Discrete-Space Social Interaction Models: Stability and Continuous Limit

Takashi Akamatsu and Shota Fujishima and Yuki Takayama

Tohoku University, University of Tokyo, Tohoku University

8 May 2014

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Takashi Akamatsu† Shota Fujishima‡ Yuki Takayama§

January 9, 2015

Abstract

We study the equilibrium properties, including stability, of discrete-space social interaction models with a single type of agents, and their continuous limit. We show that, even though the equilibrium in discrete space can be non-unique for all finite degree of discretization, any sequence of discrete-space models' equilibria converges to the continuous-space model's unique equilibrium as the discretization of space is refined. Showing the existence of multiple equilibria resorts to the stability analysis of equilibria. A general framework for studying equilibria and their stability is presented by characterizing the discrete-space social interaction model as a potential game.

JEL classification: C62; C72; C73; D62; R12

Keywords: Social interaction; Agglomeration; Discrete space; Potential game; Stability; Evolutionary game theory.

1 Introduction

Beckmann's (1976) social interaction model has been an important benchmark for the study of spatial agglomeration. Considering the fact that face-to-face com-

*We are grateful to Daisuke Oyama and Chin-Sheng Chen for helpful discussions. We also would like to thank audiences at the Third Asian Seminar in Regional Science and the 27th Annual Meetings of the Applied Regional Science Conference for useful comments. Takashi Akamatsu acknowledges financial support from the Japan Society for the Promotion of Science (Grant-in-Aid for Scientific Research (B) 21360240 and 24360202). Any remaining errors are our own.

†Graduate School of Information Sciences, Tohoku University, 6-6-6 Aoba, Sendai, Miyagi 980-8579, Japan. Email: akamatsu@plan.civil.tohoku.ac.jp

‡Center for Spatial Information Science, University of Tokyo, 5-1-5 Kashiwa-no-ha, Kashiwa, Chiba 277-8568, Japan. Phone/Fax: +81 4 7136 4298, Email: sfujishima@csis.u-tokyo.ac.jp

§Graduate School of Engineering, Tohoku University, 6-6-6 Aoba, Sendai, Miyagi 980-8579, Japan. Email: takayama@civil.tohoku.ac.jp

munications are important to understand the mechanisms behind spatial distributions of economic activities, Beckmann presents a model in which people aiming to interact with others choose their locations. People can save the costs of interactions by locating close to one other, but agglomeration causes congestion such as increases in housing prices. Equilibrium population distributions, which are of interest to this paper, emerge as a result of the trade-off between the positive and negative effects of agglomeration. This type of model has been of particular interest for urban economists because the location of an urban center is not specified a priori unlike classical urban models such as the monocentric city model.[1]

Beckmann (1976) considers social interactions among households for a linear city that is represented by a real line. After Beckmann's work, Tabuchi (1986) and Mossay and Picard (2011) also consider social interactions among a single type of agents on the real line.[2] All of these studies attain symmetric unimodal population distributions as unique equilibria. The uniqueness result is compelling, and the shape of the equilibrium distribution is intuitively reasonable. Moreover, this is also a good news for policy makers because they do not have to worry about multiple equilibria when internalizing externalities.

Having said that, although the results attained in continuous-space models serve as an important theoretical benchmark, it is also important to study whether those results are robust in terms of the discretization of space. In particular, if we would like to empirically test the model, we would have to discretize it. Empirical works cannot invoke the uniqueness result of the continuous-space model, unless we can view the continuous-space model as the limit of discrete-space models in regard to the size of geographical zones. This paper provides a positive answer to this issue for a social interaction model having a single type of agents.

There are few papers on spatial social interactions using a discrete-space model. Anas and Xu (1999) present a multi-regional general equilibrium model in which every region employs labor and produces goods. Although the technology exhibits a constant return to scale, the goods are differentiated over regions and consumers

[1]See, for example, Section 3.3 of Fujita and Thisse (2013).

[2]Mossay and Picard (2011) consider consumers, whereas Tabuchi (1986) considers firms. Besides models on the real line, O'Hara (1976) considers the social interactions of firms in a square city, and Borukhov and Hochman (1977) consider the social interactions of consumers in a circular city. They also obtain a symmetric unimodal distribution as a unique equilibrium. In Borukhov and Hochman (1977), though, the cost of social interaction is not weighted by population density, so social interactions do not cause any externality.

travel to each region to purchase them, which yields an agglomeration force in the central region.[3] Although their model is useful for the evaluation of urban policies, they rely entirely on numerical simulations, forcing us to consider particular equilibrium that might be unstable in case of multiple equilibria. Turner (2005) and Caruso *et al.* (2009) consider one-dimensional discrete-space location models with neighborhood externalities in the sense that utility at a particular location depends on the population distribution of that neighborhood.[4] Caruso *et al.* (2009) rely on numerical simulations, while Turner (2005) generically attains a unique equilibrium outcome by considering an extreme type of neighborhood externalities wherein an individual located between vacant neighborhoods receives a bonus. However, because they focus on the effects of residential locations on open spaces, they abstract away from the endogenous determination of an urban center, although this remains an important feature of the model in which we are interested.[5] Moreover, we emphasize that none of the above works studies the relationship between continuous- and discrete-space models.

In this paper, we consider social interactions among consumers in the discrete space in which a finite number of cities are evenly distributed on a line segment, and we study the properties of equilibria accordingly. To this end, we begin with writing the model for a general quasi-linear utility function, invoking the fact that our model of location choice can be described as a *potential game* (Monderer and Shapley, 1996).[6] One important consequence of being a potential game is that the equilibrium can be characterized with a finite-dimensional optimization problem. Indeed, by assuming that the pair-wise interaction cost between cities is symmetric, we can identify a function, which is called a *potential function*, so that the set of equilibria coincides exactly with the set of Kurash-Kuhn-Tucker points for the maximization problem of the function. Moreover, even if multiple equilibria arise, we can conduct stability analysis with the potential function. In fact, we recognize the fact that every local maximizer of the potential function is a stable

[3]Braid (1988) considers a five-town model having a similar structure, although he abstracts away from general equilibrium effects. He shows that, depending on the degree of product differentiation, the equilibrium firm distribution can be bimodal.

[4]Caruso *et al.* (2007) considers a two-dimensional discrete space.

[5]Moreover, they make the so-called open-city assumption in which the equilibrium utility level is exogenous, whereas the total city population is endogenous.

[6]The potential function approach has been recognized as a promising analytical tool for regional science (Fujita and Thisse, 2013). See Oyama (2009a, b) and Fujishima (2013) for applications of the potential game approach to geography models.

equilibrium under a broad class of myopic evolutionary dynamics. Note that the stability of equilibria has not been addressed in continuous-space models.[7] The discretization of space reduces the dimension of stability analysis and enables us to scrutinize the properties of equilibria more closely.

After the general characterization of equilibria and their stability above, we focus on a discrete version of Mossay and Picard's (2011) model to have a closer look at equilibrium properties. Because the utility function is linear in city populations under their model, it is possible to obtain analytical results regarding equilibrium properties for an arbitrary number of cities.[8] As we mentioned above, we study the relationship of equilibrium properties between discrete and continuous spaces. In particular, we increase the number of cities while the total size of location space remains fixed, and we study the limiting properties of equilibria. We show that *any* sequence of the discrete-space model's equilibria converges to the equilibrium of the continuous-space model as the number of cities goes to infinity, or the distance between adjacent cities vanishes. This means that the set of equilibria is continuous in the number of cities at their limit because equilibrium in a continuous space is unique. Therefore, we may think that, as long as the number of geographical zones is sufficiently large, any equilibria of discrete-space model are close to the equilibrium of the limiting continuous-space model.

We claim that this result merits attention because the equilibrium in discrete space is generally not unique. We show that, as long as the interaction cost is not too small, the equilibrium is essentially non-unique in the sense that equilibria having different numbers of populated cities coexist. In particular, we can pin down a range of the interaction costs where multiple equilibria arise for *any* finite number of cities. Our result regarding the connection between discrete- and continuous-space models implies that, even if there were multiple equilibria, all of them would converge to a single equilibrium as discretization is refined.

This paper proceeds as follows. Section 2 introduces a general class of social interaction models, characterizing this class as a potential game. Section 3 examines the uniqueness and stability of equilibria. Section 4 studies the connections between discrete- and continuous-space models by increasing the number of cities. Section

[7]Naturally, continuous-space models are not always free from the problem of multiple equilibria, as we will discuss in the concluding remarks.

[8]Tabuchi (1982) considers the same class of discrete-space social interaction model, though he studies only the social planner's problem.

5 concludes the paper. Proofs omitted in the main text are provided in Appendix.

2 The Model

We start with a general class of discrete-space social interaction models that includes the discrete-space analogue of Beckmann's (1976) and Mossay and Picard's (2011) models as special cases. This description allows us to illustrate how the potential function approach generally works for the equilibrium characterization and stability analysis of discrete-space social interaction models.

2.1 Basic Assumptions

We consider a region in which K cities are evenly distributed on a line segment normalized as the unit interval $[0, 1]$. Cities are labeled by $i \in S \equiv \{1, 2, \cdots, K\}$ in order of distance from location 0, and city i's location is $x_i \equiv \frac{1}{K}\left(i - \frac{1}{2}\right) \in [0, 1]$. Each city has the same amount of land A/K so that the total amount of land in the region is fixed at A regardless of the number of cities. See Figure 1 for the structure of this region. As is common in the literature, the land is owned by absentee landlords. The opportunity cost of land is normalized to zero.

