byproduct of the 2007-2008 credit crunch, derivatives pricing and risk management are experiencing a dramatic transformation. Assumptions that were widely accepted not long ago, like absence of counterparty credit risk and the existence of a unique risk free curve available for every derivatives hedger in the derivatives replication process, are no longer accepted. Financial institutions are changing the way in which counterparty credit risk and funding risk are managed. We find ourselves in a world with multiple discounting curves for any given currency and with different adjustments to apply to the price of financial derivatives that seem difficult to hedge. The target of this book is to make a deep review of how these effects impact the derivatives valuation theory.

The book consists of 10 different chapters. In the first chapter we will tackle the impact of collateral on derivatives pricing and introduce funding value adjustment. After reviewing the traditional approach, where the existence of a tradeable risk free curve is assumed for marker participants, we will tackle the valuation of fully collateralized derivatives, where collateral is assumed to be cash denominated in the deal's currency. By doing so, we will justify OIS discounting for cash collateralized derivatives when the deal and the collateral currencies coincide. After that, we will analyze the impact of exotic collateral, that is, any type of collateral different from cash in the deal's currency (foreign cash, stocks or bonds possibly denominated in a currency different from that of the deal). After that, funding value adjustment will be introduced. In order to do so, we will initially assume no counterparty credit risk, assumption that will be relaxed in subsequent chapters. Hence, in this chapter the derivatives hedger is assumed to have access to a funding curve different from the OIS curve and we will analyze the pricing of non collateralized or partially collateralized derivatives assuming default free counterparties.

After the first chapter, we will conclude that in the new framework, for a single currency, we will end up with many current accounts and their corresponding discount factors (a different discounting curve for each collateral asset and an additional discounting curve for the funding of uncollateralized derivatives). In the second chapter, we will revisit the fundamental theorem of asset pricing under this multiple discounting curves framework.

The third chapter will be devoted to interest rate curve calibration and risk free dynamics. Special focus will be put on the calibration of cross currency swaps, since from these quotes we can obtain curves to discount derivatives collateralized in cash denominated in a different currency. Risk free dynamics of collateral basis curves and the funding curve will also be discussed.

In the fourth chapter, we will review the modeling of credit risk. We will start by reviewing the dynamic replication of credit derivatives (paying attention to default and spread risks) and justify OIS discounting for cash collateralized credit derivatives. The credit default swaps market will be fundamental in the management of counterparty credit risk, therefore the importance of this analysis. After that, we will analyze the difference between cash collateralized credit derivatives and bonds, and propose a risk free modeling approach of the Bond-CDS basis. This approach will allow us to determine implied REPO curves for bonds that can be used to value bond collateralized derivatives. This approach will also help us to value derivatives written on bonds. Finally, we will explore the management of the derivative's hedger own debt (asset liability management), a concept closely linked to FVA (funding value adjustment).

In chapter five we will derive the PDE (partial differential equation) followed by a derivative closed with a risky counterparty. Although a little bit technical, the results obtained in this section will help us to understand results obtained in later sections where counterparty credit risk hedging is analyzed.

Chapter six will be dedicated to CVA hedging. We will explore CVA hedging under complete markets (markets where a liquid CDS curve for the counterparty is available) and CVA hedging under incomplete markets (the counterparty credit curve is mapped to a CDS tradeable curve). We will conclude that unless both spread and jump to default risks are hedged (something that will only be possible under complete markets), the partially (or non partially hedged) CVA position will have a non neutral carry. Under incomplete market conditions, we will propose a hedging alternative that, while smoothing the P&L evolution, does not erode the positive carry of leaving CVA unhedged.

In chapter seven we will inspect the relationship between DVA and FVA (funding value adjustment) and study the feasibility of DVA and FVA hedging. As a result, we will propose a carry neutral (opposite to the negative carry of the CVA-DVA approach) pricing for non collateralized (or partially collateralized) derivatives.

In a CVA-FVA engine, due to the portfolio effect, we will be forced to work with deals and collateral amounts denominated in different currencies. We might ask ourselves about the risk neutral drift of FX rates used to convert all these amounts to the same valuation currency. Chapter eight will be used for that purpose.

Chapter nine will be devoted to default correlation models to calculate CVA-FVA on a portfolio of credit derivatives. We will explore the limitations of traditional copula approaches and suggest a particular case of the Marshall-Olkin copula that does not suffer from these limitations. Nevertheless, this approach is not practical for more than three credit references. An alternative in high dimensions will be proposed and analyzed.

In chapter ten, we will review the different approaches to wrong way risk modeling, analyzing the limitations of each of these.

Why an online book?

As with traditional books, the main target of this book is to share our experience with other researchers and practitioners. We believe that compared to traditional books, the online format is more dynamic, in the sense that the reader can have access to updates (revisions and new chapters) whenever they are available. Therefore revised or expanded versions of the chapters and new chapters will be made available from time to time. The software developed in order to generate examples, charts will also be available in the near future.

Access to updates and feedback

As we have mentioned, the main purpose of this initiative is to share our experiences with other researchers and practitioners, therefore we believe that a feedback (positive and negative) on the book contents is fundamental. Should you have any doubts, suggestions, please let us know.

We can be contacted at freequants@gmail.com for any feedback. Whenever we have any new update on the book (or software in the future), we will send an email to a distribution list. If you want to be included in the list, please let us know.

Current status of the book

The status of this book version is the following: Some chapters are in a mature status, some other lack some text/formulae revision, examples... and some other are in progress. The following table summarizes the status of the different chapters.

Chapter #	Last Revision	Status
1	Feb 2015	Mature
2	Feb 2015	Mature
3	Feb 2015	Pending Revision
4	Feb 2015	Mature
5	Feb 2015	Mature
6	Feb 2015	Mature
7	Feb 2015	Mature
8	Feb 2015	Pending Revision
9	Feb 2015	In Progress
10	Feb 2015	In Progress

Contents

Chapter 1

Valuation of fully collateralized derivatives and introduction to FVA

1.1 Derivation of the classical pricing formula

In the classical quantitative finance literature it is assumed that the hedger of a financial derivative borrows and lends cash at a theoretical risk free curve.

Let's denote the short term risk free rate by r_t^{rf} and assume that we wanted to price a derivative written on a particular asset whose price at time t is represented by S_t. We assume that the underlying asset pays continuous dividends q_t and that under the real world measure $\mathbb{P}$

$$\frac{dS_t}{S_t} = \mu_t^{\mathbb{P}} dt + \sigma_t dW_t^{\mathbb{P}}$$

Where $\mu_t^{\mathbb{P}}$ represents the real world drift, $W_t^{\mathbb{P}}$ a $\mathbb{P}-$Brownian motion, σ_t the volatility of the process.

The replication formula of a derivative V_t will be given in this context by

$$V_t = \alpha_t S_t + \beta_t^{\mathrm{rf}} \tag{1.1}$$

Where α_t represents the number of shares of S_t to purchase (or sell if $\alpha_t < 0$) and β_t^{rf} the value of the risk free current account.

The differential change of the risk free current account is given by:

$$\frac{d\beta_t^{\mathrm{rf}}}{\beta_t^{\mathrm{rf}}} = r_t^{\mathrm{rf}} dt$$

So that if we apply Itô's Lemma to (1.1)

$$\left(\frac{\partial V_t}{\partial t}+\frac{1}{2}\sigma_t^2 S_t^2\frac{\partial^2 V_t}{\partial S_t^2}\right)dt+\frac{\partial V_t}{\partial S_t}dS_t=\alpha_t\left(dS_t+q_tS_tdt\right)+\left(-\alpha_tS_t+V_t\right)r_t^{\mathrm{rf}}dt$$

So that in order to be hedged

$$\alpha_t=\frac{\partial V_t}{\partial S_t}$$

Which implies

$$\frac{\partial V_t}{\partial t}+\underbrace{\left(r_t^{\mathrm{rf}}-q_t\right)}_{\text{Drift}}S_t\frac{\partial V_t}{\partial S_t}+\frac{1}{2}\sigma_t^2S_t^2\frac{\partial^2 V_t}{\partial S_t^2}=\underbrace{r_t^{\mathrm{rf}}}_{\text{Discounting}}V_t \tag{1.2}$$

If we assume that V_t has no cash flows until its maturity T, the solution of (1.2) with terminal condition $V_T=g(S_T)$ will be given by

$$\frac{V_t}{\beta_t^{\mathrm{rf}}}=E_{\mathbb{Q}}\left[\frac{V_T}{\beta_T^{\mathrm{rf}}}\middle|\mathcal{F}_t\right] \tag{1.3}$$

Where $\mathbb{Q}$ is a measure equivalent to $\mathbb{P}$ such that $\mu_t^{\mathbb{Q}}=r_t^{\mathrm{rf}}-q_t$.
Under $\mathbb{Q}$, both $\frac{V_t}{\beta_t^{\mathrm{rf}}}$ and $\frac{S_t\exp\left(\int_{s=0}^t q_sds\right)}{\beta_t^{\mathrm{rf}}}$ behave as martingales.
r_t^{rf} was practically assumed to be the interbank curve (LIBOR curve). This consideration was due to the following reasons:

- Derivative hedger's were interbank counterparties.
- Financial institutions could funds themselves at LIBOR levels.
- Financial institutions were believed to have very low default risk.
- There was a single LIBOR curve bootstrapped from interbank deposits (short term) and swaps (whose values were assumed independent of the tenor of the floating leg).

With the exception of the first, the other are no longer valid.

1.2 Valuation of fully cash collateralized derivatives

The number of collateralized derivatives transaction has increased dramatically during the last years. The increase is due to counterparty default risk concerns. Nowadays, most (if not all) of interbank derivatives transactions are collateralized.

In this section we will tackle the pricing of collateralized derivatives with the following characteristics:

- Are collateralized in cash denominated in the deal's currency.
- Collateral is symmetrical (both counterparties post collateral with symmetrical rules) and with no frictions (collateral to be posted is equal to the value of the derivative).
- Daily margining (collateral amounts are re balanced on a daily basis), although we will theoretically assume that margining is done continuously.

Being able to value collateralized interbank deals will be key, since the valuation of non collateralized instruments will imply adjustments to put on top of the valuation of these interbank deals.
We will assume that one of the two counterparties acts as the investor (risk taker) and the other as the hedger (risk hedger).
In establishing the valuation equation we will assume that the default of any of the counterparties does not imply a jump in the price of the underlying asset (or assets) of the derivative being priced.
A fully collateralized derivative transaction in cash, is such that if counterparty A enters into a transaction with counterparty B with a value of V_t (that we assume positive) from A's perspective:

- A pays V_t to B.
- B posts collateral in cash to A with a value of V_t.
- At time $t + dt$ A pays to B interest on collateral at a predetermined rate.
- The amount posted as collateral is rebalanced reflecting the change in value experienced by the derivative between t and $t + dt$. That is, B posts $dV_t = V_{t+dt} - V_t$

Notice that the net cash flow at trade date is always zero. If V_t was negative, it will be A the counterparty posting collateral.
The interest rate paid on collateral accounts is reflected in the contract. The usual choice is the OIS (overnight index swap) rate for the currency in which the collateral asset is denominated. r_t will represent the short term interest rate paid on cash posted as collateral.
We are going to assume that there is a liquid REPO (repurchase agreement or securities lending for equities) market written on the underlying asset, so that we can buy the asset funding the position at the asset's short term REPO rate r_t^S.
The replication formula in this context will be given by

$$V_t = \alpha_t S_t + \beta_t^c + \beta_t^r \tag{1.4}$$

Where the right hand side of (1.4) represents the hedging portfolio and $-V_t$ the value of the derivative being priced, both seen from the hedger's perspective.

β_t^c represents amounts posted as collateral by the hedger due to the cash collateralized transaction. As already mentioned:

$$\frac{d\beta_t^c}{\beta_t^c} = r_t dt$$

β_t^r represents amounts lent by the hedger's though the REPO transaction written on the underlying asset. Therefore

$$\frac{d\beta_t^r}{\beta_t^r} = r_t^S dt$$

Obviously $\beta_t^c = V_t$ and $\beta_t^r = -\alpha_t S_t$, so that applying Itô's Lemma to (1.4)

$$\left(\frac{\partial V_t}{\partial t} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2}\right) dt + \frac{\partial V_t}{\partial S_t} dS_t = \alpha_t \left(dS_t + q_t S_t dt\right) - \alpha_t S_t r_t^S + V_t r_t dt \tag{1.5}$$

Again, in order to be hedged

$$\alpha_t = \frac{\partial V_t}{\partial S_t}$$

Which implies

$$\frac{\partial V_t}{\partial t} + \underbrace{\left(r_t^S - q_t\right)}_{\text{Drift}} S_t \frac{\partial V_t}{\partial S_t} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 V_t}{\partial S_t^2} = \underbrace{r_t}_{\text{Discounting}} V_t \tag{1.6}$$

The solution to (1.6) with terminal condition $V_T = g(S_T)$ is the same as calculating the following expected value:

$$\frac{V_t}{\beta_t^C} = E_{\mathbb{Q}}\left[\frac{V_T}{\beta_T^C}\middle|\mathcal{F}_t\right]$$

Where $\mathbb{Q}$ is equivalent to $\mathbb{P}$ such that the $\mathbb{Q}$-drift of S_t is $r_t^S - q_t$.

Notice that under this new context, even if $q_t = 0$, the growth rate of the underlying asset (r_t^S) and the discounting rate of the derivative collateralized in cash (r_t) are not the same.

Under $\mathbb{Q}$, both $\frac{V_t}{\beta_t^c}$ and $\frac{S_t \exp\left(\int_{s=0}^t q_s ds\right)}{\beta_t^r}$ behave as martingales.

Notice that, contrary to what happens in the classical quantitative finance literature, for the different self financing portfolios to behave as martingales, they have to be deflated by the current account that accrues at the rate at which each portfolio can be funded. The fundamental theorem of asset pricing needs to be revisited.

1.3 Valuation of exchange traded derivatives (futures)

In this section we point out the differences between fully cash collateralized transactions and exchange traded derivatives such as futures.

As already seen in the previous section, for fully cash collateralized transactions, if counterparty A sees a value of the derivative $V_t > 0$ at a given time t, the other counterparty B posts collateral. At time $t + dt$, A pays to B interest on the collateral posted with a value equal to $r_t V_t dt$. If $V_t < 0$, then A posts collateral and receives interest on it. At time $t + dt$ the collateral is rebalanced to reflect V_{t+dt}.

In the case of futures, things are different. In the following discussion we will leave aside initial margins.

Let's assume that the t value of a future is F_t. If we wanted to trade on the future taking, for example, a long position we would not need to pay (neither receive) any extra cash. At time $t + dt$ the future will be settled, meaning that we will receive $dF_t = F_{t+dt} - F_t$ (or pay if negative).

Therefore, the net cash flow for the fully cash collateralized trade at time $t + dt$ is:

$$d\pi_t = dV_t - r_t V_t dt$$

Whereas for an exchange traded future it is

$$d\pi_t = dF_t$$

Which is equivalent to considering exchange traded futures as fully collateralized deals with collateral rate equal to 0.

Therefore, the PDE followed by an exchange traded future written on an asset S_t that can be repoed at a short term repo rate r_t^S is:

$$\frac{\partial F_t}{\partial t} + \left(r_t^{\mathrm{S}} - q_t\right) S_t \frac{\partial F_t}{\partial S_t} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 F_t}{\partial S_t^2} = 0 \tag{1.7}$$

Which implies

$$F_t = E_{\mathbb{Q}}\left[F_T \exp\left(-\int_{s=t}^{T} 0 ds\right) \middle| \mathcal{F}_t\right] = E_{\mathbb{Q}}\left[F_T \middle| \mathcal{F}_t\right]$$

In a measure under which S_t evolves with a drift $r_t^S - q_t$. Notice that this is consistent with the classical result of future rates being martingales (irrespective of their maturities) under the spot martingale measure.

1.3.1 Fully collateralized derivatives with exotic collateral

In this section we analyze the valuation of fully collateralized derivatives with a collateral asset different from cash denominated in the deal's currency.

We will assume that the amount posted as collateral coincides with the replication value of the derivative. We will see that the asset used as collateral has an impact in the replication cost. Hence, we assume that the amount posted as collateral reflects this impact.

The most general situation would be using an asset as collateral (could be a stock or bond) denominated in a currency different from that of the deal. The deal currency will be refered to as currency D, whereas the collateral currency will be represented by F.

We will use the following notation:

- r_t^D will represent the OIS rate in currency D.
- r_t^F the OIS rate in currency F.
- r_t^C the REPO rate of the collateral asset.
- C_t the collateral price at time t.
- X_t the FX rate expressed in D/F.

We will assume V_t to be the time t derivative's value from the investor's standpoint measured in D.

Assuming that V_t is positive, the hedger would have a positive amount V_t in cash in currency D available as a byproduct of the dynamic replication strategy.

V_t should be posted by the hedger to the investor in the form of the collateral asset denominated in currency F. Therefore the hedger will have to buy the collateral asset. By doing so, the hedger will be left with a long position in an asset denominated in currency F. Both the FX risk and the exposure to the collateral asset price changes will have to be hedged by the derivatives hedger.

Therefore, the hedger will have to enter into these transactions at a generic time step t:

- Exchange V_t in cash denominated in D for cash denominated in F in the spot FX market.

- With the cash obtained from the FX spot transaction, the hedger will buy the collateral asset spot and sell it forward (with maturity $t + dt$) through a REPO transaction. Under the REPO transaction the hedger will deliver at time t $\frac{V_t}{X_t}$ in cash denominated in F in exchange of collateral asset shares with the same value [1].

- These shares in the collateral asset will be posted as collateral to the investor.

- At time $t+dt$ the investor will give the collateral back (with a value of $\frac{V_t C_{t+dt}}{X_t C_t}$ measured in currency F) to the hedger, who will give it back to the REPO counterparty.

- At time $t + dt$ the hedger will receive $\frac{V_t}{X_t}\left(1 + r_t^C dt\right)$ from the REPO counterparty in cash denominated in F.

- In order to hedge the FX risk of the last amount, since it is denominated in F, at time t the hedger should sell this amount forward (with maturity $t + dt$) receiving at time $t + dt$ cash in currency D with a value equal to the amount to be paid in currency F ($\frac{V_t}{X_t}\left(1 + r_t^C dt\right)$) multiplied by the forward FX rate $X_t \frac{\left(1+r_t^D\right)}{\left(1+\left(r_t^F+b_t\right)\right)}$ seen at time t with maturity $t + dt$.

We assume that forward rates cannot be inferred by the spot FX rate and the OIS rates in both currencies, so that an adjustment needs to be made in the F rate. Notice that this adjustment represents the short term cross currency basis and will be represented by b_t.

Both cash transactions (in currencies D and F) and collateral asset transactions occurring at times t and $t + dt$ are represented in figure 1.1. Notice that if V_t was negative, the trades will be right the opposite.

[1]We assume no haircut in the REPO transaction

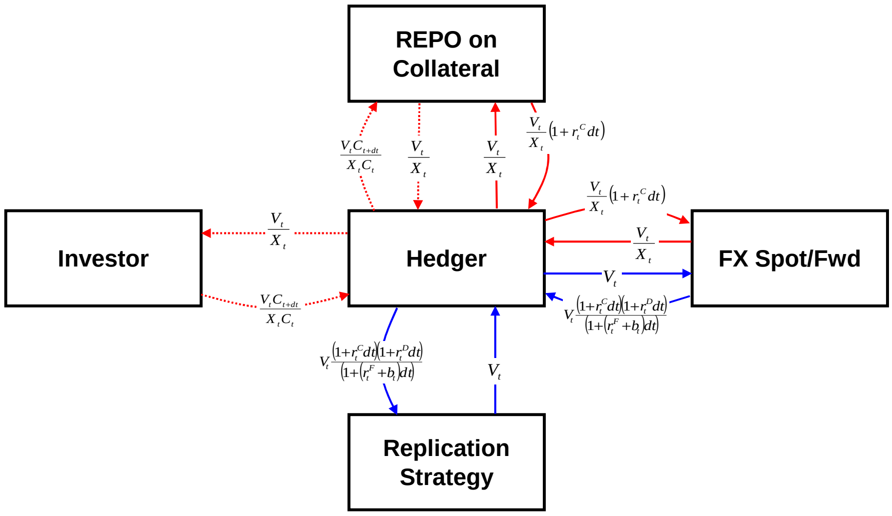


Figure 1.1: Continuous lines represent cash transactions whereas discontinuous ones represent asset transactions. Blue lines indicate amounts denominated in currency D, whereas red ones represent cash or asset transactions denominated in currency F. Straight lines refer to initial transactions, that take place at time t, and curved lines to final transactions taking place at time $t + dt$.

So that from t to $t + dt$ the value of the funds posted as collateral experiences a change equal to:

$$V_t \left(r_t^D + r_t^C - r_t^F - b_t \right) dt \tag{1.8}$$

Notice that the interest rate in (1.8) would be equal to:

- r_t^D if the collateral was cash in D.
- r_t^C if the collateral was an asset denominated in D.
- $r_t^D - b_t$ if the collateral was cash in F.
- $r_t^D + r_t^C - r_t^F - b_t$ if the collateral was an asset denominated in F.

We have seen that collateralizing deals in assets different from cash denominated in the currency of the deal implies additional risks (FX and collateral price changes risks), that once hedged imply that funds posted as collateral accrue at a rate that differs generally from the OIS rate of the deal's currency.

Therefore, for deals denominated in a single currency, different collateralization schemes imply different accrual rates for funds posted as collateral, so that we can end up with different current accounts that accrue at different rates and their corresponding discount factors.

1.4 Valuation of collateralized derivatives. Conclusions:

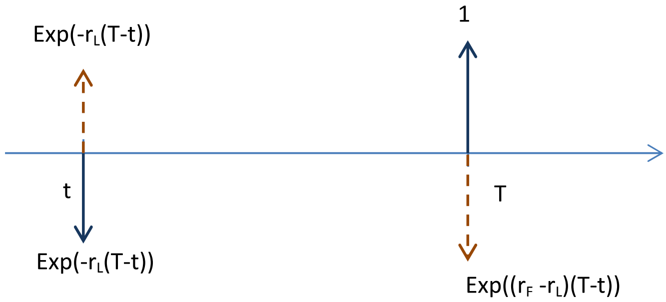
- The concept of a unique risk free rate per currency is both theoretical and invalid.
- Under the spot martingale measure, each self financing portfolio accrues at the interest rate at which it is funded.
  - Non dividend paying stocks/bonds accrue at their REPO rates.
  - Cash collateralized derivatives accrue at the collateral rate.
  - Exchange traded futures are driftless.
- Any type of non standard collateral implies an accrual rate different from the OIS rate of the deal's currency. The new accrual rate will have the following components:
  - The OIS rate of the deal's currency (eg. EONIA if EUR).
  - The REPO-OIS basis of the collateral asset (eg. REPO rate minus Fed Funds if a bond denominated in USD).
  - The cross currency basis.

1.5 Introduction to Funding Cost: A Case Example

Before the credit crunch, it was common practice for banks to borrow money at the libor rate. In this environment, the spread between libors with different tenor were fairly small. This libor rate was used as the risk-free rate as it was the rate at which banks *too big to fail* funded their business. In the post-Lehmann era, the libor rate can not longer used as the risk-free rate. While the spread between the libor and the overnight rate was around 10 bp before 2008, it grew up over the 364 bp in 2008. We know that in order to get today's price for a derivative we must discount its future's flows, but in this new environment,

What is the rate we must use for discounting?

In order to introduce the FVA concept, let us consider the following case: Let's suppose a non-collateralized European digital option with 0 strike. We are sure the option will expiry in the money and we will receive the option's notional at maturity. We denote by r^F our constant funding rate and r^L will denote the constant risk-free rate (the one we observed before 2008).



We next see the price of such a derivative before and after the 2008 crisis.
Before the crisis: we would price this derivative by discounting the *almost* sure flow at maturity at the risk-free rate $B(t,T) = e^{-r^F(T-t)}$. To be able to pay that money to our counterparty we should borrow $B(t,T)$ at our funding rate $r^F = r^L$. At maturity we would receive the money from our counterparty and pay back the money borrowed to the market.
After the crisis: If we discount the future cash-flow at the risk-free rate (e.g the overnight rate might be used as a proxy.) and we borrow money at our funding rate r^F, where $r^F > r^L$ at maturity we should face

$$P\&L = 1 - e^{\left(r^F - r^L\right)(T-t)} < 0$$

So we would end up losing money.
It seems reasonable in this case to use our funding rate for discounting, but

- is it always reasonable to do so?
- Does a collateralized derivative have the same price as a non-collateralized one?

1.6 Pricing of non collateralized derivatives

Let us denote the price of a derivative V_t written on an underlying S_t that pays a continuous dividend yield q_t.
Let us assume the dynamics of such an underlying to be under the real measure $\mathbb{P}$,

$$\frac{dS_t}{S_t} = \mu_t^{\mathbb{P}} dt + \sigma_t dW_t^{\mathbb{P}} \tag{1.9}$$

Let us assume the existence of,

- $H_t = H(t, S_t)$: Derivative written on S_t to hedge the sensitivity of V to S_t and fully cash-collateralized.

- f_t: The rate at which we can borrow money in the money market.
- i_t: The rate at which we can lend money in the money market.
- c_t: collateral rate paid on posted collateral.
- r_t: Repo rate for underlying S_t.

Let us focus on the flows that would take place in an infinitesimal time interval, between t and $t + dt$, when hedging and funding the derivative.

- The Hedger must enter at t into a derivative whose price is V_t, as seen from the investor point of view. The Hedger will fund the derivative at time t (fund if $V_t < 0$, invest if $V_t > 0$), and pay the loan back plus interests at time $t + dt$. The Hedger will pay f_t for the money borrowed and earn i_t for the money lent.
- At time t the Hedger will pay/receive the price of the derivative whether positive/negative and he will receive the derivative's value at $t + dt$.
- In order to hedge variations in V_t, as the underlying S_t moves, the hedger enters into a cash-collateralized derivative H_t with notional α_t.

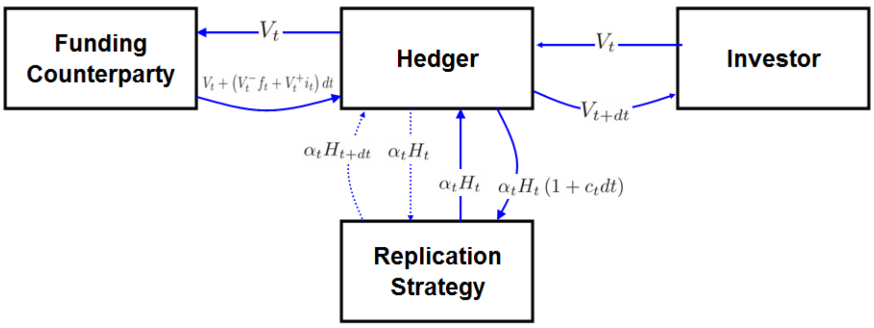


Figure 1.2: Non-Collateralized derivative strategy P&L

If we denote by ϕ_t the gain process at time t, consequence of all the flows intervening in the hedging/funding of V_t, it can be seen that,

$$d\phi_t = -dV_t + \alpha_t \left[dH_t - H_t c_t dt\right] + \left[V_t^- f_t + V_t^+ i_t\right] dt \tag{1.10}$$

The only uncertainty in $d\phi_t$ is dS_t. So we will eliminate such uncertainty by properly choosing α_t.
By choosing the notional to be,

$$\alpha_t = \frac{\frac{\partial V_t}{\partial S_t}}{\frac{\partial H_t}{\partial S_t}} \tag{1.11}$$

applying Ito to V_t and H_t in (1.10),

$$\begin{aligned} dV_t &= \frac{\partial V_t}{\partial t}dt + \frac{\partial V_t}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}\sigma_t^2 S_t^2 dt \\ dH_t &= \frac{\partial H_t}{\partial t}dt + \frac{\partial H_t}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 H_t}{\partial S_t^2}\sigma_t^2 S_t^2 dt \end{aligned} \tag{1.12}$$

we obtain

$$\frac{\partial V_t}{\partial t} + \frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}\sigma_t^2 S_t^2 - \alpha_t \underbrace{\left[\frac{\partial H_t}{\partial t} + \frac{1}{2}\frac{\partial^2 H_t}{\partial S_t^2}\sigma_t^2 S_t^2 - H_t c_t\right]}_{-(r_t - q_t)S_t\frac{\partial H_t}{\partial S_t}} = V_t^- f_t + V_t^+ i_t \tag{1.13}$$

So (1.13) becomes,

$$\boxed{\frac{\partial V_t}{\partial t} + (r_t - q_t)\, S_t \frac{\partial V_t}{\partial S_t} + \frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}\sigma_t^2 S_t^2 = V_t^- f_t + V_t^+ i_t} \tag{1.14}$$

If we define the funding spreads as spreads over the collateral rate,

$$s_t^f = f_t - c_t, \qquad s_t^i = i_t - c_t$$

We can express (1.14) as

$$\begin{aligned} &\frac{\partial V_t}{\partial t} + (r_t - q_t)\, S_t \frac{\partial V_t}{\partial S_t} + \frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}\sigma_t^2 S_t^2 = c_t V_t + V_t^- s_t^f + V_t^+ s_t^i \\ &s.t. \quad V(T) = \psi(T) \end{aligned} \tag{1.15}$$

By the Feyman-Kac theorem, we can express the price of V_t as,

$$\begin{aligned} V_t = &\underbrace{E_t^{\mathbf{Q}}\left[e^{-\int_t^T c_s ds}\ \psi_T\right]}_{\text{Perfect Collat. price}} - \underbrace{\int_t^T E_t^{\mathbf{Q}}\left[e^{-\int_t^u c_s ds} s_u^f V_u^-\right] du}_{\text{Funding Cost}} \\ &- \underbrace{\int_t^T E_t^{\mathbf{Q}}\left[e^{-\int_t^u c_s ds} s_u^i V_u^+\right] du}_{\text{Funding Benefit}} \end{aligned} \tag{1.16}$$

Where $\mathbf{Q}$ is the risk neutral measure under which cash-collateralized deals grow at the collateral rate and S_t at the repo rate.

Just notice the *recursive* nature of (1.16) as in the funding cost/benefit adjustments the *full* derivative's price appears, including again the funding cost/benefit.

Notice that when $i_t = f_t = r_F(t)$ we can simplify (1.16) as

$$\boxed{V_t = E_t^{\mathbf{Q}}\left[e^{-\int_t^T r_F(u)du}\ \psi(T)\right]} \tag{1.17}$$

Conclusions:

- The FVA is either a cost or a benefit that arises by the need of funding the derivative along its life. This cost will be a benefit when the derivative generates a positive flow that the Hedger can use to reduce our funding needs.
- We can express the price of a non collateralized derivatives in term of the fully collateralized price plus an Add-on (FVA). This add-on depends on the derivative's price itself including FVA.
- Under absence of credit risk, the FVA term is the difference between the non collateralized price and the perfectly collateralized one.
- In order to price non-collateralized derivatives, the Hedger must discount them at his funding rate (assuming the funding and the investment rate are the same).

1.7 Pricing of partially collateralized derivatives.

Let us suppose we want to calculate the price of a derivative with the same terminal pay-off as the one in the previous section. Let us assume,

- The derivative is partially collateralized with a generic derivative B_t.
- The collateral can be re-hypotecated.
- There exists a liquid repo market for B_t with a standard haircut rate h_t and repo rate r_t^B.

Let us focus on the flows that would take place in an infinitesimal time interval, between t and $t + dt$, when hedging and funding the derivative, V_t.

- At time t, the Hedger will have to pay/receive the price of the derivative whether positive/negative. In exchange, the Hedger will receive/post some collateral, B_t, to face part of the derivative's value. At time $t+dt$, the Hedger will receive the value of the derivative as seen at $t + dt$ and the Hedger will set/get back the collateral posted at t.

- The Hedger will repo the collateral in the repo market to make some money to fund the derivative. The Hedger will deliver/receive the collateral in exchange for $\frac{B_t}{1+h_t}$ cash. At $t+dt$ the Hedger will get/give back the collateral plus the repo rate r_t^B.
- The Hedger still has to fund/invest $\left[V_t - \frac{B_t}{1+h_t}\right]$. In exchange, at $t+dt$ he will pay/receive his funding/investing rate on the amount borrowed/lent.
- The Hedger will hedge the exposure of V_t to S_t by entering into a cash-collateralized derivative. The Hedger will buy/sell α_t units of derivative H_t.

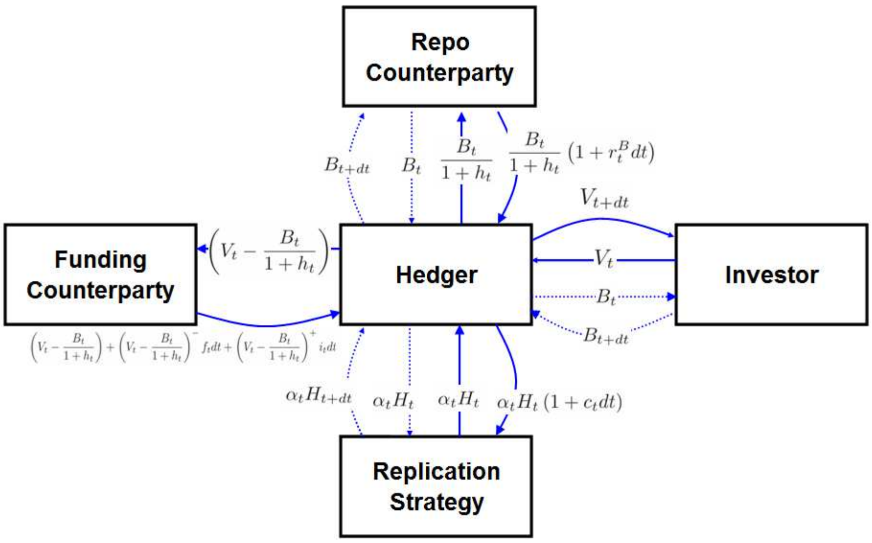


Figure 1.3: Partially Collateralized strategy P&L

If we denote by ϕ_t the gain process at time t, consequence of all the flows intervening in the hedging/funding of V_t, it can be easily seen that,

$$\begin{aligned} d\phi_t =& - dV_t + \alpha_t \left[dH_t - H_t c_t dt\right] + \frac{B_t}{(1+h_t)} r_t^B dt \\ &+ \left[V_t - \frac{B_t}{1+h_t}\right]^- f_t dt + \left[V_t - \frac{B_t}{1+h_t}\right]^+ i_t dt \end{aligned} \tag{1.18}$$

In order to eliminate the dependence on the underlying S_t we choose

$$\alpha_t = \frac{\frac{\partial V_t}{\partial S_t}}{\frac{\partial H_t}{\partial S_t}}$$

so the equation above becomes,

$$\frac{\partial V_t}{\partial t}+\frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}\sigma_t^2 S_t^2-\alpha_t\underbrace{\left[\frac{\partial H_t}{\partial t}+\frac{1}{2}\frac{\partial^2 H_t}{\partial S_t^2}\sigma_t^2 S_t^2-H_t c_t\right]}_{-(r_t-q_t)S_t\frac{\partial H_t}{\partial S_t}}$$
$$=\frac{B_t}{1+h_t}r_t^B+\left[V_t-\frac{B_t}{1+h_t}\right]^- f_t+\left[V_t-\frac{B_t}{1+h_t}\right]^+ i_t \tag{1.19}$$

By arranging terms,

$$\frac{\partial V_t}{\partial t}+(r_t-q_t)\,S_t\frac{\partial H_t}{\partial S_t}+\frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}\sigma_t^2 S_t^2$$
$$=\frac{B_t}{1+h_t}r_t^B+\left[V_t-\frac{B_t}{1+h_t}\right]^- f_t+\left[V_t-\frac{B_t}{1+h_t}\right]^+ i_t \tag{1.20}$$

If we define the funding spreads as spreads over the collateral rate,

$$s_t^f=f_t-c_t,\qquad s_t^i=i_t-c_t,\qquad s_t^B=r_t^B-c_t$$

We can express (1.20) as

$$\frac{\partial V_t}{\partial t}+(r_t-q_t)\,S_t\frac{\partial V_t}{\partial S_t}+\frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}\sigma_t^2 S_t^2$$
$$=V_t c_t+\frac{B_t}{1+h_t}s_t^B+\left[V_t-\frac{B_t}{1+h_t}\right]^- s_t^f+\left[V_t-\frac{B_t}{1+h_t}\right]^+ s_t^i \tag{1.21}$$

By applying Feyman-Kac theorem, we obtain that,

$$V_t=\underbrace{E_t^{\mathbf{Q}}\left[e^{-\int_t^T c_s ds}\ \psi(T)\right]}_{\text{Fully Collateralized price}}$$
$$-\underbrace{\int_t^T E_t^{\mathbf{Q}}\left[e^{-\int_t^u c_s ds}\ \left(V_t-\frac{B_t}{1+h_t}\right)^-\ s_u^f\right]du}_{\text{Funding Cost Adjustment}}$$
$$-\underbrace{\int_t^T E_t^{\mathbf{Q}}\left[e^{-\int_t^u c_s ds}\ \left(V_t-\frac{B_t}{1+h_t}\right)^+\ s_u^f\right]du}_{\text{Funding Benefit Adjustment}}$$
$$-\underbrace{\int_t^T E_t^{\mathbf{Q}}\left[e^{-\int_t^u c_s ds}\ \frac{B_t}{1+h_t}s_u^B\right]du}_{\text{Repo Adjustment}} \tag{1.22}$$

Just note, again, that **in the *FVA* terms appear the value of the derivative that also accounts itself for the FVA terms (recursive term)**

But we can remove such *recursivity* under one assumption ...!!

In order to remove the *recursive* term above, let us assume $f_t = i_t$, so we can re-express (1.20) as,

$$\begin{aligned} &\frac{\partial V_t}{\partial t} + (r_t - q_t)\, S_t \frac{\partial V_t}{\partial S_t} + \frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2}\sigma_t^2 S_t^2 = V_t f_t - \frac{B_t}{(1+h_t)}\left(f_t - r_t^B\right) \\ &s.t \qquad V_T = \psi(T) \end{aligned} \tag{1.23}$$

And by applying Feyman-Kac theorem, we obtain that,

$$\boxed{V_t = \underbrace{E_t^{\mathbf{Q}}\left[e^{-\int_t^T f_s ds}\ \psi(T)\right]}_{\text{Uncollateralized price}} + \underbrace{\int_t^T E_t^{\mathbf{Q}}\left[e^{-\int_t^u f_s ds}\ \frac{B_u}{(1+h_u)}\left(f_u - r_u^B\right)\right] du}_{\text{Collateral Adjustment}}} \tag{1.24}$$

That is, by assuming $f_t = i_t$ we have simplified the price of a partially collateralized derivative, removing the recursive dependence on the value of the derivative.

<u>Conclusions:</u>

- In partially collateralized derivatives the funding cost/benefit arises from the need to fund part of the derivative (the excess of the derivatives value over the collateral).
- In the case the collateral is an asset (no cash), we should incorporate an extra term as seen in equation (1.22). This term accounts for the *extra cost* of funding the non standard collateral over cash-collateral.
- The *recursive* nature is avoidable in the case of imperfect collateralization, by assuming no *bid-offer* in the Hedger's funding rate. By doing so, we can decompose the derivative's value into the price of the same non-collateralized derivative plus a collateral adjustment.

Chapter 2

The fundamental theorem of asset pricing revisited

2.1 Introduction

We have seen that in the new framework, for a given currency, each asset used as collateral implies a different accrual rate for funds posted as collateral.

In the absence of counterparty credit risk, the funding rate represents another accrual rate.

Therefore, there will be different current accounts that accrue at different rates with their corresponding discount factors.

Under the spot martingale measure $\mathbb{Q}$, each self financing portfolio has to be deflated by the current account that accrues at the rate at which the self financing portfolio is funded.

The fundamental theorem of asset pricing has to be revisited.

2.2 Model Assumptions

We assume that for a particular currency there are two different collateralization schemes (two different assets accepted as collateral). We will refer to one of them as the standard one and to the other as the non standard.

Notice that establishing what collateral scheme is the standard one is completely arbitrary, although for developed currencies (USD, GBP, CHF, JPY, EUR), standard collateral refers to cash denominated in the deal's currency.

In the remaining of the section, every equation will be in matrix form. Sometimes we will point out the dimensions of the different matrices involved. In the equations where this is done, variables with no indication are scalar variables (1×1 matrices).

$B(t,T)$ represents the discount factor curve used to discount cash flows collateralized under the standard scheme. Therefore, $B(t,T)$ represents the value at time t of receiving one currency unit at time T but collateralized with the standard collateral.

$$B(t,T) = \exp\left(-\int_{s=t}^{T} f(t,s)ds\right) \tag{2.1}$$

Where $f(t,T)$ is the instantaneous forward curve for the standard collateral.

Similarly $\widetilde{B}(t,T)$ represents the discount factor curve used to discount cash flows collateralized under the non standard scheme and $\widetilde{f}(t,T)$ its instantaneous forward curve.

$$\widetilde{B}(t,T) = \exp\left(-\int_{s=t}^{T} \widetilde{f}(t,s)ds\right) \tag{2.2}$$

So that $r_t = f(t,t)$ and $\widetilde{r}_t = \widetilde{f}(t,t)$ are the short term interest rates at which funds posted as collateral accrue under each collateralization scheme.

We assume that under the real world measure $\mathbb{P}$, the evolutions of $f(t,T)$ and $\widetilde{f}(t,T)$ follow

$$\begin{aligned} df(t,T) &= \mu^f(t,T)dt + \underbrace{\sigma^f(t,T)}_{1\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} \\ d\widetilde{f}(t,T) &= \mu^{\widetilde{f}}(t,T)dt + \underbrace{\sigma^{\widetilde{f}}(t,T)}_{1\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} + \underbrace{\widetilde{\sigma}^{\widetilde{f}}(t,T)}_{1\times m}\underbrace{dZ_t^{\mathbb{P}}}_{m\times 1} \end{aligned} \tag{2.3}$$

Where $W_t^{\mathbb{P}}$ and $Z_t^{\mathbb{P}}$ represent vectors of independent Wiener processes under $\mathbb{P}$ of dimensions n and m respectively. $\mu^f(t,T)$ and $\mu^{\widetilde{f}}(t,T)$ are real world drifts of the two processes and $\sigma^f(t,T)$, $\sigma^{\widetilde{f}}(t,T)$, $\widetilde{\sigma}^{\widetilde{f}}(t,T)$ their volatilities. $W_t^{\mathbb{P}}$ and $Z_t^{\mathbb{P}}$ are also independent of each other.

The evolutions of the current accounts that accrue at r_t and $\widetilde{r}_t$ are governed by the following differential equations:

$$\begin{aligned} dC_t &= r_t C_t dt \\ d\widetilde{C}_t &= \widetilde{r}_t \widetilde{C}_t dt \end{aligned} \tag{2.4}$$

In this section we will only analyze the effect of multiple discounting curves, letting aside the tenor basis [1]. Therefore we assume that the tenor basis is non stochastic.

[1] Basis due to different tenors of floating references.

In the following sections we will try to price derivatives with standard and non standard collateral. E_t will represent the time t value of a derivative with standard collateral and $\widetilde{E}_t$ the value of a derivative with non standard collateral. We will also assume that the cashflows of E_t only depend on interest rate indexes referenced to $B(t,T)$, therefore E_t will only depend on $W_t^{\mathbb{P}}$. On the other hand, $\widetilde{E}_t$ will depend on both $W_t^{\mathbb{P}}$ and $Z_t^{\mathbb{P}}$.

Hence, Itô's Lemma, together with (2.3) imply

$$
\begin{aligned}
dE_t &= \mu_t^E dt + \underbrace{\sigma_t^E}_{1\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} \\
d\widetilde{E}_t &= \mu_t^{\widetilde{E}} dt + \underbrace{\sigma_t^{\widetilde{E}}}_{1\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} + \underbrace{\widetilde{\sigma}_t^{\widetilde{E}}}_{1\times m}\underbrace{dZ_t^{\mathbb{P}}}_{m\times 1}
\end{aligned}
\tag{2.5}
$$

μ_t^E and $\mu_t^{\widetilde{E}}$ are the real world drifts of both processes and σ_t^E, $\sigma_t^{\widetilde{E}}$ and $\widetilde{\sigma}_t^{\widetilde{E}}$ their volatilities.

In order to replicate E_t, we will use a set of n interest rate derivatives collateralized under the standard scheme and whose cashflows only depend on $B(t,T)$. H_t will be a $n \times 1$ vector representing the prices at time t of these. The stochastic differential equation followed by H_t under the real world measure will be given by:

$$
\underbrace{dH_t}_{n\times 1} = \underbrace{\mu_t^H}_{n\times 1} dt + \underbrace{\sigma_t^H}_{n\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1}
\tag{2.6}
$$

Where the size of the different matrices has been pointed out.

In order to replicate $\widetilde{E}_t$ we will use H_t plus m additional instruments collateralized under the non standard collateral [2] due to the dependence of $\widetilde{E}_t$ on $\widetilde{B}(t,T)$. $\widetilde{H}_t$ represents the t price of this set of additional hedging instruments. The stochastic differential equation followed by $\widetilde{H}_t$ under the real world measure will be given by:

$$
\underbrace{d\widetilde{H}_t}_{m\times 1} = \underbrace{\mu_t^{\widetilde{H}}}_{m\times 1} dt + \underbrace{\sigma_t^{\widetilde{H}}}_{m\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} + \underbrace{\widetilde{\sigma}_t^{\widetilde{H}}}_{m\times m}\underbrace{dZ_t^{\mathbb{P}}}_{m\times 1}
\tag{2.7}
$$

We would like to point out that μ_t^E, $\mu_t^{\widetilde{E}}$, μ_t^H and $\mu_t^{\widetilde{H}}$ are real world drifts.

[2] We could have assumed that $\widetilde{E}_t$ is hedged with $n+m$ derivatives collateralized with non standard collateral obtaining the same conclusions. We have chosen this alternative for didactic reasons.

2.3 Valuing derivatives under the spot martingale measure

In this section we will deal with the valuation of both E_t and $\widetilde{E}_t$ under the spot martingale measure, that is the measure associated with current accounts as numeraire.

2.3.1 Derivatives with standard collateral

The hedging formula will be the following.

$$E_t = \underbrace{\alpha_t}_{1\times n}\underbrace{H_t}_{n\times 1} + C_t \tag{2.8}$$

C_t represents funds posted as collateral by the hedger [3].

The risk hedger trades $\alpha_t H_t$ with interbank counterparties paying its value and receiving it as collateral from the same interbank counterparties. H_t represents the value of the hedging instruments from the hedger's perspective, α_t is a vector that contains the amounts to invest in each one of the components of H_t in order to hedge the risks of E_t. E_t represents the value of the derivative to be replicated from the risk taker's perspective (which implies that the value from the risk hedger's perspective is $-E_t$).

Taking into account the stochastic differential equations followed by E_t and H_t, the replication equation in differential form will be given by

$$\mu_t^E dt + \sigma_t^E dW_t^{\mathbb{P}} - E_t r_t dt = \alpha_t \left(\mu_t^H dt + \sigma_t^H dW_t^{\mathbb{P}} - H_t r_t dt\right) \tag{2.9}$$

Where we have taken into account that fact that C_t accrues at r_t and that $C_t = E_t - \alpha_t H_t$

In order to be hedged α_t must be chosen so that the terms in $dW_t^{\mathbb{P}}$ in both sides of (2.8) are canceled. For this to happen, α_t must be the solution of the following system of linear equations:

$$\sigma_t^E = \alpha_t \sigma_t^H \tag{2.10}$$

So that the real world drifts must follow:

$$\mu_t^E - E_t r_t = \alpha_t \left(\mu_t^H - H_t r_t\right) \tag{2.11}$$

[3] If $E_t > 0$, the hedger receives E_t from the risk taker and posts it as collateral (and the opposite if $E_t \leq 0$) to the risk taker

Being in a complete market [4] together with the absence of arbitrage opportunities [5] implies both (2.10) and (2.11).