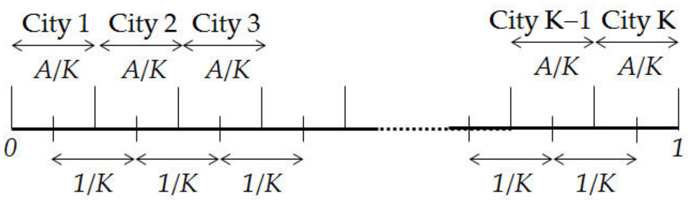


Figure 1: The regional structure

There are a unit mass of identical consumers in this region. Let $n_i \in [0, 1]$ be the mass of consumers in city i and let $\Delta \equiv \left\{\boldsymbol{n} = (n_1, \cdots, n_K) \in \mathbb{R}^K_+ : \sum_{i=1}^K n_i = 1\right\}$ denote the set of consumers' spatial distributions. Each consumer travels to every other consumer for social interaction. In each city, they have the same preference $u_i(z_i, y_i)$ for residential land y_i and for the composite good z_i which is chosen as the numéraire. Given land rent r_i and population distribution $\boldsymbol{n} \in \Delta$, the utility maximization problem of consumers in city i is expressed as

$$\max_{z_i, y_i} \left\{ u_i(z_i, y_i) \,|\, z_i + r_i y_i + T_i(\boldsymbol{n}) \leq Y,\ i \in S \right\}, \tag{1}$$

where r_i denotes the land rent in city i and Y is the fixed income. $T_i(\boldsymbol{n})$ is the total cost of traveling to other consumers from city i, which is defined as

$$T_i(\boldsymbol{n}) \equiv \tau \sum_{j=1}^{K} d_{ij} n_j, \tag{2}$$

where τd_{ij} denotes the travel cost from city i to j. We assume that $\boldsymbol{D} = (d_{ij})$ fulfills the following four conditions: (i) $d_{ii} = 0$ for all $i \in S$; (ii) $d_{ij} = d_{ji}$ for any $i, j \in S$; (iii) $\boldsymbol{D}$ is conditionally negative definite; and (iv) $d_{ij} + d_{jk} \leq d_{ik}$ for any $i < j < k$.[9] In the terminology of spatial statistics, the first three conditions imply that d_{ij} is an isotropic *variogram*. This class of travel costs includes the exponential cost ($d_{ij} = e^{|x_i - x_j|} - 1$) and the linear cost ($d_{ij} = |x_i - x_j|$), both of which are commonly assumed in the literature of spatial interaction.

The utility function $u_i(z_i, y_i)$ is assumed to be quasi-linear:

$$u_i(z_i, y_i) = z_i + f_i(y_i), \tag{3}$$

where $f_i(x)$ is a strictly increasing, concave, and twice differentiable function for $x > 0$. We also assume that $\lim_{x \to 0} f_i'(x) = \infty$. Note that f_i can be city-specific. If $f_i(x) = \alpha \ln x$ [resp. $f_i(x) = -\frac{\alpha}{2x}$] where $\alpha > 0$ is a constant, we obtain the discrete-space analogue of Beckmann's (1976) [resp. Mossay and Picard's (2011)] model.

2.2 Spatial Equilibrium and Potential Games

Having elaborated the structure of the model, we will now define the equilibrium. Because our model includes the location choice of consumers, the equilibrium conditions require that a consumer chooses a city that gives him the highest utility, in addition to choosing an optimal allocation in his city.

Definition 1. *An* equilibrium *is a collection of allocations* $(z_i^*, y_i^*)_{i=1}^K$, *land rents* $(r_i^*)_{i=1}^K$,

[9] An $n \times n$ matrix $\boldsymbol{M}$ is *conditionally negative definite* if $\boldsymbol{x}'\boldsymbol{M}\boldsymbol{x} < 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$ such that $\sum_{i=1}^n x_i = 0$. See, e.g., Bapat and Raghavan (1997) for properties of conditionally negative definite matrices.

and a population distribution $\boldsymbol{n}^* \in \Delta$ *such that*

1. *Given* r_i^* *and* $\boldsymbol{n}^* \in \Delta$, (z_i^*, y_i^*) *solves problem* (1) *for all* $i \in S$;
2. *For all* $i \in S$, *the land market clears whenever* $n_i > 0$;
3. *Given* $(r_i^*)_{i=1}^K$ *and* $\boldsymbol{n}^* \in \Delta$, *no one has incentive to change his location. That is, there exists* $u^* \in \mathbb{R}$ *such that*

$$\begin{cases} u^* = u_i(z_i^*, y_i^*) & \text{if} \quad n_i^* > 0, \\ u^* \geq u_i(z_i^*, y_i^*) & \text{if} \quad n_i^* = 0, \end{cases} \qquad \forall i \in S. \tag{4}$$

In particular, we call an equilibrium population distribution $\boldsymbol{n}^* \in \Delta$ a *spatial equilibrium*. Under the quasi-linear utility function specified in (3), the first-order condition for the utility maximization problem (1) is

$$f_i'(y_i) \leq r_i \qquad \forall i \in S, \tag{5}$$

where the equality holds whenever $y_i > 0$. However, because the marginal utility of residential land is infinity at $y_i = 0$ by assumption, we must have $y_i > 0$. Therefore, $f_i'(y_i) = r_i$ for all $i \in S$. For $y_i > 0$, let $g_i(f_i'(y_i))$ be the inverse function of $f_i'(y_i)$ (i.e., $g_i(f_i'(y_i)) = y_i$).[10] Then, $g_i(r_i)$ is the per-capita demand for the residential land in city i, and the indirect utility of consumers in city i is

$$\begin{aligned} v_i(r_i, Y - T_i(\boldsymbol{n})) &\equiv \max_{z_i, y_i} \left\{ u_i(z_i, y_i) \,|\, z_i + r_i y_i + T_i(\boldsymbol{n}) \leq Y,\ i \in S \right\} \\ &= Y - T_i(\boldsymbol{n}) - r_i g_i(r_i) + f_i(g_i(r_i)). \end{aligned} \tag{6}$$

The equilibrium land rent is determined so that the land market clears, as long as consumers are willing to pay more than the opportunity cost of land that is assumed to be zero. Let $\bar{r}_i$ be the land rent at which the total demand $n_i g_i(r_i)$ of the residential land in city i is equal to the total land supply A/K. Then,

$$r_i^* = \max\{\bar{r}_i, 0\} \qquad \forall i \in S. \tag{7}$$

If $\bar{r}_i < 0$, land is used for non-residential purpose, and we necessarily have $y_i^* = 0$. However, it follows from $r_i = f_i'(y_i) > 0$ that this does not occur. Therefore, the

[10]From the assumption that $f(x)$ is a strictly increasing function, the inverse function exists for $x > 0$.

equilibrium condition (7) reduces to

$$g_i(r_i^*) = \frac{A}{n_i^* K} \qquad \forall i \in S. \tag{8}$$

Let

$$h_i(n_i) = f_i\left(\frac{A}{n_i K}\right) - \frac{A}{n_i K} f_i'\left(\frac{A}{n_i K}\right). \tag{9}$$

Because $r_i = f_i'(\frac{A}{n_i K})$, this is the net utility from land consumption. The argument above then leads to the following lemma.

Lemma 1. $\boldsymbol{n}^* \in \Delta$ *is a spatial equilibrium if and only if there exists* $v^* \in \mathbb{R}$ *such that*

$$\begin{cases} v^* = v_i(\boldsymbol{n}^*) & \text{if} \quad n_i^* > 0, \\ v^* \geq v_i(\boldsymbol{n}^*) & \text{if} \quad n_i^* = 0, \end{cases} \qquad \forall i \in S, \tag{10}$$

where $v_i(\boldsymbol{n})$ *is the indirect utility function in city* i *defined by*

$$v_i(\boldsymbol{n}) \equiv v_i\left(f_i'\left(\frac{A}{n_i K}\right), Y - T_i(\boldsymbol{n})\right) = Y - T_i(\boldsymbol{n}) + h_i(n_i). \tag{11}$$

Writing the indirect utilities in a vector form, we have

$$\boldsymbol{v}(\boldsymbol{n}) \equiv (v_i(\boldsymbol{n}))_{i=1}^K = Y\mathbf{1} - \boldsymbol{T}(\boldsymbol{n}) + \boldsymbol{h}(\boldsymbol{n}) \tag{12}$$

where $\boldsymbol{T}(\boldsymbol{n}) = (T_i(\boldsymbol{n}))_{i=1}^K (= \boldsymbol{D}\boldsymbol{n})$, $\boldsymbol{h}(\boldsymbol{n}) = (h_i(n_i))_{i=1}^K$, and $\mathbf{1}$ is a vector of ones with an appropriate dimension. People prefer to agglomerate to reduce the social interaction costs that are summarized by $\boldsymbol{T}(\boldsymbol{n})$. On the other hand, people prefer to disperse and avoid the congestion from land consumption that is summarized by $\boldsymbol{h}(\boldsymbol{n})$ because $h_i'(n_i) = \frac{A^2}{n_i^3 K^2} f_i''(\frac{A}{n_i K}) < 0$. As we will see, a spatial equilibrium is attained as a result of tradeoffs between the agglomeration force represented by $\boldsymbol{T}(\boldsymbol{n})$ and the dispersion force represented by $\boldsymbol{h}(\boldsymbol{n})$.

In what follows, to characterize spatial equilibria and their stability, we invoke the properties of a *potential game* that is introduced by Monderer and Shapley (1996). Note that, because we are interested in the spatial equilibrium, our model may be viewed as a game in which the set of players is $[0, 1]$, the (common) action set is S, and the payoff vector is $(v_i)_{i=1}^K$ by Lemma 1.[11] Moreover, as is evident from the

[11] A game with a continuum of anonymous players is called a *population game* (Sandholm, 2001).

definition, a spatial equilibrium is actually a Nash equilibrium of the game. Thus, let us denote our game by $G = (v_i)_{i=1}^K$. We then define that G is a potential game if $(v_i)_{i=1}^K$ allows for a continuously differentiable function W such that

$$\frac{\partial W(\boldsymbol{n})}{\partial n_i} - \frac{\partial W(\boldsymbol{n})}{\partial n_j} = v_i(\boldsymbol{n}) - v_j(\boldsymbol{n}) \quad \forall \boldsymbol{n} \in \Delta, \forall i, j \in S \tag{13}$$

where W is defined on an open set containing Δ so that its partial derivative is well-defined on Δ. If the condition above holds, W is called a *potential function*.

Suppose, for the moment, that G is a potential game with the potential function W. As mentioned in the introduction, the equilibria of a potential game are characterized with the optimization problem of an associated potential function. Indeed, let us consider the following problem:

$$\max_{\boldsymbol{n} \in \Delta} \ W(\boldsymbol{n}). \tag{14}$$

Let μ be a Lagrange multiplier for the constraint $\sum_{i=1}^K n_i = 1$. Then, the first-order condition is $\frac{\partial W(\boldsymbol{n})}{\partial n_i} \leq \mu$ where the equality holds whenever $n_i > 0$. Then, by (13), we have $v_i(\boldsymbol{n}) = v_j(\boldsymbol{n})$ for any populated cities i and j, and $v_k(\boldsymbol{n}) \leq v_i(\boldsymbol{n})$ if $n_k = 0$ and $n_i > 0$. Therefore, $\boldsymbol{n}$ is a spatial equilibrium. By similar reasoning, it follows that the converse is also true.[12] That is, if $\boldsymbol{n}$ is a spatial equilibrium, it satisfies the necessary condition for problem (14). Therefore, *the equilibrium set of G exactly coincides with the set of Kurash-Kuhn-Tucker (KKT) points of problem* (14).

The necessary and sufficient condition for the existence of a potential function is the *triangular integrability* (see, e.g., Hofbauer and Sigmund, 1988), which, in our model, is stated as

$$d_{ij} + d_{jk} + d_{ki} = d_{ik} + d_{kj} + d_{ji} \quad \text{for any } i, j, k \in S. \tag{15}$$

Recall that our travel costs are pair-wise symmetric (i.e., $d_{ij} = d_{ji}$ for any $i, j \in S$). Hence, the condition above necessarily holds, and our game is a potential game. Indeed, the following lemma explicitly constructs a potential function for $(v_i)_{i=1}^K$.

In our game, players are anonymous in that the payoff depends on only strategy distributions.

[12] See Proposition 3.1 of Sandholm (2001).

Lemma 2. *G is a potential game with the potential function*

$$W(\boldsymbol{n}) \equiv \tau W_1(\boldsymbol{n}) + W_2(\boldsymbol{n}) \tag{16}$$

where

$$W_1(\boldsymbol{n}) = -\oint \boldsymbol{T}(\boldsymbol{n}')\mathrm{d}\boldsymbol{n}' = -\frac{1}{2}\sum_{i=1}^{K}\sum_{j=1}^{K} d_{ij}n_i n_j, \tag{17}$$

$$W_2(\boldsymbol{n}) = \oint \boldsymbol{h}(\boldsymbol{n}')\mathrm{d}\boldsymbol{n}' = \sum_{i=1}^{K} n_i f_i\left(\frac{A}{n_i K}\right). \tag{18}$$

$\oint$ denotes the line integral over a path in Δ connecting $\mathbf{0}$ to $\boldsymbol{n}$. Because $d_{ij} = d_{ji}$ for any $i, j \in S$, it is guaranteed that the line integrals are path-independent.

Observe that, in our potential game, we can recognize the tradeoff between centrifugal and centripedal forces as the tradeoff between the concavity and convexity of the potential function. Indeed, W_2 is strictly concave because f_i's are strictly concave, whereas W_1 is quasiconvex because $\boldsymbol{D}$ is nonnegative and conditionally negative definite.[13] If the concavity of W_2 dominates so that W is strictly concave, a dispersed population distribution (i.e., an interior point in Δ) is attained as a unique equilibrium. On the other hand, if the convexity of W_1 dominates, equilibrium population distributions would be more agglomerated. Therefore, W_1 represents the centripedal force whereas W_2 represents the centrifugal force.

2.3 Stability

2.3.1 Adjustment Dynamics

We are interested in the stability of equilibria particularly because our model generally includes multiple equilibria, as shown in the next section. Specifically, we are interested in whether we can justify an equilibrium through the existence of a learning process that makes players settle down in their equilibrium strategies. In this paper, we describe players' learning process with an *evolutionary dynamics*, or a (set-valued) dynamical system V that maps population distribution $\boldsymbol{n}^0 \in \Delta$ to

[13]See, for example, Theorem 4.4.6 of Bapat and Raghavan (1997).

a set of Lipschitz paths in Δ that starts from $\boldsymbol{n}^0$.[14] Although we usually consider a specific evolutionary dynamics for stability analysis, we will see that a more general analysis is possible due to the existence of a potential function. That is, the stability of equilibria can be characterized under a broad class of dynamics. In particular, we consider the class of *admissible* dynamics defined below:

Definition 2. *An evolutionary dynamics V is* admissible *for $G = (v_i)_{i=1}^K$ if for almost all $t \geq 0$ and for all $\boldsymbol{n}^0 \in \Delta$, it satisfies the following conditions:*

(PC) $\dot{\boldsymbol{n}}(t) \neq 0 \Rightarrow \dot{\boldsymbol{n}}(t) \cdot v(\boldsymbol{n}(t)) > 0$ *for all* $\boldsymbol{n}(\cdot) \in V(\boldsymbol{n}^0)$*,*
(NS) $\dot{\boldsymbol{n}}(t) = 0 \Rightarrow \boldsymbol{n}(t)$ *is a Nash equilibrium of G for all* $\boldsymbol{n}(\cdot) \in V(\boldsymbol{n}^0)$*.*

To interpret condition (PC), which is called *positive correlation*, we rewrite it as

$$\dot{\boldsymbol{n}}(t) \cdot v(\boldsymbol{n}(t)) = \sum_{i=1}^{K} \dot{n}_i(t) \left(v_i(\boldsymbol{n}(t)) - \frac{1}{K} \sum_{j=1}^{K} v_j(\boldsymbol{n}(t)) \right). \tag{19}$$

In general, it would be reasonable to expect that each term in the summation over i is positive: if the payoff from city i is higher than the average payoff (i.e., $v_i(\boldsymbol{n}(t)) - \frac{1}{K}\sum_{j=1}^K v_j(\boldsymbol{n}(t)) > 0$), then the mass of consumers choosing city i should increase (i.e., $\dot{n}_i(t) > 0$), and vice versa. Condition (PC) only requires that this be true in the aggregate. Therefore, in learning periods, it is possible that the mass of consumers choosing city i increases even though it yields a less-than-average payoff. Condition (NS), which is called *Nash stationary*, states that if there is a profitable deviation, some consumers change their locations. Under condition (PC), the converse is also true.[15] Therefore, under conditions (PC) and (NS), $\dot{\boldsymbol{n}}(t) = 0$ if and only if $\boldsymbol{n}(t)$ is a Nash equilibrium of G.

Specific examples of admissible dynamics include the *best response dynamics* (Gilboa and Matsui, 1991), the *Brown-von Neumann-Nash (BNN) dynamics* (Brown, 1950), and the *projection dynamics* (Dupuis, 1993).[16] One important remark is that the *replicator dynamics* (Taylor, 1978), which is often used in spatial economic models (e.g., Fujita *et al.*, 1999), is *not* admissible. Under the replicator dynamics, a rest point

[14]Considering a general dynamical system allows us to include set-valued dynamics such as the best-response dynamics which is important from the game-theoretic point of view.
[15]See Proposition 4.3 of Sandholm (2001).
[16]See Sandholm (2005) for more examples.

is always attained on the boundary, but the boundary points are not always Nash equilibria. Thus, condition (NS) does not hold under the replicator dynamics.[17]

2.3.2 Stability Condition of Equilibrium

The admissible dynamics are closely connected to the potential function, and thereby to the stability of Nash equilibria. Given a dynamics, we say that a population distribution $\boldsymbol{n} \in \Delta$ is *stable* if there exists a neighborhood $U \subseteq \Delta$ of $\boldsymbol{n}$ such that $\boldsymbol{n}(t) \to \boldsymbol{n}$ for any trajectory $\boldsymbol{n}(\cdot)$ of the dynamics with $\boldsymbol{n}(0) \in U$. In particular, if we can consider Δ for U, $\boldsymbol{n}$ is *globally stable*. $\boldsymbol{n} \in \Delta$ is *unstable* if it is not stable.

To understand how the admissible dynamics are related to the potential function, let us consider our game $G = (v_i)_{i=1}^K$ with the potential function W given by (18). Note that, by conditions (PC) and (NS), any trajectory $\boldsymbol{n}(\cdot)$ of an admissible dynamics monotonically ascends the potential function until it reaches a Nash equilibrium because

$$\dot{W}(\boldsymbol{n}(t)) = \sum_{i=1}^{K} \frac{\partial W(\boldsymbol{n}(t))}{\partial n_i} \dot{n}_i(t) = \sum_{i=1}^{K} v_i(\boldsymbol{n}(t)) \dot{n}_i(t) > 0 \tag{20}$$

whenever $\dot{\boldsymbol{n}}(t) \neq 0$.[18] Therefore, if Nash equilibrium $\boldsymbol{n}^*$ does not locally maximize W, we can perturb $\boldsymbol{n}^*$ so that the trajectory ascends W and goes away from the equilibrium. In other words, assuming that each Nash equilibrium is isolated, *a Nash equilibrium is stable under any admissible dynamics if and only if it locally maximizes an associated potential function.*[19] Therefore, if a game has a potential function, we can characterize the stability of equilibria under admissible dynamics by looking at the shape of the potential function.