On the other hand, Girsanov theorem guarantees that when we perform a change of measure from real world measure $\mathbb{P}$ to an equivalent measure $\mathbb{Q}$, $\mathbb{P}$ and $\mathbb{Q}$ Wiener processes are related through

$$\begin{aligned} \underbrace{dW_t^{\mathbb{Q}}}_{n\times 1} &= \underbrace{dW_t^{\mathbb{P}}}_{n\times 1} - \underbrace{\gamma_t}_{n\times 1} dt \\ \underbrace{dZ_t^{\mathbb{Q}}}_{m\times 1} &= \underbrace{dZ_t^{\mathbb{P}}}_{m\times 1} - \underbrace{\widetilde{\gamma}_t}_{m\times 1} dt \end{aligned} \tag{2.12}$$

Where γ_t and $\widetilde{\gamma}_t$ are non anticipative processes of dimensions n and m that describe the change of measure.

Girsanov Theorem also implies that under $\mathbb{Q}$ the drift of H_t will be given by

$$\underbrace{\mu_t^H}_{n\times 1} - \underbrace{\sigma_t^H}_{n\times n}\underbrace{\gamma_t}_{n\times 1}$$

If we wanted to change to a measure $\mathbb{Q}$ where the drift of H_t was given by $H_t r_t$, γ_t will be the solution to:

$$\underbrace{\mu_t^H}_{n\times 1} - \underbrace{\sigma_t^H}_{n\times n}\underbrace{\gamma_t}_{n\times 1} = \underbrace{H_t}_{n\times 1} r_t \tag{2.13}$$

Up to this point we will have no condition for $\widetilde{\gamma}_t$, although it will be revealed in the next subsection.

Now we will explore what the drift of E_t is under $\mathbb{Q}$. Girsanov theorem implies:

$$\underbrace{\mu_t^E}_{1\times 1} - \underbrace{\sigma_t^E}_{1\times n}\underbrace{\gamma_t}_{n\times 1} \tag{2.14}$$

Plugging (2.10) and (2.11) into (2.14) and taking into account (2.13)

$$\underbrace{\mu_t^E}_{1\times 1} - \underbrace{\sigma_t^E}_{1\times n}\underbrace{\gamma_t}_{n\times 1} = \alpha_t\left(\mu_t^H - H_t r_t\right) + E_t r_t - \alpha_t \sigma_t^H \gamma_t = E_t r_t \tag{2.15}$$

So that under $\mathbb{Q}$ any interest rate derivative with standard collateral follows

$$dE_t = E_t r_t dt + \sigma_t^E dW_t^{\mathbb{Q}}$$

Which implies that

[4] The value of a derivative can be replicated with a set of hedging instruments.

[5] The value of a derivative must coincide with the value of the replicating portfolio.

$$E_t = E_{\mathbb{Q}}\left[E_T \exp\left(-\int_{s=t}^{T} r_s ds\right)\Big|\mathcal{F}_t\right] \;\Rightarrow\; \frac{E_t}{\beta_t} = E_{\mathbb{Q}}\left[\frac{E_T}{\beta_T}\Big|\mathcal{F}_t\right]$$

Where $\beta_T = \exp\left(\int_{s=0}^{T} r_s ds\right)$ represents the current account that accrues at the standard collateral rate r_t.

Notice that nothing new has been obtained in this section. We have just confirmed the fundamental theorem of asset pricing in a collateralization framework as was already obtained in [3].

2.3.2 Derivatives with non standard collateral

In this case, the hedging equation will be

$$\widetilde{E}_t = \underbrace{\alpha_t}_{1\times n}\underbrace{H_t}_{n\times 1} + \underbrace{\epsilon_t}_{1\times m}\underbrace{\widetilde{H}_t}_{m\times 1} + C_t + \widetilde{C}_t \tag{2.16}$$

H_t and $\widetilde{H}_t$ are the values of the hedging instruments from the hedger's perspective. $-\widetilde{E}_t$ is the value of the derivative to be replicated also from the hedger's point of view. α_t and ϵ_t are the amounts to invest in each component of H_t and $\widetilde{H}_t$ respectively. C_t and $\widetilde{C}_t$ represent amounts posted as collateral by the hedger in the standard and non standard collateralization schemes respectively.

Notice that due to the fact that only H_t is collateralized under the standard scheme and both $\widetilde{H}_t$ and $\widetilde{E}_t$ under the non standard the following must hold:

$$\begin{aligned} C_t &= -\alpha_t H_t \\ \widetilde{C}_t &= \widetilde{E}_t - \epsilon_t \widetilde{H}_t \end{aligned} \tag{2.17}$$

The hedging equation under $\mathbb{P}$ in differential form will be:

$$\begin{aligned} \mu_t^{\widetilde{E}} dt + \sigma_t^{\widetilde{E}} dW_t^{\mathbb{P}} + \widetilde{\sigma}_t^{\widetilde{E}} dZ_t^{\mathbb{P}} - E_t \widetilde{r}_t = {} & \alpha_t \left(\mu_t^H dt + \sigma_t^H dW_t^{\mathbb{P}} - H_t r_t dt\right) \\ & + \epsilon_t \left(\mu_t^{\widetilde{H}} dt + \sigma_t^{\widetilde{H}} dW_t^{\mathbb{P}} + \widetilde{\sigma}_t^{\widetilde{H}} dZ_t^{\mathbb{P}} - \widetilde{H}_t \widetilde{r}_t dt\right) \end{aligned} \tag{2.18}$$

In order to be hedged, terms in $dW_t^{\mathbb{P}}$ and $dZ_t^{\mathbb{P}}$ in (2.18) should be canceled. Therefore α_t and ϵ_t must be the solution to the following system of linear equations:

$$\begin{aligned} \sigma_t^{\widetilde{E}} &= \alpha_t \sigma_t^H + \epsilon_t \sigma_t^{\widetilde{H}} \\ \widetilde{\sigma}_t^{\widetilde{E}} &= \epsilon_t \widetilde{\sigma}_t^{\widetilde{H}} \end{aligned} \tag{2.19}$$

So that the condition followed by the drifts under the real world measure is

$$\mu_t^{\widetilde{E}} - \widetilde{E}_t\widetilde{r}_t = \alpha_t \left(\mu_t^H - H_t r_t\right) + \epsilon_t \left(\mu_t^{\widetilde{H}} - \widetilde{H}_t\widetilde{r}_t\right) \tag{2.20}$$

In the previous section we imposed a change of measure from real world measure $\mathbb{P}$ to the spot martingale measure $\mathbb{Q}$ by imposing that the $\mathbb{Q}$ drift of H_t becomes $H_t r_t$.

In this section we are analyzing the hedge of $\widetilde{E}_t$, which carries a collateralization scheme different from the standard one. Since we find ourselves in unexplored territory, let's leave the drift of $\widetilde{H}_t$ under $\mathbb{Q}$ as $\widetilde{H}_t z_t$, where z_t will be determined thereon. So that once z_t is known, $\widetilde{\gamma}_t$ will be given by the solution to the following system of linear equations (notice that γ_t was obtained in the last subsection):

$$\underbrace{\mu_t^{\widetilde{H}}}_{m\times 1} - \underbrace{\sigma_t^{\widetilde{H}}}_{m\times n}\underbrace{\gamma_t}_{n\times 1} - \underbrace{\widetilde{\sigma}_t^{\widetilde{H}}}_{m\times m}\underbrace{\widetilde{\gamma}_t}_{m\times 1} = \underbrace{\widetilde{H}_t}_{m\times 1}\underbrace{z_t}_{1\times 1} \tag{2.21}$$

So that the change of measure performed on $\widetilde{E}_t$ implies a new drift that is equal to

$$\mu_t^{\widetilde{E}} - \sigma_t^{\widetilde{E}}\gamma_t - \widetilde{\sigma}_t^{\widetilde{E}}\widetilde{\gamma}_t \tag{2.22}$$

Plugging (2.19) and (2.20) into (2.22) and taking into account both (2.13) and (2.21) imply

$$\begin{aligned} \mu_t^{\widetilde{E}} - \sigma_t^{\widetilde{E}}\gamma_t - \widetilde{\sigma}_t^{\widetilde{E}}\widetilde{\gamma}_t &= \widetilde{E}_t\widetilde{r}_t + \alpha_t\left(\mu_t^H - H_t r_t\right) + \epsilon_t\left(\mu_t^{\widetilde{H}} - \widetilde{H}_t\widetilde{r}_t\right) \\ &\quad - \left(\alpha_t\sigma_t^H + \epsilon_t\sigma_t^{\widetilde{H}}\right)\gamma_t - \epsilon_t\widetilde{\sigma}_t^{\widetilde{H}}\widetilde{\gamma}_t \\ &= \widetilde{E}_t\widetilde{r}_t + \epsilon_t\widetilde{H}_t\left(z_t - \widetilde{r}_t\right) \end{aligned} \tag{2.23}$$

Notice that if $z_t = \widetilde{r}_t$ the drift of $\widetilde{E}_t$ becomes $\widetilde{E}_t\widetilde{r}_t$. Any other value of z_t will imply a drift of $\widetilde{E}_t$ under $\mathbb{Q}$ that depends on the particular characteristics of the contract being replicated (which are reflected in ϵ_t) and is therefore useless from a pricing perspective. Hence, under $\mathbb{Q}$ the growth rate of every derivative with standard collateral (either H_t or E_t) becomes r_t and the growth rate of any derivative with non standard collateral (either $\widetilde{H}_t$ or $\widetilde{E}_t$) becomes $\widetilde{r}_t$.

$$dE_t = E_t r_t dt + \sigma_t^E dW_t^{\mathbb{Q}}$$
$$d\widetilde{E}_t = \widetilde{E}_t\widetilde{r}_t dt + \sigma_t^{\widetilde{E}} dW_t^{\mathbb{Q}} + \widetilde{\sigma}_t^{\widetilde{E}} dZ_t^{\mathbb{Q}}$$

That are equivalent to

$$\begin{aligned} E_t &= E_{\mathbb{Q}}\left[E_T \exp\left(-\int_{s=t}^T r_s ds\right)\Big|\mathcal{F}_t\right] \;\Rightarrow\; \tfrac{E_t}{\beta_t} = E_{\mathbb{Q}}\left[\tfrac{E_T}{\beta_T}\Big|\mathcal{F}_t\right] \\ \widetilde{E}_t &= E_{\mathbb{Q}}\left[\widetilde{E}_T \exp\left(-\int_{s=t}^T \widetilde{r}_s ds\right)\Big|\mathcal{F}_t\right] \;\Rightarrow\; \tfrac{\widetilde{E}_t}{\widetilde{\beta}_t} = E_{\mathbb{Q}}\left[\tfrac{\widetilde{E}_T}{\widetilde{\beta}_T}\Big|\mathcal{F}_t\right] \end{aligned} \tag{2.24}$$

Where $\beta_T = \exp\left(\int_{s=0}^T r_s ds\right)$ represents the current account that accrues at the standard collateral rate r_t and $\widetilde{\beta}_T = \exp\left(\int_{s=0}^T \widetilde{r}_s ds\right)$ represents the current account that accrues at the non standard collateral rate $\widetilde{r}_t$.

Notice that under measure $\mathbb{Q}$ there seems to be two different numeraires: the standard collateral current account β_t used to deflate derivatives with standard collateral and the non standard collateral current account $\widetilde{\beta}_t$ used to deflate derivatives with non standard collateral. This result was obtained, for example, in [4].

We could also have written

$$\begin{aligned} \tfrac{\widetilde{E}_t}{\beta_t} &= E_{\mathbb{Q}}\left[\exp\left(\int_{s=t}^T (r_s - \widetilde{r}_s)\, ds\right) \tfrac{\widetilde{E}_T}{\beta_T}\Big|\mathcal{F}_t\right] \\ \tfrac{E_t}{\widetilde{\beta}_t} &= E_{\mathbb{Q}}\left[\exp\left(\int_{s=t}^T (\widetilde{r}_s - r_s)\, ds\right) \tfrac{E_T}{\widetilde{\beta}_T}\Big|\mathcal{F}_t\right] \end{aligned} \tag{2.25}$$

This last expression will be analyzed in subsection 2.4.4.

In the next section we will generalize the results obtained so far to a numeraire different from current accounts (such as discount factors, annuities...)

2.4 Change of numeraire

In this section we assume that we use as numeraire a derivative with standard collateral whose cash flows are referenced to the curve $B(t, T)$. Therefore we will assume that any of the components of H_t whose value cannot vanish is used as numeraire, so that under the real world measure $\mathbb{P}$ the evolution of the numeraire N_t will be governed by

$$dN_t = \mu_t^N N_t dt + N_t \underbrace{\sigma_t^N}_{1\times n} \underbrace{dW_t^{\mathbb{P}}}_{n\times 1}$$

N_t could, for example, be annuities or discount factors collateralized under the standard scheme.

2.4.1 Derivatives with standard collateral agreement

Again, the hedging equation will be

$$E_t = \underbrace{\alpha_t}_{1\times n}\underbrace{H_t}_{n\times 1} + C_t \tag{2.26}$$

We divide every term by the numeraire N_t, so that we define

$$\begin{aligned} e_t &:= \tfrac{E_t}{N_t} \\ \underbrace{h_t}_{n\times 1} &:= \underbrace{H_t}_{n\times 1}\underbrace{\frac{1}{N_t}}_{1\times 1} \\ c_t &:= \tfrac{C_t}{N_t} \end{aligned} \tag{2.27}$$

So that the hedging equation, once every term has been divided by the numeraire, is

$$e_t = \alpha_t h_t + c_t \tag{2.28}$$

And in differential form

$$\mu_t^e dt + \underbrace{\sigma_t^e}_{1\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} = \underbrace{\alpha_t}_{1\times n}\left(\underbrace{\mu_t^h}_{n\times 1} dt + \underbrace{\sigma_t^h}_{n\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1}\right) + \mu_t^c dt + \underbrace{\sigma_t^c}_{1\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} \tag{2.29}$$

μ_t^e, μ_t^h and μ_t^c are the $\mathbb{P}$ drifts of the deflated processes and σ_t^e, σ_t^h and σ_t^c their volatilities.

Notice that c_t has a diffusion different from 0 since C_t has been divided by a numeraire with non zero diffusion.

In order to be hedged, α_t must be the solution to

$$\sigma_t^e = \alpha_t \sigma_t^h + \sigma_t^c \tag{2.30}$$

So that the real world drifts must follow in this complete market / no arbitrage environment

$$\mu_t^e = \alpha_t \mu_t^h + \mu_t^c \tag{2.31}$$

Let's now apply a change of measure from $\mathbb{P}$ to an equivalent martingale measure $\mathbb{N}$ associated with N_t that vanishes the drift of every component of h_t

$$\begin{aligned} \underbrace{\mu_t^h}_{n\times 1} - \underbrace{\sigma_t^h}_{n\times n}\underbrace{\gamma_t}_{n\times 1} &= \underbrace{0}_{n\times 1} \\ \underbrace{\mu_t^c}_{1\times 1} - \underbrace{\sigma_t^c}_{1\times n}\underbrace{\gamma_t}_{n\times 1} &= \underbrace{0}_{1\times 1} \end{aligned} \tag{2.32}$$

Notice that the first equation in (2.32) will not be enough to determine γ_t, since N_t will be a component of H_t, so that $\frac{N_t}{N_t}$ will have null drift under every measure. We must also impose that the current account that accrues at the collateral rate r_t divided by the numeraire has also zero drift. This is reflected in the second equation in (2.32), so that both expressions help us determine γ_t.

The drift of e_t under $\mathbb{N}$ will be given by

$$\mu_t^e - \sigma_t^e \gamma_t \tag{2.33}$$

Plugging (2.30) and (2.31) into (2.33) and taking into account (2.32) implies

$$\mu_t^e - \sigma_t^e \gamma_t = \alpha_t \mu_t^h + \mu_t^c - \left(\alpha_t \sigma_t^h + \sigma_t^c\right) \gamma_t = 0 \tag{2.34}$$

So that μ_t^e has also zero drift under $\mathbb{N}$. This implies that

$$\frac{E_t}{N_t} = E_{\mathbb{N}} \left[\frac{E_T}{N_T} \middle| \mathcal{F}_t \right] \tag{2.35}$$

Notice that in this subsection we have just confirmed the change of numeraire result in a collateralization framework. In the next subsection we analyze the effect of the change of measure introduced in this section in derivatives collateralized with the non standard collateral.

2.4.2 Derivatives with non standard collateral agreement

The hedging equation will be given by

$$\widetilde{E}_t = \underbrace{\alpha_t}_{1\times n} \underbrace{H_t}_{n\times 1} + \underbrace{\epsilon_t}_{1\times m} \underbrace{\widetilde{H}_t}_{m\times 1} + C_t + \widetilde{C}_t \tag{2.36}$$

We divide every component in (2.36) by N_t, so that we define the following terms

$$\begin{aligned} \widetilde{e}_t &:= \tfrac{\widetilde{E}_t}{N_t} \\ \underbrace{h_t}_{n\times 1} &:= \underbrace{H_t}_{n\times 1} \underbrace{\frac{1}{N_t}}_{1\times 1} \\ \underbrace{\widetilde{h}_t}_{m\times 1} &:= \underbrace{\widetilde{H}_t}_{m\times 1} \underbrace{\frac{1}{N_t}}_{1\times 1} \\ c_t &:= \tfrac{C_t}{N_t} \\ \widetilde{c}_t &:= \tfrac{\widetilde{C}_t}{N_t} \end{aligned} \tag{2.37}$$

So that once it has been divided by N_t, the hedging equation becomes

$$\widetilde{e}_t = \alpha_t h_t + \epsilon_t \widetilde{h}_t + c_t + \widetilde{c}_t \tag{2.38}$$

And in differential form

$$\begin{aligned}
\mu_t^{\widetilde{e}} dt + \underbrace{\sigma_t^{\widetilde{e}}}_{1\times n} \underbrace{dW_t^{\mathbb{P}}}_{n\times 1} + \underbrace{\widetilde{\sigma}_t^{\widetilde{e}}}_{1\times m} \underbrace{dZ_t^{\mathbb{P}}}_{m\times 1} = \underbrace{\alpha_t}_{1\times n} &\left(\underbrace{\mu_t^h}_{n\times 1} dt + \underbrace{\sigma_t^h}_{n\times n} \underbrace{dW_t^{\mathbb{P}}}_{n\times 1} \right) \\
&+ \underbrace{\epsilon_t}_{1\times m} \left(\underbrace{\mu_t^{\widetilde{h}}}_{m\times 1} dt + \underbrace{\sigma_t^{\widetilde{h}}}_{m\times n} \underbrace{dW_t^{\mathbb{P}}}_{n\times 1} + \underbrace{\widetilde{\sigma}_t^{\widetilde{h}}}_{m\times m} \underbrace{dZ_t^{\mathbb{P}}}_{m\times 1} \right) \\
&+ \mu_t^c dt + \underbrace{\sigma_t^c}_{1\times n} \underbrace{dW_t^{\mathbb{P}}}_{n\times 1} + \mu_t^{\widetilde{c}} dt + \underbrace{\sigma_t^{\widetilde{c}}}_{1\times n} \underbrace{dW_t^{\mathbb{P}}}_{n\times 1}
\end{aligned}$$

Again, the drifts in the last equation are the real world measure drifts of the deflated processes. Notice that both c_t and $\widetilde{c}_t$ have non zero diffusions. Also notice that due to the fact that N_t solely depends on $W_t^{\mathbb{P}}$, neither c_t nore $\widetilde{c}_t$ depend on $Z_t^{\mathbb{P}}$.

In order to be hedged, α_t and ϵ_t must be obtained from

$$\begin{aligned}
\sigma_t^{\widetilde{e}} &= \alpha_t \sigma_t^h + \epsilon_t \sigma_t^{\widetilde{h}} + \sigma_t^c + \sigma_t^{\widetilde{c}} \\
\widetilde{\sigma}_t^{\widetilde{e}} &= \epsilon_t \widetilde{\sigma}_t^{\widetilde{h}}
\end{aligned} \tag{2.39}$$

So that terms in $dW_t^{\mathbb{P}}$ and $dZ_t^{\mathbb{P}}$ are canceled, which yields a relationship between the real world drifts

$$\mu_t^{\widetilde{e}} = \alpha_t \mu_t^h + \epsilon_t \mu_t^{\widetilde{h}} + \mu_t^c + \mu_t^{\widetilde{c}} \tag{2.40}$$

Now let's assume that we perform the same change of measure that was discussed in the last subsection and that produced zero drifts for both h_t and e_t.

Since we are again in an unexplored territory, due to the fact that both $\widetilde{E}_t$ and $\widetilde{H}_t$ are collateralized with the non standard collateral, we assume that $\mathbb{N}$ implies a drift of $\widetilde{h}_t z_t$ in $\widetilde{h}_t$, where z_t will again be determined thereon.

$$\mu_t^{\widetilde{h}} - \sigma_t^{\widetilde{h}} \gamma_t - \widetilde{\sigma}_t^{\widetilde{h}} \widetilde{\gamma}_t = \widetilde{h}_t z_t \tag{2.41}$$

Notice that (2.41) will help us determine $\widetilde{\gamma}_t$ once z_t is known (γ_t has already been determined in subsection 2.4.1).

Let's analyze the relationship between the drifts of c_t and $\widetilde{c}_t$ under $\mathbb{N}$. If we apply Itô's Lemma to c_t under $\mathbb{N}$

$$c_t = \frac{C_t}{N_t} \Rightarrow dc_t = c_t \left(r_t dt - \mu_t^{N,\mathbb{N}} dt - \sigma_t^N dW_t^{\mathbb{N}} + (\sigma_t^N)^2 dt \right) \tag{2.42}$$

Where $\mu_t^{N,\mathbb{N}}$ is the $\mathbb{N}$ drift of N_t.
Doing the same to $\widetilde{c}_t$

$$\widetilde{c}_t = \frac{\widetilde{C}_t}{N_t} \Rightarrow d\widetilde{c}_t = \widetilde{c}_t \left(\widetilde{r}_t dt - \mu_t^{N,\mathbb{N}} dt - \sigma_t^N dW_t^{\mathbb{N}} + (\sigma_t^N)^2 dt \right) \tag{2.43}$$

So that if $\mu_t^{c,\mathbb{N}} = 0$ (as imposed in 2.4.1), $\mu_t^{\widetilde{c},\mathbb{N}}$ will be given by

$$\mu_t^{c,\mathbb{N}} = 0 \Rightarrow \mu_t^{N,\mathbb{N}} = r_t + (\sigma_t^N) \Rightarrow \mu_t^{\widetilde{c},\mathbb{N}} = \widetilde{c}_t \left(\widetilde{r}_t - r_t \right) \tag{2.44}$$

If we apply Girsanov's theorem to $\widetilde{e}_t$, its drift under $\mathbb{N}$ is given by

$$\mu_t^{\widetilde{e}} - \sigma_t^{\widetilde{e}} \gamma_t - \widetilde{\sigma}_t^{\widetilde{e}} \widetilde{\gamma}_t \tag{2.45}$$

Plugging (2.39) and (2.40) in the last equation and taking into account (2.32) and (2.41)

$$\begin{aligned} \mu_t^{\widetilde{e}} - \sigma_t^{\widetilde{e}} \gamma_t - \widetilde{\sigma}_t^{\widetilde{e}} \widetilde{\gamma}_t &= \alpha_t \mu_t^h + \epsilon_t \mu_t^{\widetilde{h}} + \mu_t^c + \mu_t^{\widetilde{c}} \\ &\quad - \left(\alpha_t \sigma_t^h + \epsilon_t \sigma_t^{\widetilde{h}} + \sigma_t^c + \sigma_t^{\widetilde{c}} \right) \gamma_t - \epsilon_t \widetilde{\sigma}_t^{\widetilde{h}} \widetilde{\gamma}_t \\ &= \underbrace{\mu_t^{\widetilde{c}} - \sigma_t^{\widetilde{c}} \gamma_t}_{\mu_t^{\widetilde{c},\mathbb{N}}} + \epsilon_t \widetilde{h}_t z_t \end{aligned} \tag{2.46}$$

And taking into account (2.44)

$$\mu_t^{\widetilde{e}} - \sigma_t^{\widetilde{e}} \gamma_t - \widetilde{\sigma}_t^{\widetilde{e}} \widetilde{\gamma}_t = \widetilde{c}_t \left(\widetilde{r}_t - r_t \right) + \epsilon_t \widetilde{h}_t z_t \tag{2.47}$$

Since $\widetilde{C}_t = \widetilde{E}_t - \epsilon_t \widetilde{H}_t$, then $\widetilde{c}_t = \widetilde{e}_t - \epsilon_t \widetilde{h}_t$, so that

$$\mu_t^{\widetilde{e}} - \sigma_t^{\widetilde{e}} \gamma_t - \widetilde{\sigma}_t^{\widetilde{e}} \widetilde{\gamma}_t = \widetilde{e}_r \left(\widetilde{r}_t - r_t \right) + \epsilon_t \widetilde{h}_t \left(z_t - \left(\widetilde{r}_t - r_t \right) \right) \tag{2.48}$$

Notice that unless $z_t = \widetilde{r}_t - r_t$, the drift of $\widetilde{e}_t$ would depend on the particular characteristics of $\widetilde{E}_t$, so that the only valid drift for valuation purposes would be

$$\mu_t^{\widetilde{e}} - \sigma_t^{\widetilde{e}} \gamma_t - \widetilde{\sigma}_t^{\widetilde{e}} \widetilde{\gamma}_t = \widetilde{e}_r \left(\widetilde{r}_t - r_t \right) \tag{2.49}$$

Which implies

$$\frac{\widetilde{E}_t}{N_t} = E_{\mathbb{N}} \left[\exp \left(\int_{s=t}^{T} \left(r_s - \widetilde{r}_s \right) ds \right) \frac{\widetilde{E}_T}{N_T} \Big| \mathcal{F}_t \right] \tag{2.50}$$

2.4.3 Using numeraires with non standard collateral

In subsections 2.4.1 and 2.4.2 we chose as numeraire a derivative whose payments depend on the standard collateral discounting curve $B(t,T)$, that was collateralized with the standard collateral and whose price cannot vanish.

Notice that if we had assumed that the numeraire is collateralized under the non standard scheme and that its cashflows just depended on $\widetilde{B}(t,T)$, the situation would be exactly symmetrical as the one analyzed in 2.4.1 and 2.4.2, so that we would have obtained:

$$\frac{\widetilde{E}_t}{\widetilde{N}_t} = E_{\widetilde{\mathbb{N}}}\left[\frac{\widetilde{E}_T}{\widetilde{N}_T}\Big|\mathcal{F}_t\right] \tag{2.51}$$

$$\frac{E_t}{\widetilde{N}_t} = E_{\widetilde{\mathbb{N}}}\left[\exp\left(\int_{s=t}^{T}(\widetilde{r}_s - r_s)\,ds\right)\frac{E_T}{\widetilde{N}_T}\Big|\mathcal{F}_t\right] \tag{2.52}$$

2.4.4 The zero vol FX analogy

So far, we have obtained the following

$$\begin{aligned}
&\tfrac{E_t}{\beta_t} = E_{\mathbb{Q}}\left[\tfrac{E_T}{\beta_T}\Big|\mathcal{F}_t\right] && \tfrac{\widetilde{E}_t}{\beta_t} = E_{\mathbb{Q}}\left[\exp\left(\textstyle\int_{s=t}^{T}(r_s - \widetilde{r}_s)\,ds\right)\tfrac{\widetilde{E}_T}{\beta_T}\Big|\mathcal{F}_t\right]\\
&\tfrac{\widetilde{E}_t}{\widetilde{\beta}_t} = E_{\mathbb{Q}}\left[\tfrac{\widetilde{E}_T}{\widetilde{\beta}_T}\Big|\mathcal{F}_t\right] && \tfrac{E_t}{\widetilde{\beta}_t} = E_{\mathbb{Q}}\left[\exp\left(\textstyle\int_{s=t}^{T}(\widetilde{r}_s - r_s)\,ds\right)\tfrac{E_T}{\widetilde{\beta}_T}\Big|\mathcal{F}_t\right]\\
&\tfrac{E_t}{N_t} = E_{\mathbb{N}}\left[\tfrac{E_T}{N_T}\Big|\mathcal{F}_t\right] && \tfrac{\widetilde{E}_t}{N_t} = E_{\mathbb{N}}\left[\exp\left(\textstyle\int_{s=t}^{T}(r_s - \widetilde{r}_s)\,ds\right)\tfrac{\widetilde{E}_T}{N_T}\Big|\mathcal{F}_t\right]\\
&\tfrac{\widetilde{E}_t}{\widetilde{N}_t} = E_{\widetilde{\mathbb{N}}}\left[\tfrac{\widetilde{E}_T}{\widetilde{N}_T}\Big|\mathcal{F}_t\right] && \tfrac{E_t}{\widetilde{N}_t} = E_{\widetilde{\mathbb{N}}}\left[\exp\left(\textstyle\int_{s=t}^{T}(\widetilde{r}_s - r_s)\,ds\right)\tfrac{E_T}{\widetilde{N}_T}\Big|\mathcal{F}_t\right]
\end{aligned} \tag{2.53}$$

Notice that the expressions in (2.53) would appear in a cross currency setting where deals with standard collateral were denominated in the local currency and deals with non standard collateral in a foreign currency, such that r_t is the domestic short rate, $\widetilde{r}_t$ the foreign short rate and the spot FX rate ζ_t expressed in D/F followed under any measure [6] the following stochastic differential equation:

$$d\zeta_t = (r_t - \widetilde{r}_t)\,\zeta_t dt \Rightarrow \zeta_T = \zeta_t \exp\left(\int_{s=t}^{T}(r_s - \widetilde{r}_s)\,ds\right)$$

In such a framework, the change of measure between the two spot martingale measures $\mathbb{Q}$ (domestic) and $\widetilde{\mathbb{Q}}$ (foreign) would be innocuous, since the Radon-Nikodym derivative would be given by:

[6] Having zero diffusion implies no drift change due to a measure change.

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}(t,T) = \frac{\widetilde{\beta}_T \zeta_T}{\beta_T} \frac{\beta_t}{\widetilde{\beta}_t \zeta_t} = \frac{\widetilde{\beta}_T \beta_t}{\widetilde{\beta}_t \beta_T} \exp\left(\int_{s=t}^{T} (r_s - \widetilde{r}_s)\, ds\right) = 1$$

So that we could rewrite (2.53)

$$\begin{aligned}
\tfrac{E_t}{\beta_t} &= E_{\mathbb{Q}}\left[\tfrac{E_T}{\beta_T}\Big|\mathcal{F}_t\right] & \tfrac{\widetilde{E}_t}{\beta_t} &= E_{\mathbb{Q}}\left[\exp\left(\textstyle\int_{s=t}^{T} (r_s - \widetilde{r}_s)\, ds\right) \tfrac{\widetilde{E}_T}{\beta_T}\Big|\mathcal{F}_t\right] \\
\tfrac{\widetilde{E}_t}{\widetilde{\beta}_t} &= E_{\widetilde{\mathbb{Q}}}\left[\tfrac{\widetilde{E}_T}{\widetilde{\beta}_T}\Big|\mathcal{F}_t\right] & \tfrac{E_t}{\widetilde{\beta}_t} &= E_{\widetilde{\mathbb{Q}}}\left[\exp\left(\textstyle\int_{s=t}^{T} (\widetilde{r}_s - r_s)\, ds\right) \tfrac{E_T}{\widetilde{\beta}_T}\Big|\mathcal{F}_t\right] \\
\tfrac{E_t}{N_t} &= E_{\mathbb{N}}\left[\tfrac{E_T}{N_T}\Big|\mathcal{F}_t\right] & \tfrac{\widetilde{E}_t}{N_t} &= E_{\mathbb{N}}\left[\exp\left(\textstyle\int_{s=t}^{T} (r_s - \widetilde{r}_s)\, ds\right) \tfrac{\widetilde{E}_T}{N_T}\Big|\mathcal{F}_t\right] \\
\tfrac{\widetilde{E}_t}{\widetilde{N}_t} &= E_{\widetilde{\mathbb{N}}}\left[\tfrac{\widetilde{E}_T}{\widetilde{N}_T}\Big|\mathcal{F}_t\right] & \tfrac{E_t}{\widetilde{N}_t} &= E_{\widetilde{\mathbb{N}}}\left[\exp\left(\textstyle\int_{s=t}^{T} (\widetilde{r}_s - r_s)\, ds\right) \tfrac{E_T}{\widetilde{N}_T}\Big|\mathcal{F}_t\right]
\end{aligned} \tag{2.54}$$

And if we take into account that current accounts are particular cases of numeraires with standard and non standard collaterals:

$$\begin{aligned}
\tfrac{E_t}{N_t} &= E_{\mathbb{N}}\left[\tfrac{E_T}{N_T}\Big|\mathcal{F}_t\right] & \tfrac{\widetilde{E}_t}{N_t} &= E_{\mathbb{N}}\left[\exp\left(\textstyle\int_{s=t}^{T} (r_s - \widetilde{r}_s)\, ds\right) \tfrac{\widetilde{E}_T}{N_T}\Big|\mathcal{F}_t\right] \\
\tfrac{\widetilde{E}_t}{\widetilde{N}_t} &= E_{\widetilde{\mathbb{N}}}\left[\tfrac{\widetilde{E}_T}{\widetilde{N}_T}\Big|\mathcal{F}_t\right] & \tfrac{E_t}{\widetilde{N}_t} &= E_{\widetilde{\mathbb{N}}}\left[\exp\left(\textstyle\int_{s=t}^{T} (\widetilde{r}_s - r_s)\, ds\right) \tfrac{E_T}{\widetilde{N}_T}\Big|\mathcal{F}_t\right]
\end{aligned} \tag{2.55}$$

For generic numeraires, the Radon-Nikodym derivative expression can be obtained from either

$$E_t = N_t E_{\mathbb{N}}\left[\frac{E_T}{N_T}\Big|\mathcal{F}_t\right] = \widetilde{N}_t E_{\widetilde{\mathbb{N}}}\left[\exp\left(\int_{s=t}^{T} (\widetilde{r}_s - r_s)\, ds\right) \frac{E_T}{\widetilde{N}_T}\Big|\mathcal{F}_t\right] \tag{2.56}$$

or

$$\widetilde{E}_t = N_t E_{\mathbb{N}}\left[\exp\left(\int_{s=t}^{T} (r_s - \widetilde{r}_s)\, ds\right) \frac{\widetilde{E}_T}{N_T}\Big|\mathcal{F}_t\right] = \widetilde{N}_t E_{\widetilde{\mathbb{N}}}\left[\frac{\widetilde{E}_T}{\widetilde{N}_T}\Big|\mathcal{F}_t\right] \tag{2.57}$$

and would be given by

$$\frac{d\widetilde{\mathbb{N}}}{d\mathbb{N}}(t,T) = \frac{\widetilde{N}_T \zeta_T}{N_T} \frac{N_t}{\widetilde{N}_t \zeta_t} = \frac{\widetilde{N}_T N_t}{\widetilde{N}_t N_T} \exp\left(\int_{s=t}^{T} (r_s - \widetilde{r}_s)\, ds\right)$$

The zero volatility FX analogy has already been found in [11] using other arguments.

2.5 Practical example

Assume that in the EUR economy, we have deals collateralized:

- In EUR.
- In USD.
- In X (asset denominated in GBP).

We have 3 different current accounts that accrue at different rates:

$$d\beta_t^{€€} = c_t^{€} \beta_t^{€€} dt$$

$$d\beta_t^{€\$} = \left(c_t^{€} + b_t^{€\$}\right) \beta_t^{€\$} dt$$

$$d\beta_t^{€X} = \left(c_t^{€} + r_t^X - c_t^{£} + b_t^{€£}\right) \beta_t^{€X} dt$$

Where c_t^H is the OIS rate for currency H, b_t^{HI} the short term cross currency basis between currencies H and I and r_t^X the REPO rate for asset X.
With their corresponding discount factors:

$$B^{€€}(t,T),\ B^{€\$}(t,T),\ B^{€X}(t,T)$$

The situation would be similar to having 3 sub currencies:

- EUR collateralized in EUR.
- EUR collateralized in USD.
- EUR collateralized in X (denominated in GBP).

With FX rates:

$$\frac{€€}{€\$}(T) = \frac{€€}{€\$}(t) \exp\left(-\int_{s=t}^{T} b_s^{€\$} ds\right)$$

$$\frac{€€}{€X}(T) = \frac{€€}{€X}(t) \exp\left(-\int_{s=t}^{T} \left(r_t^X - c_t^{£} + b_t^{€£}\right) ds\right)$$

2.6 Stochastic funding curve modeling

In this section we assume that we want to price a non collateralized interest rate transaction. Its value at time t from the investor's (risk taker) perspective will be denoted by $\hat{E}_t$.

In pricing the non collateralized deal, we will make the following assumptions:

- The non collateralized derivative is closed with a counterparty with no default risk, so that funding issues are analyzed in isolation from counterparty credit risk.
- As assumed in [9], the hedger is not concerned about the changes in the derivative upon his own default, but is concerned about the changes experienced by the derivative due to changes in his own funding curve.

$\hat{B}(t,T)$ represents the value at time t of a zero coupon bond issued by the derivative's hedger.

$$\hat{B}(t,T) = 1_{\{\tau>t\}} \exp\left(-\int_{s=t}^{T} \hat{f}(t,s)ds\right) + R(t,T)1_{\{\tau\leq t\}} \tag{2.58}$$

τ represents the default time of the derivative's hedger, $\hat{f}(t,T)$ the instantaneous forward curve associated to the hedger's funding curve and $R(t,T)$ the recovery rate for a zero coupon bond maturing at T.

We assume that under the real world measure $\mathbb{P}$, the evolution of $\hat{f}(t,T)$ is given by

$$d\hat{f}(t,T) = \mu^{\hat{f}}(t,T)dt + \underbrace{\sigma^{\hat{f}}(t,T)}_{1\times n}\underbrace{dW_t^{\mathbb{P}}}_{n\times 1} + \underbrace{\hat{\sigma}^{\hat{f}}(t,T)}_{1\times m}\underbrace{dZ_t^{\mathbb{P}}}_{m\times 1} \tag{2.59}$$

Obviously, after τ, $\hat{f}(t,T)$ is no longer meaningful. Therefore, (2.59) only makes sense before default.

Regarding the short term financing of the derivative's hedger, its evolution will be given by

$$d\hat{C}_t = \hat{r}_t\hat{C}_t dt + (1-R_t)C_t dN_t^{\mathbb{P}} \tag{2.60}$$

Where $\hat{r}_t$ is the short term funding rate, R_t is the recovery rate for short term debt and $N_t^{\mathbb{P}} = 1_{\{\tau<t\}}$ a Poisson counting process with real world intensity $\lambda_t^{\mathbb{P}}$.

As hedging instruments the hedger will use the set of vanilla instruments H_t since the product cash flows could depend on $B(t,T)$ and also a set of discount factors associated with his funding curve. The set of funding discount factors will be denoted by $\hat{H}_t$. The set of funding discount factors is necessary for the hedger to become immune to changes in his funding curve.

$$\hat{E}_t = \alpha_t H_t + C_t + \epsilon_t \hat{H}_t + \hat{C}_t$$

Notice that $\alpha_t H_t + C_t = 0$ since every component in H_t is collateralized.

The fact that $\hat{E}_t = \epsilon_t \hat{H}_t + \hat{C}_t$ is what in [9] is called the self financing condition. That is, incoming funds from uncollateralized derivatives are used to buy back issued debt and outgoing funds from uncollateralized derivatives need to be funded. In either case, the net issuance or buy back is such that the spread sensitivity of the uncollateralized derivative matches the sensitivities with respect to the funding curve of the debt issuance / buy back.

So that in every path in which the hedger remains not defaulted

$$\begin{aligned}\mu_t^{\hat{E}} dt + \sigma_t^{\hat{E}} dW_t^{\mathbb{P}} + \hat{\sigma}_t^{\hat{E}} dZ_t^{\mathbb{P}} - E_t \hat{r}_t = {} & \alpha_t \left(\mu_t^H dt + \sigma_t^H dW_t^{\mathbb{P}} - H_t r_t dt\right) \\ & + \epsilon_t \left(\mu_t^{\hat{H}} dt + \sigma_t^{\hat{H}} dW_t^{\mathbb{P}} + \hat{\sigma}_t^{\hat{H}} dZ_t^{\mathbb{P}} - \hat{H}_t \hat{r}_t dt\right)\end{aligned} \tag{2.61}$$

Notice that (2.61) is equivalent to the hedging formula obtained is previous sections for derivatives with non standard collateral. Therefore, $\hat{\beta}_t$ (funding current account) and $\hat{B}(t,T)$ can also be seen as self financing portfolios denominated in a fictitious foreign currency with the spot FX rate (ζ_t) expressed in D/F following

$$d\zeta_t = (r_t - \hat{r}_t)\,\zeta_t dt$$

Notice that under the assumption of the hedger not being concerned to what happens upon his own default, $\hat{B}(t,T)$ and $\hat{\beta}_t$ behave as risk free (there is no default dependence in their risk neutral dynamics) and can be used as numeraires.

2.7 Conclusions

- Multiple collateral schemes for deal denominated in a given currency imply multiple discounting curves (and their corresponding current accounts) for that currency.
- Current accounts, annuities and discount factors belonging to any collateral scheme can be used as numeraires.
- From a pricing perspective, having N different collateral schemes for the same currency is equivalent to having $N-1$ additional currencies with zero volatilities and whose drift is equal to the difference between the collateral rate chosen as the standard one and the collateral rate of each of the other schemes.
- Assuming that the derivatives hedger is not concerned with his own default while hedging derivatives, the hedger's funding curve represents an additional discounting curve that can be modeled the same way as the other discounting curves.

Chapter 3

Interest rate curve calibration and non arbitrage

3.1 OIS Curve Construction

An overnight indexed swap, OIS,is a contract between two parties in which one party pays a fixed rate (the OIS rate) against receiving the compounded overnight rate over the term of the contract. OIS indexed swaps are traded with maturities ranging from 1 week up to 30 years. In the USD market from one week to twelve months, an OIS swap has a single floating payment. For two years onwards OIS swaps have annual coupons. OIS indexed swaps are traded with maturities ranging from 1 week up to 40 or even 60 years.

There is a strong tendency in the market, to **use an OIS curve to price (cash) collateralized derivatives transactions**.

Let us assume a swaplet within a OIS swap that accrues between dates t_j and t_{j+1}. The payment of such a swaplet at t_{j+1} will be

$$V(t_{j+1}) = \underbrace{\prod_{i=t_j}^{t_{j+1}} (1 + \delta_i \, c_i)}_{\approx e^{\int_{t_j}^{t_{j+1}} c_s ds}} - 1 - K\delta_{j+1} \tag{3.1}$$

where, δ_i is normally one day-length , $(t_{j+1} - t_j)$ is the number of days in the accrual period and N the number of days in a year. K is the strike of such a swaplet and δ_{j+1} is the day count fraction for the swaplet.

In the post credit-crunch world, swaps are generally collateralized under the a ISDA Master Agreement, with collateral rates being Fed Funds (USD), Eonia (EUR), Sonia (GBP), etc.

Let us define $C(t,T) = e^{-\int_t^T c_s ds}$ as the stochastic discount factor that accrues at the collateral rate and let us assume a collateralized OIS swap with maturity T_M that pays at times $\{t_j\}_{j=1,\ldots,M}$ the compounded overnight rate against a fixed rate K_M. Its price today is

$$\begin{aligned} V_M(t) &= \sum_{j=1}^{M} E_t^{\mathbf{Q}}\left[C(t,t_j)\left(\prod_{i=t_{j-1}}^{t_j}(1+\delta_i c_i) - 1 - K_M\delta_j\right)\right] \quad (3.2)\\ &\approx \sum_{j=1}^{M} E_t^{\mathbf{Q}}\left[C(t,t_j)\left(e^{\int_{t_{j-1}}^{t_j} c_s ds} - 1 - K_M\delta_j\right)\right]\\ &= \sum_{j=1}^{M}\left[B(t,t_{j-1}) - B(t,t_j) - K_M\delta_j B(t,t_j)\right]\\ &= B(t,t_0) - B(t,t_M) - K_M A_{t,t_M}(t) \quad (3.3) \end{aligned}$$

where $B(t,t_0)$ is the discount factor to the effective date and can be determined by the overnight rate.

We have denoted by $\mathbf{Q}$ the risk neutral measure associated to the bank account that accrues at the collateral rate. And,

$$B(t,T) = E_t^{\mathbf{Q}}\left(e^{-\int_t^T c_s ds}\right)$$

$$A_{t_0,t_M}(t) = \sum_{j=1}^{j=M} B(t,t_j)\delta_j$$

By solving for the last discount factor in equation (3.3),

$$B(t,t_j) = \frac{B(t,t_0) - K_j \sum_{i=1}^{j-1} \delta_i B(t,t_i)}{1 + K_j\delta_j} \quad (3.4)$$

If we would observe in the market N consecutive OIS swaps (each IRS only has an additional payment respect the previous one), by using (3.4), we might calculate iteratively the discount factors intervening in the N OIS swaps.

In general terms, this will not be the case and we will observe N interest rate swaps in the market that will depend on M different discount factors. These N IRS can be expressed as a linear combination of the M discount factors at different maturities.

$$V_i(t) = \sum_{j=1}^{M} c_{i,j} B(t,t_j) \quad \forall i = 1,\ldots,N$$

Let $T_1, T_2, \dots T_N$ denote the maturities of the N market IRS. In this case we must have $t_j > T_i$ (that is T_i indicates the OIS Swap start date).
In general, $M > N$ so we will need to solve an undetermined linear system of equations on discount factors such as

$$\mathbf{V} = \mathbf{cB} \tag{3.5}$$

In general, in order to solve for (3.5) we have several options:

- **Raise the number of equations** by introducing synthetically more instruments. (Interpolation on market instruments ..?)
- **Lower the number of equations** by choosing some pillar dates and interpolate in between.
- **Parametrization** of the interest rate curve.
- We might add constraints to the problem and look for a feasible solution (**Least Square problem**).

What should we ask to the constructed OIS Curve ?
The discount curve extracted from the OIS market will be mainly used for discounting future cash-flows. We will not estimate forward rates with such curve, so the requirements for the construction of the OIS curve will be lower than those for the estimation curve, as we will see later on.
We will require to the constructed OIS discount factor curve,

- **Completeness**: Reproduce all the relevant market instrument prices.
- The constructed discount factor curve to be both **continuous and differentiable**.
- **Local curve**: If an input is changed, does the constructed curve only changes nearby...?
- **Local Hedges**: Does most of the delta risk get assigned to the hedging instruments that have maturities close to the given tenors, or does a material amount leak into other regions of the curve?

A general bootstrapping algorithm might be,

Algorithm

1. Let $B(t, t_j)$ be known for $t_j \leq T_{i-1}$.
2. Make a guess for $B(t, T_i)$.
3. Use an interpolation method to fill $B(t, t_j)$, $T_{i-1} < t_j < T_i$.
4. Compute V_i from the values of $B(t, t_j)$, $t_j \leq T_i$.
5. If v_i equals the value observed in the market, stop. Otherwise return to step 2.
6. If $i < N$, set $i = i + 1$ and repeat.

Just remains to choose the interpolation rule ... !!!

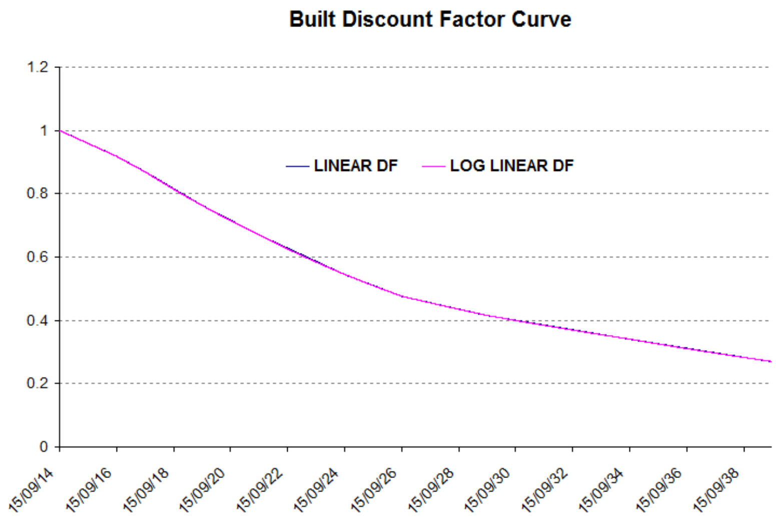


Figure 3.1: Discount Factor curve built up from Call Money Swaps.