2.4 Examples

We illustrate the potential function approach through examples. We consider two models: Beckmann's (1976) model in which $f_i(x) = \alpha \ln x$ and Mossay and Picard's (2011) model in which $f_i(x) = -\frac{\alpha}{2x}$. We assume $K = 3$, $\alpha = A = 4$, and

[17] The replicator dynamics belongs to the class of *strict myopic adjustment dynamics* due to Swinkels (1993) where Nash stationary is not imposed.

[18] Recall that $\dot{\boldsymbol{n}}(t) = 0$ if and only if $\boldsymbol{n}(t)$ is a Nash equilibrium.

[19] See Sandholm (2001) for a formal argument about this.

$d_{ij} = |x_i - x_j|$. Thus, the total number of cities is three. Under these parameter values, Figures 3 and 2 depict contour plots of each model's potential function, respectively. In these figures, the background color represents the value of potential function: regions where the value is largest are red, while regions where the value is smallest are blue. To characterize equilibria with these figures, we invoke the fact that a local maximizer of potential function is a stable equilibrium, whereas any other KKT points are unstable equilibria.

Looking at Figure 2, we can see that, when $\tau = 0.4$, the potential function is strictly concave, and thus there exists a unique interior equilibrium that is stable. However, when $\tau = 2.0$, the potential function fails to be concave, and five equilibria arise while the interior equilibrium vanishes. Stable equilibria are full agglomerations in which only one city is populated. Looking at Figure 3, we can see that equilibria of Beckmann's model exhibit qualitatively similar properties to those of Mossay and Picard's model.[20] In the next section, we analytically study equilibrium properties while mostly focusing on Mossay and Picard's model.

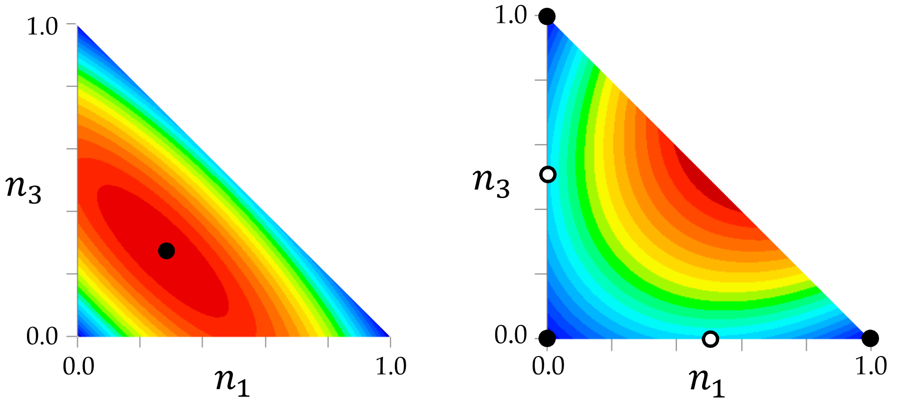
3 Equilibrium Analysis

3.1 Instability of Population Distributions

In view of the previous section, we investigate the relationship between interaction cost τ and the instability of spatial equilibria. We elaborate this point by obtaining a sufficient condition under which a population distribution could not be stable even if it were a spatial equilibrium.

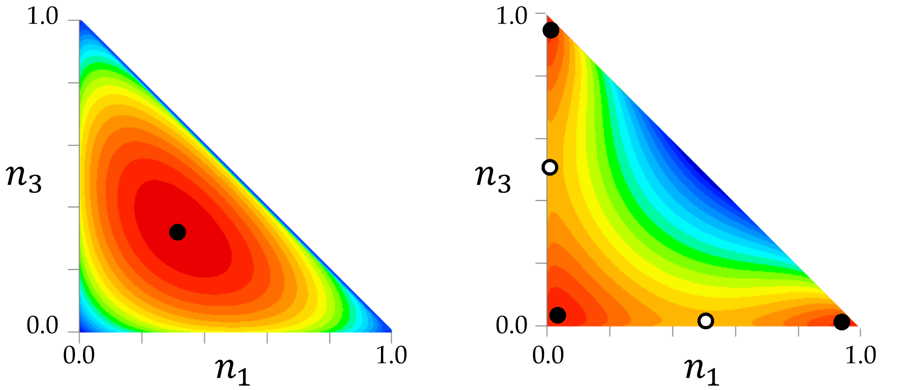
Let $\boldsymbol{n} \in \Delta$ be a spatial equilibrium such that $\operatorname{supp} \boldsymbol{n} = L \subseteq S$ where $\operatorname{supp} \boldsymbol{n}$ is the support of $\boldsymbol{n}$ (i.e., $\operatorname{supp} \boldsymbol{n} = \{i \in S : n_i > 0\}$). We denote the cardinality of L by $|L|$. Because a stable spatial equilibrium locally maximizes potential function W, we may investigate its Hessian $\boldsymbol{H}$, while we have to consider the fact that trajectories of admissible dynamics stay in Δ. To this end, let G_L be the matrix of the active constraints' gradients corresponding to L. For example, if $L = S \setminus \{1\}$, $G_L = \left(\begin{smallmatrix} 1 & 1 & \cdots & 1 \\ -1 & 0 & \cdots & 0 \end{smallmatrix} \right)'$, where the prime means the transpose of matrix, because the active constraints are $\sum_{i=1}^{K} n_i = 1$ and $-n_1 \leq 0$. Let Z_L be a G_L's null-space matrix. Then,

[20] However, two models' bifurcation patterns might not completely be identical. Although the case of three equilibria in which having two populated cities is stable exists for Mossay and Picard's model, such a case could not be found for Beckmann's model in our numerical exercises.



(a) $\tau = 0.4$ (b) $\tau = 2.0$

Figure 2: Contour plot of the potential function of Mossay-Picard's model (•: stable, ◦: unstable)



(a) $\tau = 1$ (b) $\tau = 10$

Figure 3: Contour plot of the potential function of Beckmann's model (•: stable, ◦: unstable)

the second-order necessary condition implies that $\boldsymbol{n}$ does not locally maximize W if $\boldsymbol{H}_L \equiv Z_L' \boldsymbol{H} Z_L$ is not negative semi-definite, and this boils down to showing that the largest eigenvalue of $\boldsymbol{H}_L$ is positive.[21]

Choosing reference city $k \in L$, let $\boldsymbol{D}_L$ be the submatrix of $\boldsymbol{D}$ representing travel costs within $L \setminus \{k\}$ and $\boldsymbol{d}_{kL} = (d_{ki})_{i \in L \setminus \{k\}}$. Then, we can take Z_L so that

$$\boldsymbol{H}_L = \tau \boldsymbol{H}_1 + \boldsymbol{H}_2 \tag{21}$$

[21] $\boldsymbol{H}_L$ is called the *reduced* Hessian. See, for example, Griva *et al*. (2009).

where

$$\boldsymbol{H}_1 = \boldsymbol{d}_{kL} \otimes \mathbf{1} + (\boldsymbol{d}_{kL} \otimes \mathbf{1})' - \boldsymbol{D}_L, \tag{22}$$

$$\boldsymbol{H}_2 = \operatorname{diag}[(h_i'(n_i))_{i\in L\setminus\{k\}}] + h_k'(n_k)\mathbf{1}'\mathbf{1}. \tag{23}$$

In the formula above, $\otimes$ denotes the Kronecker product, $\mathbf{1}$ is a vector of ones with an appropriate dimension, and $\operatorname{diag}(\boldsymbol{x})$ is the diagonal matrix having $\boldsymbol{x}$ as its diagonal elements. For analytical convenience, we choose the left end city in support of $\boldsymbol{n}$ as a reference city. Note that every matrix and vector is defined for support L which is generally a subset of S. However, to simplify notations, we sometimes suppress subscript L when no confusion arises.[22]

In the following analysis, we exploit the fact that a support of spatial equilibrium can be considered a downsized replica of the full support. Specifically, populated cities in a spatial equilibrium are congregated (i.e., there is no vacant city between any populated cities) as shown in the following lemma:

Lemma 3. *Suppose $\boldsymbol{n} \in \Delta$ is a spatial equilibrium. Then,* $\operatorname{supp} \boldsymbol{n} \in \mathcal{S}_C$ *where*

$$\mathcal{S}_C = \left\{\{i_1, ..., i_a\} \subseteq S : i_{j+1} = i_j + 1, 1 \le j \le a-1, a \in S\right\}. \tag{24}$$

Proof. All proofs are relegated to the Appendix. □

As a result, the properties of $\boldsymbol{D}$ carry over to $\boldsymbol{D}_L$. As we will see in further sections, this significantly simplifies the analysis and enables us to obtain analytical insights.

To attain a threshold value of τ above which the largest eigenvalue of $\boldsymbol{H}_L$ is positive, we invoke Weyl's inequality that says

$$\lambda_{\max}(\boldsymbol{H}_L) \equiv \lambda_{|L|-1}(\boldsymbol{H}_L) \ge \tau\lambda_{|L|-j}(\boldsymbol{H}_1) + \lambda_j(\boldsymbol{H}_2) \tag{25}$$

for $2 \le j \le |L|-1$ where $\lambda_i(M)$ is the i-th smallest eigenvalue of matrix M.[23] Although we made some adjustments to account for feasibility constraints, we can see that $\boldsymbol{H}_1$ corresponds to agglomeration force W_1 whereas $\boldsymbol{H}_2$ corresponds to dispersion force W_2. Indeed, because $\boldsymbol{D}_L$ is conditionally negative definite as $\boldsymbol{D}$ is by Lemma 3, it

[22]For example, $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ should have been written as $\boldsymbol{H}_{1L}$ and $\boldsymbol{H}_{2L}$.

[23]Weyl's inequality states that $\lambda_p(B+C) \le \lambda_{p+q}(B) + \lambda_{n-q}(C)$ for $q \in \{0, 1, 2, ..., n-p\}$ and $\lambda_p(B+C) \ge \lambda_{p-q+1}(B) + \lambda_q(C)$ for $q \in \{1, 2, ..., p\}$ where B and C are $n \times n$ symmetric matrices. See Theorem 4.3.1 and Corollary 4.3.3 of Horn and Johnson (2013).

follows that $\boldsymbol{H}_1$ is positive definite, and thus all of its eigenvalues are also positive. Therefore, $\boldsymbol{H}_1$ acts as the destabilizing force against interior distribution. On the other hand, because h_i is a decreasing function, all of $\boldsymbol{H}_2$'s eigenvalues, except for one zero eigenvalue, are negative, and thus $\boldsymbol{H}_2$ acts as the stabilizing force. The threshold value is attained when those two forces are balanced:

Proposition 1. *A population distribution* $\boldsymbol{n} \in \Delta$ *such that* $\operatorname{supp} \boldsymbol{n} = L$ *cannot be a stable spatial equilibrium if* $\tau > \min_{2 \le j \le |L|-1} \lambda_{j-1}(\operatorname{diag}[(|h_i'(n_i)|)_{i \in L \setminus \{k\}}]) / \lambda_{|L|-j}(\boldsymbol{H}_1)$.

To closely examine the instability condition above, we consider the linear cost ($d_{ij} = |x_i - x_j|$) and the exponential cost ($d_{ij} = e^{|x_i - x_j|} - 1$). Moreover, to abstract away from the spatial variation of $h_i'(n_i)$, we assume $h_i'(n_i) = -\alpha K/A$ for any $i \in S$.[24] Then, we can see that $\boldsymbol{H}_L$ is independent of the population distribution, and $\boldsymbol{H}_L = \boldsymbol{H}_{L'}$ whenever $L, L' \in \mathcal{S}_C$ and $|L| = |L'|$. Thus, we may focus on the number of populated cities in a spatial equilibrium. The following corollaries give the explicit expressions of threshold values of τ for each case:

Corollary 1.1. *Suppose* $h_i'(n_i) = -\alpha K/A$ *and* $d_{ij} = |x_i - x_j|$. *Then, a population distribution* $\boldsymbol{n} \in \Delta$ *having R populated cities cannot be a stable spatial equilibrium if*

$$\tau > \tau^l(R) \equiv \left(1 - \cos\frac{2\pi}{2R+1}\right)\frac{\alpha K^2}{A}. \tag{26}$$

Corollary 1.2. *Suppose* $h_i'(n_i) = -\alpha K/A$ *and* $d_{ij} = e^{|x_i - x_j|} - 1$. *Then, a population distribution* $\boldsymbol{n} \in \Delta$ *having R populated cities cannot be a stable spatial equilibrium if*

$$\tau > \tau^e(R) \equiv \frac{1}{e^{2/K} - 1}\left(1 + e^{2/K} - 2e^{1/K}\cos\frac{2\pi}{R-1}\right)\frac{\alpha K}{A}. \tag{27}$$

There are two remarks here. First, because $\tau^e(R)$ and $\tau^l(R)$ are decreasing in R, *the maximum possible number of populated cities that might constitute a stable spatial equilibrium is decreasing in* τ in either of exponential and linear cases. Second, it follows that $\tau^e(K)$ and $\tau^l(K)$ are increasing in K whereas $\tau^e(K) \to \frac{\alpha}{2A}(1 + 4\pi^2)$ and $\tau^l(K) \to \alpha\pi^2/(2A)$ as $K \to \infty$. Therefore, *if* τ *is sufficiently large, a population distribution with full support cannot be a stable spatial equilibrium for any finite K.*

[24] This can be induced through Mossay and Picard's (2011) model. See Section 3.

3.2 Multiplicity of Spatial Equilibria

We have seen how the potential function approach generally works for discrete-space social interaction models, and, as an illustration, we obtained an instability condition with respect to τ. In deriving a sufficient condition for the statement that a population distribution *cannot* be a stable spatial equilibrium, we do not have to guarantee that a population distribution is indeed a spatial equilibrium. However, if we are interested in equilibrium properties such as the multiplicity and stability of equilibria, we have to demonstrate that population distributions under consideration are actually spatial equilibria. Therefore, there would be no hope for attaining analytical observations under a general environment.

Thus, in what follows, to get clear insights into the equilibrium properties of the discrete-space model, we adopt Mossay and Picard's (2011) specification in which $f_i(x) = -\frac{\alpha}{2x}$ and $d_{ij} = |x_i - x_j|$, and exploit its linear structure. Indeed, under these assumptions, we have

$$h_i(n_i) = f_i\left(\frac{A}{n_i K}\right) - \frac{A}{n_i K} f_i'\left(\frac{A}{n_i K}\right) = -\alpha n_i K/A \tag{28}$$

for all $i \in S$, and therefore the net utility from land at equilibrium is linear in n.

In this section, we compare the equilibrium properties of our model with those of Mossay and Picard's continuous-space model. As mentioned in the introduction, a symmetric unimodal population distribution is attained as the unique spatial equilibrium in their model. Invoking the argument above, we would like to see whether the uniqueness result is robust in terms of the discretization of space.

Because the total number of cities is exogenously given, we are particularly interested in the *essential* multiplicity of equilibria in terms of population distributions restricted to the support of spatial equilibria. As we observed, all spatial equilibria exhibit the same population distribution over their supports, as long as the number of populated cities is the same. For example, although there are three stable equilibria in Figure 2(b), they should be regarded as qualitatively the same. Therefore, we aim to find cases in which spatial equilibria with different numbers of populated cities simultaneously exist.

Now that we are interested in the *existence* of multiple equilibria, we need to examine equilibrium conditions. Note that, because $\boldsymbol{v}(\boldsymbol{n}) \equiv (v_i(\boldsymbol{n}))_{i=1}^K$ is linear in $\boldsymbol{n}$, the distribution over the support of a spatial equilibrium solves a system of linear equations. To simplify notations, we focus on population distribution having full support without loss of generality. Then, observing that (2) can be expressed in matrix form $\tau \boldsymbol{D}\boldsymbol{n}$, payoff vector $\boldsymbol{v}(\boldsymbol{n})$ is written as

$$\boldsymbol{v}(\boldsymbol{n}) = Y\mathbf{1} - \tau \boldsymbol{D}\boldsymbol{n} - \frac{\alpha K}{A}\boldsymbol{E}\boldsymbol{n} = Y\mathbf{1} - \boldsymbol{C}\boldsymbol{n} \tag{29}$$

where $\boldsymbol{E}$ is the identity matrix with an appropriate dimension and

$$\boldsymbol{C} = \tau \boldsymbol{D} + \frac{\alpha K}{A}\boldsymbol{E}. \tag{30}$$

Because $\boldsymbol{n}$ is a spatial equilibrium, there exists $v^* \in \mathbb{R}$ such that $v_i(\boldsymbol{n}) = v^*$ for all $i \in S$. Furthermore, the equilibrium value of $w \equiv Y - v^*$ is given by $(\mathbf{1}'\boldsymbol{C}^{-1}\mathbf{1})^{-1} \in \mathbb{R}$ because

$$w\,\mathbf{1} = \boldsymbol{C}\boldsymbol{n} \Rightarrow w\,\mathbf{1}'\boldsymbol{C}^{-1}\mathbf{1} = \mathbf{1}'\boldsymbol{n} = 1$$

where the prime means the transpose of vector or matrix. Therefore, $\boldsymbol{n}$ solves

$$\boldsymbol{C}\boldsymbol{n} = \left(\mathbf{1}'\boldsymbol{C}^{-1}\mathbf{1}\right)^{-1}\mathbf{1}. \tag{31}$$

Note that the analogue argument holds for support $L \subseteq S$ if matrices and vectors are restricted to L. This implies that a spatial equilibrium with support L is generically unique if it exists.[25] Thus, the number of equilibria is at most one for each $L \subseteq S$, and therefore, the set of spatial equilibria is finite. Furthermore, recall that populated cities in a spatial equilibrium are congregated by Lemma 3. Therefore, we can see that the number of spatial equilibria having R populated cities is $K - R + 1$ if they exist where $1 \leq R \leq K$. By invoking index theory, we then obtain the following result:

Lemma 4. *If there is a spatial equilibrium* $\boldsymbol{n}$ *such that* $|\operatorname{supp} \boldsymbol{n}| < K$, *then there is another spatial equilibrium* $\boldsymbol{n}'$ *such that* $|\operatorname{supp} \boldsymbol{n}'| \neq |\operatorname{supp} \boldsymbol{n}|$.

Thus, if a spatial equilibrium having some unpopulated cities exists, then there is necessarily another spatial equilibrium that is essentially different from the equilibrium. Therefore, the only situation in which the (essential) multiplicity of equilibria will not arise is when the spatial equilibrium with full support uniquely exists.

[25]For spatial equilibria, we have to address unpopulated cities in addition to (31).

We illustrate the multiplicity of spatial equilibria by finding cases when a spatial equilibrium with full support cannot be stable even if it exists. Since every admissible evolutionary dynamics converges to a spatial equilibrium, if the spatial equilibrium with full support exists but is unstable, an admissible dynamic starting in the unstable manifold converges to another equilibrium that must have a different number of populated cities.