3.2 Dynamics for the OIS rate

Let us set the non arbitrage dynamics for the OIS curve in a HJM framework. Let us denote the instantaneous forward rate,

$$c(t, T) = -\frac{\partial \log B(t, T)}{\partial T}$$

Since $B(t, T)$ is a continuous function, it seems reasonable the following approximation,

$$c(t, T) = -\frac{\partial \log B(t, T)}{\partial T} = \frac{1}{\Delta} \lim_{\Delta \to 0} \log \left(\frac{B(t, T - \Delta)}{B(t, T)} \right) \approx \frac{1}{\Delta} \left(\frac{B(t, T - \Delta)}{B(t, T)} - 1 \right)$$

That is, $c(t,T)$ is a martingale under the forward measure $\mathbf{Q}_T$

$$c(t,T) = E_t^{\mathbf{Q}_T}(c(T,T)) \tag{3.6}$$

Let us assume the dynamics for $c(t,T)$ under $\mathbb{Q}$ to be,

$$dc(t,T) = \mu^{\mathbb{Q}}(t,T)dt + \sigma(t,T) \cdot dW^{\mathbb{Q}}(t)$$

where $\sigma(t,T)$ is a N-dimensional adapted process, and $dW^{\mathbf{Q}}(t)$ is a N-dimensional Brownian motion under $\mathbf{Q}$.

We look for the non arbitrage $\mu^{\mathbb{Q}}(t,T)$. By (3.6) we can express the dynamics for the OIS rate under $\mathbf{Q}_T$ as,

$$dc(t,T) = \sigma(t,T) \cdot dW^{\mathbf{Q}_T}(t)$$

And we know from the Cameron-Girsanov theorem, the relationship between $\mathbf{Q}$ and $\mathbf{Q}_T$ is

$$dW^{\mathbf{Q}_T}(t) = dW^{\mathbf{Q}}(t) + \left(\int_t^T \sigma(t,u)du\right) dt \tag{3.7}$$

so

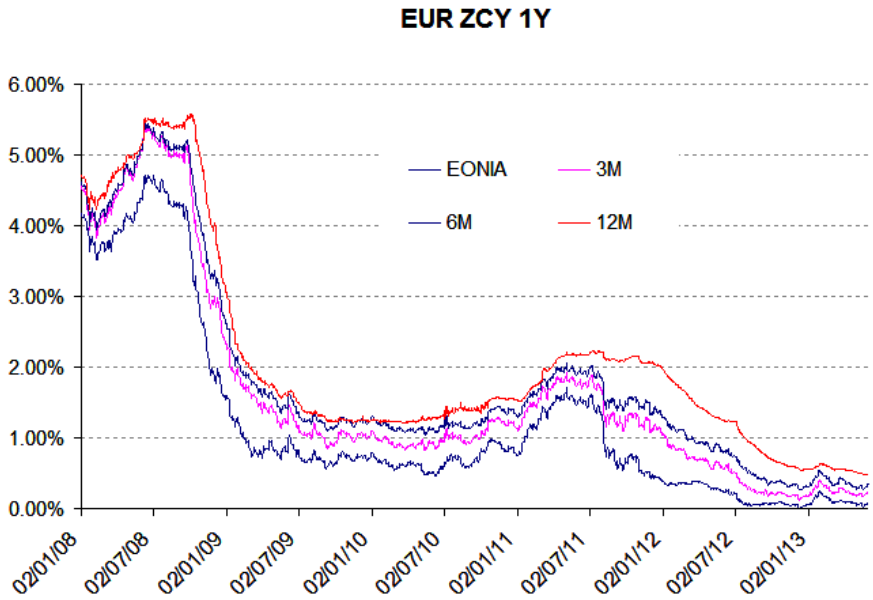
$$\mu^{\mathbb{Q}}(t,T) = \sigma(t,T) \cdot \int_t^T \sigma(t,u)du$$

what defines the dynamics for $c(t,T)$ under $\mathbf{Q}$ to be,

$$\boxed{dc(t,T) = \left(\sigma(t,T) \cdot \int_t^T \sigma(t,u)du\right) dt + \sigma(t,T) \cdot dW^{\mathbf{Q}}(t)} \tag{3.8}$$

3.3 Tenor Swaps Curves

A consequence of the 2007 credit crunch was the divergence of rates that until then closely chased each other. We observe in the market, a set of IRS with different maturities that pays a floating reference with different tenors. This swaps may have either fixed/floating or floating/floating legs.



From these market instruments, we should be able to estimate the different forward curves for the different tenors (e.g 1m, 3m, 6m, 1y).

To set notation, let us define a pseudo discount curve built from these instruments by $B_{(m)}(t,T)$ $\forall m = 0,\dots,M$, where $B_{(0)}(t,T)$ is reserved to the discount curve taken from the *OIS* market.

Let us assume all these instruments to be cash-collateralized in the currency in which the instrument is denominated. Let us focus on a fixed/floating swap with floating reference $F_j^{(m)}$ and maturity T_N.

Its price today will be given by,

$$V_N^{(m)}(t) = \sum_{j=1}^{N} E_t^{\mathbf{Q}}\left[C(t,t_j)\left(F_j^{(m)}(t_{j-1}) - K_N^{(m)}\right)\right]\delta_j \tag{3.9}$$

Let us zoom in the price of the j-th floating payment.
That is,

$$E_t^{\mathbf{Q}}\left(C(t,t_j)F_j^{(m)}(t_{j-1})\right) = B^{(0)}(t,t_j)E_t^{\mathbf{Q}_{T_j}}\left(F_j^{(m)}(t_{j-1})\right) := B^{(0)}(t,t_j)F_j^{(m)}(t)$$

Where $F_j^{(m)}(t)$ is the strike that makes the value of the j-th FRA, that pays at t_j the libor with tenor m (that fixed at t_{j-1}), to be worth zero. That is, $F_j^{(m)}(t)$ is the j-th forward with tenor m, that will be defined later.

So that we may express (3.9) as,

$$V_N^{(m)}(t) = \sum_{j=1}^{N} B^{(0)}(t,t_j) F_j^{(m)}(t)\delta_j - K_N^m A_{t,t_N}^{(0)}(t) \tag{3.10}$$

In order to solve for (3.10) the **only unknown** are $F_j^{(m)}(t)$ as $B^{(0)}(t,t_j)$ has been previously obtained from the *OIS* market calibration.
Just notice that we may arbitrary define the pseudo discount factor curve $B^{(m)}(t,t_j)$ through the relationship

$$F_j^{(m)}(t) := \left[\frac{B^{(m)}(t,t_{j-1})}{B^{(m)}(t,t_j)} - 1\right]\frac{1}{\delta_j^{(m)}} \tag{3.11}$$

Just notice that equation (3.11) is an **arbitrary definition** not resting on replicating arguments, as it was the case in the *mono-curve* world where we might replicate the floating payment by going long and short in two deposits with different maturities.
As in the case of the OIS construction, we might find the tenor curve, by solving a either a non-linear system of equations on discount factors or a linear one in forwards.

What should we ask to the constructed Tenor-Basis Curve ?

- **Completeness**: Reproduce all the relevant market instrument prices.
- **Smooth forward curve**.
- **Local curve**: If an input is changed, does the constructed curve only changes nearby...?
- **Stable forward curve**: We can quantify the degree of stability by looking for the maximum basis point change in the forward curve given some basis point change (up or down) in one of the inputs.
- **Local Hedges**: Does most of the delta risk get assigned to the hedging instruments that have maturities close to the given tenors, or does a material amount leak into other regions of the curve?

Fixed Point iteration algorithm

Let's explicitly denote the dependence of the N-th IRS price on the different rates[1] by,

[1]We talk about rates in an abstract way. These might be Zero coupon rates, libor rates, etc.

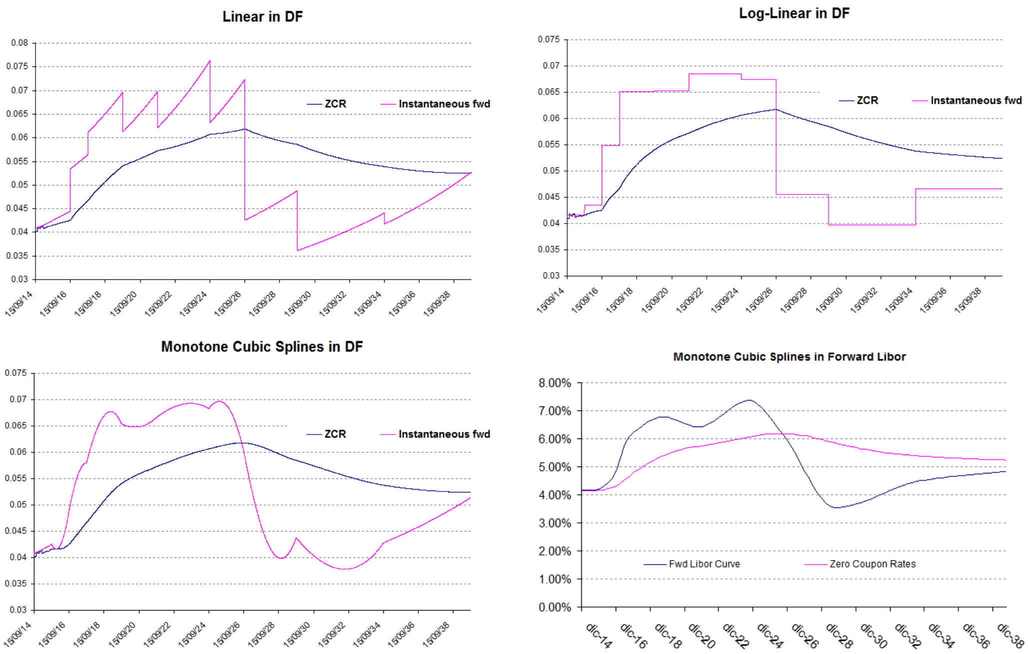


Table 3.1: Zero coupon and instantaneous forward rates for different interpolations. From top to bottom and left to right: Linear in DF, Log-Linear in DF, Hyman Cubic Splines in DF interpolation, Hyman Local Cubic Splines in Libor.

$$V_N = f(R_1, \ldots, R_N)$$

Where $\{t_j\}_{\{i=1,\ldots,N\}}$ denotes the pillar dates in the calibration.

Guess $\{R_j\}_{\{j=1,\ldots,N\}}$ **to start the iteration**
$K = 1$
Do {
  For Each $IRS_j \; \forall j = 1, \ldots, N$
    Find R_j^K such that $V_j = f\left(R_1^K, \ldots, R_j^K\right)$

  $\mathbf{R} := \mathbf{R}^K$
  $K = K + 1$
  }**While** $\left(\frac{1}{N}\sum_{j=1}^N \left[R_j - R_j^K\right]^2 < \epsilon\right)$

3.4 Dynamics for the Forward Curve

Let us introduce the instantaneous forward rate $f^{(m)}(t,T)$ defined by the pseudo discount factor curve $B^{(m)}(t,T)$

$$B^{(m)}(t,T) = e^{-\int_t^T f^{(m)}(t,u)du}$$

Let us define the instantaneous forward spread over the OIS rate as

$$s^{(m)}(t,T) = f^{(m)}(t,T) - f^{(0)}(t,T)$$

Let us assume the dynamics for the spread under $\mathbf{Q}$ to be,

$$ds^{(m)}(t,T) = \mu_s^{\mathbf{Q}}(t,T)dt + \sigma^{(m)}(t,T) \cdot dW^{\mathbf{Q}}(t)$$

We aim at obtaining $\mu_s^{\mathbf{Q}}(t,T)$.

For this, let us focus on some non-arbitrage condition. We know from the previous section that,

$$f^{(0)}(t,T) = E_t^{\mathbf{Q}_T}\left(f^{(0)}(T,T)\right), \quad f^{(m)}(t,T) = E_t^{\mathbf{Q}_T}\left(f^{(m)}(T,T)\right) (^2)$$

what implies that the spread is also martingale,

$$s^{(m)}(t,T) = E_t^{\mathbf{Q}_T}\left(s^{(m)}(T,T)\right)$$

What means that we can express the dynamics for the spread under the forward measure as,

$$ds^{(m)}(t,T) = \sigma^{(m)}(t,T) \cdot dW^{\mathbf{Q}_T}(t)$$

And by (3.7),

$$\boxed{ds^{(m)}(t,T) = \left(\sigma^{(m)}(t,T) \cdot \int_t^T \sigma^{(0)}(t,u)du\right) dt + \sigma^{(m)}(t,T) \cdot dW^{\mathbf{Q}}(t)}$$

Proof: $f^{(m)}(t,T) = E_t^{\mathbf{Q}_T}\left(f^{(m)}(T,T)\right)$
We have seen that,

$$F_j^{(m)}(t) = \left(\underbrace{\frac{B^{(m)}(t,t_{j-1})}{B^{(m)}(t,t_j)}}_{\xi_j(t)} - 1\right)\delta_j^{-1} \tag{3.12}$$

[2] This will be demonstrated below.

must be a martingale under $\mathbf{Q}_j$ (the measure associated to $B^{(0)}(t,t_j)$). This will imply that also,

$$\xi_j(t) = e^{\int_{t_{j-1}}^{t_j} f^{(m)}(t,s)ds}$$

must be also be a martingale. So we can express the dynamcis for $\xi_j(t)$ as,

$$\frac{d\xi_j(t)}{\xi_j(t)} = \phi_j(t) dW^{\mathbf{Q}_j}(t)$$

So by applying Ito to

$$df^{(m)}(t,t_j) = d\left(\frac{\partial \log \xi_j(t)}{\partial t_j}\right)$$

we obtain,

$$\boxed{df^{(m)}(t,t_j) = \phi_j(t) dW^{\mathbf{Q}_j}(t)}$$

3.5 Cross Currency interest rate curves

Let us imagine two economies denominated in currencies A and B. We have seen, so far, two particular type of derivatives denominated in both currencies and perfectly collateralized in the deal's currency: Call money swaps, and tenor interest rate swaps

From these instruments we have extracted the following information,

- **OIS IRS Market**: Discount factors for cash-collateralized derivatives $B_{(0)}^{A,A}(t,T)$ and $B_{(0)}^{B,B}(t,T)$
- **Tenor IRS Market**: Pseudo Discount factors for estimation of floating reference for cash-collateralized contracts $B_{(m)}^{A,A}(t,T)$ and $B_{(m)}^{B,B}(t,T)$.

Where in $B_{(k)}^{A,C}(t,T)$, A denotes the currency in which $B_k(\cdot,\cdot)$ is denominated, C denotes the currency of collateralization and (m) denotes the tenor from which the pseudo-discount factor curve has been built up.

What happens when a derivative is cash-collateralized in other currency than that of the derivative ...?

Let us assume a derivative denominated in currency A, that pays at T an uncertain amount ψ_T^A of currency A, to be perfectly collateralized in currency B.

The price of such a derivative at time t is

$$\boxed{V_t^A = E_t^{\mathbb{Q}_A}\left[e^{-\int_t^T (c_s^A + b_s^{A,B})ds}\ \psi_T^A\right]} \tag{3.13}$$

where $b_t^{A,B}$ is the cross currency basis.

Is there any market instrument providing us with such an information (derivatives denominated is currency A and collateralized in currency B) so that we can infer both curves $B_{(0)}^{A,B}(t,T)$ and $B_{(m)}^{A,B}(t,T)$?

The answer is yes ..!. The cross currency market gives us prices for cross currency swaps that are cash-collateralized in only one currency.

Let's denote a par floating/floating cross currency Swap to swap flows in currency A plus a spread s, in exchange for flows in currency B with maturity T_N and collateralized (both legs) in currency B by $CCS^{A,B}(t,T_N)$.

The price today, at t, of the B-denominated leg (collateralized in B), can be expressed as,

$$\begin{aligned} V_N^{B,B}(t) &= \sum_{j=1}^{N} E_t^{\mathbf{Q}^B}\left(C_B(t,t_j)\ F_j^{(m),B}(t_{j-1})\right)\delta_j^B \\ &= \sum_{j=1}^{N} B^{B,B}(t,t_j)F_j^{(m),B}(t)\delta_j^B \end{aligned} \tag{3.14}$$

Where in the last equation **everything is known** as $B^{B,B}(t,t_j)$ has been bootstrapped from the OIS IRS market and $F_j^{(m),B}(t)$ has been calculated from B-denominated tenor IRS, liquidly quoted in the market.

What happens to the A-denominated leg ...?

The price today of the A-denominated leg (collateralized in B), can be expressed as,

$$\begin{aligned} V_N^{A,B}(t) &= \sum_{j=1}^{N} E_t^{\mathbf{Q}^A}\left[e^{-\int_t^{t_j}(c_u^A + b_u^{A,B})du}\left(F_j^{(m),A}(t_{j-1}) + s\right)\right]\delta_j^A \\ &= \sum_{j=1}^{N} B_{(0)}^{A,B}(t,t_j)\left[\underbrace{E_t^{\mathbf{Q}_{t_j}^{A,B}}\left(F_j^{(m),A}(t_{j-1})\right)}_{F_j^{(m),A,B}(t)} + s\right]\delta_j^A \end{aligned} \tag{3.15}$$

where use has been made of the Radon-Nikodyn derivative in moving to the forward measure,

$$\frac{d\mathbf{Q}^A}{d\mathbf{Q}^{A,B}_{t_j}}(t_j) = \frac{e^{\int_t^{t_j}(c_u^A + b_u)du}}{B^{A,B}_{(0)}(t_j,t_j)/B^{A,B}_{(0)}(t,t_j)}$$

Where the measure $\mathbf{Q}^{A,B}_{t_j}$ is the one associated to the Zero Coupon Bond denominated in A and collateralized in B.

$F_j^{(m),A,B}(t)$ denotes the strike that makes zero the price of a FRA denominated in A and collateralized in B. That is, the A-denominated forward with tenor m of a FRA contract collateralized in currency B.

Notice that in (3.15), we have **2 unknowns, neither** $B^{A,B}_{(0)}(t,t_j)$ **nor** $F_j^{(m),A,B}(t)$ **are known** and we must find them.

But we just have only one equation ...!!!

Let us reduce the dimensionality by assuming independence ...!!!

3.6 Independence between $b_t^{A,B}$ and c_t^A

If $b_t^{A,B}$ and c_t^A are independent,

$$B^{A,B}(t,t_j) = E_t^{\mathbf{Q}}\left(e^{-\int_t^{t_j}(c_u^A + b_u^{A,B})du}\right) = E_t^{\mathbf{Q}}\left(e^{-\int_t^{t_j} c_u^A du}\right)\underbrace{E_t^{\mathbf{Q}}\left(e^{-\int_t^{t_j} b_u^{A,B} du}\right)}_{\phi^{A,B}(t,t_j)}$$
$$= B^{A,A}_{(0)}(t,t_j)\phi^{A,B}(t,t_j) \tag{3.16}$$

And

$$E_t^{\mathbf{Q}^{A,B}_{t_j}}\left[F_j^{(m),A}(t_{j-1})\right] = E_t^{\mathbf{Q}^{A,A}_{t_j}}\left[F_j^{(m),A}(t_{j-1})\,\frac{d\mathbf{Q}^{A,B}_{t_j}}{d\mathbf{Q}^{A,A}_{t_j}}(t_j)\right]$$
$$= \frac{B^{A,A}_{(0)}(t,t_j)}{B^{A,B}_{(0)}(t,t_j)} E_t^{\mathbf{Q}^{A,A}_{t_j}}\left[e^{-\int_t^{t_j} b_s ds} F_j^{(m),A}(t_{j-1})\right] \;\; = F_j^{(m),A,A}(t)$$

Where both $B^{A,A}(t,t_j)$ and $F_j^{(m),A,A}(t_{j-1})$ have been calculated from the Call Money Swap Market and the Tenor IRS, and the only unknown is $\phi^{A,B}(t,t_j)$.

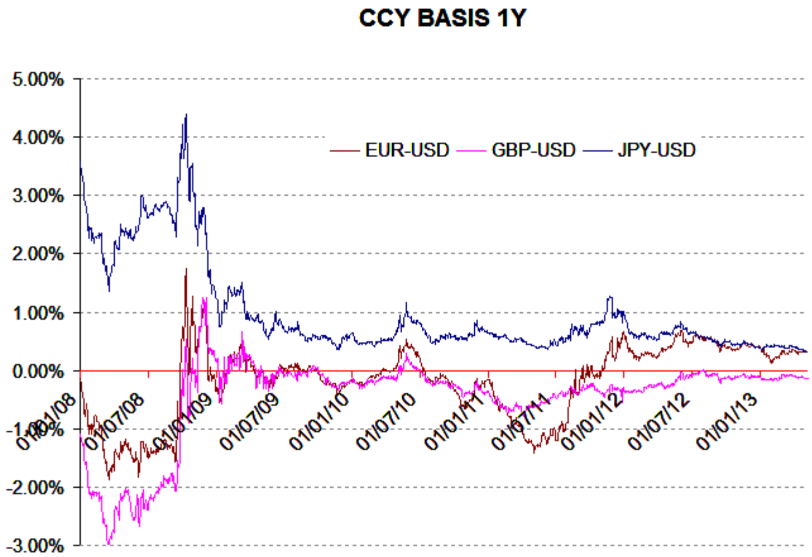
As in the case of (3.5) we should solve for a linear system of equations in ϕ.

Notice that under the assumption of independence between $b_t^{A,B}$ and c_t^A, the only unknown is $\phi_{t,T}^{A,B}$ as the forward tenor becomes independent of the collateralization scheme.

Conclusions:

- The Cross currency market provides information about the cross currency basis, once the OIS and tenor basis has been constructed in both currencies.

- Under independence, the forward libor does not depend on the collateralization currency, and all the Xccy basis information lies on one of the discount curves. The Xccy basis information is easily separable from this discount factor curve (term $\phi^{A,B}_{t,T}$).



What should we ask to the constructed Xccy Curve $\phi^{A,B}_{t,T}$?
The Xccy curve will be used for discounting purposes, so the requirements for the construction of such a curve will be the same as those required to the OIS curve.
We will require to the constructed OIS Xccy curve,

- **Completeness**: Reproduce all the relevant market instrument prices.

- The constructed Xccy Curve should be both **continuous and differentiable**.

- **Local curve**

- **Local Hedges**

3.7 What happens when we want to price derivatives denominated in B but collateralized in A ?

So far, we have been able to find from market instruments,

- **OIS Market**: $B^{A,A}(t,T)$ and $B^{B,B}(t,T)$
- **Tenor Market**: $B^{A,A}_{(m)}(t,T)$ and $B^{B,B}_{(m)}(t,T)$
- **XCcy Market**: $B^{A,B}_{(0)}(t,T)$, $F_j^{(m),A,B} = F_j^{(m),A,A}$
- $B^{B,A}_{(0)}(t,T)$ .. ? **is there any any Market to retrieve it from ..?**:

Imagine we would want to calculate $B^{B,A}_{(0)}(t,T)$. For this, we would need an instrument denominated in currency B and collateralized in A in order to recover,

$$B^{B,A}_{(0)}(t,T) = E_t^{\mathbf{Q}_B}\left(e^{-\int_t^T\left(c_s^B + b^{B,A}(s)\right)ds}\right)$$

As there might not be such an instrument, **we will be *forced to take some assumptions***.
Before exploring this, we formulate the question:

is there any relation between $b_t^{B,A}$ and $b_t^{A,B}$? .

The answer is **yes**. For this, let us think of an over-night FX forward. The price at inception of this contract is zero, so there is not need for posting collateral along the life of such contract (i.e one day). This will imply that the forward of such contract will be independent of the collateralization mechanism and hence,

$$\boxed{b_t^{A,B} = -b_t^{B,A}} \tag{3.17}$$

The easiest assumption we might take to infer $B^{B,A}_{t,T}$ in a illiquid market where the only input available is $B^{A,B}_{t,T}$ is to assume that **the price of a Cross currency instrument is independent of the collateralization currency** (i.e the FX forwards are independent of the collateralization scheme).
Under this assumption, **how would $\phi^{B,A}_{t,T}$ look like ..?**
To see this, let us think of a FX forward to exchange 1 unit of currency A in exchange of K units of currency B at a future time, T. (Just note that K is the forward rate B/A)
<u>**Collateralization in currency A**</u>:

Assuming that the collateralization is made in currency A, the forward rate becomes,

$$K^{(B/A),A} = X_t^{B/A} \frac{B^{A,A}(t,T)}{B^{B,A}(t,T)} \tag{3.18}$$

<u>**Collateralization in currency** B</u>:
Assuming that the collateralization is made in currency B, the forward rate becomes,

$$K^{(B/A),B} = X_t^{B/A} \frac{B^{A,B}(t,T)}{B^{B,B}(t,T)} \tag{3.19}$$

Under the assumption $K^{(B/A),A} = K^{(B/A),B}$, we have the equality

$$\frac{B^{A,A}(t,T)}{B^{B,A}(t,T)} = \frac{B^{A,B}(t,T)}{B^{B,B}(t,T)}$$

Which implies,

$$\boxed{\underbrace{E_t^{\mathbb{Q}_T^{A,A}}\left[e^{-\int_t^T b_s^{A,B} ds}\right]}_{\phi_{t,T}^{A,B}} = \frac{1}{\underbrace{E_t^{\mathbb{Q}_T^{B,B}}\left[e^{-\int_t^T b_s^{B,A} ds}\right]}_{\phi_{t,T}^{B,A}}}} \tag{3.20}$$

Where, $\mathbb{Q}_T^{A,A}$ and $\mathbb{Q}_T^{B,B}$ are the measures associated to the numeraires $B^{A,A}(t,T)$ and $B^{B,B}(t,T)$ respectively.
As we have seen, $b_t^{A,B} = -b_t^{B,A}$, what implies that equation (3.20) will only be true when,

$$\phi_{t,T}^{A,B} = e^{-\int_t^T b_s^{A,B}} ds$$

$$\phi_{t,T}^{B,A} = e^{\int_t^T b_s^{A,B}} ds$$

Or what is the same,

$$\boxed{\textbf{as long as } b_t^{A,B} \textbf{ IS DETERMINISTIC ..!!}}$$

<u>**Proof:**</u>
We will see under which assumptions equation (3.20) is true. For this, let us expand

$$E_t^{\mathbb{Q}_T^{B,B}}\left[e^{-\int_t^T b_s^{B,A} ds}\right] = \frac{B_{t,T}^{A,A}}{\bar{X}_t^{A/B} B_{t,T}^{B,B}} E_t^{\mathbb{Q}_T^{A,A}}\left[\bar{X}_T^{A/B} e^{-\int_t^T b_s^{B,A} ds}\right] \tag{3.21}$$

Where $\bar{X}_t^{A/B}$ is a non-standard FX such as,

$$\frac{d\bar{X}_t^{A/B}}{\bar{X}_t^{A/B}} = \left(c_t^A - c_t^B\right) dt + \sigma_t^{A/B} dW^{\mathbb{Q}_A}(t)$$

By substituting into (3.20),

$$\frac{B_{t,T}^{A,A}}{\bar{X}_t^{A/B} B_{t,T}^{B,B}} E_t^{\mathbb{Q}_T^{A,A}} \left[e^{-\int_t^T b_s^{A,B} ds}\right] E_t^{\mathbb{Q}_T^{A,A}} \left[\bar{X}_T^{A/B} e^{-\int_t^T b_s^{B,A} ds}\right] = 1 =$$
$$\frac{B_{t,T}^{A,A}}{\bar{X}_t^{A/B} B_{t,T}^{B,B}} \left[E_t^{\mathbb{Q}_T^{A,A}} \left(\bar{X}_T^{A/B}\right) - Cov^{A,A} \left(e^{-\int_t^T b_s^{A,B} ds} \bar{X}_T^{A/B}, e^{-\int_t^T b_s^{B,A} ds}\right)\right] \tag{3.22}$$

By taking into account that,

$$E_t^{\mathbb{Q}_T^{A,A}} \left(\bar{X}_T^{A/B}\right) = \bar{X}_t^{A/B} \frac{B_{t,T}^{B,B}}{B_{t,T}^{A,A}}$$

identity (3.22) is fulfilled as long as

$$Cov^{A,A} \left(e^{-\int_t^T b_s^{A,B} ds} \bar{X}_T^{A/B}, e^{-\int_t^T b_s^{B,A} ds}\right) = 0$$

But this covariance is zero only in the case the volatility for the Xccy basis is zero. So the Xccy basis must be deterministic in order to make the FX forward independent of the collateralization scheme.

Under non-zero volatility, the different FX forward will depend on the collateralization currency and will be related by,

$$\boxed{\frac{K^{(B/A),B}}{K^{(B/A),A}} = \left[1 - \frac{B_{t,T}^{A,A}}{\bar{X}_t^{A/B} B_{t,T}^{B,B}} Cov^{A,A} \left(e^{-\int_t^T b_s^{A,B} ds} \bar{X}_T^{A/B}, e^{-\int_t^T b_s^{B,A} ds}\right)\right]} \tag{3.23}$$

That is, **the FX forward under collateralization in currency B is equal to the FX forward under collateralization in currency A plus a convexity adjustment**.

3.7.1 Summary:

- In general terms, **we will not observe market instruments that give information about $B_{t,T}^{B,A}$**. So in order to recover these prices **we will be forced to take some assumptions**.

- One simple assumption is to **consider market FX forwards to be independent of the collateral currency**. Under this assumption, we recover prices for $B^{B,A}_{t,T}$ very easily.

$$B^{B,A}_{t,T} = B^{B,B}_{t,T} \frac{1}{\phi^{A,B}_{t,T}}$$

- This assumption is only consistent with **deterministic Xccy basis spreads.**

- Under non-deterministic Xccy spreads **forwards will depend on the collateralization strategy**, through a **convexity adjustment**.

3.8 What happens when we want to price derivatives denominated in A but collateralized in C from $B^{A,B}(t,T)$ and $B^{C,B}(t,T)$?

It is common to see different **cross currency swaps to be collateralized in the same currency** (usually the USD). In our case, imagine we observe in the market, cross currency swaps on (A, B) and on (C, B) **both collateralized in** B. These cross currencies give us information about $B^{A,B}_{(0)}(t,T)$ and $B^{C,B}_{(0)}(t,T)$.

Is it possible to infer a relationship for $b^{A,C}_t$?

The answer is a clear **yes** in the **Over-night** FX market. We know that an **overnight** market FX forward **is independent of the collateralization scheme**. This will imply that the following relationship between forwards must be fulfilled:

$$X^{A/C}_{t,t+dt} = \frac{X^{A/B}_{t,t+dt}}{X^{C/B}_{t,t+dt}} \tag{3.24}$$

Where $X_{t,t+dt}$ denotes the over-night market FX forward.
Expanding terms in (3.24),

$$X^{A/C}_t \frac{1 + (c^A_t + b^{A,C}_t)dt}{1 + c^C_t dt} = X^{A/B}_t \frac{1 + (c^A_t + b^{A,B}_t)dt}{1 + c^B_t dt} \frac{1}{X^{C/B}_t} \frac{1 + c^B_t}{1 + \left(c^C_t + b^{C,B}_t\right) dt} \tag{3.25}$$

What implies that

$$\boxed{b_t^{A,C} = b_t^{A,B} - b_t^{C,B}}$$

Note, that this identity must be fulfilled in order to preclude arbitrage opportunities in the over-night FX forward market.

In order to get a relationship between long-term cross currency basis, let us imagine a FX forward to exchange currencies A and C where both legs are perfectly collateralized in currency B.

In this case, the identity

$$X_{t,T}^{A/C} = \frac{X_{t,T}^{A/B}}{X_{t,T}^{C/B}} \tag{3.26}$$

is true **as long as every FX forward is collateralized in currency** B. As we did before, if we make the FX forward independent of the collateralization scheme, equation (3.26) will be true always.

So let us make the following assumption,

Let us assume the FX Forward to be independent of the collateralization scheme ...!

Under this assumption, we obtain the following relationship between the different Xccy basis spreads,

$$\boxed{\phi_{t,T}^{A,C} = \frac{\phi_{t,T}^{A,B}}{\phi_{t,T}^{C,B}}} \tag{3.27}$$

Where $\phi_{t,T}^{X,Y} = E_t^{\mathbb{Q}_T^{X,X}}\left[e^{-\int_t^T b_s^{X,Y} ds}\right]$

Under this relationship we can *triangulate* discount factors by the identity,

$$\boxed{B_{t,T}^{A,C} = B_{t,T}^{C,C}\frac{B_{t,T}^{A,B}}{B_{t,T}^{C,B}}} \tag{3.28}$$

But we should notice that,

$$\boxed{\textbf{relationship } (3.27) \textbf{ will only be satisfied for deterministic XCcy basis}}$$

3.8.1 Summary:

- In general terms, **we will not observe market instruments that give information about** $B_{t,T}^{A,C}$. So in order to recover these prices **we will be forced to take some assumptions**.

- One simple assumption is to **consider market FX forwards to be independent of the collateral currency**. Under this assumption, we recover prices for $B_{t,T}^{A,C}$ very easily.

$$B_{t,T}^{A,C} = B_{t,T}^{C,C} \frac{B_{t,T}^{A,B}}{B_{t,T}^{C,B}}$$

- This assumption is only consistent with **deterministic Xccy basis spreads.**

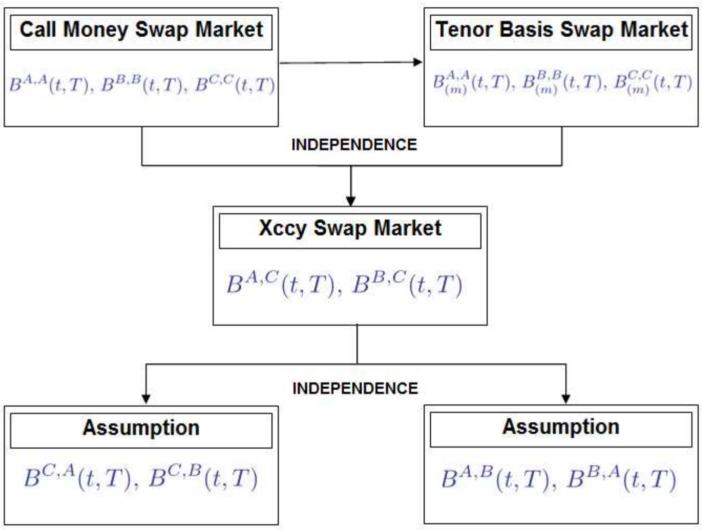


Figure 3.2: Xccy Curve Calibration Diagram.

3.9 Dynamics for the cross currency basis spread.

It this section we derive the non arbitrage dynamics for the instantaneous forward cross currency basis. For this, let us remind the price of Zero Coupon Bonds denominated in A-currency and collateralized in both A and B to be,

$$\begin{aligned} B^{A,A}(t,T) &= E_t^{\mathbf{Q}}\left(C_A(t,T)\right) \\ B^{A,B}(t,T) &= E_t^{\mathbf{Q}}\left(C_A(t,T)\, e^{-\int_t^T b_s ds}\right) = B^{A,A}(t,T) E_t^{\mathbf{Q}_T}\left(e^{-\int_t^T b_s ds}\right) \\ &= B^{A,A}(t,T)\phi^{A,B}(t,T) \end{aligned} \tag{3.29}$$

Just notice that no independence assumption has been made in the last expression.

On the other hand, we have seen that we can define the dynamics of $B^{A,A}(t,T)$ and $B^{A,B}$,

$$\begin{aligned} \frac{dB^{A,A}(t,T)}{B^{A,A}(t,T)} &= c_A(t)dt - \Sigma_{A,A}(t,T)\cdot dW^{\mathbf{Q}}(t) \\ \frac{dB^{A,B}(t,T)}{B^{A,B}(t,T)} &= \left(c_A(t)dt + b_t\right)dt - \Sigma_{A,B}(t,T)\cdot dW^{\mathbf{Q}}(t) \end{aligned} \tag{3.30}$$

for arbitrary N-dimensional adapted processes $\Sigma_{A,A}(t,T)$ and $\Sigma_{A,B}(t,T)$.

Let us assume the dynamcis of $\phi_{t,T}^{A,B}$ to be under $\mathbb{Q}$,

$$\frac{d\phi_{t,T}^{A,B}}{\phi_{t,T}^{A,B}} = \mu_t^{\mathbb{Q}} dt + \Sigma_\phi(t,T)\cdot dW_t^{\mathbb{Q}}$$

By taking differentials to both sides of (3.29), and taking into account (3.30), we obtain

$$\begin{aligned} &B_{t,T}^{A,B}\left[\left(c_t^A + b_t\right)dt - \Sigma_{A,B}(t,T)\cdot dW_t^{\mathbb{Q}}\right] = \\ &\quad B_{t,T}^{A,B}\left[\left(c_t^A + \mu_\phi^{\mathbb{Q}}(t) + \Sigma_{A,A}(t,T)\cdot\Sigma_\phi(t,T)\right)dt - \left(\Sigma_{A,A}(t,T) + \Sigma_\phi(t,T)\right)\cdot dW_t^{\mathbb{Q}}\right] \end{aligned} \tag{3.31}$$

what forces the dynamcis of $\phi^{A,B}(t,T)$ to be,

$$\frac{d\phi^{A,B}(t,T)}{\phi^{A,B}(t,T)} = \left[b_t - \Sigma_{A,A}(t,T)\cdot\Sigma_\phi(t,T)\right]dt - \Sigma_\phi(t,T)\cdot dW^{\mathbf{Q}}(t) \tag{3.32}$$

where we have denoted

$$\Sigma_\phi(t,T) = \left(\Sigma_{A,A}(t,T) - \Sigma_{A,B}(t,T)\right)$$

if we express $\phi^{A,B}(t,T)$ in terms of the instantaneous cross currency basis forward $b^{A,B}(t,T)$ by,

$$b^{A,B}(t,T) := -\frac{\partial \log \phi^{A,B}(t,T)}{\partial T}$$

it can be seen that,

$$\begin{aligned} db^{A,B}(t,T) = &[(\sigma_\phi(t,T) + \sigma_{A,A}(t,T)) \cdot \Sigma_\phi(t,T) + \sigma_b(t,T) \cdot \Sigma_{A,A}(t,T)]\, dt \\ &+ \sigma_\phi(t,T) \cdot dW^{\mathbf{Q}}(t) \end{aligned} \tag{3.33}$$

where we have denoted $\sigma_X(t,T) = \frac{\partial \Sigma_X(t,T)}{\partial T}$

Chapter 4

Review of dynamic credit modeling: CDSs and Bonds

4.1 Introduction

Counterparty credit risk (CVA) and funding (FVA) imply the management of the credit risk of our counterparties and also of our own credit risk [1]. Therefore, a section reviewing credit risk modeling and management is crucial.

In the following chapters, we will deal with credit default swaps and bonds of our counterparties, but also bonds issued by ourselves (derivatives hedger). We tackle each of these.

Derivatives hedgers replicate non collateralized derivatives in different currencies. Therefore it is also mandatory to analyze the relationships of funding spreads for a given debt issuer in different currencies.

4.2 Collateralized credit derivatives

In this section our aim is to derive the PDE followed by collateralized credit derivatives written on a generic credit reference.

We assume that under the real world measure $\mathbb{P}$ the short term CDS spread h_t follows

$$dh_t = \mu_t^{\mathbb{P}} dt + \sigma_t dW_t^{\mathbb{P}} \quad \text{Short term CDS spread}$$

We will explore the hedging of both components of credit risk (spread risk and jump to default risk).

E_t will represent the value of the collateralized credit derivative from the investor perspective.

[1] Assuming that we are in the position of a derivatives hedger.

Obviously $E_t(t, h_t, N_t^{\mathbb{P}})$
$N_t^{\mathbb{P}} = 1_{\{\tau \le t\}}$ represents the default indicator function where τ is the default time of the underlying reference credit.
We assume $N_t^{\mathbb{P}}$ to have a real world default intensity $\lambda_t^{\mathbb{P}}$ (will it be relevant?).
Therefore

$$dE_t = \underbrace{\frac{\partial E_t}{\partial t}dt + \frac{1}{2}\sigma_t^2\frac{\partial^2 E_t}{\partial h_t^2}dt}_{\text{Theta}} + \underbrace{\frac{\partial E_t}{\partial h_t}dh_t}_{\text{Spread risk}} + \underbrace{\Delta E_t dN_t^{\mathbb{P}}}_{\text{Jump to default risk}}$$

Where ΔE_t represents the change in E_t on default.
The two sources of randomness (dh_t and $dN_t^{\mathbb{P}}$) will have to be hedged with two different credit derivatives.
$\star$ One of them will be a short term credit default $CDS(t, t+dt)$. h_t will be such that $CDS(t, t+dt) = 0$. Its differential change will be given by:

$$dCDS(t, t+dt) = h_t dt - (1-R)dN_t^{\mathbb{P}}$$

R will represent the recovery rate.
$\star$ The hedger should also trade on another cash collateralized credit derivative (such as a CDS) H_t (NPV as seen by the hedger) such that

$$dH_t = \frac{\partial H_t}{\partial t}dt + \frac{\partial H_t}{\partial h_t}dh_t + \frac{1}{2}\sigma_t^2\frac{\partial^2 H_t}{\partial h_t^2}dt + \Delta H_t dN_t^{\mathbb{P}}$$

Where ΔH_t represents the change in H_t on default.
The hedging equation will be

$$E_t = \alpha_t H_t + \gamma_t CDS(t, t+dt) + \beta_t$$

Where β_t represents cash held in collateral accounts. α_t and γ_t the amounts to trade on each one of the hedging instruments.
We assume both E_t and H_t to be collateralized in cash, so that:

$$d\beta_t = c_t E_t dt - c_t \alpha_t H_t dt \quad c_t \text{ represents the collateral accrual rate}$$

c_t is typically the OIS rate.
So that the hedging equation in differential form is

$$\begin{aligned}
&\tfrac{\partial E_t}{\partial t}dt + \tfrac{\partial E_t}{\partial h_t}dh_t + \tfrac{1}{2}\sigma_t^2\tfrac{\partial^2 E_t}{\partial h_t^2}dt + \Delta E_t dN_t^{\mathbb{P}} = \\
&\alpha_t\left(\tfrac{\partial H_t}{\partial t}dt + \tfrac{\partial H_t}{\partial h_t}dh_t + \tfrac{1}{2}\sigma_t^2\tfrac{\partial^2 H_t}{\partial h_t^2}dt + \Delta H_t dN_t^{\mathbb{P}}\right) \\
&+\gamma_t\left(h_t dt - (1-R)dN_t^{\mathbb{P}}\right) \\
&+c_t E_t dt - c_t\alpha_t H_t dt
\end{aligned} \tag{4.1}$$

In order to be hedged, the random terms dh_t and $dN_t^{\mathbb{P}}$ should be canceled. In order to do so

$$\alpha_t = \frac{\frac{\partial E_t}{\partial h_t}}{\frac{\partial H_t}{\partial h_t}} \quad \gamma_t = \alpha_t\frac{\Delta H_t}{1-R} - \frac{\Delta E_t}{1-R}$$

So that

$$\frac{\frac{\partial E_t}{\partial t} + \frac{1}{2}\sigma_t^2\frac{\partial^2 E_t}{\partial h_t^2} + \frac{h_t}{1-R}\Delta E_t - c_t E_t}{\frac{\partial E_t}{\partial h_t}} = \frac{\frac{\partial H_t}{\partial t} + \frac{1}{2}\sigma_t^2\frac{\partial^2 H_t}{\partial h_t^2} + \frac{h_t}{1-R}\Delta H_t - c_t H_t}{\frac{\partial H_t}{\partial h_t}} \tag{4.2}$$

Adding $\mu_t^{\mathbb{P}}$ and dividing by σ_t both sides of the last equation we obtain what could be interpreted as the expected excess return of the derivative over the collateral rate divided by the the derivatives volatility factor, therefore

$$\frac{\frac{\partial E_t}{\partial t} + \mu_t^{\mathbb{P}}\frac{\partial E_t}{\partial h_t} + \frac{1}{2}\sigma_t^2\frac{\partial^2 E_t}{\partial h_t^2} + \frac{h_t}{1-R}\Delta E_t - c_t E_t}{\sigma_t\frac{\partial E_t}{\partial h_t}} = \frac{\frac{\partial H_t}{\partial t} + \mu_t^{\mathbb{P}}\frac{\partial H_t}{\partial h_t} + \frac{1}{2}\sigma_t^2\frac{\partial^2 H_t}{\partial h_t^2} + \frac{h_t}{1-R}\Delta H_t - c_t H_t}{\sigma_t\frac{\partial H_t}{\partial h_t}} = M(t, h_t) \tag{4.3}$$

Since the ratio must be valid for any credit derivative (H_t and E_t are two generic payoffs), then it must be just a function of t and h_t. $M_t = M(t, h_t)$ will be called the market price of credit risk.

Therefore, the PDE followed by any credit derivative must be

$$\frac{\partial E_t}{\partial t} + \underbrace{\left(\mu_t^{\mathbb{P}} - \sigma_t M_t\right)}_{\text{Drift}}\frac{\partial E_t}{\partial h_t} + \frac{1}{2}\sigma_t^2\frac{\partial^2 E_t}{\partial h_t^2} + \underbrace{\frac{h_t}{1-R}}_{\text{Default intensity}}\Delta E_t - \underbrace{c_t}_{\text{Discounting}}E_t = 0 \tag{4.4}$$

Notice that $\lambda_t^{\mathbb{P}}$ has disappeared from (4.4).

That equation implies that the cash flows of cash collateralized credit derivatives should be discounted at the OIS rate c_t. This reflects the fact that we can finance the purchase of cash collateralized credit derivatives at the OIS rate.

In the case of a credit default swap, the NPV would be given by:

$$NPV_t^{\text{CDS}} = \underbrace{SE_{\mathbb{Q}}\left[\sum_{j=1}^{n} \gamma_j 1_{\{\tau > t_j\}} \exp\left(-\int_{s=t}^{t_j} c(s)ds\right) \Big| \mathcal{F}_t\right]}_{\text{Premium leg}} \\ -(1-R)\underbrace{E_{\mathbb{Q}}\left[\int_{s=t}^{t_n} \exp\left(-\int_{u=t}^{s} c(u)du\right) 1_{\{\tau \in (s,s+ds]\}} \Big| \mathcal{F}_t\right]}_{\text{Default leg}} \tag{4.5}$$

In a measure $\mathbb{Q}$ under which the default intensity is given by $\frac{h_t}{1-R}$ and the drift of h_t is $\mu_t^{\mathbb{P}} - \sigma_t M_t$.
The last equation is used to obtain risk neutral default intensities from CDS quotes.

4.3 Bonds

When dealing with bonds, things are a little bit different. First we have to establish a relationship between the short term financing rate f_t and the short term CDS rate h_t. In order to do so, we compare two different strategies:

- Selling protection at time t with maturity $t + dt$.
- Buying a bond at t maturing at time $t + dt$ through a REPO transaction maturing also at time $t + dt$.

Both strategies imply a net cash flow at time t equal to 0. At time $t + dt$, the net cash flows are (assuming $\tau > t$):

$$\text{CDS:} \quad h_t dt - (1-R)1_{\{\tau \leq t+dt\}}$$

$$\text{REPO:} \quad (1 + f_t dt)1_{\{\tau > t+dt\}} + R1_{\{\tau \leq t+dt\}} - (1 + r_t dt) =$$

$$= (1 + f_t dt) - (1 + r_t dt) - (1 - R + f_t dt)1_{\{\tau \leq t+dt\}} =$$
$$= (f_t - r_t)dt - (1-R)1_{\{\tau \leq t+dt\}}$$

Where r_t is a short term REPO rate on a short term bond maturing at time $t + dt$. Therefore:

$$\underbrace{h_t}_{\text{Short term CDS spread}} = \underbrace{f_t}_{\text{Short term financing rate}} - \underbrace{r_t}_{\text{Short term REPO rate}} \tag{4.6}$$

Which yields

$$\underbrace{z_t := f_t - c_t}_{\text{Short term funding spread over Eonia}} = \underbrace{h_t}_{\text{Short term CDS spread}} + \underbrace{(r_t - c_t)}_{\text{REPO / OIS basis}}$$

So that the bond CDS basis is determined by the difference in the rates at which credit derivatives and bonds can be financed.

$$\underbrace{z_t - h_t}_{\text{Bond / CDS basis}} = \underbrace{r_t - c_t}_{\text{REPO / OIS basis}}$$

Therefore, in order to obtain the PDE followed by defaultable bonds we should keep in mind that collateralized credit derivatives are financed at the collateral rate, whereas bonds purchases are financed at REPO rates that might differ between different bonds.