However, in view of Corollary 1.1, we already know that a population distribution with full support cannot represent a stable spatial equilibrium if $\tau > \tau^l(K)$ where $\tau^l(K)$ is given by (26). Therefore, we conclude the following result:

Proposition 2. *Suppose $h_i'(n_i) = -\alpha K/A$ and $d_{ij} = |x_i - x_j|$. Then, the spatial equilibrium is essentially non-unique if $\tau > \tau^l(K)$.*

As we observed, $\tau^l(K)$ is increasing in K but converges to $\frac{\alpha\pi^2}{2A}$ as $K \to \infty$. Thus, if $\tau > \frac{\alpha\pi^2}{2A}$, the spatial equilibrium is essentially non-unique for *any* finite K.

4 The Limit of Discrete-Space Models

We investigated the equilibrium properties of discrete-space model in the previous section, but we have not studied any potential connections between discrete-space and continuous-space models. In particular, a natural question to ask is whether a sequence of the discrete-space model's spatial equilibria converges to the unique equilibrium of a continuous-space model as the number of cities goes to infinity while the size of a region is fixed (or the distance between adjacent cities vanishes). In this section, we provide a positive answer to this question. In fact, we show that *any* sequence of spatial equilibria in a discrete space converges to a single equilibrium in a continuous space.

In Mossay and Picard's (2011) model, the unique equilibrium has $(-b, b) \subseteq \mathbb{R}$ as its support where $b = \frac{\pi}{2}\sqrt{\frac{\alpha}{2\tau}}$. To make our analysis compatible with theirs, we assume that the region is given by $[-c, c]$ where $b < c$ and the location of city i is $x_i^K = \frac{2c}{K}\left(i - \frac{1}{2}\right) - c$ for $i \in S$.[26] Moreover, because they assume that the land density is uniformly one, we let $A = 2c$.

[26]While Mossay and Picard consider the real line for the region, we consider a finite interval. One might think that the boundaries of our region then affect equilibrium. However, what we actually assume is that our region is finite but large enough to contain Mossay and Picard's equilibrium support. As long as the region is large enough in this sense, its boundaries do not affect equilibrium.

We start with a continuous-space model and denote the population at location x by $\phi(x)$. Mossay and Picard (2011) characterize the equilibrium conditions as

$$\phi(x) + \frac{\alpha}{2\tau}\phi''(x) = 0, \tag{32}$$

$$\phi(-b) = 0, \phi(b) = 0, \int_{-b}^{b} \phi(x)\mathrm{d}x = 1. \tag{33}$$

Note that, because the general solution of (32) is an even function, $\phi(-b) = 0 \Leftrightarrow \phi(b) = 0$. Hence, it suffices to impose $\phi(-b) = 0$. Moreover, integrating both sides of (32) over $[-b, b]$ and invoking the population constraint $\int_{-b}^{b} \phi(x)\mathrm{d}x = 1$, we have $\phi'(-b) - \phi'(b) = 2\tau/\alpha$. Then, because $\phi'(x)$ is an odd function, $\phi'(-b) = -\phi'(b) = \tau/\alpha$. Therefore, the conditions reduce to:

$$\phi(x) + \frac{\alpha}{2\tau}\phi''(x) = 0, \tag{34}$$

$$\phi'(-b) = \frac{\tau}{\alpha}, \tag{35}$$

$$\phi(-b) = 0. \tag{36}$$

We would like to show that the equilibrium conditions of a discrete-space model converge to the above ones as $K \to \infty$. To this end, let us take a sequence of spatial equilibria, and let $\boldsymbol{n}^K$ be the population distribution restricted to the support of spatial equilibrium when the total number of cities is K. By Lemma 3, we may assume that the support of the equilibrium is

$$L^K = \left\{\bar{\ell}^K, \bar{\ell}^K + 1, \bar{\ell}^K + 2, ..., \bar{\ell}^K + R^K - 1\right\}$$

where $\bar{\ell}^K, \bar{\ell}^K + R^K - 1 \in S$. Let $\varepsilon = 2c/K$. In what follows, we approximate $\phi(x_i^K)$ by $\phi_i^K \equiv n_i^K/\varepsilon$ that is interpreted as the population density in city i. The following lemma summarizes equilibrium conditions that $\boldsymbol{n}^K$ has to satisfy:

Lemma 5. *Suppose that* $\boldsymbol{n}^K$ *is a population distribution over* $L^K \subseteq \{1, 2, ..., K\}$ *that is the* *support of a spatial equilibrium. Then, it solves*

$$\phi_j^K + \frac{\alpha}{2\tau\varepsilon^2}(\phi_{j-1}^K - 2\phi_j^K + \phi_{j+1}^K) = 0 \quad \text{for } j \in \left\{\bar{\ell}^K + 1, \bar{\ell}^K + 2, ..., \bar{\ell}^K + R^K - 2\right\}, \tag{37}$$

$$\varepsilon\phi_{\bar{\ell}^K}^K + \frac{\alpha}{2\tau\varepsilon}(\phi_{\bar{\ell}^K+1}^K - \phi_{\bar{\ell}^K}^K) = \frac{1}{2}, \tag{38}$$

$$\phi_{\bar{\ell}^K}^K + \phi_{\bar{\ell}^K+R-1}^K \leq \frac{2\tau\varepsilon}{\alpha}. \tag{39}$$

Note that, because $x_{j+1}^K - x_j^K = \varepsilon$, (37) becomes (34), whereas (38) becomes

$$\phi'(x_{\bar{\ell}}) = \frac{\tau}{\alpha} \tag{40}$$

as K goes to infinity or ε goes to zero where $x_{\bar{\ell}} = \lim_{K\to\infty} x_{\bar{\ell}^K}^K$. Moreover, because each of $\phi_{\bar{\ell}^K}^K$ and $\phi_{\bar{\ell}^K+R-1}^K$ are nonnegative, (39) becomes

$$\phi(x_{\bar{\ell}}) = 0 \tag{41}$$

as $K \to \infty$. Therefore, the limiting population distribution solves differential equation (34) with boundary conditions (40) and (41). Thus, the equilibrium conditions of population distribution with support L^K converge to the equilibrium conditions in the continuous space only when $x_{\bar{\ell}} = -b$.

However, it follows that this is always true as long as we take a sequence of spatial equilibria. Indeed, if $x_{\bar{\ell}} \neq -b$, the solution to differential scheme (34), (40), and (41) does not satisfy the population constraint (i.e., $\int \phi(x)\mathrm{d}x \neq 1$). This means that the population constraint does not hold either when K is sufficiently large, but this contradicts the fact that we are taking a sequence of spatial equilibria. In other words, we cannot take a sequence of spatial equilibria such that the support does not converge to $(-b, b)$. Therefore, equilibrium conditions (37)-(39) converge to equilibrium conditions (34)-(36) as $K \to \infty$.

In general, though, the convergence of a discrete scheme to a differential scheme does not necessarily imply that the solution also converges.[27] However, by solving scheme (37)-(39), we can verify that the solution of scheme (37)-(39) converges to that of scheme (34)-(36) as $K \to \infty$. We thus obtain the following result:

Proposition 3. $\max_{1\leq i\leq K} \left|\phi(x_i^K) - \phi_i^K\right| \to 0$ *as* $K \to \infty$.

[27] The mathematics literature including the *finite difference method* addresses the relationship between difference and differential equations. See, for example, LeVeque (2007).

Observe that, in the argument above, the sequence of spatial equilibria is arbitrary. Thus, any sequence of spatial equilibria converges to the unique equilibrium of the continuous-space model. Recall that spatial equilibrium in a discrete space is generally not unique. In particular, when τ is large, a spatial equilibrium is essentially non-unique whenever K is finite (Proposition 2). Nevertheless, each equilibrium converges to the single equilibrium as $K \to \infty$. This means that the set of spatial equilibria parametrized by K is upper hemi-continuous at the limit. Furthermore, because the spatial equilibrium in the continuous space is unique, the lower hemi-continuity is implied by the upper hemi-continuity. Therefore, *the set of spatial equilibria is continuous in K at the limit.*

5 Conclusion

We studied the properties of discrete-space social interaction models by using the potential game approach. We showed that any sequence of the discrete-space model's equilibria converges to the unique equilibrium of the continuous-space model as the distance between adjacent cities vanishes. It is worth pointing out that this result holds even though the equilibrium of discrete-space model can be non-unique for any finite number of cities.

In this paper, we considered social interactions among a single type of agents. Thus, a natural extension is to consider multiple types of agents. There is a rich literature on (continuous-space) social interaction models having both consumers and firms.[28] Because of general equilibrium effects, the properties of equilibrium is more complex than the class of models considered here. In particular, equilibrium is generally not unique even in the continuous-space model, although the stability of equilibria has not been explored. It is difficult to determine the stability of equilibria in the continuous-space model, but we may be able to address this by approximating the model with a discrete-space model.[29]

Finally, although we did not engage in policy discussions, the spatial equilibrium of our model is generally not efficient because social interactions cause externalities. Indeed, population distribution is more concentrated at social opti-

[28]See Chapter 6 of Fujita and Thisse (2013) and references therein.

[29]Blanchet *et al*. (2013) use the potential function(al) approach to characterize equilibria of continuous-space models. Although their work has a great potential for further investigations of continuous-space models, they still abstract away from stability analysis.

mum than at market equilibrium. This is a consequence of positive externalities in social interactions, which yields under-agglomeration. Thus, to achieve a social optimum, it is necessary that the planner internalize those externalities. However, because the equilibrium under such an intervention is not necessarily unique as in a laissez-faire case, there may exist a stable equilibrium besides social optima. Therefore, in contrast to the continuous world, the policy design to achieve a social optimum in the discrete world is not straightforward because of the multiplicity of equilibria. This is an important subject of future research.[30]

Appendix

Lemma A1. $\boldsymbol{e}_i \in \Delta$ *is a spatial equilibrium if and only if* $\tau \geq \alpha K^2/A$.