So that if we wanted to replicate a cash collateralized credit derivative (E_t) with a coupon paying bond ($B^C(t,T)$) and with a short term bond ($B(t,t+dt)$), the hedging equation will be given by:

$$E_t = \alpha_t B^C(t,T) + \gamma_t B(t,t+dt) + \beta_t$$

β_t will be comprised of:

- E_t due to the cash collateralized derivative accruing at c_t.
- $-\alpha_t B^C(t,T)$ due to the short term REPO to finance the bond purchase that accrues at r_t^T (short term REPO rate on bond $B^C(t,T)$).
- $-\gamma_t B(t,t+dt)$ due to the short term REPO to finance the bond purchase that accrues at $r_t := r_t^{t+dt}$ (short term REPO rate on bond $B(t,t+dt)$).

In order to simplify the algebra, we assume that the only stochastic factor affecting the price of bonds is h_t, although changes in REPO rates will also affect those prices. We assume that both c_t and r_t^T are deterministic. Therefore, the hedging equation will be given by:

$$\begin{aligned}
&\tfrac{\partial E_t}{\partial t}dt + \tfrac{\partial E_t}{\partial h_t}dh_t + \tfrac{1}{2}\sigma_t^2\tfrac{\partial^2 E_t}{\partial h_t^2}dt + \Delta E_t dN_t^{\mathbb{P}} = \\
&\alpha_t\left(\tfrac{\partial B^C(t,T)}{\partial t}dt + \tfrac{\partial B^C(t,T)}{\partial h_t}dh_t + \tfrac{1}{2}\sigma_t^2\tfrac{\partial^2 B^C(t,T)}{\partial h_t^2}dt + \Delta B^C(t,T)dN_t^{\mathbb{P}}\right) \\
&+\gamma_t B(t,t+dt)\left(f_t dt - (1-R)dN_t^{\mathbb{P}}\right) \\
&+c_t E_t dt - r_t^T \alpha_t B^C(t,T)dt - r_t\gamma_t B(t,t+dt)dt
\end{aligned} \tag{4.7}$$

Where we have taken into account that $\frac{dB(t,t+dt)}{B(t,t+dt)} = f_t dt - (1-R)dN_t^{\mathbb{P}}$

$\Delta B^C(t,T)$ is the jump on $B^C(t,T)$ on default.
In order to be hedged

$$\alpha_t = \frac{\frac{\partial E_t}{\partial h_t}}{\frac{\partial B^C(t,T)}{\partial h_t}} \qquad \gamma_t B(t,t+dt) = \alpha_t \frac{\Delta B^C(t,T)}{1-R} - \frac{\Delta E_t}{1-R} \tag{4.8}$$

Plugging (4.8) into (4.7) and taking into account (4.6)

$$\frac{\frac{\partial E_t}{\partial t} + \mu_t^{\mathbb{P}} \frac{\partial E_t}{\partial h_t} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 E_t}{\partial h_t^2} + \frac{h_t}{1-R}\Delta E_t - c_t E_t}{\sigma_t \frac{\partial E_t}{\partial h_t}} = \frac{\frac{\partial B^C(t,T)}{\partial t} + \mu_t^{\mathbb{P}} \frac{\partial B^C(t,T)}{\partial h_t} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 B^C(t,T)}{\partial h_t^2} + \frac{h_t}{1-R}\Delta B^C(t,T) - r_t^T B^C(t,T)}{\sigma_t \frac{\partial B^C(t,T)}{\partial h_t}} = M(t,h_t) \tag{4.9}$$

So that the PDE followed by bonds is:

$$\frac{\partial B^C(t,T)}{\partial t} + \underbrace{\left(\mu_t^{\mathbb{P}} - \sigma_t M_t\right)}_{\text{Drift}} \frac{\partial B^C(t,T)}{\partial h_t} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 B^C(t,T)}{\partial h_t^2} + \underbrace{\frac{h_t}{1-R}}_{\text{Default intensity}} \Delta B^C(t,T) - \underbrace{r_t^T}_{\text{Discounting}} B^C(t,T) = 0 \tag{4.10}$$

(4.10) implies that the cash flows of bonds should be discounted at the particular bond short term REPO rate r_t^T. This reflects the fact that we can finance the purchase of bonds at REPO rates.
So that the price of a bond would be given by

$$\begin{aligned} B^C(t,T) = & \underbrace{E_{\mathbb{Q}}\left[C\sum_{j=1}^{n}\gamma_j 1_{\{\tau>t_j\}} \exp\left(-\int_{s=t}^{t_j} r_s^T ds\right) + \exp\left(-\int_{s=t}^{t_n} r_s^T ds\right) 1_{\{\tau>t_n\}} \middle| \mathcal{F}_t\right]}_{\text{Bond coupons \& notional}} \\ & + \underbrace{E_{\mathbb{Q}}\left[\int_{s=t}^{t_n} R(s,T)\exp\left(-\int_{u=t}^{s} r_u^T du\right) 1_{\{\tau\in(s,s+ds]\}} \middle| \mathcal{F}_t\right]}_{\text{Recovery leg}} \end{aligned} \tag{4.11}$$

In a measure $\mathbb{Q}$ under which the default intensity is given by $\frac{h_t}{1-R}$ and the drift of h_t is $\mu_t^{\mathbb{P}} - \sigma_t M_t$.
$R(s,T)$ is the recovery rate of the bond maturing at time T if $\tau = s$.

4.3.1 Implied REPO curve calibration

If we assume that the short term REPO rate for a given bond issuer does not depend on the particular bond, the implied REPO curve for a given issuer could be obtained in the following way.

- Obtain the risk neutral default intensity through OIS quotes and CDS quotes with the help of the CDS pricing equation, assuming a given recovery and a given interpolation assumption.

- Use the risk neutral default intensity curve together with bond quotes and the bond pricing formula to bootstrap the implied REPO curve. An interpolation assumption would also be needed.

The implied REPO curve:

- Should be used to discount bond cash flows (after adjusting the survival and default cashflows by their corresponding risk neutral probabilities).
- Should be used to discount cash flows of derivatives collateralized in bonds (taking into account the probability of default of the bond issuer and the implications in the collateral scheme [2]).
- Help us to model the bond-cds basis in a possibly dynamic pricing framework.

4.4 Own debt

The results obtained in section 4.3 are valid when we are trading on someone else's debt.
When trading on our own debt:

- We will have no access to the CDS market written on our debt (We won't be able to sell protection on ourselves).
- We will have no access to the REPO market (We won't be able to get financing leaving our own bonds as collateral.)
- We will have no access to the recovery lock market written on our debt.

Therefore the risk neutral dynamics imposed by (4.10), that depend on magnitudes implied by markets to which we do not have access, seem not to work when we are managing our own debt.
What do we mean by managing our own debt?
Cash flow matching of our assets and liabilities such that the bank meets its current and future cash-flow obligations and collateral needs (assets / liabilities management).
Let's assume that a bank has issued debt with both short term maturity ($B(t, t+dt)$) and long term maturity $B^C(t, T)$.
Let's assume that we needed to issue (or buy back) debt with a given coupon and maturity S with a notional N.

[2] What happens with collateral if the asset used as collateral defaults

Can we dynamically replicate the issuance (or buy back) of a bond with maturity $S \neq T$ with a net issuance (or bay back) in $B(t, t+dt)$ and $B^C(t,T)$?
In a one factor world, yes. In a n factor world, we will have to trade on $n+1$ issued bonds.
The hedging equation would be:

$$NB^C(t,S) = N\left(\omega_t B(t,t+dt) + \Omega_t B^C(t,T)\right) \tag{4.12}$$

In (4.12), $N > 0$ represents a buy back and $N < 0$ an issuance.
The differential change of both sides of the hedging equation under the real world measure would be given by

$$\frac{\partial B^C(t,S)}{\partial t}dt + \frac{\partial B^C(t,S)}{\partial h_t}dh_t + \frac{1}{2}\sigma_t^2\frac{\partial^2 B^C(t,S)}{\partial h_t^2}dt + \Delta B^C(t,S)dN_t^{\mathbb{P}}$$

and

$$\begin{aligned}&\omega_t B(t,t+dt)\left(f_t dt - (1-R)dN_t^{\mathbb{P}}\right) + \\ \Omega_t &\left(\frac{\partial B^C(t,T)}{\partial t}dt + \frac{\partial B^C(t,T)}{\partial h_t}dh_t + \frac{1}{2}\sigma_t^2\frac{\partial^2 B^C(t,T)}{\partial h_t^2}dt + \Delta B^C(t,T)dN_t^{\mathbb{P}}\right)\end{aligned}$$

Notice that in (4.12) there is only one free parameter. Therefore we won't be able to hedge both the spread and the jump to default risks simultaneously.
In addition, the jump to default risk will not be experienced by ourselves. Therefore, leaving the $dN_t^{\mathbb{P}}$ term unhedged is not a concern.
We will remain hedged on every path under which we remain not defaulted.

$$\begin{aligned}&\tfrac{\partial B^C(t,S)}{\partial t}dt + \tfrac{\partial B^C(t,S)}{\partial h_t}dh_t + \tfrac{1}{2}\sigma_t^2\tfrac{\partial^2 B^C(t,S)}{\partial h_t^2}dt \\ &= \omega_t B(t,t+dt)f_t dt + \Omega_t\left(\tfrac{\partial B^C(t,T)}{\partial t}dt + \tfrac{\partial B^C(t,T)}{\partial h_t}dh_t + \tfrac{1}{2}\sigma_t^2\tfrac{\partial^2 B^C(t,T)}{\partial h_t^2}dt\right)\end{aligned} \tag{4.13}$$

In order to hedge the spread risk:

$$\Omega_t = \frac{\frac{\partial B^C(t,S)}{\partial h_t}}{\frac{\partial B^C(t,T)}{\partial h_t}}$$

Which together with (4.12) and (4.13) imply

$$\frac{\frac{\partial B^C(t,T)}{\partial t} + \mu_t^{\mathbb{P}}\frac{\partial B^C(t,T)}{\partial h_t} + \frac{1}{2}\sigma_t^2\frac{\partial^2 B^C(t,T)}{\partial h_t^2} - f_t B^C(t,T)}{\sigma_t\frac{\partial B^C(t,T)}{\partial h_t}} = \frac{\frac{\partial B^C(t,S)}{\partial t} + \mu_t^{\mathbb{P}}\frac{\partial B^C(t,S)}{\partial h_t} + \frac{1}{2}\sigma_t^2\frac{\partial^2 B^C(t,S)}{\partial h_t^2} - f_t B^C(t,S)}{\sigma_t\frac{\partial B^C(t,S)}{\partial h_t}} = M^{\mathrm{OD}}(t,h_t) \tag{4.14}$$

Where $M^{\text{OD}}(t,h_t)$ represents the market price of risk of our own debt. So that the PDE followed by our bonds is:

$$\frac{\partial B^C(t,T)}{\partial t} + \underbrace{\left(\mu_t^{\mathbb{P}} - \sigma_t M_t^{\text{OD}}\right)}_{\text{Drift}} \frac{\partial B^C(t,T)}{\partial h_t} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 B^C(t,T)}{\partial h_t^2} - \underbrace{f_t}_{\text{Discounting}} B^C(t,T) = 0 \tag{4.15}$$

Notice that the risk free dynamics of our own debt reflected in (4.15) do not depend on REPO rates, recovery rates and has no default indicators.
So that the price of a particular bond would be given by

$$B(t,T) = \underbrace{E_{\mathbb{Q}}\left[C\sum_{j=1}^{n}\gamma_j \exp\left(-\int_{s=t}^{t_j} f_s ds\right) + \exp\left(-\int_{s=t}^{t_n} f_s ds\right)\Big|\mathcal{F}_t\right]}_{\text{Bond coupons \& notional}} \tag{4.16}$$

Equation that can be used in order to bootstrap the derivatives hedger's funding curve.

4.5 Funding rates in different currencies

In this section we will establish a relationship between the funding rates of a credit reference in different currencies.
Let's assume that the short term funding rate in a given currency D is represented by f_t^D. c_t^D will represent the short term OIS rate in D. Therefore, the spread over the OIS rate in currency D will be given by

$$s_t^D = f_t^D - c_t^D$$

4.5.1 Quanto CDSs

Before exploring the relationship of funding rates of a credit reference in different currencies we will tackle the relationship of CDS spreads.
Let's assume that there are short term CDSs written on the credit reference and denominated in currencies D and F with short term premiums h_t^D and h_t^F. $CDS^D(t,t+dt)$ and $CDS^F(t,t+dt)$ represent the net present values of both trades in their respective currencies D and F. h_t^D and h_t^F are such that both NPVs are 0 at t.
We will analyze the most general situation under which the default of the credit reference has an impact on the FX rate between currencies D and F. If X_t represents the FX rate expressed in D/F.

We will assume that under the real world measure $\mathbb{P}$ the evolution of X_t follows

$$dX_t = \underbrace{\mu_t^{X,\mathbb{P}} X_t dt + \sigma_t^X X_t dW_t^{X,\mathbb{P}}}_{d\widetilde{X}_t} + (J_t^{\mathbb{P}} - 1) X_t dN_t^{\mathbb{P}}$$

Where $\mu_t^{X,\mathbb{P}}$ is the real world drift, σ_t^X the FX volatility, $N_t^{\mathbb{P}} = 1_{\{\tau \leq t\}}$ the default indicator of the credit reference and $J_t^{\mathbb{P}} - 1$ the jump experienced by the FX rate upon default of the reference credit. Therefore:

$$X_\tau = X_{\tau^-} J_\tau^{\mathbb{P}}$$

Notice that $J_t^{\mathbb{P}}$ is uncertain (an additional risk factor). Let's see how we can hedge this risk factor.

The changes in both $CDS^D(t,T)$ and $CDS^F(t,T)$ measured in D and F respectively are given by

$$dCDS^D(t, t+dt) = h_t^D dt - (1-R) dN_t^{\mathbb{P}}$$
$$dCDS^F(t, t+dt) = h_t^F dt - (1-R) dN_t^{\mathbb{P}}$$

The NPV of $CDS^F(t,T)$ measured in D is given by $X_t CDS^F(t,T)$. Therefore the differential change of $CDS^F(t,T)$ measured in D can be obtained applying Itô's Lemma for jump diffusion processes:

$$d\left(X_t CDS^F(t, t+dt)\right) = X_t h_t^F dt - (1-R) X_t J_t^{\mathbb{P}} dN_t^{\mathbb{P}}$$

If we sold protection with a notional of D 1 in the short term CDS denominated in D and bought protection with a notional of F $1/X_t$ (D 1) in the short term CDS denominated in F, the differential change of the whole portfolio expressed in D would be:

$$d\pi_t = \left(h_t^D - h_t^F\right) dt - (1 - J_t^{\mathbb{P}})(1-R) dN_t^{\mathbb{P}} \tag{4.17}$$

Therefore, with opposite trades (and equal notional) in short term CDSs denominated in the two different currencies, we can hedge the sudden change experienced by the FX trade upon default of the credit reference.

Let's now explore the relationship between CDS premiums in both currencies. In order to do so, we will assume that we want to replicate a cash collateralized credit derivative written on a reference credit and denominated in the foreign currency F with credit derivatives denominated in the domestic currency D. E_t will represent the NPV of the derivative to replicate in currency F. Therefore, the NPV expressed in D will be given by

$$X_t E_t$$

So that its differential change will be given by (applying Itô's Lemma for jump diffusion processes)

$$d\left(X_tE_t\right) = X_t d\widetilde{E}_t + E_t d\widetilde{X}_t + d\widetilde{X}_t d\widetilde{E}_t + \Delta\left(X_tE_t\right) dN_t^{\mathbb{P}} \tag{4.18}$$

Where $d\widetilde{X}_t$ and $d\widetilde{E}_t$ represent the continuous change of X_t and E_t respectively. $\Delta\left(X_tE_t\right)$ represents the jump of X_tE_t upon default.

Again, in order to simplify the algebra, we will assume a one factor world, that is, we assume that the dynamics of the CDSs curves in both currencies are governed by a single factor, the short term CDS spread in currency D. Nevertheless, the results obtained also apply in a more general setting. In this one factor world, we must assume that the short term CDS spread h_t^F is a deterministic function of both t and h_t^D, that is

$$h_t^F = G(t, h_t^D)$$

The most simple form of G would be $h_t^F = J_t^{\mathbb{Q}} h_t^D$, where $J_t^{\mathbb{Q}}$ is a non negative deterministic function of time.

So that the differential change of X_tE_t would be given by

$$d\left(X_tE_t\right) = X_t\left(\frac{\partial E_t}{\partial h_t^D} dh_t^D + \mathcal{L}_h E_t dt\right) + E_t d\widetilde{X}_t + \rho_t \sigma_t^h X_t \sigma_t^X \frac{\partial E_t}{\partial h_t^D} dt + \Delta\left(X_tE_t\right) dN_t^{\mathbb{P}} \tag{4.19}$$

Where $\mathcal{L}_h = \frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial (h_t^D)^2}(\sigma_t^h)^2$ and ρ_t the instantaneous correlation between h_t^D and X_t.

Let's assume that we just used as hedging portfolio two cash collateralized credit derivatives denominated in D (one of them being a short term CDS). The hedging portfolio would be:

$$E_tX_t = \alpha_t H_t + \gamma_t CDS^D(t, t+dt) + \beta_t^D + X_t\beta_t^F$$

Where H_t is the NPV of the credit derivative denominated in D and used as a hedging instrument, α_t the notional to trade on H_t, $CDS^D(t, t+dt)$ the short term CDS denominated also in D, γ_t the notional to trade on $CDS(t, t+dt)$, β_t^D and β_t^F amounts posted by the hedger as collateral in both currencies.

The values of the amounts posted as collateral by the hedger in currencies D and F are given by

$$\begin{aligned} \beta_t^D &= -\alpha_t H_t \\ \beta_t^F &= E_t \end{aligned} \tag{4.20}$$

Notice that we assume $CDS^D(t, t+dt) = 0$.

The differential change of H_t is given by

$$dH_t = \frac{\partial H_t}{\partial h_t^D} dh_t^D + \mathcal{L}_h H_t dt + \Delta H_t dN_t^{\mathbb{P}} \tag{4.21}$$

Where $\Delta H_t dN_t^{\mathbb{P}}$ is the change in H_t upon default of the underlying credit reference.

The differential changes in $CDS^D(t, t+dt)$ and β_t^D are

$$\begin{aligned} d\beta_t^D &= c_t^D \beta_t dt \\ dCDS^D(t, t+dt) &= h_t^D dt - (1-R) dN_t^{\mathbb{P}} \end{aligned} \tag{4.22}$$

And the differential change in $X_t \beta_t^F$

$$\begin{aligned} d(\beta_t^F X_t) &= \beta_t^F d\widetilde{X}_t + X_t d\widetilde{\beta}_t^F + \Delta(X_t \beta_t^F) \\ &= \beta_t^F d\widetilde{X}_t + c_t^F X_t \beta_t^F dt + (J_t^{\mathbb{P}} - 1) X_t \beta_t^F dN_t^{\mathbb{P}} \end{aligned} \tag{4.23}$$

So that the hedging equation in differential form would be given by

$$\begin{aligned} &X_t \left(\tfrac{\partial E_t}{\partial h_t^D} dh_t^D + \mathcal{L}_h E_t dt \right) + E_t d\widetilde{X}_t + \rho_t \sigma_t^h X_t \sigma_t^X \tfrac{\partial E_t}{\partial h_t^D} dt + \big(X_t (J_t^{\mathbb{P}} - 1)(E_t + \Delta E_t) + X_t \Delta E_t \big) dN_t^{\mathbb{P}} \\ &= \alpha_t \left(\tfrac{\partial H_t}{\partial h_t^D} dh_t^D + \mathcal{L}_h H_t dt + \Delta H_t dN_t^{\mathbb{P}} \right) \underbrace{- c_t^D \alpha_t H_t dt}_{d\beta_t^D} + \gamma_t \underbrace{\left(h_t^D dt - (1-R) dN_t^{\mathbb{P}} \right)}_{dCDS^D(t,t+dt)} \\ &+ \underbrace{E_t d\widetilde{X}_t + c_t^F X_t E_t dt + (J_t^{\mathbb{P}} - 1) X_t E_t dN_t^{\mathbb{P}}}_{d(X_t \beta_t^F)} \end{aligned} \tag{4.24}$$

In the last expression there are 4 different risk factors (dh_t^D, $d\widetilde{X}_t$, $J_t^{\mathbb{P}}$, $dN_t^{\mathbb{P}}$). Notice that $d\widetilde{X}_t$ is canceled since the FX risks of the derivative E_t and the money posted in the collateral account in F are matched. There are 3 risk factors remaining and just two degrees of freedom α_t and γ_t, so that we will not be hedged. We need an additional hedging instrument that also has exposure to $J_t^{\mathbb{P}}$. This additional instrument could be a short term credit default swap denominated in F, so that the hedging equation would be

$$E_t X_t = \alpha_t H_t + \gamma_t CDS^D(t, t+dt) + \epsilon_t X_t CDS^F(t, t+dt) + \beta_t^D + X_t \beta_t^F$$

And in differential form:

$$
\begin{aligned}
&X_t\left(\frac{\partial E_t}{\partial h_t^D}dh_t^D+\mathcal{L}_hE_tdt\right)+E_td\widetilde{X}_t+\rho_t\sigma_t^hX_t\sigma_t^X\frac{\partial E_t}{\partial h_t^D}dt+\left(X_t(J_t^{\mathbb{P}}-1)(E_t+\Delta E_t)+X_t\Delta E_t\right)dN_t^{\mathbb{P}}\\
&=\alpha_t\left(\frac{\partial H_t}{\partial h_t^D}dh_t^D+\mathcal{L}_hH_tdt+\Delta H_tdN_t^{\mathbb{P}}\right)\underbrace{-c_t^D\alpha_tH_tdt}_{d\beta_t^D}+\gamma_t\underbrace{\left(h_t^Ddt-(1-R)dN_t^{\mathbb{P}}\right)}_{dCDS^D(t,t+dt)}\\
&+\epsilon_t\underbrace{\left(X_th_t^Fdt-(1-R)X_tJ_t^{\mathbb{P}}dN_t^{\mathbb{P}}\right)}_{d(X_tCDS^F(t,t+dt))}\\
&+\underbrace{E_td\widetilde{X}_t+c_t^FX_tE_tdt+(J_t^{\mathbb{P}}-1)X_tE_tdN_t^{\mathbb{P}}}_{d(X_t\beta_t^F)}
\end{aligned}
\tag{4.25}
$$

Canceling and rearranging terms:

$$
\begin{aligned}
&X_t\left(\frac{\partial E_t}{\partial h_t^D}dh_t^D+\mathcal{L}_hE_tdt-c_t^FE_tdt\right)+\rho_t\sigma_t^hX_t\sigma_t^X\frac{\partial E_t}{\partial h_t^D}dt+X_tJ_t^{\mathbb{P}}\Delta E_tdN_t^{\mathbb{P}}\\
&=\alpha_t\left(\frac{\partial H_t}{\partial h_t^D}dh_t^D+\mathcal{L}_hH_tdt+\Delta H_tdN_t^{\mathbb{P}}-c_t^DH_tdt\right)+\gamma_t\left(h_t^Ddt-(1-R)dN_t^{\mathbb{P}}\right)\\
&+\epsilon_t\left(X_th_t^Fdt-(1-R)X_tJ_t^{\mathbb{P}}dN_t^{\mathbb{P}}\right)
\end{aligned}
\tag{4.26}
$$

In order to be hedged:

$$
\begin{aligned}
&X_t\frac{\partial E_t}{\partial h_t^D}=\alpha_t\frac{\partial H_t}{\partial h_t^D}\\
&0=\alpha_t\Delta H_t-(1-R)\gamma_t\\
&\Delta E_t=-(1-R)\epsilon_t
\end{aligned}
\tag{4.27}
$$

Which implies

$$
\frac{\mathcal{L}_hE_t-c_t^FE_t+\Delta E_t\frac{h_t^F}{1-R}+\rho_t\sigma_t^h\sigma_t^X\frac{\partial E_t}{\partial h_t^D}}{\frac{\partial E_t}{\partial h_t^D}}=\frac{\mathcal{L}_hH_t+\Delta H_t\frac{h_t^D}{1-R}-c_t^DH_t}{\frac{\partial H_t}{\partial h_t^D}}
\tag{4.28}
$$

Adding the real world drift of h_t^D to both sides of the equation and dividing both sides by σ_t^h

$$
\begin{aligned}
&\frac{\frac{\partial E_t}{\partial t}+\mu_t^{h,\mathbb{P}}\frac{\partial E_t}{\partial h_t^D}+\frac{1}{2}(\sigma_t^h)^2\frac{\partial^2E_t}{\partial(h_t^D)^2}+\Delta E_t\frac{h_t^F}{1-R}+\rho_t\sigma_t^h\sigma_t^X\frac{\partial E_t}{\partial h_t^D}-c_t^FE_t}{\sigma_t^h\frac{\partial E_t}{\partial h_t^D}}\\
&=\frac{\frac{\partial H_t}{\partial t}+\mu_t^{h,\mathbb{P}}\frac{\partial H_t}{\partial h_t^D}+\frac{1}{2}\sigma_t^2\frac{\partial^2H_t}{\partial(h_t^D)^2}+\frac{h_t}{1-R}\Delta H_t-c_t^DH_t}{\sigma_t^h\frac{\partial H_t}{\partial h_t^D}}=M^D(t,h_t^D)
\end{aligned}
\tag{4.29}
$$

Where $M(t, h_t^D)$ is the market price of credit risk in D.
Therefore, the PDE followed by collateralized credit derivatives denominated in F is given by

$$\frac{\partial E_t}{\partial t}+\left(\mu_t^{h,\mathbb{P}} - M^D(t,h_t^D)\sigma_t^h + \rho_t\sigma_t^h\sigma_t^X\right)\frac{\partial E_t}{\partial h_t^D}+\frac{1}{2}(\sigma_t^h)^2\frac{\partial^2 E_t}{\partial (h_t^D)^2}+\Delta E_t\frac{h_t^F}{1-R}-c_t^F E_t = 0$$

The solution to the last PDE for a credit default swap with the corresponding terminal condition is

$$\begin{aligned} NPV_t^{\text{CDS},F} &= \underbrace{SE_{\mathbb{Q}^f}\left[\sum_{j=1}^{n}\gamma_j 1_{\{\tau>t_j\}}\exp\left(-\int_{s=t}^{t_j} c_s^F ds\right)\Big|\mathcal{F}_t\right]}_{\text{Premium leg}} \\ &\quad -(1-R)\underbrace{E_{\mathbb{Q}}\left[\int_{s=t}^{t_n}\exp\left(-\int_{u=t}^{s} c_u^F du\right)1_{\{\tau\in(s,s+ds]\}}\Big|\mathcal{F}_t\right]}_{\text{Default leg}} \end{aligned} \tag{4.30}$$

In a measure $\mathbb{Q}^f$ under which the drift of h_t^D is given by $\mu_t^{h,\mathbb{P}} - M^D(t,h_t^D)\sigma_t^h + \rho_t\sigma_t^h\sigma_t^X$ and the default intensity is given by $\frac{h_t^F}{1-R}$. It is important to emphasize that compared to $\mathbb{Q}$, under $\mathbb{Q}^f$:

- The drift of h_t^D changes by the regular Quanto adjustment.
- The default intensity changes from $\frac{h_t^D}{1-R}$ to $\frac{h_t^F}{1-R}$.

At this point it is convenient to gain some intuition relative to the meaning of the difference between h_t^F and h_t^D. In order to do so, we go back to the portfolio consistent on selling protection in currency D and buying protection in currency F with equal notional measured in D. The NPV at time t of this portfolio is 0, therefore the expected value of its change under measure $\mathbb{Q}$ must also be 0. Therefore

$$\begin{aligned} 0 &= E_{\mathbb{Q}}\left[\left(h_t^D - h_t^F\right)dt - (1-J_t)(1-R)dN_t\Big|\mathcal{F}_t\right] \\ &= \left(h_t^D - h_t^F\right)dt - \left(1 - E_{\mathbb{Q}}\left[J_t\Big|\mathcal{F}_t\right]\right)(1-R)\frac{h_t^D}{1-R}dt \end{aligned}$$

Where we have taken into account that under $\mathbb{Q}$ the default intensity is given by $\frac{h_t^D}{1-R}$. Which implies

$$J_t^{\mathbb{Q}} = E_{\mathbb{Q}}\left[J_t \middle| \mathcal{F}_t\right] = \frac{h_t^F}{h_t^D}$$

That is, the ratio $\frac{h_t^F}{h_t^D}$ is related with the expected value of the jump experienced by the FX rate upon default of the credit reference.

Under measure $\mathbb{Q}$, the following ratio should behave as a martingale

$$\frac{\beta_t^F X_t}{\beta_t^D}$$

Therefore, the risk neutral drift $\mu_t^{\mathbb{Q},X}$ of X_t is obtained by imposing the martingale condition

$$E_{\mathbb{Q}}\left[d\left(\frac{\beta_t^F X_t}{\beta_t^D}\right)\middle|\mathcal{F}_t\right] = E_{\mathbb{Q}}\left[\frac{\beta_t^F X_t}{\beta_t^D}\left((c_t^F - c_t^D + \mu_t^{\mathbb{Q},X} + (J_t - 1)dN_t^{\mathbb{Q}}\right)\middle|\mathcal{F}_t\right]$$

Which implies

$$\mu_t^{\mathbb{Q},X} = c_t^D - c_t^F + (1 - J_t^{\mathbb{Q}})\frac{h_t^D}{1 - R}$$

4.5.2 Spread and REPO rates under different currencies

In order to explore the relationship, we compare the following alternatives to get short term financing in currency F:

- Fund in currency F.
- Fund in currency D, change the received amount to D through a spot FX transaction and hedge the FX risk through a forward transaction.

Under the first option the reference credit bears a funding cost of f_t^F.

Under the second option, we assume that the market discounts and relative appreciation of the foreign currency with respect to the domestic upon default of the counterparty. This reflects the fact that the reference credit plays a key role in the D economy. Therefore $J_t^{\mathbb{P}} \geq 1$ although with an uncertain value.

Therefore, the counterparty of the forward transaction will be left with wrong way risk. The way this risk can be hedged is by trading in CDSs denominated in both currencies with equal notionals measured in the same currency and opposite signs. The trades are represented in figures 4.1, 4.2.

Therefore the funding rate through the second alternative (and for no arbitrage opportunities to exist, it must be the same under both alternatives) is:

$$f_t^F = f_t^D + (c_t^F - c_t^D) + b_t + (h_t^F - h_t^D)$$

Therefore

$$s_t^F = f_t^F - c_t^F = s_t^D + b_t + (h_t^F - h_t^D)$$

With the help of the last equation we can also establish the relationship between REPO rates in both currencies: $r_t^F = r_t^D + (c_t^F - c_t^D) + b_t$

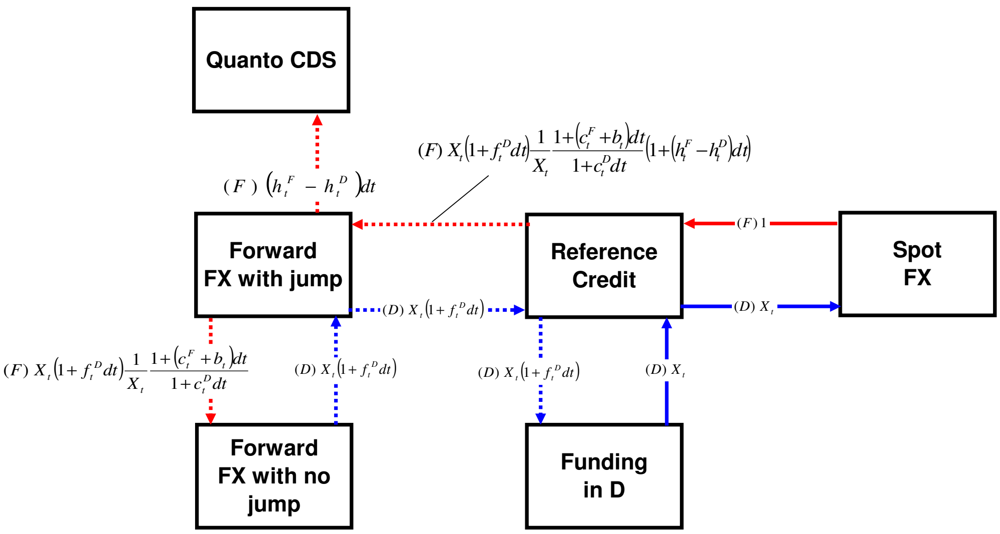


Figure 4.1: Continuous lines represent cash flows at time t whereas discontinuous ones represent cash flows at time $t + dt$. Blue lines indicate amounts denominated in currency D, whereas red ones represent amounts denominated in currency F

4.6 Conclussions:

- Hedging credit risk of a reference credit implies hedging 2 different components (spread risk and default risk).
- Cash collateralized credit derivatives should be discounted at OIS rates.
- Bonds should be discounted at REPO rates.
- Implied REPO rates could be obtained from both bond and CDS quotes.
- Funding spreads in different currencies are related through the cross currency basis and quanto CDSs.

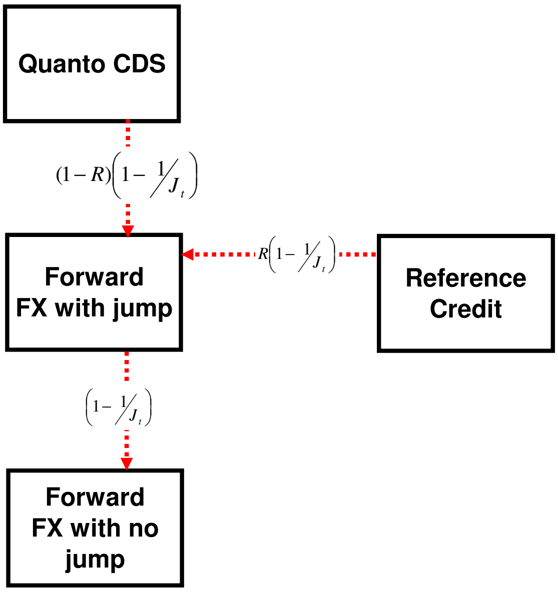


Figure 4.2: Cash flows upon default

Chapter 5

PDE for risky derivatives

5.1 Replicating portfolio for risky derivatives

We have seen how the price of a derivative depends on the credit of the counterparty (we assume the hedger is default-free). If we want to hedge the derivative, we will have to hedge both the market risk due to variations on the underlying and also the credit risk.

Let us consider for instance an equity derivative. The risks to hedge are:

- Equity movements
- Counterparty credit spread movements
- Counterparty default

Since we have three risks we will need three instruments in our hedging portfolio. We will chose these instruments to be the equity spot, a CDS C_t of maturity T on the counterparty, and a CDS of differential maturity (denoted by Θ_t) on the counterparty.

The hedging portfolio is:

$$V_t = \alpha_t S_t + \gamma_t C_t + \epsilon_t \Theta_t + \beta_t$$

where β_t is a bank account. Matching the sensitivities of the risky derivative with those of the hedging portfolio, we get:

$$\alpha = \frac{\partial V_t}{\partial S_t} \qquad \gamma_t = \frac{\frac{\partial V_t}{\partial h_t}}{\frac{\partial C_t}{\partial h_t}} \qquad \epsilon_t = \frac{\Delta V_t - \gamma_t \Delta C_t}{(1 - R)}$$

Differentiating the hedging equation and having these equalities in mind, we get:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \frac{1}{2}\sigma_h^2 \frac{\partial^2 V}{\partial h_t^2} + \rho\sigma_h\sigma_S S_t \frac{\partial^2 V}{\partial S_t \partial h_t} + (r_t - q_t) S_t \frac{\partial V}{\partial S_t} + \frac{h}{1-R}\Delta V_t - r_t V_t$$
$$= \frac{\frac{\partial V_t}{\partial h_t}}{\frac{\partial C_t}{\partial h_t}} \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma_h^2 \frac{\partial^2 C}{\partial h_t^2} + \frac{h}{1-R}\Delta C_t - r_t C_t \right)$$

where q_t is the dividend yield of the underlying equity. It is important to remember that C_t is a CDS, and therefore it should fulfill the PDE of a credit derivative. With this in mind, we can write:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma_h^2 \frac{\partial^2 C}{\partial h_t^2} + \frac{h}{1-R}\Delta C_t - r_t C_t = -(\mu_t^h - \sigma_t^h M_t)\frac{\partial C_t}{\partial h_t}$$

We can finally write the PDE for the risky derivative as:

$$\begin{aligned} \frac{\partial V_t}{\partial t} &+ (\mu_t^h - \sigma_t^h M_t)\frac{\partial V_t}{\partial h_t} + (r_t - q_t) S_t \frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma_S^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} \\ &+ \frac{1}{2}\sigma_h^2 \frac{\partial^2 V}{\partial h_t^2} + \rho\sigma_h\sigma_S S_t \frac{\partial^2 V}{\partial S_t \partial h_t} + \underbrace{\frac{h_t}{1-R}}_{\text{inst. default prob.}} \underbrace{\Delta V_t}_{\text{Jump to default}} = r_t V_t \\ s.t \qquad & V(T \wedge \tau) = V_T \mathbf{1}_{\{\tau > T\}} + \pi_\tau \mathbf{1}_{\{\tau \le T\}} \end{aligned} \tag{5.1}$$

Where $\pi_\tau = (V^{rf})_\tau^- + R(V^{rf})_\tau^+$ is the amount to exchange at counterparty's default (Note that we have chosen a risk-free close-out).

We can see how the CVA appears naturally when we express the solution of this PDE in the form of expected value.

To avoid working with too long expressions, we will define:

$$\begin{aligned} \mathcal{L}V_t \equiv & \frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + + \frac{1}{2}\sigma_h^2 \frac{\partial^2 V}{\partial h_t^2} + \rho\sigma_h\sigma_S S_t \frac{\partial^2 V}{\partial S_t \partial h_t} + \\ & + (\mu_t^h - \sigma_t^h M_t)\frac{\partial V_t}{\partial h_t} + (r_t - q_t) S_t \frac{\partial V}{\partial S_t} \\ X_u \equiv & e^{-\int_t^u r_s ds} V_u \mathbf{1}_{\{\tau > u\}} \end{aligned}$$

Applying Itô to X_u, we get:

$$\begin{aligned} dX_u = & e^{-\int_t^u r_s ds} \mathbf{1}_{\{\tau > u\}} \left[\left(\frac{\partial V}{\partial u} + \frac{1}{2}\sigma_S^2 S_u^2 \frac{\partial^2 V}{\partial S_u^2} + + \frac{1}{2}\sigma_h^2 \frac{\partial^2 V}{\partial h_u^2} + \rho\sigma_h\sigma_S S_u \frac{\partial^2 V}{\partial S_t \partial h_u} \right) du \right. \\ & \left. + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial h} dh - r_u V_u - V_u dN_u \right] \end{aligned} \tag{5.2}$$

Now, under the measure $\mathbf{Q}$ where the drift of the equity is $(r_t - q_t)S_t$ and the drift of the credit spread is $\mu_t^h - \sigma_t^h M_t$, we can write:

$$dX_u = e^{-\int_t^u r_s ds}\mathbf{1}_{\{\tau>u\}}\left(\mathcal{L}V_u du - r_u V_u du - V_u dN_u + \sigma_S S_u dW_u^S + \sigma_h dW_u^h\right)$$

From the PDE for the risky derivative we know that:

$$\mathcal{L}V_u du - r_u V_u = -\frac{h}{1-R}\Delta V_u$$

and from here, if we take expected values and integrate:

$$E^{\mathbf{Q}}\left[\int_t^T dX_u \,\middle|\, \mathcal{F}_t\right] =$$
$$= E^{\mathbf{Q}}\left[\int_t^T e^{-\int_t^u r_s ds}\mathbf{1}_{\{\tau>u\}}\left(\left((dN_u - \tfrac{h}{1-R}du\right)\Delta V_u - (V_u + \Delta V_u)dN_u\right) \,\middle|\, \mathcal{F}_t\right]$$

The term $dN_u - \frac{h}{1-R}$ vanishes because it is nothing but the integral of a Cox process less its compensator. We arrive to:

$$E^{\mathbf{Q}}\left[X_T \mid \mathcal{F}_t\right] - X_t = -E^{\mathbf{Q}}\left[\int_t^T e^{-\int_t^u r_s ds}\mathbf{1}_{\{\tau>u\}}(V_u + \Delta V_u)dN_u \,\middle|\, \mathcal{F}_t\right]$$

and from here:

$$\begin{aligned} V_t = E^{\mathbf{Q}}\left[e^{-\int_t^T r_s ds}V_T \,\middle|\, \mathcal{F}_t\right] - E^{\mathbf{Q}}\left[e^{-\int_t^T r_s ds}V_T\mathbf{1}_{\{\tau\le T\}} \,\middle|\, \mathcal{F}_t\right] + \\ + E^{\mathbf{Q}}\left[\int_t^T e^{-\int_t^u r_s ds}\mathbf{1}_{\{\tau>u\}}(V_u + \Delta V_u)dN_u \,\middle|\, \mathcal{F}_t\right] \end{aligned} \tag{5.3}$$

where the second term in the left hand side of the equation is:

$$\begin{aligned} E^{\mathbf{Q}}\left[e^{-\int_t^T r_s ds}V_T\mathbf{1}_{\{\tau\le T\}} \,\middle|\, \mathcal{F}_t\right] &= E^{\mathbf{Q}}\left[\int_t^T e^{-\int_t^T r_s ds}V_T\mathbf{1}_{\{\tau>u\}}dN_u \,\middle|\, \mathcal{F}_t\right] = \\ &= E^{\mathbf{Q}}\left[E^{\mathbf{Q}}\left[\int_t^T e^{-\int_t^T r_s ds}V_T \,\middle|\, \mathcal{F}_u\right]\mathbf{1}_{\{\tau>u\}}dN_u \,\middle|\, \mathcal{F}_t\right] \\ &= E^{\mathbf{Q}}\left[\int_t^T e^{-\int_t^u r_s ds}V_u^{rf}\mathbf{1}_{\{\tau>u\}}dN_u \,\middle|\, \mathcal{F}_t\right] \end{aligned}$$

With this, and taking into account that:

$$\Delta V_u = (V^{rf})_u^- + R(V^{rf})_u^+ - V_u$$

$$\boxed{V_t = \underbrace{E^{\mathbf{Q}}\left[e^{-\int_t^T r_s ds} V_T \;\middle|\; \mathcal{F}_t\right]}_{\text{Risk-Free price}} - \underbrace{(1-R)E^{\mathbf{Q}}\left[\int_t^T e^{-\int_t^u r_s ds}\mathbf{1}_{\{\tau>u\}}(V^{rf})_u^+ dN_u \;\middle|\; \mathcal{F}_t\right]}_{\text{CVA}}}$$

In former chapters we have obtained the former expression for CVA by discounting flows on default. Here we have obtained the same expression from replication arguments.

Conclusions:

- As long as we can hedge all risks, the price for the risky derivative derived from replicating arguments, coincide with the one resting on discounting flows.
- The price of a risky derivative can be express as a sum of two terms: One that is the risk-free price of the derivative, assuming a non-defaultable counterparty, and other term that uniquely accounts for the counterparty credit risk.

Chapter 6

Hedging CVA

CVA can be seen as a derivative. In fact, derivatives paying the NPV of another derivative if an entity defaults exist and are called *contingent CDS*. Therefore we can build a replicating portfolio to hedge CVA. That is the usual way of working: rather than hedging the risky derivative, the corresponding desk hedges the risk-free derivative, while the CVA desk hedges CVA.

The motivation to hedge CVA is:

- Mitigate losses in default of a counterparty
- Lowering the volatility of the bank's results
- Diminishing capital requirements

Let's consider a stand-alone swap. We will also assume that both the interest rate curve and counterparty's credit spread curve are driven by one stochastic factor each one.

The hedging portfolio should contain:

- Market risks: movements of the IR curve. Hedged with α_t units of a IR derivative B_t
- Movements of counterparty's credit spread curve. Hedged with γ_t units of a CDS of maturity T_1, C_t.
- Counterparty default. Hedged with $\tilde{\gamma}_t$ units of a CDS of maturity T_2, $\tilde{C}_t$.
- A bank account β_t

We will be able to hedge all these risks if there is liquidity in the CDS with the counterparty as credit reference (complete market). If there are no CDS on the counterparty, we have built a credit curve by means of a proxy (CDS index, etc). We will be able to hedge the credit spread movements, but not the jump to default.

6.1 Hedging CVA in complete markets

To get the amount of each of the hedging instruments to be bought, we only need to match CVA sensitivities to those of the hedging portfolio.

$$
\begin{aligned}
\frac{\partial CVA_t}{\partial r_t} &= \alpha_t \frac{\partial B_t}{\partial r_t} + \gamma_t \frac{\partial C_t}{\partial r_t} + \tilde{\gamma}_t \frac{\partial \tilde{C}_t}{\partial r_t} \\
\frac{\partial CVA_t}{\partial s_t} &= \gamma_t \frac{\partial C_t}{\partial s_t} + \tilde{\gamma}_t \frac{\partial \tilde{C}_t}{\partial s_t} \\
\Delta CVA_t &= \gamma_t \Delta C_t + \tilde{\gamma}_t \Delta \tilde{C}_t
\end{aligned}
$$

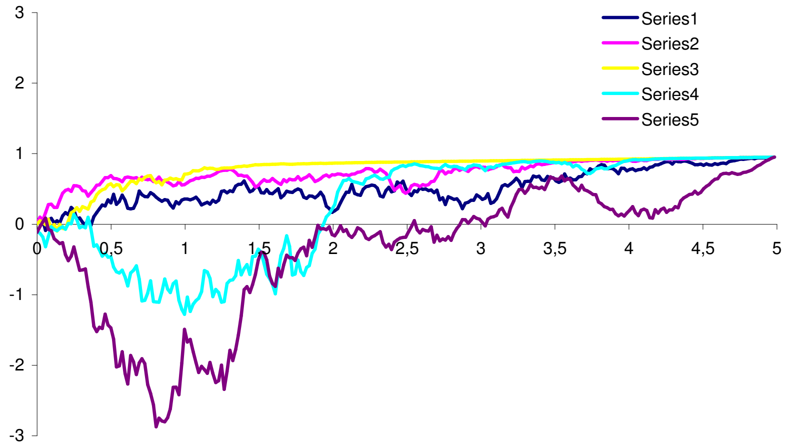
From here we obtain :

$$
\begin{aligned}
\gamma_t &= \frac{\frac{\frac{\partial CVA_t}{\partial s_t}}{\frac{\partial \tilde{C}_t}{\partial s_t}} - \frac{\Delta CVA_t}{\Delta \tilde{C}_t}}{\frac{\frac{\partial C_t}{\partial s_t}}{\frac{\partial \tilde{C}_t}{\partial s_t}} - \frac{\Delta C_t}{\Delta \tilde{C}_t}} \\
\tilde{\gamma}_t &= \frac{\frac{\frac{\partial CVA_t}{\partial s_t}}{\frac{\partial C_t}{\partial s_t}} - \frac{\Delta CVA_t}{\Delta C_t}}{\frac{\frac{\partial \tilde{C}_t}{\partial s_t}}{\frac{\partial C_t}{\partial s_t}} - \frac{\Delta \tilde{C}_t}{\Delta C_t}} \\
\alpha_t &= \frac{\frac{\partial CVA_t}{\partial r_t} - \gamma_t \frac{\partial C_t}{\partial r_T} - \tilde{\gamma}_t \frac{\partial \tilde{C}_t}{\partial r_t}}{\frac{\partial B_t}{\partial r_t}}
\end{aligned}
$$

Here we have studied the case of a stand-alone deal, but CVA is managed for the whole bank's portfolio.

6.2 Illustration: CVA Hedge in complete markets

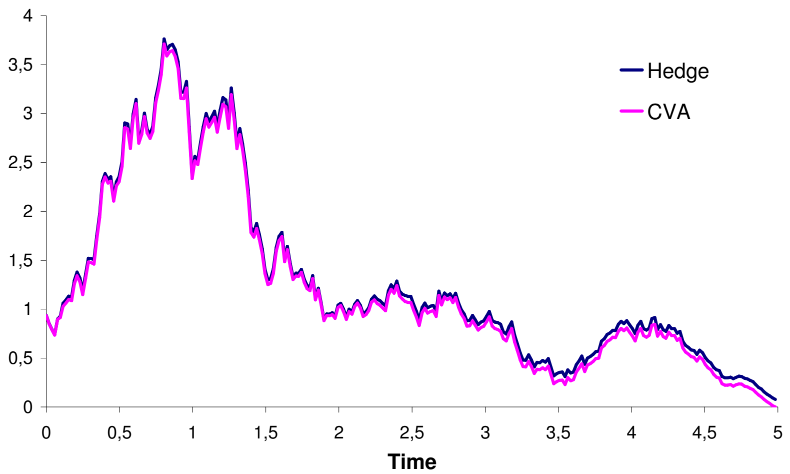
To better understand the concepts we will have a look at a FX forward. We show the evolution of unhedged CVA Desk position (initial CVA changed accrued at the OIS rate minus CVA at every future date) conditional on no default



Evolution of unhedged CVA Desk position (Initial CVA changed accrued at the OIS rate minus CVA at every future date) conditional on no default. Five paths plotted.

- Under no default, initial CVA charged will turn into a profit (accrued), so that P&L will not depend on spreads and credit exposures throughout time (Is this good news?).
- Before maturity, the CVA desk is exposed to P&L variance. Increases in spreads and increases in credit exposures will imply losses before maturity.

If we hedge all risks, we get (for path #5):



We stress here that a complete credit spread hedge is not possible since in general we will have as many stochastic factors as liquid CDS maturities. If that number is N, we need $N+1$ CDS to hedge the N factors plus the jump to default. But in the market there are only N CDS, so the $N-th$ principal component will always be open.

6.3 Hedging CVA in incomplete markets

If there is no liquid CDS market for the counterparty, we relate its credit to a proxy. We can buy CDS on that proxy, but with that we only get protection to spread movements. The jump to default cannot be hedged. To get rid of uncertainty in the event of default, we can force our hedging portfolio to be insensitive to default of the underlying credit name for the CDS. That is, we require:

$$\underbrace{\frac{\partial \mathrm{CVA}_t}{\partial S_t} = \frac{\partial \mathrm{Hedge}_t}{\partial S_t}}_{\text{Market hedge}} \qquad \underbrace{\frac{\partial \mathrm{CVA}_t}{\partial h_t} = \frac{\partial \mathrm{Hedge}_t}{\partial h_t}}_{\text{Spread hedge}} \qquad \underbrace{\mathrm{JTD}_t^{\mathrm{CVA}} = 0}_{\text{Jump to default hedge}}$$

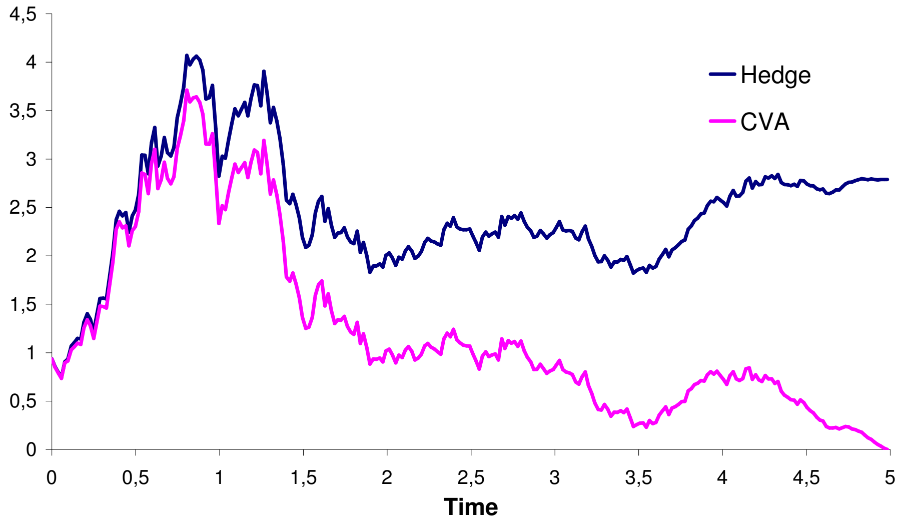


Figure 6.1: Evolution of CVA and its hedge (assuming spread and market hedge) for path #5

Conditional on no default, for the particular path, there seems to be a positive carry of the spread and market hedge over CVA. This positive carry can also be seen by analyzing weekly changes.

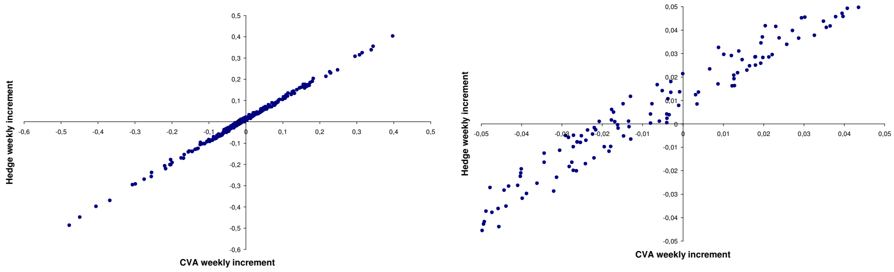
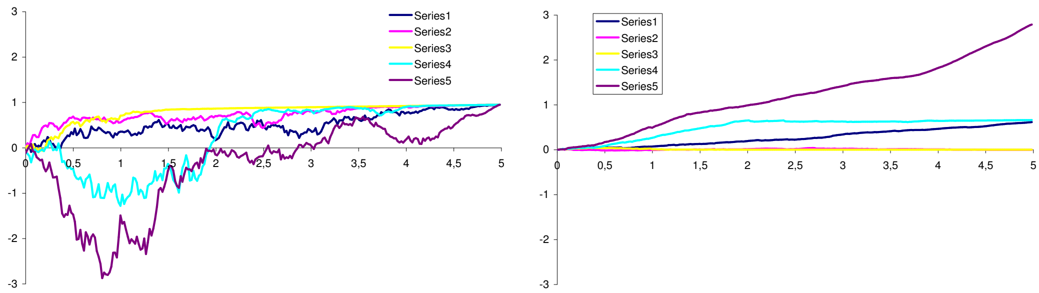


Figure 6.2: Scatter plot of weekly changes of CVA (x-axis) vs Hedge (y-axis).

6.3.1 No hedge and spread and market hedge compared

From the former study for path #5 it seems that there is a positive carry when we do not hedge jump to default leaving our hedging portfolio insensitive to default. Let us see what happens with the different paths:



Evolution of unhedged (left) and spread & market hedged (right) CVA desk position

- Scenarios that underperform (conversely outperform) before maturity under no hedge outperform (underperform) before maturity under spread and market hedge.
- Spread and market hedge smooths the evolution of P&L with a positive carry under most market conditions.
- We can end up with a higher P&L at maturity by hedging spread and market risk that under no hedge (but we can also end up with a smaller P&L).

It seems that the proposed hedge helps to smooth the results while keeping on average a positive carry. We are left with the following question: What is the main driver of spread and market hedge P&L?. To answer this question, let's have a look to the full CVA hedge equation:

$$d\left(\text{Hedge}_t^{\text{Full}} - \text{CVA}_t\right) = 0$$

So that

$$d\left(\text{Hedge}_t^{\text{Spd\&Mkt}} + \text{Hedge}_t^{\text{JtD}} - \text{CVA}_t\right) = 0$$
$$\Downarrow$$
$$d\left(\text{Hedge}_t^{\text{Spd\&Mkt}} - \text{CVA}_t\right) = -d(\text{Hedge}_t^{\text{JtD}})$$

So the difference between CVA and the hedge portfolio when we leave the hedging portfolio insensitive to jump to default is precisely the jump to default hedge. To hedge this JtD we need an overnight CDS (that is not sensitive to spread changes) with a notional such that its jump to default offsets that of CVA:

$$\underbrace{N(1-R)}_{\text{Notional times JTD of o/n CDS}} = \underbrace{\text{JTD}_t^{CVA}}_{\text{JTD of CVA}} \Rightarrow N = \frac{\text{JTD}_t^{CVA}}{(1-R)}$$

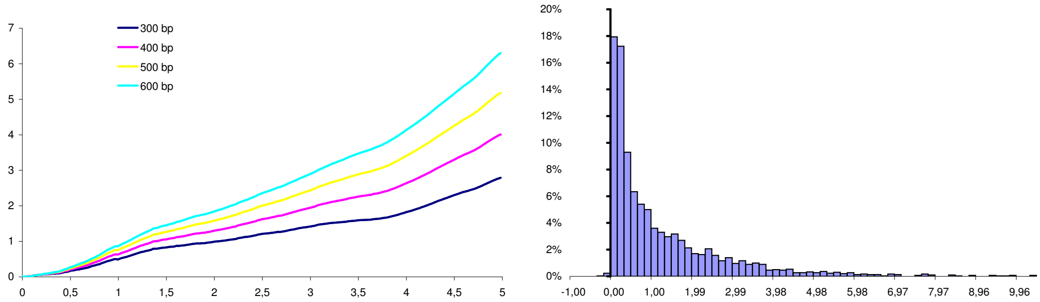
Where $\text{JTD}_t^{\text{CVA}} = (1-R)\text{NPV}_t^+ - \text{CVA}_{t^-}$
If the counterparty survives to time t, that is if $\tau > t$:

$$-d(\text{Hedge}_t^{\text{JtD}}) = \frac{h_t^{o/n}}{1-R}\text{JTD}_t^{\text{CVA}}$$

Where $h_t^{o/n}$ is the short term CDS spread of the counterparty.
We see that the carry of the spread and market hedged position is due to the short term CDS premium that we are not paying while being unhedged to the jump experienced by CVA upon default of the counterparty.
So that the carry of the spread and market hedged position is greater under:

- High spreads.
- Hight credit exposures.

That is, the scenarios where defaults are more probable (high spreads) and more harmful (high exposures). One could argue that this carry is not always positive, since JTD^{CVA} can become negative if $\text{CVA}_{t^-} > (1-R)\text{NPV}_t^+$. This happens, for example, always that $\text{NPV}_t < 0$.
However, the magnitude of negative JTD in CVA is quite limited whereas the magnitude of positive JTD in CVA can become huge. To illustrate that, in the next figure we show the evolution of CVA position (hedging portfolio minus CVA) hedged to both spread and market risks for scenario #5 as a function of the clean spread curve level (left plot), and the histogram of partial hedge P&L at maturity of the forward contract (right).
The positive carry is given by the integral of the overnight CDS spread times the JtD of CVA. We can see this by looking at the CVA PDE. To find what's the PDE



Left: Evolution of CVA position (hedging portfolio minus CVA) hedged to both spread and market risks for scenario #5 as a function of the clean spread curve level.
Right: Histogram of partial hedge P&L at maturity of the forward contract.

that CVA must satisfy, we only need to take the CVA for the risky derivative and express:

$$V_t = V_t^{rf} + CVA_t$$

By doing that, and taking into account that the risk-free equity derivative satisfies:

$$\frac{\partial V_t^{rf}}{\partial t} + (r_t - q_t)S_t\frac{\partial V^{rf}}{\partial S_t} + \frac{1}{2}\sigma_S^2 S_t^2\frac{\partial^2 V^{rf}}{\partial S_t^2} = r_t V_t^{rf}$$

We get that:

$$\frac{\partial CVA_t}{\partial t} + (\mu_t^h - \sigma_t^h M_t)\frac{\partial CVA_t}{\partial h_t} + (r_t - q_t)S_t\frac{\partial CVA}{\partial S_t} + \frac{1}{2}\sigma_S^2 S_t^2\frac{\partial^2 CVA_t}{\partial S_t^2} +$$
$$+\frac{1}{2}\sigma_h^2\frac{\partial^2 CVA_t}{\partial h_t^2} + \rho\sigma_h\sigma_S S_t\frac{\partial^2 CVA}{\partial S_t \partial h_t} + \frac{h}{1-R}\Delta CVA_t = r_t CVA_t$$

which is not surprising since it is the PDE of a hybrid equity-credit derivative. The portfolio we have is:

$$\pi_t = CVA_t - \alpha_t^{CVA} S_t - \gamma_t CDS_t - \beta_t$$

where $\gamma_t CDS_t$ stands for a linear combination of N different CDS instruments. The number $N =$ is equal to $M+1$, where M is the number of factors of the credit spread that we are hedging. We need one more CDS to ensure that the hedging portfolio is not sensitive to default.

$$d\pi_t = \mathcal{L}_{Sh}CVA_t + \Delta CVA_t dN_t - \mathcal{L}_h CDS_t - (r_t - q_t)\alpha_t^{CVA} S_t - r_t\gamma_t CDS_t - r_t CVA_t$$

where we have defined:

$$\mathcal{L}_h CDS(t,M) = \frac{\partial CDS(t,M)}{\partial t} + \frac{1}{2}\left(\sigma_t^h\right)^2 \frac{\partial^2 CDS(t,M)}{\partial {h_t}^2}$$
$$\mathcal{L}_{Sh} CVA_t = \frac{\partial V_t}{\partial t} + \frac{1}{2}\frac{\partial^2 V_t}{\partial S_t^2} S_t^2 (\sigma_t^S)^2 + \frac{1}{2}\frac{\partial^2 V_t}{\partial {h_t}^2}(\sigma_t^h)^2 + \frac{\partial^2 V_t}{\partial S_t \partial h_t} S_t \sigma_t^S \sigma_t^h \rho_t$$

Taking into account the PDEs for CVA and CDS, we get:

$$d\pi_t = \Delta CVA_t \left(\frac{h_t}{1-R} dt - dN_t^{\mathbb{P}}\right) + c_t \pi_t dt$$

But we have already seen that $\lambda_t^{\mathbb{Q}} := \frac{h_t}{1-R}$. Therefore, conditional on the default time of the counterparty being greater than the maturity T of the longer derivative for the counterparty

$$d\pi_t | \{\tau > T\} = \Delta CVA_t \lambda_t^{\mathbb{Q}} dt + c_t \pi_t dt$$

If we define X_t as the discounted P&L conditional on survival

$$X_t := e^{-\int_{s=0}^t c_s ds} \pi_t | \{\tau > T\}$$

Differentiating the last expression

$$dX_t = -c_t e^{-\int_{s=0}^t c_s ds} \pi_t | \{\tau > T\} dt + e^{-\int_{s=0}^t c_s ds} d\pi_t | \{\tau > T\} = e^{-\int_{s=0}^t c_s ds} \Delta CVA_t \lambda_t^{\mathbb{Q}} dt$$

Which yields

$$X_T - \underbrace{X_0}_{=0} = e^{-\int_{s=0}^T c_s ds} \pi_T | \{\tau > T\} = \int_{s=0}^T dX_s = \int_{s=0}^T e^{-\int_{h=0}^s c_h dh} \Delta CVA_s \lambda_s^{\mathbb{Q}} ds$$

To conclude, when we do not have a liquid market of CDS for the counterparty we can hedge by building a hedging portfolio insensitive to jump to default. This implies the following:

- CVA will be marked to market reflecting in a more accurate way the price of liquidating the positions before maturity and avoiding cumbersome situations in which illiquid and liquid CVA positions are handled in a meaningless asymmetrical way.

- Under the CVA spread and market hedge, the CVA position will evolve smoothly and with a positive carry whenever the jump to default of CVA is positive (CVA increases upon default of the counterparty). Although possible, the probability and magnitude of a negative carry is quite limited.

- The CVA spread and market hedge does not have a negative carry compared to leaving non liquid CVA positions completely unhedged.

- Under high spread and high credit exposures environments the carry of the CVA market hedge will increase, compensating the default events that the hedger will more likely be exposed to in such environments.

- There's no need to be too sophisticated in the curve mapping methodology. Our target is to maximize the carry experienced by the partially hedged CVA position under scenarios with a high number of defaults. A good alternative is to map illiquid CDS curves to volatile indexes whose spread might most likely be affected by the default events of illiquid CVA counterparties.

Chapter 7

CVA, DVA and FVA

7.1 Introduction

In former sections we have studied CVA hedge on complete and incomplete markets. But we already now that the risky price of the derivative also has corrections due to FVA and DVA. In this section we will see how can we hedge these contributions, paying special attention to the link between DVA and FVA.

In traditional regulation and also in classical valuation, the concept of fair price is always present. A derivative has a unique value, so that hedger and investor can agree on its price. This has been one of the arguments to include DVA in the price, since otherwise we can never agree on it. But the price of a derivative should reflect all its hedging costs, and this costs are different for each entity. For instance, a corporate has different capital requirements to those of a bank, each entity has its own funding spread, etc. So it does not seem illogical that each entity has its own cost and therefore its own price. Deals are closed because the investor has appetite for the risk, and it is willing to pay a fee for it, while the hedger may not have appetite for that risk.

We will make the following assumptions in order to find the price of a derivative taking into account default risk of both hedger and investor:

- The price of a derivative should reflect all of its hedging costs.
- Since nowadays a very high percentage (if not all) of uncollateralized transactions imply a counterparty acting as an investor (risk taker) and a hedger (risk hedger), the derivative's price should just reflect the hedging costs borne by the hedger.
- The hedger will only be willing to hedge the fluctuations in the derivative's price that he will experience while being alive, that is, while not having defaulted.

- There is neither CVA nor FVA to be made to fully collateralized derivatives (with continuous collateral margining in cash, symmetrical collateral mechanisms and no thresholds, minimum transfer amounts, ...).

Market assumptions:

- There is a liquid CDS (credit default swap) curve for the investor.

- There is a liquid curve of bonds issued by the hedger.

- There is a liquid market of cash collateralized derivatives written on the same underlying asset as that of the uncollateralized derivative being priced.

- Continuous hedging is possible, unlimited liquidity, no bid-offer spreads, no trading costs.

- Recovery rates are either deterministic or there are recovery locks available so that recovery risk is not a concern.

Model assumptions:

- Both the hedger and the investor are defaultable. Simultaneous default is not possible.
- The underlying asset follows a diffusion process under the real world measure.
- The derivative's underlying asset is unaffected by the default event of any of the counterparties.
- Both the credit spreads of the investor and of the hedger are stochastic following correlated diffusion processes under the real world measure.

Funding costs arise due to asymmetries between the collateral characteristics of derivatives traded with investors (that could be non collateralized or partially collateralized) and those of the hedging instruments (usually traded in the interbank market, where deals are fully collateralized). Hence, we will assume that market risk is hedged with a fully collateralized derivative.

7.2 The hedgeable risks while managing uncollateralized derivatives

We will assume that under the real world measure $\mathbb{P}$, the evolution of the relevant market variables are governed by:

$$\begin{array}{ll} dS_t = \mu_t^S S_t dt + \sigma_t^S S_t dW_t^{S,\mathbb{P}} & \text{Underlying asset} \\ dh_t^I = \mu_t^I dt + \sigma_t^I dW_t^{I,\mathbb{P}} & \text{Investor's short term CDS spread} \\ dh_t^H = \mu_t^H dt + \sigma_t^H dW_t^{H,\mathbb{P}} & \text{Hedger's short term CDS spread} \end{array} \tag{7.1}$$

With time dependent correlations

$$\rho_t^{S,I} dt = dW_t^{S,\mathbb{P}} dW_t^{I,\mathbb{P}}, \quad \rho_t^{H,I} dt = dW_t^{H,\mathbb{P}} dW_t^{I,\mathbb{P}}, \quad \rho_t^{S,H} dt = dW_t^{S,\mathbb{P}} dW_t^{H,\mathbb{P}}$$

The other two sources on uncertainty are the default indicator processes:

$$\begin{array}{ll} N_t^{I,\mathbb{P}} = 1_{\{\tau_I \leq t\}} & \text{with real world default intensity } \lambda_t^{I,\mathbb{P}} \\ N_t^{H,\mathbb{P}} = 1_{\{\tau_H \leq t\}} & \text{with real world default intensity } \lambda_t^{H,\mathbb{P}} \end{array} \tag{7.2}$$

τ_I and τ_H will represent the default times of the investor and the hedger.
The derivatives price will depend on every risk factor:

$$V_t = V(t, S_t, h_t^I, h_t^H, N_t^{I,\mathbb{P}}, N_t^{H,\mathbb{P}})$$

V_t represents the derivative's value from the investor's perspective
Therefore

$$dV_t = \underbrace{\frac{\partial V_t}{\partial S_t} dS_t}_{\text{Delta risk}} + \underbrace{\frac{\partial V_t}{\partial h_t^I} dh_t^I}_{\text{Spread risk to I}} + \underbrace{\frac{\partial V_t}{\partial h_t^H} dh_t^H}_{\text{Spread risk to H}} + \underbrace{\Delta V_t^I dN_t^{I,\mathbb{P}}}_{\text{Default risk to I}} + \underbrace{\Delta V_t^H dN_t^{H,\mathbb{P}}}_{\text{Default risk to H}} + \underbrace{O(dt)}_{\text{Theta}} \tag{7.3}$$

ΔV_t^I Jump in the value of the derivative if the investor defaults at t.
ΔV_t^H Jump in the value of the derivative if the hedger defaults at t.
The hedger will only be exposed to:

- $\frac{\partial V_t}{\partial S_t} dS_t$
- $\frac{\partial V_t}{\partial h_t^I} dh_t^I$
- $\frac{\partial V_t}{\partial h_t^H} dh_t^H$
- $\Delta V_t^I dN_t^{I,\mathbb{P}}$

Which of the components of (7.3) can actually be hedged?

- $\frac{\partial V_t}{\partial S_t} dS_t$: Yes. With either a REPO or a cash collateralized derivative on S_t.

- $\frac{\partial V_t}{\partial h_t^I} dh_t^I$ and $\Delta V_t^I dN_t^{I,\mathbb{P}}$: Yes. In a one factor world, with two CDSs written on the investor and with different maturities (two sensitivities to cancel, two CDSs to trade).

- $\frac{\partial V_t}{\partial h_t^H} dh_t^H$ and $\Delta V_t^H dN_t^{H,\mathbb{P}}$ altogether: No. Should trade on two CDSs written on the hedger (to kill both sensitivities). In general terms, the hedger will benefit from his default. The hedge should have a negative JtD. No counterparty will be willing to act as a counterparty under the deal is over-collateralized. Is trying to hedge the JtD the real problem?

- $\frac{\partial V_t}{\partial h_t^H} dh_t^H$: Yes, as we will see.

The same source of risk that the hedger will not be able to hedge is the same source of risk whose cash flow will never be paid or received by the hedger since he will already be defaulted.
Is there really an incentive to hedge $\Delta V_t^H dN_t^{H,\mathbb{P}}$?

- Employees working for the hedger: No. They will have lost their jobs after the default event.

- Equity holders: No. They will have lost their investment after the default event.

- Bond holders: No. Trying to hedge this component will imply posting extra collateral that will reduce the recovery rate left for the bond holders. In addition, this extra collateral can accelerate the default.

7.3 The replication strategy

As we have already seen, the hedger will hedge the risk factors that he is exposed to on every path under which he finds himself not defaulted (that are in fact the only ones that are hedgeable). These risk factors are:

- Market risk due to changes in S_t.

- Investor's spread risk due to changes in h_t^I.

- Investor's default event.
- Hedger's spread risk due to changes in h_t^H.

We explore how to hedge the different risk factors.

7.4 Hedging the market risk due to changes in S_t:

We will hedge the market risk by trading in a cash collateralized derivative written on S_t. H_t will represent the value of this derivative from the hedger's perspective.

⋆ **Hedging the spread and jump to default risks of the investor:**

Notice that regarding counterparty credit risk, the hedger is exposed to two different sources of uncertainty (default risk and credit spread risk). Therefore the hedger will have to trade on two CDSs written on the investor with different maturities.

Had we assumed an n factor model for the dynamic of the hedger's credit curve, the hedger would have to trade on $n+1$ CDSs.

CDS(t,t+dt) is the value of an overnight credit default swap (with unit notional) under which the protection buyer pays a premium at time $t+dt$ equal to $h_t^I dt$. We will assume that $h_t^I dt$ is such that $CDS(t, t+dt) = 0$.

CDS(t,T) is a credit default swap maturing on a later date $T > t$. In general $CDS(t, T) \neq 0$. CDS(t,T) will be collateralized in cash.

⋆ **Hedging spread risks of the hedger (hedging liquidity risk):**

If $V_t > 0$, the hedger will have excess cash with which he will be able to buy back his own debt. If $V_t < 0$, the hedger will have to issue new debt.

In either case the hedger will have to decide the spread duration (sensitivity to spread changes) of the debt issued/bought back.

When a new uncollateralized derivative is replicated, the hedger will see that the spread duration of his debt is altered unless he imposes that the spread duration of the incoming uncollateralized derivative is perfectly matched with the spread duration of the bonds issued/bought back.

Notice that the hedger can match the spread duration of the uncollateralized derivative by trading on bonds with two different maturities while imposing that the net buyback is equal to V_t (issuance if V_t is negative).

We will assume that the hedger trades on bonds that mature on a future date T ($B(t, T)$) and also on short term bonds that mature on $t+dt$ ($B(t, t+dt)$). Had we assumed an n factor model for the dynamic of the hedger's credit curve, the hedger would have to trade on $n+1$ bonds issued by himself.

Ω_t and ω_t will represent the amounts to purchase (or issue if negative) in $B(t, t+dt)$ and $B(t, T)$ respectively.
In order to ensure that the self financing condition holds:

$$V_t = \Omega_t B(t, t+dt) + \omega_t B(t, T)$$

Which implies

$$\Omega_t = \frac{V_t - \omega_t B(t, T)}{B(t, t+dt)}$$

The hedging equation will be

$$\begin{aligned} V_t = \alpha_t H_t + \beta_t + \gamma_t CDS(t, T) + \epsilon_t \underbrace{CDS(t, t+dt)}_{=0} + \frac{V_t}{B(t, t+dt)} B(t, t+dt) + \\ + \omega_t \underbrace{\left(B(t, T) - \frac{B(t, T)}{B(t, t+dt)} B(t, t+dt) \right)}_{=0} \end{aligned}$$

β_t represents cash held in collateral accounts.
The change in β_t will be given by:

$$d\beta_t = -c_t \alpha_t H_t dt - c_t \gamma_t CDS(t, T) dt$$

In every path in which the hedger has not defaulted before $t+dt$ and conditional on both the investor and the hedger being alive at time t the change in V_t will be given by

$$dV_t = \mathcal{L}_{SIH} V_t dt + \frac{\partial V_t}{\partial S_t} S_t \sigma_t^S dW_t^S + \frac{\partial V_t}{\partial h_t^I} \sigma_t^I dW_t^I + \frac{\partial V_t}{\partial h_t^H} \sigma_t^H dW_t^H + \Delta V_t^I dN_t^{I,\mathbb{P}}$$

Where

$$\begin{aligned} \mathcal{L}_{SIH} V_t = & \frac{\partial V_t}{\partial t} + \mu_t^S S_t \frac{\partial V_t}{\partial S_t} + \mu_t^H \frac{\partial V_t}{\partial h_t^H} + \mu_t^I \frac{\partial V_t}{\partial h_t^I} + \frac{1}{2} \frac{\partial^2 V_t}{\partial S_t^2} S_t^2 (\sigma_t^S)^2 + \frac{1}{2} \frac{\partial^2 V_t}{\partial {h_t^H}^2} (\sigma_t^H)^2 \\ & + \frac{1}{2} \frac{\partial^2 V_t}{\partial {h_t^I}^2} (\sigma_t^H)^2 + \frac{\partial^2 V_t}{\partial S_t h_t^H} S_t \sigma_t^S \sigma_t^H \rho_t^{S,H} + \frac{\partial^2 V_t}{\partial S_t h_t^I} S_t \sigma_t^S \sigma_t^I \rho_t^{S,I} + \rho_t^{I,H} \sigma_t^I \sigma_t^H \frac{\partial^2 V_t}{\partial h_t^I h_t^H} \end{aligned}$$

The differential change in H_t

$$dH_t = \mathcal{L}_S H_t dt + \frac{\partial H_t}{\partial S_t} S_t \sigma_t^S dW_t^S$$

where

$$\mathcal{L}_S H_t = \frac{\partial H_t}{\partial t} + \mu_t^S S_t \frac{\partial H_t}{\partial S_t} + \frac{1}{2} S_t^2 \left(\sigma_t^S\right)^2 \frac{\partial^2 H_t}{\partial S_t^2}$$

The differential change in $CDS(t, t+dt)$ and in $B(t, t+dt)$

$$dCDS(t, t+dt) = h_t^I dt - (1 - R_I) dN_t^{I,\mathbb{P}}$$

$$dB(t, t+dt) = f_t^H B(t, t+dt) dt$$

Notice that the jump to default component of $B(t, t+dt)$ has been omitted since it will not be experienced by the hedger. f_t^H represents the hedger's funding rate.
The differential change of CDS(t,T)

$$dCDS(t,T) = \mathcal{L}_I CDS(t,T) dt + \frac{\partial CDS(t,T)}{\partial h_t^I} \sigma_t^I dW_t^I + \Delta CDS(t,T) dN_t^{I,\mathbb{P}}$$

with

$$\mathcal{L}_I CDS(t,T) = \frac{\partial CDS(t,T)}{\partial t} + \mu_t^I \frac{\partial CDS(t,T)}{\partial h_t^I} + \frac{1}{2} \left(\sigma_t^I\right)^2 \frac{\partial^2 CDS(t,T)}{\partial {h_t^I}^2}$$

And finally

$$dB(t,T) = \mathcal{L}_H B(t,T) dt + \frac{\partial B(t,T)}{\partial h_t^H} \sigma_t^H dW_t^H$$

where

$$\mathcal{L}_H B(t,T) = \frac{\partial B(t,T)}{\partial t} + \mu_t^H \frac{\partial B(t,T)}{\partial h_t^H} + \frac{1}{2} \left(\sigma_t^H\right)^2 \frac{\partial^2 B(t,T)}{\partial {h_t^H}^2}$$

Again, we have omitted the jump component in $B(t,T)$ since it will not be experienced by the hedger.
So that the hedging equation in differential form will be given by

$$
\begin{aligned}
&\mathcal{L}_{SIH}V_t dt + \tfrac{\partial V_t}{\partial S_t} S_t \sigma_t^S dW_t^S + \tfrac{\partial V_t}{\partial h_t^I}\sigma_t^I dW_t^I + \tfrac{\partial V_t}{\partial h_t^H}\sigma_t^H dW_t^H + \Delta V_t^I dN_t^{I,\mathbb{P}} = \\
&= V_t^+ f_t^H dt + V_t^- f_t^H dt - c_t\alpha_t H_t dt - c_t\gamma_t CDS(t,T)dt \\
&+\alpha_t\left(\mathcal{L}_S H_t dt + \tfrac{\partial H_t}{\partial S_t} S_t\sigma_t^S dW_t^S\right) \\
&+\gamma_t\left(\mathcal{L}_I CDS(t,T)dt + \tfrac{\partial CDS(t,T)}{\partial h_t^I}\sigma_t^I dW_t^I + \Delta CDS(t,T) dN_t^{I,\mathbb{P}}\right) \\
&+\epsilon_t\left(h_t^I dt - (1-R_I)dN_t^{I,\mathbb{P}}\right) \\
&+\omega_t\left(\mathcal{L}_H B(t,T)dt + \tfrac{\partial B(t,T)}{\partial h_t^H}\sigma_t^H dW_t^H - f_t^H B(t,T)dt\right)
\end{aligned}
\tag{7.4}
$$

In order to be hedged

$$
\begin{aligned}
\alpha_t &= \frac{\frac{\partial V_t}{\partial S_t}}{\frac{\partial H_t}{\partial S_t}} \\
\gamma_t &= \frac{\frac{\partial V_t}{\partial h_t^I}}{\frac{\partial CDS(t,T)}{\partial h_t^I}} \\
\epsilon_t &= \gamma_t \tfrac{\Delta CDS(t,T)}{1-R_T} - \tfrac{\Delta V_t^I}{1-R_I} \\
\omega_t &= \frac{\frac{\partial V_t}{\partial h_t^H}}{\frac{\partial B(t,T)}{\partial h_t^H}}
\end{aligned}
\tag{7.5}
$$

So that every risk factor disappears from the hedging equation

$$
\begin{aligned}
&\widetilde{\mathcal{L}}_{SIH}V_t = V_t^+ f_t^H + V_t^- f_t^H \\
&+\alpha_t\left(\widetilde{\mathcal{L}}_S H_t - c_t H_t\right) \\
&+\gamma_t\left(\widetilde{\mathcal{L}}_I CDS(t,T) - c_t CDS(t,T)\right) \\
&+\epsilon_t h_t^I \\
&+\omega_t\left(\widetilde{\mathcal{L}}_H B(t,T) - f_t^H B(t,T)\right)
\end{aligned}
$$

Where

$$\widetilde{\mathcal{L}}_{SIH}V_t = \tfrac{\partial V_t}{\partial t} + \tfrac{1}{2}\tfrac{\partial^2 V_t}{\partial S_t^2}S_t^2(\sigma_t^S)^2 + \tfrac{1}{2}\tfrac{\partial^2 V_t}{\partial {h_t^H}^2}(\sigma_t^H)^2 + \tfrac{1}{2}\tfrac{\partial^2 V_t}{\partial {h_t^I}^2}(\sigma_t^H)^2$$

$$+\tfrac{\partial^2 V_t}{\partial S_t h_t^H}S_t\sigma_t^S\sigma_t^H\rho_t^{S,H} + \tfrac{\partial^2 V_t}{\partial S_t h_t^I}S_t\sigma_t^S\sigma_t^I\rho_t^{S,I} + \tfrac{\partial^2 V_t}{\partial h_t^I h_t^H}\sigma_t^I\sigma_t^H\rho_t^{I,H}$$

$$\widetilde{\mathcal{L}}_S H_t = \frac{\partial H_t}{\partial t} + \frac{1}{2}S_t^2\left(\sigma_t^S\right)^2\frac{\partial^2 H_t}{\partial S_t^2}$$

$$\widetilde{\mathcal{L}}_I CDS(t,T) = \frac{\partial CDS(t,T)}{\partial t} + \frac{1}{2}\left(\sigma_t^I\right)^2\frac{\partial^2 CDS(t,T)}{\partial {h_t^I}^2}$$

$$\widetilde{\mathcal{L}}_H B(t,T) = \frac{\partial B(t,T)}{\partial t} + \frac{1}{2}\left(\sigma_t^H\right)^2\frac{\partial^2 B(t,T)}{\partial {h_t^H}^2}$$

Substituting ϵ_t by its value and grouping terms

$$\widetilde{\mathcal{L}}_{SIH}V_t + \tfrac{h_t^I}{1-R_I}\Delta V_t^I = V_t^+ f_t^B + V_t^- f_t^C$$

$$+\alpha_t\left(\widetilde{\mathcal{L}}_S H_t - c_t H_t\right)$$

$$+\gamma_t\left(\widetilde{\mathcal{L}}_I CDS(t,T) + \tfrac{h_t^I}{1-R_I}\Delta CDS(t,T) - c_t CDS(t,T)\right)$$

$$+\omega_t\left(\widetilde{\mathcal{L}}_H B(t,T) - f_t^H B(t,t+dt)\right)$$

H_t is a cash collateralized derivative written on S_t, therefore it must meet the following PDE as seen in [15]

$$\widetilde{\mathcal{L}}_S H_t + \left(r_t - q_t\right)S_t\frac{\partial H_t}{\partial S_t} - c_t H_t = 0$$

$CDS(t,T)$ is a collateralized credit derivative written on I, therefore it must follow

$$\widetilde{\mathcal{L}}_I CDS(t,T) + \left(\mu_t^I - M_t^I\sigma_t^I\right)\frac{\partial CDS(t,T)}{\partial h_t^I} + \frac{h_t^I}{1-R_I}\Delta CDS(t,T) - c_t CDS(t,T) = 0$$

And $B(t,T)$ must follow

$$\widetilde{\mathcal{L}}_H B(t,T) + \left(\mu_t^H - M_t^H\sigma_t^H\right)\frac{\partial B(t,T)}{\partial h_t^H} - f_t^H B(t,T) = 0$$

So that the hedging equation is given by

$$
\begin{aligned}
&\widetilde{\mathcal{L}}_{SIH}V_t + \tfrac{h_t^I}{1-R_I}\Delta V_t^I = V_t^+ f_t^H + V_t^- f_t^H \\
&+\frac{\frac{\partial V_t}{\partial S_t}}{\frac{\partial H_t}{\partial S_t}}\left(-\left(r_t - q_t\right) S_t \frac{\partial H_t}{\partial S_t}\right) \\
&+\frac{\frac{\partial V_t}{\partial h_t^I}}{\frac{\partial CDS(t,T)}{\partial h_t^I}}\left(-\left(\mu_t^I - M_t^I \sigma_t^I\right) \frac{\partial CDS(t,T)}{\partial h_t^I}\right) \\
&+\frac{\frac{\partial V_t}{\partial h_t^H}}{\frac{\partial B(t,T)}{\partial h_t^H}}\left(-\left(\mu_t^H - M_t^H \sigma_t^H\right) \frac{\partial B(t,T)}{\partial h_t^H}\right)
\end{aligned}
$$

Simplifying terms

$$
\widehat{\mathcal{L}}_{SIH}V_t + \tfrac{h_t^I}{1-R_I}\Delta V_t^I = V_t^+ z_t^H + V_t^- z_t^H + c_t V_t = f_t^H V_t
$$

Where $z_t^H = f_t^H - c_t$ represents the short term funding spread of the hedger over the OIS rate and

$$
\begin{aligned}
\widehat{\mathcal{L}}_{SIH}V_t = &\tfrac{\partial V_t}{\partial t} + (r_t - q_t)S_t \tfrac{\partial V_t}{\partial S_t} + (\mu_t^H - M_t^H \sigma_t^H)\tfrac{\partial V_t}{\partial h_t^H} + (\mu_t^I - M_t^I \sigma_t^I)\tfrac{\partial V_t}{\partial h_t^I} \\
&+\tfrac{1}{2}\tfrac{\partial^2 V_t}{\partial S_t^2} S_t^2 (\sigma_t^S)^2 + \tfrac{1}{2}\tfrac{\partial^2 V_t}{\partial {h_t^H}^2}(\sigma_t^H)^2 + \tfrac{1}{2}\tfrac{\partial^2 V_t}{\partial {h_t^I}^2}(\sigma_t^H)^2 \\
&+\tfrac{\partial^2 V_t}{\partial S_t h_t^H} S_t \sigma_t^S \sigma_t^H \rho_t^{S,H} + \tfrac{\partial^2 V_t}{\partial S_t h_t^I} S_t \sigma_t^S \sigma_t^I \rho_t^{S,I} + \tfrac{\partial^2 V_t}{\partial h_t^I h_t^H} \sigma_t^I \sigma_t^H \rho_t^{I,H}
\end{aligned}
\tag{7.6}
$$

The solution to (7.6) with terminal condition given by $V_T = g(S_T)$ is equal to calculating the following expected value

$$
\begin{aligned}
V_t = &\underbrace{E_{\mathbb{Q}}\left[V_T \exp\left(-\int_{s=t}^{T} c_s ds\right)\Big|\mathcal{F}_t\right]}_{\text{Fully collateralized price}} \\
&\underbrace{-E_{\mathbb{Q}}\left[\int_{s=t}^{T} 1_{\{\tau^I > s\}} \exp\left(-\int_{h=t}^{s} c_h dh\right)\left(z_s^H V_s^+ + z_s^H V_s^-\right) ds\Big|\mathcal{F}_t\right]}_{\text{Funding value adjustment}} \\
&+\underbrace{E_{\mathbb{Q}}\left[\int_{s=t}^{T} 1_{\{\tau^I > s\}} \exp\left(-\int_{h=t}^{s} c_h dh\right)(R_I - 1)\left(V_s^{rf}\right)^- dN_s^{I,\mathbb{Q}}\Big|\mathcal{F}_t\right]}_{\text{CVA}}
\end{aligned}
\tag{7.7}
$$

In a measure $\mathbb{Q}$ in which the drifts of S_t, h_t^H and h_t^I are given by $(r_t - q_t)S_t$, $\mu_t^H - M_t^H \sigma_t^H$ and $\mu_t^I - M_t^I \sigma_t^I$ respectively. Under this measure, the default intensity of the default event of the investor is $\frac{h_t^I}{1-R_I}$.

We have assumed a risk free close out, that is $V_{\tau^I} = R(V_{\tau^I}^{rf})^- + (V_{\tau^I}^{rf})^+$. V_t^{rf} represents the NPV of the derivative if it was fully collateralized with cash.

z_t^H represents the short term funding spread for the hedger.

Notice that the last formula is recursive.

V_t can also be expressed in a non recursive way

$$V_t = \underbrace{E_{\mathbb{Q}}\left[V_T \exp\left(-\int_{s=t}^{T} f_s^H ds\right)\Big|\mathcal{F}_t\right]}_{\text{Price with funding adjustment and no counterparty credit risk}}$$

$$+ \underbrace{E_{\mathbb{Q}}\left[\int_{s=t}^{T} 1_{\{\tau^I > s\}} \exp\left(-\int_{h=t}^{s} f_h^H dh\right)\left((V_s^{rf})^- R_I + (V_s^{rf})^+ - V_s^f\right) dN_s^{I,\mathbb{Q}}\Big|\mathcal{F}_t\right]}_{\text{CVA over price with funding}} \tag{7.8}$$

$$V_s^f := E_{\mathbb{Q}}\left[V_T \exp\left(-\int_{s=t}^{T} f_s^H ds\right)\Big|\mathcal{F}_s\right]$$

f_t^H is the short term funding rate for the hedger.

7.5 DVA hedging vs FVA hedging: a simplified example

In this section we explore DVA vs FVA hedging in a simplified framework. We will assume:

- We want to replicate a forward on a particular underlying asset (S_t) such that at maturity (5 years) the investor (risk taker) receives $S_T - K$.
- The underlying asset pays no dividends.
- Interest rates are assumed to be zero (OIS and REPO rates).
- The investor is default free.
- The hedger is defaultable with a short term funding spread z_t.
- The recovery rate for the hedger is 0.

- The underlying asset follows Black-Scholes.
- z_t^H follows an Ornstein-Uhlenbeck process.
- We assume no correlation between S_t and z_t.

So that the SDEs of the two processes under the real world measure are:

$$dS_t = \mu_t^{\mathbb{P}} S_t dt + \sigma_t^S dW_t^{S,\mathbb{P}}$$

$$dz_t = \kappa \left(\theta_t^{\mathbb{P}} - z_t\right) dt + \sigma_t^z dW_t^{z,\mathbb{P}}$$

$$dW_t^{S,\mathbb{P}} dW_t^{z,\mathbb{P}} = 0$$

We have chosen the following set of parameters:

$\mu_t^{\mathbb{P}}$	10%
σ_t^S	20%
$\theta_t^{\mathbb{P}}$	4%
κ	0.5
σ_t^z	1%
S_0	1
z_0	3%
K	1

We assume that at $t = 0$, the funding curve is flat at a level of 3%.
Before exploring the effects of hedging, we will analyze the sensitivities with respect to spread changes of both approaches (Risk free price + FVA vs Risk free price + DVA).

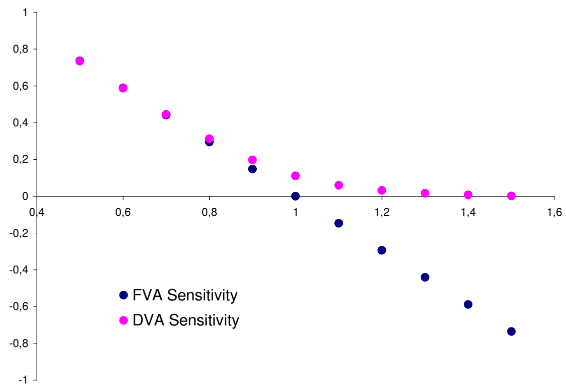
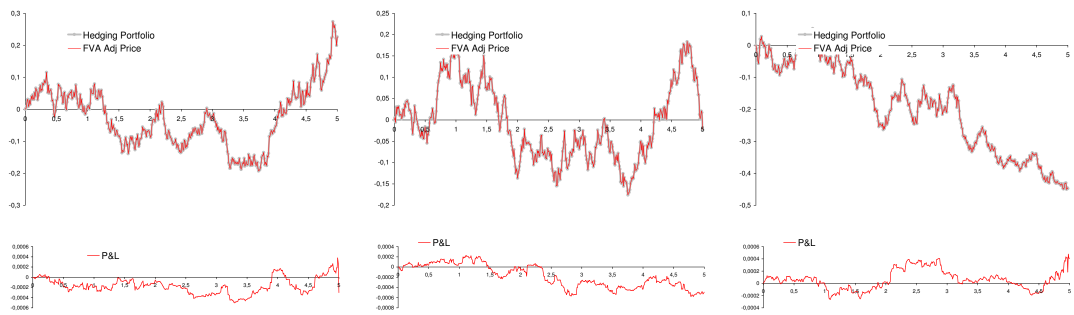


Figure 7.1: Sensitivities to spread changes of both FVA and DVA adjusted price. x-axis represents K (fix payment to be paid by the risk taker in the forward contract.)

Notice that the sensitivity of the DVA adjusted price is always positive (the well known effect of DVA), whereas for the case of the FVA adjusted price, the sensitivity is positive when $K < S_0$ and negative when $K > S_0$.

- Under a DVA approach, the hedger always benefits from an increase in his funding spread.
- Under a FVA approach, the hedger benefits from an increase in the spread when the NPV is positive for the risk taker, that is, the hedger borrows funds from the client.
- Under a FVA approach, the hedger experiences a loss from an increase in the spread when the NPV is negative for the risk taker, that is, the hedger lends funds to the client.

7.6 FVA Hedging

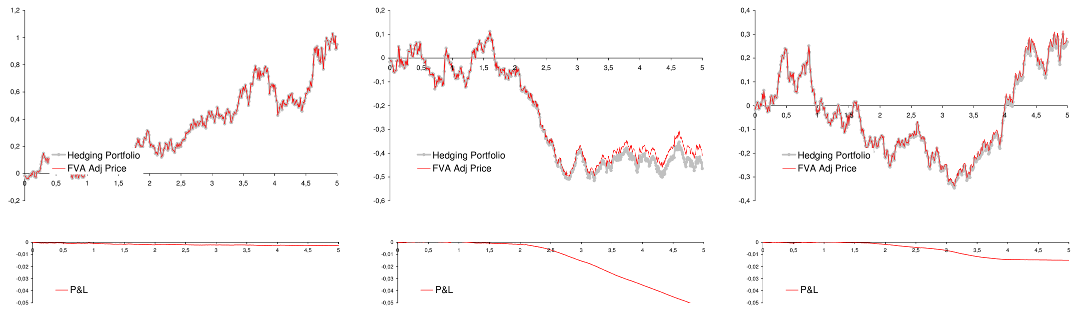


We can observe the evolution of both the FVA adjusted price and the hedging portfolio. We can see that the P&L is negligible and deviations from zero seem noisy and due to the discrete rebalancing frequency of the hedge. This confirms the theoretical results that we have seen in the previous sections.

7.7 DVA Hedging

Now we explore DVA hedging.
As a hedging strategy we use the same one used to try to hedge FVA. That is, if the price for the risk taker is positive, we receive funds from the risk taker with which we buy back our own debt.
If the price is negative, we have to issue new debt.

In either case, we impose that the sensitivity to spread changes of the debt issued (or bought back) matches that of the incoming derivative. In next figure we show the adjusted price vs hedging portfolio (graphs above) and P&L (graphs below)



We observe that the P&L becomes negative (although the evolution is smooth) and that it depends on the path. It seems that there is a theta mismatch between the DVA adjusted price and the hedging portfolio.

In next figure we plot many different scenarios and see that the P&L seems to be always negative, and that it is path dependent.

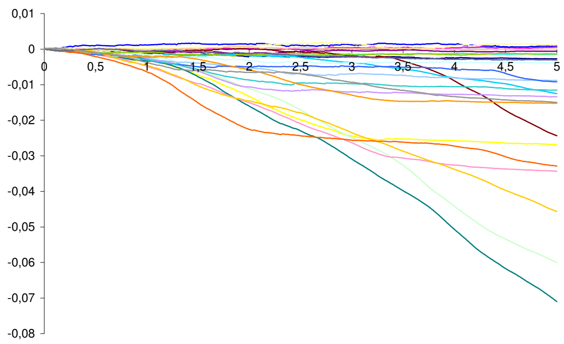


Figure 7.2: P&L for DVA adjusted price hedging (various paths.)