Proof. $v_i(\boldsymbol{e}_i) - v_j(\boldsymbol{e}_i) = h(1) - h(0) + d_{ji}$. Therefore,

$$v_i(\boldsymbol{e}_i) \geq v_j(\boldsymbol{e}_i) \text{ for any } j \neq i \Leftrightarrow h(0) - h(1) \leq \min_{j \neq i} d_{ji} = \tau/K. \qquad \square$$

Proof of Lemma 3. Suppose to the contrary that there exists an equilibrium $\boldsymbol{n}$ in which, for some $i, j \in \operatorname{supp} \boldsymbol{n}$ with $j-i \geq 2$, $n_\ell = 0$ for all $i < \ell < j$. Let $k \in \{i+1, .., j-1\}$. Then, because $d_{i\ell} + d_{ik} \leq d_{k\ell}$ and $d_{k\ell} + d_{jk} \leq d_{j\ell}$ for $\ell \leq i$,

$$\sum_{\ell=1}^{i} d_{i\ell} n_\ell + d_{ik} \sum_{\ell=1}^{i} n_\ell \leq \sum_{\ell=1}^{i} d_{k\ell} n_\ell \leq \sum_{\ell=1}^{i} d_{j\ell} n_\ell - d_{jk} \sum_{\ell=1}^{i} n_\ell. \tag{42}$$

Similarly,

$$\sum_{\ell=j}^{K} d_{j\ell} n_\ell + d_{jk} \sum_{\ell=j}^{K} n_\ell \leq \sum_{\ell=j}^{K} d_{k\ell} n_\ell \leq \sum_{\ell=j}^{K} d_{i\ell} n_\ell - d_{ik} \sum_{\ell=j}^{K} n_\ell. \tag{43}$$

[30] Sandholm (2007) and Fujishima (2013) consider Pigouvian tax policies in the presence of multiple equilibria.

Without loss of generality, suppose $\sum_{\ell=j}^{K} n_\ell \leq \sum_{\ell=1}^{i} n_\ell$. Then,

$$\sum_{\ell=1}^{K} d_{j\ell} n_\ell \geq \sum_{\ell=1}^{K} d_{j\ell} n_\ell + \sum_{\ell=j}^{K} (d_{i\ell} - d_{j\ell} - d_{ik} - d_{jk}) n_\ell \tag{44}$$

$$\geq \sum_{\ell=1}^{K} d_{j\ell} n_\ell + \sum_{\ell=j}^{K} (d_{i\ell} - d_{j\ell} - d_{ik}) n_\ell - d_{jk} \sum_{\ell=1}^{i} n_\ell \geq \sum_{\ell=1}^{K} d_{k\ell} n_\ell. \tag{45}$$

Therefore,

$$h_i(n_i) - h_j(n_j) - \sum_{\ell=1}^{K} (d_{i\ell} - d_{j\ell}) n_\ell > h_i(n_i) - h_j(0) - \sum_{\ell=1}^{K} (d_{i\ell} - d_{j\ell}) n_\ell$$

$$\geq h_i(n_i) - h_k(0) - \sum_{\ell=1}^{K} (d_{i\ell} - d_{k\ell}) n_\ell \geq 0 \quad \because \; v_i(\boldsymbol{n}) \geq v_k(\boldsymbol{n}).$$

But this contradicts $i, j \in \operatorname{supp} \boldsymbol{n}$ (i.e., $v_i(\boldsymbol{n}) = v_j(\boldsymbol{n})$). □

Proof of Proposition 1. Because $\boldsymbol{D}$ is conditionally negative definite, it follows from Lemma 3 that $\boldsymbol{D}_L$ is also conditionally negative definite, and this further implies that $\boldsymbol{H}_1$ is positive definite. Thus, all of $\boldsymbol{H}_1$'s eigenvalues are positive. On the other hand, the eigenvalues of $h'_k(n_k)\mathbf{1}'\mathbf{1}$ are $(|L|-1)h'_k(n_k)$ and 0, so the matrix has exactly one negative eigenvalue because $h'_i(n) < 0$ for any $i \in S$. Thus, by Weyl's inequality, $\lambda_i(\boldsymbol{H}_2) \geq \lambda_{i-1}(\operatorname{diag}[(h'_i(n_i))_{i \in L \setminus \{k\}}])$.

Then, by invoking Weyl's inequality for $\tau \boldsymbol{H}_1 + \boldsymbol{H}_2$, we obtain

$$\lambda_{\max}(\boldsymbol{H}_L) \equiv \lambda_{|L|-1}(\boldsymbol{H}_L) \geq \tau \lambda_{|L|-j}(\boldsymbol{H}_1) + \lambda_j(\boldsymbol{H}_2) \tag{46}$$

$$\geq \tau \lambda_{|L|-j}(\boldsymbol{H}_1) + \lambda_{j-1}(\operatorname{diag}[(h'_i(n_i))_{i \in L \setminus \{k\}}]) \tag{47}$$

where $2 \leq j \leq |L| - 1$. Because $n_i > 0$ for all $i \in L$, $\lambda_{j-1}(\operatorname{diag}[(h'_i(n_i))_{i \in L \setminus \{k\}}]) \in (-\infty, 0)$ for $2 \leq j \leq |L| - 1$. Therefore, we obtain the stated result. □

Proof of Corollaries 1.1 and 1.2. Suppose $h'_i(n_i) = -\alpha K / A$. Because the Hessian does not depend on population distribution, we may focus on the number of populated cities by letting $\boldsymbol{H}_L = \boldsymbol{H}_R$ for any $L \in \mathcal{S}_C$ such that $|L| = R$.

Suppose $d_{ij} = |x_i - x_j|$. In this case, we can directly compute the inverse of $\boldsymbol{H}_1$ as

$$\boldsymbol{H}_1^{-1} = \frac{K}{2}\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{pmatrix}. \tag{48}$$

This is an $(R-1)\times(R-1)$-dimensional tridiagonal Toeplitz matrix where the lower right corner is perturbed. Yueh and Cheng (2008) attain explicit expressions for the eigenvalues of this class of matrices. Invoking their results, it follows that $\lambda_p(\boldsymbol{H}_1^{-1}) = K\left(1 - \cos\frac{2p\pi}{2R+1}\right)$. Thus, $\lambda_p(\boldsymbol{H}_1) = \frac{1}{K}\left(1 - \cos\frac{2(R-p)\pi}{2R+1}\right)^{-1}$. Then, because $\lambda_{\max}(\boldsymbol{H}_R) \geq \tau\lambda_{R-1}(\boldsymbol{H}_1) - \alpha K/A$, we obtain $\tau^l(R)$.

Next, suppose $d_{ij} = e^{|x_i - x_j|} - 1$, and let $\gamma = \exp(1/K)$. Then, $\boldsymbol{D}_R = \boldsymbol{\Gamma}_R - \mathbf{1}'\mathbf{1}$ where

$$\boldsymbol{\Gamma}_R = \begin{pmatrix} 1 & \gamma & \gamma^2 & \cdots & \gamma^{R-2} \\ \gamma & 1 & \gamma & \cdots & \gamma^{R-3} \\ \gamma^2 & \gamma & 1 & \cdots & \gamma^{R-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma^{R-2} & \gamma^{R-3} & \gamma^{R-4} & \cdots & 1 \end{pmatrix}. \tag{49}$$

Unfortunately, the eigenvalues of $\boldsymbol{H}_1$ are no longer easily attainable, as opposed to the linear case. Thus, instead, we obtain the eigenvalues of $\boldsymbol{\Gamma}_R$, and attain a condition stronger than the one in Proposition 1.

The inverse of $\boldsymbol{\Gamma}_R$ is

$$\boldsymbol{\Gamma}_R^{-1} = \frac{1}{1-\gamma^2}\begin{pmatrix} 1 & -\gamma & & & \\ -\gamma & 1+\gamma^2 & -\gamma & & \\ & \ddots & \ddots & \ddots & \\ & & -\gamma & 1+\gamma^2 & -\gamma \\ & & & -\gamma & 1 \end{pmatrix}. \tag{50}$$

This is a tridiagonal Toeplitz matrix where the upper left and lower right corners are perturbed. On the basis of the results of Yueh and Cheng (2008), we have $\lambda_p(-\boldsymbol{\Gamma}_R^{-1}) =$

$\frac{1}{\gamma^2-1}\left(1+\gamma^2-2\gamma\cos\frac{(p-1)\pi}{R-1}\right)$, and thus $\lambda_p(-\boldsymbol{\Gamma}_R) = (\gamma^2-1)\left(1+\gamma^2-2\gamma\cos\frac{(R-p-1)\pi}{R-1}\right)^{-1}$. On the other hand, the eigenvalues of $\mathbf{1}'_{R-1}\mathbf{1}_{R-1}$ are $R-1$ and 0. Thus, the matrix does not have a negative eigenvalue, and hence $\lambda_i(-\boldsymbol{D}_R) \geq \lambda_i(-\boldsymbol{\Gamma}_R)$.