What is the main driver of the path dependent P&L?

The hedging portfolio minus the DVA adjusted price is given by (we assume that we are initially hedged)

$$\alpha_t H_t + \beta_t + \Omega_t B(t, t + dt) + \omega_t B(t, T) - V_t$$

The differential change is given by (assuming the hedger does not default):

$$
\begin{aligned}
d\Pi_t = \alpha_t & \left(\mathcal{L}_s H_t dt + \tfrac{\partial H_t}{\partial S_t} dS_t - c_t H_t dt\right) \\
& + \omega_t \left(\mathcal{L}_{sh} B(t,T) dt + \tfrac{\partial B(t,T)}{\partial S_t} dS_t + \tfrac{\partial B(t,T)}{\partial h_t} dh_t - f_t B(t,T) dt\right) \\
& - \left(\mathcal{L}_{sh} V_t dt + \tfrac{\partial V_t}{\partial S_t} dS_t + \tfrac{\partial V_t}{\partial h_t} dh_t - f_t V_t dt\right)
\end{aligned}
\tag{7.9}
$$

Where

$$
\begin{aligned}
\mathcal{L}_S &= \tfrac{\partial}{\partial t} + \tfrac{1}{2} S_t^2 \left(\sigma_t^S\right)^2 \tfrac{\partial^2}{\partial S_t^2} \\
\mathcal{L}_h &= \tfrac{\partial}{\partial t} + \tfrac{1}{2} \left(\sigma_t^h\right)^2 \tfrac{\partial^2}{\partial {h_t}^2}
\end{aligned}
\tag{7.10}
$$

$$
\mathcal{L}_{Sh} = \tfrac{\partial}{\partial t} + \tfrac{1}{2} \tfrac{\partial^2}{\partial S_t^2} S_t^2 (\sigma_t^S)^2 + \tfrac{1}{2} \tfrac{\partial^2}{\partial {h_t}^2} (\sigma_t^h)^2 + \tfrac{\partial^2}{\partial S_t \partial h_t} S_t \sigma_t^S \sigma_t^h \rho_t
$$

Where we have taken into account that $V_t = \Omega_t B(t, t+dt) + \omega_t B(t,T)$. That is, funds exchanged with the investor are matched with the issuance or buy back of debt.

In order to be hedged to the two risk factors on every scenario under which the hedger has not defaulted:

$$
\begin{aligned}
d\Pi_t = & \tfrac{\frac{\partial V_t}{\partial S_t}}{\frac{\partial H_t}{\partial S_t}} \left(\mathcal{L}_s H_t dt - c_t H_t dt\right) \\
& + \tfrac{\frac{\partial V_t}{\partial h_t}}{\frac{\partial B(t,T)}{\partial h_t}} \left(\mathcal{L}_h B(t,T) dt - f_t B(t,T) dt\right) \\
& - \left(\mathcal{L}_{sh} V_t dt - f_t V_t dt\right)
\end{aligned}
\tag{7.11}
$$

We assume that DVA is discounted at the Eonia rate, therefore its PDE would be:

$$
\mathcal{L}_{Sh} V_t + (r_t - q_t) S_t \frac{\partial V_t}{\partial S_t} + \left(\mu_t^h - M_t^h \sigma_t^h\right) \frac{\partial V_t}{\partial h_t} + \Delta V_t \frac{h_t}{1-R} - c_t V_t = 0 \tag{7.12}
$$

The PDEs followed by H_t and $B(t,T)$

$$
\mathcal{L}_S H_t + (r_t - q_t) S_t \frac{\partial H_t}{\partial S_t} - c_t H_t = 0 \tag{7.13}
$$

$$
\mathcal{L}_h B(t,T) + \left(\mu_t^h - \sigma_t^h M_t^h\right) \frac{\partial B(t,T)}{\partial h_t} + \frac{h_t}{1-R} \Delta B(t,T) - r_t^T B(t,T) = 0 \tag{7.14}
$$

So that

$$\begin{aligned} d\Pi_t = &-\frac{\frac{\partial V_t}{\partial S_t}}{\frac{\partial H_t}{\partial S_t}}\left(r_t - q_t\right) S_t \frac{\partial H_t}{\partial S_t} dt \\ &+\frac{\frac{\partial V_t}{\partial h_t}}{\frac{\partial B(t,T)}{\partial h_t}}\left(-\left(\mu_t^h - \sigma_t M_t\right)\frac{\partial B(t,T)}{\partial h_t} - \frac{h_t}{1-R}\Delta B(t,T) + r_t^T B(t,T) - f_t B(t,T)\right) dt \\ &-\left(-\left(r_t - q_t\right) S_t \frac{\partial V_t}{\partial S_t} - \left(\mu_t^h - M_t^h \sigma_t^h\right)\frac{\partial V_t}{\partial h_t} - \Delta V_t \frac{h_t}{1-R} + c_t V_t - f_t V_t\right) dt \end{aligned} \tag{7.15}$$

Canceling terms:

$$\begin{aligned} d\Pi_t = &\frac{\frac{\partial V_t}{\partial h_t}}{\frac{\partial B(t,T)}{\partial h_t}}\left(-\frac{h_t}{1-R}\Delta B(t,T) + r_t^T B(t,T) - f_t B(t,T)\right) dt \\ &+\left(\Delta V_t \frac{h_t}{1-R} + z_t V_t\right) dt \end{aligned} \tag{7.16}$$

Reordering terms

$$d\Pi_t = \underbrace{\frac{h_t}{1-R}\left(\Delta V_t - \frac{\frac{\partial V_t}{\partial h_t}}{\frac{\partial B(t,T)}{\partial h_t}}\Delta B(t,T)\right)}_{\text{Jump to default mismatch}} + \underbrace{\left(z_t V_t - (f_t - r_t^T) B(t,T) \frac{\frac{\partial V_t}{\partial h_t}}{\frac{\partial B(t,T)}{\partial h_t}}\right)}_{\text{Funding mismatch}} \tag{7.17}$$

If $\Delta B(t,T) = (1-R)B(t,T),\ r_t^T = r_t$

$$d\Pi_t = +\left(\Delta V_t \frac{h_t}{1-R} + z_t V_t\right) dt$$

$$\begin{aligned} &\Delta V_t = R(V_t^{rf})^+ + (V_t^{rf})^- - V_t = V_t^{rf} - (1-R)V_t^{rf} - V_t \\ &= DVA_t - (1-R)V_t rf = -JTD_{DVA} \end{aligned} \tag{7.18}$$

$$\Downarrow$$

$$d\Pi_t = \left(\Delta V_t \frac{h_t}{1-R} + z_t V_t\right) dt = -\left(\frac{JTD_{DVA}}{1-R} + z_t V_t\right) dt$$

The first term is due to the jump to default component of DVA that cannot be hedged. The second term is due to the funding adjustment not made in the pricing.

Which is generally negative.

In our case:

$R = 0;\ r_t^T = 0;\ f_t = z_t = h_t$. Therefore

$$d\Pi_t = \min(V_t^{rf}, 0) z_t dt$$

In figure 7.3 we compare the evolution of $\int_{s=0}^{t} \exp\left(\int_{u=0}^{s} z_s ds\right) \min(V_s^{rf}, 0) z_s ds$

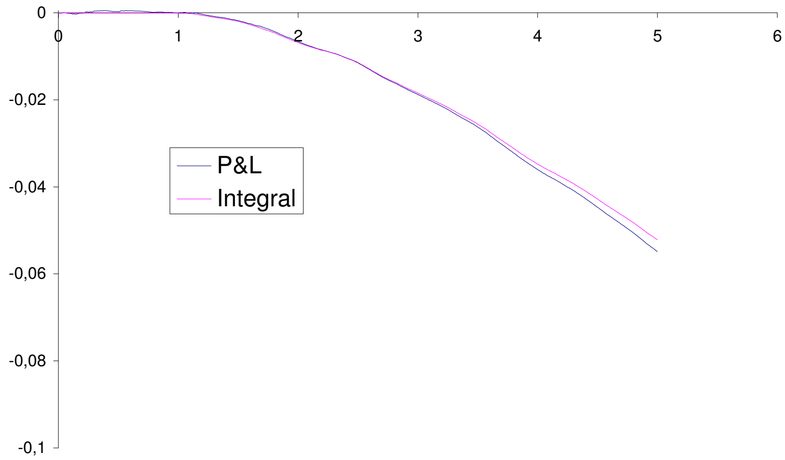


Figure 7.3: P&L vs integral

7.8 Conclusions

- We have followed a full replication approach.
- The following sources of risk are eliminated:
  - Market risk.
  - Default risk of the investor.
  - Spread risk of the investor.
  - Spread risk of the hedger.
- We got rid of the hedger's jump to default risk. It is unhedgeable and no one is really willing to hedge it (traders will already been fired, equity holders will have lost all of its investment, bond holders will never want it to be hedged).
- The only components to incorporate in the price are:
  - The risk free price.
  - A CVA component equivalent to considering the hedger default free.
  - A FVA component equivalent to considering the hedger default free.
- CVA can be hedged by trading on two different CDSs (under a 1 factor model).

- FVA can be hedged by trading on two bonds issued by the hedger (under a 1 factor model) while maintaining the self financing condition.
- It does not seem to make sense to pay for both funding benefit and DVA.
- In order to satisfy accountants, let's just call DVA to funding benefit.
- We have seen that DVA, even if we made an attempt to hedge its fluctuations, implies a negative carry (laugh today, cry tomorrow).

Chapter 8

CVA In a Multi Currency Framework

8.1 Funding rates in different currencies

An institution might fund its activity in either domestic currency, at the instantaneous domestic funding rate f_t^D, or in foreign currency, at the foreign funding rate f_t^F. Both rates must respect some relationship in order to preclude arbitrage opportunities. We will study such a relationship by looking at the transactions that takes place in the funding strategy. We will follow the argument line in [**?**].

Let us assume the hedger must fund a derivative that is denominated in the foreign currency F. We might do so twofold,

8.1.1 Funding in Domestic currency

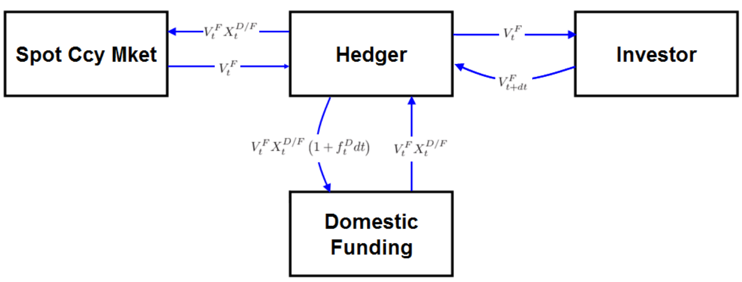


Figure 8.1: Funding Scheme in domestic currency.

To fund the derivative denominated in foreign currency, whose value we denote by V_t^F, the hedger would proceed as follows: She would borrow at time t, from the domestic capital market, the domestic amount $V_t^F X^{D/F_t}$ that would be converted into V_t^F units of foreign currency (in the FX spot market) that would be used, eventually, to fund the foreign derivative. As consequence of this strategy, the hedger, at time $t + dt$, would return the money borrowed plus interests (that accrues at the domestic funding rate f_t^D) and would hold a derivative worth V_{t+dt}^F (consequence of the replicating portfolio). This strategy is represented in figure (8.1).

In overall, by adding all the terms in this strategy, it can be seen that variations in the price of the derivative, expresed in domestic currency, follow

$$d\left(V_t^F X_t^{D/F}\right) = V_t^F X_t^{D/F} f_t^D dt \tag{8.1}$$

8.1.2 Funding in Foreign currency

The hedger might also have funded the foreign derivative, by borrowing V_t^F units of foreign currency at time t. At time $t + dt$ she would had returned the money borrowed plus interests (at the foreign funding rate f_t^F). To be able to repay the loan, she would enter in a standard overnight FX forward to exchange at $t + dt$: $V_t^F \left(1 + f_t^F dt\right)$ in foreign currency against $V_t^F X_t^{D/F} \left(1 + f_t^F dt\right) \frac{1+c_t^D dt}{1+(c_t^F+b_t^{F,D})dt}$ in domestic one. Note that c_t^X indicates the over-night rate denominated in currency X, while $b^{F,D}$ indicates the instantaneous cross currency basis.

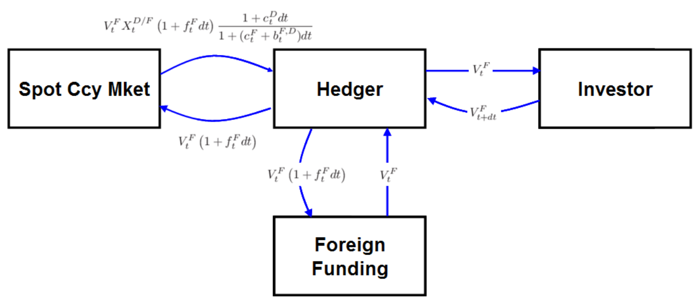


Figure 8.2: Funding Scheme in foreign currency.

In overall, by adding all the terms in this strategy, it can be seen that variations in the price of the derivative, expresed in domestic currency, follow

$$d\left(V_t^F X_t^{D/F}\right) = V_t^F X_t^{D/F} \left(f_t^F - (c_t^F + b_t^{F,D}) + c_t^D\right) dt \tag{8.2}$$

Just note that (8.1) = (8.2), so we find the relationship between the domestic and foreign rate to be,

$$\boxed{\left(f_t^D - c_t^D\right) = f_t^F - \left(c_t^F + b_t^{F,D}\right)} \tag{8.3}$$

8.2 Different dynamcis for the FX:

So far, we are used to think of FX as a price to translate unities of one currency into another independently of the collateralization underlying the contract. This is not longer the case, and we might face different FX to move from one currency to another because of the collateraliation underneath.

Let us illustrate this by looking at a FX forward contract by which two parties exchange one unit of currency A against K units of currency B as the figure (8.3) illustrates.

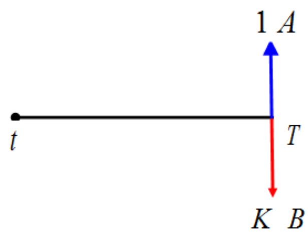


Figure 8.3: Forward FX

Under different collateralization schemes the FX forward (that K that makes the contract to be worth zero today) will vary.

Collateralization in A

By assuming the forward contract to be collateralized in currency A, the FX forward becomes,

$$K = X_{t,T}^{A/B,A} = X_t^{A/B} \frac{B_{t,T}^{B,A}}{B_{t,T}^{A,A}} \tag{8.4}$$

In order to be consistent with this forward, the dynamics of the FX spot must fulfill,

$$E_t^{\mathbb{Q}_A}\left[\frac{dX_t^{A/B}}{X_t^{A/B}}\right] = \left(c_t^A + b_t^{A,B} - c_t^B\right) dt \tag{8.5}$$

Leg A collateralized in A and Leg B collateralized in B

Such an instrument will not be observed in the market as multicurrency derivatives are collateralized in only one currency.
By assuming this collateraliztion scheme, the FX forward becomes,

$$K = X_{t,T}^{A/B,AB} = X_t^{A/B} \frac{B_{t,T}^{B,B}}{B_{t,T}^{A,A}} \tag{8.6}$$

So, in order to be consistent with this forward, the dynamics of the FX spot must fulfill,

$$E_t^{\mathbb{Q}_A} \left[\frac{dX_t^{A/B}}{X_t^{A/B}} \right] = \left(c_t^A - c_t^B \right) dt \tag{8.7}$$

That is, **this FX does not carry Xccy basis ..!!**

There are different FX dynamcis to move from economy A into economy B depending on the collateralization scheme ...

8.3 Risky pricing of a derivative partially collateralized in cash

Let us assume a derivative with maturity T, denominated in currency D to be a function of an underlying denominated in D, S_t^D, whose price we denote, at time t, by V_t^D. Let us assume the existence of a perfectly collateralized derivative written on S_t^D whose price, at time t, we denote by H_t^D.
As it can be seen in [9], we can replicate the derivative through the self-financing portfolio,

$$V_t^D = \alpha_t H_t^D + \beta_t^D + \gamma_t CDS^D(t,T) + \epsilon_t \, CDS^D(t, t+dt) + \underbrace{\left(\Omega_t B(t, t+dt) + \omega_t B(t,T) \right)}_{\left(V_t^D - C_t^D \right)} \tag{8.8}$$

Where,

- V_t^D: Price of the D-denominated derivative as seen from the counterparty point of view.
- $CDS^D(t,T)$: Price of a perfectly collateralize CDS (denominated in D) written on the counterparty, as seen from the Hedger's point of view.
- $CDS^D(t, t+dt)$: Price of an overnight CDS written on the counterparty and denominated in D, as seen from the Hedger's point of view.

- $B(t,T)$: Hedger's bond with maturity T, as seen from the Hedger's point of view.
- $B(t,t+dt)$: Hedger's over-night bond, as seen from the Hedger's point of view.
- C_t^D: Cash in currency D posted as collateral (assumed a bilateral setting).
- $\Omega_t B(t,t+dt)+\omega_t B(t,T)$: Self-financing condition. (See [9] for more details.)
- β_t^D: Bank account denominated in D to finance both the derivative and the hedging portfolio.

The bank account, at time t, will contain the following terms,

$$\beta_t^D = -\alpha_t H^D(t) - \gamma_t CDS^D(t,T) + C_t^D \tag{8.9}$$

So the variation for the bank account, in a infinitesimal time interval, will be,

$$d\beta_t = \left[c_t^D C_t^D - \alpha_t c_t^D H_t^D - \gamma_t c_t^D CDS^D(t,T)\right] dt \tag{8.10}$$

Where,

- c_t^D: is the instantaneous D-denominated over-night rate.
- f_t^D: is the instantaneous D-denominated funding/investment rate for the hedger.

According to [9], it can be seen the price of the derivative to fulfill with the following PDE:

$$\begin{aligned} &\mathcal{L}V_t^D + \frac{h_t^D}{(1-R)}\Delta V_t^D = \left(V_t^D - C_t^D\right) f_t^D + c_t^D C_t^D \\ &s.t \qquad V(T\wedge\tau) = V_T \mathbf{1}_{\{\tau>T\}} + \pi_\tau^D \mathbf{1}_{\{\tau\leq T\}} \end{aligned} \tag{8.11}$$

Where

- h_t^D: is the over-night CDS rate for the Counterparty.
- R: is the expected recovery for the counterparty.
- $\mathcal{L}$: is the differential operator (See [9]).
- τ: Is the first time at which the counterparty defaults.

- π_τ^D: Net flows exchanged at the time the counterparty defaults. By assuming a Risk-Free close-out,

$$\pi_t^D = -\left[V_t^{RF} + (1-R_C)\left(\left(V_t^{RF}\right)^- - \left(C_t^D\right)^-\right)^- + (1-R_{Col})\left(\left(V_t^{RF}\right)^+ - \left(C_t^D\right)^+\right)^-\right] \tag{8.12}$$

Where R_{Coll} is the collateral recovery in case of the counterparty'sdefault, V_t^{RF} is the risk-free value of the derivative. That is, the price assuming the derivative is perfectly collateralized under a standard collateralization scheme (eg. an IRS denominated in EUR will be assumed to be perfectly cash-collateralized in EUR).

- **Discounting at the domestic over-night interest rate:**

By applying the *Feyman-Kac* theorem, it can be seen that V_t^D can be represented in the following form

$$\begin{aligned} V_t^D =& E_t^{\mathbb{Q}_D}\left[e^{-\int_t^T c_s^D ds} V_T^D \ \mathbf{1}_{\{\tau>T\}}\right] \\ &+ E_t^{\mathbb{Q}_D}\left[e^{-\int_t^\tau c_s^D ds}\left(\pi_\tau^D\right)\mathbf{1}_{\{\tau<T\}}\right] \\ &- \int_t^T E_t^{\mathbb{Q}_D}\left[e^{-\int_t^s c_u^D du}\ \mathbf{1}_{\{\tau>s\}}\left(f_s^{H,D} - c_s^D\right)\left(V_s^D - C_s^D\right)\right] ds \end{aligned} \tag{8.13}$$

By adding and subtracting $E_t^{\mathbb{Q}_D}\left(e^{-\int_t^\tau c_s^D ds} V_T^D \ \mathbf{1}_{\{\tau<T\}}\right)$, we obtain

$$\begin{aligned} V_t^D =& \underbrace{E_t^{\mathbb{Q}_D}\left[e^{-\int_t^T c_s^D ds} V_T^D\right]}_{V_t^{D,c^D}} \\ &+ \underbrace{E_t^{\mathbb{Q}_D}\left[e^{-\int_t^\tau c_s^D ds}\left(\pi_\tau^D - V_\tau^{D,c^D}\right)\mathbf{1}_{\{\tau<T\}}\right]}_{\text{CVA}} \\ &- \underbrace{\int_t^T E_t^{\mathbb{Q}_D}\left[e^{-\int_t^s c_u^D du}\ \mathbf{1}_{\{\tau>s\}}\left(f_s^{H,D} - c_s^D\right)\left(V_s^D - C_s^D\right)\right] ds}_{\text{FVA}} \end{aligned} \tag{8.14}$$

Where V_t^{D,c^D} is the price of the derivative obtained by discounting its future flows at c_t^D.

Just note that this is a recursive formula ..!!

- **Discounting at the domestic Hedger's funding rate :**

  In order to avoid the recursive nature in the price of the derivative, this can be also represented in a more convenient way as,

$$
\begin{aligned}
V_t^D = & \underbrace{E_t^{\mathbb{Q}_D}\left[e^{-\int_t^T f_s^{H,D} ds} V_T^D\right]}_{V_t^{D,f}} \\
& + \underbrace{E_t^{\mathbb{Q}_D}\left[e^{-\int_t^\tau f_s^{H,D} ds}\left(\pi_\tau^D - V_\tau^{D,f}\right)\mathbf{1}_{\{\tau<T\}}\right]}_{\text{CVA adjusted for FVA}} \\
& + \underbrace{\int_t^T E_t^{\mathbb{Q}_D}\left[e^{-\int_t^s f_u^{H,D} du}\, \mathbf{1}_{\{\tau>s\}}\left(f_s^{H,D} - c_s^D\right) C_s^D\right] ds}_{\text{Collateral Adjustment}}
\end{aligned}
\tag{8.15}
$$

The formula becomes simpler ..!!

Just note, that **in order to implement this formula we will need two different pricers**. One to calculate the **close-out amount made up of perfectly collateralized derivatives**, and the other one needed **to price derivatives not collateralized at all** and, hence, discounted at the Hedger's domestic funding rate.

Conclusions:

- **The choice of the discount rate to use is very relevant**, when pricing a derivative. By choosing the overnight rate, we get a recursive formula difficult to solve for. While by choosing the Hedger's funding rate, the pricing formula becomes simpler. **There is a clear choice to take ...!!**

- In general terms, we will need two different pricers to calculate the CVA term. One for the close-out and the other one that will depend on the discounting rate chosen.

- When choosing the over-night rate as discounting rate, we can disentangle the pricing formula in a pure CVA term and FVA term, though we do not know how to solve for this FVA term. When using the Hedger's funding rate, we obtain a hybrid term accounting for both CVA and FVA.

8.4 Risky pricing of a foreign derivative partially collateralized in cash in another currency

In this section, we will assume a derivative denominated in a foreign currency F and partially collateralized in currency G. We will look at the price of such a derivative from the point of view of a domestic hedger (i.e as seen in currency D).

The replicating portfolio will be,

$$\begin{aligned} V_t^F X_t^{D/F} =&\alpha_t H_t^F X_t^{D/F} + \beta_t^D + \gamma_t CDS^D(t,T) + \epsilon_t CDS^D(t,t+dt) \\ &+ \underbrace{(\Omega_t B(t,t+dt) + \omega_t B(t,T))}_{\left(V_t^F X_t^{D/F} - C_t^G X_t^{D/G}\right)} \end{aligned} \tag{8.16}$$

Where H_t^F is a foreign derivative that depends on the same foreign underlying as the one we are exposed to in the derivative.

The bank account in domestic currency β_t^D will be made up of,

$$\beta_t^D = -\alpha_t H_t^F X_t^{D/F} - \gamma_t CDS^D(t,T) + C_t^G X_t^{D/G} \tag{8.17}$$

so variations in an infinitesimal interval in the bank-account will be,

$$d\beta_t^D = \left[\left(c_t^D + b_t^{D,G}\right) C_t^G X_t^{D/G} - \alpha_t \left(c_t^D + b_t^{D,F}\right) H_t^F X_t^{D/F} + \gamma_t c_t^D CDS^D(t,T)\right] dt \tag{8.18}$$

Where $b_t^{D,F}$, $b_t^{D,G}$ are the instantaneous cross currrency basis observed in the overnight forex market. Note that both collateral accounts (the associated to both C_t^G and H_t^F) are **effectively remunerated to the domestic overnight rate plus the Xccy basis spread so as to eliminate the additional FX risk**.

Following the same steps as in [9], and taking into account that

$$\begin{aligned} \frac{\partial\left(H_t^F X_t^{D/F}\right)}{\partial t} dt + \mu_{H^F X}^{\mathbb{Q}_D}(t) \frac{\partial\left(H_t^F X_t^{D/F}\right)}{\partial S_t^F X_t^{D/F}} dt + \frac{1}{2} \frac{\partial^2\left(H_t^F X_t^{D/F}\right)}{\partial (S_t^F X_t^{D/F})^2} \left[d\left(S_t^F X_t^{D/F}\right)\right]^2 \\ = \left(c_t^D + b_t^{D,F}\right) H_t^F X_t^{D/F} dt \end{aligned} \tag{8.19}$$

It can be seen the PDE that $V_t^F X_t^{D/F}$ must fulfill is,

$$\begin{aligned} &\mathcal{L}\left(V_t^F X_t^{D/F}\right) + \frac{h_t^D}{(1-R)} \Delta\left(V_t^F X_t^{D/F}\right) = \left(V_t^F X_t^{D/F} - C_t^G X_t^{D/G}\right) f_t^{H,D} + \left(c_t^D + b_t^{D,G}\right) C_t^G X_t^{D/G} \\ &s.t \qquad V_{t'}^F X_{t'}^{D/F} = V_T^F X_T^{D/F} \mathbf{1}_{\{\tau > T\}} + \pi_\tau^F X_T^{D/F} \mathbf{1}_{\{\tau \leq T\}} \quad , \qquad t' = (T \wedge \tau) \end{aligned} \tag{8.20}$$

Where

$$\pi_t^F = -V_t^{F,RF} - (1-R_C)\left[\left(V_t^{F;RF}\right)^- - \left(C_t^G X_t^{F/G}\right)^-\right]^-$$
$$-(1-R_{Col})\left[\left(V_t^{F,RF}\right)^+ - \left(C_t^D X_t^{F/G}\right)^+\right]^- \tag{8.21}$$

The **equivalent PDE as seen in F units** becomes,

$$\mathcal{L}V_t^F + \frac{h_t^D}{(1-R)}\Delta V_t^F = \left(V_t^F - C_t^G X_t^{F/G}\right) f_t^{F,D} + \left(c_t^F + b_t^{F,G}\right) C_t^G X_t^{F/G}$$
$$s.t \qquad V_{t'}^F X_{t'}^{D/F} = V_T^F X_T^{D/F}\mathbf{1}_{\{\tau>T\}} + \pi_\tau^F X_T^{D/F}\mathbf{1}_{\{\tau\le T\}} \quad , \qquad t' = (T\wedge\tau) \tag{8.22}$$

Where π_t^F is the one seen in (8.21).

- **Discounting at the domestic over-night rate plus the cross currency basis.**

$$\begin{aligned} V_t^F X_t^{D/F} = & \underbrace{E_t^{\mathbb{Q}_D}\left[e^{-\int_t^T (c_s^D + b_s^{D,F})ds}\, V_T^F X_T^{D/F}\right]}_{V_t^{D,(c^D+b^{D,F})}} \\ & + \underbrace{E_t^{\mathbb{Q}_D}\left[e^{-\int_t^\tau (c_s^D + b_s^{D,F})ds}\left(\pi_\tau^F X_\tau^{D/F} - V_\tau^{D,(c^D+b^{D,F})}\right)\mathbf{1}_{\{\tau<T\}}\right]}_{\text{CVA}} \\ & - \underbrace{\int_t^T E_t^{\mathbb{Q}_D}\left[e^{-\int_t^s (c_u^D + b_u^{D,F})du}\mathbf{1}_{\{\tau>s\}}\left(f_s^{H,D} - (c_s^D + b_s^{D,F})\right) V_s^F X_s^{D/F}\right] ds}_{\text{FVA}} \\ & + \underbrace{\int_t^T E_t^{\mathbb{Q}_D}\left[e^{-\int_t^s (c_u^D + b_u^{D,F})du}\mathbf{1}_{\{\tau>s\}}\left(f_s^{H,D} - (c_s^D + b_s^{D,G})\right)\ C_s^G X_s^{D/G}\right] ds}_{\text{Collateral Adjustment}} \end{aligned} \tag{8.23}$$

Again, a recursive formula ... !!

- **Discounting at the domestic over-night rate.**

$$\begin{aligned} V_t^F X_t^{D/F} = & \underbrace{E_t^{\mathbb{Q}_D}\left[e^{-\int_t^T c_s^D ds}\, V_T^F X_T^{D/F}\right]}_{V_t^{D,c^D}} \\ & + \underbrace{E_t^{\mathbb{Q}_D}\left[e^{-\int_t^\tau c_s^D ds}\left(\pi_\tau^F X_\tau^{D/F} - V_\tau^{D,c^D}\right)\mathbf{1}_{\{\tau<T\}}\right]}_{\text{CVA}} \\ & - \underbrace{\int_t^T E_t^{\mathbb{Q}_D}\left[e^{-\int_t^s c_u^D du}\mathbf{1}_{\{\tau>s\}}\left(f_s^{H,D} - c_s^D\right)\right] V_s^F X_s^{D/F}\Big) ds}_{\text{FVA}} \\ & + \underbrace{\int_t^T E_t^{\mathbb{Q}_D}\left[e^{-\int_t^s c_u^D du}\mathbf{1}_{\{\tau>s\}}\left(f_s^{H,D} - (c_s^D + b_s^{D,G})\right) C_s^G X_s^{D/G}\right] ds}_{\text{Collateral Adjustment}} \end{aligned} \tag{8.24}$$

We have modified the discount rate and the FVA integrand terms.

- **Discounting at the foreign over-night rate.**

  We might also price the derivative in the foreign currency and translate it at the domestic currency with the help of the FX spot, by applying *Feyman-Kac* to (8.22).

$$
\begin{aligned}
V_t^F X_t^{D/F} =& X_t^{D/F} \underbrace{E_t^{\mathbb{Q}_F}\left[e^{-\int_t^T c_s^F ds} V_T^F\right]}_{V_t^{F,c^F}} \\
&+ \underbrace{X_t^{D/F} E_t^{\mathbb{Q}_F}\left[e^{-\int_t^\tau c_s^F ds}\left(\pi_\tau^F - V_\tau^{F,c^F}\right)\mathbf{1}_{\{\tau<T\}}\right]}_{\text{CVA}} \\
&- \underbrace{X_t^{D/F}\int_t^T E_t^{\mathbb{Q}_F}\left[e^{-\int_t^s c_u^F du}\mathbf{1}_{\{\tau>s\}}\left(f_s^{H,F} - c_s^F\right)V_s^F\right] ds}_{\text{FVA}} \\
&+ \underbrace{X_t^{D/F}\int_t^T E_t^{\mathbb{Q}_F}\left[e^{-\int_t^s c_u^F du}\mathbf{1}_{\{\tau>s\}}\left(f_s^{H,F} - (c_s^F + b_s^{F,G})\right)C_s^G X_s^{F/G}\right] ds}_{\text{Collateral Adjustment}}
\end{aligned}
\tag{8.25}
$$

- **Discounting at the domestic funding rate.**

  As in the last section, we can remove the recursivity in (8.23) by re-expressing,

$$
\begin{aligned}
V_t^F X_t^{D/F} =& \underbrace{E_t^{\mathbb{Q}_D}\left[e^{-\int_t^T f_s^{H,D} ds} V_T^F X_T^{D/F}\right]}_{V_t^{D,f^{H,D}}} \\
&+ \underbrace{E_t^{\mathbb{Q}_D}\left[e^{-\int_t^\tau f_s^{H,D} ds}\left(\pi_\tau^F X_\tau^{D/F} - V_\tau^{D,f^{H,D}}\right)\mathbf{1}_{\{\tau<T\}}\right]}_{\text{CVA Adjusted for funding}} \\
&- \underbrace{\int_t^T E_t^{\mathbb{Q}_D}\left[e^{-\int_t^s f_u^{H,D} du}\,\mathbf{1}_{\{\tau>s\}}\left(f_s^{H,D} - (c_s^D + b_s^{D,G})\right)C_s^G X_s^{D/G}\right] ds}_{\text{Collateral Adjustment}}
\end{aligned}
\tag{8.26}
$$

The recursive term has vanished ...

- **Discounting at the foreign funding rate.**

  If the derivative was funded in currecny F,

$$
\begin{aligned}
V_t^F X_t^{D/F} =& X_t^{D/F} \underbrace{E_t^{\mathbb{Q}_F}\left[e^{-\int_t^T f_s^{H,F} ds}\, V_T^F\right]}_{V_t^{F,f^{H,F}}} \\
&+ X_t^{D/F} \underbrace{E_t^{\mathbb{Q}_F}\left[e^{-\int_t^\tau f_s^{H,F} ds}\left(\pi_\tau^F - V_\tau^{F,f^{H,F}}\right)\mathbf{1}_{\{\tau<T\}}\right]}_{\text{CVA Adjusted for funding}} \\
&- X_t^{D/F} \underbrace{\int_t^T E_t^{\mathbb{Q}_F}\left[e^{-\int_t^s f_u^{F,D} du}\,\mathbf{1}_{\{\tau>s\}}\left(f_s^{H,F} - (c_s^F + b_s^{F,G})\right) C_s^G X_s^{F/G}\right] ds}_{\text{Collateral Adjustment}}
\end{aligned}
\tag{8.27}
$$

Where we have used the non arbitrage relationship,

$$
\left(f_s^{H,F} - (c_s^F + b_s^{F,G})\right) = \left(f_s^{H,D} - (c_s^D + b_s^{D,G})\right)
$$

- A question must arise,

Which dynamics should the FX follow ?

For this, just note that equations (8.26) and (8.27) are different representations of the same price, and hence they must be equal. In order for both equations to coincide, the Radon-Nikodyn's derivative that relates both economies is forced to be,

$$
\frac{dQ^F}{dQ^D}(t) = \frac{X_t^{D/F} e^{\int_0^t f_s^F ds}}{X_0^{D/F} e^{\int_0^t f_s^F ds}} = \frac{X_t^{D/F} e^{\int_0^t c_s^F ds}}{X_0^{D/F} e^{\int_0^t (c_s^D + b_s^{D,F}) ds}} \tag{8.28}
$$

What implies that the non-arbitrage dynamics for the FX must be,

$$
\boxed{E_t^{\mathbb{Q}_D}\left[\frac{dX_t^{D/F}}{X_t^{D/F}}\right] = \left(c_t^D + b_t^{D,F} - c_t^F\right) dt} \tag{8.29}
$$

That, is

FX must be simulated with basis for all terms ... !!
But this will not be always the case ...

Conclusions:

- **The choice of the discount rate to use is very relevant**, when pricing a derivative. By choosing the overnight rate, we get a recursive formula difficult to solve for. While by choosing the Hedger's funding rate, the pricing formula becomes simpler. **There is a clear choice to take ...!!**
- In general terms, we will need two different pricers to calculate the CVA term. One for the close-out and the other one that will depend on the discounting rate chosen.
- When choosing the over-night rate as discounting rate, we can disentangle the pricing formula in a pure CVA term and FVA term, though we do not know how to solve for this FVA term. When using the Hedger's funding rate, we obtain a hybrid term accounting for both CVA and FVA.
- **The simulated FX will carry a Xccy basis in its drift**.

8.5 Risky pricing of a portfolio of derivatives

In order to look at the whole picture, we will assume a **Collateral Set** made up of a portfolio with N derivatives where each derivative is denominated in a different currency.

We will assume the collateral to be composed of a portfolio of bonds [1], R_t, denominated in a different currency to those of the derivative.

In this set-up we will consider $N+1$ FX, $\{X^{D,C_j}\}_{j=0,\dots,N}$, where we keep the index 0 for the collateral currency.

The **replicating portfolio** would be made up of,

$$\underbrace{\sum_{j=1}^{N} V_t^{C_j} X_t^{D/C_j}}_{\mathbf{V}_t^D} = \sum_{j=1}^{N} \alpha_t^{(j)} \underbrace{H_t^{C_j} X_t^{D/C_j}}_{H_t^{D_j}} + \beta_t^D + \gamma_t CDS^D(t,T) + \epsilon_t\, CDS^D(t,t+dt)$$
$$+ \underbrace{(\Omega_t B(t,t+dt) + \omega_t B(t,T))}_{\sum_{j=1}^{N} V_t^{C_j} X_t^{D/C_j} - R_t^{C_0} X_t^{D/C_0}} \tag{8.30}$$

The bank account is composed of,

$$\beta_t^D = -\sum_{j=1}^{N} \alpha_t^{(j)} H_t^{C_j} X_t^{D/C_j} - \gamma_t CDS^D(t,T) + C_t^{C_0} X_t^{D/C_0} \tag{8.31}$$

[1]we will assume no joint default counterparty and bond's reference

so variations, in an infinitesimal time interval, in the bank-account will be,

$$\begin{aligned} d\beta_t^D = & \left[\left(r_t^R + b_t^{D,C_0}\right) R_t^{C_0}\right] X_t^{D/C_0} dt \\ & - \left[\sum_{j=1}^{N} \alpha_t^{(j)} \left(c_t^D + b_t^{D,C_j}\right) H_t^{D_j} + \gamma_t c_t^D CDS^D(t,T)\right] dt \end{aligned} \tag{8.32}$$

being r_t^R the instantaneous repo rate associated to the bond R.
As it can be seen in the Appendix, the pricing PDE becomes

$$\begin{aligned} & \mathcal{L}\mathbf{V}_t^D + \frac{h_t^D}{(1-R)}\mathbf{V}_t^D = \left(\mathbf{V}_t^D - C_t^{C_0} X_t^{D/C_0}\right) f_t^{H,D} + \left[\left(r_t^R + b_t^{D,C_0}\right) R_t^{C_0}\right] X_t^{D/C_0} \\ & s.t \qquad \mathbf{V}_{t'}^D = \mathbf{V}_T^D \mathbf{1}_{\{\tau > T\}} + \pi_\tau^D \mathbf{1}_{\{\tau \le T\}} \quad , \qquad t' = (T \wedge \tau) \end{aligned} \tag{8.33}$$

By applying Feyman-Kac,

- **<u>Discounting at the domestic over-night rate flat</u>**:

$$\begin{aligned} \mathbf{V}_t^D = & \underbrace{E_t^{\mathbb{Q}_D}\left(e^{-\int_t^T c_s^D ds}\, \mathbf{V}_T^D\right)}_{\mathbf{V}_t^{D,c^D}} \\ & + \underbrace{E_t^{\mathbb{Q}_D}\left(e^{-\int_t^\tau c_s^D ds}\left(\pi_\tau^D - \mathbf{V}_\tau^{D,c^D}\right)\mathbf{1}_{\{\tau<T\}}\right)}_{\text{CVA}} \\ & - \underbrace{\int_t^T E_t^{\mathbb{Q}_D}\left(e^{-\int_t^s c_u^D du}\, \mathbf{1}_{\{\tau>s\}}\left(f_s^{H,D} - c_s^D)\right)\mathbf{V}_s^D\right) ds}_{\text{FVA}} \\ & + \underbrace{\int_t^T E_t^{\mathbb{Q}_D}\left(e^{-\int_t^s c_u^D du}\, \mathbf{1}_{\{\tau>s\}}\left(f_s^{H,D} - (c_s^D + b_s^{D,C_0})\right) M_s^{C_0} X_s^{D/C_0}\right) ds}_{\text{Collateral Adjustment}} \\ & + \underbrace{\int_t^T E_t^{\mathbb{Q}_D}\left(e^{-\int_t^s c_u^D du}\, \mathbf{1}_{\{\tau>s\}}\left(f_s^{H,D} - (r_s^D + b_s^{D,C_0})\right) R_s^{C_0} X_s^{D/C_0}\right) ds}_{\text{Repo Adjustment}} \end{aligned} \tag{8.34}$$

Where

$$\mathbf{V}_t^{D,c^D} = \sum_{j=1}^{N} E_t^{\mathbb{Q}_D}\left[e^{-\int_t^T c_s^D}\; V_T^{C_j} X_T^{D/C_j}\right]$$

$$\begin{aligned} \pi_t^F = & -\mathbf{V}_t^{D,RF} - (1-R_C)\left[\left(\mathbf{V}_t^{D,RF}\right)^- - \left(C_t^{C_0} X_t^{D/C_0}\right)^-\right]^- \\ & - (1-R_{Col})\left[\left(\mathbf{V}_t^{D,RF}\right)^+ - \left(C_t^{C_0} X_t^{D/C_0}\right)^+\right]^- \end{aligned} \tag{8.35}$$

$$\mathbf{V}_t^{D,RF} = \sum_{j=1}^{N} V_t^{C_j,c_j} X_t^{C_j/D}$$

Note that the **Raw price, $\mathbf{V}_t^{D,c^D}$, is under the assumption that the operative is perfectly cash-collateralized in currency** D. And V_t^{RF,C_j} is a C_j-denominated derivative with standard collateralization.

- **Discounting at the Hedger's domestic funding rate**:

$$\begin{aligned}
\mathbf{V}_t^D = & \underbrace{E_t^{\mathbb{Q}_D}\left(e^{-\int_t^T f_s^{H,D} ds}\, \mathbf{V}_T^D\right)}_{\mathbf{V}_t^{D,f^{H,D}}} \\
& + \underbrace{E_t^{\mathbb{Q}_D}\left(e^{-\int_t^\tau f_s^{H,D} ds}\left(\pi_\tau^D - \mathbf{V}_\tau^{D,f^{H,D}}\right) X_\tau^{D/F} \mathbf{1}_{\{\tau<T\}}\right)}_{\text{CVA adjusted for funding}} \\
& + \underbrace{\int_t^T E_t^{\mathbb{Q}_D}\left(e^{-\int_t^s f_u^{H,D} du}\, \mathbf{1}_{\{\tau>s\}}\left(f_s^{H,D} - (c_s^D + b_s^{D,C_0})\right) M_s^{C_0} X_s^{D/C_0}\right) ds}_{\text{Collateral Adjustment}} \\
& + \underbrace{\int_t^T E_t^{\mathbb{Q}_D}\left(e^{-\int_t^s f_u^{H,D} du}\, \mathbf{1}_{\{\tau>s\}}\left(f_s^{H,D} - (r_s^D + b_s^{D,C_0})\right) R_s^{C_0} X_s^{D/C_0}\right) ds}_{\text{Repo Adjustment}}
\end{aligned} \tag{8.36}$$

8.6 Summary

- The only pricing formula we know how to calculate is when the **bank-account that accrues at the hedger's funding rate is used as numeraire**.
- Under this numeraire, we know how to calculate the price of the portfolio, though we do not know how to disentangle CVA and FVA terms.
- In order to calculate the CVA, **two different pricers will be needed**. One to use in the **close-out** calculation and other that uses the **funding curve to discount** the pay-off.
- FX involved in the calculation **always takes a Xccy basis spreads in its drift**.
- It will be **difficult to conciliate CVA figures** across *"geographies"*.

8.7 Traditional Approach to the CVA term.

In this section, we look at the traditional approach **CVA/DVA**, where these metrics are accounted for by discounting the future flows in the derivative.

Let's define a derivative that pays out at different times within the time-grid $\mathbb{T} = \{t_0, \dots, t_n = T\}$ an uncertain quantity denoted by $CF(t_i)$ in currency F. Let's define the value at any instant t, as seen in the domestic currency, by

$$\Pi^D(t,T) = X^{D/F}(t)\ \underbrace{N^F(t)\, E_t^{\mathbb{N}_F}\left(\sum_{i=I(t)}^{I(T)} \frac{CF^F(t_i)}{N^F(t_i)}\right)}_{\Pi^F(t,T)}$$

Where $I(t)$ makes reference to the index i such that $t_i \geq t$, and $\Pi^D(t,T)$ is as seen from the counterparty's point of view.

Let us assume the existence of a bilateral CSA agreement. The collateral at t will be denoted by C_t^G (cash denominated in currency G).

We next derive the general pricing formula for this derivative under the possibility of only the counterparty being default-able.

Let's denote the risky premium by $V^D(t)$, and τ the time at which the counterparty defaults.

$$\begin{aligned} V^D(t) =& E_t^{\mathbb{N}_D}\left[\mathbf{1}_{\{\tau > T\}}\Pi^D(t,T)\right] + E_t^{\mathbb{N}_D}\left[\mathbf{1}_{\{\tau < T\}}\Pi^D(t,\tau)\right] + \\ & N^D(t) E_t^{\mathbb{N}_D}\left[\mathbf{1}_{\{\tau<T\}}\left(\frac{(\Pi^D(\tau,T) - C_\tau^G X_\tau^{D/G})^+ + R(\Pi(\tau_C,T) - C_\tau^G X_\tau^{D/G})^- + C_\tau^G X_\tau^{D/G}}{N^D(\tau)}\right)\right] \end{aligned} \tag{8.37}$$

After straight forward algebra, it can be shown that

$$\begin{aligned} V^D(t) =& \Pi^D(t,T) - \\ & \underbrace{N^D(t)\,(1-R)\,E_t^{\mathbb{N}_D}\left[\mathbf{1}_{\{\tau<T\}}\left(\frac{(\Pi^D(\tau,T) - C_\tau^G X_\tau^{D/G})}{N^D(\tau)}\right)^-\right]}_{\text{CVA}} \end{aligned} \tag{8.38}$$

In order to compare this CVA term with the others obtained before, we set as domestic (foreign) numeraire the one associated to the domestic (foreign)risk-free bank-account (i.e $\mathbb{N}_D = \mathbb{Q}_D$).

Let us think of a portfolio denominated in currency F so that

$$\Pi^D(t,T) = X_t^{D/F} V_t^{F,c^F} := X_t^{D/F} E_t^{\mathbb{Q}_F}\left[e^{-\int_t^T c_s^F ds} V_T^F\right]$$

The CVA term will become

$$CVA_t^D = (1-R)\, E_t^{\mathbb{Q}_D}\left[e^{-\int_t^T c_s^D ds} X_\tau^{D/F}\left(V_\tau^{F,c^F} - C_\tau^G X_\tau^{F/G}\right)^- \mathbf{1}_{\{\tau<T\}}\right] \tag{8.39}$$

Note that **this term is not the same as the one in** (8.34) **where two different pricers were needed**.
Again a question is in order,

Which dynamics should we impose to the FX ..?

For this, we should note that the CVA term in (8.39), should be equivalent to the one obtained by measuring the conterparty risk in currecny F and then translate it into currency D.
That is,

$$CVA_t^D = X_t^{D/F}(1-R)\, E_t^{\mathbb{Q}_F}\left[e^{-\int_t^T c_s^F ds}\left(V_\tau^{F,c^F} - C_\tau^G X_\tau^{F/G}\right)^- \mathbf{1}_{\{\tau<T\}}\right] \tag{8.40}$$

As (8.39) = (8.40), this will imply

$$\frac{dQ^F}{dQ^D}(t) = \frac{X_t^{D/F} e^{\int_0^t c_s^F ds}}{X_0^{D/F} e^{\int_0^t c_s^D ds}} \tag{8.41}$$

So in this case, $X_t^{D/F}$ **does not have the cross currency basis in its drift**,

$$\boxed{E_t^{\mathbb{Q}_D}\left[\frac{dX_t^{D/F}}{X_t^{D/F}}\right] = (c_t^D - c_t^F)dt}$$

Notice, that **under this approach the CVA figure calculated in two different currencies will agree ..!!**

8.8 Conclusions

- When non-arbitrage arguments are applied, the dynamcis for the FX carries a instantaneous cross currency basis in its drift. This is no the case when calculating the CVA term by the discounting-flows approach where the FX appearing in the CVA term does not carry Xccy basis.
- This definition of CVA is completely **an arbitray choice that makes this figure independent of the currency chosen to express the CVA**.

8.9 Appendix: Risky Pricing PDE

In this appendix we follow the route in [9] to demonstrate equation (8.33).
Let us make explicit the dependence of the portfolio's value,

$$\mathbf{V}_t^D = \mathbf{V}^D(t, H^{D_j}, h_t^C, h_t^H, N_t^{C,\mathbb{P}})$$

We remind the replicating portfolio to be

$$\begin{aligned}\mathbf{V}_t^D =& \sum_{j=1}^{N} \alpha_t^{(j)} H_t^{D_j} + \beta_t^D + \gamma_t CDS^D(t,T) + \epsilon_t\, CDS^D(t,t+dt) \\ &+ \underbrace{(\Omega_t B(t,t+dt) + \omega_t B(t,T))}_{\sum_{j=1}^{N} V_t^{C_j} X_t^{D/C_j} - R_t^{C_0} X_t^{D/C_0}}\end{aligned} \tag{8.42}$$

and the bank-account

$$\beta_t^D = -\sum_{j=1}^{N} \alpha_t^{(j)} H_t^{D_j} - \gamma_t CDS^D(t,T) + R_t^{C_0} X_t^{D/C_0} \tag{8.43}$$

By applying differentiating to both sides in (8.42),
<u>LHS</u>:

$$\begin{aligned}d\mathbf{V}_t^D =& \frac{\partial \mathbf{V}_t^D}{\partial t} dt + \sum_{j=1}^{N} \frac{\partial \mathbf{V}_t^D}{\partial S_t^j} dS_t^j + \frac{1}{2} \sum_{j,k=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 \mathbf{V}_t^D}{\partial S^j \partial S^k} \sigma_{S_j} \sigma_{S_k} \rho_{S_j,S_k}(t) dt \\ &+ \frac{\partial \mathbf{V}_t^D}{\partial h_t^C} dh_t^C + \frac{1}{2} \frac{\partial^2 \mathbf{V}_t^D}{\partial (h_t^C) 2} \sigma_{h^C}^2 dt + \sum_{j=1}^{N} \frac{\partial^2 \mathbf{V}_t^D}{\partial S^j \partial h^C} \sigma_{S_j} \sigma_{h^C} \rho_{S_j,h^C}(t) dt + \Delta \mathbf{V}_t^D dN_t^{\mathbb{P}} \\ &+ \frac{\partial \mathbf{V}_t^D}{\partial h_t^H} dh_t^H + \frac{1}{2} \frac{\partial^2 \mathbf{V}_t^D}{\partial (h_t^H) 2} \sigma_{h^C}^2 dt + \sum_{j=1}^{N} \frac{\partial^2 \mathbf{V}_t^D}{\partial S^j \partial h^H} \sigma_{S_j} \sigma_{h^H} \rho_{S_j,h^C}(t) dt \\ &+ \frac{\partial^2 \mathbf{V}_t^D}{\partial h^C \partial h^H} \sigma_{h^C} \sigma_{h^H} \rho_{h^C,h^C}(t) dt\end{aligned} \tag{8.44}$$

RHS:

$$
\begin{aligned}
d\mathbf{V}_t^D =& \sum_{j=1}^N \alpha_t^{(j)} dH_t^{D_j} + \gamma_t dCDS^D(t,T) + \epsilon_t\, dCDS^D(t,t+dt) \\
&+ \frac{\left(\mathbf{V}_t^D - C_t^D\right) - \omega_t B(t,T)}{B(t,t+dt)} dB(t,t+dt) + \omega_t dB(t,T) \\
&- \left[\sum_{j=1}^N \alpha_t^{(j)} H_t^{D_j}\left(c_t^D + b_t^{D,j}\right) + \gamma_t CDS^D(t,T)c_t^D\right] dt \\
&+ \left[M_t^{C_0}\left(c_t^D + b_t^{D,G}\right) + R_t^{C_0}\left(r_t^D + b_t^{D,G}\right)\right] dt
\end{aligned}
\tag{8.45}
$$

where,

$$
\begin{aligned}
dCDS(t,T) &= \frac{\partial CDS(t,T)}{\partial t} dt + \frac{\partial CDS(t,T)}{\partial h_t^C} dh_t^C + \frac{1}{2}\frac{\partial^2 CDS(t,T)}{\partial (h_t^C)^2}\sigma_{h^C}^2 dt + \Delta CDS(t,T) dN_t^{\mathbb{P}} \\
dCDS(t,t+dt) &= h_t^C dt - (1-R) dN_t^{\mathbb{P}} \\
dB(t,T) &= \frac{\partial B(t,T)}{\partial t} dt + \frac{\partial B(t,T)}{\partial h_t^H} dh_t^H + \frac{1}{2}\frac{\partial^2 B(t,T)}{\partial (h_t^H)^2}\sigma_{h^H}^2 dt \\
dB(t,t+dt) &= B(t,t+dt) f_t^H dt
\end{aligned}
\tag{8.46}
$$

and $dH_t^{D_j}$ has been defined in (8.19).
By choosing the coefficicients,

$$\alpha_t^{(j)} = \frac{\frac{\partial \mathbf{V}^D}{\partial SD_j}}{\frac{\partial H^{D_j}}{\partial SD_j}} \tag{8.47}$$

$$\gamma_t = \frac{\frac{\partial \mathbf{V}^D}{\partial h^C}}{\frac{\partial CDS(t,T)}{\partial h^C}} \tag{8.48}$$

$$\epsilon_t = \frac{\gamma_t \Delta CDS(t,T) - \Delta \mathbf{V}^D}{(1-R)} \tag{8.49}$$

$$\omega_t = \frac{\frac{\partial \mathbf{V}^D}{\partial h^H}}{\frac{\partial B(t,T)}{\partial h^H}} \tag{8.50}$$

$$\Omega_t = \frac{\left(\mathbf{V}_t^D - C_t^D\right) - \omega_t B(t,T)}{B(t,t+dt)} \tag{8.51}$$

and substituting into (8.42), the **RHS** becomes,

$$
\begin{aligned}
\mathbf{RHS} = & \sum_{j=1}^{N} \alpha_t^{(j)} \left[\frac{\partial H^{D_j}}{\partial t} + \frac{1}{2} \frac{\partial^2 H^{D_j}}{\partial (S^j)2} \sigma_{S^j}^2 - H_t^{D_j} \left(c_t^D + b_t^{D,j} \right) \right] \\
& + \gamma_t \left[\frac{\partial CDS(t,T)}{\partial t} + \frac{1}{2} \frac{\partial^2 CDS(t,T)}{\partial (h^C)2} \sigma_{h^C}^2 - CDS(t,T) c_t^D + \frac{h_t^C}{(1-R)} \Delta CDS(t,T) \right] \\
& - \frac{h_t^C}{(1-R)} \Delta \mathbf{V}_t^D + \omega_t \left[\frac{\partial B(t,T)}{\partial t} + \frac{1}{2} \frac{\partial^2 B(t,T)}{\partial (h^H)2} \sigma_{h^H}^2 - B(t,T) f_t \right] \\
& + \left(\mathbf{V}_t^D - C_t^{C_0} X_t^{D/C_0} t \right) f_t + \left[M_t^{C_0} \left(c_t^D + b_t^{D,0} \right) + R_t^{C_0} \left(r_t^R + b_t^{D,0} \right) \right] X_t^{D/C_0}
\end{aligned}
\tag{8.52}
$$

And finally, by substituting it into (8.42), we arrive to the fundamental pricing PDE,

$$
\begin{aligned}
& \mathcal{L}\mathbf{V}_t^D + \frac{h_t^D}{(1-R)} \mathbf{V}_t^D = \left(\mathbf{V}_t^D - C_t^{C_0} X_t^{D/C_0} \right) f_t^{H,D} + \left[\left(r_t^R + b_t^{D,C_0} \right) R_t^{C_0} \right] X_t^{D/C_0} \\
& s.t \qquad \mathbf{V}_{t'}^D = \mathbf{V}_T^D \mathbf{1}_{\{\tau > T\}} + \pi_\tau^D \mathbf{1}_{\{\tau \leq T\}} \quad , \qquad t' = (T \wedge \tau)
\end{aligned}
\tag{8.53}
$$

Chapter 9

CVA for Credit Derivatives

9.1 Introduction

CVA is a credit contingent option written on a set of derivatives that are written on different assets.

If the derivatives are written on market risk instruments (equities, interest rates, fx, commodities), the following ingredients should be in place:

- Volatilities, even if the derivatives are linear (CVA is an option).
- Correlations between the different market risk factors (will affect the variance of the NPV of the whole portfolio).
- Volatility of the credit curve of the counterparty and correlations between this and the market risk factors (soft wrong way risk).
- Jumps in market risk factors (FX) upon default of the counterparty (hard wrong way risk).

If some derivatives are written on credit, there are more ingredients to be added:

- Volatilities of the different credit curves and correlations between them and other market risk factors.
- Default correlation (underlying credit references and counterparty (and own default for DVA supporters)).

In this section we will tackle the default correlation issue.

9.2 Risky Pricing for a Credit Derivative

We will assume the existence of three agents intervening in a credit derivative transaction: The **Hedger** or risk taker, the **counterparty** and the **reference** of the credit derivative.

Let us assume the Hedger enters into a partially cash-collateralized credit derivative, written on reference R, with counterparty C. The collateral amount at time t will be denoted by C_t (as seen from the counterparty's point of view), we will denote the price of such derivative [1], at time t, by,

$$V_t = V(t, h_t^R, h_t^C, h_t^H, N_t^{\mathbb{P},R}, N_t^{\mathbb{P},C}) \tag{9.1}$$

Where, h_t^X is the over-night CDS par spread for party X. $N_t^{\mathbb{P},X} = \mathbf{1}_{\{\tau^X \leq t\}}$ is the process that counts the unique default of party X, under the historical measure $\mathbb{P}$. We omit the dependence of V_t on the Hedger's default as the Hedger will be in charge of setting the price of such derivative and, thus, he will not be worried about V_t once he has defaulted.

In a one dimension world,

$$\begin{aligned} dV_t =& \frac{\partial V_t}{\partial t}dt + \sum_j \frac{\partial V_t}{\partial h_t^j} dh_t^j + \frac{1}{2}\sum_{j,k} \frac{\partial^2 V_t}{\partial h_t^j \partial h_t^k} dh_t^j dh_t^k \\ &+ \Delta V_t^R dN_t^{\mathbb{P},R} + \Delta V_t^C dN_t^{\mathbb{P},C} + \Delta V_t^{R,C} dN_t^{\mathbb{P},R} dN_t^C \quad j = R, C, H \end{aligned} \tag{9.2}$$

Where ΔV^X denotes the jump to default upon party X's default.

We will assume the existence of a fully cash-collateralized CDS written on both parties R and C, that will be used in the hedging portfolio. We will also assume that the Hedger is able to issue debt in the Bond Market, as well as the existence of a credit derivative with sensitivity to the (counterparty and credit reference) joint default, whose price we denote by $J(t, t + dt)$. We will further assume this credit instrument to pay at $t + dt$ (as seen from t) a premium $h_t^{R,C} dt$ against $(1 - R_{R,C})$ in case both parties, R, C defaults.

In a one factor world, we will assume the following replicating portfolio,

$$\begin{aligned} V_t =& \sum_{j=1}^{2} \alpha^{(j)}(t) CDS^R(t, T_j) + \gamma_t CDS^C(t, T) + \underbrace{\epsilon_t CDS^C(t, t + dt)}_{=0} \\ &+ \underbrace{\phi_t J(t, t + dt)}_{=0} + \underbrace{(\Omega_t B(t, t + dt) + \omega_t B(t, T))}_{(V_t - C_t)} + \beta_t \end{aligned} \tag{9.3}$$

[1] We will assume, without loss of generality, that the derivative is denominated in the domestic currency

Where V_t is seen from the counterparty's point of view, while the hedging portfolio is as seen from the Hedger's point of view.

It can be seen that the bank-account at time t is made up of,

$$\beta_t = \underbrace{-\gamma_t CDS^C(t,T) - \sum_{j=1}^{2} \alpha^{(j)}(t) CDS^R(t,T_j) + C_t}_{\text{Collateral account}} \tag{9.4}$$

And its variation in a infinitesimal time interval,

$$d\beta_t = \left[-\gamma_t CDS^C(t,T) - \sum_{j=1}^{2} \alpha^{(j)}(t) CDS^R(t,T_j) + C_t \right] c_t dt \tag{9.5}$$

where c_t is the domestic over-night rate at which the collateral is remunerated.

The self-financing condition will imply,

$$\begin{aligned} dV_t =& \sum_{j=1}^{2} \alpha^{(j)}(t) dCDS^R(t,T_j) + \gamma_t dCDS^C(t,T) + \epsilon_t dCDS^C(t,t+dt) \\ &+ \phi_t dJ(t,t+dt) + \left[\frac{(V_t - C_t) - \omega_t B(t,T)}{B(t,t+dt)} \right] dB(t,t+dt) + \omega_t dB(t,T) + d\beta_t \end{aligned} \tag{9.6}$$

By applying Ito to the different instruments, it can be seen

$$\begin{aligned} dCDS^X(t,T) &= \frac{\partial CDS^X}{\partial t} + \frac{\partial CDS^X}{\partial h_t^x} dh_t^x + \frac{1}{2} \frac{\partial^2 CDS^X}{\partial (h_t^x)^2} (dh_t^x)^2 + \Delta CDS^X dN_t^{\mathbb{P},X} \\ dCDS^C(t,t+dt) &= h_t^C dt - (1-R^C) dN_t^{\mathbb{P},C} \\ dB(t,T) &= \frac{\partial B(t,T)}{\partial t} + \frac{\partial B(t,T)}{\partial h_t^H} dh_t^H + \frac{1}{2} \frac{\partial^2 B(t,T)}{\partial (h_t^H)^2} \left(dh_t^H \right)^2 \\ dB(t,t+dt) &= B(t,t+dt) f_t^H dt \\ dJ(t,t+dt) &= \left(h_t^{R,C} dt - (1-R_{R,C}) dN_t^{\mathbb{P},R} dN_t^{\mathbb{P},C} \right) \end{aligned} \tag{9.7}$$

Just note again that in the last two equations in (9.7) there is not dependence on $N_t^{\mathbb{P},H}$ as the hedger is not interested on the world once he is not alive to stare it.

By choosing as hedging coefficients,

$$\alpha_t^{(1)} = \frac{\Delta V^R - \alpha_t^{(2)} \Delta CDS^R(t,T_2)}{\Delta CDS^R(t,T_2)} \tag{9.8}$$

$$\alpha_t^{(2)} = \frac{\left[\frac{\partial V}{\partial h_t^R} - \frac{\Delta V^R}{\Delta CDS^R(t,T_1)} \frac{\partial CDS^R(t,T_1)}{\partial h_t^R} \right]}{\left[\frac{\Delta CDS^R(t,T_2)}{\partial h_t^R} - \frac{CDS^R(t,T_2)}{\Delta CDS^R(t,T_1)} \frac{\partial CDS^R(t,T_1)}{\partial h_t^R} \right]} \tag{9.9}$$

$$\gamma_t = \frac{\frac{\partial V}{\partial h_t^C}}{\frac{\partial CDS^C(t,T)}{\partial h_t^C}} \tag{9.10}$$

$$\epsilon_t = \frac{\gamma_t \Delta CDS^C(t,T) - \Delta V_t^C}{(1-R^C)} \tag{9.11}$$

$$\omega_t = \frac{\frac{\partial V}{\partial h_t^H}}{\frac{\partial B(t,T)}{\partial h_t^H}} \tag{9.12}$$

$$\phi_t = -\frac{\Delta V_t^{R,C}}{(1-R_{R,C})} \tag{9.13}$$

and substituting them back into (9.6), we obtain

$$\begin{aligned}&\frac{\partial V_t}{\partial t}dt + \sum_j \frac{\partial V_t}{\partial h_t^j}\left(\mu^{\mathbb{P}}(t,h_t^j) - M_t^j\sigma(t,h_t^j)\right)dt + \frac{1}{2}\sum_{j,k}\frac{\partial^2 V_t}{\partial h_t^j \partial h_t^k}\sigma(t,h_t^j)\sigma(t,h_t^k)dt\\&\sum_i \frac{h_t^i}{(1-R^i)}\Delta V^i dt + \frac{h_t^{R,C}}{(1-R_{R,C})}\Delta V_t^{R,C}dt = (V_t - C_t)\, f_t dt + C_t c_t dt \qquad \forall j,k = R,C,H, \quad \forall i = R,C.\end{aligned} \tag{9.14}$$

where M_t^j denotes the market price of risk for h_t^j.

We will next relate the price of the derivative that fulfills with the former equation with an expected value. For this, we define

$$X_t = \underbrace{\mathbf{1}_{\{\tau^C>t\}}\mathbf{1}_{\{\tau^R>t\}}e^{-\int_0^t f_s ds}}_{Y_t} V_t \tag{9.15}$$

Applying Ito,

$$\begin{aligned}dX_t =& - f_t Y_t V_t + Y_t V_t^{R,C} dN_t^R dN_t^C + Y_t\left[dN_t^R + dN_t^C\right]V_t\\&+ Y_t\left[\frac{\partial V_t}{\partial t}dt + \sum_j \frac{\partial V_t}{\partial h_t^j}dh_t^j + \frac{1}{2}\sum_{j,k}\frac{\partial^2 V_t}{\partial h_t^j \partial h_t^k}\sigma(t,h_t^j)\sigma(t,h_t^k)dt + \Delta V_t^{R,C} dN_t^R dN_t^C\right]\end{aligned} \tag{9.16}$$

By taking expected values under the risk-neutral measure,

$$\begin{aligned}&E_t^{\mathbb{Q}}\left[dX_t\right] = Y_t V_t^{R,C}\lambda_t^{R,C}dt + Y_t\left[\lambda_t^R + \lambda_t^C\right]V_t dt\\&+ Y_t\left[\frac{\partial V_t}{\partial t} + \sum_j \frac{\partial V_t}{\partial h_t^j}\left(\mu^{\mathbb{P}}(t,h_t^j) - M_t^j\sigma(t,h_t^j)\right) + \frac{1}{2}\sum_{j,k}\frac{\partial^2 V_t}{\partial h_t^j \partial h_t^k}\sigma(t,h_t^j)\sigma(t,h_t^k) + \Delta V_t^{R,C}\lambda_t^{R,C} - f_t V_t\right]dt\end{aligned} \tag{9.17}$$

By taking into account (9.14),

$$E_t^{\mathbb{Q}}[dX_t] = Y_t\left[V_t^{R,C}\lambda_t^{R,C} + V_t^R\lambda_t^R + V_t^C\lambda_t^C\right]dt - (f_t - c_t)\,C_t dt \tag{9.18}$$

And, by integrating between t_0 and T and taking expected values conditioned to the information revealed at t_0,

$$\begin{aligned} V_{t_0} =& E_{t_0}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau^C>T\}}\mathbf{1}_{\{\tau^R>T\}}e^{-\int_{t_0}^T f_s ds}\, V_T\right] \\ &- \int_{t_0}^T E_t^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau^C>s\}}\mathbf{1}_{\{\tau^R>s\}}e^{-\int_{t_0}^s f_u du}\, V_s^R\lambda_s^R\right]ds \\ &- \int_{t_0}^T E_{t_0}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau^C>s\}}\mathbf{1}_{\{\tau^C>s\}}e^{-\int_{t_0}^s f_u du}\, V_s^C\lambda_s^C\right]ds \\ &- \int_{t_0}^T E_{t_0}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau^C>s\}}\mathbf{1}_{\{\tau^{R,C}>s\}}e^{-\int_{t_0}^s f_u du}\, V_s^{R,C}\lambda_s^{R,C}\right]ds \\ &+ \int_{t_0}^T E_{t_0}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau^C>s\}}\mathbf{1}_{\{\tau^{R,C}>s\}}e^{-\int_{t_0}^s f_u du}\, (f_s - c_s)\,C_s\right]ds \end{aligned} \tag{9.19}$$

Risky price of a Credit derivative (CD) by using the hedger's funding rate to discount flows

$$\begin{aligned} V_{t_0} =& \underbrace{E_{t_0}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau^R>T\}}e^{-\int_{t_0}^T f_s ds}\, V_T\right]}_{\text{Survival Payments } (V_{t_0}^{f,Surv})} - \underbrace{E_{t_0}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau^R\le T\}}e^{-\int_{t_0}^{\tau^R} f_s ds}\, V_{\tau_R^R}\right]}_{\text{Default Payments } (V_{t_0}^{f,Dft})} \\ &- \underbrace{E_{t_0}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau^C\le T\}}\mathbf{1}_{\{\tau^R>\tau^C\}}e^{-\int_{t_0}^{\tau^C} f_s ds}\left(V_{\tau^C}^C - (V_{\tau^C}^{f,Surv} - V_{\tau^C}^{f,Dft})\right)\right]}_{\text{CVA Adjusted for Funding}} \\ &- \underbrace{E_{t_0}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau^C=\tau^R\le T\}}e^{-\int_{t_0}^{\tau} f_s ds}\, V_\tau^{R,C}\right]}_{\text{Joint default CVA}} \\ &+ \underbrace{\int_{t_0}^T E_{t_0}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau^C>s\}}\mathbf{1}_{\{\tau^R>s\}}e^{-\int_{t_0}^s f_u du}\, (f_s - c_s)\,C_s\right]ds}_{\text{Collateral Adjustment}} \end{aligned} \tag{9.20}$$

Where

$$V_{\tau^C}^C = R_C\left(CD_{\tau^C}^c - C_{\tau^C}\right)^- + \left(CD_{\tau^C}^c - C_{\tau^C}\right)^+ + C_{\tau^C}$$

$$V_\tau^{R,C} = -(1 - R_C)\left(CD_\tau^{Dfl} - C_\tau\right)^- + CD_\tau^{Dfl}; \qquad \tau = \tau^C = \tau^R$$

Where $CD_{\tau^C}^c$ is the price of the credit derivative at counterparty's default by discounting both the survival and default leg with the domestic over-night rate c_t. CD_τ^{Dfl} is the payment of the CD at the reference's default.

Just note that the sum of the first two terms in (9.20) is the price of the CD by assuming an immortal counterparty. The third term is the CVA adjusted for funding and accounts for the amount that the hedger might loose in the case the counterparty defaults. In this case, there might be a **contagion effect** by which the reference's credit spread upon the counterparty's default is pushed down. The fourth term accounts for the event in which both the counterparty and the CD reference jointly defaults.

Risky price of a CD by using the over-night rate to discount flows

We might use as discounting rate the domestic over-night rate. In this case equation (9.20) would become,

$$\begin{aligned}
V_{t_0} = & \underbrace{E^{\mathbb{Q}}_{t_0}\left[\mathbf{1}_{\{\tau^R>T\}} e^{-\int_{t_0}^{T} c_s ds}\, V_T\right]}_{\text{Survival Payments } (V^{f,Surv}_{t_0})} - \underbrace{E^{\mathbb{Q}}_{t_0}\left[\mathbf{1}_{\{\tau^R\le T\}} e^{-\int_{t_0}^{\tau^R} c_s ds}\, V_{\tau^R_R}\right]}_{\text{Default Payments } (V^{f,Dft}_{t_0})} \\
& \underbrace{- E^{\mathbb{Q}}_{t_0}\left[\mathbf{1}_{\{\tau^C\le T\}}\mathbf{1}_{\{\tau^R>\tau^C\}} e^{-\int_{t_0}^{\tau^C} c_s ds}\left(V^C_{\tau^C} - (V^{c,Surv}_{\tau^C} - V^{c,Dft}_{\tau^C})\right)\right]}_{\text{CVA}} \\
& \underbrace{- E^{\mathbb{Q}}_{t_0}\left[\mathbf{1}_{\{\tau^C=\tau^R\le T\}} e^{-\int_{t_0}^{\tau} c_s ds}\, V^{R,C}_{\tau}\right]}_{\text{Joint default CVA}} \\
& \underbrace{+ \int_{t_0}^{T} E^{\mathbb{Q}}_{t_0}\left[\mathbf{1}_{\{\tau^C>s\}}\mathbf{1}_{\{\tau^R>s\}} e^{-\int_{t_0}^{s} c_u du}\, (f_s - c_s)(V_s - C_s)\right] ds}_{\text{FVA}}
\end{aligned} \tag{9.21}$$

In this case, note that the third term in equation (9.21) is a pure term of counterparty credit risk that is discounted at the over-night rate. Again, in this term there might be a contagion effect by which the CD value upon the counterparty's default might be pushed down. The fifth term is the FVA term that is conditional on both counterparty and the CDS reference being alive. Just note that the Value of the CDS in this term already includes CVA and funding terms, making this term hard to solve.

So far, we have seen that the CVA of a CDS will be determined by mainly two effects (apart from the credit dynamcis, etc):

- **Contagion Effect**: At the time the counterparty defaults, the CDS spreads might experiment a subit rise. This fact might increase/decrease ?? the risky price of a partially collateralized CDS. This effect will tend to disappear for perfectly collateralized credit derivatives.
- **Joint default**: At the time both parties (the counterparty and the credit reference) default, the hedger will face a huge loss. This effect will persist even in the case of perfectly collateralized derivatives.

Two of the main drivers of the risky price of a credit derivative (**contagion** and **joint default**) will depend on the Copula chosen to relate both the Counterparty and the CDS reference's times to default. In the next section we get deeper into this.

9.3 Traditional default correlation approach

Default correlation is usually modeled with the help of copulae in the following way:

- Risk neutral survival probability curves are obtained from market instruments (CDS) where available.
- Stochastic credit spreads could be considered.
- A set of uniform random variables (one for each credit reference) are simulated with a given copula (Gaussian has been a popular choice, although there are other alternatives).
- The uniform variables are projected on the inverse of the (possibly stochastic) survival probability curves.

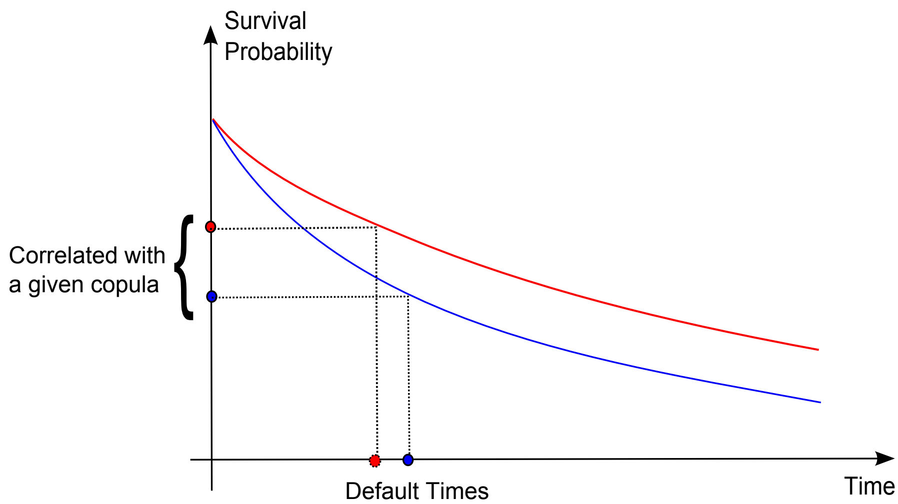


Figure 9.1: Default correlation under traditional copula approach.

9.3.1 Drawbacks of the traditional default correlation approach

The traditional approach exhibits some important drawbacks (independently of the copula being used (with one exception!!!)) that forces us to search for other alternatives.

In order to examine these drawbacks, we will work in a simplified framework with the following assumptions:

- We will analyze the CVA of a single CDS.
- Therefore, only 2 credit references will be considered (the CDS underlying reference and the counterparty).
- We will assume no CSA.
- The CDS is assumed to pay a continuous premium.
- Non stochastic spreads.
- Flat interest rate and credit curves.

CVA might be non monotonic wrt correlation

We compute the CVA of a single CDS. To start with, we will use the Gaussian Copula to simulate defaults. CVA will be computed for different correlation values.

Let's assume the following:

Underlying spread	2.00%
Counterparty spread	2.00%
Recoveries	40.00%
Int. rates	1.00%
Notional protection sold	-1
CDS Premium	2.00%
CDS maturity (yrs)	5
Integration points / year	50
Copula model	Gaussian

Notice that the two spreads are the same.

Notice that CVA is increasing with respect to correlation (in line with intuition), since the counterparty credit risk of a protection payer CDS is closely related with the joint default probability of the underlying credit reference and the counterparty.

But what happens if the credit spreads are different? Let's assume that the spread of the counterparty is smaller than the spread of the CDS underlying reference (seems to make sense):

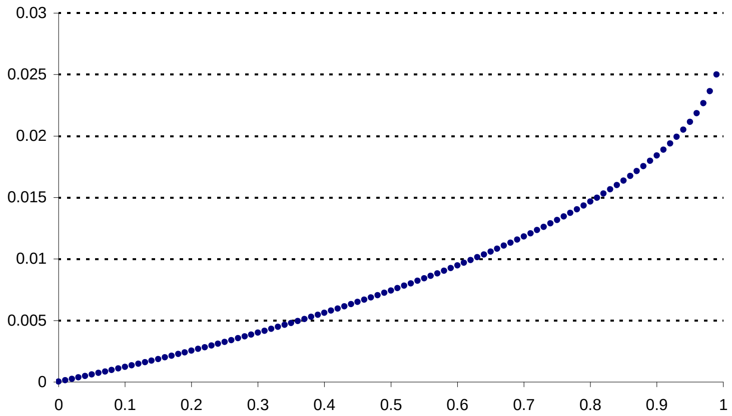


Figure 9.2: CVA of a single CDS (protection bought) as a function of correlation (Gaussian Copula). Underlying reference and counterparty spreads equal to 2%.

Underlying spread	2.00%
Counterparty spread	0.50%
Recoveries	40.00%
Int. rates	1.00%
Notional protection sold	-1
CDS Premium	2.00%
CDS maturity (yrs)	5
Integration points / year	50
Copula model	Gaussian

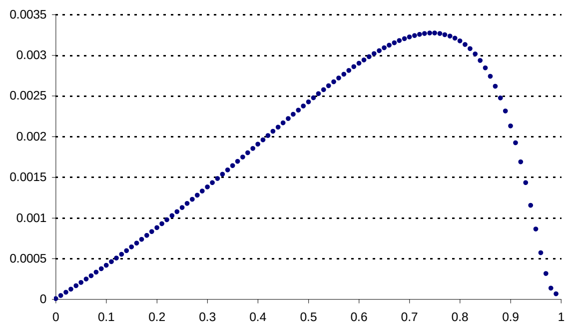


Figure 9.3: CVA of a single CDS (protection bought) as a function of correlation (Gaussian Copula). Underlying reference spread: 2%. Counterparty spreads: 0.5%.

Notice that the CVA is non monotonic with respect to correlation.

What is the reason behind this behavior?

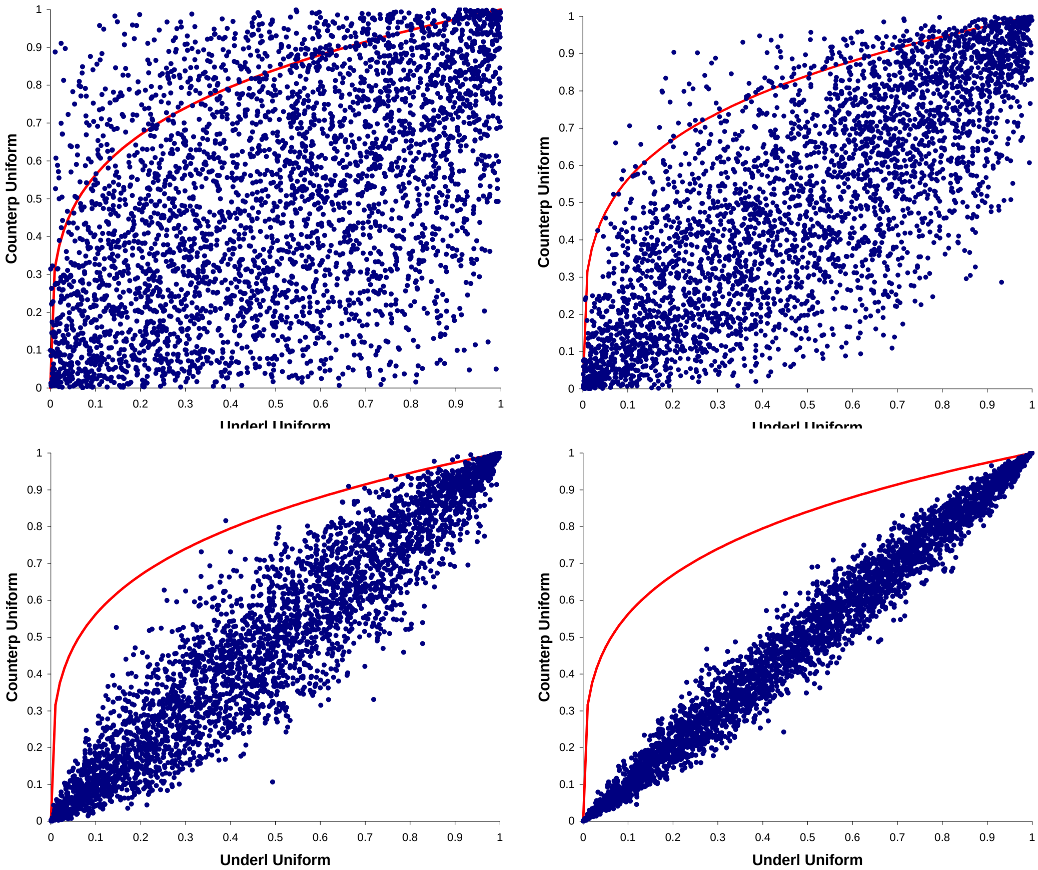


Table 9.1: Gaussian copula for different levels of correlation and the joint default line (red). From top to bottom and left to right: $\rho = 0.5$, $\rho = 0.8$, $\rho = 0.95$, $\rho = 0.99$

The joint default line in table of figures 9.1 is obtained by imposing $\tau_C = \tau_U$, where τ_C and τ_U represent the default times of the counterparty and underlying respectively, which yields:

$$U_C = \exp\left(-\lambda_C \tau_C\right), \ U_U = \exp\left(-\lambda_U \tau_U\right) \ \Rightarrow \ U_C = U_R^{\frac{\lambda_C}{\lambda_R}}$$

Where λ_X stands for the default intensity of $X \in \{C, U\}$.
Notice that due to the fact that the default intensities are different, increasing correlation decreases the probability of joint defaults.
This feature arises several issues:

- We lose intuition with respect to the meaning (and impact) of the correlation parameter.
- More difficult to assess whether the price is aggressive or conservative.
- Does the model cover the whole range of arbitrage free prices?
- Is the maximum CVA copula dependent?

Maximum CVA is copula dependent

Is the feature described in the previous section a drawback of the Gaussian Copula? What happens with other copulae?
Let's repeat the exercise with different copulae:

Underlying spread	2.00%
Counterparty spread	0.50%
Recoveries	40.00%
Int. rates	1.00%
Notional protection sold	-1
CDS Premium	2.00%
CDS maturity (yrs)	5
Integration points / year	50
Copula model	Gaussian, Clayton, Frank

In table of figures 9.2 we see that the maximum CVA value depends on the copula choice (model risk).
Once a copula model has been chosen, we are not sure that we can reach the maximum arbitrage free CVA.

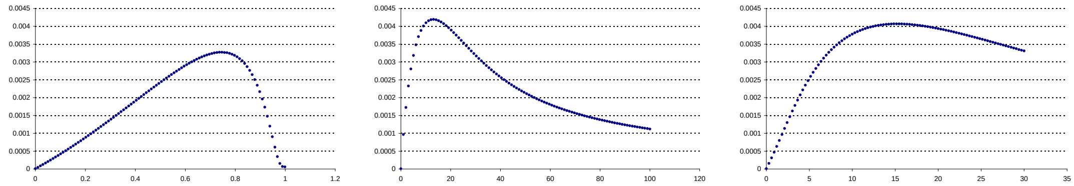


Table 9.2: Gaussian copula for different levels of correlation and the joint default line (red). From left to right: Gaussian copula,Clayton copula, Frank copula

Non realistic joint default probabilities and contagion

In this section we force the 3 copulae (Gaussian, Clayton and Frank) to infer the same CVA quote (0.24569%).

Underlying spread	2.00%
Counterparty spread	0.50%
Recoveries	40.00%
Int. rates	1.00%
Notional protection sold	-1
CDS Premium	2.00%
CDS maturity (yrs)	5
Integration points / year	50
Copula model & correl params	Gaussian (0.504545), Clayton (3.26795), Frank (4.78549)

We consider a time grid of 50 time steps per year. In the following picture we plot the joint default probabilities under the 3 models.

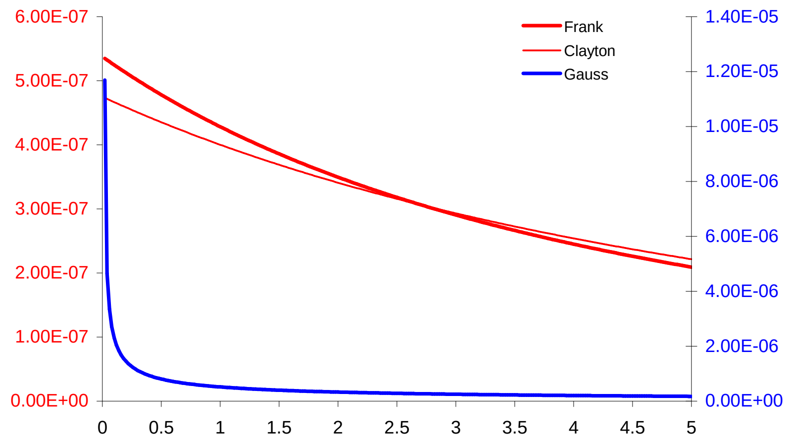


Figure 9.4: Joint default probabilities

Notice that the joint default probability is greater in the near future. This effect is specially abrupt for the Gaussian copula.

This modeling framework implies a spread widening of the surviving credit references upon the default of another (default contagion), that contributes to CVA. How can this spread widening be calculated?

The CDS NPV at a future time step t_{j+1} conditional on the counterparty having defaulted in (t_j, t_{j+1}) is going to be a byproduct of the following survival probabilities:

$$P\left[\tau_U > T \middle| \tau_U > t_{j+1}, \tau_C \in (t_j, t_j+1)\right] = \frac{P\left[\tau_U > T, \tau_U > t_{j+1}, \tau_C \in (t_j, t_j+1)\right]}{P\left[\tau_U > t_{j+1}, \tau_C \in (t_j, t_j+1)\right]}$$
$$= \frac{C(U^C_{t_j}, U^U_T) - C(U^C_{t_{j+1}}, U^U_T)}{C(U^C_{t_j}, U^U_{t_{j+1}}) - C(U^C_{t_{j+1}}, U^U_{t_{j+1}})}$$

Where $C(,)$ represents the copula and $U^X_t = \exp\left(-\int_{s=0}^t \lambda^X_s ds\right) \quad X \in \{C, U\}$

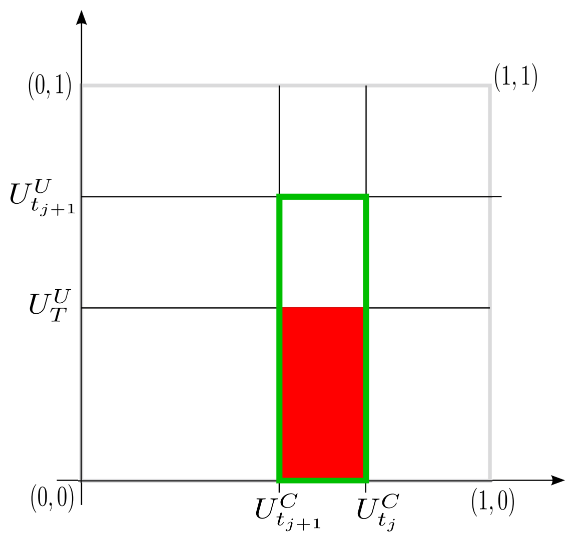


Figure 9.5: $P\left[\tau_U > T \middle| \tau_U > t_{j+1}, \tau_C \in (t_j, t_j+1)\right]$ is the integral of the red region divided by the integral surrounded in green.

The contagion effect is also greater in the near future. This effect is again specially abrupt for the Gaussian copula.

Inconsistency between forward and future spot scenarios

At a given time t, we price the CVA of a CDS with a given copula and a given correlation parameter. At a future time T, we use the same copula model and

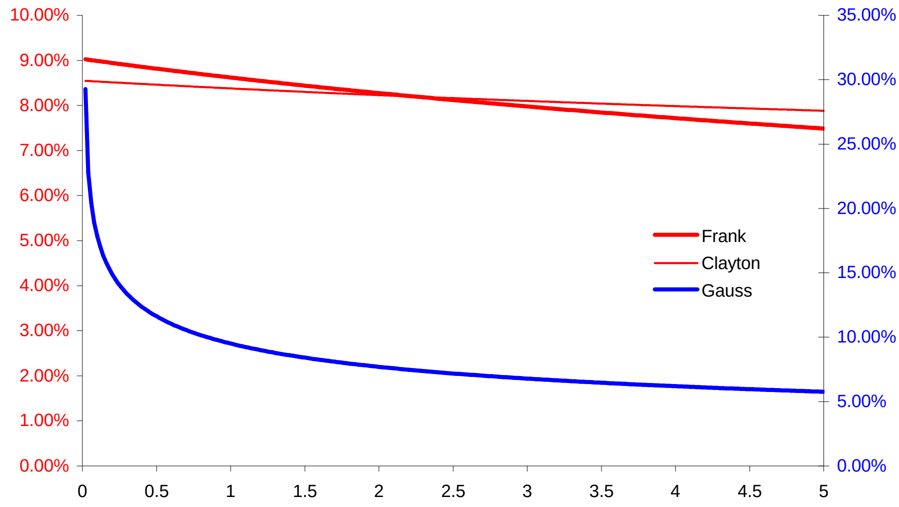
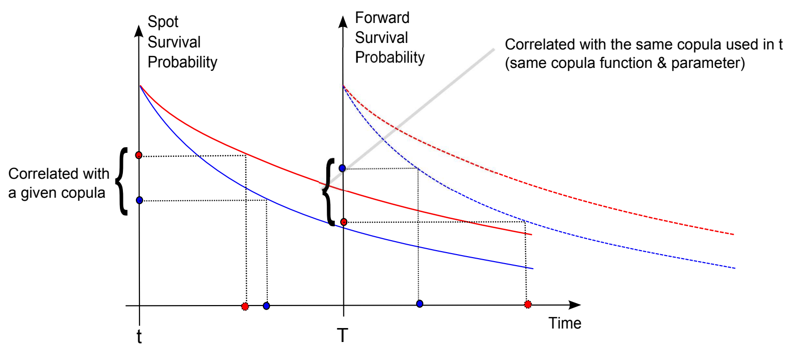


Figure 9.6: Underlying credit spread at t_j conditional on the counterparty defaulting in (t_{j-1}, t_j). Set of grid dates generated with 50 steps per year.

assume that the correlation parameter was kept constant. Are the T forward scenarios inferred by the model at t consistent with the spot scenarios implied by the model at T? Again, we assume non stochastic default intensities.



We compare the joint default probabilities and the spread widening of the underlying reference credit upon default of the counterparty for two different scenarios:

- The scenarios implied the copula at time $t = 0$ conditional on both references having survived at time $T = 3$.
- The scenarios implied by the copula at time $T = 3$.
- Under both scenarios, we use the same copula with the same correlation parameter.

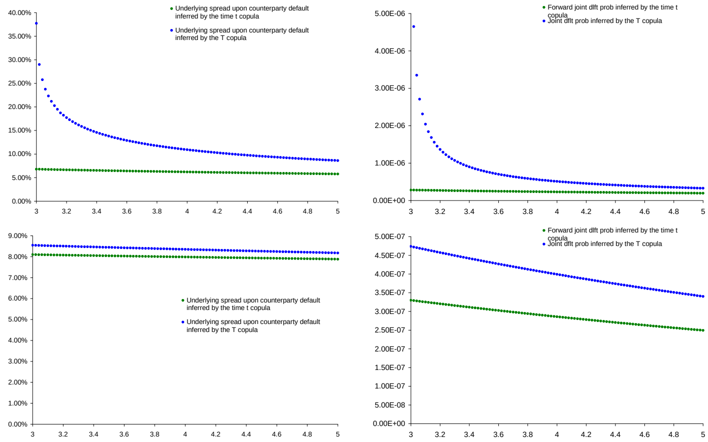


Table 9.3: Forward and future spot scenarios comparison in terms of joint default probabilities and spread of the underlying upon default of the counterparty.
Top left: Gaussian Copula: Spread contagion. Top right: Gaussian Copula: Joint default probability. Bottom left: Clayton Copula: Spread contagion. Bottom right:Clayton Copula: Joint default probability.

We see can the forward scenarios imply a smaller joint default probability and contagion than the future spot scenarios.

Although the copula model name and the model parameter is kept constant throughout time, the model is being changed on a day to day basis.

We can also see this effect in the evolution if the CVA NPV: We compute the initial NPV of a forward starting contingent CDS that starts at $T = 2.5$ with maturity $T = 5$ (same as the underlying CDS). We plot the evolution of the NPV in time under two assumptions:

- Being consistent with the initial copula
- By naively using the original copula with the same parameter at every future time (rolling copula).
- We perform the analysis for the Gaussian and Clayton copulae.

In table of figures 9.4 we see that time 0 copula under estimates the forward starting CVA.

All these effects are due to the fact that the rolling copula is not consistent with the scenarios implied by the initial copula.

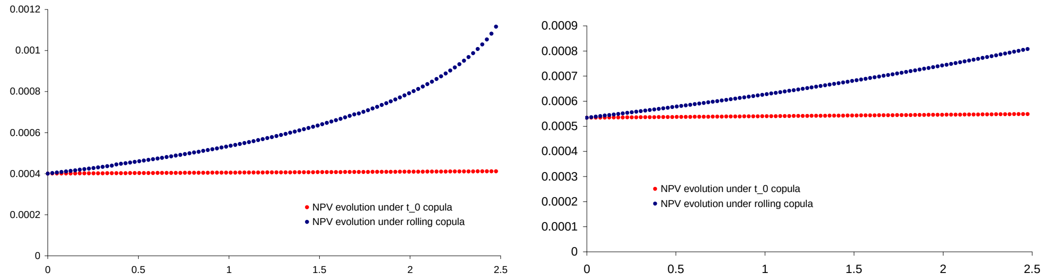


Table 9.4: Forward and future spot scenarios comparison in terms of joint default probabilities and spread of the underlying upon default of the counterparty. Left: Gaussian Copula: Spread contagion. Right: Gaussian Copula: Joint default probability.