The eigenvalues of $\boldsymbol{d}_{kL}\otimes\mathbf{1}+(\boldsymbol{d}_{kL}\otimes\mathbf{1})'$ are $\sum_{j\in L\setminus\{k\}} d_{kj} \pm \sqrt{R-1}\sqrt{\sum_{j\in L\setminus\{k\}} d_{kj}^2}$ and 0. By Hölder's inequality, $\sum_{j\in L\setminus\{k\}} d_{kj} \leq \sqrt{R-1}\sqrt{\sum_{j\in L\setminus\{k\}} d_{kj}^2}$, thus the matrix has at most one negative eigenvalue. Hence, $\lambda_i(\boldsymbol{H}_1) \geq \lambda_{i-1}(-\boldsymbol{D}_R)$. Then, because $\lambda_{\max}(\boldsymbol{H}_R) \geq \tau\lambda_{R-2}(\boldsymbol{H}_1) + \lambda_1(\mathrm{diag}[(h_i'(n_i))_{i\in L\setminus\{k\}}]) \geq \tau\lambda_{R-3}(-\boldsymbol{\Gamma}_R) - \alpha K/A$, we obtain $\tau^e(R)$. □

Proof of Lemma 4. Suppose, to the contrary, that every spatial equilibrium has R populated cities where $R < K$. To show the result, we use index theory. Define the index of a spatial equilibrium having L as its support by

$$\mathrm{ind}_L = \begin{cases} -1 & \text{if } \det\boldsymbol{H}_L > 0, \\ 0 & \text{if } \det\boldsymbol{H}_L = 0, \\ 1 & \text{if } \det\boldsymbol{H}_L < 0, \end{cases} \tag{51}$$

where $\det\boldsymbol{H}_L$ is the determinant of $\boldsymbol{H}_L$. Then, indices of each spatial equilibria must sum up to one by the index theorem of Simsek *et al.* (2007).[31] However, because $\boldsymbol{H}_L = \boldsymbol{H}_{L'}$ whenever $L, L' \in \mathcal{S}_C$ and $|L| = |L'|$, it follows from Lemma 3 that the total value of indices of spatial equilibria having R populated cities is either $K-R+1$, $-(K-R+1)$, or 0. In either case, it is not one. □

Proof of Lemma 5. To simplify notations, we omit superscript k of $L^K, \bar{\ell}^K$, and R^K. Recall from Section 3.1 that $\boldsymbol{n}^K$ must solve a system of linear equations. Specifically, multiplying both sides of $\boldsymbol{C}_L\boldsymbol{n}^K = w^K\mathbf{1}$ by $\frac{1}{\tau}\boldsymbol{D}_L^{-1}$ from the left, $\boldsymbol{n}^K$ solves

$$\left(\boldsymbol{E} + \frac{\alpha}{\tau\varepsilon}\boldsymbol{D}_L^{-1}\right)\boldsymbol{n}^K = w^K\boldsymbol{D}_L^{-1}\mathbf{1} \tag{52}$$

[31] Simsek *et al.* (2007) establish the index theorem that is applicable to the KKT set of nonlinear programming (See, in particular, Proposition 5.2). Their theorem is relevant to us because the set of spatial equilibria coincides with that of KKT points of the potential's maximization problem.

where $w^K = (\mathbf{1}'\boldsymbol{C}_L^{-1}\mathbf{1})^{-1}$. Note that, because $|L| = R$,

$$\boldsymbol{D}_L^{-1} = \frac{1}{2\varepsilon}\begin{pmatrix} \frac{1}{R-1} - 1 & 1 & & & & \frac{1}{R-1} \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ \frac{1}{R-1} & & & & 1 & \frac{1}{R-1} - 1 \end{pmatrix}. \tag{53}$$

Then, we have

$$\varepsilon\phi_j^K + \frac{\alpha}{2\tau\varepsilon}\left(\phi_{\kappa(j)}^K - \phi_j^K + \frac{1}{R-1}(\phi_{\bar{\ell}}^K + \phi_{\bar{\ell}+R-1}^K)\right) = \frac{1}{\tau\varepsilon}\frac{1}{R-1}w^K \text{ for } j \in \{\bar{\ell}, \bar{\ell}+R-1\}, \tag{54}$$

$$\phi_j^K + \frac{\alpha}{2\tau\varepsilon^2}(\phi_{j-1}^K - 2\phi_j^K + \phi_{j+1}^K) = 0 \quad \text{for } j \in \{\bar{\ell}+1, \bar{\ell}+2, ..., \bar{\ell}+R-2\}, \tag{55}$$

where $\kappa(\bar{\ell}) = \bar{\ell} + 1$ and $\kappa(\bar{\ell} + R - 1) = \bar{\ell} + R - 2$. Summing the first and last rows of $\boldsymbol{C}_L\boldsymbol{n}^K = w^K\mathbf{1}$ in each of left-hand and right-hand sides, we have

$$w^K = \frac{\tau\varepsilon}{2}(R-1) + \frac{\alpha}{2}(\phi_{\bar{\ell}}^K + \phi_{\bar{\ell}+R-1}^K). \tag{56}$$

Substituting this into (54), we obtain

$$\varepsilon\phi_{\bar{\ell}}^K + \frac{\alpha}{2\tau\varepsilon}(\phi_{\bar{\ell}+1}^K - \phi_{\bar{\ell}}^K) = \frac{1}{2}. \tag{57}$$

The analogue relationship holds for $j = \bar{\ell} + R - 1$. Moreover, because $\bar{\ell} - 1, \bar{\ell} + R \notin \operatorname{supp}\boldsymbol{n}^K$, $\sum_{j=1}^R jn_{\bar{\ell}-1+j}^K \geq w^K$ and $\sum_{j=1}^R(\bar{\ell}+1-j)n_{\bar{\ell}-1+j}^K \geq w^K$. Hence, by (56),

$$2w^K - \tau\varepsilon(R+1) = \alpha(\phi_{\bar{\ell}}^K + \phi_{\bar{\ell}+R-1}^K) - 2\tau\varepsilon \leq 0. \tag{58}$$

Therefore, the equilibrium conditions are summarized as (37)-(39). □

Proof of Proposition 3. Multiplying the LHS of (37) by $2\tau\varepsilon^2/\alpha$, we get

$$\phi_{i+1}^K - 2a^K\phi_i^K + \phi_{i-1}^K = 0. \tag{59}$$

where $a^K = 1 - \tau\varepsilon^2/\alpha$. It follows that the solution property crucially depends on the sign of $D^K = (a^K)^2 - 1$. Because we are interested in the case where K is sufficiently large, we assume $D^K < 0$. Then, the solution is represented as

$$\phi_i^K = C_1^K \cos(\omega^K i) + C_2^K \sin(\omega^K i) \tag{60}$$
$$= C^K \cos(\omega^K i - \xi^K) \tag{61}$$

where $C^K = \sqrt{(C_1^K)^2 + (C_2^K)^2}$, $\omega^K = \cos^{-1}(a^K)$ with $0 \leq \omega^K < 2\pi$, and ξ^K satisfies $C^K \cos \xi^K = C_1^K$ and $C^K \sin \xi^K = C_2^K$. Because ϕ_i^K, which is given by cosine function (61), is nonnegative for all $i \in L$, we assume $[\omega^K \bar{\ell} - \xi^K, \omega^K(\bar{\ell} + R - 1) - \xi^K] \subseteq [-\frac{\pi}{2}, \frac{\pi}{2}]$ without loss of generality.

By symmetry, $\phi_{\bar{\ell}}^K = \phi_{\bar{\ell}+R-1}^K$. Thus, $\xi^K = \frac{\omega^K(2\bar{\ell}+R-1)}{2}$. Then, because $\varepsilon = \frac{x_{\bar{\ell}+R-1}^K - x_{\bar{\ell}}^K}{R-1}$,

$$\frac{1}{C^K}\frac{\alpha}{2\tau\varepsilon}(\phi_{\bar{\ell}+1}^K - \phi_{\bar{\ell}}^K) = \frac{\alpha}{2\tau\varepsilon}\left(\cos(\omega^K(\bar{\ell}+1) - \xi^K) - \cos(\omega^K\bar{\ell} - \xi^K)\right) \tag{62}$$
$$= \frac{\alpha\omega^K(R-1)}{2\tau(x_{\bar{\ell}+R-1}^K - x_{\bar{\ell}}^K)} \frac{\sin\left(\frac{\omega^K(R-2)}{2}\right)\sin\left(\frac{\omega^K}{2}\right)}{\omega^K/2}. \tag{63}$$

Note that $\lim_{K\to\infty} \phi_{\bar{\ell}+R-1}^K = 0 \Rightarrow \lim_{K\to\infty} \omega^K(R-1) = \pi$. Moreover, $\lim_{K\to\infty} \omega^K = 0$ because $a^K \to 1$ as $K \to \infty$. Therefore, by (38), $C^K \to \frac{\tau(x_{\bar{\ell}+R-1} - x_{\bar{\ell}})}{\alpha\pi}$ as $K \to \infty$.

Now, fix location x. Then,

$$C^K \cos\left[\omega^K\left(\frac{1+K}{2} + \frac{x}{\varepsilon}\right) - \xi^K\right] = C^K \cos\left[\frac{\omega^K}{\varepsilon}\left(x - \frac{x_{\bar{\ell}+R-1}^K + x_{\bar{\ell}}^K}{2}\right)\right] \tag{64}$$
$$\to \frac{\tau(x_{\bar{\ell}+R-1} - x_{\bar{\ell}})}{\alpha\pi} \cos\left(\frac{\pi}{2}\frac{2x - x_{\bar{\ell}+R-1} - x_{\bar{\ell}}}{x_{\bar{\ell}+R-1} - x_{\bar{\ell}}}\right) \tag{65}$$

as $K \to \infty$. Then, because $x_{\bar{\ell}+R-1} = b$ and $x_{\bar{\ell}} = -b$, we obtain $\frac{\pi}{4b}\cos\left(\frac{\pi}{2b}x\right)$. This is the solution to scheme (34)-(36), and we thus complete the proof. □

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