9.4 The Marshall Olkin Copula

What is the problem of the traditional approach? basically that the following does not hold for $(S_1, S_2 > T > t)$:

$$\underbrace{P\left[\tau_C > S_1, \tau_U > S_2 \middle| \tau_C > t, \tau_U > t\right]}_{C_t\left(e^{-\int_{s=t}^{S1}\lambda_s^C ds}, e^{-\int_{s=t}^{S2}\lambda_s^U ds}\right)} = \underbrace{P\left[\tau_C > T, \tau_U > T \middle| \tau_C > t, \tau_U > t\right]}_{C_t\left(e^{-\int_{s=t}^{T}\lambda_s^C ds}, e^{-\int_{s=t}^{T}\lambda_s^U ds}\right)} \underbrace{P\left[\tau_C > S_1, \tau_U > S_2 \middle| \tau_C > T, \tau_U > T\right]}_{C_T\left(e^{-\int_{s=T}^{S_1}\lambda_s^C ds}, e^{-\int_{s=T}^{S_2}\lambda_s^U ds}\right)} \tag{9.22}$$

Where C_t and C_T are the copulae used at times t and T. Notice that under independence $C(u, v) = uv$, the condition holds.
Let's try to find the copula that meets (9.22) apart form the independence case.
Lets assume that $t < S_1 < S_2$ and compare the copulas at times t and $t + dt$. By imposing (9.22):

$$C_t\left(e^{-\int_{s=t}^{S1}\lambda_s^C ds}, e^{-\int_{s=t}^{S2}\lambda_s^U ds}\right) = C_t\left(e^{-\int_{s=t}^{t+dt}\lambda_s^C ds}, e^{-\int_{s=t}^{t+dt}\lambda_s^U ds}\right) C_{t+dt}\left(e^{-\int_{s=t}^{S_1}\lambda_s^C ds}, e^{-\int_{s=t}^{S_2}\lambda_s^U ds}\right)$$

The term $C_t\left(e^{-\int_{s=t}^{t+dt}\lambda_s^C ds}, e^{-\int_{s=t}^{t+dt}\lambda_s^U ds}\right)$ represents the probability of both credit references surviving at $t + dt$. We assume that there is a joint default intensity $\lambda_t^{UC} \leq \min(\lambda_t^U, \lambda_t^C)$. Notice that this assumption is the most general. Then

$$C_t\left(e^{-\int_{s=t}^{t+dt}\lambda_s^C ds}, e^{-\int_{s=t}^{t+dt}\lambda_s^U ds}\right) = 1 - \left(\lambda_t^C + \lambda_t^U - \lambda_t^{UC}\right) dt$$

So that

$$C_t = \left(1 - \left(\lambda_t^C + \lambda_t^U - \lambda_t^{UC}\right) dt\right) C_{t+dt} \Rightarrow \frac{dC_t}{C_t} = \left(\lambda_t^C + \lambda_t^U - \lambda_t^{UC}\right) dt$$

Notice that we know that $C_{S_1} = e^{-\int_{S_1}^{S_2} \lambda_s^U ds}$, therefore

$$\int_{C_t}^{C_{S_1}} \frac{dC_t}{C_t} = \int_{s=t}^{S_1} \left(\lambda_s^C + \lambda_s^U - \lambda_s^{UC}\right) ds$$
$$\Downarrow$$
$$C_t = e^{-\int_{S_1}^{S_2} \lambda_s^U ds} e^{-\int_{s=t}^{S_1} \left(\lambda_s^C + \lambda_s^U - \lambda_s^{UC}\right) ds}$$

Which can be written in the following way:

$$C_t\left(e^{-\int_{s=t}^{S1} \lambda_s^C ds}, e^{-\int_{s=t}^{S2} \lambda_s^U ds}\right) = e^{-\int_{s=t}^{S1} \left(\lambda_s^C - \lambda_s^{UC}\right) ds} e^{-\int_{s=t}^{S2} \left(\lambda_s^U - \lambda_s^{UC}\right) ds} e^{-\int_{s=t}^{\max(S1,S2)} \lambda_s^{UC} ds}$$

This is a particular case of the Marshall Olkin copula.
Notice that under this approach, in the time interval $(t, t + dt)$:

- C will default with probability $\lambda_t^C dt$.
- U will default with probability $\lambda_t^U dt$.
- Both U and C will default with probability $\lambda_t^{UC} dt$
- $\lambda_t^{UC} = K \min(\lambda_t^C, \lambda_t^U)$ where $0 \leq K \leq 1$.
- The probability that C defaults without U defaulting is $\lambda_t^C dt - \lambda_t^{UC} dt$.
- The probability that U defaults without C defaulting is $\lambda_t^U dt - \lambda_t^{UC} dt$.
- If $k = 0$, we have the independence case.
- There is no contagion effect (no spread widening of the surviving references upon default of)

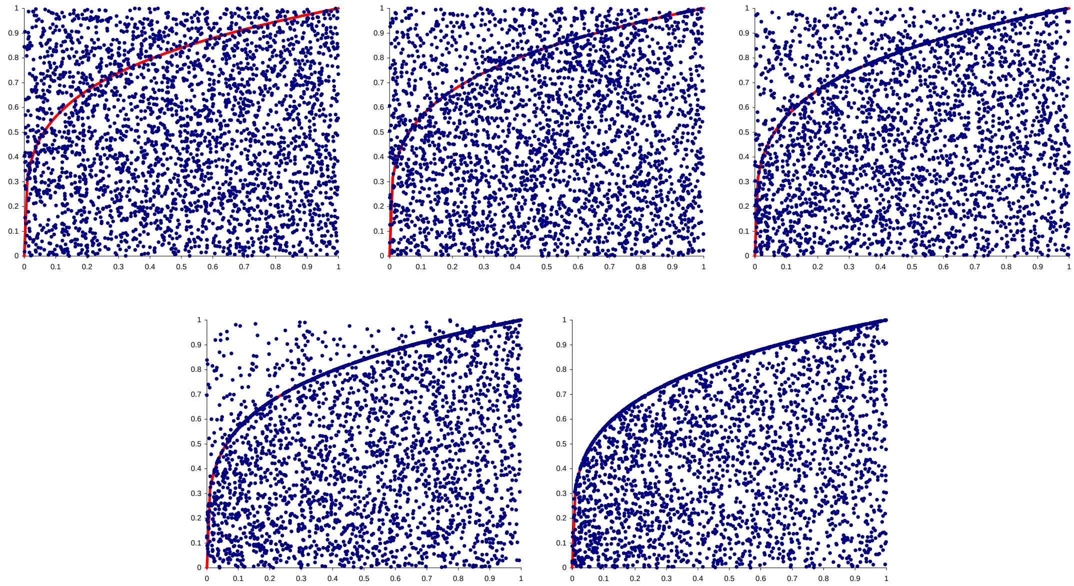


Table 9.5: Marshall Olkin copula with different correlations. From top to bottom and left to right: $k = 0$, $k = 0.25$, $k = 0.5$, $k = 0.75$, $k = 1$.

In table of figures 9.5, we see that, contrary to the traditional approach, as k increases, the probability of a joint default also increases.

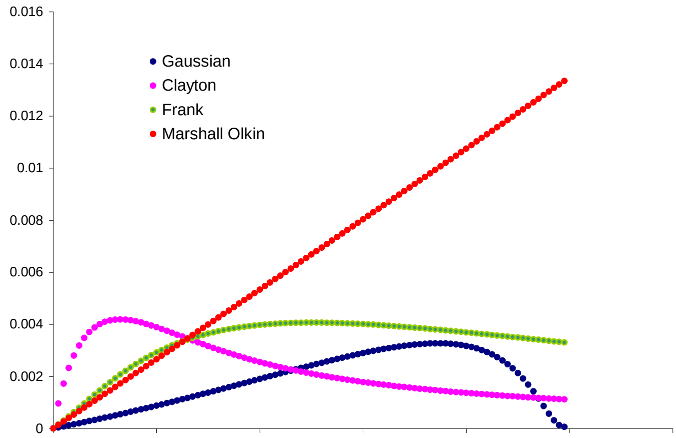


Figure 9.7: Different copulae as a function of their correlation parameters (Gaussian: 0-0.99, Clayton: 0-100, Frank: 0-30, Marshall-Olkin: 0-0.99).

In figure 9.7 we observe that:

- Under the Marshall-Olkin copula, CVA is increasing with respect to the correlation parameters.
- Under the Marshall-Olkin copula, the maximum CVA attained is much larger than for any of the other copulae.

9.4.1 Problem with the Marshall-Olkin copula in high dimensions

- When dealing with 2 credit references, we have to input the joint default intensity λ_t^{12}.
- With 3 credit references, we have to input λ_t^{12}, λ_t^{13}, λ_t^{23}, λ_t^{123}.
- With N credit references, we have to input $2^N - 1 - N$ joint default intensities.

We have to somehow reduce the number of joint default parameters to input.

9.4.2 The Lévy-frailty model

In this section we introduce the Lévy-frailty model. Under this model, in order to simulate default events, we do the following (assuming n credit references):

- Assume that the different credit references had flat default intensities $\lambda_1, \ldots, \lambda_n$.
- We consider a subordinator (increasing Lévy process) and simulate its path.
- We draw independent exponential random variables $\tau_1^*, \ldots, \tau_n^*$ with parameters $\lambda_1^*, \ldots, \lambda_n^*$.

- The default time of reference j will be given by the first time in which the subordinator is greater than τ_j^*.
- Obviously $\lambda_j^*,\ j = 1, \ldots n$ will be determined so that the marginals are preserved $(\tau_j \sim \exp(\lambda_j))$.

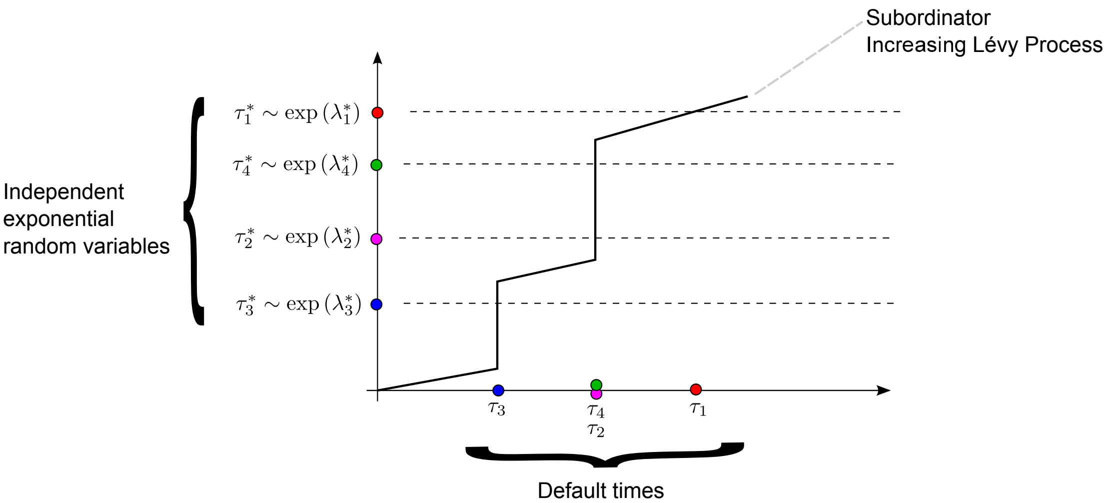


Figure 9.8: Simulation of default events under the Lévy-frailty model

How can we preserve the marginals? By imposing:

$$P[\tau_j > T] = e^{-\lambda_j T}$$
$$\Updownarrow$$
$$P[X_T < \tau_t^*] = e^{-\lambda_j T} \tag{9.23}$$

But

$$P[X_T < \tau_t^*] = P\left[X_T < -\frac{\log(U)}{\lambda_j^*}\right] = P\left[U < e^{-X_T \lambda_j^*}\right] = E\left[e^{-X_T \lambda_j^*}\right] = \Phi_{X_T}(\lambda_j^*) \tag{9.24}$$

Where we have taken into account that $\tau_j^* \sim \exp(\lambda_j^*)$.
$\Phi_{X_T}(s) = E\left[e^{-X_T s}\right]$ is the Laplace transform of the positive random variable X_T. Notice that $\Phi(0) = 1$ and that $\Phi_{X_T}(s)$ is decreasing in s.
(9.23) and (9.24) imply:

$$e^{-\lambda_j T} = \Phi_{X_T}(\lambda_j^*) \ \Rightarrow \lambda_j^* = \Phi_{X_T}^{-1}\left(e^{-\lambda_j T}\right)$$

Obviously λ_j^* will depend on the subordinator X_T

9.4.3 Nested Archimedean copulae

The model described in the previous section allows us to correlate defaults obtaining default times under flat spread curves.

What if the spread curves are non flat (or stochastic)?

Then, if we want to preserve the marginals on every time interval $(t, t+dt)$, we will have to impose that:

$$e^{-\lambda_t^j dt} = \Phi_{dX_t}(\lambda_t^{j*}) \ \Rightarrow \lambda_t^{j*} = \Phi_{dX_t}^{-1}\left(e^{-\lambda_t^j dt}\right)$$

Obtaining a different λ_t^{j*} for each t.

Under this situation (which is the most realistic), the only solution is to consider a fine time grid $t_1, \ldots, t_n$. And proceed in the following way:

- For a given time interval (t_k, t_{k+1}), obtain the default intensity of each non defaulted reference $\lambda_{t_k}^j$.
- Compute $\lambda_{t_k}^{j*} = \Phi_{(X_{t_{k+1}} - X_{t_k})}^{-1}\left(e^{-\lambda_{t_k}^j (t_{k+1}-t_k)}\right)$
- Draw the exponential τ_j^* for every non defaulted entity.
- Draw the Lévy subordinator increment.
- Test the default condition $\{\Delta X_{t_k} > \tau^*\}$.

Let's work out the default condition:

$$\{\Delta X_{t_k} > \tau_j^*\} = \left\{\Delta X_{t_k} > -\frac{\log(U_j)}{\lambda_{t_k}^{j*}}\right\} = \left\{\Phi_{\Delta X_{t_k}}^{-1}\left(e^{-\lambda_{t_k}^j \Delta t_k}\right) > -\frac{\log(U_j)}{\Delta X_{t_k}}\right\} = \left\{\Phi_{\Delta X_{t_k}}\left(-\frac{\log(U_j)}{\Delta X_{t_k}}\right) > e^{-\lambda_{t_k}^j \Delta t_k}\right\}$$

If we define the random variable $V_j := \Phi_{\Delta X_{t_k}}\left(-\frac{\log(U_j)}{\Delta X_{t_k}}\right)$, let's see how it is distributed:

$$\begin{aligned} P[V_j < v_j] &= P\left[\Phi_{\Delta X_{t_k}}\left(-\frac{\log(U_j)}{\Delta X_{t_k}}\right) < v_j\right] = P\left[U_j < e^{-\Delta X_{t_k}\Phi_{\Delta X_{t_k}}^{-1}(v_j)}\right] \\ &= E\left[e^{-\Delta X_{t_k}\Phi_{\Delta X_{t_k}}^{-1}(v_j)}\right] = \Phi_{\Delta X_{t_k}}\left(\Phi_{\Delta X_{t_k}}^{-1}(v_j)\right) = v_j \end{aligned}$$

Obviously, it is uniformly distributed.

The variables $V_1, \ldots, V_n$ will be a set of uniform random variables correlated through an Archimedean copula with $\phi_{\Delta X_{t_k}}^{-1}(\ )$ being its generator.

We are applying nested Archimedean copulae.

Under this model:

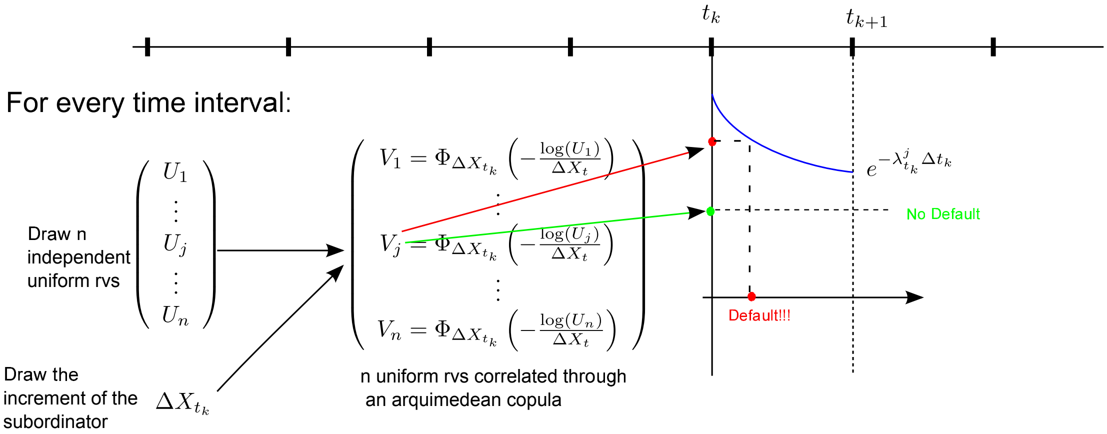


Figure 9.9: Simulation of default events under nested Archimedean copulas

- The default of one credit reference in (t_k, t_{k+1}) does not condition the surviving firms after t_{k+1} (the Lévy subordinator is Markovian). No contagion!!!.
- The probability of joint defaults is $O(dt)$ even if the credit references have different default intensities. This was not the case with traditional copulae.
- Default correlation depends on the Lévy subordinator being chosen and is reflected through joint defaults.
- It is nothing but a n dimensional Marshall-Olkin copula with a reduced number of parameters.
- Once a Lévy subordinator has been chosen, we just have to plug its parameters.

9.4.4 Correlation smile implied by the model

In order to gain some intuition with respect to effect of the Lévy subordinator, we compute the correlation smile implied by the model.

Notice that a Lévy subordinator is the sum of a drift component plus a set of Poisson processes.

We start by the most simple Lévy subordinator: drift plus single Poisson process:

Process	Parameter (drift/jump size)	Parameter (intensity)
Drift	1	
Poisson	100	0.001

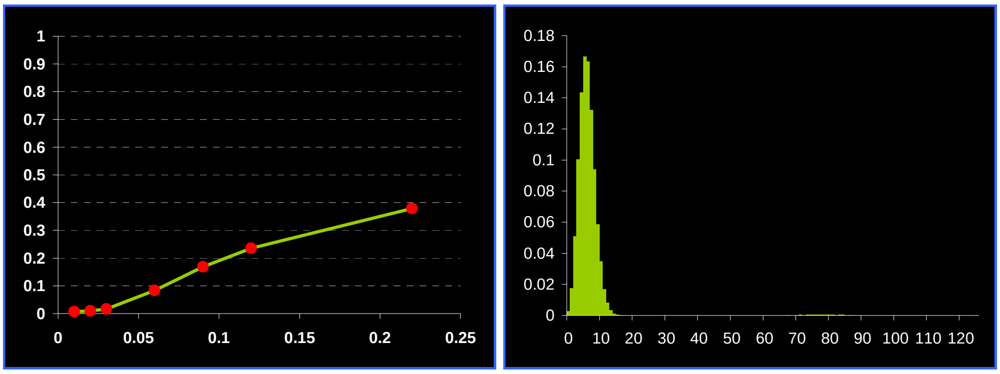


Figure 9.10: Correlation smile and loss density

We consider an homogeneous portfolio with 125 references, a spread of 1.00% and a 5 years maturity.
If we increase the jump size:

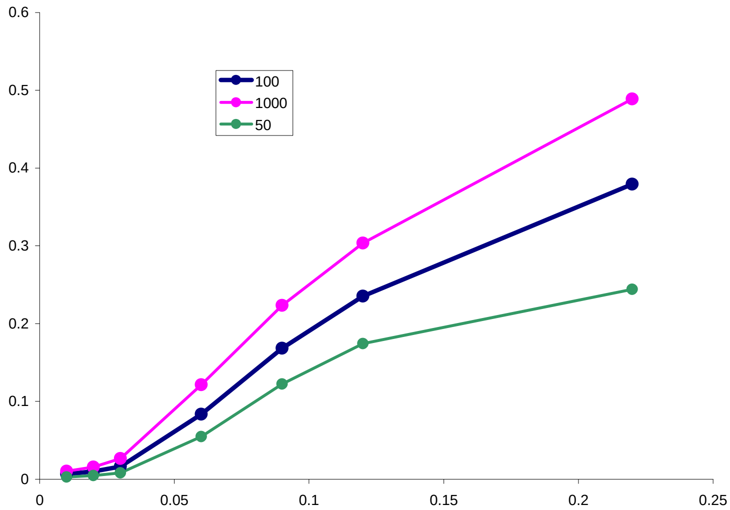


Figure 9.11: Correlation smile with Poisson intensity of 0.001 and changing the jump size.

Now we increase the Poisson intensity:

Process	Parameter (drift/jump size)	Parameter (intensity)
Drift	1	
Poisson	100	0.1

and changing the jump size:
Now with an intensity between the last two cases:

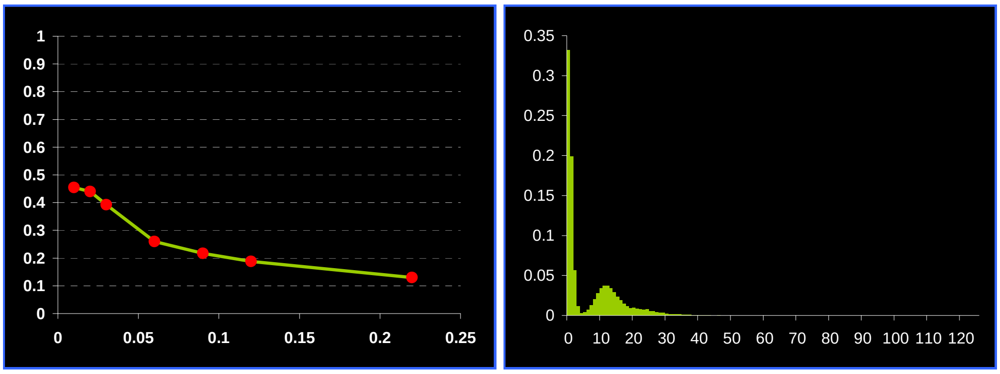


Figure 9.12: Correlation smile and loss density

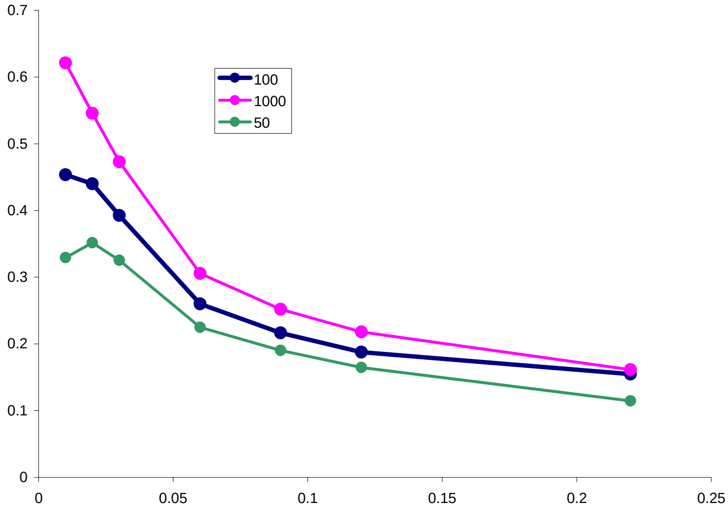


Figure 9.13: Correlation smile with Poisson intensity of 0.1 and changing the jump size.

Process	Parameter (drift/jump size)	Parameter (intensity)
Drift	1	
Poisson	100	0.01

And changing the jump size:
We can conclude the following:

- With low intensities, we control the correlation for high strikes.
- With high intensities, we control the correlation for low strikes.
- With intermediate intensities, we control the correlation for intermediate strikes.

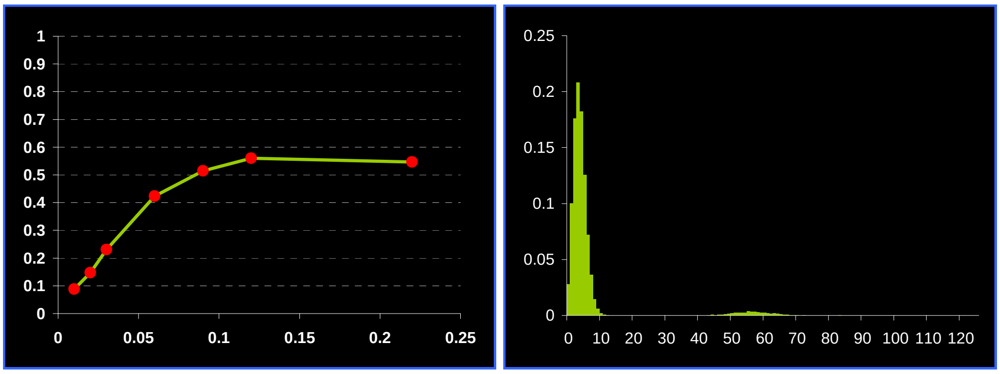


Figure 9.14: Correlation smile and loss density

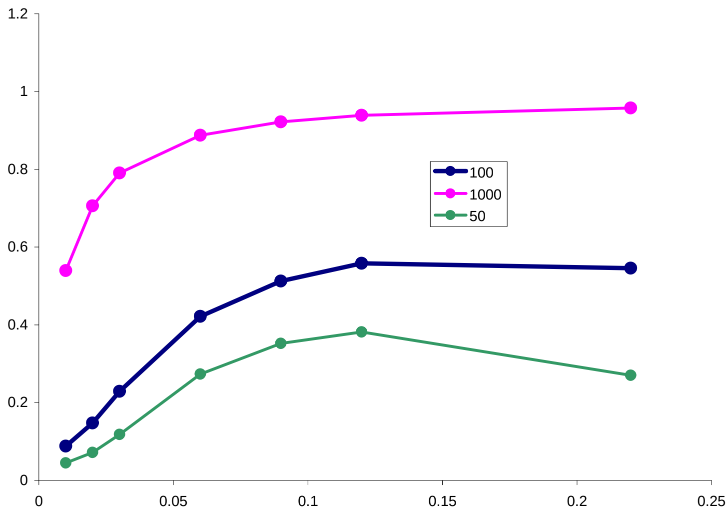


Figure 9.15: Correlation smile with Poisson intensity of 0.01 and changing the jump size.

- The larger the jump size, the greater the correlation.

With several processes, we can produce very rich smile shapes:

Process	Parameter (drift/jump size)	Parameter (intensity)
Drift	1	
Poisson	1000	0.001
Poisson	1000	0.01
Poisson	40	0.1

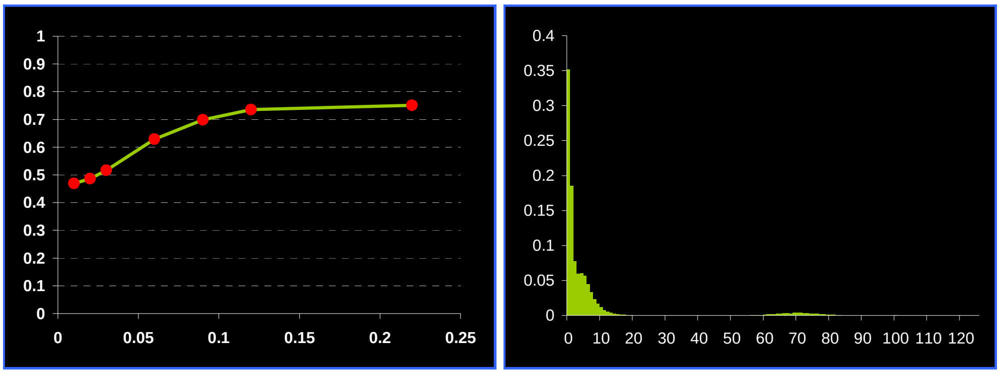


Figure 9.16: Correlation smile and loss density.

9.5 Conclusions

- Traditional copula approaches exhibit important drawbacks while calculating the CVA of credit derivatives.
- In low dimensions, the Marshall-Olkin copula is a much better approach.
- In high dimensions we have to reduce the number of parameters under the Marshall-Olkin model.
- Simulating defaults through Lévy subordinators while maintaining the marginals on every time interval is equivalent to using nested archimedean copulae.
- Default correlation is imposed by choosing a given subordinator.
- The model implies rich correlation smiles even for canonical Lévy processes.

Chapter 10

Wrong/Right Way Risk

We have already seen what the Wrong/Right way risk is and the different ways to take it into account. In this section we will get deeper into the matter and we will see how the diferent ways affect to the CVA as well as to its hedge.

We have seen that at time t the CVA for a derivative with maturity T, whose price at time t we denote by $V(t)$, can be expressed as,

$$\begin{aligned} CVA(t) &= (1-R)\, M(t) E_t^{\mathbf{Q}_M} \left(\frac{V^+(\tau)}{M(\tau)} \mathbf{1}_{\{\tau<T\}} \right) \\ &= (1-R)\, M(t) \int_t^T E_t^{\mathbf{Q}_M} \left(\frac{V^+(s)}{M(s)}\, dN(s) \right) \\ &= (1-R)\, M(t) \int_t^T E_t^{\mathbf{Q}_M} \left(\frac{V^+(s, \mathbf{X}_s)}{M(s, \mathbf{X}_s)} |\, dN(s) = 1 \right) E_t^{\mathbf{Q}_M} (dN_s) \end{aligned} \tag{10.1}$$

where N_t is a stopped count process with first jump at τ such that L_t

$$L_t = N_t - \int_0^{t\wedge\tau} \lambda(s) ds$$

is a martingale.

In this section we will see different ways to induce correlation between the NPV and the time to default of the counterparty so as to solve for,

$$E_t^{\mathbf{Q}_M} \left(\frac{V^+(s, \mathbf{X}_s)}{M(s, \mathbf{X}_s)} |\, dN(s) = 1 \right) \tag{10.2}$$

10.1 Copula Approach

Without loss of generality we will assume the conditional expected value we must solve for, is

$$E_t^{\mathbf{Q}_M}\left(\frac{V^+(s,X_s)}{M(s,X_s)}|\,\tau=s\right)=\int_\Omega \frac{V^+(s,X_s)}{M(s,X_s)}\eta_{X_s|Y}(x_s)dx_s \tag{10.3}$$

Le us assume we have got the distribution for $\frac{V^+(s,X_s)}{M(s,X_s)}$ by having simulated by Monte Carlo $N+1$ paths. Lets us denote the ordered distribution by

$$\frac{V^+(s,X_s)}{M(s,X_s)}(\omega_0)\leq\frac{V^+(s,X_s)}{M(s,X_s)}(\omega_1)\leq\ldots\leq\frac{V^+(s,X_s)}{M(s,X_s)}(\omega_N)$$

We might integrate (10.3) numerically as,

$$\sum_{j=1}^{N}\underbrace{\frac{V^+(s,X_s)}{M(s,X_s)}}_{\bar V^+(s,X_s)}(\omega_j)P_r\left(\underbrace{\bar V^+(s,X_s)(\omega_{j-1})}_{\bar V^+_{j-1}}<\underbrace{\bar V^+(s,X_s)}_{\bar V^+_s}\leq\underbrace{\bar V^+(s,X_s)(\omega_j)}_{\bar V^+_j}\right)$$

where we have defined the empirical distribution by,

$$P_r\left(\bar V^+_s\leq\bar V^+_j\right)=\frac{1}{N+1}\sum_{j=0}^{N}\mathbf{1}_{\{\bar V^+_s\leq\bar V^+_j\}}$$

We need to relate the event $(\tau=s)$ with the empirical distribution of exposures. We can make this by projecting both variables into a standard bivariate normal random variable.

Let us assume that

$$\begin{aligned}(\tau_s=s)\Leftrightarrow Y_s&=N^{-1}\left(Pr(\tau<s)\right)\qquad Y_s\overset{d}{\sim}N(0,1)\\(\bar V_s\leq\bar V_j)\Leftrightarrow X_s&\leq N^{-1}\left(\bar V_s\leq\bar V_j\right)\qquad X_s\overset{d}{\sim}N(0,1)\end{aligned}\tag{10.4}$$

So that, we assume X_s and Y_s have correlation ρ. Under this assumption we can express (10.3) as,

$$E_t^{\mathbf{Q}_M}\left(\frac{V^+(s,X_s)}{M(s,X_s)}|\,\tau=s\right)\approx\sum_{j=1}^{M}\bar V^+_j\left(N\left(\frac{x_j-y_s\rho}{\sqrt{1-\rho^2}}\right)-N\left(\frac{x_{j-1}-y_s\rho}{\sqrt{1-\rho^2}}\right)\right)$$

for

$$x_j=N^{-1}\left(\bar V_s\leq\bar V_j\right),\qquad y_s=N^{-1}\left(Pr(\tau<s)\right)$$

So the CVA would become

$$\boxed{CVA(t)=(1-R)\sum_{k,j=1}^{M,N}\bar V^+_j\left(N\left(\frac{x_j(t_k)-y_{t_k}\rho}{\sqrt{1-\rho^2}}\right)-N\left(\frac{x_{j-1}(t_k)-y_{t_k}\rho}{\sqrt{1-\rho^2}}\right)\right)Pr\left(t_{k-1}<\tau\leq t_k\right)}$$

10.2 A case Example: WWR for a FX Forward.

We will next focus on the CVA of a non-collateralized FX forward with maturity T that pays.

$$V_T = \left(X_T^{D/F} - K\right) N^F$$

The price at time t of such a contract is,

$$V_t = \bar{B}^F(t,T)X_t - K\bar{B}^D(t,T)$$

Where,

$$\bar{B}^F(t,T) = E_t^{\mathbf{Q}_F}\left(e^{-\int_t^T (c^F(u)+s_u^F)du}\right) \qquad \bar{B}^D(t,T) = E_t^{\mathbf{Q}_D}\left(e^{-\int_t^T (c^D(u)+b_u+s_u^F)du}\right)$$

where c_u is the instantaneous collateral rate, b_u denotes the instantaneous cross currency basis and s_u^F is the spread over collateral rate to fund currency B.
In the following, we define the dynamics for the FX,

$$\frac{dX_t^{D/F}}{X_t^{D/F}} = (c_D(t) - c_F(t) - b_t + m_X(t))\,dt + \sigma_X(t)C_X(t)\cdot dW^{\mathbb{P}}(t)$$

where $C_X(t)\cdot C_X(t) = 1$. $c_k(t)$ makes reference to the instantaneous collateral rate for $k = D, F$ domestic and foreign currencies respectively. m_X will denote the market price of risk for the FX under $\mathbb{P}$.

Conditioned to default we will be interested on the risk-free price of the derivative, so we will assume $s_t^F = b_t = 0$.
In order to calculate the CVA for such an instrument, we will have to solve for

$$\begin{aligned} &E_t^{\mathbf{Q}_M}\left(\frac{V^+(s,Z_s)}{M(s,Z_s)}|\,\tau = s\right) \\ &= B^D(t,T)E_t^{\mathbf{Q}_T}\left(V^+(s,Z_s)|\,Y_s = y^*\right) \\ &= B^D(t,T)\int_\Omega V^+(s,z_s)\eta_{Z_s|Y}(z_s)dz_s \end{aligned} \tag{10.5}$$

Where Z_s is an unidimensional standard Normal random variable. And we have mapped the default event $(\tau = s)$ to a standard normal random variable Y_s. So, given a survival probability curve $P_r(\tau > t$ for our counterparty

$$(\tau = s) \quad \Leftrightarrow \quad Y_s = y^* = N^{-1}\left(P_r(\tau \le s)\right)$$

We will correlate the exposure with the time to default of our counterparty by correlating the X_s, Y_s by a one-factor Gaussian copula with parameter ρ

$$Z_s = \rho\, Y_s + \sqrt{1-\rho^2}\,\epsilon_s$$

Then (10.5) reduces to,

$$\begin{aligned} & E_t^{Q_T}\left((X_{s,T}-K)^+ \,|\, Y_s = y^*\right) \\ &= \frac{1}{\sqrt{2\pi}} \int_{d_1}^{\infty} \left(X_{t,T} e^{-\frac{1}{2}\Sigma_{t,s}^2 + \Sigma_{t,s}\left(\rho y^* + \sqrt{1-\rho^2} u\right)} - K \right) e^{-\frac{1}{2}u^2} du \end{aligned} \tag{10.6}$$

Where [1]

$$\Sigma_{s,t}^2 = \int_t^s \sigma_X^2(s) ds$$

$$X_{t,T} = X_t \frac{B^F(t,T)}{B^D(t,T)}$$

$$d_1 = \frac{\log \frac{K}{X_{t,T}} + \frac{1}{2}\Sigma_{s,t}^2 - \Sigma_{s,t}\rho y^*}{\Sigma_{s,t}\sqrt{1-\rho^2}}$$

By a little of algebra it can be seen that the conditioned expected value above has analytical solution equal to,

$$BSC(t,s,T,K,\rho) = e^{\Sigma_{s,t}\rho\left(y^* - \frac{1}{2}\Sigma_{s,t}\rho\right)} X_{t,T} N(\hat{d}_1) - KN(-d_1) \tag{10.7}$$

for

$$\hat{d}_1 = \frac{\log \frac{X_{t,T}}{K} - \frac{1}{2}\Sigma_{s,t}^2 + \Sigma_{s,t}\rho y^* + \Sigma_{s,t}^2\sqrt{1-\rho^2}}{\Sigma_{s,t}\sqrt{1-\rho^2}}$$

So the CVA at time t can be expressed as

$$\boxed{CVA(t) = (1-R)\, B(t,T) \int_t^T BSC(t,s,T,K,\rho) E_t^{\mathbf{Q}_M}(dN_s)}$$

[1] We will assume while pricing that interest rates are deterministic.

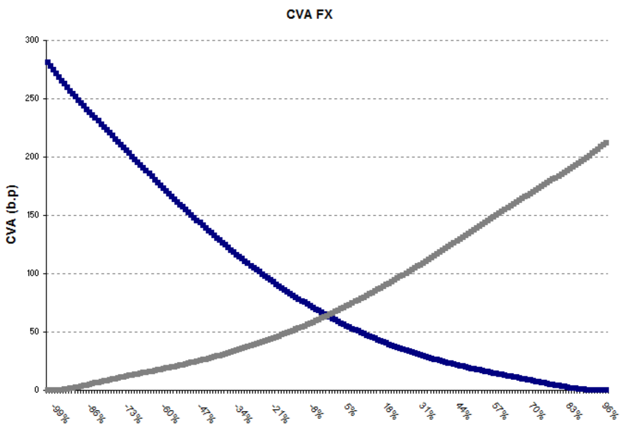


Figure 10.1: WWR: Copula approach for an ATM FX forward

10.3 Hedging with the Copula

In this section we will have a look to the $P\&L$ resulting from the hedging strategy of the CVA of the FX Forward above, when WWR is taken into account by the Copula approach.

We briefly comment on the portfolio needed to dynamically hedge CVA's first order risks, (i.e market delta and credit deltas, both on survival and on default).

For this, we first introduce an economy where interest rates, FX, default intensities and default times are stochastic with the following general dynamics under the real measure, $\mathbb{P}$.

$$c_k(t) = x_k(t) + \varphi_{x_k}(t); \tag{10.8}$$
$$dx_k(t) = (-\kappa_{x_k} x_k(t) + m_{x_k}(t))\, dt + \sigma_{x_k}(t) C_{x_k}(t) \cdot dW^{\mathbb{P}}(t) \quad \forall k = D, F \tag{10.9}$$

$$\lambda(t) = y(t) + \varphi_y(t) \tag{10.10}$$
$$dy(t) = (-\kappa_y y(t) + m_y(t))\, dt + \sigma_y(t) C_y(t) \cdot dW^{\mathbb{P}}(t) \tag{10.11}$$

and the dynamics for the FX are the same as in (10.5).
where $C_i(t) \cdot C_k(t) = \rho_{i,k}$. $c_k(t)$ makes reference to the instantaneous collateral rate for $k = D, F$ domestic and foreign currencies respectively and $\lambda(t)$ makes reference to the instantaneous default intensity for the counterparty. m_k will denote the market price of risk for risk factor k.

We will look for a hedging portfolio that mimics the CVA at every instant, so as to get

$$CVA(t) = H(t) \tag{10.12}$$

where the hedge may be expressed as the self-financing portfolio,

$$H(t) = \beta(t) + \alpha(t)X(t,T) + \gamma(t)CDS(t,T_U) + \epsilon(t)CDS(t,T_L) \tag{10.13}$$

where $X(t,T)$ is the FX forward used in the hedge of the market risk.
The self-financing requirement for (10.13) is given by,

$$dH(t) = d\beta(t) + \alpha(t)dX_{t,T} + \gamma(t)dCDS(t,T_U) + \epsilon(t)dCDS(t,T_L) \tag{10.14}$$

where,

$$\begin{aligned}
dX(t,T) &= \frac{\partial X(t,T)}{\partial t}dt + \mathcal{L}_{X_t}X(t,T) \\
dCDS(t,T_U) &= \frac{\partial CDS(t,T_U)}{\partial t}dt + \mathcal{L}_{\lambda_t}CDS(t,T_U) + \Delta CDS(t,T_U)dN(t) \\
dCDS(t,T_L) &= \frac{\partial CDS(t,T_L)}{\partial t}dt + \mathcal{L}_{\lambda_t}CDS(t,T_L) + \Delta CDS(t,T_L)dN(t) \\
d\beta(t) &= c^D(t)\left(CVA(t) - \alpha(t)X(t,T) - \gamma(t)CDS(t,T_U) - \epsilon(t)CDS(t,T_L)\right)dt
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_{X_t}(\cdot) &= \frac{\partial(\cdot)}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2(\cdot)}{\partial X_t^2}X_t^2\sigma_X^2(t)dt \\
\mathcal{L}_{\lambda_t}(\cdot) &= \frac{\partial(\cdot)}{\partial \lambda_t}d\lambda_t + \frac{1}{2}\frac{\partial^2(\cdot)}{\partial \lambda_t^2}\sigma_{\lambda_t}^2 dt
\end{aligned}$$

On the other hand, we know that,

$$dCVA_t = \frac{\partial CVA_t}{\partial t}dt + \mathcal{L}_{X_t}CVA_t + \mathcal{L}_{\lambda_t}CVA_t + \mathcal{L}_{X_t,\lambda_t}CVA_t + \Delta CVA_t dN(t) \tag{10.15}$$

For (10.12) to be true along time, we must impose,

$$dCVA(t) = dH(t) \tag{10.16}$$

what implies that in order to be dynamically hedged we should rebalance the hedging portfolio according to the following coefficients,

$$\begin{aligned}
\epsilon(t) &= \frac{\left(\frac{\partial CVA(t)}{\partial \lambda_t} - \frac{\Delta CVA(t)}{\Delta CDS(t,T_U)} \frac{\partial CDS(t,T_U)}{\partial \lambda_t}\right)}{\left(\frac{\partial CDS(t,T_L)}{\partial \lambda_t} - \frac{\Delta CDS(t,T_L)}{\Delta CDS(t,T_U)} \frac{\partial CDS(t,T_U)}{\partial \lambda_t}\right)} \\
\gamma(t) &= \frac{\Delta CVA(t) - \epsilon_t \ \ \Delta CDS(t,T_L)}{\Delta CDS(t,T_U)} \\
\alpha(t) &= \frac{\left(\frac{\partial CVA(t)}{\partial X_t}\right)}{\left(\frac{\partial X(t,T)}{\partial X_t}\right)}
\end{aligned} \tag{10.17}$$

so we are hedged to first order movements (both market and credit sensitivities)
Under the parameterization stated above, we are are now in position to calculate the CVA's greeks

CVA jump to default:

$$\Delta CVA(t) = \big((1-R)\ V(t)^{+} - CVA(t^{-})\big) \tag{10.18}$$

FX forward delta:

$$\frac{\partial X(t,T)}{\partial FX_t} = B^F(t,T) \tag{10.19}$$

CDS interest rate delta:

$$\begin{aligned}
\frac{\partial CDS(t_i,T,x_{t_i},y_{t_i})}{\partial x_{t_i}} =& - K \sum_{j=1}^{m} \delta_j G(\kappa_x, t_i, t_j^{'}) E^{\mathbb{Q}} \left(e^{-\int_{t_i}^{t_j^{'}} (c^D(u)+\lambda(u))du} | x_{t_i}, y_{t_i} \right) \\
& - LGD \sum_{k=1}^{l} G(\kappa_x, t_i, t_k^{''}) E \left(e^{-\int_{t_i}^{t_k^{''}} c^D(u)du} \mathbf{1}_{\{t_{k-1}^{''} < \tau_C \le t_k^{''}\}} | x_{t_i}, y_{t_i} \right)
\end{aligned} \tag{10.20}$$

CDS credit delta:

$$\begin{aligned}
\frac{\partial CDS(t_i,T,x_{t_i},y_{t_i})}{\partial y_{t_i}} = & \quad -c \textstyle\sum_{j=1}^{m} \delta_j G(\kappa_y, t_i, t_j^{'}) E^{\mathbb{Q}} \left(e^{-\int_{t_i}^{t_j^{'}} (c^D(u)+\lambda(u))du} | x_{t_i}, y_{t_i} \right) \\
& -LGD \textstyle\sum_{k=1}^{l} G(\kappa_y, t_i, t_k^{''}) E \left(e^{-\int_{t_i}^{t_k^{''}} c^D(u)du} \mathbf{1}_{\{\tau_C \ge t_k^{''}\}} | x_{t_i}, y_{t_i} \right) \\
& -LGD \textstyle\sum_{k=1}^{l} G(\kappa_y, t_i, t_{k-1}^{''}) E \left(e^{-\int_{t_i}^{t_k^{''}} c^D(u)du} \mathbf{1}_{\{\tau_C \ge t_{k-1}^{''}\}} | x_{t_i}, y_{t_i} \right)
\end{aligned}$$

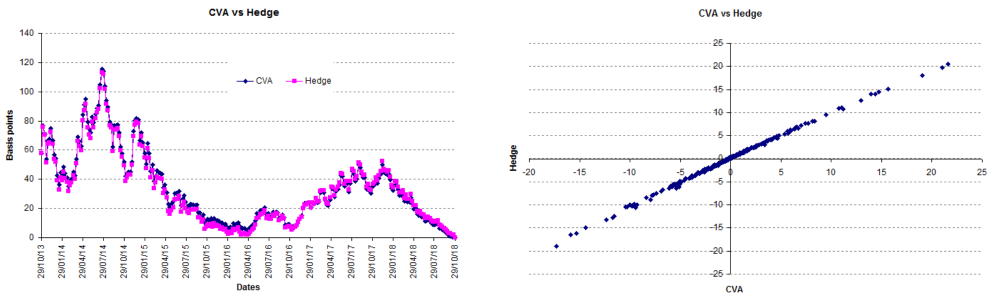


Figure 10.2: WWR: Hedging strategy for a 5y Mty ATM FX Forward with $\rho = 0$

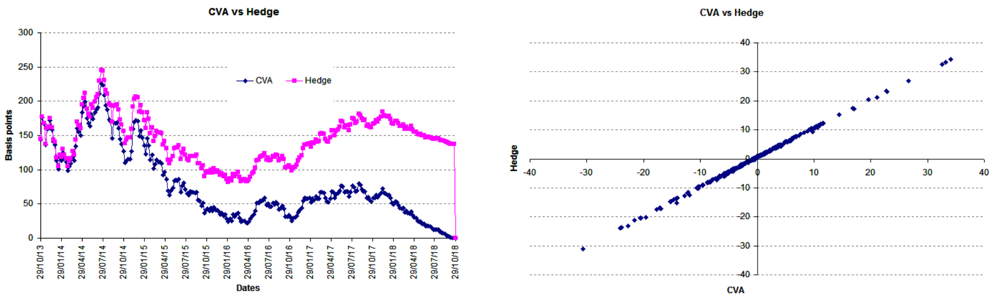


Figure 10.3: WWR: Hedging strategy for a 5y Mty ATM FX Forward with $\rho = -50\%$

CDS jump to default:

$$\Delta CDS(t_i, T) = -LGD_C \left(1 - \sum_{k=1}^{l} E\left(e^{-\int_{t_i}^{t_k''} c^D(u)du} \mathbf{1}_{\{t_{k-1}'' < \tau_C \le t_k''\}} | r_{t_i}, \lambda_{t_i}\right)\right) - c \sum_{j=1}^{m} \delta_j E^{\mathbb{Q}} \left(e^{-\int_{t_i}^{t_j'} (c^D(u) + \lambda(u))du} | r_{t_i}, \lambda_{t_i}\right) \quad (10.21)$$

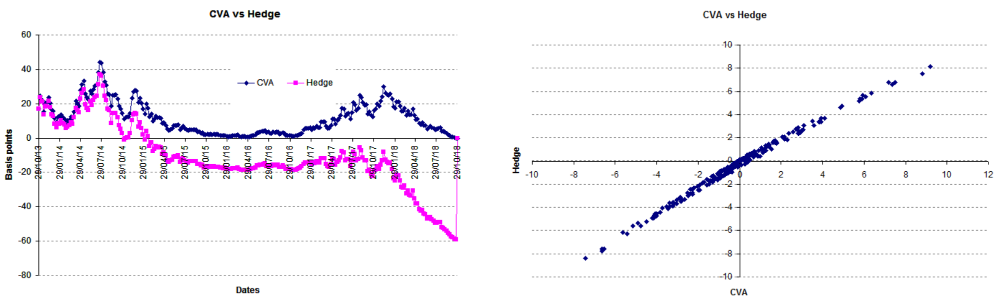


Figure 10.4: WWR: Hedging strategy for a 5y Mty ATM FX Forward with $\rho = +50\%$

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